



Collaborative inquiry into mathematics teaching

Developing a partnership in researching
practice in primary grades and teacher
education

Jónína Vala Kristinsdóttir

Dissertation submitted in partial fulfilment of a Ph.D.-degree



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and teacher education*

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Abstract

Over the last two decades Icelandic teachers have been under growing pressure to adapt their work to changes in the new curriculum guidelines and laws for schools. Teachers are now expected to meet the needs of diverse groups of children and improve their teaching competence.

The main purpose of the study was to deepen the understanding of teacher development in mathematics teaching. The investigation is in two parts.

The goal in the first part is self-study of teacher education practices and focussed on how a teacher educator, who is the researcher, reviewed her own understanding of learning theories and practitioner research, and how this influenced and shaped the second part of study. This study took the form of autobiographical study where the teacher educator used retrospective data to reflect on how her understanding of constructivist and socio-cultural theories, have affected her teaching for forty years. This journey through her experience of practitioner research resulted in her choosing the methodology of developmental research for the second part and positioning it within the socio-cultural theoretical framework.

In the second part the researcher recorded and analysed the progress made in a developmental study in which seven teachers in primary grades and a mathematics teacher educator collaborated on investigating their teaching. In particular, the aim was to investigate how the teachers and the teacher educator collaborated in researching their own practice, and the ways in which this collaboration impacted the work of both parties. The goal was to identify approaches to teacher education that could support pre-service and in-service teachers in meeting the needs of diverse learners in the mathematics classroom.

A model of a developmental research cycle was used as the framework for the research. In this model there are two interconnected cycles of development and research that model a linked dialectical growth of theory and practice. As the collaborative group moved through the developmental cycle teachers reflected on their work in terms of learning theories. The research cycle was guided by well-known socio-cultural theories (often called global theories) and by proposing and working with theories of teacher development emerging in school (often called local theories).

The global theories that guided the development of a learning community are socio-cultural: (1) learning is understood as social participation, (2) inquiry and critical reflection are the means for developing practice, (3) boundaries between the communities of practice the teachers belong to within their schools are expected to affect the extent to which the teachers will align themselves to a new community, (4) participants collaborate in inquiring in mathematics teaching and learning, (5) meaning is mediated through discussions and collaborative analysis of mathematical problems and of pupils ways of solving them, and (6) participants support each other in developing their mathematics teaching, the potential for improving practice is cultivated and constraints resolved.

The local theories with which the group worked were based on the researcher's former findings from working with teachers. The findings revealed that teachers lack confidence in cultivating their pupils' competences in mathematical thinking, expressing themselves in terms of mathematics, making mathematical connections and communicating their thinking.

For three years the teachers and the teacher educator participated in 17 collaborative workshops where they: (1) solved mathematical problems together, discussed their own understanding of them and what children might learn from solving them (2) reflected on their own teaching and pupils' learning in their classrooms, and (3) discussed research on mathematics teaching and learning and what could be learned from them. The workshops were planned on the outcomes of previous workshops and specific requests by teachers.

The analysis of the results from the developmental project was based on a grounded theory approach to coding. A narrative inquiry was adopted in analysing the development of the four teachers who participated in all steps of the project. When analysing her own learning from planning and carrying out the project the teacher educator adopted a methodology of self-study of teacher education practices.

Through the project a learning community was established in which teachers and a teacher educator negotiated their understanding of mathematics teaching and learning through collaborative inquiry. Diverse views of effective ways of inquiring into one's own practice resulted in teachers using methods with which they were confident. Conflicts about effective ways of learning mathematics were resolved by careful considerations of children's diverse ways of learning. When the teachers were empowered to rethink their own way of solving mathematical

problems they started to pay attention to the communication in their classrooms and focus on the learning of their pupils. In tackling concerns on diversity the participants refined their perception of the meaning and practice of inclusion in activities in the mathematics classroom. As the project progressed the teachers gradually took the lead in deciding what to focus on at the workshops.

The results of the collaborative inquiry in the workshops showed that partnership between teachers in schools and teacher educators, where the knowledge both parties bring into the project is mutually respected, can add to our understanding of teacher development.

In the narrative inquiry it was found that the teachers gained confidence in reflecting on their practices and in researching into their teaching. Their teaching habits changed as by the end of the project they placed more emphasis on inquiry based approaches as opposed to their emphasis on the outset on highlighting fluency in carrying out traditional algorithms. One major concern was how to resolve conflicts concerning established norms about mathematics teaching and learning. The results of this project showed that in future research conflicts from the advent of new communities can be resolved through close collaboration with the communities of practices already present within the teachers' schools.

The main conclusion of the self-study into my own practice was that through using the developmental process carefully I realised that it was essential for teachers to be aligned with the professional communities within their schools and this obligation impacts their potential for professional learning. The learning gained from this three year long project, carried out with teachers, also confirmed my belief about teacher development that aims at inclusive practices and mutual understanding. Teachers require opportunities to develop and enhance their knowledge about teaching and learning in an environment that reflects the very same aspects they are expected to foster in their own classrooms.

The overall results indicate that teachers are professionals who can work at developing their mathematics teaching in order to cultivate inquiry in mathematics within their classrooms when provided with support on discussing and interpreting their work in classrooms. The findings support the view that teachers' opportunities for further empowerment to participate in educational research needs to be facilitated. Their contribution to knowledge about educational practices should be incorporated into the general body of knowledge about teaching and learning mathematics.

Ágrip

Námssamfélag um stærðfræðikennslu: Að þróa samvinnurannsókn um kennslu í grunnskóla og kennaramenntun

Undanfarna tvo áratugi hafa kennarar á Íslandi þurft að laga starf sitt að breyttum áherslum í námskrám og lögum um skóla. Þeim er ætlað að mæta þörfum ólíkra nemendahópa og þróa kennsluhætti sína til að geta tekist á við þær kröfur sem til þeirra eru gerðar.

Megintilgangur rannsóknarinnar var að afla þekkingar og öðlast skilning á starfsþróun kennara sem endurskoða stærðfræðikennslu sína. Rannsóknin er í tveimur hlutum.

Markmiðið með fyrri hlutanum er rannsókn á eigin starfi í kennaramenntun þar sem sjónum var beint að því hvernig kennari í stærðfræðimenntun, sem er rannsakandinn, endurskoðaði eigin skilning á námskenningum og starfendarannsóknum og áhrif þess á og mótun seinni hluta rannsóknarinnar. Aðferðum lífssögurannsókna var beitt við að greina gögn frá fyrri rannsóknum og ígrunda hvernig skilningur á kenningum um hugsmíði og félags- og menningarkenningar hafa haft áhrif á kennslu rannsakandans á fjörutíu ára kennsluferli. Þessi könnunarleiðangur leiddi til þess að aðferðafræði þróunarrannsókna (e. developmental research) var valin fyrir seinni hluta rannsóknarinnar og byggt á kenningagrundi félags- og menningarkenninga.

Í síðari hlutanum greinir rannsakandinn frá samvinnurannsókn sinni og sjö kennara í grunnskóla og niðurstöðum þróunarferlis þeirra. Meginmarkmiðið var að kanna hvernig grunnskólakennarar og kennari í stærðfræðimenntun unnu saman að því að rannsaka eigið starf og með hvaða hætti samvinnan hafði áhrif á starf þeirra. Stefnt var að því að bera kennsl á leiðir til að styðja kennara við að mæta þörfum ólíkra nemenda við stærðfræðináms- og öðlast skilning á hvernig nýta megi þær til að bæta kennaramenntun.

Stuðst var við líkan um hringferli þróunarrannsókna sem byggir á samtvinnum ferlum rannsóknar og þróunar sem endurspeglar rökræn tengsl milli fræða og framkvæmdar. Í þróunarferlinu ígrunduðu þátttakendur sameiginlega verk sín með tilvísun til kenninga um stærðfræðináms. Rannsóknarferillinn byggði annars vegar á víðtækum kenningum

um nám í samfélagi og hins vegar á kenningum um fagmennsku kennara sem hafa þróast í tengslum við skólastarf.

Byggt var á grunni félags- og menningarkenninga við að þróa náms-samfélag: (1) litið er á nám sem þáttöku í samfélagi, (2) rýnt er í nám og kennslu og ígrundað hvað af því má læra, (3) gert er ráð fyrir að þau námssamfélög sem kennarar tilheyra hafi áhrif á mótun nýs námssamfélags, (4) unnið er saman að því að rannsaka og greina stærðfræðináms- og kennslu, (5) samræður og sameiginleg greining á stærðfræðiverkefnum og lausnum nemenda eru notaðar sem tæki til að miðla merkingu og (6) þátttakendur styðja hver annan við að þróa stærðfræðikennslu og tækifæri eru sköpuð til bæta starf sitt og takast á við hindranir.

Við skipulagningu þróunarverkefnisins tók rannsakandi mið af niðurstöðum fyrri rannsókna sinna sem leiddu í ljós að kennurum finnst erfitt að nálgast stærðfræðikennslu sem stuðlar að því að nemendur öðlist hæfni til stærðfræðilegrar hugsunar, geti tjáð sig um stærðfræði, sett viðfangsefni fram á fjölbreyttan hátt og komið auga á tengsl í stærðfræði.

Í sameiginlegum vinnustofum, samtals 17, á þriggja ára tímabili, var nám þátttakenda byggt á þremur megin þáttum: (1) sameiginleg glíma við stærðfræðiþrautir og umræða um hvað nemendur læra af að takast á við verkefni, (2) ígrundun um eigin kennslu og nám nemenda í skólastofum kennaranna, (3) umræða um rannsóknir á stærðfræðinámi og -kennslu og hvað af þeim má læra. Við skipulag hvernar vinnustofu var byggt á reynslu frá fyrri vinnustofum og óskum kennara um hvað þeir vildu takast á við.

Við greiningu á niðurstöðum á þróunarferlinu var stuðst við grundaða kenningu. Frásagnarrýni var beitt við að greina þróun fjögurra kennara sem þátt tóku í öllum þrepum rannsóknainnar. Við greiningu á þróun rannsakandans við að skipuleggja og framkvæma rannsóknina var aðferðum sjálfsrýni í kennararamenntun beitt.

Þátttakendur þróuðu með sér námssamfélag þar sem samvinnurýni var beitt við að ígrunda ólíkan skilning á stærðfræðinámi- og kennslu. Ólík sjónarhorn á hvaða aðferðir henta við að rannsaka eigið starf leiddu til þess að farnar voru leiðir þar sem kennararnir fundu til öryggis. Leyst var úr ágreiningi um árangursríkar leiðir við stærðfræðináms með því að skoða og ígrunda fjölbreyttar námsleiðir barna. Þegar kennararnir efdust í að endurskoða eigin leiðir við að leysa stærðfræðiþrautir fóru þeir að beina sjónum sínum að samskiptum í skólastofum sínum og námi nemenda sinna. Þátttakendur skerptu skilning sinn á hvernig tryggja megi aðgengi ólíkra nemenda að stærðfræði með því að ræða vandamál sem upp komu í

kennslustundum þeirra. Þegar samvinnuverkefnið þróaðist tóku kennararnir smám saman frumkvæði í að ákveða hvað fengist var við á vinnustofunum.

Niðurstöður samvinnurannsóknarinnar í vinnustofunum sýna fram á að þróun námssamfélags kennara í grunnskóla og í kennaramenntun, þar sem þekkingu beggja er sýnd gagnkvæm virðing, getur aukið skilning á kennarapróun.

Af niðurstöðum rýni í starf kennaranna má ráða að þeir öðluðust öryggi í að ígrunda starf sitt og rannsaka kennslu sína. Kennsla þeirra breyttist á þá leið að þeir lögðu áherslu á rannsóknarnálgun við stærðfræðiverkefni í lokin í stað þjáfunar í reikningi eftir fyrirfram ákveðinni forskrift í upphafi. Mikilvægt er að finna leiðir til að leysa úr ágreiningi um hefðir sem hafa skapast í stærðfræðinámi- og kennslu. Leggja þarf áherslu á samstarf milli þeirra sem vinna að því að þróa námsamfélag um stærðfræðinámi og þeirra samfélaga sem kennararnir tilheyra í skólum sínum.

Meginniðurstöður mínar um hvað ég lærði af þróunarferlinu eru að ég gerði mér grein fyrir að það er mikilvægt fyrir kennara að finna að þeir tilheyra faglegu samfélagi innan skóla sinna og þessi skuldbinding hefur áhrif á tækifæri þeirra til faglegrar þróunar. Það sem ég lærði af þessari þriggja ára samvinnu við kennara hefur líka styrkt hugmyndir mínar um gefa þurfi þeim svigrúm til að þróa starf sitt. Markmiðið er að þeir geti mætt þörfum ólíkra nemenda við stærðfræðinámi og skapað þeim aðstæður til að verða virkir þáttakendur í skólastarfinu. Kennarar þurfa að fá tækifæri til að þróa og bæta við þekkingu sína á námi og kennslu í aðstæðum sem endurspeglar það umhverfi sem þeim er ætlað að skapa í eigin kennslu.

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1 Introduction

Over the last two decades Icelandic teachers have been under growing pressure to adapt their work to changes in accordance with new curriculum guidelines, laws, and regulations for schools. Teachers are expected to meet the needs of diverse groups of children due for instance to growth in immigration and the closure of special education schools. As a result, a rising number of children with special education needs attend mainstream schools (Lög um grunnskóla, 2008).

The main purpose of this study is to deepen the understanding of teacher development in mathematics teaching. In particular, the aim is to investigate how teachers in schools and mathematics teacher educators collaborate in researching their own practice, and the ways in which this collaboration, impacts the work of both parties. The goal was to identify approaches to teacher education that support teachers in facing these challenges. The goal with the research project was to examine how teachers meet new cultural and mathematical challenges in their classrooms, and how participation in a learning community with their colleagues and myself as a teacher educator, could lead to valuable changes in their work. As a teacher educator, my aim was to unfold the ways in which my earlier experiences of working with pre- and in-service teachers affected my understanding of how learning develops, individually and collectively. Furthermore, I sought to reveal how the undertaking of this research project will supplement my earlier comprehension of mathematics teaching and learning.

In 1999, a new national curriculum guide in mathematics for compulsory education¹ was put into effect in Iceland. The reformed guide was oriented under the influence of the National Council of Teachers of Mathematics Standards (Bjarnadóttir, 2010). For the first time, mathematical processes were given the same attention as mathematical content (Menntamálaráðuneytið, 1999). The guide was revised in 2007 (Menntamálaráðuneytið, 2007) with minor changes. New curriculum guidelines (Mennta og menningarmálaráðuneytið, 2013) were implemented in 2013 with a strong focus on mathematical proficiency (Kilpatrick, 2004) and mathematical competences (Niss, 2004).

¹ In Iceland, education is compulsory for ages 6 to 16, a total of 10 years

A consequence of the curriculum reform efforts from 1999 and 2013 is that the teaching of mathematics has become more demanding than before, because they call for new teacher competencies and different instructional practices (Kilpatrick, 2004; Niss, 2004). In addition to the abovementioned mathematics guidelines, there has emerged a strong focus on diversity and access to meaningful learning for all children in schools. In the 2013 guidelines, the focus on inclusive practices (Ainscow, 1995; 2007) is articulated along with six fundamental pillars: literacy, sustainability, democracy and human rights, equity, health and welfare, and creativity. These six aspects are set to support the democratic focus in the law for compulsory education in Iceland (Lög um grunnskóla, 2008).

My fellow teacher educators and I have found that if teachers are given opportunities to collaboratively investigate 'with' mathematics and solve mathematical problems, they discover how the different experiences they bring into the community can contribute to their own understanding of the mathematics involved, as well as how individuals learn mathematics (Guðjónsdóttir & Kristinsdóttir, 2006; 2007a; 2007b; 2011; Gunnarsdóttir, Kristinsdóttir, & Pálsdóttir, 2008; 2013). In our work with pre- and in-service teachers, we found that they must be offered opportunities to experience learning that enhances inclusive education. Our results correspond with those of Bredcamp (2004) and Moore (2005), who emphasised that if teachers' work is expected to be aimed at diversity and mutual understanding, they require the opportunity to develop and enhance their knowledge about teaching and learning in an environment that reflects the very same aspects that they are expected to foster in their own classrooms.

I had the chance to take part in the writing of the mathematics guidelines of the national curriculum that were brought into effect in both in 1999 and in 2013. My role in the process varied. When participating in the former I was a primary school teacher, voicing the views of teachers from the field, and in doing so, I built on reflective experiences regarding my own mathematics teaching. When writing and editing the 2013 guide, I worked as a mathematics teacher educator. My responsibility was to ensure the quality of the guidelines, and I called on specialists from the field, both university teachers and teachers in schools. In leading the writing of the mathematics chapter, I emphasised collaboration and indicated that the conclusions should reflect all participants' understanding of what is important to attend to in mathematics learning and teaching.

In the years between writing the guidelines, I finished my master's degree in education (2003), participated in writing textbooks for grades five

to seven in compulsory schools (2000-2007), based on the guidelines from 1999, and in 2003 took on a position as a mathematics teacher educator. My three-year experience of studying my own mathematics teaching in early primary grades resulted in the writing my master's thesis. Moreover, collaborative research with my colleagues within the teacher education prompted me to enter a doctoral program in mathematics education. My aim with the doctoral studies was to improve as a mathematics teacher educator. I wanted a study that would explore teacher development in mathematics teaching. However, writing a doctoral thesis is an individual task that is carried out with the support of supervisors and teachers within the doctoral program, but the research process itself and the responsibility for the outcomes belong solely to the doctoral student. When reflecting on my experience in research with young children and colleagues, I realized that learning as a collective endeavour and a collaborative study was necessary for fulfilling my goals.

In preparing for the new research, I looked for what I had learned at previous stages about teacher development. Throughout my work as a teacher in primary education, and as a teacher educator, I had realised that many teachers find it difficult to teach mathematics in diverse classrooms in ways that correspond to the goals of the national curriculum guidelines. My own observations in classrooms and through discussions with pre-service teachers and primary school teachers revealed that interactions among students in mathematics classrooms were limited. Teachers saw themselves more as guiders and supporters than investigators and facilitators of environments that foster learning. In their own experience as mathematics learners, they had typically been passive receivers who had practiced rules and procedures, introduced by teachers and textbooks. Teachers lacked experience of focusing on mathematical processes such as investigating, communicating, reasoning and making connections. Additionally, they felt incompetent in using these approaches in inclusive schools where emphasis is placed on the perspective that everyone is respected and noticed, their participation is valued and an opportunity is created for them to achieve and show their strengths (Guðjónsdóttir & Kristinsdóttir, 2011). In inclusive schools learning communities are created that work against discrimination, welcome diversity, and aim to educate all pupils in successful ways (Ainscow & Miles, 2008; Fullan, 1999; Slee, 2011; UNESCO, 1994, 2001).

My concern that all children should be offered opportunities for meaningful mathematics learning prompted me to work with a group of teachers with the aim of assisting them in reflecting on the mathematics learning in their classrooms. The reflection should concern both their

students' learning and their own learning, with regard to which I encouraged critical awareness. I contacted two neighbouring schools with diverse groups of students. The study thus involved, firstly, a) seven primary school teachers who examined their own practice as mathematics teachers, with my support. And secondly, b) myself, where I focussed on the collaborative process itself, as a whole, as well as the development I underwent throughout the research process, as a teacher and a researcher. The teachers had no prior experience of researching their own practice but they had expressed a desire to develop and improve their own teaching in order to become better mathematics teachers in these schools. Over a period of three years, we met at workshops on a monthly basis where we solved mathematical problems and discussed and reflected on our collaborative investigations. We also discussed the teachers' stories from their classrooms and reflected on their students' learning, as well as discussing how their experiences reflected findings from other research on mathematics teaching and learning and on teachers' professional development.

During my doctoral studies my understanding of theories of learning and methodologies for educational research was developing and when reflecting on my interpretation of these theories and methodologies, I reconceptualised my thinking about them. Participating in courses and conferences as well as conducting research with colleagues along with this collaborative research study, I wrote about my reflections and discussed them with colleagues and co-learners within doctoral courses as well as participants in the conferences I attended. When writing the thesis I realised that these reflections were an essential part of the study, and the thesis had to be written in two connected parts to answer the questions that were emerging.

The main research question for both parts of the study is:

- In what way has my understanding of teacher development in mathematics teaching and learning changed through working with teachers at improving their own practice?

Research question for Part I

- In what way has my understanding of theories of learning and methodologies of practitioner research developed over the last 20 years through studying my own practice as educator and researcher?

To answer this research question, I first present an overview of the researches on mathematics teacher development that I have conducted in the last two decades, mainly together with colleagues. I then discuss the learning theories and methodologies that underlie the process of researching one's own practice, that have informed my practice, as a teacher, teacher educator and a researcher during this time. Finally, I present my interpretation of how my understanding of these theories and practitioner research have developed and affected my practice.

Research question for Part II:

- What learning processes emerge through long-term collaborative inquiry undertaken by classroom teachers and a mathematics teacher educator?
- When teachers are participants in a project based on collaborative inquiry how do they perceive this as being reflected in their mathematics teaching?
- In what way do I interpret that my own learning from carrying out a collaborative inquiry project with teachers has influenced my practice as a teacher educator and a researcher?

Part II is devoted to the collaborative study with the teachers where I discuss the research process, the methodology and theories it was based on, present my findings and discuss the implication for further research. The third question discussed in Part II, relates to both parts of the study since it revolves around my process of learning, both before and after, but also while, the study was carried out.

In Chapter 2 common to both parts, I situate my study within the cultural community in which it was carried out by accounting for research on classroom cultures in Icelandic schools. I also offer an overview of mathematics teaching and teacher development in other countries that have influenced my development as a mathematics teacher and teacher educator over the last two decades. I then delineate approaches in educational research in the context of my own research.

Part I, starts with Chapter 3, where I present an overview of my background and discuss research projects that have affected the planning of this study. In Chapter 4 I outline the learning theories that have influenced my work as a teacher and impacted the development of this study. In Chapter 5 I will elucidate the structure of practitioner research. Finally in Chapter 6 I will analyse how taking on a self-study into my practice as a teacher educator has affected my approach to the collaborative study.

Part II addresses the collaborative research with teachers. In Chapter 7 I start by discussing the theories that I called upon in the execution of the study. In Chapter 8 I describe the methodology of developmental research and the methods used in collecting and analysing data. The findings are presented and discussed in Chapter 9 and Chapter 10. In Chapter 11, I summarise the findings from Part I and II and comment on the implications they might have for teacher education practices and research approaches into mathematics teacher development.

Choice of terms in this thesis

- Pupil: I use this word when I talk about children in primary grades and the pupils that attend the classes of the teachers that participated in the study.
- Student: I use this term when I talk about learners in secondary school and university. When I refer to other texts I use the term used by the authors.
- Learner: I use this term when I talk about learners in general. When I refer to texts where the authors use the term learner I use that term.
- Teacher: I use the term teacher for professionals who teach in schools.
- Student teacher/pre-service teacher: I use the terms student teacher and pre-service teacher when I refer to students who are enrolled in a teacher education program and have not gained a degree in teaching.
- Post graduate students: This term applies to students who have earned a university degree and have returned to university studies.
- In-service course: I use this term for courses that are planned for teachers who work in schools and are not a part of an academic study program.
- Study/project: I use these both these terms when I talk about the study that I took on as a part of my doctoral studies.
- Manipulatives/hands-on: I use these terms for objects that teachers make available to their pupils in mathematics classrooms to support their learning.
- Mathematical models: This term I use for objects that have been structured with the goal of supporting the understanding of mathematical concepts. Examples of these are base-10 blocks, number lines and fraction models.
- Teacher educator: This term applies to teachers who teach within a teacher education program. When I refer to my practice as a teacher educator I use this term
- Researcher: Researchers are people who conduct research. In this study I refer to my role in researching into the study as a researcher.

2 Situating the research in Icelandic context and relating to international perspectives in mathematics education

As I wrote about my research and discussed ideas with people from other countries, there emerged a need to present an outline of where this research is situated. I will start by shedding light on findings from studies on practices within mathematics teaching in Iceland. Particularly, studies that focus on mathematics teaching and learning, in order to provide information about and insight into the environment (i.e. circumstances and conditions) in which the study was carried out. Then, I will delineate various other studies on mathematics teaching and learning that have impacted my work as a mathematics teacher and a teacher educator, as well as influenced my understanding of what is important to pay attention to in awakening pupils' and teachers' awareness of how their learning develops. Finally, I will discuss educational research and professionalism in relation to my own research.

2.1 Classroom cultures and teacher development in Iceland

At the outset of this study, the research literature on classroom cultures in Iceland was meagre, particularly with regard to mathematics classrooms. In a study on classroom cultures, Sigurgeirsson (1992) observed instruction throughout the country, focusing on the use of curriculum materials in 4th to 6th grade classrooms in Icelandic schools. He found that what he labelled as seatwork dominated the lessons he observed, in mathematics as well as others. Most lessons began with the teacher indicating the source of the activities to be dealt with in the lesson, often just noting the name of the book that the pupils were to use. There might be a short review of homework, background and context or the rules relevant to completing the assignment. Then the pupils started working and the teacher monitored their work, either by walking round and attending to individuals or by sitting at the teacher's desk and calling on the pupils to come to the desk if they needed help. Now and then during the lesson, the teacher might call for the attention of the class for a short while. The teacher would then give pointers and discuss problems, working procedures, rules and content, or praise or reprimand the children if they were noisy. There was no common closing discussion or round up of the topic at the end of lessons. Sigurgeirsson's findings present a pessimistic picture of the culture in Icelandic

classrooms at the end of the 20th century, given that the interaction and discussions about the subject of learning seemed to be rare, and in reality, both the teachers and the children had a passive role in the classroom.

In a study of the practice of six general education teachers in compulsory schools in Iceland, who had been recognized by the heads of their schools for innovative practices, Guðjónsdóttir (2000) arrived at more promising results. Their classrooms included students with identified disabilities, such as physical, mental, or learning disabilities, and they all received individual curriculum based to their needs. She identified the teachers as what she labels 'responsive professional educators', who understand child development and individual differences, and who have a knowledge base, which enables them to differentiate between students. According to Guðjónsdóttir, responsive teachers go beyond acknowledging and respecting differences as they create curricula. These teachers focus on what the children bring into school and respond to individual children's differences as they create the learning environment. The teachers were skilled in creating a curriculum of learning activities and an environment in which all students had opportunities to succeed.

More recently, studies in mathematics classrooms in Iceland have increased. In a study where structures of 9th grade mathematics classroom lessons in Iceland and Finland were compared, Savola (2010) found that in contrast to Finnish teachers, who according to him are rather traditional and pedagogically conservative in the classroom, Icelandic teachers have adopted what he calls 'learner-based instructional strategies'. His concerns about the Icelandic classrooms have to do with the lack of communication in the classrooms, as well as limited guidance in mathematics learning. Many students received only minimal attention from their teacher. Savola's findings indicate that independent learning dominates Icelandic mathematics classrooms, where students work individually at their own pace with problems in textbooks.

The findings from Sigurgeirsson's and Savola's studies are similar in the sense that the teachers seem to have difficulties in taking the lead in promoting learning communities in the classrooms. Guðjónsdóttir's study, however, offers an image of the teachers as being skilled in creating learning activities and an environment that favours all students' learning opportunities. My own research with colleagues (Guðjónsdóttir & Kristinsdóttir, 2006; 2007a; 2007b; 2011; Gunnarsdóttir, Kristinsdóttir, & Pálsdóttir, 2008; 2013; Kristinsdóttir, 2010a; 2010b) indicate that if teachers collaboratively reflect on their practices in communities their abilities to create

learning communities, in their own classrooms, improve. Moreover, an action research study carried out by Angantýsdóttir (2010), where she worked with four primary school teachers in Iceland, supports our findings. Her results suggest that through project participation, the teachers developed further as professionals and were more aware than before of the development of their students' understanding of mathematics. The teachers learned more in terms of applying diverse ways of teaching and assessing children's work by visiting and observing each other's classrooms and discussing research on mathematics teaching and learning. Friðriksdóttir and Aðalbjarnardóttir (2010) also underline the importance that teachers reflect on their own learning. By relating life story and pedagogical vision, they found that teachers' own experience of learning mathematics affects their beliefs regarding both teaching and learning the subject, and their work with the students in mathematics classrooms.

Findings from a recent large-scale study, conducted in 20 schools in four municipalities, focusing on teaching in compulsory schools in Iceland, indicate that more emphasis is placed on direct instruction and less on discussions, group work and project work, all of which are recommended in the national curriculum guidelines (Sigurgeirsson, Björnsdóttir, Óskarsdóttir, & Jónsdóttir, 2014). These results correspond with Sigurgeirson's findings from 1992. In their analysis of data from observation protocols from the 51 mathematics lessons that were observed in this large scale study Gunnarsdóttir and Pálsdóttir (2014) found similar results for the mathematics lessons. Their findings indicated that there was a strong focus on individual seatwork where students worked mostly with problems in textbooks, and the teachers seemed passive and relied almost solely on the textbooks for the instruction. Students were active in their work and the teachers circulated around them as they guided them through the problems or exercises in the books. There were some examples of teachers creating opportunities for whole-class discussions about topics or ways of working. Still there were many lessons with no public interaction or opportunities to discuss with others mathematical concepts and connections where their ideas could have been challenged.

Two studies on Icelandic primary and lower secondary student teachers reveal that certain aspects of their mathematical knowledge and understanding of mathematical concepts require improvement. Diego and Jónsdóttir (2012) found that many of the students who had mathematics as major in the teacher education program displayed poor understanding of mathematical concepts and were in need of further support in honing their abilities to reflect mathematically. Jóhannsdóttir studied the mathematical

content knowledge of prospective teachers in primary grades (Jóhannsdóttir & Gísladóttir, 2014). The findings of Jóhannsdóttir's study brought to light that these prospective teachers' level of mathematical knowledge is based on recollection and reproduction of basic skills and concepts, and that they had difficulties with evaluating alternative solution methods. The authors concluded that some of these student teachers did not have the sufficient understanding of concepts and underlying principles in order to teach mathematics. The researchers in both of these studies (Diego & Jónsdóttir, 2012; Jóhannsdóttir & Gísladóttir, 2014) moreover questioned whether the teacher education programs offered these student teachers learning experiences that were sufficient enough to teach mathematics in schools. They noted that in both studies they measured cumulative knowledge and that poor prior knowledge in mathematics may partly explain these adverse results. It is important to keep their conclusions in mind when planning future education programs for prospective and in-service mathematics teachers, and find ways to support them in taking responsibility for their mathematical knowledge, in the manner called for by Ball, Thames and Phelps (2008).

Findings from a recent study on Swedish and Icelandic teachers' use of teacher guides to mathematics textbooks (Ahl, Gunnarsdóttir, Koljonen, & Pálsdóttir, 2015) suggest that teachers who used what the authors called 'educative guides' saw them as fruitful tools for learning and they were more likely to apply diverse methods in designing their lessons. Moreover, these teachers more frequently reflected on the quality of their lesson design than teachers who used traditional teacher guides. These educative guides supported teachers in seeing the larger curricular picture by connecting different mathematical ideas and urging them to reflect on their lesson design. The teachers who used traditional guides were satisfied with having a manual, which prescribed ideas on how to work with the problems in the textbooks and structure the classroom work. The researchers wondered whether the teachers who used the traditional guides were satisfied with them because they were unfamiliar with other designs of teacher guides. The authors also noted that further research should be pursued which would explore whether the teachers' choice of textbooks and the accompanying teacher guides, reflect their beliefs on mathematics teaching and learning.

The research studies on classroom cultures in Iceland described in this section are somewhat contradictory. Results from studies in the 1990s and the beginning of this century were based on teachers who were observed in their teaching and not taking part in the research process. These results

give a picture of classroom teaching in which classroom interaction does not involve learning communities. Similar results were found in a recent large-scale study in Icelandic schools, both for the study as a whole and the mathematics lessons in particular. The research studies into teaching in schools where teachers' reflection on their practice was central to the research are more promising and imply that teachers are capable of creating learning environments for all children. The results from the study of teachers' use of teacher guides indicate that a guide in, which teachers are supported in connecting different mathematical ideas urges teachers to reflect on their lesson design and use varied approaches in their teaching. The research on pre-service teachers' mathematical content knowledge and pedagogical content knowledge reveals that teacher educators need to attend to the mathematical knowledge for teaching in their work with pre- and in-service teachers.

As a teacher educator, I wanted to learn more about how we can work with teachers in the field in order to create communities in mathematics classrooms that support students in engaging with meaningful mathematics learning. I looked for research within the field of mathematics education that highlighted the significance of collaboration. In the following section, I will discuss some of the work that I have found helpful in organising my work with teachers in pre-, in-service and graduate courses.

2.2 Mathematics teacher development in other countries

International research literature on mathematics teaching details a broader picture of cultures and collaboration in mathematics classrooms. I will discuss research projects that have influenced my work as a mathematics teacher and teacher educator and that directly relate to the study of my own teaching and collaboration with others.

One of the research projects that impacted my mathematics teaching in primary grades, and my work as teacher educator, was the Cognitively Guided Instruction (CGI), which grew from research on children's thinking about whole numbers. The research's point of departure was explicit knowledge about the development of children's mathematical thinking (Carpenter, 1985; Carpenter, Fennema, & Franke, 1995; Carpenter & Moser, 1984) which was used as a context to study teachers' knowledge of students' mathematical thinking (Carpenter, Fenema, Peterson, & Carey, 1988) and the way teachers might use this knowledge in making instructional decisions (Carpenter, Fennema, Peterson, Chiang, & Franke, 1989). The project was built on an integrated research program that

focused on the development of students' mathematics thinking; on instruction that influenced the development; on teachers' knowledge and beliefs that influenced their instructional practices, and; on the ways that teachers' knowledge, beliefs, and practices were influenced by their understanding of students' mathematical thinking. When children began to show signs of increased learning, the teachers implemented new methodologies that resulted in improved learning, and so on, in a cyclical way (Fennema, Carpenter, Franke, & Carey, 1993). As a result, the teachers learned to learn from their students' thinking and were able to continue to independently improve their practice once the external support ended.

Developing an understanding of children's thinking provides a basis for change, but true change occurs as teachers attempt to apply their knowledge with the aim of understanding their own students. The findings from the CGI study provide strong evidence that knowledge of children's thinking is a powerful tool that enables teachers to use and transform this knowledge, which thereby benefits more diverse instruction. It also appears that this knowledge is not static and only extracted outside the classrooms in workshops, but rather is dynamic and ever-growing, and can probably only be acquired within the context of teaching mathematics (Fennema et al., 1993; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996).

Similarly, in the Purdue Problem-Centred Mathematics Project, the researchers noticed how the teachers collaborating with them learned from participating in the project (Cobb, Wood, & Yackel, 1992). The focus of the project was on children's communication while dealing with mathematical problems. The researchers reported on their findings from their collaboration with one of the teachers, and realized that not only were the children learning by engaging in the activities, the same held for their teacher (Wood, Cobb, & Yackel, 1991). The teacher's main concern was how to change her way of teaching in accordance with her change of beliefs. She struggled with shifting the focus from verbal instructions on how to calculate, to respecting the children's way of explaining their own procedures, and accepting them even if they got the wrong answers to the problems. This conflict between change of beliefs about teaching mathematics and the actions carried out in the classroom was also reflected in my former research with teachers (Kristinsdóttir, 2010a; 2010b), as well as in the CGI project (Fennema et al., 1993; Fennema et al., 1996).

Carpenter and his co-researchers continued their research on children's mathematical understanding (Carpenter, Levi, Franke, & Zeringue, 2005) and teacher development (Kazemi & Franke, 2004). In a study performed in

19 elementary urban schools in one of the lowest performing school districts in California, the researchers adapted their focus on student thinking to accommodate the mathematical content and the teachers' identities in relation to the content. They also took notice of the fact that they were working in urban, low-performing elementary schools. The professional development included school-based work-group meetings and on-site support. The teachers explored the development of students' algebraic reasoning and how that reasoning could then support students' understanding of arithmetic. They were encouraged to tell stories about the mathematics their students could do instead of focusing on what they could not do (Jacobs, Franke, Carpenter, Levi, & Battey, 2007). In Chapters 3 and 4 I will discuss further how the CGI project impacted my own and my colleagues' teaching in primary grades and my work as a teacher educator.

Conclusions from the TIMSS² video study of 8th grade mathematics classrooms in Germany, Japan and the United States, have greatly impacted research on mathematics teacher education and developmental projects in many countries. The main results of the study were that differences in teaching varied across cultures but varied little within cultures (Stiegler & Hiebert, 1999). In classrooms in the United States, teachers presented definitions of terms and demonstrated procedures for solving specific problems. Students were then asked to memorise the definitions and practice the procedures. The focus was therefore on learning terms by heart and practicing procedures, and less on the mathematics. In Germany, the emphasis was put on developing advanced procedures. The teacher owned the mathematics and parcelled it out to students and defined facts and explanations at the appropriate times. In Japan, the focus was directed on structured problem solving. Teachers allowed students to invent their procedures for solving demanding problems procedurally and conceptually. They designed and orchestrated lessons so that students were likely to use procedures that had been developed recently in class. These findings influenced my colleagues and my own teaching within teacher education in Iceland (Guðjónsdóttir, Kristinsdóttir, & Óskarsdóttir, 2007; 2009; 2010; Gunnarsdóttir et al., 2008; 2013). We recognise the importance of giving student teachers the opportunity to focus on their own learning and development of mathematical thinking, and engaging in mutual discussions about their understanding, as opposed to practicing procedures explained by others. This is further discussed in Chapters 3 and 4.

² The Third International Mathematics and Science Study

Stigler and Hiebert argue (2004) that the idea of teaching as a cultural activity might explain why teaching has been resistant to change. They find it is necessary to recognize the cultural nature of teaching and they further stress that the results from the TIMSS study produced new insights into what we need to do if we wish to improve mathematics teaching in schools. According to them, current efforts to improve the quality of teaching have focused on the teacher, how the profession could recruit more qualified teachers and how to remedy deficiencies in the knowledge base of teachers. As a response, they propose the Japanese Lesson Study approach, which takes the form of developmental work where groups of teachers meet regularly over long periods of time to work on one or several research lessons. Moreover, the Japanese approach is also a way of improving teaching and the methods that teachers use in the classroom (Stiegler & Hiebert, 2004). Watanabe (2002) argued that educators in the United States could learn some of the strategies of Japanese research lessons, including anticipating students' responses, learning to observe lessons carefully, and developing a culture in which teachers observe each other's lessons. In 2004, Lewis, Perry and Hurd discussed results from a successful teacher-led lesson study initiative. They concluded that lesson study goes far beyond simply improving a lesson, it also challenges teachers to improve their classroom instruction (Lewis, Perry, & Hurd, 2004). My colleagues and I have introduced the lesson study approach to student teachers and they have adopted it in their teacher training in schools (Gunnarsdóttir et al., 2008; 2013). We have seen that the co-learning that develops as the student teachers plan their teaching together, and reflect on it collaboratively, has facilitated the pre-service teachers in rethinking their understanding of mathematics teaching and learning.

In recent years, there has been a shift in mathematics education research. From focusing on what works to why and how teachers do what they do (Sfard, 2005; Sherin, Sherin, & Madanes, 2000). This new emphasis relates to my own experience of working with teachers (Kristindóttir, 2010a; 2010b) and has helped me focus on relevant issues in research with teachers, particularly the way that these researchers rely on the teachers' work and their collaboration in the research. Such collaboration allows teachers to explain how their own teaching is grounded and how their beliefs and experiences have impacted their teaching.

Similarly, the focus of many articles in *The Journal of Mathematics Teacher Education (JMTE)* has been on research where teachers' voices are expressed. In an editorial, Wood and Berry argued "for the importance of generating and sharing knowledge about the complexity in mathematics

teaching” (Wood & Berry, 2003, p. 195). The journal has played an important role in building up a knowledge base for teacher educators who wish to improve the education of mathematics teachers. The collaboration of teachers, teacher educators and students has been a central issue in many of the research studies reported in the journal. Teamwork, learning communities, networks and design research are often suggested as ways to help teachers tackle and meet the complexity of teaching mathematics (Jaworski, 2005; 2006a; Krainer, 2003; Wood, 2002; Wood & Berry, 2003). In her editorial to the JTME, Jaworski (2007c) summarized the development of content in the journal over the decade it had been running. Her conclusion was that increased weight had been given to relationships between research and development, through research collaboration between teachers and teacher educators as researchers. Jaworski notes that there is specific need to account for learning in classrooms at both micro and macro level. Furthermore, she calls for seminars on the topic and theoretical papers that explore the complexities of learning and teaching in institutions and society. The emphasis on collaboration between teachers, teacher educators and researchers, that discussed in the JMTE, has encouraged my own research process with teachers and inspired me to learn about how their work develops when they focus on their own learning, as well as on their pupils learning.

The view presented above, namely, that professional development should be a collective endeavour, is shared by many other researchers and teacher educators. Drawing on data from program sessions, where teachers discussed with each other their lesson designs and reflections on their teaching, Brodie and Shalem (2011) found that challenges and solidarity were important aspects in developing the type of conversations among teachers that helped them improve their teaching. Coles (2013) examined a study, which revolved around a teacher developmental project, where teachers watched video clips from mathematics lessons in a group and discussed what could be learned from them. The teachers that participated in the research found this more useful than lesson observation. Miyakawa and Winsløw (2013), on the other hand, reported on open lessons in Japanese schools that, according to them, create a professional learning community of teachers. They concluded that the discussion following the open lessons “relates the lesson to more theoretical aspects of the mathematics curriculum as such, and even to more general pedagogical and societal aims of the school” (p. 204), as in the case of lesson study discussed above. These diverse findings have helped me realise that teachers’ experiences of professional development varies and in my work with

teachers I need to pay attention to their vision with regard to their own professional development.

The growing interest over the past three decades in understanding and responding to inequitable mathematical experiences, has for example been discussed by Walshaw (2015). In this context, Walshaw suggested that observations of the changing nature of our mathematics classrooms that increasingly cater for diverse groups of learners, have sparked new kinds of research projects and initiated a wide range of student-centred interventions. There also emerges a need to challenge established norms in terms of which we assess children's learning competences and highlight their strengths instead of looking for their weaknesses (Dalvang & Lunde, 2006). As Walshaw (2015) has emphasised, an understanding of equitable arrangements within mathematics education is vital as teachers confront challenges and problems, being caught up in changes within the wider educational apparatus. Moreover, she calls for analyses of these contexts that would allow us to create a space where new possibilities might be envisioned.

Holgeirsson (2010) conducted a study where teacher educators worked with primary- and lower secondary-school teachers who wanted to develop their mathematics teaching in inclusive schools. In workshops, the teachers solved open-ended tasks and discussed their students' way of learning mathematics. The conclusions he drew from his findings was that through the workshop participation, the teachers developed an awareness of student reasoning and thinking, which in turn helped them include all children in their classrooms in meaningful mathematics learning. These conclusions figure into my intentions for the collaborative research with teachers. Holgeirsson's results echo Ainscow's (2007) findings, who insists that schools that make progress towards inclusive ways of working, evolve a capacity within the school for teachers to learn from one another, where they share ideas and practices, and spend time talking about how teaching can be improved.

Sullivan (2001) addressed the growing interest in describing the knowledge that teachers need for teaching mathematics, namely, subject matter knowledge and pedagogical content knowledge. He called for research into processes and experiences that assist teachers in gaining that knowledge. In 2014, a special issue of *Nordic Studies in Mathematics Education* was devoted to the theme 'mathematical knowledge for teaching' (MKT), and included a wide variety of perspectives. Dragset (2014) examined the types of knowledge teachers use when leading

mathematical discourses, in relation to which he describes three areas that frame mathematical knowledge for teaching: doing, guiding and requesting. He lists the different ways teachers do the mathematical work themselves, how they guide their students and the different ways they ask students to explain or contribute to a discourse. A learning study conducted by a group of Swedish teacher educators who intended to identify critical features concerning the teaching, and learning of MKT, is reported in Bommel (2014). Their findings uncover that teacher students need to be better prepared for specific aspects of teaching, such as formulating proper goals for a lesson, outlining a lesson plan in detail, shifting the role from being a teacher to being a mathematics teacher, and understanding the underlying mathematics of the lesson topic at hand. Central to my planning of the present collaborative research project, is the focus on the teachers' mathematical knowledge and their reflections on their own understanding of the mathematics they are teaching, as well as their way of working with their students in gaining mathematical competences.

Recent research on mathematics teacher development in the Nordic countries includes for instance the Learning Communities in Mathematics project (LCM) in Norway (Jaworski, 2005; 2006b; Jaworski, 2007a, 2007b). The project sought to explore ways in which classrooms could provide better learning outcomes for pupils in mathematics. The project was based on collaboration between teachers in schools and didacticians³ at the University of Agder. The participants developed an awareness of what is possible both from their own points of view and from those of their colleagues. Collaborative learning was evidenced and the participants made progress in creating a learning community. However, there were tensions between the perspectives of the teachers and the didacticians, but these contributed to the growth of awareness, explained by Jaworski as follows: "An important part of the analytical reporting that is central to the research in this project is recognition of tensions and issues and of the ways in which the project learns through them" (Jaworski 2007b, p. 23). As an example of these tensions she mentioned that at the initial stages of the project the teachers expected a clear leadership role from the didacticians. One teacher expressed that she had anticipated that the didacticians would explain to the teachers how to run their mathematics teaching and was disappointed when they did not. During the three years the project was running, the participants learned how to resolve dissonance in their expectations and further develop their collaboration.

³ Professionals responsible for theorising teaching

Following in the footsteps of this project, another mathematics development project was established in Norway, entitled Teaching Better Mathematics (TBM). It is both an extension and development of the LCM project and the ICT and Mathematics Learning (ICTML) project that ran simultaneously to the LCM project at the University of Agder (Breiteig & Goodchild, 2010). The TBM project also ran in four university colleges in Norway, and consequently the research period was extended. Berg (2012) reported on a developmental research she conducted within the project. Her collaboration with lower secondary school mathematics teachers focused on mathematics, while also preparing and conducting activities with them and the collaboration between teachers and researchers according to a co-learning agreement. She argued that:

... results emerging from this research offer guidelines for the organisation of developmental research projects where a trustworthy collaboration between in-service mathematics teachers and researchers and a clear focus on the subject-matter need to come to the fore. (Berg, 2012, p. 155)

Results from these two developmental research projects in Norway have contributed to our knowledge about how teachers and researchers can collaborate to improve teaching in schools with the goal of facilitating pupils' meaningful mathematic learning. While preparing for and conducting this collaborative research project, I drew on their design of these projects and below I will expound further, in Part II, how it has contributed to my developing of this research.

The research projects discussed above have impacted my understanding of mathematics teaching and learning and how teachers' capabilities to develop their mathematics teaching can be supported through collaborative reflection on their work. I now move on to discuss effects of educational research and their relation to practice in schools.

2.3 Educational research and professionalism

Mathematics education researchers engage in practices and the successful enactment of knowledge in different situations. Boaler, Ball and Even (2004) emphasised that researchers must develop attitudes that foster a sceptical approach, being open to surprise, and display a will to prove one's ideas wrong and consider alternatives. They also found it important to account for the ways in which theory may support work in the scholarship of mathematics education, and equally worthy of consideration, the ways in

which theory may impact the practices of mathematics teaching and learning. Educational researchers are working in an applied field and our work is judged by the extent to which it is able to improve educational practice. In this section, I will discuss goals for educational research in general and relate this to my study of the research process conducted with mathematics teachers.

The contradictory aims of education, and the different things that education does for state and society, place pressures on teachers which results in their work being complex and unstable. According to Ozga (2000), education is an opportunity for teachers to experience and enrich the life of their students, while policy makers are occupied with managing the teaching workforce and the demand on education that it should contribute to economic growth. Correspondingly, Sykes, Schneider and Ford (2009) argue that policy discussions about economic growth and national competitiveness commonly focus on the importance of human capital, an educated workforce and the need to expand and improve the educational opportunities for all children. Education is thus no longer just about what happens in classrooms and schools, instead, attention is now cast to the process through which rules and regulations are adopted, and the consequences they have on teaching and learning.

At the turn of the century, Hammersley (2002) critiqued trends in research in England and Wales, in particular the idea that research is capable of improving educational policymaking and practice, as well as the contention that education determines national economic success. He maintained that the meaning of the term policy depends on the perspective of the researcher. Those who understand policy in terms of the actions of government, with the aim of securing particular outcomes, tend to rely on evidence-based research and believe that practice should be based on evidence. For example, there have been recurrent demands in the US in recent decades for the “back to the basics” in schools and the “no child left behind” policy both of which presume an evidence-based practice. Moreover, systematic research reviews are considered efficient approaches in educational research.

In support of an evidence based-approach, Boreman (2009) argues that randomised experiments in education provide results that could help researchers and policymakers understand the proposed effects of an intervention, when it is implemented beyond the laboratory or in the “real world” of schools and classrooms. According to Boreman, the central element that distinguishes a true experiment from all other methods of

research is the random assignment of alternative treatments to the individuals or groups involved in the study. He expressed the view that there are complexities entailed in carrying out these random assignments and a guarantee must be in place, ensuring that the treatment group actually receives the intended treatment different from the one received by the control group.

Hammersley (2002) writes that there is a scope for improvement within the field of educational research and policymaking and practice. He advocates for further analyses of the roles that research has played, as well as encouraging researchers to ask theoretical questions about the roles research can play and ought to play. Later, Hammersley (2005) claimed that evidence-based policymaking and practice relies on the myth that research is essential in rendering the public sector as a whole more accountable. In fact, the aim has been to reshape educational research so that it can serve this function more effectively, and to establish procedures to control its performance in that role. Additionally, Hammersley argued that the consequences of this myth are likely damaging, not only for research in general, but also for policy and practice. Similarly, Biesta (2007) opposes the idea that education should become evidence-based practice and that teaching should become an evidence-based profession. He raised his concerns in face of the growing enthusiasm for evidence-based practice, and in response, asserts that evidence-based education seems to favour a technocratic model where the only relevant research questions are about the effectiveness of educational means and techniques. The focus on “what works” makes it difficult to ask questions about what it should work for and who should have a say in determining what is educationally desirable.

It is important for policymakers in education, both at the macro and micro level, to be aware of the different interpretations of research intentions, how studies are carried out, and what kind of effect they can have, or even should have, on teaching in schools. Taking for granted that evidence-based research gives us the answers to what works in schools, and what and how we should teach, undermines the role of the teachers as professionals; capable of reflecting on what research can tell us and how research findings can be used in schools. In carrying out research with teachers and supporting them in researching their own practice, my wish is that they will learn to reflect on what truly demands attention in relation to learning in schools, and critically weigh findings from educational research that revolve around what works in schools.

In 2014, the Ministry of Culture and Education in Iceland published a White Paper on education reform (Mennta- og menningarmálaráðuneytið, 2014). The paper sets targets in several areas reflecting the Ministry's concern with falling performance scores in the PISA⁴ survey. Nevertheless, this concern does indeed reflect a reality that must be taken seriously and responded to immediately. It is considered important for educators to take advantage of research studies and the findings from pilot projects, in order to improve teaching. The Ministry insisted that the accountability for student achievement belonged to schools, municipalities and central authorities. The proposals made in the paper underline that progress should be based on students' own capacity and goals, comparisons with other school and municipalities, and performance, relative to the best results achieved at home and abroad.

It is not easy to find an evidence-based approach to planning and policy in education in Iceland. A counter-example to the approach is put forward in a project called Beginning Literacy, which began in 2006. Compulsory schools were offered to implement an interactive literacy approach in collaboration with the Centre for School Development (CSD) at the University of Akureyri. The approach is directed at literacy education in the 1st and 2nd grades. It is implemented by means of a two-year staff development programme led by consultants from the CSD and draws on current research and experience in the field of professional development. The project however met with criticism from the Minister, who lamented it for not carrying out an evidence-based evaluation.

The interest in mathematics education has grown due to international comparative studies such as TIMSS and PISA. Sfard (2005) wrote that, in spite of the ongoing efforts toward reform in mathematics education, many countries found the results of these international measurements of their students' achievements rather disappointing. In the United States, an answer to the criticism took the form of appeals for teaching grounded in scientifically based research and for instructional methods, that drew on reliable and empirically proven evidence. Sfard argues that a scientifically based research approach, where measurements are prioritised, fail to address what has been measured, the content of measurement. She further contends that, in the eyes of a politician, the idea of measurement is irresistibly appealing. Instead of relying on results from quantitative research and scientific reliability, and the attempt to arrive at a mechanistic

⁴ Programme for International Student Assessment

view of what works in the classroom, Sfard proposed a qualitative, participatory approach where the focus is directed on how things work. She added that the researcher's message has usually come to the teacher in the form of a policy document, a textbook or an external examination. Moreover, the teachers are rarely presented the rationale for what is suggested and neither do these documents reflect the overall spirit of the researcher's advice. Sfard stresses that the responsibility for progress in mathematics teaching ought to be the researchers' and their influence of schools practice should stem from collaboration with teachers.

Noddings (2007) asserts that teachers need to be critical of research findings and ask questions about how these findings correspond to life in schools, and whether their premises, methods, and conclusions hang together in a convincing way. Additionally, she claims that the question should be asked about who counts as an authority in educational research and whether they better serve the interests of researchers rather than the interests of participants or the targets of the research. Skott (2009) argues that researchers have not always succeeded in giving teachers access to their research findings in a form that is meaningful for them and applicable. He adds that teachers are often incapable of or reluctant to change, if their current priorities and practices appear incompatible with the observer's research recommendations. As a result, he claims that this could lead to an image of the teacher as someone who lacks knowledge or beliefs, in contrast to what research has actually suggested. Skott (2010) calls for change in mathematics education research. He states that research is only likely to have an impact on instruction in ordinary classrooms if the contextual character of teaching and learning is acknowledged. He further highlights the importance of unpacking how different patterns of participation, stemming from different social practices, are related and together form instructional activity.

In recent years, there has been growing interest in teacher research such as action research. Stiegler and Hiebert (2004) reason that teachers have a central role to play in building a useful knowledge base for their profession. They argue that teachers need to analyse what happens when they try something new in their own teaching and record what they are learning and share that knowledge with their colleagues. Artigue (2009) notes that it is important that research conducted within schools take into account factors internal to the development of the field itself. Additionally, she argues that research developments have made it more and more evident that research methodologies must include a way to organize the relationships with the situational, institutional and cultural dimensions of

learning and teaching processes. She maintains that research projects within schools where teachers are active participants in the research process meet these requirements.

Goos and Geiger (2010) also address the need for researchers to work with teachers. They pose a number of questions in relation to what factors could guide future socio-culturally oriented research into the learning and development of mathematics teaching. In this context, they propose that mathematics teaching requires a social turn toward change in which the emphasis is placed on collaborative research with teachers in the field. Moreover, their questions concern for example the ways in which researchers' professional contexts structure their interactions with the prospective and practicing teachers with whom they work, what activities and tools researchers could access to change themselves, and how researchers who work with teachers could develop new knowledge, beliefs, awareness, goals and practices. My own interest in researching with teachers reflects the growing interest in collaboration between teachers in the field and researchers within mathematics education.

In her explanation of the framework for the LCM project discussed in Section 2.2, Jaworski (2008b) describes how teachers and educators worked collaboratively on developmental projects. She writes that, traditionally, teacher education has been seen as a transfer of knowledge from educators to teachers. The educators either transferred knowledge to teachers or worked with them on developmental programs. However, in the framework for the LCM project, teacher knowledge of students and schools is viewed as having the same level of importance as the educators' knowledge of theory, research and systems. In fact, together they produce a shared knowledge. Jaworski proclaims that the knowledge of the teachers and teacher educators was deeply related, though there were obvious differences, both regarding specialist areas, individuals and groups. Both groups are concerned with the mathematical knowledge of students and seek to know more about how to meet the diversity of learning needs. Lastly, both teachers and educators bring knowledge to the enterprise and which was tested in practical settings.

Knowledge about the relationships between actions and consequences can only provide us with possibilities, not with certainties. Decisions about educational actions and arrangements always have to be taken in clear view of the desirability of what such actions and arrangements are supposed to bring about. While making decisions about rules and regulations for schools, such as the law and the curriculum, policymakers must have access

to work within the field of education and educational research. Teachers are responsible for teaching in schools and it is therefore important that their voices are heard in educational research. Research on teachers, where the teachers themselves do not directly participate in the research process, can create an image of their teaching that does not reflect their own view of their practice.

My goal in researching with teachers into their practices is in accordance with the views presented above on research with teachers. In researching with them I respect their values and offer them a voice to report about their work. Many teachers that I have worked with have found it difficult to relate their work to research findings that they cannot align with their own practice. By researching their own practice and participating in research with experienced researchers, the teachers' reflections on their teaching, and on the narratives they themselves build in relation to their own practice, will contribute to development in teaching at all levels. Throughout this process, I also learn about my own development as a teacher educator and a researcher.

2.4 Summary

In locating my research within the field of mathematics education, I have reviewed and examined educational research in the Icelandic context, and in doing so, sketched out the landscape in which my research was carried out. In addition, I have traced how research in other countries has affected my interpretation of how mathematics teaching and learning develops, as well as my vision for how to support teachers in primary grades to include all children in meaningful mathematics learning. By placing my research within the field of education I want to explore how the research can add to the growing interest in collaborative research between teachers in schools and teacher educators.

The goal with accounting for my experience in Chapter 3 is to trace the aspects that have influenced my practice. I provide the background for my research and an overview of research projects in which I have been involved. In Chapters 4 and 5 I will discuss further how I interpret the impact that my former experience of researching my practice has had on my understanding of theories of learning and research into own practice.

Part I: Developing as a professional: teacher, teacher educator, researcher and doctoral student

When taking the initial steps of planning for my doctoral thesis and identifying what knowledge I found important to gain from the study, I drew on my former experience as a teacher, teacher educator and a researcher. As I participated in doctoral courses within the University of Iceland and courses arranged within the Nordic Graduate School in Mathematics Education, I was urged to process my experiences and relate them to the literature I was reading about theories of teaching and learning and methodologies within the field of educational research. When I joined the doctoral studies, I saw it as an opportunity to develop as a teacher educator and a researcher with the goal of improving my practice, and so the study of this process became an essential part of my study. In this part of the thesis, I will analyse how my studies have impacted my understanding of theories of learning and teaching and methodologies of researching my own practice, and relate my interpretation of the process to my experience as a teacher, teacher educator and a researcher. During the study of my own learning, I adopted the methodology of self-study of teacher education practices when answering the research question for Part I:

- In what way has my understanding of theories of learning and methodologies of practitioner research developed over the last 20 years through studying my own practice as educator and researcher?

Part I begins with Chapter 3, where I present an overview of my background as a teacher, teacher educator and researcher, and of research studies that I have carried out and which have influenced my thinking about teacher development. In Chapter 4 I outline the theories of learning that have influenced my work as a teacher and impacted the development of my interpretation of said theories. From grounding my work in interpretations of constructivist theories to my emerging understanding of learning as a social construct. In Chapter 5 I will elucidate the structure of practitioner research; who initiates the study and the part played by theory in the research process. Additionally I will explicate the ramifications of self-study within teacher education practices. Finally in Chapter 6 I will analyse how writing an autobiographical account describing the development of my

experiences of reflecting on my own understanding, has affected my approach to the collaborative study.

In Part II of the study, I will discuss further my development as a teacher educator and a researcher while carrying out the collaborative research project with teachers, accounted for in Part II.

3 Professional background and former research

After I finished my B.Ed. degree (1975), I took on a position as a classroom teacher in a compulsory school for children from the age of 6 to 16, and which was at the time a 'practice school' to the Iceland University of Education. During the first years of my practice, I was a general classroom teacher in primary grades. I had the opportunity to join developmental projects together with my colleagues even though I did not participate in the education of prospective teachers to begin with. Following the five years of teaching in primary grades, I lived in Sweden for three years (1980-1983), where I studied pedagogy at the University of Uppsala. Returning to the practice school in Iceland, I was offered the position of 'practice teacher' (from 1985). As such I had a dual role, namely, as a teacher in primary grades and a teacher educator for pre-service teachers at the Iceland University of Education. My main role in teacher education was to assist student teachers to prepare for their work as teachers and collaborate with them and the teachers who mentored them in the school. When they came back from their practice periods to the university, we met and discussed their experiences and reflected on how it related to the literature they were reading within the teacher education program. I often had student teachers practicing in my own primary classes and thus experienced the role of mentoring pre-service teachers in their practice teaching at school.

I also taught courses on pedagogy and teaching methods, and my background as both a teacher and a pedagogue, aided me in teaching student teachers about teaching in schools. During a period of four years (1988-1992), I was the main organiser of the teaching practice for students at the Iceland University of Education and collaborated with teachers and heads of schools throughout the country. At that time, I organised courses about mentoring student teachers intended for teachers in schools that received our students. During this time, and as I increasingly supervised student teachers in their practice period, my interest grew in teacher education practices and collaboration in the field.

The teaching of mathematics and science was my main subject during my teacher education studies, and with support from colleagues within the teacher education where I was a practice teacher, I gradually started to teach courses about mathematics teaching and learning at the Iceland

University of Education. This inspired me to reflect on my experiences of mathematics teaching in primary grades. Consequently, I wrote notes and gathered data from my teaching, such as children's work and video-recordings in classrooms. As I taught student teachers about children's mathematical learning, I often related to my own pupils' learning and presented examples of their own work to shed light on our discussions about the theories of mathematics learning, which they read about in the course syllabus. Collaboration with colleagues who taught the same courses also urged me to reflect further on my experiences. When studying for a M.Ed. degree in the late nineties, I decided to further develop my experiences of reflecting on my practice and consequently conducted a research on my own mathematics teaching in primary grades, lasting three years, which culminated in the writing of my M. Ed. thesis.

In 2003, I was employed as a mathematics teacher educator at the Iceland University of Education and left my position at the compulsory school. My experience of studying my own teaching in primary grades encouraged me in reflecting on my practice within the university. Together with colleagues, I have conducted several research projects, which focus on my own practice. I will elucidate this further below, where my experience of these studies will be tied with the ways in which I interpret the development of my understanding with relation to theories of learning and methodologies of studying my own practice. The Iceland University of Education merged with the University of Iceland in 2008, and since then, I have been employed as a mathematics teacher educator at the School of Education, University of Iceland.

When I wrote the proposal for the doctoral thesis, I wrote a summary of four research projects that I had previously conducted which I considered a fruitful way to provide a background for the research I was planning to conduct. I present this summary here and a brief summary of two research projects I carried out alongside the research project discussed in Part II.

3.1 The teacher learning in the classroom

An important event in mathematics education in Iceland occurred in 1995, when a summer course was held on the Cognitively Guided Instruction (CGI) research project (Carpenter et al., 1995). The course was organised by the Iceland University of Education, in collaboration with 'Flötur', the mathematics teacher association in Iceland. The teacher was Donald Chambers, one of the researchers who had participated in the CGI project.

When the course finished, the participants were determined to build a community for discussing mathematics learning and teaching, in order to properly process what they had learned at the course, and as a result, they decided to meet on a regular basis. The developmental process of two of these teachers and their colleagues will be accounted for in the following section. My involvement in the course and the continued discussions with the participants added urge me to study my own teaching in primary grades as discussed above.

In studying my own mathematics teaching in primary school, my aim was to make sense of my students' thought processes as they constructed their own knowledge of mathematics. I wanted to gain the ability to learn from my experiences in the classroom, interpret what I observed, and use my analyses to make decisions about my teaching. Later, I realised that what I was aiming at was an attempt to understand how my interaction with the students proceeded, and interpret what I learned as it progressed and thus the focus of the study changed. In a similar way, Loughran and Norfield (1998) describe their study in which Norfield's focus on the learning in his classroom led him to recast the focus toward the process of his own experiential learning. In cases where the children brought new ideas into the classroom or noticed different aspects of the problems than I had noticed myself, I used it as an opportunity for us to learn something new together. When I planned new projects, I used the children's ideas to guide me both in what kind of mathematics to focus on and how to formulate the tasks. I often responded spontaneously to the children's ideas and used the ideas as a chance to pinpoint important mathematical aspects (Kristinsdóttir, 2003; 2006; 2007). I was not only using my knowledge of children's learning and mathematics education in my teaching, but I was also making decisions about my teaching based on my reflection on action, and in action as theorised by Schön (1983; 1987). Moreover, Mason (2002) helped me in realising this point by describing the process as follows:

The more you listen to students working together in groups, the more you realize the complexity of being 'taught'. The more you probe children's thinking, the more you realize how sophisticated and powerful children's thinking can be. (Mason, 2002, p. 27)

Mason also stressed the risk of making habits of how you respond to your students, instead of responding sensitively to situations. He maintains that we frequently react according to established patterns without realising it and we continue to believe that we act freshly all the time, when in fact

much of the time we are reacting rather than responding. However, when we have acted freshly and appropriately, we feel a sense of freedom and these moments keep us moving forward. Reflecting on my teaching helped me to avoid making habits of my responses to the children's ideas. All their powerful ideas about mathematics helped me understand how important it is to be sensitive to the student's ways of learning mathematics.

3.2 Teacher development through reflective discussions

Over the course of several years, I worked with a team of four teachers who were in the process of developing their mathematics teaching in primary grades. Their collaboration was not intentional to begin with, however, their willingness to cooperate emerged from their enthusiasm in learning to understand the development of their students' mathematical learning. Their cooperation started when two of them attended the CGI course on children's mathematics learning, discussed above (Carpenter et al., 1995). They were eager to use what they learned about children's development of mathematics thinking to bolster their ability to interpret their pupil's learning, and in that way, support them in their teaching. At weekly meetings with their colleagues, they discussed their work and inspired their co-teachers to collaborate. In order to be able to develop their work further, they later asked school authorities for support in terms of time allotted for collaborative work as well as for professional guidance from the Iceland University of Education. Given that I had met them several times to discuss our teaching in primary grades, as discussed above, I agreed to work with them, now in the role of a teacher educator. I then participated in some of their meetings and also observed some of their lessons. When this formal collaboration began, we collectively decided to research their practice and collect data from their classrooms and our meetings.

The four teachers assisted each other in building the confidence to construct a conceptual portrait of all the students in their classes. Their collaborative learning helped change their beliefs about mathematics teaching and learning as well as their classroom practices. When they saw that their students were able to find their own solution strategies when given meaningful problems to work on, they were more willing than before to rely on the children's thinking. This is in accordance with the teachers in the CGI studies (Fennema et al., 1996). Developing an understanding of the children's thinking provided a basis for change, and change occurred as the teachers attempted to apply their knowledge to understand their own students. Systematic reflection on mathematical interactions that focuses

on student's learning and understanding of processes, as well as on one's own interaction behaviour, represents an essential professional competence of teachers (Mason, 2002).

The findings indicate that teacher discussions about their students' way of learning mathematics, and their reflections on their own teaching, can influence teaching in diverse classrooms. Their reflection became an important part of community building, paving the way for the emergence of common ways of thinking and for developmental concepts to grow within a community (Jaworski, 2006b). These findings are in harmony with Goos's (2004) work, who discovered that dialogue participation about their beliefs on teaching and learning mathematics, allowed teachers to build a learning community of inquiry. In such communities, teachers support each other in creating classroom cultures that promote understanding and help all children make sense of mathematics (Hiebert et al., 1997).

The teachers who participated in this project were all working fulltime as primary school teachers and did not see themselves as researchers. Their reflective discussions and the co-learning community that they managed to build facilitated them in developing their practice. This collaboration urged them to collect data from their work, and in addition, they have written several reports, reported their work at teacher conferences and supported other teachers both within their own school and outside. Together we have reported on their work at educational conferences and I have published two papers about their work (Kristinsdóttir, 2010a, 2010b). One teacher used her sabbatical year for further studies on mathematics education, and with my support, wrote an article about their developmental work (Skúladóttir, 2009).

Working with these teachers offered me strong insights into how teachers' enthusiasm to learn about the ways their students learn mathematics can result in inclusive education in schools that restructure themselves to embrace all children (Ainscow, 1995) and include all their students in a community of learning.

3.3 Grappling with the identity of becoming a teacher

The experiences from studying my own teaching in primary school impacted my teaching in teacher education programs. I urge my students to both reflect on their own mathematics learning in primary school and to engage reflectively with the literature they read about children's mathematics learning. When they learn about new studies that are devoted to children's ways of learning mathematics and different approaches to

mathematics teaching, they become inspired to teach their students in a way that affords all children the opportunity to apply themselves to mathematics learning in meaningful ways. As a teacher educator my responsibility is to guide them in this process. Reflecting on their experience as mathematics learners and relating to their studies, three of my students wrote that the teacher described the procedures for calculating numbers, the traditional algorithm. They then practiced the algorithms individually and did not recall having used mathematical models, discussed their work or explored relationships between numbers. The focus was on memorising and rote learning and the problems were without context.

When the student teachers look back this is what they recall, and that this is the way we learn mathematics seems to be so deeply rooted in our culture. According to their experience, Icelandic classrooms seem to resemble classrooms in other Western countries, as outlined in Stiegler and Hiebert (1999; 2004). Stiegler and Hiebert's description of teaching as a cultural activity which is resistant to change, has made me aware of how difficult it is for undergraduate students to find their identities as teachers.

When I asked the student teachers about other things they did at school, they recalled playing games along with exploring together many fields in science, arts and crafts, where they used mathematics as a tool to measure, calculate, and reason, etc. Nevertheless, they did not view these experiences as relating to mathematics learning. These student teachers perceived themselves as having been active learners. Yet the image they create of their own learning is a stereotype of mathematics learning.

Another excerpt from working with student teachers highlights the stereotypes we have of school culture. While assisting two students write their final examination as primary school teachers, we discussed their experience of learning mathematics in primary school. The student teachers had chosen to explore how young children learn mathematics through play and games. They observed and interviewed teachers who had long experience in creating space for their students where play was seen as essential feature of creating opportunities to learn mathematics. In their first draft to the introduction chapter they wrote:

Much has changed in mathematics teaching in primary schools the last decades and we see this as a positive development. When we went to school, the mathematics teaching mainly consisted of pupils writing answers to the problems in the textbooks. Today the goal is to create a diverse approach to teaching. Pupils are motivated to rely on their own thinking and approach the work on their own premises. Mathematical games and play make the mathematics enjoyable and interesting. (Alda & Rúna, pre-service teachers, March 2010)

I was interested in learning more about their experience and asked them to tell me what they remembered from their mathematics classrooms at the primary level. Both of them remembered writing answers to problems in textbooks and had no recollection of different approaches. They were both impressed by what they had learned from their observations and interviews and expressed their desire to take after the teachers they observed and talked to. During our discussion Rúna mentioned that one of these teachers, had been her teacher in first grade and she remembered her as a loving and caring teacher. When I asked her what she remembered from her mathematics classes, she said that this teacher only taught her in first grade and then proceeded to say: “We were always playing, I don’t remember doing any schoolwork”. This led to a conversation about the nature of the playing and this teacher’s possible goals with relation to why she planned her teaching the way she did. What Rúna recollected from her mathematics learning in school, mainly concerned problem solving in textbooks. Even though Rúna, had at that point come to grasp how young children learn from participating in playing with mathematics Rúna was incapable of connecting this to her own learning experiences as a child.

The conflicting stories of my students’ memories from school have impelled me to seek further explanations. Wenger (1998) noted the dissonance between how we teach in school and how children learn: “To assess learning we use tests with which students struggle in one-on-one combat, where knowledge must be demonstrated out of context, and where collaboration is considered cheating” (Wenger, 1998, p. 3). As a result, much of our teaching and training is perceived as irrelevant, and many children come out of school feeling that learning is boring and difficult, and that school is not for them. Björn, one of my students in a pre-service teacher education program, expressed his feelings about his experiences from school in the following way:

One of my main problems in mathematics is really how lazy I am. I believe it is because I got away with not doing my best in school. I only did enough to get good grades and nothing more. This is a general problem in schools. It has to do with how we evaluate our students. We give them tests and they get their grades. This gives them the message that it’s the grades that are most important in school. We need to find a more effective way to help the students learn to appreciate their learning more than the grades. (Björn, student teacher, May 2006)

Björn had been successful in school and was, at the time he wrote the above passage, preparing to become a primary and lower secondary school teacher. He analysed his learning habits and drew the conclusions that he had received insufficient challenges in primary and lower secondary school, where he only carried out the minimum of work required to accomplish good grades. In upper secondary school, Björn felt that the learning habits from compulsory school prevented him in succeeding in school. He explained further how he had failed in mathematics and science, and as a teenager had the sense that school was not for him. He later realised that he had to rethink his learning styles and gradually discovered that learning became interesting as soon as he devoted his time to his studies and worked diligently. In the teacher education program, he gradually became aware of how important it is to share his thinking with his fellow students, and how much he enjoyed discussing mathematics learning and exploring mathematics together with them. He was adopting a different perspective to learning than he had become accustomed to in his earlier studies, one that places learning in the context of our lived experience of participation in the world, as described in Wenger (1998).

Wenger also explains the kind of insight we can extract from the idea that learning is a social phenomenon with regard to how learning takes place and what is required to facilitate it. According to him knowing is a matter of active engagement in the world; learning revolves around honing our ability to experience the world and our engagement with it in meaningful ways. He stresses that a perspective is not a recipe; it does not tell us what to do, rather “it acts as a guide about what to pay attention to, what difficulties to expect, and how to approach problems” (Wenger, 1998, p. 9).

In his final remarks on the end-of-term assignment Björn wrote that he found the assignment helpful and that it is important to take time to think what one has been doing during the winter. He had discovered things about himself, which he had some vague ideas about, but he found important to write down because they then somehow become more real. He emphasised that his reflection on his own way of studying mathematics helped me to understand his way of learning not only mathematics but in general. The assignment was individual, but Björn wrote about his reflections on learning in a community together with his fellow student teachers. Exploring mathematics with others and discussing with them the research literature on children’s mathematics learning, helped him reflect on how mathematics learning gradually became meaningful to him (Gunnarsdóttir et al., 2008).

The experience from working with pre-service teachers and supporting them in reflecting on their own mathematics learning, as well as drawing on Mason's (1999) writings concerning confidence-building in studying mathematics and reflective practices (Mason, 2002), has aided me in arriving at an understanding of how important it is for pre- and in-service teachers to engage in such reflection.

3.4 Teaching mathematics in diverse classrooms

As a teacher educator, I have taught many in-service courses on mathematics teaching and learning. Many of the teachers who attend the courses expect to learn how to solve problems in textbooks. When they are urged to engage in meaningful explorations using mathematics as a tool and reflect on their students' way of learning mathematics, they are often reluctant to do so. I plan the courses over a period of several months, and ask the teachers to share their experience from their classrooms with the group, looking for a common ground for our mutual learning (West & Staub, 2003). They find this difficult and are reluctant to expound on their classroom practices, which indicated a certain resistance to change, as noted in Rodriguez (2005). When the courses end, I often sense that the teachers have not developed much in their role as mathematics teachers. I was therefore prompted to learn more about the ways in which teachers can be facilitated in developing their competences in teaching mathematics, which I will discuss further below.

With graduate students who enter the university after several years of teaching in schools, the results often are more positive. While they study research on mathematics learning and teaching, they feel inspired to rethink their own way of teaching, and they are also willing to try an investigative approach to mathematics themselves. For several years, I have taught a course on mathematics teaching in diverse classrooms together with two colleagues (Guðjónsdóttir & Kristinsdóttir, 2006; 2007a, 2007b; 2011; Guðjónsdóttir et al, 2007; 2010). From the onset, we have studied the way we approach the teaching of the course and reflected collectively on our work. Our analysis and interpretations have influenced the ongoing development of the course and the focus has moved from a traditional focus on diagnosing deficiencies, to an emphasis on children's mathematical development (Dalvang & Lunde, 2006). Central to the course is a focus on teachers' capacity to evaluate and support students' learning through analysis of engagement in authentic mathematical problems, as well as their reflections on the learning in their classrooms (Fennema et al.,

1993; 1996, Jacobs et al., 2007). The teachers' responses to the activities in the course and their discussions, imply that they gradually become a part of our community of inquiry (Jaworski, 2003).

We, the teacher educators, were aware that teachers need support to develop their practice, both within their own schools and by participating in professional courses. Teachers' professional development involves making explicit the beliefs and values that underlie their actions and practices and contribute to their learning.

All our work, our pedagogy and the trust we have managed to build while teaching the course, enables us to collaborate during our teaching, and demonstrate the possibilities of the ways in which student teachers can teach (Russell, 1997). The process of writing about our research enhances our understanding of how our collaboration and team-teaching (Crow & Smith, 2003) has grown to develop into community of inquiry, where we collectively reflect on our work (Jaworski, 2003). It has also affected the learning community that we have developed together with the teachers. In our experience, the teachers' confidence in teaching mathematics is considerably enhanced through the activity of collectively reflecting on ways to solve mathematics problems. By the same token, their understanding of how children use diverse ways to solve mathematical problems expands. Our findings thus echo those of Boero, Dapuzo and Parenti (1996) and Crespo and Sinclair (2008), who show that problem-solving activity may help teachers to experience and discuss difficulties similar to those met by students in class, and to understand the importance of evaluating the processes entailed in mathematical activities.

3.5 Other projects running alongside with the doctoral studies

The fact that I have been working full time as a teacher educator alongside my doctoral studies, means that I have had other obligations as a teacher and a researcher. Two main projects that I carried out with my colleague Hafdís Guðjónsdóttir have impacted my understanding of the theories and methodologies that I will discuss in the following two chapters. However, I will offer a brief account of these theories and methodologies below.

When we were given the chance to write about mathematics teaching in pre- and primary grades, we decided to report on the findings (Kristinsdóttir & Guðjónsdóttir, 2015) from our former work with teachers in preschool and primary school. In an effort to understand more about teachers' professionalism in teaching mathematics to young children, we decided to collaborate with nine teachers in pre-schools and primary schools, with

whom we had worked on different projects. We created focus groups, which allowed the teachers to meet and discuss their evolving experiences. Through the study, we learned about responsive teachers who create learning environments that foster mathematical understanding and creativity for all children, during their transition from pre-school to primary school. The teachers claimed that they realised efficacy of this experience and stated that the participation in the research project had opened up opportunities to participate in professional dialogue (Cochran-Smith & Lytle, 2009). The teacher education community can, through partnership with teachers, acquire understanding and knowledge about the aspects that teachers need in order to develop and grow. These findings need to be considered both in practice and in teacher education. The next step for this particular research topic could consist in establishing a continued relationship with this group as they become more aware of how to extract learning from the transition itself (Kristinsdóttir & Guðjónsdóttir, 2015).

We conducted another research project together with a teacher, with whom we had collaborated over a period of several years. We focused on how, during the action research period, she developed her strategies for responding to pupils who had difficulties with mathematics. Our goal with this research was to build a bridge between theories that concern mathematics teaching and learning and the practice within schools. Teachers are engaged in working with children and may not have the time or the experience to research their own practice and write about their work. The educators were responsible for the broader data collection through narratives and their analyses (Kristinsdóttir, Reimarsdóttir, & Guðjónsdóttir, 2016).

The work that I engaged in with these experienced mathematics teachers fostered in me a sense of respect for teachers as professionals capable of researching and developing their practice. The influence of collaboration and co-learning in communities, where professionals with diverse background come together and share experiences, prevailed in these studies and all the participants expressed that their work had been strongly impacted by the opportunity to share their experiences.

3.6 Summary

The research projects discussed above have all impacted my understanding of and beliefs about teaching and learning, and significantly influenced my work as a teacher educator and researcher. The experience I gained from participating in and carrying out these studies, along with my work as a

teacher educator, has motivated me to learn more about how teachers can improve their teaching of mathematics in primary grades.

- The teachers I worked with about mathematics teaching and learning in courses for in-service teachers have found it difficult reflect on their work and discuss the learning that takes place in their classrooms.
- In working with teachers and student teachers in formal education settings, both in pre-service teacher education and graduate courses for teachers, I have seen the improvement of their abilities to reflect on their own way of learning mathematics as well as their pupils' learning.
- This experience inspired me to work with teachers that had not participated in courses about mathematics teaching and learning with the intention to learn more about how their participation in a community with colleagues and a mathematics teacher educator has affected their practice.

In the following chapter, I will discuss how this experience has affected my understanding of the theories that I grounded my work on and how my understanding of theories of learning and teacher professionalism has developed through collaborating with children in primary grades, teachers, student teachers and my colleagues and in turn affected my work as a teacher educator.

4 Theoretical perspectives on learning and development

In this chapter, I will explain the ways in which I interpret and understand the impact that theories of learning have had on my approach to teaching, with reference to my previous studies of my own teaching and collaboration with teachers, discussed in Chapter 3. I will begin with my understanding of the concept of learning, drawing on Piaget's theories of child development, and my experience as teacher in primary grades. This I will couple with my interpretation of children's development in mathematical thinking, which emerged through my participation in professional development courses. I then elucidate Vygotsky's theories of learning and how my understanding of sociocultural theories developed during my work as teacher educator, working with teachers and student teachers, as well as participating in courses on theories of mathematics teaching and learning during my doctoral studies. Additionally, I will address communities of practice and teacher learning within such communities, followed by an examination of on how reflective discussions can develop within collaborative communities. Lastly, I will discuss how the process of reflecting on these theories has affected my work as teacher educator.

4.1 Constructivist perspectives

In order to make sense of how my understanding of theories of learning has developed, I will account for how constructivist thinking has affected my way of understanding developments in teaching and learning.

During my teacher education in the early seventies, I was introduced to Piaget's (1969) theories of child development. My work with young children in a preschool, alongside my studies, helped me process my understanding of Piaget's ideas, which in turn offered me a tool through which I could interpret the children's development. This trajectory of relating theory to practice developed further in discussions with my fellow students in the education program. In my work as teacher educator I always emphasise the importance of this relation and I encourage student teachers to connect their learning within the teacher education program to their experiences of working with children.

When I began my teacher career, working with children in primary grades, I relied considerably on Piaget's theories. Throughout my teaching

career, I have found it important to return to my understanding of these ideas in my interpretations of the development of my thinking about learning and how my early acquaintance with his work influenced my teaching.

4.1.1 Piaget's theories of child development

Piaget's (1969) description of the stages of cognitive development, the concepts of assimilation and accommodation, and conservation, affected my teaching from early on. Piaget identifies four stages of development, namely, the sensory-motor, preoperational, concrete operational and the formal operational stage. However, the age at which children accomplish these stages can vary and it is difficult to specify the exact moment when it appears: "What one actually finds is a remarkably smooth succession of stages, each marking a new advance, until the moment when the acquired behaviour presents characteristics that one or another psychologist recognizes as those of intelligence" (Piaget, 1969, p. 4–5).

Piaget describes how assimilation and accommodation mechanisms progress at the sensory-motor level. The child gradually assimilates inputs from reality by modifying them and integrating into an existing schema. He described his findings in the following: "The filtering or modification of the input is called assimilation; the modification of internal schemes to fit reality is called accommodation" (Piaget, 1969, p. 6). Piaget believes that this pure assimilation is inherently pleasurable and categorises this kind of exercising as belonging to play. He maintains that as children enter school, most of them are in the process of passing through the preoperational stage and accessing the concrete operational stage. "The clearest indication of the existence of a preparatory period ... is the absence of conservation until the age of seven or eight" (Piaget, 1969, p. 97), and the children seem to reason only about states or static configurations at this stage, overlooking transformations. Lovell (1979) argues that Piaget's studies of the move from pre-operational thought to concrete operational thought, and from the latter to formal operational thought, have undoubtedly framed teacher's view of his work.

When I taught first graders, together with my colleagues, we drew vastly on our knowledge of Piaget's theories. We organised science stations in the classroom and asked the children to experiment with for instance comparing the weight stones. Moreover, prediction figured an essential part of these experiments and from the children's predictions and our

discussions with them about the topic, we made sense of their developmental stages in light of Piaget's theories.

One of my main concerns, both regarding my teaching of young children and in teaching student teachers about mathematics teaching, was to discover how to bolster children's development of mathematical thinking and how teachers could reinforce this development. I found Piaget's developmental theories of intellectual development useful in resolving these issues as well as his distinction between the two aspects of child's intellectual development. More specifically:

- One is the psychological aspect, that is, everything the child receives from without and learns in general from family, school and educative transmission.
- The other is the development aspect, which can be called spontaneous, that is, the lengthy period of time during which the child learns to make sense of the world by him/herself.

Piaget also emphasised that thinking precedes language and that the child uses language to modify and support thinking and to communicate her/his inner thoughts to others (Piaget, 1969). By the same token, as in the project discussed above, I sought to strengthen the children's psychological and spontaneous development by organising opportunities for them to experiment with things in their surroundings and creating an environment where they could discuss their work with their classmates and their teacher.

These ideas then developed further through professional developmental courses and collaboration with colleagues. In the mid-nineties, I was invited to participate in a workshop in Iceland, which was inspired by the Cognitively Guided Instruction (CGI) project, discussed in 2.2 and 3.1. What I learned there strongly inspired in my work as a teacher in primary grades, for example the analysis of the development of addition and subtraction concepts and skills, reflected in children's solution of different types of word problems (Carpenter et al. 1989), affected.

The findings of studies conducted by Carpenter and his co-researchers (1989) correlated with Piaget's findings about the move from the concrete operational to the formal operational stage. They discovered that children's solution strategies developed from concrete operations, modelling the actions described in the problem while finding a solution, to a more sophisticated approach. As children's experiences with modelling solutions develop, they begin to rely on counting strategies, and in this way, they

gradually gain the ability to use flexible approaches to solving problems. Moreover, they derive at solutions based on their knowledge of facts about numbers they learned through their explorations. The findings from the CGI project encouraged me to interact more with the children in my classroom and I started to record my observations of their work. These observations guided my decisions regarding the choice of content for the following lessons and what kinds of problems to use, so that all the children in the classroom would gain from participating in solving them, as described in Section 3.1.

Even though the age differed for when the children in the CGI studies passed through these stages, the researchers found remarkably consistent findings across a number of studies and drew similar conclusions about how children solve different kinds of problems. They also described in detail how the children's use of solution strategies varied depending on the complexity of the problem. Children, who used derived facts when solving problems of a familiar type with low numbers, might need to model the situation in a more complex problem. These findings accord with Piaget's description of the lag of operations that can create an obstacle to the generalisation of stages and introduce considerations of caution, and limitation (Piaget, 1973). The lags, according to Piaget, characterise the repetition or the reproduction of the same formative operation at various levels, both horizontally and vertically. These findings from the CGI studies, as well as my former experience as teacher in lower primary grades, and along with my knowledge of Piaget's theories, all helped me in deciding how to respond when children seemed to have regressed in their solving problems development (Kristinsdóttir, 2003; 2006; 2007).

My experience of teaching children mathematics has strengthened my capabilities in working with pre- and in-service teachers, and below I will discuss further how my reflections on this experience have informed my teaching as teacher educator. I will start by discussing constructivist theories based on Piaget's theories of learning and how my interpretation of these ideas, have affected my work as a teacher and teacher educator.

4.1.2 Constructivism

Modern constructivism can be seen as deriving directly from Piaget's work in genetic epistemology (Copeland, 1974; Confrey, 1994; Jaworski, 1994). "For Piaget knowledge does not exist in static form but is a constructive process characterized by its origin and development" (Inhelder, 1976, p. 1). Glasersfeld (1995) argues that Piaget was not the first to suggest that we

construct our concepts and the image of the world we live in, but he was the first to pursue the developmental approach. He criticises how Piaget's theories were interpreted by psychologists in the late 19th century and claims that many psychological books provided an incomplete view of Piaget's theory. Glasersfeld asserts that Piaget's thinking and ideas never ceased to develop and reading only few of his articles would only offer a limited perspective of his work. He also criticises researchers in mathematics education who espoused the notion that children progressively build up their cognitive structures, but disregarded the fact that Piaget changed the concept of knowledge, stating that knowledge is not a picture of the real world. Glasersfeld built further on Piaget's conception of knowledge and put forth a radical model of constructivism, based on his understanding of how we interpret our world:

- knowledge is not passively received but built up by the cognizing subject;
- the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality. (Glasersfeld, 1995, p. 18)

This radical description of constructivism does not presume any relation to social influence in the learning processes, even though, Glasersfeld (1995) often referred to social interaction in his discussion about learning. Since social inference is part of the world around us it is implicitly included and not ignored. Confrey (1995c) highlights this point and writes that in constructivist terms, the self develops through experiences in the physical world through communicating with others.

Ernest (2010b) differentiates between individualistic and social construction of knowledge and understanding. He argues that individualistic learning is seen as internally constructed, based on personal interpretation of experiences and pre-existing knowledge. He sees social learning and knowledge construction as taking place within social relationships, even if its end products are internalised individually. Ernest (2010a) notes there is danger entailed in an overemphasis on individuality in concepts of knowledge construction, especially if this emphasis disregards the social basis for interpersonal communications within a shared space of feelings and constructs. He voices a concern that this view of individuality might lead to a position where interpersonal relations are seen as being competitive. In his discussion of social constructivism, Ernest (2010b) argues that, according to social constructivism perspective, individual and collective learning are intertwined. Thus, mind is viewed as social and conversational and individual thinking is formed by internalised

conversation. He points out that these assumptions stem from Vygotsky's account of the origins of language, namely, language is something that the individual internalises by participating in social activities. Ernest stresses that in mathematics education, social constructivism places emphasis on teacher-learner and learner-learner interactions, including negotiation, collaboration and discussion, and the role of language, texts and semiotics.

Goodchild (2001) also raises concerns about the constructivist notion of the individual, who interprets his/her experiences of the physical world, neglecting the communication with the outer world and with other people. In reviewing the differences between Vygotsky's and Piaget's theories, he states that they differ fundamentally, particularly in terms of Vygotsky's idea of knowledge as social and cultural, in contrast to Piaget's understanding, according to which cognitive development is primarily dependent on biological maturation and only secondary to interaction with the world. In 4.2.1 I will discuss further my understanding of Vygotsky's theories of the development of language and social participation.

In the above descriptions regarding interaction with children in my classes, based on my knowledge of Piaget's theories, and the findings from the CGI studies, social interactions were emphasised and interactions with the children were planned both for their benefit and my own. Organising opportunities for children to solve mathematical problems and discuss their work with their classmates and their teacher, was an attempt to create opportunities for them to interact with others and the outer world, with the aim of facilitating the construction of their understanding of mathematics. From my interactions with them, I observed their overt activities and in my discussions with them I could probe for further information to support me in making sense of their development in mathematical thinking, and thus plan my future activities with them. In this relation, Noddings (1990) argues that if we want to be able to teach well, we need to know what our students are thinking. However, she insists that the cognitive premises of constructivism can only furnish us with guidelines for good teaching: "We cannot derive from them, any more than we can from any other cognitive position, specific teaching methods" (Noddings, 1990, p. 15).

Steffe & Thompson (2000) argue that in order to learn to engage in productive mathematics teaching, researchers need to conduct teaching experiments where they seek to bolster productive mathematical learning. Steffe (2004) based his research on the concept of a hypothetical trajectory, presented in Simon (1995). According to Steffe (2004), it is hypothetical because the actual learning trajectory is not known in advance; it is

constructed during and after the experience through intense interaction with children. He further explains that:

By building an understanding of children's mathematical concepts and operations and how a teacher can engage children to bring forth changes in those concepts and operations, a vision of children's mathematics education can emerge in which children engage in productive mathematical learning and teachers engage in productive mathematical thinking. (Steffe, 2004, p. 130)

Similarly, Clements and Sarama (2004) developed learning trajectories for young children's mathematical learning based on several research projects with the aim of promoting children's achievement of specific mathematical goals. In defining the trajectories, they referred to Simon (1995) and his definition of learning trajectories. This view of teaching as an approach built on constructivism is criticised by Lerman (2014), who explains that Piaget's theory of constructivism, is a theory of learning, not of teaching. According to Lerman, teachers' actions in the classroom cannot be described as constructivist, or anything else, and it is not sufficient for a teacher or a researcher to claim that they are taking a constructivist approach to teaching.

My studies of constructivism have prompted me to look deeper into the ways in which constructivist theories of learning figured into my teaching in early primary grades and as well as the teacher collaborations. At the CGI course, discussed in 3.1 and 4.1.1, we were offered to engage with ideas on how children's intuitive mathematical ideas form the basis for the development of more formal concepts and procedures. However, we were never given any instruction about how to teach children to further their development. This was also the case in the CGI studies on teachers' development, in which the focus was directed at helping teachers build relationships between their research-based model on children's thinking and the thinking of the pupils in the teacher's own classes. The teachers were encouraged to reflect on how the model could be interpreted in light of their own pupils and classrooms and at workshops they discussed their experience with other teachers and the researchers (Fennema, et al., 1996). The teachers were thus expected to be able to use the findings from the research study and make decisions about their teaching of children, based on their own interpretation of how these findings could prove fruitful in their classrooms. This was also the case for us who participated in the CGI course in Iceland in 1995. We decided to meet and aid each other in making sense of what we noticed in our classrooms and how the learning of the

children in our classes could be interpreted in view of the CGI findings. At this time, I was researching my own practice and preparing for my master's thesis and I was therefore searching for writings that could animate my interpretations of my own professional learning.

When I read Glasersfeld's description of teachers' interpretation of children's learning it resonated with my own experience at the time. He wrote:

The teacher's assessment of a student's conceptual structures does not have to be a blind conjecture. If one starts from the assumption that students generally try to make sense of their experience, it is usually possible to get some idea of how they think. The more experience with learners a teacher has gathered, the better the chance to make an educational guess about what a particular student's thinking might be and to hypothesize what Vygotsky aptly called 'the zone of proximal development'. (Glasersfeld, 1995, p. 187)

Glasersfeld maintains that teachers might gain confidence in their conceptual portrait of students after working with them for a considerable time, which would thus provide them with the necessary tools to support them at the level where they were maturing. This was in line with my experience, as well as the teacher's with whom I collaborated. We told each other what we had noticed in our classrooms and reflected on how we, as teachers, could support our pupils in developing their mathematical thinking, with reference to the findings from the CGI model. From our discussions, we found that visits to each other classrooms could be helpful in this process and we also began to videotape our lessons, which furnished the opportunity to reflect together on what we saw and heard. Results from our reflections, individual and collaborative, supported us in making sense of the potential the children in our classes had for developing their mathematical thinking.

Jaworski (2006a; 2008b) has, like Lerman (2014), criticised the constructive frame that was common in teacher educational programs in the 1980s and 1990s. She argues that educators recognised teachers as independent 'cognisers', that is, teachers construct knowledge of mathematics teaching through their experiences. In consequence, they provided the teachers with relevant experiences to construct the knowledge they require for their teaching. Thus, when the providers of the experience, both the teachers and the educators, are not satisfied with the apparent constructions that emerge, a conflict of interest may arise.

In our teaching we used the findings from the CGI project, which is knowledge about children's construction of mathematical understanding gained from others, however, we did not see this as preventing us in developing in our work. On the contrary, we felt that together we managed to build a learning community where we could share our mutual experiences and knowledge, developed through our critical discussions, and by means of individual and collective reflections. During this process, we related the CGI findings to our experience of working with children, both our former experience and new learning acquired from observing the children's learning in our classes, through their lenses. Our experience thus was more in line with Rogoff's (2003) description of sociocultural approaches, where individuals change their ways of understanding by building on the practices and traditions of communities. In a similar vein, we collaboratively reflected on our practice and together made sense of the learning we experienced was taking place in our classrooms. Through this experience, I discovered that my understanding of sociocultural theories of learning was developing, and I consequently wanted to learn more about how teachers can learn together by means of collectively reflecting on their own practice and relating to these reflections to theories of learning (Kristinsdóttir, 2006; 2007; 2010a; 2010b).

In her critique of the constructive perspective, common to approaches to the teaching and learning of mathematics, Jaworski (2008b) writes that teachers and educators must find fruitful ways that foster environments in which learners can construct of relevant knowledge for the learners. She underlined the importance of collaboration between teachers and mathematics teacher educators in relation to how to create an environment where meaningful mathematics learning can bloom. Moreover, she claims that even though constructive theoretical positions recognise the importance of social interactions in promoting growth of knowledge through inquiry, they prioritise the individual learner rather than attending to the wider scope of learning in classrooms (Jaworski, 2006a).

Jaworski (2008b) emphasises the shift in paradigms that has been visible in the literature on mathematics education lately, and claims that this shift is not from the constructivist to the sociocultural, but rather to a recognition that different lenses on practice can afford different ways of seeing and doing, and that we can learn from them all. Her discussion has informed my understanding in relation to how my approach to teaching and learning mathematics has developed in concert to these trends, and how my attention on the individual constructing knowledge of mathematics has shifted to the social interaction, where teachers and students learn

together and contribute to each other learning. In 3.1, I discussed how my focus was to being with directed at individual children's learning, and through reflections with them, I learned that we were learning together. This notion developed when I met other teachers of young children and we collectively discussed our learning as described in 3.2. In sections 3.3 to 3.5, I will further account for how collaboration with colleagues within the teacher education challenged me to focus on our collective learning, as well as on what we learned through our communication with our students.

My focus on collaboration between teachers in schools and mathematics educators, developed further during the first years of this century, in conjunction with the notion of how both partners could gain from such collaboration. In reflecting further on my experiences of working with colleagues, as well as in-service and student teachers, I have come to understand this steady development of my opinion concerning how teachers in teaching mathematics should be supported. With respect of the fact that people needed to develop their own understanding of teaching I also learned that social interaction is an essential part of such development. I will discuss this further in the following section.

4.2 Sociocultural perspectives

Sociocultural theories build on the conception that individual and cultural processes mutually constitute people's development. Rogoff (2003) argues that the sociocultural approach has shifted our understanding of cognition from prioritising individuals' thoughts, addressing only active processes of individuals as they engage in the activities of cultural communities, and then to collective thinking. Cognitive development thus is not the acquisition of knowledge or skills; rather it consists of individuals changing their ways of understanding, perceiving, noticing, and thinking, in shared efforts with other people, and building on the cultural practices as well as traditions of communities.

Lerman (2000a) discusses the social turn in mathematics education research and the emergence of theories that see meaning, thinking and reasoning as products of social activity, and therefore integral elements to the mathematics education research community. He refers to his former critique of constructivist theories, discussed above, and the notion that social interactions provide a spark that generates or stimulates an individual's internal meaning-making activity. According to him, a major challenge for the social turn theories "is to account for individual cognition and difference, and to incorporate the substantial body of research on

mathematical cognition, as product of social activity” (Lerman, 2000a, p. 23). Furthermore, Lerman argues that Lave’s (1988) description of how grocery shoppers and dieters used mathematics in their practices challenged constructivism and transfer theory in mathematics learning. According to him, her description raised fundamental questions about the conceptions that view mathematics practices in out-of-school situations merely as the application of school techniques. Lave (1988) emphasises that these practices had to be seen as situated within, and as products of those situations. Thus, the strategies and decision-making procedures that people used in those situations should be seen as a part of what one is ‘becoming’ in that practice.

4.2.1 Vygotsky’s theories of child development

Another key element in sociocultural theories in mathematics education, according to Lerman (2000a), is the work of Vygotsky and his colleagues. Sociocultural theories are historically linked to the work of Vygotsky (Confrey, 1995b; Daniels, 2001; Goodchild, 2000; Chaiklin & Hedegård, 2005). Vygotsky was the first to attempt to relate Marx’s theory of society to concrete psychological questions, and in that relation elaborates on Engels’ concept of human labour and tool use as the means through which man changes nature and transforms himself (Cole & Scribner, 1978; Confrey, 1995a; Engeström & Miettinen, 1999).

Vygotsky maintains that the internalisation of culturally produced sign systems brings about behavioural transformations and forms a bridge between early and later forms of individual development. Thus, for Vygotsky, the mechanism of individual developmental change is rooted in society and culture (Cole & Scribner, 1978), and where there was no separation of the individual and its social environment (Cole, 1985). Vygotsky distinguished between two developmental processes:

- One is the elementary that is of a biological origin
- The other is the higher psychological function of sociocultural origin.

“The history of child behaviour is born from interweaving these two lines” (Vygotsky, 1978, p. 46). To study the higher psychological functions, one must look for the developmental roots of the use of tools and speech that arise during infancy. The work of Vygotsky and his followers have had major influence on our understanding not only of child development, but also on how teachers develop their identities, by way of participating in sociocultural practices.

From his research on children's acquiescence of language, Vygotsky (1978) theorizes that, prior to mastering their own behaviour, children begin to master their surroundings with the help of speech, which not only facilitates the children's effective manipulation of objects but also controls their own behaviour. From his observations with colleagues he concluded that children solve practical tasks aided by their speech, as well as their eyes and hands. The child plans how to solve the problem through speech and then carries out the prepared solution through overt activity. In this sense, there is a dynamic relation between speech and action in children's development. At an early stage, speech accompanies the child's actions but later it moves more and more to the starting point of the process. Then it acts as an aid for planning what has been conceived but not yet realised in behaviour. According to Bruner (1985), speech, in Vygotsky's sense, is a way of sorting out one's thoughts about things, and that thought is a mode of organising perception and action. He refers to Vygotsky's writings about higher concepts transforming the meaning of lower concepts, where Vygotsky gave an example of an adolescent who has mastered algebraic concepts and arriving at this level of understanding enables him to see arithmetic concepts in a broader perspective than before.

When I organise the space for teachers at workshops with the purpose of solving problems and discussing their solution strategies, my hope is that as they think about their own understanding of mathematics, they will achieve this level of higher reflection. By means of the process of discussing their thoughts about their approach to solving problems, they may realise how they can create a learning space for their own students that in turn may allow them to think about their own understanding of mathematical concepts and view them from a broader perspective.

Vygotsky (1978) uses the concept of 'psychological tools' as a way to describe the mechanism through which mental processes are mastered. He saw them as artificial, not organic or of individual origin. Examples of tools are language, systems for counting, algebraic symbol systems, works of art, conventional signs, writing, schemes, diagrams and maps (Confrey, 1995b; Wertsch & Toma, 1995; Daniels, 2001). In his view, tools and signs as mediating functions and claimed that they may, therefore, from the psychological perspective, be subsumed under the same category:

The specifically human capacity for language enables children to provide for auxiliary tools in the solution of difficult tasks, to overcome impulsive action, to plan a solution to a problem prior to its execution, and to master their own behaviour. Signs and words serve children first and foremost as a means of social contract with other people. (Vygotsky, 1978, p. 28)

Lerman (2000b) claims that cultural tools, such as the natural numbers, transform us internally, like physical tools, because they form and transform the world and enable us to act and see the world differently. The teacher, which may be a peer or a textbook, is central in providing these tools, which they then use to mediate the world to the learner.

To give an example of how I interpret the ways cultural tools, such as language and other signs, can serve as means to mediating thinking, I will provide an example from my classroom with fourth graders. The children had been assigned the following problem to solve at home and discuss with their family.

A family of five likes to change seats at the dinner table. Each evening one member of the family changes seat with another member to take a different seat each time. How many days will it take until they have all changed seat with each other?

Pétur told us how he and his father had discussed the problem as they drew a picture of their solution. I asked him to show us how they made the picture. He drew a circle on the blackboard and then placed five dots around it representing the family sitting around the table. He then drew four lines from one dot to the other four dots and kept on connecting the other dots to represent how the family members could change place with each other. When he drew the picture he told us how he and his father had reasoned while solving the task together.

From his story it was evident that they had mediated their thinking about the problem with the help of language and other cultural signs such as drawing. Pétur then mediated their interpretation of the problem to his classmates and his teacher with the help of language and drawing, and we then discussed their interpretation, each of us making our own mental images of the solution, as we interpreted his explanation. When Pétur had finished describing their collective solution to the problem, there was a picture of a pentagon on the blackboard and a star with five arms inscribed (Kristinsdóttir, 2007). My interpretation based on Pétur's narrative, was that his mathematical thinking developed under his father's guidance, as I will discuss further.

Based on his notion of mediation, Vygotsky describes the zone of proximal development (ZPD) as:

... the distance between the actual developmental level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance and collaboration with more capable peers. The zone of proximal development defines those functions that have not yet matured but are in the process of maturation, functions that will mature tomorrow but are currently in an embryonic state. ... The actual developmental level characterizes mental development retrospectively, while the zone of proximal development characterizes mental development prospectively. (Vygotsky, 1978, p. 86)

Vygotsky adds that: “human learning presupposes a special social nature and a process by which children grow into the intellectual life of those around them” (Vygotsky, 1978, p. 88). He also states that learning awakens developmental processes that are only come to fruition when the child is interacting and collaborating with peers and other people in his/her environment. He also emphasises that good learning is solely that which advances development. The focus on the teacher, that may also be a parent, a more informed peer or anyone else that can support the child in the learning process, is central in Vygotsky’s description of the learning process. Later in this section, I will further discuss the ZPD in connection with teacher development.

In my story of Pétur’s solution to the dinner table problem, his father assisted him in making sense of the problem. From his story of their discussions it was obvious that his father had not told him how to solve the problem, but rather asked him questions and helped him create his own diagram of the situation as Pétur himself envisaged it. During this process, Pétur’s understanding of the problem developed and he was later able to explain his understanding of the problem to us. In Vygotsky’s words, under adult guidance his thinking had matured from his actual developmental level to the level of his potential development.

In Bruner’s interpretation of this process, Pétur’s father was scaffolding him in this process and Pétur then in turn was scaffolding us. Bruner (1985) describes how he puzzled for many years about Vygotsky’s definition of the zone of proximal development. He saw a contradiction in Vygotsky’s proposition that in learning situations children achieve consciousness and control, since consciousness and control only come after one has already achieved a proper grasp of a function and spontaneously mastered it. He writes about his understanding of how good learning could be in advance of development:

If the child is enabled to advance by being under tutelage of an adult or more competent peer, then the tutor or the aiding peer serves the learner as a vicarious form of consciousness until such a time as the learner is able to master his own action through his own consciousness and control. When the child achieves the conscious control over a new function or conceptual system, it is then that he is able to use it as a tool. Up to the point, the tutor in effect performs the critical function of “scaffolding” the learning task to make it possible for the child, in Vygotsky’s word, to internalize external knowledge and convert it into a tool for conscious control. (Bruner, 1985, pp. 24-25)

Bruner’s definition of the scaffolding process has affected my understanding of teaching and learning, as I have explained above, and the same holds for Vygotsky’s definition of the zone of proximal development.

Confrey (1995b) contends that when an adult or a peer scaffolds a child, the latter is enabled to solve a problem, carry out a task or achieve a goal, which the child could not accomplish without assistance. She notes that scaffolding requires the learner to be able to recognise a solution to a particular class of problems before he himself can produce the steps leading to an unaided solution. Confrey argues that, for Vygotsky, imitation is not a simple or mechanical process. To be able to imitate one must possess the means, which allow one to step from something known to something that is new. She maintains that this view of a central role of imitation led Vygotsky to argue for the ZPD. In the case of Pétur’s solution this was evident. He was capable of drawing the picture of his solution independently and he was not mechanically copying a picture that he had drawn at home with the support of his father. From this experience, I also learned that Pétur and his father were scaffolding me in improving my practice, since it aided the development my geometry teaching, as will be discussed in the following section.

In the collaborative research with teachers, accounted for in Part II, I sought to enable them to rethink their understanding of learning mathematics by offering them problems to work with, and by creating communities for scaffolding each other. By reflecting on their stories from their classrooms, we could mediate our understanding of the experiences they brought to us in our workshops. The main tool we utilised in mediating our discussions was language, but examples of the children’s work also served as tools for mediating the learning that the teachers interpreted as taking place in their classrooms. As the teachers discussed the classrooms learning we would collectively support their reflections and in this way, provided

them scaffolding for further collaboration with children in their classrooms and creating space for them to develop within their zone of proximal development.

4.2.2 Adopting the zone theory to work with teachers

In my search for more ways of supporting teachers in developing their mathematics teaching, I was inspired by Goos's (2008) exposition of Valsiner's idea of the zone. The theory offered me new insights into how I could interpret the teachers' learning and my own from a sociocultural perspective.

Valsiner (1997) re-interprets and extends Vygotsky's notion of the zone of proximal development (ZPD) to include two additional zones of interaction, namely, the zone of free movement (ZFM) and the zone of promoted action (ZPA). In Valsiner's zone theory of child development, the ZPD is regarded as a set of possibilities for development that are in the process of becoming realised. To explain how individuals negotiate their relationships with their learning environment and the people in it, he incorporates two additional zones to the model of social setting and the goals and actions of participants.

- Valsiner describes the ZFM from a developmental perspective, as a restrictive psychological mechanism that the adult creates to constrain the freedom of the child's choices of thinking and acting. Hence, the ZFM structures an individual's access to different areas of the environment and objects within it as well as the ways an individual is enabled or permitted to act with the objects within the accessible environment.
- He defined the ZPA as a set of activities, objects, or areas in the environment by which the child's actions are promoted, where the child is not obliged to accept the ZPAs.

Valsiner (1997) emphasises that the ZFM and ZPA are dynamic and inter-dependent constructs that are continually being reorganised through the adult-child in learning interactions. Valsiner's theory thus suggests that when a particular experience is promoted, some other event is excluded. Placing Valsiner's theory into the classroom context, the teacher's decisions about what to promote and what to allow defines the learning opportunities therein.

Goos (2005; 2008) developed a research program that applied Valsiner's zone theory to pre-service and in-service teacher learning and development. Valsiner's approach defined all zones from the perspective of

the teacher as learner, in order to offer a dynamic way of viewing teacher learning as identity formation.

- The teachers' zones of proximal development (ZPD) were identified as a set of developmental possibilities that are influenced by their knowledge and beliefs about the teaching and learning of mathematics.
- The ZFM is interpreted as constraints within the teacher professional context. These included students' behaviour and motivation and access to resources, such as teaching materials, curriculum and assessment requirements, organizational structures and cultures.
- The ZPA, on the other hand, represents teaching approaches like those promoted by pre-service teacher education, professional development activities, or informal interaction with colleagues.

In order to ensure teachers' learning, Goos (2008) concludes that the zone of promoted assistance (ZPA) must engage with their possibilities for development (ZPD) and promote actions that they see as viable within their professional context (ZFM).

Goos (2008) further elaborated the use of Valsiner's zone theory and its application in the context of the teacher-educator-as-learner, to which she proposes a third layer. In order to understand how zone theory might help analyse this dual role as mathematics teacher educators and researchers, Goos (2008) sketches out what such an analysis might look like. As a researcher, she saw academic structures and cultures, within and beyond her own university, as constraints to her ZFM. Her initial training as a researcher shaped her ZPA, which included participation in conferences and other activities associated with research projects, as well as being mentored by more experienced colleagues. Her possibilities for developing as a researcher (ZPD) were moulded by this complex ZFM/ZPA, which defines what is allowed and what is promoted. As a teacher educator, her zone of free movement was thus constrained by her students' characteristics, curriculum and assessment requirements, limited access to technology resources, reduction of hours for teaching methods courses, difficulties in finding practicum placements, as well as her academic colleagues' perception that teacher education is low status work. She found that her ZPA, in this role, were less clearly defined and it was difficult to identify people or activities that empowered her development as a teacher educator. Therefore, she struggled with describing the ZFM/ZPA complex that shaped her teacher education practice. Nevertheless, she discussed how, from a sociocultural perspective, she was able to identify the aspects

in her own teacher education research that acted as a ZPA and informed her practice as a mathematics teacher educator.

My intention with urging the teachers in my new study (Part II) to attend to their own mathematics learning, as well as to their pupils' mathematical learning, was to add to their zone of promoted action. I planned the workshops in collaboration with them, and in choosing what features to address, our zone of free movement was consequently restricted. It is important to be aware of these factors and how each step taken in a developmental process affects the communities we are working within, along with the range of free movement and promoted actions. In interpreting the learning I extracted in the process of teachers' collaboration, I also bring to bear Goos's conception of the zone of proximal development for the teacher educator.

In the following section, I discuss communities of practice, students and teachers' goals for mathematics learning, and how teacher knowledge about mathematics teaching can impact activities and the learning in their classrooms.

4.3 Communities of mathematics learning in schools

Teachers work in schools, which are their communities of practice in which their learning is situated. Building a community of mathematical practice with their pupils requires teachers to take the lead in establishing expectations and norms, where the active participation of all participants is acknowledged.

My interpretation of communities of practice is grounded in Wenger's definition of the four components that underlie the social process of learning and knowing: meaning, practice, community and identity. Moreover, Wenger contends that a fruitful social theory of learning must integrate these characteristics (Wenger, 1998). Individuals become what they are through the negotiation of meaning and interplay of participation and reification. Neither participation nor reification can be thought of in terms of contrasts of individual versus collective, or private versus public. According to Wenger, participation is both a social process and a personal experience. Reification helps us coordinate our actions and shapes our perceptions of the world and ourselves and is therefore both collective and individual.

4.3.1 Situated practice

Lave and Wenger (Lave, 1998; Lave & Wenger, 1991) define learning as a situated activity, or a process which they call legitimate peripheral participation. It refers both to the development of identities in practice and to the reproduction and transformation of communities of practice. Thus, learners participate in a community of practitioners. What is more, the demands of the mastery of knowledge and skill, prompts the newcomers toward full active participation in the sociocultural practices of the community. According to this view, knowledge is about competence in real-life settings, and has to be understood relationally, that is, as a relation between people and settings. Lerman (2000a) argues that this understanding of learning could create particular problems for mathematics education, as “the notion of transfer of knowledge, present as decontextualized mental objects in the minds of individual, from one situation to another, becomes perhaps untenable but at the very least require reformulation” (Lerman, 2000a, p. 26). Lerman sees the relational understanding of knowledge as being consistent with the practices within mathematics as a discipline, since mathematical modelling is the application of apparently decontextualized knowledge to almost any situation.

In the school classroom, pupils do not choose how to work with mathematics, and even though the act of learning to read mathematical tasks seems to imply decontextualized thinking, it is an overt feature of the practice of the school mathematics. The teacher may want to engage her pupils in what she interprets as mathematical ways of thinking and acting but their goal might be altogether different. The pupils’ background affects how they understand and interpret the mathematical problems and how to approach them. Thus, the agents of the apprenticeship are not only the teacher and the texts, but also the acceptance or acquiescence of the pupils who become apprenticed. In an attempt to approach the mismatch between these goals, Lerman called for analysis of the nature of these goals and of classroom practices. I will discuss this issue in more detail below, in relation to the studies that have shed light on classroom practices and the diverse goals that participants carry with them into classroom communities.

Bernstein’s (2000) description of how control and power are manifested in pedagogical relations has added to our understanding of the access pupils have to communities and different discourses in mathematics classrooms, depending on their social background. He distinguishes between what he calls the ‘classification rules’, which refer to the boundaries between discourses, such as school curriculum, and the ‘framing rules’ that

are in the hands of those who take control of the discourse. He defines pedagogical discourse as the process of moving a practice from its original or specialised knowledge site, and relocating it at a pedagogical site. The original site for mathematics is the practice of mathematicians and application of mathematics in everyday life takes places, whereas the pedagogical site concerns school mathematics. The mathematics practiced at school is therefore not the same as the mathematics used in practices outside the school.

In the preceding chapter, I discussed an event from my own fourth grade classroom where Pétur exposes his solution to a problem that he solved at home with his father. He drew a picture on the blackboard while he described their thinking, and the completed drawing on the blackboard revealed a picture of a pentagon. His drawing prompted me to relate this method to other projects we had worked at. As mentioned in the previous section, Pétur was scaffolding the problem for me through the drawing the pentagon on the blackboard. We had done research on regular polygons and studied which of them could tessellate. We had yet to work with pentagons so I decided to use this opportunity to bring in regular pentagons and see at if they would tessellate, bringing the pentagon into our mathematics discussion, and thereby moving it to the pedagogical site of the school. The children were quick to observe that the pentagons did not tessellate and explored ways to use other polygons to fill the empty spaces between them. When we returned to the classroom after a break the same day, the children brought in the soccer ball they had been playing with, and pointed out that it was made of pentagons. Together we observed the ball and discovered that it was made of both pentagons and hexagons. In this way, the children brought knowledge from the original site into the mathematics classroom, where in turn they discovered how mathematics knowledge is used in everyday life.

I asked the children why the soccer ball was made of both pentagons and hexagons and not only of either pentagons or hexagons. They realised quickly that it was impossible to make the ball with only hexagons because they tessellate. "There is no space between them and therefore it will never get curved" Dóra explained. Later I brought a dodecahedron into the classroom and the children studied how many pentagons were needed to make the solid. The children saw that the surface of the dodecahedron was not as 'smooth' as the surface of the soccer ball and realised how smart it is to use both hexagons and pentagons to make the ball (Kristinsdóttir, 2007).

Later, when reading Bernstein's description of the classification rules and framing rules, and his thoughts on pedagogical discourse, I recalled this instance and realized how we had been moving between the original site of mathematics and the pedagogical site. As a teacher, I saw the opportunity to adapt our discussion to the school curriculum in geometry, acting according to the classification rule. I framed and took control of the discourse as it was transferred to the pedagogical site, by the same token, the children took control when they brought in their observations of how mathematics is used in everyday life.

Boaler's (1999; 2002) research within different communities of mathematics classrooms, has enriched our understanding of the diverse factors that impact mathematics learning in schools. Her interpretation, framed in terms of a situated cognition perspective of the different cultures in two schools, informed my understanding of how this perspective lends itself to an interpretation of mathematical behaviour in schools and the interactions between the participants involved (Boaler 1999). Her findings from a research in two schools reveal that the traditional demonstration and practice methods that were used in one of the schools encouraged students to develop mathematical beliefs and practices that were effective in the mathematics classroom but ineffective in other places. In the other school, the project-based methods of teaching encouraged the students to develop mathematical beliefs and practices that were more consistent with the demands both outside of school and within the classroom (Boaler, 2002). She emphasises that situated perspectives turn the focus away from individual attributes and towards broader communities. Furthermore, she cautions that the implications of this shift within mathematics education research toward communities or activity systems generated within mathematics classrooms, does not imply that we should ignore the cognitive processes that students produce within such systems.

In the following section, I offer an example of how I have attended to the diverse background of teachers in pre-service, graduate and in-service courses by emphasising explorative work and investigation in mathematics. My vision is that through building and participation in learning communities and engaging in collective reflection on such problems, the teachers will gain confidence in working with their pupils in similar ways.

4.3.2 Creating communities of mathematics learning

The teacher is responsible for building communities in mathematics classrooms that enable pupils develop their mathematical thinking and capa-

bility to use it as a tool in their everyday life and work as well as in further studies. To be able to enhance learning in mathematics classrooms, teachers need to have a knowledge base that supports them in their teaching approaches in such a way that all participants in the classroom community will gain from it.

In mathematics classrooms, teaching needs to be directed towards facilitating student's achievement of learning goals and the teacher is responsible for creating classroom interactions among teachers and students around content (Hiebert & Grouws, 2007). This impelled me to engage further with my inquiries into knowledge for teaching as a means to investigate how collaboration in teacher developmental projects can lead to improvement in their teaching.

To situate this discussion on the site of my experience in teaching pre-service teachers, I will offer an example of my work with them in which I have advised them to reflect on their work. Drawing from Mason's notion of building confidence and the importance of convincing yourself and others (Mason, 1999), I gave them problems to solve and discuss with each other. In planning this work I relied on the ideas of Mason and Johnston-Wilder (2006), who emphasised that learners enter lessons with natural, or innate, powers:

- to imagine and detect patterns;
- to express those patterns in words, pictures, actions and/or symbols;
- to choose special cases of generalities in order to try to see what is going on;
- to re-generalize for themselves;
- to make conjectures;
- to modify those conjectures in order to try to convince themselves and others. (Mason, & Johnston-Wilder, 2006, p. 34)

Following his work with patterns and generalisations and discussions with his fellow students, Björn, one of my pre-service students, proceeded to work on the problem at home and then shared his thinking with us in a later lesson. The pre-service teachers had been looking for patterns and expressed those patterns in words, pictures, actions and/or symbols. They were then urged to choose special cases of generalities in order to try to gauge these patterns and then to re-generalize for themselves and their co-learners. Together they made conjectures and found that the sum of integers could be expressed as: $1+2+3+\dots+n = n(n+1)/2$. In their search for generalities in the pattern of the sum of squared integers, they had made conjectures but proved unsuccessful in modifying them in order to convince

themselves and others. Björn was not satisfied with his contribution to the discussions in class and continued to think about the problem at home and in doing so motivated his core mathematical awareness (Mason, 2008). By sharing his thinking, he supported his fellow students in developing ways to make sense of mathematics, and collectively we all gained a new understanding of mathematical investigations in schools. Mason used the term 'core mathematical awareness' to underline the point that learners "need to experience and integrate into their functioning various actions which lie at the core of mathematical topics and themes" (Mason, 2008, p. 41). According to him, each mathematical topic is based on core actions, which learners carry out under instruction and these actions constitute the core awareness around which the topic is built.

Shulman and his colleagues proposed a special domain of teacher knowledge that they termed pedagogical content knowledge (Ball, Thames & Phelps, 2008; Shulman, 1986). In his discussion of knowledge growth in teaching, Shulman (1986) distinguishes between content knowledge and pedagogical knowledge. He describes the knowledge that grows in the minds of teachers, with special emphasis on content, as 'content knowledge in teaching'. He goes on to define three discerning categories: "(a) subject matter content knowledge, (b) pedagogical content knowledge, and (c) curricular knowledge" (Shulman, 1986, p. 9). His definition of pedagogical content knowledge has been widely used and influenced teacher education in Iceland, which I have been a part of for over three decades and. I will therefore elaborate on my understanding of pedagogical content knowledge and account for how my understanding of knowledge for teaching mathematics has affected my work with teachers.

Shulman (1986) characterised pedagogical content knowledge as the knowledge of subject matter for teaching, i.e. a particular form of content knowledge that embodies the aspect of content that is relevant for teaching. It also includes an understanding of what makes the learning of specific topics easy or difficult, and the strategies most likely to be fruitful in organising teaching with the goal of making the subject matter comprehensible to learners. Shulman called for a coherent theoretical framework for teacher knowledge, and since then, the term pedagogical content knowledge (often called PCK) has become a common theoretical resource. Ball, Thames, & Phelps (2008) worked at developing a practice-based theory of content knowledge for teaching, grounded on Shulman's notion of pedagogical content knowledge. They investigated the nature of subject matter knowledge in mathematics by studying mathematics teaching and identifying mathematical knowledge for teaching based on

analyses of the mathematical problems that arise in teaching. In their research, they identified two subdomains of pedagogical content knowledge, that is, knowledge of content and students and knowledge of content and teaching. They also identified specialized content knowledge for teaching, which they differentiate from common content knowledge. Moreover, they called for a more research on this domain of content knowledge that is unique to the work of teaching.

In our work with teachers in graduate course, my colleagues and I have tried to address pedagogical content knowledge by urging our students to participate in problem solving in our workshops and discuss their diverse ways of approaching the problems (Guðjónsdóttir & Kristinsdóttir, 2007a; 2007b). Björk, a preschool teacher, who participated in one of our graduate courses, reflected on her experience of participating in the course and wrote the following:

What I found interesting was when we solved the problem in different ways. Then I thought: If we can solve problems in different ways then the children must be able to do it and even in more different ways than we do. They find a strategy that they understand and can even explain how they did it. Their understanding develops and if they invent something themselves they will remember it. Then they will use this, even without knowing it, in their lives (learning) and get used to investigating and exploring “hidden” sides of something. They will be more open, because their understanding was respected. (Björk, June, 2005)

In her writing, Björk addressed both her own learning and children’s learning. From her experience of participating in discussions about her understanding of mathematics, she realised that her knowledge of content and teaching improved. She then related this to her knowledge of children’s development in reframing her understanding of knowledge of content and the students.

Ball et al., (2008) include Schulman’s definition of curricular knowledge in their description of pedagogical content knowledge, but distinguished between knowledge of content and students, on one hand, and of content and teaching, on the other. Their research findings reveal that when teachers choose a task to assign to students they must also predict what their students will find interesting and motivating, both in anticipating what they are likely to do with it and as the teachers interpret students’ thinking about the task. When I choose problems to explore with teachers, I try to anticipate what they will find motivating and what kind of problems will

inspire them to rethink their way of approaching mathematical problems. In the case of Björk, described above, it was apparent that we, the teacher educators, had succeeded in awakening her interest, and in turn, she envisioned how she could awake the interest of the children in her preschool classes.

In the Nordic countries, the interest in knowledge for teaching has grown in recent years. In 2002, the Danish Ministry of Education published recommendations for mathematics teaching in schools, called the KOM-project (Niss & Højgaard-Jensen, 2002; Niss, 2004; Niss & Højgaard, 2011) where eight specific mathematical competences were identified. They form two clusters; the ability to ask and answer questions in and with mathematics and the ability to deal with mathematical language and tools. The project group also outlined a model for mathematics teacher competency, which included the ability to develop one's competency as a mathematics teacher as well as the competencies of working with students and working with others towards professional development (Niss & Højgaard, 2011). It is important to note that the KOM project emphasised that the development of teaching in classrooms is dependent both on the teachers' knowledge and their ability to learn together with others, with both students and colleagues.

In my work with four teachers, who were in the process of developing their practice, discussed in Section 3.2, I discovered that their competencies of working with others improved through collaboration and which simultaneously bolstered their competencies of working with children in their classes. As a consequence, the learning communities they created in their classrooms changed and their students were afforded new opportunities to learn mathematics in potentially meaningful ways (Kristinsdóttir, 2010a; 2010b). It is important to note that the mathematics chapter in the Icelandic national curriculum guide from 2013 was influenced by the recommendation in the KOM-project (Niss & Højgaard-Jensen, 2002). The teachers are expected to be able to support their pupils in asking and answering questions in and with mathematics, and to deal with mathematical language and tools.

Chapman (2013) criticises the trend in mathematics education research to investigate the nature of the knowledge teachers possess, and consequential approaches applied in order to support further development of the knowledge they presumably hold or lack. She notes that this kind of research is important, however, she advises us to seek approaches that include broader ways of making sense of that knowledge. For example, how

did the teachers' knowledge develop, how does it impact their classroom actions and students' learning, and not least, how it can be facilitated, changed, or enhanced. She calls for new ways of thinking about the development of teacher knowledge, where teachers are seen as active in their pursuit of knowledge instead of thinking of teacher knowledge as knowledge generated by others. I agree with Chapman in that a broader scope of teacher knowledge is needed, and in working with teachers as they examine the trajectory of their own learning in their classrooms, I hope to gain an understanding of how the diverse ways in which they develop as activate participants in the making of mathematics teaching knowledge. Chapman (2013) emphasises that educational researchers also need to consider how their own learning is impacted by their participation in mathematics professional development settings. In that context, I will attend to what my own participation has taught me.

In the following section, I will focus on teachers' reflection on their practices as means of negotiating their beliefs about and understanding of mathematics teaching and learning. In the communities, the participants support each other in making sense of what constitutes as classroom cultures that promote mathematics learning in classrooms with diverse groups of pupils.

4.4 Communities of reflective practice

Focusing on collaboration and reflection in this section is my way of processing my own understanding of how collaborative debate and discussion of mathematics teaching and learning can lead to changes that are valuable for those who participate in a collaborative project. In working with teachers, I have experienced that learning to learn from one's own practice requires active engagement and reflection in communities with others, as discussed above (Kristinsdóttir, 2010a; 2010b; Kristinsdóttir & Guðjónsdóttir, 2015; Kristinsdóttir et al., 2016). When teaching mathematics in schools, teachers need to attend to the cultures they create in their classrooms and how the communities within their classrooms are shaped by their approaches to their teaching.

4.4.1 Professional practice in classroom communities

An essential part of teacher development is the activity of investigating into one's own teaching and reflection with regard to pupils' learning. Boaler (2002) contends that a focus on mathematical practices in classrooms helped researchers and teacher researchers to understand the identities

that their students developed and the different ways they learned to know and use mathematics. She argues that the idea of identity builds directly from studies of practice: students develop identities through the practices with which they engage in communities with others. The classrooms are the communities, which facilitate the students' mathematics learning, and by the same token, in focusing on the mathematical practices of their students, the teachers learn about their students' learning of mathematics. Boalers' (2002) research findings influenced my understanding of classroom communities, which emphasises the interaction between teacher-learner and learner-learner and the ways the teacher learns through paying careful attention to the mutual learning that takes place in the classroom. An important component in their own development as teachers, consist in their focus on their students' development of identities and their co-learning in the classroom.

In a summary of four research projects on learning mathematics with understanding, Hiebert and his co-researchers (1997) argue that teachers need to create classroom communities in which all students can both reflect on mathematics and communicate their thoughts and actions. They conclude that classroom cultures that promote mathematical learning afford all students a voice and an environment through which they can develop their understanding of mathematics, through exploring, investigating, discussing, reflecting and drawing conclusions. In such communities, not only do the students learn, so do their teachers. In planning the new collaborative research project with teachers, discussed in Part II, my intention was to create communities where our identities could develop by sharing the meaning of mathematics learning, and through our engagement in the workshops, which would benefit and strengthen our mutual learning. The notion of participation as both a social and personal experience has been at the core of my work with teachers and student teachers and I will discuss further how my understanding of teacher development has steadily emerged from my early experiences as a teacher educator.

In my position as practice teacher in the late eighties, I had the opportunity to participate in a course together with my colleagues, led by Gunnar Handal. Through this course my notion of teacher professionalism grew and as well as my notion of how student teachers could be supported in reflecting on their understanding of the teacher profession. Handal with Lauvås (1982) created a model for developing teacher professionalism, in relation to which they stress that teachers require support which is grounded in their knowledge and beliefs. They maintain that implicit or

practical theory forms the backdrop of everything teachers do. When preparing for teaching, teachers not only need to reflect on their theory and practice based knowledge, they also need to reflect on the ethics of teaching. Accordingly, when teachers reach the stage where they ask themselves why they teach the way they do and can justify to themselves and others what and how they teach, they are working according to their own professional theory.

Their model echoes that of Cochran-Smith and Lytle (1993; 1999), who assert that teachers are not only recipients and implementers of knowledge generated by professional researchers, they are also capable of reflecting critically on their theoretical and practical knowledge, and thus able to contribute to teaching and learning in schools. Dalmau and Guðjónsdóttir (2002) build on Handal and Lauvås's model and Cochran-Smith and Lytle's theories of professional teachers, in their work, which seeks to frame professional discourse with teachers. They developed a model, based on their research with teachers, and they called 'professional working theory', with the core elements of practice, theory and ethics. Their model includes three levels of reflective questions couched in the relational analysis of experimental, systemic, and socio-cultural factors. The teachers with whom they worked asked themselves questions about their practice, which related to what they had witnessed in their daily work, the aspects that impact their work and what broad connections they are aware of. Similarly, their questions about theory related to how they explain what they do and what theoretical frames of meaning form the basis of their understanding. They also asked themselves about the sources of their ethics and values, as well as which cultural and social aspects had influenced them. Dalmau and Guðjónsdóttir (2002) concluded that the reflective approach to teaching assisted teachers in avoiding an interpretation of situations in ways that reinforce existing perceptions. Instead, they focused on the problematic features of their daily teaching and the social and cultural influences on their work. This supported them in viewing their relationships in a new light, as well as relationship with the larger community of education in general.

Teachers bring different beliefs and attitudes and their cultural background into the developmental programs they participate in, as observed in Robertson (2008). Robertson found that when teachers participate in collaborative projects, they became increasingly open to new learning; engaged together as professionals, equally committed to facilitating each other's learning development, and in doing so, gaining understanding of the work of professionals. She emphasises the value of

reflection in such communities and that collaborative reflection may help participants to purposefully think about their own values and beliefs. Similarly, St Claire-Ostwald (2007) emphasised the importance of understanding the processes involved in the different ways people negotiate social interaction, and the effects of the culture from which they come. Accordingly, the awareness of what might be unconscious and invisible is important, as well as developing skills necessary to negotiate ways of interacting with others who may not share the same values.

In planning the new research with teachers, I hoped that by discussing stories from their classes, the teachers would be provided an opportunity to visualise their experience. I also hoped that the process of reflecting on their values and discussing them with others, would help them make decisions that mirrored their own values and beliefs. Teachers need to build on their own cultural background and experiences of teaching in developmental projects, and reflective discussions become an essential component for the establishment of a community where teachers feel safe to discuss their practice.

4.4.2 Reflection on and in teaching

In researching my own practice of teaching mathematics in primary grades, my understanding of reflective practices grew. Schön's (1983; 1987) definition of reflection and action affected my understanding of reflective practices and how my reflection on and in action have bolstered and facilitated my development in teaching mathematics. Jaworski (1998), drawing on Schön's definition, stresses that in order to confidently claim that a teacher is developing in action, the teacher will have to move from the position where she is able to use her knowledge in the classroom to a situation where she is fully capable reflecting on her actions in the classroom, both during and after teaching, as well as being able to make decisions based on her analysis of what she notices in the classroom.

Reflection can be both an individual and social task, and in that sense, the process of reflecting on our work with others may benefit the development of how we notice what takes place in the classroom. Mason (2002; 2008; 2011) advises practitioners who seek to research their own practice to use the concept of 'noticing'. He refers to Schön's definition of reflective practitioners and argues that the routine practice of reflection does not, on its own, necessarily furnish an understanding of the developmental process. Mason emphasises that in preparing to respond to opportunities when they arrive and react sensitively to situations, we need

to learn how to notice and that noticing is an essential part of researching one's own practice. "Noticing as a research is designed to take as its domain of concern the locus and focus of attention: what am I attending to, and what is the structure of that attention" (Mason, 2002, p. 183). The danger here is that the attention to noticing could turn studies focused on other people and situations into studies of learning about oneself, if the content and structure of the study solely consists in our own actions and choices. Mason cautioned against the risk of making a habit of how we respond to our students, that is, instead of responding sensitively to situations, without realising it we frequently react according to established patterns. Accordingly, we continue to believe that we act freshly all the time, when in fact much of the time we react rather than respond. In order to prevent us from becoming stuck in reaction, rather than response, we must reflect on our actions and thereby become aware of how we respond.

Writing about one's reflections is an essential part of reflecting on one's actions. When we write about our experience, we can both describe what we have tried in our practice and also use the descriptions as a tool for reflection. Mason (2002; 2011) differentiates between accounting-of and accounting-for when writing about one's experiences. He explains that an account-of "describes as objectively as possible what happened by minimising emotive terms, elaboration, judgement and explanation" (Mason, 2002, p. 40). A written account-of offers a brief but vivid description of what happened, which others can then interpret and relate to their own experiences. The accounts can be written in general terms, providing a description of how one usually responds in a given situation, or more specific terms, where the exact behaviour of the actor is described. We account-for our experiences in order to learn from the reflection on what happened. We offer interpretation, explanation, value-judgement, justification and criticism. When we account-for, we ask ourselves why incidents occur and why one has noticed a particular aspect. Mason discusses how asking why could take different forms, from barely justifying our actions and classifying them prematurely, to the process of probing for more details with the intention of seeking an understanding of what lies beneath our observations. When we analyse what we account for, we need to be explicit about positive, negative and interesting features of specific acts. The analysis will then help us make informed choices in the future instead of doing things without thinking about how we respond.

Mason compacts his advice for teachers who want to learn to notice with the goal of researching their practice into four interconnected actions.

- *Systematic reflection*: collecting brief-but-vivid accounts-of salient incidents, working on them so that others recognise something from their own experience; developing sensitivities by seeking threads among those accounts, and preparing oneself to notice more detail in the future.
- *Preparing and noticing*: imagining oneself acting in some desired manner, using the power of mental imagery to direct and harness emotions, and gradually noticing more and more opportunities; reflecting on the past by re-entering situations as vividly as possible and preparing to notice in the future by imaging oneself choosing to act.
- *Recognising choices* by accumulating alternative actions and by working at bringing the moment of noticing into the present; being on the lookout to notice alternative behaviours or acts (in other people accounts, in texts and articles, while observing others in practice) which you would like to incorporate into your practice;
- *Labelling* salient incidents and alternative acts so that they begin to form a rich web of interconnected experiences associated with particular collections of incidents, and linking these labels with specific incidents so as both to enrich the moments and to empower the labels to act as triggers to notice fresh opportunities to act in the future. (Mason, 2002, p. 87)

An essential part of the process is validating the actions of noticing. How can we recognise the validity of our assessment of what we have noticed? Mason proposes personal and collective validation as a means to make sense of what one has noticed. A part of the personal validation entails laying selective strands of our own experience alongside each other to compare them and test with regard to whether they sharpen sensitivities, conform to each other, and inform practice. The collective validation, on the other hand, concerns interweaving strands of our own experiences with others, constantly seeking resonance, negotiating similarities and differences. It is a means of locating issues, understandings and possible behaviour to engage with in the future.

In my findings from the research with four teachers, accounted for in Section 3.2, the collective validation of the teachers noticing in their classrooms was apparent (Kristinsdóttir, 2010a; 2010b). In their weekly meetings, they discussed what they had noticed and compared their experiences. They also collaborated in planning their teaching and their pupils' visits to each other's classes, and in that way, became familiar with the children in other classes. They were so engaged with their reflections on noticing in their classrooms that they used every opportunity to discuss their work, to the extent that other teachers in their school felt that they

ignored them, because they did not have time to participate in the social discussion during coffee breaks.

In my former work with teachers, I have together with my colleague (Guðjónsdóttir & Kristinsdóttir, 2006; 2011), urged them to write reflections from their practice and we introduced them to case and commentary writing, as explained in Kruger & Cherednichenko (2006). The case and commentary has some common features with Mason's discipline of noticing. It consists of five iterative stages, case writing and four dimensions of praxis inquiry:

- *Case writing*: Professional stories that teachers write in order to stimulate their inquiry and analysis on the real challenges and dilemmas of their practices. The four dimensions of the protocol are used as a scaffold to analyse the case.
- *Practice described*: The participant describes the persons in the case and their social situation in sufficient detail to enable them to better understand the action.
- *Practice explained*: The participant, in describing practice, has adopted an explicit discourse or discourses for interpreting the action. As they interpret the practice, they ask themselves what professional explanations they can find to assist them to understand and explain what is happening.
- *Practice theorized*: Participants construct their personal theory of the practice described. At this stage they relate theory and professional working theory; ground their practice in theory and ask questions like: As I incorporate these understandings who am I becoming as a professional? What are my significant professional practices, beliefs and theories?
- *Practice changed*: Theorized practice presents practitioners with opportunities to propose and trial new practices. At this last step, practitioners make decisions or conclusions, and develop and improve their practice. Describing their practice they adopt discourses for interpreting the action and construct their personal theory of the practice described (Kruger & Cherednichenko, 2006).

According to Dalmau & Guðjónsdóttir (2002) and Guðjónsdóttir, Cacciattolo, Dakich, Davis, Kelly, & Dalmau (2007), teachers learn from former cycles through refinement of their teaching and 'spirals of experience' emerge while building new cycles. Drawing on such an interchange cultivates a collegial relationship between the participants as they share and analyse information, critique each other's ideas, and solve problems with each other's help.

In his work with student teachers, Korthagen (2004; 2013) used a framework for core reflection that is grounded on the idea "that the use of

one's core qualities creates flow and a natural and rapid type of learning" (Korthagen, 2013, p. 24). The concept of flow has its origins in positive psychology and was described by Csikszentmihalyi (1990) as a state of absolute being in the here-and-now. According to Korthagen, individuals can use their own personal qualities to act optimally in the world outside, and their actions are therefore both effective and personally fulfilling. The flow creates a rapid process of learning from within and this learning is contrasted to the learning that results from outside pressure. Korthagen (2004; 2013) develops his framework for core reflection further with reference to what he called the 'onion model'. The onion model describes the inner world of the person and the experience of flow and non-flow that can ensue as result of a mismatch between ideals and what is accomplished. The onion model has six layers: environment, behaviour, competences, beliefs, identity and mission. Korthagen raises the concern that the three outermost layers (environment, behaviour, competences) often receives the most attention in teacher education practices, as well as in the teachers' competences in dealing with the class and the circumstances at school. However, the inner layers may have the most influence on the outer layers since the teachers' beliefs, identities and mission, in regard to learning and teaching, determine their actions. The focus on the individual in Korthagen's model reflects a constructivist view of learning and it is worth considering whether he disregards a social basis in favour of interpersonal communications (Ernerst, 2010b; Goodchild, 2001; Lerman, 1996). In other words, that which takes place at the outer layers can impede the flow between the layers.

With the aim of assisting student teachers in reflecting on their core qualities, Korthagen (2013) claims that teacher educators should guide their student teachers to focus on their identity and mission, and thus promote awareness of the cognitive, emotional and motivational aspects embedded in the environmental and behavioural layers. In addition, he insisted that student teachers be afforded reinforcement in identifying and dealing with internal obstacles, i.e. regarding disharmonious elements between the onion layers. Moreover, student teachers should also receive support in acting out their inner potential in relation to the situation which the reflective scrutiny is directed at, and thereby gain autonomy through the use of core reflection. Korthagen (2004) writes that core reflection should be encouraged in projects with experienced teachers. This could prevent teachers from losing sight of their ideals and prevent them from experiencing as a lack of support when it comes to the realisation of those ideals. Teacher educators, who seek to promote core reflection in student

teachers, ought to be actively engaged in such reflection. In core reflection, professional development becomes the centre of focus: its depth reflected in the process of tapping into one's inner potential for the benefit of professional growth.

West and Staub (2003) also insist that it is imperative to recognise the teacher educators' awareness of their own beliefs about learning and biases about pedagogy. They emphasised that teacher educators must respect the student teachers' beliefs, whatever they may be, when engaging in dialogue about their underlying beliefs on teaching and learning. Therefore, it might be helpful to look for a common ground that can provide openings for collaboration. The key to this, they argue, is to place evidence of student learning to the fore. They further emphasised that the teacher educator must make sure they give teachers leeway in their search for a teaching style that suits them and results in rich, deep, and flexible learning. The teacher educators' insistence on a particular pedagogical stance might lead the teachers to dig in their heels rather than being willing to reconsider their beliefs. In the same way, Rodriguez (2005) asserts that teachers might show resistance to ideological change and prefer to lecture and assign individual work to students. They might lack the awareness, confidence or knowledge and skills to implement a more culturally responsive and socially relevant curriculum. They might believe that students only requisite for success in mathematics or in life in general is to work hard, no matter their language abilities, gender, ethnic background or socioeconomic status. Many teachers criticise professional development for being too theoretical, as noted in Knight (2007). If they want to focus on the outer layers in Korthagen's (2004; 2013) onion model, and if the teacher educator wants to pay more attention to their core beliefs, identity and mission, based on theories of learning, this might be result of dissonance in their understanding of the goal for their collaboration. In choosing what to base the collaborative activities on, both parts must negotiate on the focus.

In communities of practice, all participants should be able to learn from taking part in the process. Using as a point of departure Lave and Wenger's view of learning as situated activity, and a process of peripheral participation in communities, I have suggested that collaborative reflection on teachers' practice can facilitate teachers in their development as teachers. If the teacher is to be able to learn from her experience, she needs to reflect on her responses, both the positive and the negative ones, and be aware of her core qualities. It is not only important to be sensitive to your surroundings and how you respond to others, you need as well to be able to justify your actions.

4.5 Summary

Above, I have discussed how my early understanding of constructivism, arising from my experience as a teacher in compulsory school in Iceland, affected my work as a teacher educator and how I began to interpret children's development in mathematical thinking, through my participation in professional development courses and my studies for a master's degree. This was followed by the delineation of new understanding of theories of learning that developed during my time as a teacher educator working with pre- and in-service teachers, and while I participated in courses on theories of mathematics teaching and learning during my doctoral studies. In structuring a new study that I carried out as a part of my doctoral studies (see Part II), I built on my thoughts about mathematics teaching and learning that had changed in response to working with sociocultural theories of learning.

In the following, I present an overview of how I have been influenced at different times during my practice by engaging with theory, the theories that have affected my understanding of learning and teacher development, particularly in relation to young children's mathematical learning.

Constructivist perspectives: In my early career as a teacher in primary grades, I relied much on Piaget's theories of child development, which I became acquainted with in my early teacher training. I emphasised creating learning opportunities that would support my pupils in furthering their mathematical thinking and acquiring an understanding of mathematical concepts. When I had the opportunity to participate in a course where I had access to findings from research on children's thinking about numbers and operations, I discovered that these findings correlated with my understanding of children's development of mathematical thinking.

Sociocultural perspectives: My understanding of how shared experiences and collective reflection regarding learning facilitated our own learning, grew considerably through my participation in courses and collaborations with colleagues. This applied both to our learning as individuals and to our collective understanding of how to support children in learning mathematics. Taking on the position as a mathematics teacher educator, I drew on this experience and my understanding of Vygotsky's theories of child development, in my teaching about mathematics teaching, emphasising discussions and collaboration.

Communities of mathematics learning in schools: The role of the teacher in creating supportive learning communities and attending to their pupils' diverse background, has been at the core in my teaching of teachers. I

needed to make myself more cognisant of teacher competences in teaching mathematics, and how to support them in acquiring mathematical knowledge for teaching.

Communities of reflective practice: In order to gain competence in teaching mathematics, the focus must be directed at professional discourse. Collaborative reflection on their learning within teacher education programs helps teachers and prospective teachers in preparing for leading discussions in their classrooms. An issue that needs to be addressed within teacher education programs is teachers' reluctance to change their way of teaching. It was precisely this aspect that I wanted to learn more about in researching together with teachers in primary grades.

To be able to develop their practice, teachers need to actively engage in reflecting on their teaching and make sense of how it is developing. Researching their own practice is an essential factor in that process, and in the following chapter, I will discuss different approaches to researching one's own practice. In my discussion, I will refer to both the study of my own development as a teacher, teacher educator and researcher, and to the research conducted with teachers in schools.

5 Practitioner research

Educational research has long been criticised for its weak link with practice and the lack of collaboration between researcher and the actors within the field of education. Teachers often engage actively in developing and researching their practice and their initiative shown in doing such research needs to be acknowledged by educational researchers. Cochran-Smith and Lytle (2009) emphasise the importance of teacher initiatives in research and in their work. They argue that practitioners are deliberate intellectuals who constantly theorise practice as a part of practice itself, and the goal of teacher learning initiatives is the joint construction of local knowledge, the questioning of common assumptions, and thoughtful critique of the usefulness of research generated by others, both inside and outside contexts of practice.

It may be useful to distinguish between pedagogical developments and pedagogical research (Norton, 2009). Norton argues that pedagogical developments include activities that have a practical focus and aim at generating practical information that teachers may find useful in their daily practice. Pedagogical research, on the other hand, has a more theoretical focus and is more a formal inquiry with an accepted research methodology. It aims at generating theories that may work within schools but are not likely to have effects on school culture if they are not presented to teachers in a way that is accessible to them.

The relation between theory and practice and how researchers and teachers can collaborate for the benefit of both, has been my main concern as a teacher educator. To further reflect this concern, I will discuss approaches to teacher initiative research and practitioner research where collaboration between practitioners and researchers is built into the structure of the research process itself. Together with colleagues within the teacher education in Iceland, I have studied my practice as a teacher educator and supported teachers and student teachers in researching their practice. In conducting these studies, we have used methods of practitioner research such as action research, lesson-study and self-study of teacher education practices. I will start by discussing action research and lesson-study, and relate them to my work with teachers and student teachers. Then I move on to discuss research where collaboration between teachers and educational researcher is more formally built into the research process, such as learning-study and design research. Finally, I will discuss the process of studying one's own teaching as a teacher educator.

5.1 Action Research

Action research can be defined as a spiral of self-reflective cycles of planning a change. These cycles are: acting and observing the process and consequences of the change; reflecting on these processes and consequences; and in turn re-planning; and so on (Kemmis, 1999). The core element is reflection and action, as described in Schön (1983; 1987), characterised by a progression from knowing-in-action, reflecting-on-action to reflecting-in-action that traces a development of awareness and understanding and an overt growth of knowledge about practice, as discussed in the previous chapter in relation to reflection.

Action research is a participatory and social process with six key features that are no less important than the cycle as whole. Kemmis and Wilkinson (1998) define action research as:

- A social process, where people individually and collectively seek to improve and understand the processes of teaching and learning in the classroom.
- Participatory; it engages people in examining their knowledge and the ways they interpret themselves and their actions and people can only do action research 'on' themselves – individually or collectively. It is not research done 'on' other people.
- Practical and collaborative; it is a process that engages people in examining their acts collaboratively.
- Emancipatory; it is a process where people explore the ways in which their practices are constrained and aims at helping them recover themselves from the constraints.
- Critical and helps people to release themselves from their modes of work. It is a process in which people deliberately set out to contest and reconstitute irrational ways of working and relating to others.
- Recursive (reflective, dialectical); it aims to help people to investigate reality in order to change it. (Kemmis & Wilkinson, pp. 23-24)

In our collaborative action research (Kristinsdóttir et al., 2016; Reimarsdóttir, 2015), discussed in Section 3.5, a special education teacher researched her practice, together with two teacher educators, as she was reflecting on her former work when planning new ways of working with children, parents and colleagues.

- The teacher collected data on her work that facilitated her ability to reflect on former cycles. In connection to her analysis of the children's thinking, she reflected on her understanding of theories of learning, and in our discussions, we collectively discussed what could be concluded from her interviews with the children.

- In consequence, we (the teacher educators) rethought our former understanding of those theories. Although we were not present when she was researching her practice, we discussed her data with her and thus helped her interpret her own experience and what she learned. In this process, we studied our practice as researchers in collaboration with her in conducting research into her own practice as a teacher. We then collectively wrote about our learning extracted from the participation in this process.

Our approach to this action research project can be interpreted as clinical partnership as described in Wagner (1997), since we were working together to improve knowledge about and within schools and educational practice and reported our work at conferences and in educational journals.

McNiff (2010) emphasises that the idea of self-reflection is central in action research. It is an inquiry into one's own work that is open-ended and does not begin with a fixed hypothesis. Therefore, the research process is the developmental process of following through on an idea that people want to develop. It can thus be seen as a form of self-evaluation and the researchers need to be constantly aware that what they are doing has a positive effect on their work. McNiff points out that the reflection and self-evaluation can lead to unforeseen loops in the action cycle.

These unforeseen loops in the action cycles were apparent in the work of graduate students who conducted action research on their teaching under my tutelage, as a part of their master's degree studies. Even though they began with clearly defined cycles, which they planned to implement in their teaching, this process was disturbed by incidents in their daily work and they were prompted to attend to these unforeseen loops. The teachers experienced these disruptions to the plan as constraining, and in that situation, and their tutor's support was vital in resolving these incidents.

In his description of an action research project he carried out with teachers, Krainer (1999) notes how the systematic reflection of practitioners on action was used as framework for improving professional practice in mathematics teaching. He sought new ways of mediating between theory and practice, of collaborating with teachers on different levels, and of accounting for the culture in which they live and work. Moreover, Krainer notes that an inevitable part of professional practice calls for the redefinition of teacher change and teacher education.

In another action research project, Krainer focused on the development of one teacher and how she interacted with the school community where she was working. He concluded that focusing on the individual as a learner, is insufficient, instead, we need to build a bridge between classroom

development and the development of schools, and the educational system on the whole (Krainer, 2001). Collaborative action-research, where actors at different stages collaborate, can potentially broaden our scope regarding the development of the educational system in general.

In her action research project with mathematics teachers in secondary schools, Jaworski (1998) found that, despite their unsystematic research approach, she observed the teachers develop through their research, and an analysis of her data shows evidence of reflection on, in and for action. The teachers were inexperienced researchers and their research process could not be labelled as neat, regular, or well planned cycles. She used the term evolutionary action research to define the methodology that emerged through their collaboration.

Jaworski's (1998) experience resembles my own experience of working with teachers in researching their practice. Despite the fact that teachers learn about research methods in the teacher education program, the main emphasis is on teaching and learning in schools, and they therefore have little experience in conducting research and collaborating with an experienced researcher who can support them in the process. In Part II, I will further discuss how I drew on my former experience of action research methods in planning a new collaborative research with teachers.

In the following section, I will discuss lesson-study, a form of teacher-led research into their practice.

5.2 Lesson-study

Lesson-study, a form of collaborative action research, has been common in the past century in Japan and is strongly recommended there by education authorities. It is a teacher-led development process, focusing on the role of the teacher and the learning environment in the classroom. The participants are in charge of both planning and implementing teaching and making decisions about the processes based on their reflection on former experiences. Fernández (2010) defines lesson study as a process that brings a group of teachers together to collaboratively design and investigate a 'research lesson'. The lesson is developed to meet a specified overarching student-learning goal, and includes cycles composed of several phases, these are: collaborative planning, lesson observation by colleagues and other knowledgeable advisors, analytic reflection, and ongoing revision.

One important feature of lesson-study is that the participants deepen their knowledge of the content and possible teaching approaches. They communicate, do research, work together, take decisions, plan teaching

and experience the advantages of participating in a learning community. As part of the process, the lesson study groups then develop a written reflective report of their work to make their work accessible to others.

Pang and Marton (2003) argue that lesson study places a greater emphasis on teaching practices associated with a specific object of learning, like the additive property of multiplication, than does action research, where the accent is often on more general approaches to teaching and learning. Fernández (2010) also addresses this emphasis on the specific objects of learning through comparing lesson study in Japan and China, in which teachers work together to improve teaching and students' learning. Both countries take their point of departure from specific objects of learning. The idea of the object of learning will be clarified in more detail in the following section, on learning-study.

Lewis, Perry and Murata (2006) explain the research process of lesson study in cyclic form. Like in action research, each cycle is repeated and consequent reflections on the first cycle will then help formulate goals for a new cycle. The emphasis is placed on the developmental cycle for teacher improvement.

- *Study curriculum and formulate goals:* Consider long-term goals for student learning and development. Study curriculum and standards, identify topic of interest.
- *Plan:* Select or revise research lesson. Write instruction plan that includes long-term goals, anticipated student thinking, data collection plan, model of learning trajectory and rationale for the chosen approach.
- *Conduct research:* One team member conducts research lesson, others observe and collect data.
- *Reflect:* Formal lesson colloquium in which observers share data from lessons and use the data to illuminate student learning, disciplinary content, lesson and unit design and broader issues in teaching-learning. Documentation of cycle, to consolidate and carry forward learning and new questions into next cycle of lesson study. (Lewis, et al., 2006, p. 4.)

Interest in lesson study has grown in other parts of the world, since Stiegler and Hiebert (1999) recommended it as a way of improving the teaching and the methods that teachers use in the classroom. According to Lewis et al. (2006), Japanese educators make public their ideas about instructional design in the form of research lessons that are observed and discussed by local and outside educators, often including university based educators. Consequently, widely shared norms about teaching and learning begin to change when observing educators closely scrutinise both the

teaching and the learning process and its rationale. School-based teacher researchers and university researchers thus make sense of these processes collaboratively, through discussions, sometimes reshaping their own practice and research lessons as a result. As a consequence, both the teacher researchers and university researchers can actively enter into research process.

At my institution we have planned a lesson study with our students in one of their practice periods (Gunnarsdóttir et al., 2008; 2013). A group of student teachers plans a lesson together and one of the teacher educators assumes the role of an expert who, together with the other student teachers, observes a lesson taught by one of lesson study groups. The collective planning has resulted in more in-depth discussions of the experiences from their practice than before. Through deliberation on the shared experience, the student teachers improve the lesson plan, which is then taught by another student teacher and observed by the others in the group.

The pre-service teachers' experience of participating in a lesson study process strengthened their competences, particularly in terms of analytic thinking and enabled them to combine their theoretical and practical knowledge. By means of lesson-study participation, the student teachers experienced that learning to teach is a collective process, which prepares them for active participation in lifelong learning projects. The teacher educators learned from this experience that collaborative planning and reflective discussions on teaching practice is a fruitful way of learning to teach mathematics.

Lewis et al. (2006) argue, that research should contribute to instructional improvement and warned that faddism was a risk in educational research. They were concerned that summative trials of lesson-study, while little is still known about its nature and mechanisms, might contribute toward making it a fad. They also worried that controlled experimental research on undeveloped versions of lesson study could lead us to the conclusions that it is impossible to implement it. Drawing on examples from lesson study, both in Japan and the United States, they propose three types of research might prevent lesson-study from being discarded before being fully understood or properly implemented.

- The first type of research was expansion of the descriptive knowledge base on lesson study to avoid misinterpretation of the approach.
- Secondly, explication of the innovation mechanism was needed to make the innovation mechanism more visible. Models could be useful to enhance

conversations about the essential features of lesson study and stimulate sharing data and models across sites as well as model improvement. They might also enable innovators to adapt a thoughtful and flexible approach to innovation and accompanying research.

- Thirdly, they proposed design-based research cycles that enable researchers to progressively hone an innovation while also building theory about how it works, in order to develop theories, rather than merely tuning it empirically with what works.

A variation of lesson-study called learning-study will be discussed in the next section. Following which I will discuss design research and explain how theory building is an integral part of the research cycle.

5.3 Learning-study

Learning-study, a variation of lesson study, has received wide international attention. However, definitions of the approach vary slightly depending on the context of the discussion. According to Lo (2009), learning-study is a special type of classroom research that originated in Hong Kong, and has developed into an approach that has affected the teaching and learning in schools both locally and internationally:

- Holmqvist (2011) defines learning-study as a research method that is a fusion between lesson-study and a design experiment
- Lo (2009) described learning-study as an approach that is closely affiliated with the research areas lesson-study, and action research.
- According to Goodchild (2008), it is a form of developmental research in which the researcher is concerned with the coordination of development and research into the developmental process.
- Pang and Marton (2003) describe learning study as an alternative to lesson-study that builds on design experiments and systematic in-depth studies of particular lessons.

Accordingly, learning-study aims to build innovative learning environments and conduct research into innovations grounded in theory, where the central emphasis is placed on the object of learning, not teaching methods.

Lo (2009) recounts a meeting of researchers in Hong Kong, where Marton introduced a learning perspective based on his work on phenomenography and variation theory to his collaborators. They focused on learning as the ability to discern aspects of a phenomenon other than those one has been capable of discerning before. To ensure that students

experienced variation in studying the object of learning, the teachers first had to study the object in detail in order to tease out its critical features, as Lo explained:

He or she should then ascertain the limited number of qualitatively different ways in which students may understand it, which will subsequently become a useful resource in lesson planning. ... Attention should also be paid to what students should be able to do with the object of learning, and the capability that can be developed as a result of learning it. (Lo, 2009, p. 170)

The teachers are thus expected to reflect in-depth on what their students are expected to learn and plan their lesson accordingly, and here the similarities with lesson study are apparent.

Pang and Marton (2003) defined five steps of a learning study, namely: (1) choosing the object of learning; (2) ascertaining students' pre-understandings; (3) planning and implementing the lesson(s); (4) evaluating and revising the lessons; (5) reporting and disseminating the results. Lo (2009) points out that in contrast to most teaching innovations, where teaching is centralised, in learning-study approaches it is learning that is placed to the fore. For Lo, it is imperative that teachers receive support in understanding how to learn in a community of practice, whose purpose is to hone the students' ability to master the object of learning. Furthermore, he argues that, in this process, the teachers develop pedagogical content knowledge.

According to Pang and Marton (2003), learning-study is expected to be a bridge between theory and practice and between basic research and developmental work. The focus is on the participating students', teachers' and researchers' learning and is therefore threefold:

- The students are expected to learn about the object of learning and to learn better than they otherwise would have done.
- The teachers are expected to learn about handling the object of learning, not only the specific object, but the object of learning in general.
- The researchers are expected to learn about how the theory works, because every learning study is based on a particular theory and that theory is put to a test.

In learning-study, teachers become researchers who generate knowledge about their own practice. The scope of the research is wide, ranging from the teaching of particular topics in specific subject areas to the

professional development of teachers, the development of school learning communities, and theoretical insights with respect to variation theory (Lo, 2009). Pang and Marton (2003), state that central benefit of learning-study is the contribution it offers to the progression of the study itself and to the theoretical tools that are used to reach the aim.

5.4 Design research

Design research has received increased attention in recent years within the field of educational studies (Wood & Berry, 2003; Kelly, 2003) and was the topic of the 2003 edition of *Journal of Mathematics Teacher Education* 6(3) as well as the *Educational Researcher* 32(1). The Design-Based Research Collective (2003) chose the phrase ‘design-based research methods’ to avoid invoking a mistaken identification with experimental design. They point out that by grounding itself in the needs, constraints, and interactions of local practice, it could provide a lens for understanding how theoretical claims about teaching and learning could be transformed into effective learning within educational settings.

Shavelson, Philips, Towne, & Feuer (2003) list three generic questions put forth by a National Research Council Committee in order to identify within which fields design studies might be appropriate. These questions are: what is happening; is there a systematic effect; and why or how it is happening? The questions indicate that a thorough description of the research process and its effects was needed and a theoretical lens was necessary through which to analyse the process. Cobb, Confrey, diSessa, Lehrer and Schauble (2003) presented a similar view by stating: “Design experiments have both a pragmatic bent—‘engineering’ particular forms of learning—and a theoretical orientation—developing domain specific theories by systematically studying those forms of learning and the means of supporting them” (Cobb et al., 2003, p. 9).

Van den Akker, Gravemeijer, McKenney and Nieven (2006) claim that the most compelling argument for initiating design research stems from the desire to increase the relevance of research for educational policy and practice. They use the term ‘Design Research’ as a common label for a family of related research, such as design study, design experiment, developmental research, formative research and engineering research. They insist that, even though the terminology had not yet been established, it was possible to outline a number of characteristics that apply to most design studies. Building on previous works of Cobb et al. (2003); Kelly

(2003); Design-Based Research Collective, (2003); and van den Akker (1999), van den Akker et al. (2006) characterises design research as:

- Interventionist;
- Iterative;
- Process oriented;
- Utility oriented;
- Theory oriented.

The research thus aims at designing intervention in the real world, with a cyclic approach that is process-orientated, focusing on understanding and improvements. The merit of a design is thus measured by its practicality in real contexts; it is based on theoretical propositions and the testing of the design in the field contributes to theory building. In addition, van den Akker et al. (2006) point out that design researchers do not emphasise isolated variables but instead attempt to study and frame specific objects in specific contexts as integral and meaningful phenomena.

The approach to design research by Gravemeijer & Cobb (2006) is based on their experience in different contexts in two continents, which involves socio-constructivist analysis of instruction in the United States and realistic mathematics education (RME) in the Netherlands. They suggested that the underlying principle of design research is that if you want to bring about innovative forms of education you have to understand what they entail in order to be able to produce them.

According to Gravemeijer & Cobb (2006), various forms of professional instructional design may have sparked the idea for integrating design and research.

This idea was strengthened by the experience that conscious and thorough instructional design work brought about a learning process in which the designers developed valuable and well-grounded knowledge in what retrospectively might be called design experiments. (Gravemeijer & Cobb, 2006, pp. 17-18)

They discussed many attempts to define design research in mathematics education and note that Freudenthal and his colleagues were perhaps the first to propose an approach of this kind, which contained the concept of developmental research. Gravemeijer & Cobb (2006) emphasise that the focus on understanding is a salient characteristic of design research and which is different from the focus of experimental and quasi-experimental

research that aim at explanation. I will discuss developmental research further in Part II, in relation to the methodological approach used in my collaborative research with teachers.

Gravemeijer & Cobb (2006) explain that the purpose of design research is to develop theories about both the process of learning and the means designed to support learning. They may evolve either through the development of local instruction theories or by means of developing theoretical frameworks that address more extensive issues. They tried to combine the two in their approach to design research, which consisted of three phases:

- Preparing for the experiment;
- Experimenting in the classroom;
- Conducting retrospective analyses.

They define the preparation phase as the step where a local instruction theory is formulated, which in turn is elaborated and refined as the experiment proceeds. During the experimental phase, the conjectured local instruction theory is improved and tested, and an understanding of how it works is developed. This phase, according to them, consists of cumulative cyclic processes with reflexive relation between theory and experiments; hence a local instruction theory arises from the conjectured theory. Moreover, they hold that the primary goal of the retrospective analysis is to enhance the development of a local instruction theory, but it may also concern more encompassing issues, or ontological innovations.

As an example of these phases, Gravemeijer & Cobb (2006) describe the analytic process involved in the data gathering of a learning design experiment on statistics learning. They used a constant comparative approach and began by examining their data chronologically and at each point tested the current and relevant data. The first round of data analysis resulted in a sequence of conjectures and refutations that became the data for the meta-analysis of their results. This step also entailed scrutinising the conjectures and assumptions, which they had formulated at the outset. From their retrospective analysis, they learned that several key assumptions that underpinned an instructional sequence were badly founded. As a consequence, the sequence was revised and a further design experiment was conducted. The cyclic approach of design research is thus a process-oriented approach with focus on understanding and improvement, as was evident in their approach where results from one cycle were used to prepare for another cycle.

5.5 Self-study of teacher education practices

In Chapter 6, I will discuss how the process of reflecting on my own learning during my doctoral study, which took place alongside with my professional work as a teacher educator and a researcher, helped resolve my understanding of theories of learning and methodologies of practitioner research. In this section, I will account for my understanding of the methodology of self-study of teacher education practices that underpinned and informed the study of own practices.

The growth within the field of self-study of teacher education practices has largely been based on the desire of teacher educators to better understand teaching and learning of teaching and to improve teacher education in general. The method has emerged and is influenced by a range of events, couched in the methodologies of several fields, such as reflective practice, action research and practitioner research. According to Loughran (2005), self-study can in many ways be seen as a response to earlier calls for studies into ‘teaching about teaching’ that involve teacher educators themselves. The research findings have been applicable to teacher education generally and have brought to the surface aspects of teaching and learning about teaching that we had limited previous knowledge of (Loughran, 2007). Pereira and Schuck (2011) argue that teachers, including mathematics educators, have to constantly reinvent themselves in order to stay fresh, competent and committed. They proposed self-study as an approach that can help regenerate our practice, and further assert that when self-study is conducted in collaboration with others, it can aid us in reframing our practices and in challenging our assumptions. They urge mathematics teacher educators to study their own practice because it can potentially support us to face obstacles and recognise how we can improve our practice.

The methodology of self-study can be seen as composed of four integral aspects (LaBoskey, 2004).

- Self-study aims at improving and “looks for and requires evidence of the reframed thinking and transformed practice of the research, which are derived from an evaluation of the impact of those development efforts” (LaBoskey, 2004, p. 859).
- Self-study is interactive and demonstrates interactions with colleagues, students, the educational literature and our own previous work confirming or challenging our developing understandings.
- Self-study employs multiple, primarily qualitative methods, many of which are used in general educational research and some that are innovative.
- Self-study is an approach to formalise our work and make it available to our professional community for debate, further testing and judgment.

These aspects demonstrate an expectation that learning from self-study will not only be informative to the individual conducting the research but promises also to be meaningful, useful and trustworthy for those drawing on such findings for their own practice. In discussing my own learning from studying my practice as a teacher educator and a doctoral student, I attend to these aspects. I will write about my reframed thinking by providing examples from my practice with specific emphasis on collaboration with colleagues and my students.

Teaching about teaching is complex work and demands much of teacher educators. The complexity is embedded in the nature of teaching itself. Indeed, the focus on teaching about teaching necessitates a more sophisticated understanding of practice. The dynamic nature of teaching and the learning environment requires teacher educators to be responsive to their teaching and sensitive to all participants. Teacher educators must build trust by talking together and problem-solving together (Loughran, 2007). Russel (2007) argues that the common assumption attached to research finding is that when evidence is clear, and the new practices match one's own values as teachers, they inspire new practices that are easily transferable to one's teaching. He stressed that this assumption was contradictory to his findings from conducting a self-study of his teacher education practices, and that only by significant effort over several years has he been able to fully understand the potential benefits and personal consequences of a change recommended by research. Pereira (2011) presents a similar perspective in his discussions of mathematics teacher change. Through the self-study of his teacher education practices he learned that it is not sufficient for teachers to improve their mathematical content knowledge and pedagogical content knowledge. We also need to attend to teachers' emotional experiences of learning mathematics, because they will transfer most powerfully to their classrooms. In their daily work, teachers have to make judgements such as: "Should I allay student's fears and minimize discomfort? Is it time to reduce frustration by giving more clues?" (Pereira, 2011, p. 159). These are questions that teacher educators frequently ask themselves, and if they have not reflected on their own mathematical learning in their teacher education programs, there is a danger that the way they teach mathematics will only reflect what they have been taught even though they have learned new approaches to mathematics teaching and learning.

Brandenburg (2011) describes how she became aware of her own learning as a teacher educator while teaching prospective mathematics teachers. She elucidates how her pedagogical skills developed as she

changed her habit of teaching and shifted the focus toward her pre-service teachers' learning through own experiences. While analysing her data, she found that she often felt discomfort and struggled with being true to her goals. She concluded that the process of changing embedded patterns of behaviour follows a slow and often painful path. However, the initial step concerns the acknowledgement of the need for change and the hardship entailed in developing new practices, and through self-study, we do come to know and understand more about who we are as teacher educators. We thus learn about how our assumptions and practices impact our own learning as well as our students' learning. Austin and Senese (2004) similarly encourage teachers to include self-study in their list of professional expectations and responsibilities, because self-study is about who we are as teachers. They suggest that there is no better way to strengthen teaching practices than by recognising personal values and beliefs, and thereby enhancing students' learning.

Even though the term self-study carries with it the tacit message that it is an individual activity, it requires involvement of others so that the learning outcomes are more than personal constructions of meaning. It relies on interaction with colleagues, ideas and perspectives presented by others, and requires an acceptance of the need to seek alternative perspectives and data outside of the self (Russell, 2006). Dalmau and Guðjónsdóttir (2002) address this tacit message in their conclusion to their collaborative self-study with the goal of working with teachers at framing professional discourse. They found that not only had the process provided them with information about how they could improve their discourse with teachers, but it also helped raise questions about the process of learning, knowledge creation and research. They identified three broad understandings of self-study in their work: (1) exploration of their professional identity, (2) systematic understanding and transformation of their practice, and (3) creation of new knowledge and contribution to educational discourse. They argue that if they and the teachers had only focused on the 'self' they would have been trapped in stereotypic and tacit assumptions. When they extended their research focus to interact with other members of the educational community, and related their understanding to established educational theory, a deeper level of personal and professional understanding was achieved. By drawing on other perspectives and reframing situations, self-study is strengthened through analysing data, ideas and input that necessitate moving beyond the self. "Moving beyond the self also matters because a central purpose in self-study is uncovering deeper understandings of the relationship between teaching about

teaching and learning about teaching” (Loughran, 2007, p. 12). An essential part of my studies of my own practice as a teacher educator is collaboration with colleagues and my students, as well as the opportunity to collectively reflect on our teaching, as discussed in Section 3.4.

This understanding of the relationship between teaching about teaching and learning about learning was also addressed in Russell (2007). In his work as teacher educator researching his own practice he learned to understand the new teacher are required to make crucial changes of their perspective. A fundamental goal for Russell was creating and sustaining a teaching-learning relationship with each student, from which everything else followed. Through shared experiences and sharing of experiences the relationship developed as he came to understand the significant messages conveyed to students by how a teacher or teacher educator teaches. His results correspond with the findings from my previous collaborative studies, where the experience of collective reflections aimed at our experiences affected both our own teaching and the teaching practices of our participants. An example of this was discussed in section 3.4 in relation to Anna’s reflections on her experiences of participating in the course Mathematics for all. She emphasised that this experience helped her structure her teaching in new ways. Our collaborative reflections on the teachers’ experiences from their classrooms and literature they were engaging in, helped her in acknowledging the resources the children bring with them to school (Guðjónsdóttir et al., 2009).

Loughran (2008) addresses the difficulties that arise in conducting self-study research with individuals who have no prior experience with researching their practice. He argues that many self-studies have derived, from the issues, problems and concerns that emerge out of a teacher educator’s practice, yet at the same time, the drive for conducting a self-study would be stifled if there were no problems or impasses in the process. Consequently, the difficulty in attempting to coach self-study research lies in is the fact that what is compelling, interesting and curious for one teacher educator may be utterly different to another. What one teacher educator has seen in a given situation, another might not have recognised at all. If the attraction to doing a self-study is not sufficiently real, then the self-study might simply become a process of applying a ‘self-study protocol’ rather than an authentic desire to see, hear and learn about practice in new ways. Capturing a real reason for doing a self-study is therefore important and supporting others in this process can be challenging, yet rewarding.

A critical friend who is willing to listen, observe and discuss is vital for the process of studying one's own teaching. Shuck (2011) describes how a critical friend supported her in studying her work as a mathematics teacher educator. She and her colleague who was not a mathematics teacher shared experiences through mutual visits to each other's classrooms and in-depth discussions about their work. She found that the critical friendship provided her with an opportunity to reflect, discuss and grapple with ideas about teaching and learning. She learned to be more cognisant of her students' reactions to what she said to them and in revisiting her role in the classroom she had to justify what she did, both to herself and the critical friend. Similarly, in our collaborative self-study research, my colleague and I (Guðjónsdóttir & Kristinsdóttir, 2011) discovered that our different academic backgrounds, that is, in the fields of special needs education and mathematics education, brought into our teaching diverse strengths and a variety of perspectives of the subject under consideration. We were thus able to critically discuss our shared experiences from our work with teachers and to rethink our collaborative work with them. As discussed in section 3.4, our background helped us in resolving a dilemma concerning the focus of a course that originally revolved around special education needs and focusing on pupils' weaknesses. Our reflective discussions resulted in an improved course where the attention was instead directed toward seeing how all children can be included in the mathematics classrooms when given a chance to participate in explorative work.

While self-study researchers acknowledge the role of the self in the research project, the analytic accent is rather cast on the space between the self and the practice engaged with. As a result, the findings that are reported require a balance between biography and history (Bullough & Pinnegar, 2001). The balance must be in the evidence, not only in the presented data; it needs to be evident in how they have been analysed and brought together into conversation. Bullough and Pinnegar describe how autobiographical forms of self-study research can take the form of narratives and how the stories told in describing the structures of complicated settings become the narratives. In their guidelines for autobiographical forms of self-study, they emphasise the critical features that arise in carrying out the research, and I will attend to those features that have significantly figured into my discussions about my own learning in chapters 3 to 5.

- Autobiographical studies should ring true and enable connection.
- Self-study should promote insight and interpretation.
- Autobiographical self-study research must engage history forthrightly and the author must take an honest stand.
- Biographical and autobiographical self-studies in teacher education are about problems and issues that make someone an educator.
- Authentic voice is a necessary but not sufficient condition for the scholarly standing of biographical self-study.
- The autobiographical self-study researcher has an ineluctable obligation to seek to improve the learning situation not only for the self, but for the other.
- Powerful autobiographical self-studies portray character development and include dramatic action: Something genuine is at stake in the story.
- Quality autobiographical self-studies attend carefully to persons in context or setting.
- Quality autobiographical self-studies offer fresh perspectives on established truths. (Adopted from: Bullough & Pinnegar, 2001, pp. 16-18)

When I wrote about my reflections concerning theories of learning and methodologies of researching my own practice, I paid particular attention to nodal moments in my teaching and my position of being a teacher educator, with the purpose of enabling readers' insight into and understanding of my experience. In my analysis of how my thinking about these theories and methodologies developed, presented in Chapter 6, my intention was to engage history forthrightly and take an honest stand in portraying my development as a teacher educator in face of the issues attended to within these settings.

5.6 Summary

In this chapter I have discussed five types of cyclic research approaches where practitioners are active participants in the research process:

The level of participation in the research process:

- In action research, lesson-study and self-study, teachers and teacher educators initiate the research process and conduct research into their own practice.
- In learning-study and design research teachers engage actively in the research together with experienced researchers who lead the research.

The level of theory building varies between these approaches:

- Action research is an inquiry into one's own work and does not begin with a fixed hypothesis. The research process is therefore the developmental process of pursuing an idea that people want to develop in practice.
- Lesson-study is an attempt to improve learning and a way to organise teachers' professional development. Thus, lesson study might be considered developmental work and not research because even though the results are documented and shared it is not always systematic research.
- Learning-study aims to build innovative learning environments and conduct research into innovations grounded in theory where the primary focus is on an object of learning.
- The purpose of design research is to develop theories about both the process of learning and the means designed to support the learning. They may evolve either by developing local instruction theories or by developing theoretical frameworks that address more extensive issues.
- Self-study of teacher education practices aims at improving and formalising own teaching which are addressed within professional communities for debate and further judgement that might lead to theory building. It requires evidence for reframed thinking and transformed practice, which are derived from an evaluation of the impact of developmental efforts while interacting with colleagues, students, the educational literature and previous work.

The purpose of reflecting on these diverse approaches to practitioners' research was to make myself more aware of the factors that influence teachers' participation in researching into their own practice. In learning more about these approaches and how collaboration between teachers and researchers can vary, as well as the level of theory building, I concluded that the methodology of developmental research would be helpful in my own developmental research project with teachers. In Part II, Chapter 8 I will discuss these aspects in more detail.

In the following chapter, I will discuss how the process of conducting a self-study into how my experience of researching my own practice has influenced both my understanding of theories of learning and related research methodologies, and my teaching as a teacher educator.

6 Studying my practice as a teacher educator and a researcher

In the two preceding chapters, I discussed how my understanding of theories of learning and ways of researching one's own practice developed during my doctoral studies. I then related my reflections and interpretations of this process, to my former experiences as a teacher, teacher educator, and researcher. The methodology of self-study of teacher education practices guided me during this process, in relation to which I gathered examples of how my understanding was reframed through engagement with educational literature. Finally, I discussed how the self-study trajectory has affected my practice in general. My goal with this journey was to strengthen myself in making sense of how my understanding of practitioner research methodologies have affected my approach to teaching and researching, as well as in planning and conducting the collaborative research presented here. In doing so, I was formalising my work and making it available to our professional community for debate, further testing and review (LaBoskey, 2004).

In the introduction chapter, I posed research questions for both Part I and Part II. The question I wanted to answer in Part I was:

- How has my understanding of theories of learning and methodologies of practitioner research developed over the past 20 years through studying my own practice as a researcher and educator?

In Chapters 4 and 5, I addressed this question by discussing my self-study and presenting the findings in the form of autobiography. I summarise my conclusions and discuss how my interpretations of practitioner research methodologies led me toward a developmental approach in the study discussed in Part II.

6.1 Data collection and analysis

The data from the self-study was extracted from my writings at different times, both before and after I joined the doctoral program, as well as notes from my teaching at different times over a period of almost 40 years of teaching. In my dual role as a teacher in primary grades and within the teacher education program, I collected data from my work with children

and discussed with student teachers. The data consists of videotapes and pictures from classroom work, children's drawings and writings, and my reflective notes. In Chapter 3, I accounted briefly for my published writings, which are the basis of my reflection on my teaching and work with teachers. In addition, my writings about learning theories from courses I attended during the doctoral program were used as scaffolds to help me make sense of my developing understanding of aforementioned theories and methodologies. These archives and my writing at different times figure as my retrospective data, which I refer to in my journey to understand my development as a professional.

Throughout the process of analysing my texts, I revisited the data I had collected at earlier times and as I proceeded to write about them I simultaneously analysed the context in which they were gathered. Since I was particularly interested in tracing my developing understanding of constructivist and sociocultural theories of learning, I looked for what I had written at different times, both as a researcher into my practice and a doctoral student writing about theories of learning. This helped me sketch a trajectory of my temporal experience, and frame my changing understanding, given that life is experienced as a continuum, which though can lead us to neglect that the meanings we make of our experiences change continuously along the path (Bryman, 2004; Clandinin & Connelly, 2000). My autobiography (Bullough & Pinnegar, 2001) and first-person stories of my identity (Sfard & Prusak, 2005) described in chapters 3 and 4, are therefore written in chronological order. In Chapter 5, the discussion is centred on different types of practitioner research and my experience of applying three of them in my research. The goal with writing this chapter was to help me find a research method that would inform my approach to the research questions for Part II.

I have tried to bring aspects of teaching and learning about teaching to the surface, which I considered important to contemplate in the process of planning and conducting the new research with teachers (Loughran, 2007). The written account of my reframed thinking, provides examples from my practice, where collaboration with colleagues and my students is emphasised. In this process, I noted also the discomfort I felt and hard times I had in resolving my interpretation of theories of learning. Brandenburg (2011) experienced similar problems in her work, as teacher educator, an issue also was addressed in Pereira (2011) and Russell (2007). When I presented the findings, I sought to extract learning from the process that was not solely informative to myself but also meaningful, useful and credible for

those interested in drawing on such findings for their own practice, as emphasised in LaBoskey (2004).

I addressed critical instances in my journey with the purpose of offering insights into and understanding of my experience. At that stage, my intention was to present a comprehensive image of my development as a teacher educator and doctoral student. Using an authentic voice, I carefully attended to context and settings as well as illustrating and giving place to the dynamic struggle of relating theory and practice in my work as a teacher educator; hence offering fresh perspectives on established truths, summarised in Bullough & Pinnegar, 2001, as follows:

A self-study is a good read, attends to the “nodal moments” of teaching and being a teacher educator and thereby enables reader insight or understanding into self, reveals a lively conscience and balanced sense of self-importance, tells a recognizable teacher or teacher educator story, portrays character development in the face of serious issues within a complex setting, gives place to the dynamic struggle of living life whole, and offers new perspective. (Bullough & Pinnegar, 2001, p. 19)

The text is written in a language that is not my mother tongue. However, I tried to write in an authentic voice with the intention of providing an example of a teacher educator’s development. The story I have told concerns my challenges with interpreting learning theories and understanding how they have affected my teaching. The nodal moments I chose to highlight are representative for my struggle of coming to an awareness of the kind of learning taking place within the community where I was acting at different times. Other teacher educators may have similar experiences and they might be able to identify with my story, and hopefully it offers them new perspectives.

6.2 Making sense of theories of learning and teacher practices

During my doctoral studies I participated in courses about learning theories which afforded an opportunity to refresh my former knowledge of these theories, which consequently led to my understanding of them undergoing development. This was a challenging process and my revised interpretation of theories of mathematics learning urged me to reflect on my experience of working with children and teachers, building on my former understanding of the same theories. I had for several years been researching my own practice and collaborating with other teachers in developing our teaching and using knowledge about children’s mathematical thinking in

order to bolster our abilities in responding to our pupils' thinking about mathematical problems. We had become familiar with children's development of thinking about numbers and operations at the CGI course, discussed in sections 2.2, 3.1, 3.2 and 4.1. The research on the children's knowledge was framed in terms of a constructivist perspective, and as we attempted to make sense of our pupils' approach to numbers and operations, we adapted what we had observed in our own classrooms to the CGI findings, and used our knowledge to analyse the children's level of thinking. In order to make sense of the children's thought processes, we found it important to openly discuss these issues with them, individually and in groups, and collectively we managed to develop our understanding of how children learn mathematics. As a consequence, we used our learning in the classroom to plan further our work with the children (Kristinsdóttir, 2006; 2010a). Collective reflections with colleagues were a vital part of this learning process, as discussed in 3.1 and 3.2.

Our experience corresponds with what Fennema and her co-researchers (1993; 1996) found in their research on teachers that participated in their CGI workshops about teachers learning from listening to their pupils. In 4.1 I offered an example of this in my discussion about Pétur's solution to a problem about changing seats at the dinner table, and in 4.2, I described further how his solution stimulated me to discuss the pentagon with my pupils. I interpreted our experience of working with the children from a sociocultural perspective, given that I observed how our mutual experience, the teachers' and the children's, contributed to our learning in the classroom; our community of practice. I therefore had a hard time accepting the assumption that constructivist theories and sociocultural theories were incompatible (Goodchild, 2001; Lerman, 1996). When I met other teachers to discuss our work with the children and as we reflected collectively on our work, we related our experiences to findings from a constructivist research approach. Through my interpretation of the experience of our collaboration, I discovered that sociocultural theories might inform my analysis of our learning, especially in terms of how we supported each other in making sense of our own learning based on our experiences in our classrooms and knowledge of children's learning in general. I therefore found it essential to revisit my conception of these ideas, and in the two first sections of Chapter 4, I elucidated how my interpretation of these theories has developed by means of examples from my teaching in school and within teacher education programs.

What helped me in resolving this uncertainty was revisiting Lerman's writings and his criticism of those who claim that they were taking a

constructivist approach to teaching. He emphasises that Piaget's theory of constructivism is a theory of learning, not of teaching, and that teachers' actions in the classroom cannot be described as constructivist, or anything else. Furthermore, it is not sufficient for a teacher or a researcher to claim that they are taking a constructivist approach to teaching (Lerman, 2014). His main criticism was addressed to the concept of hypothetical trajectories, as introduced in Simon (1995) and adopted in Steffe (2004), who explain that the learning trajectories were hypothetical because the actual learning trajectory is not known in advance; it is constructed during and after the teachers' experience of intensively interacting with children. I then came to realise that my colleagues and I were not solely making hypothesis about our pupils' learning with reference to the findings from the CGI study, instead, we were using a knowledge base developed from a constructivist approach to support us in making sense of the children's level of understanding and uncover ways to scaffold them in their learning process. We adopted a model that was in harmony with our former experience of working with children, which further sharpened our interpretation of the learning we saw developing in our classrooms. Our aim was not to develop theories of teaching; we were developing our teaching through the process of making sense of our pupil's learning, and in that way, we were working as professional teachers.

I also revisited Bruner's writings and discussion of Piaget's and Vygotsky's theories, in terms of their similarities and differences, as well as whether their conception of the growth of mind are incompatible (Bruner, 1997). He argues that Piaget was preoccupied with the ontogenesis of causal explanation and its logical and empirical justification, while Vygotsky was concerned with the ontogenesis of interpretation and understanding. Bruner clarifies that in his analysis of how children explain and justify their explanations, Piaget ignored context, transactional dynamics, background and social variation. Vygotsky, on the other hand, took children's cultural and linguistic background into account and the context in which they find themselves. In consequence, the two approaches diverged as they matured and can in that sense, be viewed as incommensurate. Bruner concluded:

... we are enormously fortunate to have had two such rich theoretical accounts as an inheritance from our mentors, even if they prove to be incommensurate. Just as depth perception requires a disparity between two views of a scene, so in the human sciences the same may be true: depth demands disparity. ... To have had either of them as a guide would have been a gift. To have had them both is stronger

stuff, and even though it may at times seem overwhelming, we are the better for it. (Bruner, 1997, p. 72).

Bruner's conclusion echoes my experience of delving into their theories and exploring the epistemology they are based on.

In the latter part of Chapter 4, I elaborated further on my understanding of sociocultural theories of learning, communities of practice and teacher knowledge, with the aim of reflecting on which aspects I consider important to attend to in teacher research. In 4.3, I offered examples of my work with pre-service and graduate students to shed light on how their reflections on their work helped them develop their knowledge for teaching, and I made sure that I attended carefully to the texts that I had collected as data in my former research. Moreover, in 4.2 and 4.3, I thought further on how children can add to the teacher's learning in the classroom by bringing in their reflections on everyday experiences and how the teacher, by acknowledging their contribution, can build a bridge between school mathematics and the mathematics used in everyday life.

Furthermore, I discussed teacher professionalism and how my experience of working with teachers and student teachers has informed my understanding of collaboration and reflective practices. I revisited Mason's writing about the concept of noticing and addressed my understanding of his descriptions of "accounting of" "and accounting for". In my conclusions, I argued that if teachers are to be able to develop their competences they need to have opportunities to participate in collaborative development, where they can reflect on their own learning, both in terms of what they learn in their classrooms and within developmental projects.

My account in Chapter 4 is based on my previous experiences of teaching and my search for improved understanding of learning theories. My discussion of Goos's (2005; 2008) extension of Valsiner's (1997) theory was an attempt to make myself aware of features that influence collaboration in a community of practice. In my work with teachers, I have encouraged them to inquire into their practice and find ways to improve it. There are often obstacles on the path toward improved teaching, as the teachers strive to change their way of working with mathematics and within their classrooms, and this requires careful consideration. What they inquire into, during workshops, depends on what we have chosen to address, hence adding to their zone of promoted actions though at the same time restricting their zone of free movement.

6.3 Making sense of practitioner research

In order to better understand diverse approaches to practitioner research, and in the preparation of my collaborative research with teachers, I decided to revisit my former knowledge of action research and lesson-study, and also acquaint myself with related research methodologies; learning-study and design research, which I knew less about. Finally, I discussed process of the self-study of teacher education practices, which is the approach I adopted in conducting a study into my own practice as a teacher educator and doctoral student, discussed in Part I.

In 5.1, I offered an example of my experience of conducting action research with an experienced teacher and, in 5.2, of how pre-service teachers at my institution have used lesson-study in researching their teaching practice. Action research and lesson-study relies primarily on teachers' initiatives in conducting research into their own practice. The research process in action research is the developmental process of following through with proposed idea. Lesson-study is an attempt to improve learning and a way to organise teachers' professional development. Thus, action research and lesson study are often considered developmental work and not research because, even though the results are documented and shared publicly, the process of theory development is not necessarily built into the process.

When conducting learning-study and design research, teachers and researchers work together in finding ways to improve practices in schools. In learning-study, teachers aim to build innovative learning environments and conduct research into innovations grounded in theory, where the primary focus is on an object of learning. The purpose of design research is to develop theories about both the process of learning and the means designed to support the learning.

Self-study of teacher education practices is a methodology that developed within the field of teacher education. The emphasis revolves around the process of reframing thinking and transforming practice in teacher education, which can assume both an individual and/or collective character. The students of the teacher educator, both in academic programs and in-service projects, can participate in the self-study process. However, the collaboration is more commonly between colleagues.

My understanding matured through the process of reflecting on these diverse approaches to practitioners' research, and the factors that influence teachers' participation in researching into their own practice. Learning more about these approaches, and how collaboration between teachers

and researchers can vary, as well as the level of theory building, strengthened my understanding of how these factors affect the research process. I also studied developmental research that builds on a cyclic alternation of development and research, and where equal attention, is paid to both theory and practice in the research process. From my study of these diverse approaches, I drew the conclusion that developmental research would suit the purpose of my collaborative research with teachers. In this approach, the emphasis is on collaboration between practitioners and researchers and Development and research, and theory and practice, are all intertwined to an extent that that is not as implicitly built into any of the other approaches. I will discuss developmental research further in Chapter 8 and account for how I arrived at the conclusion that a developmental research approach fulfilled the goals for my research project.

6.4 Influence on teacher education practices

My first acquaintance with self-study of teacher education practices was during the time I wrote my M.Ed. thesis. I started my study from an action research cycle and during the three-year research process, I discovered how my study of classroom learning had led me to focus on my own learning in the classroom, to a degree that was not built into the design of the study. I felt that I needed further support in analysing my learning and found self-study research helpful in interpreting this process. Since then, I have, together with my colleagues, studied my own teaching as a teacher educator. This experience has supported me in reflecting on my learning process during the time of my doctoral studies.

The intention with including Part I in the thesis was to situate the three-year research I carried out with teachers within the context of teacher education practices. My decision to commence a doctoral program was to improve my practice as a teacher educator and I believe that this is an inseparable part of teacher educators practice. The cyclic study of learning theories, teacher development and practitioner research has resulted in development of my own teaching, both within the formal teacher education program and with in-service teachers. Every year, I have restructured my teaching together with my colleagues, emphasising inquiry and reflection to a greater extent than before. In this process, an essential component of the planning resides in the collaboration with the participants concerning the structure of the courses. In chapters 4 and 5, I offered examples of my students' reflections on their work during the last 20 years. This fall, my colleague and I, have once again restructured a

graduate course on inclusive mathematics teaching where the participants have diverse backgrounds. Some of them are pre-service teachers, others have extensive experience of teaching, many have focused on teaching young children, some are special education teachers and a few are mathematics teachers in older grades. Sara, an experienced special education teacher, actively discussed how the learning at this course affected her teaching and challenged her to restructure her teaching. She wrote:

I have used flash cards as a reward by the end of a lesson if the children worked well. ... I did though not find that they improved, they were not quicker in calculating mentally. They always stumbled on the same numbers. ... Then I read in the course textbook that competition and using flashcards, the way I did, could awaken anxiety amongst children and did not result in pupils' improved learning.

Sara enjoys discussing mathematical problems we deal with in the course and our diverse solutions to them, as well as discussing what the children might possibly gain from working with such problems. It has inspired her to work on such problems with her pupils, challenging them to reflect on their way of solving them. She described how she asks her pupils questions about their solutions strategies and what they anticipated would happen next. The children find this challenging, they "have difficulties with finding words to explain their thinking, as they are not used to explain their work with mathematics", Sara said.

Sara mentioned that she would like to learn more about problems that are suitable for diverse groups of children. She did not find it helpful to discuss inclusive practices in the mathematics classroom with young and un-experienced pre-service teachers within her group and felt that their ideas about teaching in school were unrealistic. Our challenge is to respond to these concerns. It has been a meaningful experience to have participants, with diverse background, attend our course about inclusive practices, and we have made an effort to structure our work in a way that offers all the students opportunities that allow them to make contributions to our collective learning. We have asked ourselves how we can support Sara in contributing to our discussions, building on her long experience of teaching. At the same time, we need to give the young people within the group a chance to reflect on their ideas and develop them further in collaboration with the other participants of the course.

I chose to present Sara's story to respond to Feldman's advice to teacher educators conducting self-study. Feldman (2003) urges us to provide an

account of the markers of change, i.e. how we, as teacher educators, become aware of our process of change, in a way that is visible to others, not only the fact of change, but also that the experience has been valuable to us. My teaching has developed successively through a constant strive for improvement. I have achieved this through the process of studying theories of mathematics learning and educational research methodologies collaborating with my colleagues and students, and reflecting on my learning individually and collectively. There were moments in my practice that helped me reframe my teaching, both positive and negative, and that often opened up for new ways of working with my students. Sara's reflections on her learning, confirms that our constant struggle for improving the course has at least had value for her.

6.5 Summary

I have discussed the findings from the study of the development of my understanding of theories of education and methodologies of researching one's own practice. The main findings from Part I are that through the process of reflecting on my understanding of constructivist and socio-cultural theories, I managed to resolve my conflicting interpretation of them, using results from research projects conducted within the framework of constructivism, in collaborative settings. My findings informed the mutual discussions I had with my pupils regarding how they made sense of the mathematics we were working on, and contributed to both their and my own learning. By drawing on our discussions, I decided how to proceed with my teaching, thus attending to their zone of proximal development. Their response to the problems we discussed, supported me in making sense of the research findings, which in turn strengthened my interpretation of their learning. During this time, I collaborated with teachers in primary grades who were reflecting on their pupils understanding of mathematics and our mutual discussions resulted in our improved understanding of the children's learning. Through this experience, we realised that in responding to our conclusions from the collaborative discussions, we adopted the approach of researching one's own practice. Through the process of this self-study, I have improved my practice as teacher educator, in teaching teachers and student teachers about mathematics teaching and teacher developmental practices, emphasising inquiry and collaboration.

In Part II, I will discuss my collaborative research with teachers. In the first two chapters 7 and 8 respectively, I will delineate how the findings from Part I informed my theoretical position and methodological approach.

Part II: Teacher development in a community of inquiry

In this second part of the study I account for the process of taking on a collaborative research project with teachers in schools. The motivation to do this study was to extend my former experience of working with colleagues and with student teachers.

Together with colleagues I had researched our teaching within the teacher education program, at both pre-service and graduate levels, and I had found that when teachers are urged to reflect collaboratively on their own way of learning mathematics their problem-solving competences grew as well as their appreciation of diverse ways of learning mathematics (Guðjónsdóttir & Kristinsdóttir, 2007b; 2011; Guðjónsdóttir et al., 2009; Gunnarsdóttir et al., 2008).

Student teachers learned about theories of learning, both in general and in mathematics particularly, and also about teacher professionalism. When collaborating on mathematics problems they related their thinking about the problems to their interpretations of the theories they were studying and to their own experiences as students and teachers. This experience assisted them in respecting their pupils' diverse ways of learning mathematics and in supporting them in gaining proficiency in mathematics. Accordingly they developed their competency of working with students as well as the competencies of working with others.

In my work with teachers at in-service courses my experience had not been as positive as within the teacher education program. I had interpreted that the reason might be teachers' reluctance to reflect on their own mathematics learning and teaching as many claimed that they did not have time for reflecting on their practice. I had not researched my teaching at in-service courses and therefore had not been able to identify approaches to collaborate with these teachers that might lead to improvement in both their and my own practice. My earlier collaboration however with teachers who collectively reflected on their own practice and connected it to their knowledge of children's mathematical learning had revealed that teachers are capable of developing their mathematics teaching for the benefit of their pupils (Kristinsdóttir, 2010a; 2010b).

I was curious to learn more about teachers' practice in primary mathematics classrooms and how they could be supported in developing their

mathematics teaching in schools with children of diverse background. Also when planning a collaborative research project with teachers I wished to gain experience that could support me in developing further my work with teachers. I was particularly interested in learning about ways of collaborating with teachers about their daily work in mathematics classrooms.

For three years I worked with seven teachers at 90-minute workshops on a monthly basis. They taught 10- to 12-year-old pupils in two neighbouring schools; four were homeroom teachers and three were support teachers who joined them in mathematics classes. The focus of the workshops was on reflection on mathematics and on mathematics teaching and learning. To help the teachers develop their own understanding of mathematics, we worked with problems that had the potential to promote mathematical activity and thinking as well as to stimulate collaboration where discussions and sharing thinking were meaningful. We also discussed research on mathematics education and stories from the teachers' classrooms, reflected on their pupils' mathematics learning and considered how their mathematical thinking developed. To learn about the teachers' visions for the project and the cultures in their mathematics classrooms I interviewed them and observed their classrooms three times: at the outset of the project, after the first year, and one year after the last workshop.

As discussed in Part I it was essential to start with my former work and reflecting on it as educator and researcher in planning this new research. Part II is about my study with teachers in primary grades. I start by discussing in Chapter 7 my theoretical stance and relate my discussion to the study of my understanding of educational theories that I have built my work on as a teacher and teacher educator and accounted for in Part I. Chapter 7 therefore has the twofold function of reporting my final conclusions of how self-study supported me in resolving my understanding of educational theories and underpinning the research study I took on as a part of my doctoral studies. In Chapter 8 I discuss developmental research and particularly the developmental research cycle that guided the research process. I also account for how I came to the conclusion that a developmental research approach fulfilled the goals for the research project. Chapter 8, like Chapter 7 serves the twofold function of reporting my findings of my self-study into my understanding and interpretation of practitioner research and then describing the methodology that guided my collaborative study with teachers. The methods used for carrying out the project and in analysing the results are then accounted for.

My guiding questions for Part II are:

- What learning processes emerge through long-term collaborative inquiry undertaken by classroom teachers and a mathematics teacher educator?
- When teachers are participants in a project based on collaborative inquiry how do they perceive this as being reflected in their mathematics teaching?
- In what way do I interpret that my own learning from carrying out a collaborative inquiry project with teachers has influenced my practice as a teacher educator and a researcher?

The main research question for both parts of the study is as follows:

- In what way has my understanding of teacher development in mathematics teaching and learning changed through working with teachers at improving their own practice?

The first research question in Part II with focus on how the collaborative research process emerged during three years is answered in Chapter 9. In Chapter 10 narratives of four teachers are presented and discussed in answering the second research question. In Chapter 11, common to both parts, I answer the main research question for both parts and the third research question in Part II as I summarise my findings. I also comment on the implications they have for my further work and teacher education practices as well as for research approaches into mathematics teacher development.

The key concepts for Part II are participation, teacher development, reflective practices, paying attention, developmental research, sociocultural theories, co-learning practices, grounded theory, narrative research, and autobiography.

7 Theoretical framework for the collaborative study

This study builds on my former research on developments in mathematics teaching, when the teachers' reflection on their practices becomes a part of their daily work. In designing this study, I drew on experiences generated from developmental projects and from my own studies, as well as other research, and I used findings from these projects in order to help me structure the new research. The study is a collaborative inquiry into mathematics teaching and learning where teachers and a teacher educator jointly inquire into their practices with the aim of building a co-learning partnership between teachers and a researcher in order to support classroom inquiry where pupils in schools learn mathematics through exploration such, as conceptualised in Jaworski (2006a). It involves a process through which teachers research their own practice with my support, and myself researching both this collaborative process as a whole and my development as a researcher.

The theories that guide the developmental process of this research study are sociocultural, in the Vygotskian sense, that individual cognition develops when people change their ways of understanding, perceiving, noticing and thinking through shared efforts with others (Vygotsky, 1978). During this development, they build on the cultural practices and traditions of communities such that participation is seen as both a social process and a personal experience (Lave, 1988; Lave & Wenger, 1991; Rogoff, 2003; Wenger, 1998). I ground my study on the Vygotskian perspective which holds that individual learning and psychological functioning derive from social processes and practices (Lave & Wenger, 1991; Wertsch, 1998), in contrast to the Piagetian perspective, where social processes are presumed to derive from individual cognitive functioning (Lovell, 1979; Piaget, 1969).

As discussed in Chapter 4, in framing the study I reflected on my understanding of learning theories and how it developed during my doctoral studies, with reference to my experience as a teacher in school, as a teacher educator and through studying my own practice, individually and collectively. In Chapters 4 and 6, I explicated my interpretation of how my understanding of constructivist and sociocultural theories of learning developed, as I reflected on my practice and on how I was motivated to rethink my understanding of the relationship between individual and social learning during my doctoral studies. My point of view has been vastly

influenced by writing about the theories that have guided me through the process of carrying out collaborative research with teachers and the process of negotiating my understanding of learning theories.

In this chapter, I discuss how I situate the collaborative study within the framework of sociocultural theories of learning as I account for the vision for improving practices in mathematics classrooms, where all learners are included in the learning processes. The four components, meaning, practice, community and identity, which Etienne Wenger (1998) has asserted must characterise the integration of a social theory of learning to a social process of learning and knowing, will be introduced in relation to my discussion of what the theoretical foundations of the developmental project. In discussing the complex relationships that people build across communities of practices, I also attend to Wenger's framework of learning in a landscape of practice that he has developed further with his wife Beverly (Wenger-Trayner & Wenger-Trayner, 2015). I also take note of a framework developed by Barbara Jaworski (2003) that can be applied to research which aims to address mathematics teaching development either from an insider or outsider perspective. Jaworski stresses that such research needs to take place within a co-learning community, involving teachers and others working collaboratively to develop teaching. Jaworski's framework has four dimensions: (1) knowledge and learning, (2) inquiry and reflection, (3) individual and community and (4) insider and outsider

I found her framework helpful when I reflected on the theories on which my collaborative research is built and I will attend to the first three components of her framework in the discussion of the theories that form the foundations for my work. The fourth component, insider and outsider, I will address in Chapter 8 where I delineate the methodological approach that I implemented in carrying out the study.

When structuring the chapter, I wanted to address features within the landscape of sociocultural theories of learning that I consider an essential issue with relation to the structure of my study.

1. In the first section I present the position I assumed as I carried out the study, by means of a discussion of learning as social participation. I pay special attention to co-learning in a community and the individual within such communities who travels through the landscape of learning.
2. In the second section I attend to inquiry and reflection as a way of developing one's own practice. I particularly address the position of inquiry I took in critically questioning established norms and accepted ways of teaching in mathematics in schools.

3. The focus of the third section is on the landscapes of learning where I frame learning both as situated in practice and in the complex landscapes of the different communities to which we belong.
4. Then I address mathematics teaching and learning as situated within communities of practice and the teacher's influence on the culture that is created in the classroom.
5. In the fifth section I will explicate mediation as a tool with specific focus on explicit mediation and on language as a tool for mediating meaning. Finally, I focus on the potentials and constraints in learning communities and the boundaries between them.

7.1 Learning as a social participation

The position I took in structuring and carrying out this collaborative research with teachers is based on the notion that learning is a social activity. When teaching mathematics in schools, teachers need to attend to the cultures they create in their classrooms and how the communities within their classrooms are shaped by their approaches to their teaching. Learning to learn from one's own practice requires active engagement and reflection in communities with others (Wenger, 1998). In order to understand how to create an environment where mathematics learning is supported, I find it essential that teachers and mathematics teacher educators engage in collaboration. In collaborative communities they share their experience and expertise in teaching mathematics in schools and teaching about mathematics teaching at universities and conducting educational research.

7.1.1 Individual and community

In communities of learning, the individual learner draws on knowledge in the community as well as on personal knowledge (Lerman, 2000a; Vygotsky, 1978; Wenger 1998). Nevertheless, in responding to diversity in classrooms the emphasis has commonly been placed on individualized learning (Askew, 2015). Moreover, students with learning difficulties in mathematics are highly likely to experience narrowly defined learning opportunities where the accent is on individual work and rote learning (Hiebert et al., 1997; Woodward, & Montague, 2002). Askew (2015) argues that in order to foster a more inclusive approach in attending to diverse learners needs, it is important to begin with learning communities, rather than taking the individual as the starting point for planning learning experiences. In these communities, teachers work with the collective construction of mathematical knowledge while still ultimately addressing

the needs of the individuals within that community. This is the position I take in working with teachers when attending to their different needs for improving their teaching and in finding ways to work with diverse groups of learners in inclusive ways.

Through collaborative activity, the community learns from the thinking, practices, and development of the individuals participating in the community. Important features of such communities are discussions about the particular approaches to mathematics in the classroom, encouraging learners to talk about mathematics and thus build their competency over time. “A challenge then is for teachers to set aside expectations of correct mathematical talk and work with the communicative resources that students do bring to school, including gestures and social resources” (Askew, 2015, p. 143). In the communities, learners listen to each other’s solutions and think about the connections they make to their solutions to help each other refine their methods and explanations. When sharing methods and trying out a particular approach and seeing whether other problems lend themselves to the same solutions, they learn about different approaches to solving mathematical tasks. Diversity is no longer an obstacle to classroom talk if learners participate in mathematical practices, in whatever way they can, even if these are mathematically limited, and use the language they are capable of using. It is thus being enriched through the diversity of learners’ contributions (Askew, 2015). The number of children with diverse backgrounds had grown in the two schools in which the participant teachers worked and we needed to attend to how to include them all in their mathematics learning communities.

When developing learning communities where the diverse background of the participants is respected, everyone’s contributions must be valued. Jaworski (2003) argues that collective learning develops through a mutually reflexive process of knowledge growth between individuals and a community in which co-learning partnership is cultivated. Thus, through the process of sharing experiences and developing norms, the community provides supportive structures for individual inquiry and acts to mediate knowledge so that knowledge grows within the community, as well as for each individual. She describes her framework for co-learning partnership:

Here, where individual and collective are concerned, I see a position in which the individual is supported in individual research through community involvement, while the community is enhanced through the learning and experiences of the individual acting within the community and society in which the community is rooted. Ultimately,

the critical focus of the framework must shift to considerations of how these various dimensions lead to enhanced learning of mathematics by students, hence to the societal expectations of mathematics teaching and learning. (Jaworski, 2003, p. 262)

Her description of collaborative research aligns with the aim of my collaboration with the teachers. In collaborating with teachers we support each other to research our practice and each of us adds to the knowledge growth of the community we build. When the teachers develop their understanding of mathematics learning and how learners can be facilitated in learning mathematics, strengthens their capability to build communities of mathematics learning in partnership with their pupils.

Within our community it was envisioned that a co-learning agreement would develop where we participated through action and reflection in processes of education and systems of schooling (Wagner, 1997). I wanted the teachers to feel that they shared ownership of the project and could contribute to the direction and development of it. There was an expectation that differences in power and authority would develop (Goodchild, 2008) since I initiated the project. As a result, I found it important to give the teachers leeway in impacting the structure of our workshops.

In creating a community with teachers to learn about our practices, I was aware that we all belong to other communities of practice and that conflicts of interest between them could arise. The teachers belonged to several communities of practice within their schools, such as communities with pupils in their classrooms, with their close collaborators in their daily teaching, and with the larger community in their schools. The communities I belonged to involved also my students, teacher educators, researchers and doctoral students. The experiences of participating in co-learning practices, in these communities, affect our work; offering challenges to the development of effective functioning of the community.

7.1.2 Learning in a landscape of practice

Wenger-Trayner and Wenger-Trayner (2015) argue, when describing their framework for learning in a landscape of practice, that most professional occupations are constituted by a complex landscape of different communities of practice. They emphasise that meaning is produced in each practice and relationships between practices are always a matter of negotiating the boundary between them where conflicts may arise due to the diversity cultures that populate the communities. In relation to our practice, we can only participate actively in a few communities and the

degree to which we contribute and are shaped by the cultures in them varies. The notion of landscape fits well with my research project with the teachers, given the fact that our project is just one of the communities they travel through on the journey across the landscape of teacher practices. I will discuss further how belonging to communities of practice and identifying oneself in a landscape of practice is reflected in my study.

Wenger (1998) describes three modes of belonging to a community of practice, namely, engagement, imagination and alignment. In Wenger-Trayner and Wenger-Trayner (2015), these are referred to as modes of identification and are described as ways of identifying oneself in a learning landscape of practice:

- *Engagement* is the most immediate relation to a landscape of practice. We do things, work on issues, use and make artefacts, talk, debate and reflect together in the communities we engage within. On our learning journey, engagement gives us direct experience, whether the result is an experience of competence or incompetence, or whether we develop an identity of participation or non-participation. When engaging directly in the communities we belong to we learn the practices of our community. Engagement can also be a way to explore a boundary if we have sufficient access to another community of practice.
- *Imagination* functions within a community as we make assumptions about each other, recall the past, and talk about our future. As we travel through a landscape we construct an image of the landscape that helps us understand who we are within it. The world provides us with many tools of imagination and these are essential to our interpretation of our participation in a landscape. Imagination can also travel without limits and is a way to experience identification beyond our immediate engagement.
- *Alignment* is a central element of a community's local competence but is also important in the functioning of broader systems. It is a two-way process of coordinating enterprises, perspectives, interpretations, and contexts and without some degree of alignment with the milieu, our engagement in practice can be ineffective. Following directions, negotiating a plan, preparing collaboration or convincing colleagues to change plans are examples of alignment that can become very deep aspects of our identity.

Wenger-Trayner and Wenger-Trayner (2015) contend that the combination of these different modes produces identification at multiple levels of scale when operating within and across practices. Teachers might for instance identify with teachers in their department, school, district, region, discipline, or country, or teachers globally and in general. Resonance could be stronger at some levels and they might dis-identify with other levels, for instance by dismissing the relevance of a regional

policy. To be able to locate ourselves in a landscape that extends beyond the practices with which we are directly involved, we need to be capable of defining our identities at multiple levels of scale.

When the teachers who took part in this study decided to join the project, they agreed to collaborate within our community of practice. At the same time, they belonged to other communities of teacher practices. The degree to which they identified with our community would differ and was partly dependent on how we would succeed in creating the kind of learning opportunities that the participants found meaningful to participate in. The cultures in the other communities in which they participated might conflict with our culture and negotiating the boundaries was therefore essential.

Jaworski (2006a) uses the term critical alignment to describe how it would be possible for participants in communities of practice to align with aspects of practice if their active participation called for critically questioning roles and purposes as a part of their participation. They would thus be critiquing the culture in their communities of practice.

I would see critical alignment to include some sense of teachers critiquing and trying to develop, improve or enhance the status quo, alongside enculturation into existing social norms. However, the significance of normal desirable states is just that they are desirable within the social practices in which they have developed. It is hard to operate against such practices, or to challenge them in practice. (Jaworski, 2006a, p. 191)

Within the teacher communities in the two schools where my study was conducted, the teachers' views of mathematics teaching might be different from what I intended to emphasise, and it could be difficult for the teachers to align with the inquiring and reflective approaches into mathematics teaching in the extent that I sought to develop within the project. Engaging in our community and aligning with the inquiring mode of learning might cause dissonance within the communities in their schools and the teachers would thus need to negotiate the boundaries between them. Jaworski (2006a; 2008a) argues that, by engaging in a questioning or inquiry mode we could bring a critical attitude to the alignment and hence would have possibilities to develop and change normal states. To move from a community of practice to a community of inquiry, participants would need to engage in existing practices, aligning to some extent with those practices, but in a questioning or inquiry mode. My vision for the project was that the teachers would succeed in bringing a critical mode of inquiry into their communities within their schools.

7.1.3 Summary

Drawing on my former experience of working with teachers on collaborative activities within teacher education, I looked further for writings that build on sociocultural theories, particularly in terms of individual and collective learning, to make myself aware of how my comprehension of them could strengthen my approach to this study. Teachers belong to several communities of practice and the metaphor of the traveller who moves through landscapes of learning and the degree to which teachers align themselves to different communities is therefore of interest for my study.

7.2 Inquiry and reflection as a mode for developing practice

To be able to develop their practice, teachers need to actively engage in reflecting on their teaching and make sense of how it is developing. In collaborative research, the participants have opportunities to engage in a dialogue about their underlying beliefs about teaching and learning, and mutual respect for each other's beliefs is important. Looking for a common ground that can provide openings for collaboration is helpful and to create such occasions it is imperative that teachers discuss their own way of approaching mathematical tasks and their student learning.

7.2.1 Reflection

Reflection on and in one's own practice (Schön, 1983; 1987) is an essential part of developing that practice. Reflection can be interpreted in terms of both its individual and its collective nature. Explained in terms of constructivism, the individual interprets personal experience and previous knowledge by internally constructing his/her image of the world (Glaserfeld, 1995). From a sociocultural perspective, reflection can be interpreted as social activity since thought is seen as a mode of organising what one has perceived and planning for action (Bruner, 1985; Vygotsky, 1978), and thus contributing to the creation of cultural processes and learning from them. By emphasising reflection in the project, I hoped that we could collectively reflect on our practices and beliefs about mathematics teaching and learning, thereby assisting each other in learning about teacher development in mathematics teaching and constructing collective images of the mathematics learning communities we wanted to create within our classes.

In the reflections on our practices we would need to attend to what was taking place in our classrooms. Mason (2002; 2011) uses the term noticing

for the mode of focusing involved in the process of researching one's own practice. He argues that by attending to our own actions, and by paying attention to the degree of sensitivity in the way we respond to people and situations as we learn about our own teaching, teachers could improve their ways of communicating in their classrooms and avoid reacting habitually according to established patterns. In the work with the teachers, I tried to support them in focusing on noticing, in relation to what happened in their classrooms and in making themselves aware of how communication with their pupils informed their learning about their teaching, as well as the children's learning. By urging them to write about their reflections I hoped that they would develop responsive practices of focusing on systematic reflection.

It is important to think about the focus of the reflection when writing about one's own experiences. In this context, Mason differentiates between accounting-of and accounting-for. In the accounts-of one writes a brief but vivid description of what happened in a way that allows other to interpret and relate to their own experiences. Conversely, one is accounting-for when the aim of the elaboration on one's own experiences and reflections on what happened is to learn from them. This is done by offering interpretation, explanation, value-judgement, justification and criticism and asking oneself why incidents occurred and why a particular aspect is noticed (Mason, 2002).

Case and commentary writing, as explained in Kruger & Cherednichenko (2006), has some common features with Mason's discipline of noticing. It consists of five iterative stages, that is, case writing and four dimensions of praxis inquiry. When writing the case, teachers are urged to offer an objective description of what happened so other people can interpret what happened. When describing and explaining their practice, they adopt discourses for interpreting the action and construct their personal theories of the practice described. Theorised practice presents practitioners with opportunities to propose and trial new practices, make decisions or conclusions, and develop and improve their practice.

In Section 4.4.2, I discussed case writing and Mason's concept of noticing in more detail with relation to my former experiences of reflective discussions with teachers. In the workshops with the teachers, I planned to discuss with them their experiences from their classrooms. I encouraged them to present detailed information of what happened when they introduced their stories so that other members of our community could be aware of what happened. We would then develop them further as the

teachers reflected on what they had noticed in their classrooms and related these discoveries to their former experience, as well as their knowledge about mathematics teaching and learning. Together we would look for common ground in interpreting what happened and discuss possible ways for the teachers to respond to the situation in their classrooms. We would use our own personal qualities when we reflected on these stories and when focusing on steps the teachers could take in their classrooms.

In my thoughts about the process of reflection, I have found Korthagen's (2004; 2013) framework of core reflection meaningful in terms of accounting for the mismatch between the ideals teachers may have and what can be accomplished. He uses what he called the 'onion model' to describe the inner world of the person, the experience of flow and the non-flow that can be the result of a discrepancy between ideals and what is accomplished. The onion model has six layers: environment, behaviour, competences, beliefs, identity and mission. Korthagen notes that the three outermost layers (environment, behaviour, competences) often receive most attention in dealing with the class and the circumstances at school. The inner layers, however, may have the more influential on the outer layers than vice versa, given that the teachers' beliefs, identities and missions, with regard to learning and teaching, determine their actions. My previous experience of working with teachers revealed that they often rather focus on the outer layers of the onion model, which means that they need support in directing their attention to the inner layers in order not to lose sight of their ideals. Pressure from established norms in the school culture often prevents teachers from fulfilling their visions for their teaching. Korthagen argues that in core reflection, professional development is in focus and the depth is reflected in the process of tapping into one's inner potential for the benefit of professional growth. The inner layers in Korthagen's model place the accent on individual thinking and can thus be interpreted in constructivist terms. It can also be seen as a way of reflecting on the conflicting cultures within the communities in their learning landscapes and a mode for solving the dissonance between them.

My vision for the project is that the teachers will actively engage in reflection on their practices within their own classrooms and as they engage in mathematical activities at the workshops. From these learning experiences they will then be empowered to present their students with opportunities to experience dialectical thinking and develop deeper their theoretical thinking in the mathematics classrooms.

7.2.2 Inquiry

Inquiring into one's own practices is essential in developmental work. Taking on an inquiry stance refers to the point in the teacher's practice where it has become a norm for them to inquire into their practice with regard to how they can relate their knowledge to their practice (Cochran-Smith & Lytle, 1999). Asking questions, making investigations, acquiring information, and searching for knowledge are all part of the inquiry process.

Inquiry is a valuable tool for teachers and educators by means of which they can enable themselves and others to engage critically with key questions and issues in practice. It could involve addressing mathematical tasks in classrooms, developing approaches to mathematics teaching or finding ways of working with teachers to promote teacher development (Jaworski, 2006a). However, the aim of inquiring into one's practice is a psychologically taxing practice and which challenges the status quo of accepted ways of teaching in schools (Goodchild, Fuglestad, & Jaworski, 2013). As discussed above, I aimed to support the teachers in questioning established norms in their schools and rethinking their way of teaching mathematics. I see the process of inquiring into one's own practice to be fundamental in the journey through the learning landscape of their practices.

Jaworski (2006a) describes three forms of inquiry communities and how she sees them as potentially contributing to learning and development. I have found her description of these forms helpful in framing my study:

- *Inquiry in mathematics*: Pupils in schools learning mathematics through exploration in tasks and problems in classrooms; teachers using inquiry as a tool to promote pupils' learning of mathematics;
- *Inquiry in teaching mathematics*: Teachers using inquiry to explore the design and implementation of tasks, problems and activity in classrooms; educators using inquiry as a tool to enable teachers to develop teaching;
- *Inquiry in research which results in developing the teaching of mathematics*: Teachers and educators researching the processes of using inquiry in mathematics and in the teaching of mathematics. (Jaworski, 2006a, p. 203)

When the teachers participate in the workshops and discuss how they solve mathematical problems and relate to their pupils' way of solving them, I expect that they will gain confidence in facilitating their pupils' mathematics learning through exploration. Following our discussions regarding what they have noticed about their pupils' learning in their

classrooms, the teachers interpret, explain and criticise their work as they ask themselves questions about how to promote pupils' learning of mathematics.

During the workshops discussion, the teachers are empowered to take risks in their mathematics teaching, such as reflecting on the nature of the tasks they choose, what kind of approaches could be used to solve them and consequently ways of planning their teaching for promoting pupil's inquiry in mathematics. When I urged the teachers to rethink their mathematics teaching, we would participate in an inquiry dialogue in which key activities and roles of the participants would take the form of questioning, exploring, investigating, and researching (Jaworski, 2008a).

The developmental aspect of the research project implied that there would be a reflexive relationship between development and inquiry in researching our collaborative project. The developmental research cycle (explained further in Chapter 8) that guided the research process includes a developmental cycle and a research cycle that are intertwined. In other words, the findings from the developmental cycle nurture the research cycle that guides the developmental process.

Jaworski (2006a, 2006b) argues that in a community of inquiry the inquiry is seen both as a tool for developing practice and as a way of being in practice, and thus, inquiry becomes a norm of a community of practice. The communities allow the participants to ask questions about improving students' opportunity to learn mathematics, and in doing so, aim to learn about their own learning. Jaworski sees inquiry as a tool that promotes critical alignment with modes of practice and corresponding development of practice. "In communities of inquiry, we all engage with inquiry as a tool to develop meta-knowing, a form of critical awareness that manifests itself in inquiry as a way of being" (Jaworski, 2008a, p. 204). Jaworski argues that when individuals are encouraged to look critically at their own practices and to modify these through their own learning-in-practice, there will be a shift from "community of practice" to "community of inquiry". Through the shift a perspective emerges in which reflective development of practice by practitioners, individually or in groups, can be seen to result in the development of community.

Inquiry, as a way of searching for knowledge and promoting learning is fundamental in developing teaching in the sense that it drives the teachers to asking questions, make investigations, acquire information and look critically at their own action.

7.2.3 Summary

The diverse background of the participants in this project opened up new learning for us as we reflected together on our understanding of teaching and learning mathematics. It required us to make sense of the processes involved in the different ways in which people negotiate social interaction and the elements of the various models of the culture from which we come. We needed to be aware of aspects that are unconscious and invisible, as well as develop skills necessary to negotiate ways of interacting with others who may not share the same values, attitudes and habits.

Being critical to the norms that emerge within communities of practice is an essential factor in developing one's own teaching. When we start to question established norms and critically reflect on our actions, we adopt an inquiry stance and question also the established norms within the communities we belong to. From a constructivist perspective, inquiry can be seen as a tool for individual cognitive development and which does not overtly address social structures and development in a broader sense. From a sociocultural perspective, where individual learning is regarded as derivative of social learning, it is however seen as a collaborative endeavour (Jaworski, 2006a).

In communities of mathematical learning, all participants are expected to have access to and contribute to cultures that develop within them.

7.3 Landscapes of learning

Mathematics teachers, work in schools and this is where their practice is situated. In their workplace, they belong to many communities of practice, such as in the classrooms where they teach with other mathematics teachers, with teachers of other subjects, the school community as a whole, mathematics teachers in other schools and teachers in the same district area. The learners in their classes also belong to different communities and teachers need to respect that they bring cultures from other communities into their classrooms.

7.3.1 Learning viewed as situated

Learning viewed as situated assumes that learners participate in communities of practitioners and the mastery of knowledge and skills requires newcomers to move toward full participation in the sociocultural practices of a community. According to this view, knowledge is about competence in life settings and has to be understood relationally, between people and

settings. Learning interpreted as situated activity is seen as a process of legitimate peripheral participation. It refers both to the development of identities in practice and to the reproduction and transformation of communities of practice (Lave, 1998; Lave & Wenger, 1991). The community that was founded within this study was new and established norms therefore did not yet exist. All the participants were expected to add to the culture and create customs that we would align to. We all found ourselves in the same situation, and although I initiated the project, my vision was that we would equally support the creation of the community's norms and learning habits. In this sense, each of us could be seen as situated centrally, transforming the culture that we established within the community and, on the periphery, adopting the norms of the community.

Lerman (2000a) warns that the interpretation of learning as a situated activity could create special problems for mathematics education. In section 4.3.1, I addressed these concerns and how the pupils' background affects their understanding and ways of working with mathematics at school. I related to Bernstein's (2000) description of pupils' differing levels of access to discourses in schools and how control and power are manifested in pedagogical relations. His description has informed my interpretation of Lerman's (2000a) writing, particularly in relation to the concepts of 'classification rules', related to boundaries between discourses, such as subjects in school curriculum, and the 'framing rules', which apply to those who takes the control of the discourse.

The cultural background of the students and the language, we use in school, can influence the students' interpretations of the mathematics they work with at school and their possibilities to participate in the pedagogical discourse at school. Solomon (2009) has looked at the ways in which learners develop identities of participation within formal learning contexts and how marginalisation in school communities can lead to the inability of acquiring access to these practices, and hence, prevent learners in becoming mathematically literate. She argues that access to mathematics literacy requires that its invisible practices are made visible which calls for a need to acknowledge the power relations of mathematical discourse, where the teacher, as an expert, is a figure of power and authority.

In Section 4.3.1, in connection to Bernstein's (2000) theorising of pedagogical discourse, I discussed how the mathematics practiced at school differs from how it is used outside the school. Solomon (2009) argues that children of the marginal classes do not have access to the pedagogical site of mathematics in schools because if they are not familiar with the

informative discourses at school, they fail to acquire the legitimate pedagogical code necessary for access. Their experience of school therefore is often an experience of the classificatory system; their place in it and the place of the subject. In the schools where the teachers in this project worked, there were children of diverse backgrounds, which necessitated us to attend to how to include them as active participants in their classrooms. In my work with the teachers, I also needed to make myself aware of their varied backgrounds and experiences when discussing their way of learning mathematics as well as their work in their classrooms.

7.3.2 Learning viewed as a journey through a landscape

Wenger-Trayner and Wenger-Trayner (2015) argue that the notion of a single community of practice misses the complexity of the field of knowledge. They see professional occupations as established by a complex landscape of different communities of practice that all have their own histories and dynamic interrelations. In addition, they maintain that in times when globalisation, travel, and new technologies continuously expand our horizons, it is increasingly possible to connect various locations in the landscape with their own local knowledge. "If a body of knowledge is the landscape of practice, then our personal experience of learning can be thought of as a journey through this landscape (Wenger-Trayner & Wenger-Trayner, 2015, p. 19). Today we have access to knowledge through various media and we can participate in professional discourses through them. We also travel to other countries and many people live in foreign countries for shorter or longer periods, and thus, we have the option to belong to communities of diverse cultures. I expected that in our collaborative project we could share experiences from our journeys through the landscape of different communities and the knowledge gained along the way we could all add to our community of practice.

Researchers from Denmark and New Zealand worked together to explore socio-cultural-political perspectives on language and school mathematics practices (Valero, Meaney, Alrø, Fairhall, Skovsmose, & Trinick, 2008) in connection with multicultural mathematics education. They sought to combine the notions of the learning landscape and school mathematical discourse and explore the relation between them. They used the term learning landscape to define mathematics education as a network of social practices. School mathematical discourse they characterise as anything that is involved in the communication and construction of meaning within the mathematics classroom. The goal with their research

was to challenge the simplification of issues in regard to mathematics learning in multicultural settings, and work towards finding ways to provide inclusive education. The main challenge was to resolve the restricted perspectives on bilingualism and the belief that students cannot participate effectively in school mathematics if they have not mastered the language of instruction. As a response to these issues, they proposed the notion of landscape of learning. According to them, the term landscape of learning can be used as a tool to guide the exploration of the complexity of the empirical field. The notion has a double meaning:

First, it represents an interpretation of (mathematics) education as a complex network of social practice that is constituted by different interrelated dimensions. Second, it makes possible to identify specific – but interrelated – dimensions of an empirical field to do research. Thus, it brings together a research perspective and a research field. (Valero, et al., 2008, pp. 72–73)

The learning landscape has nine dimensions that are considered influential with regard to the learning possibilities within multicultural settings. These dimensions relate to students, foregrounds, construction of identity, teachers perspectives of teaching, mathematical content for classroom interaction, tools and resources for learning, classroom interaction, the influence of family and friends, and public discourses about immigrants, schooling and multiculturalism.

The learning landscape as described in Valero et al., (2008) has similar features to Wenger-Trayner and Wenger-Trayner's (2015) description of learning in a landscape of practice. The nine dimensions can be interpreted as the different communities of practice that the learners participate in. The learners enter school with diverse backgrounds shaped by their participation in communities at home, in the broader society, and former schooling. If the school does not respect that they bring experience from other communities into their new communities in school their possibilities of growing are restricted. If teachers do not acknowledge the ways that knowledge of a language other than the formal language of instruction can be of an advantage in the mathematics classroom, there is a danger that children who do not master the language of instruction will be marginalised. In the two schools with which I was collaborating, the number of children with a multicultural background was growing and I was aware that we would need to attend to the teachers' concern for their student's abilities to participate in mathematical discourses.

7.3.3 Summary

The notion of learning landscape is well suited to cover the range of experiences that teachers and other learners within their classrooms bring with them into their learning communities. Pupils' background and their experience of participating in formal learning situations, and their mastering the language used in school, as well as participating in the classroom discourses which have bearing on their learning in the classroom. This also applies to the teachers in this study and as a result, I need to be mindful of their different experiences of teaching mathematics and researching their practice.

7.4 Teaching and learning in communities of practice

Building a community of mathematical learning with their pupils requires teachers to take the lead in establishing expectations and norms about mathematical knowledge, where active participation of all participants is acknowledged. They need to attend to all learners and support them in developing their understanding of mathematics through exploring, investigating, discussing, reflecting and drawing conclusions. In such communities, the teachers learn about their pupils' learning, and in reflecting on their communications with them, they learn about their own development in teaching.

The quality of mathematics learning in classrooms depends on the teachers' capability of building communities that enable learners to develop their mathematical competences. To be able to enrich learning in mathematics classrooms, teachers need to be competent in approaching their teaching in such a way that all participants in their classrooms will gain from it. Wenger-Trayner and Wenger-Trayner (2015) argue that competence includes a social dimension in the sense that a community's social negotiation of what constitutes competence results in a regime of competence. However, a regime of competence is not static given that members of communities enter into them with their own experience of practice, and in which they can shape the community as well as being shaped by it. This applies to teachers in schools as well as members of other communities of practice and even to the communities in mathematics classrooms. When newcomers enter a community, for instance teachers in schools or pupils in classrooms, they participate in a process of alignment between competence and personal experience. They may also challenge the community to reconsider its practice and thus influence the community's competence, nevertheless, their claim might also meet with refusal.

7.4.1 Teacher knowledge

Teachers use both their pedagogic knowledge about teaching and their subject matter knowledge about the subject on the agenda when planning for teaching and creating learning communities within their classrooms. Schulman (1986) distinguishes between *content knowledge* and *pedagogical knowledge*. He defines ‘content knowledge in teaching’ as the knowledge that grows in the minds of teachers, with special emphasis on content and differentiates between three categories: *subject matter content knowledge*, *pedagogical content knowledge*, and *curricular knowledge*. Ball et al. (2008) developed a practice-based theory of content knowledge for teaching based on Shulman’s notion of pedagogical content knowledge. They identified two subdomains of pedagogical content knowledge, namely, *knowledge of content and students* and *knowledge of content and teaching*. They also identify *specialized content knowledge* for teaching, which they differentiate from *common content knowledge*. This was further discussed in section 4.3.2.

In a UK study on prospective primary teachers’ mathematics subject-matter-knowledge the finding resulted in a framework, called ‘The Knowledge Quartet’. The framework is a theoretical tool for observing, analysing and reflecting on actual mathematics teaching with reference to which lessons can be discussed, with a focus on their subject matter content, and the teachers’ related knowledge and beliefs. The framework can be used for supporting mathematics teachers and teacher educators in analysing and producing feedback on mathematics teaching (Rowland, 2014).

The framework consists of four-dimensions, with one as a foundation on which the other three dimensions are based:

- *Foundation* is the basis of the framework. This dimension consists of knowledge and understanding of mathematics and of mathematics-specific pedagogy. Key factors of the foundation are beliefs concerning the nature of mathematics, the goals of mathematics education and the conditions under which students will best learn mathematics.
- *Transformation* concerns the presentation of ideas to learners in the form of analogies, illustrations, examples, explanations and demonstrations.
- *Connection* includes the sequencing of material for instruction, and an awareness of the relative cognitive demands of different topics and tasks.
- *Contingency* is the ability to make convincing, reasoned and well-informed responses to unanticipated and unplanned events (Rowland, 2014; Rowland, Huckstep, & Thwaites, 2005).

Rowland (2014) compares this framework to the practice-based theory put forth by Ball et al. (2008) about content knowledge for teaching and adds that lesser significance is placed on different kinds of mathematical knowledge in Ball's framework than in their tool.

Niss & Højgaard-Jensen (2002) developed a model for mathematics teacher competency and the ability to develop one's competency as a mathematics teacher, as well as the competencies of working with students and working with others towards professional development. The development of teaching in classrooms is seen as dependent both on the teachers' knowledge and their ability to learn together with others, both their students and colleagues. The teacher learns from participating with the learners in the classrooms, about her own learning and of the collective learning in the classroom that shapes the classroom culture.

The *competency model* consists of six main strands of didactical and pedagogical competences for professional mathematics teachers:

- *Curriculum competency*: being able to evaluate and draw up curricula.
- *Teaching competency*: being able to think and plan out, then plan and carry out teaching.
- *Competency of revealing learning*: being able to reveal and interpret students' learning.
- *Assessment competency*: being able to evaluate and characterise the students' mathematical yield and competencies.
- *Cooperation competency*: being able to cooperate with colleagues and others regarding teaching and its boundary conditions.
- *Professional development competency*: being able to develop one's own competency as a mathematics teacher (Niss & Højgaard-Jensen, 2002).

Teachers build mathematical knowledge for teaching through the process of reflecting on learning experiences and planning for future teaching. When teachers meet to reflect collaboratively on their work, they strengthen both their own competence and those of their communities of practice, the communities in their classrooms, as well as those they share with colleagues. The competence model informs the interpretations of the cyclic learning in developmental projects, as the one I carried out. Moreover, it also includes the teachers' capability to reveal and assess learning, an aspect not emphasised to the same degree in other frameworks. These frameworks for teacher knowledge supported me in interpreting my learning experiences within the project as well as in analysing the development of the teachers' learning.

7.4.2 Understanding and knowledge of mathematics

Skemp (1976) addresses the dissonance between what he called *instrumental understanding* and *relational understanding*. He refers to instrumental understanding as ‘rules without reasoning’, based on memorisation of the steps of mechanical procedures. Relational understanding on the other hand, concerns ‘knowing both what to do and why’, based on thoughtful and connected learning of principles. What is understood is associated with many other existing ideas in a meaningful network of concepts and ideas. Hiebert and Carpenter (1992) refer to ‘webs’ of interrelated ideas to describe rich understanding and differentiate between *conceptual knowledge* and *procedural knowledge*. The former is knowledge that resides in the comprehension of logical relationships constructed externally and internally in the learner’s mind as a network of ideas. Conceptual knowledge is the type of knowledge that Piaget (1973) characterises as logico-mathematical knowledge.

Procedural knowledge, on the other hand, is knowledge of rules and procedures that can be used to carry out routine mathematical tasks, like the traditional algorithm for addition, subtraction, multiplication and division, which has been the common practice in the Western world through the centuries. When carrying out the steps of this algorithm the focus is on digits, starting with the units and treating digits in other places as units. Children tend to make place value errors in carrying out these steps, “either in carrying the tens or lining up the numbers” (Fosnot & Dolk, 2005, p. 91). However, if children are invited to invent their own ways to calculate, they reflect on the meaning of the numbers in the problem and the operation involved in the calculation (Fosnot & Dolk, 2005; Hiebert et al., 1997), in furthering their conceptual understanding.

Throughout my collaboration with the teachers my vision has been to support them in gaining relational understanding of the mathematics we have dealt with, and in that way, acquire the capabilities of approaching their teaching in such a way that will benefit their pupils’ relational and conceptual understanding.

Mason and Johnston-Wilder (2006) refer to Skemp’s description of relational and instrumental understanding in their discussion of the ways in which teachers can encourage learners to use initiative when tackling mathematics problems as opposed to being passive receivers in the classroom. They argue that when learners wait to be told what to do, they are in an *accepting mode*, which might at the best, lead to instrumental understanding. However, when they ask probing questions and apply

initiative in their learning, they adopt an *asserting mode*, which is likely to lead to relational understanding. They enter an asserting mode when they take risks and learn from their mistakes, make and test conjectures and reconstruct ideas, and thus gain relational understanding. It is a challenge for teachers to create classroom cultures that support learners in asking probing questions.

Attention and *awareness* are also important features of mathematics learning. Mason, (1998; 2008) holds that teaching is fundamentally about attention and teachers can enhance pupils' attention by attending to their own awareness. When someone else points something out to us our awareness changes slightly; we become more explicitly aware of some features, and less aware of others. Thus in collaborating with colleagues, teachers are afforded the ideal conditions in which work on their own awareness, which can provide conditions for their students to experience them too.

When gaining competence in teaching mathematics teachers build on their knowledge and experience and an essential factor in this process is the participation in learning communities. In order to be able to support learners in their classrooms in acquiring competence in mathematics, teachers need to urge them to ask probing questions, take risks and learn from their mistakes.

7.4.3 Summary

To summarise my discussion of mathematics teaching and learning in communities of practice I refer to Davydov (1999) who asserts that basic learning actions are the ones in which learners have to transform the terms of a task when the task cannot be resolved in ways that have already been prescribed and learned. It requires them to make models that reflect some general relations between the elements of the tasks to be solved. According to him, learning actions also include control and evaluation which gives the learner the possibility for correcting their learning actions and determining if a general way of solving the learning task is appropriate or not. In our investigations with mathematics at the workshops, the intention was to discuss our actions and the steps taken while solving problems. We also discussed how using this kind of approach with children could help them build confidence in learning mathematics and feel safe to discuss their discoveries within their classroom instead of following rules and procedures described by teachers and in textbooks.

7.5 Mediation and tools

In mathematical learning communities, language is the main tool for communication. However, other tools can also become significant in mediating meanings and making sense of what is to be learned. The focus on collaborative investigation in mathematics learning in my work with the teachers is an attempt to help teachers realise how important it is for all mathematics learners to actively engage in collective explorations and open discussions.

7.5.1 Mediation

Mediation is central in Vygotsky's writings. In our communication with the world, our actions are mediated by sign systems, in particular, speech. Vygotsky (1978) believed that the internalisation of culturally produced sign systems brings about behavioural transformations and forms a bridge between early and later forms of individual development, from the elementary or biological origin, to the higher psychological function of sociocultural origin. According to Wertsch (2007), a hallmark for human consciousness in Vygotsky's writings is the use of tools, especially psychological tools and signs. "This means that understanding the emergence and the definition of higher mental processes must be grounded in the notion of mediation" (Wertsch, 2007, p.178). All human action can be interpreted as mediated action, in which these mediational means both shape the action in fundamental ways but are also dependent on the environment within which the action is carried out (Wertsch, 1998). For my project, I found it important to pay attention to mediational means, such as mathematical tasks, tools for supporting reflection, language and use of mathematical concepts, and signals for communication processes.

Wertsch (2007) distinguishes between implicit and explicit mediation. He argues that implicit mediation could be interpreted in terms of the role of social and inner speech in mediating human consciousness. Explicit mediation, on the other hand, is intentionally and overtly introduced into problem solving activity, often by an outside party, and the materiality of the signs involved tends to be obvious and lasting. Our discussions at the workshops include both explicit and implicit mediation, as we bring into our discussion problems but the discussion is often steered into unintentional talk based on our inner speech and that can lead the discussion along unforeseen routes.

7.5.2 Tools

Lerman (2000b) argues that cultural tools, such as the natural numbers, transform us internally, like physical tools, because they form and transform the world and enable us to act and perceive the world from a different angle. In this sense, the teacher, which may take the form of a peer or a textbook, is a central agent in providing these tools, which then mediate the world to the learner. In Vygotsky's (1999) writings about higher concepts, namely, in the sense that they transform the meaning of lower concepts, he offers an example of an adolescent who has mastered algebraic concepts, and this level of understanding then enables him to see arithmetic concepts in a broader perspective than before. My intention when I organised space for teachers at workshops on solution strategies, was to facilitate the experience of this kind of reflecting in terms of their thinking about their own understanding of mathematics. In discussing their thoughts about their approach to solving problems, they may realise how they can create a learning space for their own students that, in turn, may allow them to think about their own understanding of mathematical concepts, ultimately allowing them to see these concepts in a broader perspective.

In drawing conclusions from his research on children's acquisition of language, Vygotsky (1978) theorised that, prior to mastering their own behaviour, children begin to master their surroundings with the help of speech. According to him, speech does not only facilitate the children's effective manipulation of objects but also controls their behaviour. In other words, speech, as a communicative activity, supports children's mastering of their own behaviour. For the teacher who organises her mathematics teaching it might not seem essential to think of speech as a support for mastering one's own behaviour and surroundings, or as a mode for organising and communicating one's thinking, as in Piaget's (1969) conception. What matters is that the learners in the classroom are empowered to speak in order to share their thinking and support each other in solving problems and develop their mathematical competences.

Gee (2004) differentiates between discourses (with a lower-case d) that deal with language in everyday use and 'Discourse' (with a capitalised D) describing a wide range of representational resources, such as language, and other symbolic expressions or objects that individuals use to present particular socially situated identities. Through the participation in a 'Discourse', individuals communicate thinking, feelings, beliefs and values that one uses to identify oneself as a member of a group or a network, or to

signal a socially meaningful role. In the mathematics classroom teachers need to acknowledge that the pupils in their classes signal their way of interpreting and understanding the mathematics in diverse ways. In the workshop with the teachers, I sought to attend to their diverse ways of communicating their thinking, and in doing so, explicitly mediate the significance of teachers in becoming aware of learners' diverse ways of interacting and presenting their thinking about mathematics.

7.5.3 Summary

In the collaborative research with teachers, I wanted them to rethink their understanding of learning mathematics by offering them problems to work with and mediating their understanding of this process. By reflecting on their stories from their classrooms, we would explicitly mediate our understanding of the experiences they brought to us in our workshops. Language would be the main tool we would use in mediating our discussions but examples of the children's work could also serve as tools for mediating the learning that the teachers interpreted as taking place in their classrooms. We would collectively support the teachers in our discussions regarding the learning in their own classrooms, with the aim of supporting them in reflecting on their experiences, and thus scaffold their co-learning with the children in their classrooms, and create a space for them to develop within their zone of proximal development.

7.6 Potentials and constraints in learning communities

During the three years we collaborated within the research project, I envisaged that we would create a learning community into which we would bring our experiences and where we would support each other in gaining competence in our practices. We all belonged to multiple communities of practice, each community bearing its particular norms and traditions that we had participated in establishing and aligned to in one way or another. The boundaries between these communities could both create obstacles and be a source of learning. Meaning is created in each practice and relationships between them are always a matter of negotiating their boundary. Rather than hiding boundaries, one should focus on boundaries as learning resources (Wenger-Trayner & Wenger-Trayner, 2015).

7.6.1 Potentials for learning

I had the vision for our collaborative project that we would support each other in improving our practices. I drew on Vygotsky's (1978) description of

the zone of proximal development (ZPD) in structuring the project and planning for the workshops. When I chose problems to work with at the workshops, I used findings from former cycles within the project to reveal what would be feasible to work with in promoting the teachers' development. I also wanted the teachers to experience how guidance through a problem solving process bolstered them in using their own potentials for learning and that this would then be reflected in their own teaching. Vygotsky (1978) emphasises that only the aspects that expedite development constitute proper learning, and thus the focus on the teacher, that might be a parent or a more informed peer that can support the child, is central in Vygotsky's description of the learning process. In our workshops, I expected the teachers to support each other in solving the problems and also when discussing the cases from their classrooms that they brought into our community. During this process, the teachers also supported me in learning about their work and when challenging my approach to the project they would push the boundaries between my practice as a teacher educator and a member of our co-learning community. Bruner's (1985) definition of the scaffolding process and his description of how a child achieves conscious control over a new function or conceptual system and is then able to use it as a tool, also guided me in interpreting the learning that took place within our learning community.

Throughout our work, I was mindful of the fact that our co-learning could be influenced by boundaries between our learning community and other communities of practice we participated in. Valsiner's (1997) extension of Vygotsky's notion of the zone of proximal development (ZPD) was helpful in discerning boundary objects that might either further or restrict our development. Valsiner interprets the zone theory of child development as a set of possibilities for development that are in the process of becoming realised. He includes another two additional dynamic zones of interaction, namely, the zone of free movement (ZFM) and the zone of promoted action (ZPA). This was discussed in more detail in Section 4.2.2.

Goos (2005; 2008) applied Valsiner's zone theory in developing a research program with pre-service and in-service teachers. She frames the zones from the perspective of the teacher as learner in order to offer a dynamic way of viewing teacher learning as identity formation. She classifies the teachers' zones of proximal development (ZPD) as a set of possibilities for their development that are influenced by their knowledge and beliefs about the teaching and learning of mathematics. Similarly, the ZPA represented teaching approaches like those promoted by pre-service

teacher education, professional development activities, or informal interaction with colleagues. I will discuss Goos's definition of the ZFM in the following section, in connection to the hindrances that arise in learning communities.

Goos (2008) also looked at how zone theory might inform the analysis of her dual role as mathematics teacher educator and researcher. She claimed that her ZPA was shaped by her initial training as a researcher, participation in conferences and other activities of research associations, as well as mentoring by more experienced colleagues. Her possibilities for developing as a researcher were shaped by the complex relationship across the zones by defining what is allowed and what is promoted. From a sociocultural perspective, she could though say that her own research in teacher education acted as a ZPA that informed her practice as a mathematics teacher educator.

The zone of promoted action as theorised by Valsiner can be interpreted as boundary zone between communities. Within schools, the classrooms are communities where learners are expected to learn and the boundary objects can be seen as the experience and knowledge they brought with them into the classroom, the curriculum or the culture that has been shaped in the classroom. Similarly, in developmental programs the boundaries take the form of the teachers' knowledge and former experience from participating in teacher education programs or collaborating with colleagues. In our co-learning community, we all brought with us knowledge and experience from our former practice and other communities we belong to.

7.6.2 Constraints in the learning process

Boundaries between practices can restrict the possibilities for the learning communities to grow due to a lack of shared history or common words, and there is no guarantee that the meanings of objects are continuous across a boundary (Wenger-Trayner & Wenger-Trayner, 2015). In Valsiner's (1997) conception, the boundaries can be interpreted as hindrances in entering different areas of the environment and dealing with objects within it. Confusion and misunderstanding between boundaries can affect the ways an individual is enabled or permitted to act with the objects within the accessible environment.

In her research with teachers, Goos (2005) views the zone of free movement (ZFM) as constraints within the teacher professional context. These include students' behaviour and motivation, access to resources such

as teaching materials, curriculum and assessment requirements, organisational structures and cultures. The different cultures and regimes of these communities, and the culture the teacher wants to build in her classroom, thus collide. By the same token, she felt that as a teacher educator her zone of free movement was constrained by her students' characteristics, curriculum and assessment requirements, and the limited access to technology resources, reduction of allotted hours for teaching methods courses, difficulties in finding practicum placements, and the perception within academia that teacher education is low status work. In addition, she felt constrained by the academic structures and cultures, within and beyond her own university.

The hindrances for developing one's own practice are often to be found in the boundary between communities of practice. Teachers bring with them knowledge and experience from other communities they belong to or have belonged to and these can restrict their possibilities for changing their practice.

7.6.3 Summary

Starting with Vygotsky's zone of proximal development (ZPD), I discussed how learners could be supported by more knowledgeable people in furthering their learning within their zone of promoted action (ZPA). Similarly, I discussed how teachers' zone of free movement (ZFM) can be restricted by constraints within their zone of proximal development (ZPD). The notion of boundaries between communities of practice is an attempt to understand the different forces that influence teachers' capabilities to grow in their practices. In our community of inquiry, I predicted that the boundaries between our communities of practises would be disturbed through the process of inquiring into our practices, thereby disturbing practice on the inside and challenging the status quo of accepted ways of teaching in schools (Goodchild et al., 2013). My vision was to facilitate the teachers in challenging the established norms for teaching mathematics in their schools. However, I was aware that this process might not resonate with what they expected to gain from participating in the project and by extension, that my zone of promoted action would be disturbed. Still, I hoped that we would all engage in supporting each other in inquiring into our practices.

7.7 Rationale for the framework

The choice of the topics for the discussion above correlated with my experience of working with teachers and student teachers, as well as my interpretation of learning as a social participation. I therefore did not follow one particular framework when structuring the discussion. Rather, I used the topics as scaffolds or pillars to help me organise my interpretation of learning as a social activity. In Lester's (2005) words, I viewed the conceptual frameworks that guided my research as sources of ideas that I could appropriate and modify for my purposes as mathematics educator. In this sense, I was acting as a 'bricoleur' who uses whatever tools available in order to create everyday solutions (Gravemeijer, 1994b; Kincheloe, 2003; Lester, 2005; Lévi-Strauss, 1966). Lester suggests that rather than adhering to one particular theoretical perspective, mathematics educators should adapt ideas from a range of theoretical sources to suit their goals, and that aim to "not only to deepen our fundamental understanding of mathematics learning and teaching, but also to aid us in providing practical wisdom about problems practitioners care about" (Lester, 2005, p.466). Kincheloe (2003) suggests that researchers should avoid adopting modes of knowledge production and reasoning that come from certified processes of research and analysis, instead, they should embrace complexity. He refers to the initial speculations of Lévi-Strauss (1966) who emphasises that a knowledge producer never engages in a simple dialogue with the world, but instead interacts "with a particular relationship between nature and culture definable in terms of his particular period and civilization and the material means at his disposal" (Lévi-Strauss, 1966, p. 19). The influence of my long experience of teaching in schools and teacher education programs could not be underestimated in terms of how it has aided the undertaking of conducting research for a doctoral thesis, along with researching my own work with colleagues. Interweaving my experience with the knowledge of theories I have gained during the doctoral studies, as well as in former studies, was necessary for fulfilling my goals with the study.

7.8 Overview

I have discussed the theoretical stance I have taken in my study and related to the perspective and vision I had for the project. In drawing conclusions from the study of my former practice, and relating to my learning during the doctoral studies, accounted for in Part I, I have summarised and developed further my interpretation of learning theories. In particular, I have discussed the theories that have affected my understanding of how

mathematical learning develops and how teachers develop as professionals. I have structured the discussion by taking as a point of departure Vygotsky's theories of learning as social participation and adopting Wenger's framework of communities of practice, as well as Wenger-Trayner and Wenger-Trayner's framework of learning in a landscape of practice, and finally Jaworski's framework for teacher developmental projects. Grounded in and with reference to the abovementioned theoretical frameworks, I seek to shed light on: the individual and community, inquiry and reflection, learning landscapes, teaching and learning, mediation and tools, and finally potentials and constraints within communities of practice.

In summarising my theoretical framework, and to underline the rationale behind the choice of topics, I will relate to how it underpins the developmental research I took on in researching with the teachers.

Learning as social participation: The position I took on in carrying out the study is based on the notion that learning is a social activity and in adopting this position I refer to my former experience of teaching and learning as well as theories based on Vygotsky's research on children's development. The teachers that participate in the research project are individuals developing their own practice. They belong to different communities, within a complex landscape of learning, that all affect how they interpret the learning in their classrooms and thus their own individual development as mathematics teachers.

Inquiry and reflection as a mode for developing practice: Reflecting on one's own work is an essential feature of teacher development and in inquiring into one's practice. Reflection can be directed inwards in attending to one's identity and beliefs, and outwards in responding to people and situations in the communities we belong to. Inquiry refers to critical reflection and can be seen as a mode for critically reflecting on mathematics learning, mathematics teaching and research into the teaching of mathematics.

Landscapes of learning: The study is carried out in two neighbouring schools in Iceland and situating it there affects the process of our work. The teachers' background and the experience they bring into our community, shapes our collaborative work. The diverse background and experiences of the learners in their classrooms also influence our work as we reflect on the learning that the teachers extracted through our collaborating with them. Adopting the metaphor of a traveller, taking on a journey through the different communities of practice we belong to, has strengthened me in

interpreting the diverse factors that are at work within our collaborative community.

Mathematics teaching and learning in communities of practice: Teacher development is dependent both on their knowledge and their ability to learn together with others. This applies both to their learning within their classrooms and in collaborating with colleagues and in other communities within their landscape of teaching. Paying attention to the learning that takes place in their classrooms and being aware of the interactions within the classroom and how their pupils make sense of the mathematics being addressed is significant for their own development as teachers.

Mediation and tools: The choice of mediational means is fundamental for how cultures in communities of practice grow. My wish for this study was that we would succeed in cultivating a community where sharing meanings and critical reflection on our practices would be at the core of our collaboration. Language would be our main tool for sharing meanings but other tools such as mathematical tasks, symbols and concepts, as well as examples of children's work would also figure into the mediating of our thinking about our practice.

Potentials and constraints in learning communities: The zone of proximal development is at the centre of our work, given that my vision for the project was that we would support each other in developing our practice. However, on our route to improving our practice there emerge many hindrances within our zone of free movement as well as possibilities within our zone of promoted action. It is important be aware of how the process is affected by the boundaries between our different communities of practice within our learning landscape.

The theoretical framework discussed above guided the developmental process of my collaborative research with seven primary school teachers over a period of three years. In the following chapter, I will discuss the methodology of developmental research that I adopted for carrying out the research. The purpose of the chapter is to account for the motives of the choice of the methodology, and to describe the way I collected data and how I analysed them. In addition, it aims to make visible my beliefs and underlying premises concerning the research project.

8 Methodology and methods

In this chapter, I begin the discussion by revisiting the aims of the study and the research questions. I then discuss the methodology of developmental research and how I came to the conclusion that the developmental research cycle would be useful in guiding the project. I will delineate the methods used in collecting data, how interviews and classroom observations were carried out, as well as describe the process of writing the notes and reflections from workshops with the teachers. I will discuss how grounded coding was used in the analysis of the data as a means to interpret the data from the collaborative work with the teachers. I will also elaborate on the narrative approach that was adopted when writing the stories of the teachers and in writing about the learning that I have achieved in taking on this study. Finally, I discuss the theory of knowledge on which my work is based as well as ethical concerns related to this study.

8.1 Aims of the study

As explained in the introduction chapter, the main purpose of this study is to deepen understanding of teacher development in mathematics teaching and in particular to investigate how the collaboration between teachers in schools and a mathematics teacher educator, through the process of researching their own practice, would impact the work of both. Following this perspective, I formulated one main research question for both parts and three sub-questions for Part II to help focus on the three main threads of the study.

The main research question for both parts of the study is as follows:

- In what way has my understanding of teacher development in mathematics teaching and learning changed through working with teachers at improving their own practice?

My guiding questions for Part II are:

- What learning processes emerge through long-term collaborative inquiry undertaken by classroom teachers and a mathematics teacher educator?

- When teachers are participants in a project based on collaborative inquiry how do they perceive this as being reflected in their mathematics teaching?
- In what way do I interpret that my own learning from carrying out a collaborative inquiry project with teachers has influenced my practice as a teacher educator and a researcher?

I consider this study as an exploratory research project where a cyclic developmental process was adopted at the outset of the preparatory phase of the study and followed throughout the implementation and analytical process, as reflected in the open-ended nature of the research questions.

By answering the main research question I am in a position of articulating how my understanding of teacher development in mathematics teaching and learning changed through the process of working with teachers in carrying out this project. The first sub-question addresses the developmental nature of the project, and when answering this question, I aim at offering a detailed presentation of the cyclic process of the study. Particularly with regard to the study's interlinked aims, firstly, of deepening understanding of teacher development in mathematics teaching, and secondly, to investigate the impact of the collaboration between teachers in schools and a mathematics teacher educator in researching their own practice.

As discussed in Chapter 1, the goal with this research project with teachers is to reveal how they meet new cultural and mathematical challenges in their classrooms and how participation in a learning community with their colleagues and myself as a teacher educator, could lead to changes that might be valuable for their work. The second sub-question addresses the teachers learning in a community of inquiry and their interpretation of how it affected their teaching. In order to provide a convincing portrait of their learning, I will have to be cognisant of how they indicate that their learning has successfully entered into begun to inform their teaching. In writing narratives of their learning, I have to respect that learning is temporal and life is experienced on a continuum. This means that what they express at the initial phase of the study might have a different meaning in the final year.

The third sub-question similarly addresses my own learning in our co-learning community of inquiry and my interpretation of how it affected my understanding of teacher development. In writing about my own learning, I will also need be aware that our experience at each meeting was unique at

that time, which additionally influenced our experience and further collaboration. As a teacher educator, my aim was to unfold how my former experience of working with pre- and in-service teachers informed my understanding of how learning develops, individually and collectively, and how taking on this research project supplements my earlier comprehension of mathematics teacher development.

The study has its origins in the interest of a group of teachers who sought to improve their mathematics teaching in inclusive settings and in my enthusiasm for meaningful mathematics learning opportunities for all students. The teachers are all working fulltime as primary school teachers aiming at developing their practice and I, the researcher and teacher educator, collaborate with them on their journey to reach their shared goal.

The findings of our research are accessible to teachers in schools and researchers within the field of education, both my own writing published in academic journals and our collaborative writing published in journals for teachers. In my design of the research I looked for a methodological approach that might be useful in framing and analysing the study.

8.2 Choosing developmental research as a methodology

In Chapter 5, I wrote about my reflections on my understanding and interpretation of practitioner research, namely, action research, lesson-study, learning-study, design research and self-study of teacher education practices. I focused on who initiates the research and the level of theory building apparent in the different approaches to the various methods of researching one's own practice. These preliminary steps were essential on my route to finding a research methodology that might prove significant in analysing the process of the journey taken with the teachers into the study of our practices.

I had participated in practitioner research earlier, individually and collectively, in different roles. I now needed to find a methodology that would support me in analysing the whole process, as well as supporting the development of all the participants, both the teacher educator and participant teachers. I discovered the following aspects in summarising the findings from this process:

- In learning-study and design research, teachers and researchers work together on innovative processes that are grounded in theory in learning-study, and used to develop theories, in design research. The teachers are insiders to the research process in researching their own practice and the researchers are outsiders collecting data and analysing the process.

- In action research and lesson-study, the research process is the developmental process of teachers, following through an idea, and the research is not necessary based on theory or aimed at developing theories. Participants can be both teachers and researchers supporting the teachers in researching their own practice where teachers are insiders to their research and the researchers are outsiders to the process.
- Self-study of teacher education practices aims at improving our own teaching, formalising our work and making it available to our professional community for debate, further testing and assessment. It requires evidence of reframed thinking and transformed practice, which are derived from an evaluation of the impact of developmental efforts while interacting with colleagues, students, the educational literature and previous work. The teacher educators are insiders to their research and may conduct collective research into their practices with colleagues or their students in improving their collaborative work.

To fulfil the purposes of my study to research our development of own practice in relation to theory building, individually and collectively, throughout the process I needed a methodology where these aspects were considered. My intention with this project, reported now in Part II, was that teachers would develop competence in researching their own practice in which I would figure as an outsider in supporting them through the process of improving their practice. I was an insider to the research process, which involved our collaborative work at the workshops and researching how our work developed. I looked for further support in structuring the developmental project that I wanted to carry out and a research methodology that would serve the purpose of researching the developmental process.

8.2.1 Developmental research

When studying the methodology of design research, I became acquainted with the work of van den Akker (1999), who used the term developmental research as an umbrella term to represent a broad type of educational research in which education is primarily interpreted as a 'design science', emphasising its problem-oriented and interdisciplinary orientation. He discusses different roles of design research in several subdomains of education, amongst them teacher education and didactics. In teacher education, the improvement of professional practice is central, and according to van den Akker, action research is common within the domain. In discussing the area of didactics of school subjects he refers to Gravemeijer's (1994a) writing about developmental research within the field of mathematics education and emphasised the interactive cyclic

processes of development and research, where theoretical ideas feed development in classroom settings that in turn, may lead to local instructional theories. He wrote:

Development research is often initiated for complex, innovative tasks for which only very few validated principles are available to structure and support the design and development activities. Since in those situations the image and impact of the intervention to be developed is often still unclear, the research focuses on realizing limited but promising examples of those interventions. The aim is not to elaborate and implement complete interventions, but to come to (successive) prototypes that increasingly meet the innovative aspirations and requirements. (Van den Akker, 1999. p. 7)

According to van den Akker, the most significant knowledge gained from developmental research is therefore in the form of substantive and methodological 'design principles' to support designers in their task. This led me to look further for approaches into developmental research with teachers in schools.

Wagner (1997) notes the different steps that researchers can take in their collaboration with teachers, students and administrators when conducting research in schools. He defines three levels of collaboration in research projects within schools: data-extraction agreements, clinical partnerships, and co-learning agreements. When researchers observe the work done in schools they often work in partnership with those they are observing, and Wagner defines this kind of research collaboration as data-extraction agreements. He labels the kind of collaboration where researchers and teachers work together at improving knowledge about schools and educational practices within them, similar to collaborative action research, as clinical partnership. The third level, the co-learning agreement, Wagner defines as reflexive, systematic inquiry, which is stimulated partly by ongoing collegial communication between researchers and practitioners. Both partners participate through action and reflection through processes of education and systems of schooling. Practitioners and researcher are also involved as agents and objects of inquiry as well as being responsible for initiating changes in their own institutions by drawing on the knowledge gained through their co-operative research. Wagner's description of the co-learning agreement is in line with what I aimed at when organising the work with the teachers in my project, as was discussed in 7.1.1 in relation to the co-learning agreement that was envisioned to develop within the project.

I also revisited Freudenthal's work with which I had earlier become acquainted through curriculum development. In his writings about educational research, Freudenthal (1991) writes that developmental research means:

... experiencing the cyclic process of development and research so consciously, and reporting on it so candidly that it justifies itself, and that this experience can be transmitted to others to become like their own experience (Freudenthal, 1991, p. 161).

The developmental focus of the research I was planning was aimed at extracting knowledge from former experiences in order to guide actions taken in future steps of the process. In the work with the teachers, I anticipated a cyclic alternation of research and development and I found Freudenthal's writing about developmental research useful in that regard. He emphasises continuous awareness of what is happening in the ongoing process and that it should be recorded as much as possible, and that dissemination should not be separated from development.

Gravemeijer (1994b) develops Freudenthal's work further and emphasises that what happens in practice must be analysed, and the result of this analysis should then be used to continue the developmental work. Like Freudenthal, he argues that on a micro level this interaction between research and development is evident, but that such a relation also holds on a macro level, even though this aspect is not often noticed (Gravemeijer, 1994a; 1994b). He moreover emphasises that a certain philosophy or a global a priori theory guides developmental work.

This theory functions as a basis for a learning process by the developer that is nurtured by the cyclic alternation of thought experiment and practical experiment. Such a learning process can be interpreted as theory development: each course can be seen as the concrete sediments of local instruction theory. (Gravemeijer, 1994b, pp. 449-450)

Freudenthal (1991) and Gravemeijer (1994b) stress that there is a need for theoretical considerations of the mental processes necessary for progress and integration and that a learning sequence based solely on observable behaviours does not satisfy the requirement of theory building. Additionally, a goal oriented process of improvement and adjustment is needed in developmental research, as well as the evolutionary aspect of a process that is guided by a theory that grows during the process. An a priori theory is therefore not applied; rather the theory functions as a guideline and inspires the research as the basic theory is developing.

Making sense of what is going on in developmental research is more important than prediction. The process by which new knowledge in educational research is gained must be accounted for if we expect that it to be of use for other people. Constant awareness of the developmental process and reflective approaches must be reported in order both to make it credible and transferable (Gravemeijer, 1994a). In developmental research, the teacher working with students and confronting new ideas will also pass through a learning process that is nurtured by experience and the process of reflection (Gravemeijer, 1994b). It is not only the learning of the students in the classroom that is in focus; it also concerns the learning of the teachers who lead the teaching. Gravemeijer's description of developmental research focuses on the development of students' learning in classrooms but he also acknowledges that their teachers might learn from participating in this process.

Looking for further support, I found recourse in the description of the 'developmental research cycle' as put forth by Goodchild (2008). He discusses practitioner research from a broad perspective and draws on his own experience of multi-levelled action research. He accounts for his experience of researching his practice as a classroom teacher, teacher educator and a didactician in the Learning Communities in Mathematics (LCM) project in Norway, discussed briefly in Chapter 2. The emphasis there was placed on development, action research, inquiry and co-learning. His discussion of developmental research is based on Gravemeijer's description, though with particular emphasis on the cyclical process between development and research. He interprets and summarises Gravemeijer's ideas in a diagram he calls the 'developmental research cycle'. This diagram is presented in Figure 1.

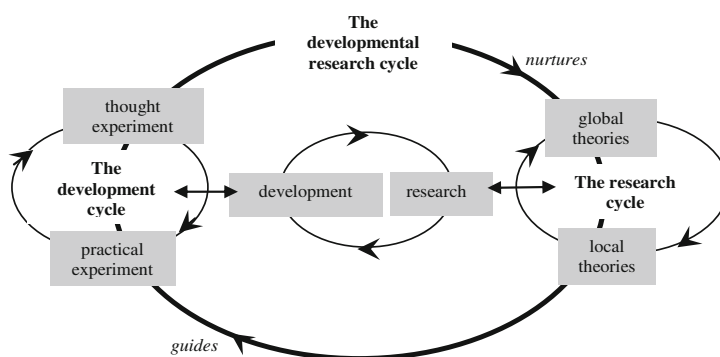


Figure 1. The developmental research cycle (Goodchild, 2008, p. 208)

The developmental research cycle represents two interconnected cycles that model the dialectical growth of both theory and practice. Components of the developmental cycle are presented as a thought experiment to accompany a practical experiment. The research cycle moves between global theories that are concretised in local theories. The cycles are interrelated; local theories are tried out in practice, when thinking through the consequences of some action and then implementing it in harmony with conclusions from the planning process. This leads to the adjustment and analysis of the local theory that informed the action, which then in turn, results in a reconstruction of the global theory. Consequently, the research cycle guides the development cycle, which in turn nurtures the research cycle (Goodchild, 2008).

8.2.2 The value of using developmental research

Goodchild and his co-researchers (2013) further described the process of the developmental research cycle in connection with their approach to the developmental research between the teachers and didacticians in the LCM project in Norway. They discuss how their approach, as described by Goodchild (2008), builds on ideas of practitioner research and promotes development simultaneous to the study of their developmental process. They perceive the idea of a community of practice as a global theory, to which they add their local theory, as inquiry and insider/outsider research, in order to produce a developmental theory within their chosen global framework. They discuss how research guides the developmental cycle through planning meetings and feedback from participating teachers, which then enables a continual adjustment of processes and implementation. They describe this process as follows:

The development cycle 'begins' with a thought experiment that is based on teachers' and didacticians' experiences, professional discourse and the exercise of imagination. Thus, the regular teaching cycle of practice evolution in which most teachers regularly engage (plan for teaching, act in the classroom, reflect on experience, feedback to regular planning) is transformed into a teaching inquiry (research) cycle by introducing to the teaching cycle: systematic observation, analysis and reporting. The inquiry cycle is thus, plan for teaching, act and observe the action, reflect upon and analyse the experience, feedback to future planning and report to the community. (Goodchild et al., 2013, pp. 397–398)

Consequently, systematic observation and analysis inform the reflective process in the inquiry cycle. Reflection and feedback provide possibilities for re-planning the teaching leading to a more knowledgeable design of teaching.

Gravemeijer (1994b) emphasises the importance of providing detailed reporting of the cyclical process of developmental research with the aim of making it intelligible for other researchers. Goodchild (2008) underlines that transmission of experience is not a simple matter and not merely for the benefit of informing a reader. The readers of the report need to be sufficiently informed so that they can work within their practice and create their own experiences. The report must facilitate reflection and support developmental processes at different sites where experiences will vary.

The dual role of writing about research that Goodchild addresses is an essential aspect of my research project. In writing the thesis, I attempt to highlight the cyclical process of our collaboration and provide detailed information regarding the steps taken. This point also needs to be reflected in my writing with the teachers about the project. In an article I wrote with one of the participants (Kristinsdóttir & Sveinbjörnsdóttir, 2015) for a journal of special education teachers in Iceland (Glæður), we attended to this by presenting detailed information of the cyclic process and what we learned from participating in it.

I found Goodchild's interpretation of developmental research useful in the process of framing my study and interpreting the research process. The connection between research and development, as well as thought experiment and practice, is well aligned with the purpose of my study. However, my study was structured differently from the LCM project, as it was not built into the project that at our workshops the teachers would develop and prepare for carrying out certain tasks in their classrooms. Rather, the intention with our collaboration at the workshops was to inspire them to try out different approaches to mathematics learning in their classrooms, which they experienced themselves at the workshops and which might help their pupils develop their mathematics competences. Additionally, the teachers were supported in the process of reflecting on their experiences. We thus intended to try these approaches out in practice, myself at the workshops and the teachers in their classrooms. We would use our local theories about what we considered helpful in teaching mathematics and in observing these actions. We would reflect upon and analyse our experiences and relate our actions to the global theories on which we grounded our work. The process of analysis would in turn lead to a revision of our local theories which would then, reciprocally, inform our future planning.

8.2.3 Summary

In section 8.2 I discussed how my former experience of practitioner research underpinned the design of this collaborative research project. The goal was to clarify how I arrived at the conclusion that the methodology of developmental research would fit in with the developmental approach I took on in this study. My vision for the project was that I would learn about the development of teachers in primary schools as they took on the role of researching their own practice and question established norms and traditions in their schools, through the process of participating in a collaborative inquiry with their colleagues and a mathematics teacher educator. I also aimed at learning about my own development in carrying out the project and reveal the ways in which the knowledge I extracted in this process supplements my earlier comprehension of mathematics teacher development.

8.3 Using the developmental research cycle

The developmental research cycle was used as a guiding tool to structure the project and inquire into our work as teachers in schools and as a mathematics educator and researcher. I will proceed by discussing the developmental research cycle and how it guided the process of my study, by attending to the main components of the cycle. I will particularly address how the two cycles, the developmental cycle and the research cycle, were woven into the cyclic process and related to the global and local theories on which I based the study, as well as accounting for the thought experiments and practical experiments we engaged in at the workshops. Additionally, I will discuss the different roles the participants took on in the project, as outsiders and insiders.

8.3.1 The research cycle

The research cycle guides the developmental cycle and the systematic analysis of the cyclic alternation of thought experiment and practical experiment then informs the development of local theories, which in turn, are guided by global theories. In the workshops, the global theories were expected to direct the developmental process as well as the local theory building (Gravemeijer, 1994b). The goal of this process was that, we, the teachers and teacher educator, would build a community where we would learn together about ways of improving our practices. By systematic observation, analysis and reporting we would inform our communities of practice, within schools, teacher education and the research community.

Global theories: As discussed in Chapter 7, the global theories that guide this study are sociocultural and the goal was that we would build a community of practice in which collegial support for improving one's own competence in teaching and learning mathematics would be facilitated. Special attention was paid to the process of inquiry into mathematics learning, teaching and researching a developmental process.

- By inquiring into our own ways of learning mathematics we would be challenging established norms and customs for solving mathematical tasks and thus questioning our own ways of teaching mathematics.
- Taking on the inquiry stance in reflecting on our practice we were challenging the status quo and accepted ways of teaching mathematics in schools and thereby disturbing our practice from the inside (Goodchild et al., 2013).
- We envisioned that we would be able to create a community of inquiry (Jaworski, 2008a) through reflecting critically together about our practices.
- Systematic observation and analysis inform the reflective process of inquiring into the developmental research cycle, and thus cultivating the research cycle as a whole.

We would progress in improving our practice through asking questions, making investigations, acquiring information and looking critically at our own action, and relating to our theoretical frames of meaning and values.

Local theories: Drawing on conclusions from educational research in Iceland, discussed in Chapters 2 and 3, I hoped the project would inspire teachers to improve their capability in interacting with their pupils, which in turn might lead to the pupils' improved competences in mathematics. The findings from research in Icelandic schools reveal that interaction in mathematics classrooms is limited and that teachers rely much on textbooks and pupils' individual work. The teachers thus enter a passive role as educators (Savola, 2010; Sigurgeirsson, 1992). Teachers were found to lack experiences of focusing on mathematical processes and relational understanding (Jóhannsdóttir, & Gísladóttir, 2014). In addition, they felt incompetent in teaching in classrooms with diverse groups of pupils (Guðjónsdóttir, & Kristinsdóttir, 2006; 2007a; Guðjónsdóttir, et al., 2007). Findings from studies with teachers where they reflected actively on their teaching and their own learning in participating with pupils, colleagues, and literature on mathematics education, indicate that participation in such learning communities helped teachers gain confidence in teaching for understanding (Angantýsdóttir, 2010; Guðjónsdóttir & Kristinsdóttir, 2006;

2007b; 2011; Guðjónsdóttir et al, 2009; 2010; Gunnarsdóttir et al, 2008; Kristinsdóttir, 2010a; 2010b).

Another goal for the project was to learn how teachers and mathematics educators could work at developing practices for the benefit of both. I sought to work with teachers in primary grades at finding ways to improve mathematics learning in their schools. This meant that we would collaborate on improving our practice. The participant teachers would improve their competences in collaborating with children in their classes with the aim of finding meaningful ways to improve their mathematical competences. I myself would improve my practice as mathematics educator in learning to collaborate with teachers toward the development of our practices and as a researcher leading a collaborative research project.

From my former experience of working with teachers at in-service courses, I had learned that many of them expect teacher educators to teach them how to solve problems in textbooks and tell them how to structure their teaching. I therefore wanted to challenge teachers to reflect on their own way of learning mathematics and hoped that our reflective discussions about their diverse ways of solving mathematical tasks could lead to improvement in the ways they attend to their pupils' diverse ways of learning mathematics. I envisaged that problem-solving activity might help teachers experience and discuss complexities similar to those met by students in class and to understand the importance of evaluating the process involved in mathematical activities. In addition, I expected that sharing reflections on their mathematics teaching and discussing them at our workshops would support the teachers in creating mathematical communities in their classrooms where children are urged to share and discuss their thinking about the mathematics attended to at each given time.

I anticipated that this process it would lead us to adopting the inquiry stance (Cochran-Smith & Lytle, 1999) when engaging in an inquiry way of being. In the inquiry process, we would be challenging the normal state in mathematics classrooms, as described in Jaworski (2008a). If the normal desirable state consists in ensuring that learners solely do what is required of them, without worrying about their understanding of what they are doing, we might be depriving them of the opportunity to understand and relate particular ideas more widely, both in mathematics and in real world applications. If we ask ourselves what we could do in classrooms to enable students to understand better the mathematics they meet in textbook exercises, we would be asking ourselves a developmental question and hence entering an inquiry or a research process.

8.3.2 The developmental cycle

The developmental cycle is guided by research and feedback from participating teachers through the process of planning meetings, which enables continual adjustment of the processes within the project. It begins with a thought experiment that is based on our experiences, professional discourse and the exercise of imagination (Goodchild et al., 2013). Practical experiments are then tried out within our workshops and in the teachers' classrooms and discussed and reflected upon with the goal of feeding back to thought experiments, which then leads to and informs global theory and local theory building.

Thought experiments: Writing and discussing cases from classroom experiences was envisioned to be an important part of the teacher's work with the aim of stimulating their inquiry and analysis on the real challenges and dilemmas of their practices. When we write about our experience we can describe what we have tried in our practice and in addition use it as a tool for reflection. In guiding the teachers through the process of writing about their reflections, my intention was that they would focus their attention on noticing the goal of learning and how their pupils' way of learning mathematics was reflected in their own actions and choices (Mason, 2002; 2011). I also had in mind Mason's description of *accounting-of* and *accounting-for* in discussing one's experiences, in the sense that when we tell stories from our classrooms, this account can provide a brief and vivid description of what happened so that other people can interpret the stories and relate them to their own experiences. When we start elaborating on our own experience and reflecting on what happened in order to learn from it, we are accounting-for our experiences.

The process of the *case and commentary writing* (Kruger & Cherednichenko, 2006) was also adopted in our work. It consists of five iterative stages, namely: case writing and four dimensions of praxis inquiry, discussed in sections 4.4.2 and 7.2.1. We learned from former cycles while building new and so-called 'spirals of experience' would emerge through refinement of their teaching and. Such an interchange cultivates a collegial relationship between the participants as they share and analyse information, critique each other's ideas, and solve problems with each other's help.

Keeping a journal of one's own teaching and experiences in classrooms, as well as reflecting on one's writing, are important features of teacher developmental projects (Dalmau & Guðjónsdóttir, 2002; Guðjónsdóttir et al., 2007). In her chapter about transformative mathematics pedagogy,

Moore (2005) concludes that teachers become empowered to practice a culturally responsive and socially relevant pedagogy as they begin to look critically at their classroom environment. The practitioner becomes the action researcher, transforming theory into practice and while researching the same practice, and once again cycling back to transformative theory.

When preparing for the workshops, I tried to be loyal to the teachers' expressed visions for the developmental project, but my own beliefs and goals for the project also played an important role. As expressed by Kise (2006), teachers need support in understanding where their strengths and beliefs lock them into practices that limit their freedom to help students learn. Their possibilities need to be enhanced in order for them to stay open to new avenues for professional growth, through engaging in deep conversations where different positions and acknowledge diverse opinions are honoured.

Practical experiments: The research process consisted of three types of collaborative activities; workshops where all the teachers participating in the project and the teacher educator/researcher participated, teacher educator/researcher's observation in teachers' classrooms, and interviews with teachers, individual and in pairs. We worked together over a period of three years and met at 17 workshops on a monthly basis for the first year, and then with longer intervals the second and third year. Before our first workshop, I observed each of the teachers' classrooms and interviewed them about their work, background as teachers and their expectations for the project. I planned the first workshop on the basis of what I learned from the observations about their teaching and their expressed vision for our collaboration, and in doing so, related to the theories that guided the project. By the end of our first workshop, we discussed what the teachers would like to work with at the second workshop, and I planned our work accordingly, again relating to the theories that underpinned the study. This cyclic approach guided the work at the workshops throughout the project. The teachers thus expressed their visions for what to participate in at the workshops. Taking notice of their wishes and the theoretical underpinnings of the project in mind I planned our collaborative work.

In the workshops, I chose and/or designed problems with the aim in mind that they would potentially promote mathematical activity and thinking and stimulate collaboration where we would engage in meaningful discussions and share our thoughts in constructive ways (Jaworski, 2007b; Mason & Johnston-Wilder, 2006). With Mason's (2008) notion, that teachers cannot make learners learn and not do the learning for their

learners in mind, I urged the teachers to investigate and discuss different ways to approach the problems. We then discussed how their pupils might approach such problems and in what ways the teachers could support their pupils' learning. Similarly, attention and awareness were discussed, drawing on Mason's discussions about what and how we attend to our surroundings and objects of learning, and how teachers can scaffold their pupils' mathematical awareness. The notion of building a spiral was important at a macro level for the operation and development of the group, but it also placed the group at a micro level, in terms of the individual learning of teachers. Mason's description of the 'spiral of building confidence' was used as a guiding tool in our explorations with the mathematics, which according to Mason consists of manipulating, getting a sense of, capturing in pictures, words and symbols, and providing fodder for further manipulation (Mason, 1999).

From my former experience (Guðjónsdóttir & Kristinsdóttir, 2007a; 2007b; Gunnarsdóttir, et al., 2008) I had learned that exploring and investigating within a group composed of people with different mathematical backgrounds is fruitful in the sense that it helps teachers to understand and be aware of their student's different paths of learning mathematics. They also correspond with Bredcamp's (2004) and Moore's (2005) findings, who insist that if teachers are expected to teach for diversity and understanding, they need to experience their own mathematics learning in an environment that reflects the one they are expected to create in their own classroom.

Watson & Mason (2007) expresses a similar view when they write:

The fundamental issue in working with teachers is to resonate with their experience so that they can imagine themselves 'doing something' in their own situation, through having particularized a general strategy for themselves, rather than relying on being given particular 'things to do'. (Watson & Mason, 2007, p. 208).

My experience correlates with their claim about teachers who are often on the lookout for something they can use in their classrooms, and tensions can therefore arise when teacher educators work with in-service teachers. When I chose or designed problems, I ensured that they were designed in a way that would allow the teachers to experience for themselves at their own level, something of what their learners might experience. Drawing on their experiences, I expected see a growth in their capability of being sensitive to their learners.

The central aim of our work at the workshops was sharing and analysing the cases from the teachers' classrooms, critiquing each other's ideas and solving problems together. Taking into consideration Robertson's (2008) emphasis on trust in collaborative relationships, I made an effort to respect the teacher's contributions to our co-learning and supported them in reflecting on their experiences. Trust takes time to develop, it cannot be assumed, and may require careful facilitation by the members of the learning community. In such relationships, it is important to recognise that colleagues act as a learning resource and that openness to new learning is important.

8.3.3 Participating as an insider and outsider in research projects

The participants in this research project are seven teachers in primary grades and me, a teacher educator and researcher. We work together at developing our practices in our roles as teachers and a teacher educator/researcher.

- The teachers are *insider* researchers into their own teaching and I am an insider into my development as a teacher educator and researcher leading the project. My role as a researcher within the development of the project is also the insider's role as I am an active participant and initiator of the research project.
- I am an *outsider* to the teachers' research into their practice and they are outsiders to my development as a teacher educator and researcher in participating in and leading the project.

As insider researchers we reflect on and inquire into our practices individually and collectively with the support of our critical friends (Schuck, 2011), who, in supporting our reflections, can take on the role as outsiders with regard to our research practice.

Jaworski (2003) discusses the different roles that the teachers and an external educator and researcher can take in collaborative research practices. According to her, the teachers are the insiders because the research is focused on their practices: "Insider research involves research by teachers into their own teaching. Individual research can take place, fruitfully, in a collaborative environment involving teachers either within a school or across a number of schools" (Jaworski, 2003, p. 259). She notes that the outsiders might take on various roles; they could help provide a community of teachers and educators in which the teachers can share their research practices and discuss their ideas. Moreover, they might themselves conduct research into classroom learning or teaching and become engaged in research into the collaborative program. They might

also research their own practices as educators while supporting teacher research, in which case they would become insiders in researching their own practice. She discusses how joint engagement in academic and research programs have resulted in co-learning for all of their participants, and writes:

It is clear that what is learned is neither the same for all, of the same form or at the same level. However, in co-learning, the learning of one is dependent on the participation and learning of others: mathematics teachers and educators learn together with different roles, goals and learning outcomes, while engaged in common activity for mutual benefit. (Jaworski, 2003, p. 259)

In developing this research project, I was aware of these different roles. The teachers were insiders in researching their own development as teachers. I was an outsider in the teachers' group and my intention was to work at building the co-learning partnership between teachers and educators in promoting classroom inquiry, and in that sense, become an insider in the project group. In addition to Jaworski's description, I see the teachers as outsiders to my development given that they support me in the process of reflecting on my own practice while leading the collaborative project.

8.3.4 Summary

I have accounted for my interpretation of developmental research, particularly the developmental research cycle, and the two intertwined cycles; the research cycle and the developmental cycle. In discussing the main features of the cyclic processes of the research, I referred to my theoretical framework that I discussed in Chapter 7, with the intention of explaining how the theories I base my study on support the developmental cycle and the alternation between thought experiments and practical experiments that figured as the main pillars on which I ground my work with the teachers. In Figure 2, I present schematically how I have made use of the developmental research cycle in carrying out the research project.

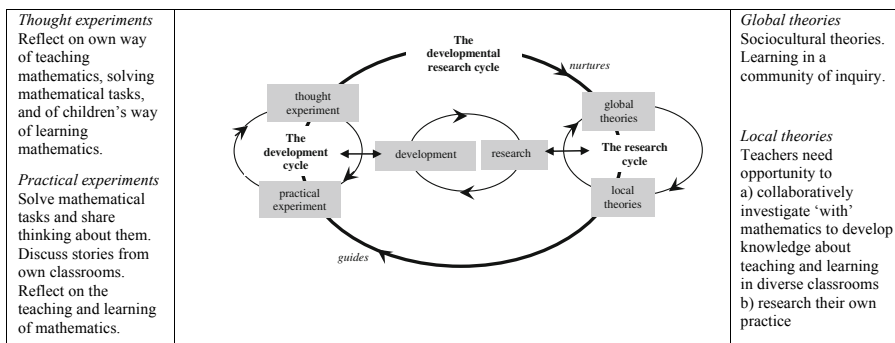


Figure 2. Application of the developmental research cycle (Goodchild, 2008, p. 208)

In order to shed light on the circumstances of the teachers who participated, in the study, I account for the process of finding participants, present an overview of their backgrounds as teachers in primary grades, and present a time frame of the study.

8.4 Choosing the settings

At the initial stage of my work I planned to work with a group of teachers who had sought my advice in structuring their professional developmental program. As I had already worked with teachers who had themselves initiated such a process, I expected it to be more challenging to work with teachers who had no experience of developing their mathematics teaching and had not specialised in teaching mathematics when they prepared for becoming teachers in primary school.

I consulted people within the teacher education field and looked for schools where mathematics teaching was not part of in-service developmental programs. I found two neighbouring schools with children of diverse background, both socioeconomic and ethnical, where results on standardised tests in mathematics revealed that there was a need for rethinking the mathematics teaching within the schools. The heads of the schools were interested in their schools taking part in the research project and consequently consulted their teachers. Seven teachers in grades five and six showed interest in the prospective collaboration, with the goal of improving their mathematics teaching and researching their work in cooperation with me. I hoped that these seven teachers would position themselves as inquirers into their own practice. My own and related research indicates that this experience would be fundamental for their

development and their own perceptions of how their work proceeds as well as for research on school development.

The participants were informed about the research through written information and a formal meeting, where the prospective research methods were introduced. The teachers and the school leaders all agreed to participate and the researcher and the leaders of the schools signed a formal contract with regard to the developmental project. Information on the project was also sent to “persónuvernd”, the Icelandic Data Protection Authority.

At the outset of our collaboration, the plan was to meet at workshops on a monthly basis for one year, from the beginning of September until May. During our workshop in April, the teachers expressed that they were concerned that our collaborative project was coming to an end and that they would like to continue meeting the following year. At the workshop in May, we agreed that we should proceed with our collaborative project, extending it for a year. When the second year was coming to an end, the teachers once again expressed that they would like to continue the collaboration the following year, and we agreed to meet at three workshops the third year. I found their enthusiasm valuable and I saw it as an opportunity for the project to develop and to cultivate our collaborative learning. I was working full time, as a teacher educator and the workload involved in finishing the doctoral thesis within strict time limits were not an obstacle. To give an overview of how the project developed, I present a time frame for the project in Table 8.1 Appendix A.

The participants in the study were seven primary teachers and a teacher educator/researcher. The teachers experience from teaching in primary grades ranged from 4 to 29 years. Their educational background was teaching in primary grades with emphasis on various differing subjects and none of them had specialised in teaching mathematics. One teacher had a background in social pedagogy and special education (see Table 8.2 in Appendix A).

The overview of the progression of the study presents information about its longitudinal aspects, in a study that was initially set to run for one year, but instead continued for two extra years. In fact, the study was prolonged to take almost four years in total in response to the teachers' interest in continuing our collaboration. In the following section, I will discuss the methods of data generation to shed light on the different sources of the data that were collected.

8.5 Methods of data generation

The data that were needed to answer the research questions were extracted using multiple methods, including interviews with teachers, observations of lessons and extensive taped records of the collaborative workshops involving the teachers and the researcher. The data which I gathered from these records covered the whole process and were collected from audiotapes of interviews, videotapes from workshops, notes from classroom observations, e-mails, samples of work from workshops, and children's work. Through the whole process, I kept a research journal and urged the teachers to do the same.

8.5.1 Interviews with teachers

The interviews were gathered at different stages of the study. At the outset of the study, I interviewed all the teachers that joined the project in autumn 2009 as well as the heads of the schools. My goal was to learn about their practices and get a sense of where support was needed in order to help me structure the workshops. For the teachers and the heads of the schools this was an opportunity to reflect on their work and discuss their beliefs, expectations and aspiration for the project. By the end of the first year of our collaborative project, I interviewed the six teachers that were at work the last days of May and beginning of June. In February 2013, nine months after our last workshop, I interviewed the four teachers that participated in the research process throughout the whole project. The interviews at later stages were planned to provide the participants space for discussing their experience and reflecting on their work. Before the interviews, I sent the teachers information about what I would like to discuss with them (Appendix D). In December 2010 and January 2011, I discussed individually with four teachers, who audiotaped their classes, their reflections on this experience.

The interviews were audio taped and transcribed. They were semi structured (Kvale, 1996; Pring, 2000; Bryman, 2004), I had a few guiding questions, tried to be flexible and responded to the direction in which the interviewees took the interview. The purpose with taking the interviews was to obtain a clear image of the teachers' beliefs and practices with respect to interpreting the descriptions of their mathematics teaching and their goals with their developmental work. The metaphor of a traveller (Kvale, 1996) who explores the many domains of a country applies to my intentions with the interviews. I critically followed the teachers' answers and asked for specifics and clarifications. My experience as a primary school

teacher and teacher educator, and the process of study of my own teaching, supported me on the journey and informed my interpretations of the stories/narratives that the teachers told. Themes generated from the narratives through coding (Maxwell, 2005) revealed where support was needed and thus guided the preparations for the workshops. These interviews were open and the participants were urged to tell their stories. Narratives emerged from the interviews in which the participants had a voice, informing others about their work (Clandinin, 1992), and the interviews therefore served as a mode for reflections. The interviews were transcribed by a professional transcriber and analysed and interpreted by me. In the analytical process, themes emerged through open coding and I used them to guide me in the co-learning process and to decide with the teachers how to structure the workshops. An overview of the interviews is presented in Table 8.3, Appendix A.

8.5.2 Observations in classrooms

I observed each of the teachers' classrooms three times in connection with the interviews taken with them. The goal with the observations on the outset of the study was to familiarise myself with the cultures in the mathematics teachers' classrooms and learn about how they interacted with their pupils. The intention here, like for the first interviews, was to gain a sense of where support was needed in order to help me structure the workshops. The goal with the later observations was to look for indicators of changes in their practices, reflected in how they structured their lessons and interacted with their pupils. My intention was not to assess their pupils' learning, but rather to make myself aware of the learning environments that the teachers created within their classrooms and consequently their pupils' opportunities for meaningful learning.

During the observations in the teachers' classrooms, I was a participant observer (Bogdan & Biklen, 1992) and sought to be mindful of the communication and the learning community established in the classroom. The themes I drew from interviews with the teachers and former observations, while conducting observations in the second and third round, helped me create structural questions to focus the observation (Spradley, 1980). When the teachers led class discussions, I was mainly a passive observer except on occasions when the teachers included me in their work by drawing their pupils' attention to my presence and when children asked me questions. While the children were working on their tasks, I walked around, observed their way of working and communicating, and discussed with them in order to gain an understanding of their way of approaching

the tasks. I only wrote a minimum of notes during the lessons, but after the lessons I promptly recorded what I had observed. I also received a copy of the children's written work and took pictures of collective work. The field notes were analysed by looking for themes that emerged in order to support me in making decisions about how to plan our work together. I wrote notes about thoughts that came up while working through the field notes and discussed them with the participants at workshops and in the process of interpreting the outcomes of the study (Taylor & Bogdan, 1998). The observations in the classrooms, like the interviews, therefore served as means for structuring the workshops with the teachers, as our co-learning within them was intended to support us in critically reflecting on our teaching, leading us to question norms and established habits within our practices. An overview of the observations is presented in Table 8.4, Appendix A.

8.5.3 Notes and reflections from workshops

I met with the teachers in 17 workshops during three years from September 2009 to May 2012. The workshops were videotaped and after each of them I wrote field notes and my reflections on our activities. I looked at the videotapes from the workshops, listened carefully to our discussions, wrote memos about important events that took place and decided on what needed further elaboration in the next workshop.

During this period of data mapping, I was adapting to the process of following the developmental research cycle, discussed in Sections 8.2 and 8.3, which guided the planning and implementation of the workshops. As discussed in Chapter 7, the global theories that inform this study are grounded in sociocultural perspectives, and I hoped that through collegial support we would build a community of practice with the aim of improving our competence in teaching and learning mathematics. Special attention was paid to the process of inquiring into mathematics learning, teaching and researching a developmental process. Adopting a local theory, which evolved from former experiences of working with teachers, guided the practical experiments at the workshops, with the intention that by participating in the problem-solving activities the teachers might experience complexities similar to those met by their students in class, which in turn would encourage them to improve their competences in interacting with their pupils. My intention was that we would ask ourselves what we could do in classrooms to enable students to understand better the mathematics they meet in textbook exercises, thus asking ourselves a developmental question, and in doing so, entering an inquiry process.

8.5.4 Summary

From my field notes, interviews and classroom observations and my emerging analysis of them, I learned to notice where the teachers needed to strengthen their knowledge and understanding in mathematics concerning both the children's developments in mathematics learning and with regard to their own exploring and investigating of mathematics. Moreover, this process informed my own approach in structuring the workshops.

8.6 Methods of data interpretation

The analysis of the data was contemporaneous with the research project. Throughout the whole study, I reflected on my data with the intention of developing the study in accordance with the participants' expressed intentions for improving their practice, feeding back to the local theories I based my work on, which I then related to the global theoretical framework discussed in Chapter 7. Themes gradually emerged as I moved through the procedure of analysing the data, which then supported me in directing the research project further.

8.6.1 Exploration of the data

The developmental nature of my study affected the ways in which I analysed the process of data gathering, as discussed in 8.5.3, in connection to the data extraction from the workshops. In answering the research question for Part II, with focus on the learning processes that emerge through collaborative inquiry between classroom teachers and a mathematics teacher educator, it was essential to analyse the process from the outset of the study. The focus of the three questions for Part II also implied that a cyclic analysis throughout the study was needed in order to account for the proposed intentions, that is, to focus on the development of the collaborative research process and the individuals that participated in the process, the seven teachers and a teacher educator. The unit of analysis was therefore the community and its collaborative development where the contribution of each the participants, the teachers and the teacher educator and researcher was traced. No predetermined themes structured the coding process of the research even though it is grounded on a framework of features that characterise studies based on the notion that people change their ways of understanding, perceiving, noticing and thinking in shared efforts with other people was the basis of the study. The

coding process was therefore based on approaches used in grounded coding in tracing the characteristics of the developmental process.

As the project proceeded, the data from our work were used to guide the progress, as learning from former stages guided further steps taken. The analysis process therefore began at the outset of the study, and took the form of looking for categories and themes that were emerging, tracing threads, which were then analysed at different levels. The spiral of analysis Creswell (2007) describes, where the researcher engages in the process of moving through analytic circles rather than using a fixed linear approach, relates to my approach.

When the data generation with the teachers came to an end after the workshops I then went through all my data again. I looked at the videos from each of the 17 workshops and transcribed each of them. During the transcriptions I wrote notes about the discussions and traced emerging themes. This was also a reliability check on my earlier interpretations. My reflective notes from each workshop reminded me of my thinking about our work at each stage of the project, and reinforced by the memos I wrote during the time the project was running. When I had transcribed the workshops activities, I looked at all the videos again, read through the transcriptions and wrote further notes about the themes that had already developed, moving to a more fine-grained analysis. In this longitudinal study, I thus examined the data chronologically, following threads that developed through time; categorically, when classifying procedures and events that took place, and then I examined the data thematically, when tracing themes that were developing within our community (Saldana, 2003).

The transcriptions were written in a spread-sheet-program, which helped in tracing codes and looking for how they emerged through time within the project. It also supported the analysis of each of the participant's development. While going through this process I made comparisons with codes in memos from my research journal, written throughout the project. This aspect of the process is further described in chapters 9 and 10, in relation to the reporting of the findings.

While coding the data, I drew on grounded theory in the sense that I used open coding (Glaser & Strauss, 1999) when reflecting on my data and made an effort to focus on the nuances and complexities of the participant's words and actions (Strauss & Corbin, 1998). I tried to be true to my sociocultural position. In this research project, we, the participants, shaped the culture within our community of inquiry as we brought into it

our diverse experiences, beliefs and visions. This view is in line with evolved grounded theory (Strauss & Corbin, 1994), which acknowledges the importance of the multiplicity of perspectives and 'truths', and as a result, the interpretations of research findings must include the diversity of perspectives and voices of the people we study.

Given that the coding started at the beginning of the project and continued throughout the whole process, my interpretations of the data shaped the emergent codes, as described in Charmaz (2000). She emphasises that grounded theorists develop analytical interpretations of their data throughout the research process in order to sharpen and direct further data collection. When I decided what to work with at the workshops, I took notice of what I had analysed by looking at the videos from the preceding workshops and noting what the teachers had expressed they wanted to learn about. In doing so, I informed and refined my developing theoretical analyses, as emphasised in Charmaz (2000), where the emergent codes derived from my former coding of the data I had collected.

Charmaz (2000) has developed an approach based on grounded theory with reference to Glaser, Strauss and Corbin. Her approach recognises that the narrowing of research questions, the creation of concepts and categories, and the integration of the theoretical framework, reflect what and how the researcher thinks and does while shaping and collecting the data. This is reflected in the position I took in situating my research within developmental research where the cyclic process between development and research is guided by global theories with relation to refining and developing local theories (see Fig. 1). The analysis emerges from the researcher's interaction within the field and questions about the data; and tells a story about people, processes and situations. This story reflects the viewer as well as the viewed (Charmaz, 2000).

Charmaz (2000) description of the cyclic analysis that arises through the researcher's interaction within the field and questions about the data corresponds to my approach to the data. In the process of analysing the data, I reflected on my interactions with the participants through the whole process, and in planning our future collaboration I respected their beliefs concerning mathematics teaching and learning, as well as their visions for teacher development, and in that way, I ensure that I was respectful toward our collaborative inquiry into mathematics teaching and learning.

8.6.2 Narratives of four teachers

To answer the second research question for Part II, which aims at the teachers' interpretation of how participation in the project became reflected in their mathematics teaching, I found that writing the narratives of the participants would serve the purpose of answering how they interpret the influence on their mathematics teaching. I wrote narratives that chronicled the experiences of the four teachers who participated in all phases of the study. These narratives are presented and discussed in Chapter 10, along with a discussion of the experience of the three teachers who participated in parts of the study. Both are presented in the final section, which summarises how the teachers perceived their participation in the project reflected in their mathematics teaching. When answering the final question for Part II in Chapter 11, about from the learning that I extracted from participating in the project, I also adopted the narrative approach, and this is further described in Section 8.6.3.

The unit of analysis for answering the question about the teachers' learning takes the form of their developmental process that each experienced in reflecting on their practices. When writing the four narratives my intention was to draw a sound portrayal for each of the individuals. This portrayal of individual teachers builds on the degree to which they each identified with our collaborative community and their participation in it. I also needed to respect that participation in our community was a part of their journey through the learning landscapes, within the multiple communities they belong to, and they therefore needed to negotiate the boundaries. Results from the process of the grounded coding of the development of the research project then supported the coding I based the narratives on. The teachers contributed to the writing of the narratives by reading drafts and commenting on the stories I wrote regarding their participation.

During the three years of our collaboration, the composition of the group changed as one teacher took on other responsibilities in her school and withdrew from the project. Another teacher retired, and a third teacher was often absent due to personal reasons. I therefore decided to write narratives only of the four teachers who had participated throughout the whole research period. I collected all the information I had gathered about each of them, notes from three visits to their classrooms, transcriptions from four interviews and of their participation in the workshops. I arranged the data for each of the teachers chronologically, from September 2009 to February 2013, to trace their development

throughout the whole trajectory. I read the data carefully and wrote notes about emergent themes, comparing them to the themes coded through the process of grounded coding which spanned the whole research process. I then rearranged the data to be able to account for their stories in my interpretation of the entire development of their participation. The narratives for each of the teachers were rewritten several times with the intention of giving a credible and convincing image of their development through the participation in the project.

In working on the narratives, I was loyal to the stories the teachers told, both at the workshops and in the interviews. I attended to the context in which they were told and the narrative flow, bearing in mind that people, perceive their lives in terms of continuity and process. The temporal sequence of peoples' stories about their lives or events around them, and the perspective of those being studied, is essential in writing narratives in relation to which the focus of attention should be directed how people make sense of what happened (Bryman, 2004; Clandinin & Connelly, 2000). Each of the four teachers read my version of their narrative, commented on it and suggested improvements, all of which were important and contributed significantly to my interpretation of their lived experiences.

Creswell (2007) argues that the narrative researcher describes the story and places it in chronological order in contrast to the grounded approach, where open coding categories are described and where a linear description is, not necessarily attended to. In this study, the focus was on the development of the individuals within the project as well as the cyclic development of the two intertwined cycles of development and research. The fact that the teachers' development could be traced within the themes that had emerged allowed me to compare the themes coded through the process of coding the whole research process, and so the work of analysing the data through time supported the writing the narratives. The themes I traced within the individual teacher's process of participation in some cases aligned with the themes that developed within the project and in others did not.

In my analysis of the teachers' participation, I tell their stories as they occurred in the data, and through that process, I tried to make sense of what was affecting their responses in the collaborative project at the time they were given. When coding themes, I saw a flow in the development of some of the teachers' understanding of their mathematics teaching through time, while with others the themes emerged at different times and thus

their stories were not chronologically ordered. This will further be attended to in Chapter 10.

When presenting a narrative, the focus is on processes. The stories that the people tell are identified as well as the aspects reflected in their participation within the projects, which paves the way for an interpretation of a broader meaning (Clandinin, 1992; Creswell, 2007). Clandinin and Connelly's (2000) emphasise that narrative is the best way of representing and understanding experience. "Experience is what we study, and we study it narratively because narrative thinking is a key form of experience and a key way of writing and thinking about it" (Clandinin & Connelly, 2000, p. 18). They argue that both individual and collective experience is temporal, and that life is experienced on a continuum. However, what we say today is attributed meaning in terms of the larger context, and this meaning will change as time passes. This echoes my experience of the collaborative project, as our experience at each meeting was unique at that time, which though also influenced our collective experience and further collaboration.

In writing the narratives I aligned my approach with that described in Clandinin (1992; 2013). According to her, narrative research involves participant observation, shared work in a practical setting, where the process is a joint living out of a researcher's and a practitioner's narratives. She describes researcher's narratives as lived, told and retold in the research process, in which the narratives of participants and researchers become shared constructions and reconstructions created through inquiring into collective processes. In Chapter 9, where I address the findings and answering the question of how the developmental process proceeded, I select examples of our collaboration and discussions, which I interpreted as being representative for the themes that emerged. I discuss the selected accounts from the workshops, analysed them in relation to the theories that underpinned the study and then retold what I had reported in the form of a narrative. In the words of Clandinin, this process can be described as lived, told and retold.

When I wrote the narratives of individual teachers I attended to what occurred in the data extracted from the workshops, observations in their classrooms and the interviews. Through the process of writing and rewriting them I tried to make sense of what was affecting their responses in the collaborative project at the time they were made. Clandinin (2013) discusses her interpretation of narrative inquiry as a relational methodology, which entails the relation between the person and her world, the past, present and future, person and a place, events and feelings. In

narrative inquiry, we intentionally enter into relation with participants, and our lived and told stories are always in relation to, or with, those of the participants. Narrative inquiry, according to her, is based on the premise that as human beings we come to understand and give meaning to our lives through stories. While writing the teachers' narratives and reading the transcriptions of the data, I looked at segments of the videos from the workshop. The intention was to confirm my interpretation of their participation in the workshops and how they expressed their feelings and visions for their work with children in their classes.

Sfard and Prusak's (2005) definition of first, second and third person helped me identifying stories from the teachers and to focus on who was telling the story each time.

- ${}_A A_C$ = an identifying story told by the identified person her-self. This story we call A's first-person identity (1st P).
- ${}_B A_A$ = an identifying story told to the identified person. This story we call A's second-person identity (2nd P).
- ${}_B A_C$ = a story about A told by a third party to a third party. This story we call A's third-person identity (3rd P). (Sfard & Prusak, 2005, p. 17)

The stories that the teachers told both in interviews and at workshops are first person-identity stories. The narratives of their participation in the project, based on my observation of their work with their pupils and in the workshops, are third-person identity stories recounted by me and they form the basis of my analysis of their development within the project, as discussed above. Each of the teachers read the narratives I wrote about their participation in the study and commented on my interpretation of their first person identity stories. As such, they became second person identity stories as the teachers made sense of my narratives and related them to their understanding of their development within the project.

8.6.3 Self-study of teacher education practices

Throughout the whole project, I discussed with myself as I went through the data, wrote my reflections and made decisions about further collaboration with the teachers. During this process, I was researching my own practice, thus adopting the process of self-study with regard to teacher education practices (LaBoskey, 2004), described in Part I, Chapter 6. Critical in this process was discussing with my critical friends, my supervisors and colleagues (Schuck, 2011). Pinnegar and Hamilton (2009) argue that in analysing self-study research, one can choose between many qualitative

methods, amongst them grounded coding. They highlight that at the beginning of the analysing process, the researcher identifies and selects the units of analysis from the data. Then, as they read through this selected work, the researchers find themes by looking at metaphors, concepts, and/or repetitions of words, dependent on the methodological tradition followed. My unit of analysis was the development of collaborative community, which traced the contribution of each of the participants, the teachers and mine. In 8.6.1, I discussed how grounded coding was adopted in tracing the development that took place within the project, and this analysis of the process forms the basis for my interpretation of my own development.

The reflections I wrote could be interpreted as stories that the storyteller addresses to herself and hence figured as my first-person self-told identity stories, as described in Sfard and Prusak (2005). They argue that these stories are likely to have the most immediate impact on our actions as being a part of our ongoing conversation with ourselves. When writing my reflections, I started the process of analysing and interpreting the process, as highlighted in Pinnegar and Hamilton (2009), who point out that in self-study research, the collection, analysis and interpretation of the data, occur in a recursive process from the onset of the study, and these processes might be difficult to distinguish when in the midst of the work. In this spiral pattern of the collection-analysis-interpretation process, the researcher looks to identify the next steps. This process correlates with the cyclic developmental research process that I entered into as I carried out this study.

The final step in the process of analysis was to write my story of the research project and of my own development through the whole project. I read the transcriptions from the interviews with the leaders of the schools and all the teachers, read the notes from my observations, and once again read the transcriptions from the 17 workshops, with the main focus on my interaction with the teachers. When I read through the data, I looked at the notes I had written at previous stages in the analysing process and continued tracing and coding themes that I had seen emerging up until that point. At this stage, I scrutinised segments extracted from the recordings from the workshops to learn more about my own learning through participating in this process.

The description of the whole research project, as I interpreted it, is my first person identity story, told by me to the readers (Sfard & Prusak, 2005). It is my autobiography, in which I attempted to write a readable and

engaging text, in which the connection to the project's time and the history is apparent, and the themes are identifiable through the conversation. Bullough and Pinnegar (2001) have emphasised that these aspects are important when self-study is framed as autobiography. In addition, I garnered evidence that I believe are sufficient for the reader to recognise the "authority of the scholarly voice, not just its authenticity" (Bullough & Pinnegar, 2001, p. 20). I present and discuss the story of my journey through carrying out this project in Chapter 11, where I also present the main conclusions and implications for further work.

8.6.4 Summary

The analysis of the data was on three levels. I analysed the data from the beginning of the research project and I looked for themes that emerged throughout the entire process. In presenting the findings, I provide detailed information about how the process developed, and discuss my analysis along with presenting nodal moments in our collaborative work, see Chapter 9. When the project finished, I wrote narratives of the five of us who participated in all the phases of the study. I wrote third person identity stories for the teachers that collaborated with me, and a first person identity story of my own interpretation of the project as a whole. In writing the narratives, I was true to the process of narrative flow, bearing in mind that people perceive their lives in terms of continuity, process and progression, and I therefore present my analysis of each of the narratives at the end of each section.

8.7 Ethical and epistemological concerns

As I arrive at the end of discussing the methodology and methods I chose in collecting and analysing data, I find it necessary to discuss the theory of knowledge and ontology on which my work is based, as well as addressing the ethical aspects entailed in carrying out educational research, in particular developmental research.

8.7.1 Theories of knowledge and existence that underpin this study

The project is goal-oriented as it seeks to build a co-learning partnership between teachers and a researcher with the intention of promoting classroom inquiry in the teachers' classrooms. The focus on teacher reflection serves to enhance their professional development as teachers, which in turn, may lead to personal development. The developmental aspect was therefore an integral part of the project.

The research approach in this study is qualitative and based on participation. It is explorative, preparatory and developmental, where the goal is not to test theories but instead to understand and interpret processes with the aim of generating themes that can be of value for teachers and teacher education in general. The emphasis is placed on process and the cyclic structure ensures flexibility. I focused on how things work rather than what works, or phrased differently, instead of trying to arrive at a mechanistic view of what worked, I tried to make myself aware of alternative possibilities, as expressed in Sfard (2005).

This study is based on human interactions, and as such, it is important part to recognise the teachers' beliefs and practices, and the potential tensions between them and my views of mathematics teaching and learning. I led the project and was conscious of my role as a stimulator who respects the participants' work by listening and supporting them in a positive way. We met on a monthly basis in workshops, where I gave the teachers problems to work with in order to facilitate a rethinking of their own way of learning mathematics, and consequently, extending the boundaries. My goal was to scaffold their reflections on their own way of learning mathematics in order to enrich their abilities of reflecting on their students' learning. My support in this process was fundamental for the teachers' development, and I had to be aware of when and how professional advice was appropriate.

I locate the study within the paradigm of phenomenology, as my approach is based on the understanding that multiple realities exist that are socially defined and constituted. Phenomenology is a philosophy that is concerned with the question of how individuals make sense of the world around them and the phenomenologist therefore seeks to see things from the participants' point of view. Accordingly, rather than seeking to explain what works, the researcher attempts to describe in rich detail how things work (Norton, 2009; Sfard, 2005). Describing and interpreting the processes of teacher learning and collaboration will always be subjective. My goal is to work in collaboration with teachers and let their voices be heard, however, I will write the dissertation and I therefore decide what is reported. Nevertheless, I will strive to present an honest depiction of their beliefs and opinions. According to Taylor and Bogdan (1998), the phenomenologist seeks understanding through qualitative methods, such as participant observation, in-depth interviewing, and others that yield descriptive data. They strive for understanding, on a perspectival level, i.e. the motives and beliefs behind people's actions.

8.7.2 Ethical issues

The fundamental principles in research on practice are according to Norton (2009) informed consent, privacy, and confidentiality and protection from harm. Consent means giving the participants sufficient enough information about the research to be able to make realistic judgments of the possible consequences of taking part before asking them to participate. These include explaining the aims of the research information with reference to the data that will be collected, where it will be stored and in which form, statements about voluntary participation and the right to withdraw as well as information on how and where the research findings will be reported. Before the project started, I met with the teachers that participated in the project and the head of their schools where I explained the purpose of the study, the research methodology and explained the procedures for the project. I gave them a written summary of the purpose of the study, stated the research questions, informed them about the methodology and presented a time frame for the study (Appendix D).

According to Norton (2009), confidentiality means making clear who has the right of access to the data provided by the participants. She argued that sometimes researchers and participants confuse confidentiality with anonymity, which means that the researcher will conceal the identity of the participants in all research findings. This however can be problematic with pedagogical action research as the research findings are part of a developmental project and data are collected with the purpose of impacting the practice of the participants. I was mindful of this and I was conscious of the psychological harm that the research might inflict on the participants if they felt that their self-esteem and academic confidence were threatened.

As an outsider to the project and a teacher educator, my role could be interpreted as the one of a mentor. In academic society a mentor is usually a knowledgeable, experienced person who provides advice and support, watches over and fosters the progress of a less experienced person. My background as a mathematics teacher and teacher educator implies that I have a solid knowledge base and understanding of mathematics teaching and learning. I also have a vision of building cultures in mathematics classrooms that promote meaningful learning opportunities for all children. I can therefore label myself as an experienced person who will guide the teachers through the learning process of developing and researching their mathematics teaching.

The teachers are experienced primary school teachers. Some of them have taught the children in their groups for several years and have therefore gained knowledge and understanding of their learning premises, as well as of their social and cultural background. They have substantial knowledge of the field that will be the forum for our discussions in the workshops. In that sense, they will be my mentors and guide me into the communities they have built in their classrooms.

The development of teachers' professional communities includes engaging in collaborative work in areas of pedagogy and curriculum and to embrace reflective discussions of both their progress and problems. A central feature in such communities is shared values about what students should learn, how faculty and students should behave and the shared aims that serve to maintain and promote the community. Campell (2005) argues that for many teachers being ethical in a professional sense has more to do with respecting the autonomy of colleagues than it has to do with the moral dimensions of working with and caring for students. Accordingly, it is the interpretation of collegiality that equates ethical treatment of colleagues with an unquestioned loyalty, group solidarity, and the belief that teachers should not interfere in the business of other teachers and criticise them. The teachers may be reluctant to observe each other's teaching and give others access to their classrooms. Resistance to open discussions and collaboration and teachers' respect for each other's autonomy may hinder the developmental process and I realised I would have to respect their values.

8.7.3 Summary

My purpose in building a learning community with the teachers is to make a space for teachers to learn together by exploring mathematical problems, gaining access to research on mathematics teaching and learning and discussing their own work. My belief is that, through discussions and collaboration, the teachers' understanding of mathematics learning will change and hence their teaching will be revised. I wanted to learn about teacher development and construct a framework for reflective discussions both on the teachers' understanding of mathematics and of their students' way of constructing knowledge. This requires an approach to the research that respects teachers as researchers.

8.8 Preparation phase and research process

In fall 2008, I contacted Sunshine School and met the head of the school for informal discussions about participation in the project. I knew that the

school was in the process of reviewing the teaching in the school and was seeking support. They were willing to participate in the project and proposed that I would work with teachers in grades five to seven. They later suggested that we ask the neighbouring school, Rainbow School, to participate in the research project with us. The head of Sunshine School contacted the head of Rainbow School who saw participating in this research project as an opportunity for the teachers in her school to develop their mathematics teaching. The heads of both schools discussed the project with the teachers in grades five to seven in their schools and asked them if they were willing to participate. In Appendix A, Table 8.2, a list of the participants and their education and experience are presented.

In late August, we had our first meeting where the teachers and heads of both schools discussed the project and I introduced the purpose of the project and my goals with regard to researching with them. I asked for permission to audiotape our meeting and mentioned that I would like their permission to videotape our workshops. The teachers introduced themselves and confirmed their willingness to join the project. We discussed when and how often we would meet and arranged my interviews with each teacher and a visit to their mathematics classes.

At this meeting, I offered to hold workshops for other teachers in these two schools even though they were not participating in the project. The teachers felt that it would support their work if their colleagues also had a chance to review and discuss their mathematics teaching. My offer resulted in five workshops with the mathematics teachers at Rainbow School (see Appendix C) in the first year of the project and one informal meeting with each of two groups of mathematics teachers at Sunshine School.

I interviewed the heads of both schools to learn about the school culture and policy concerning diversity and inclusive teaching and teachers' possibilities for in-service training and developmental activity. We also discussed the research process and collection of data. Before the meetings, I sent them information about what I wanted to discuss with them. Both schools have rules about what kinds of information about their pupils are open to people other than staff, as well as concerning filming and photographing of the children.

When the project started, we agreed to hold workshops once every month for one year. We met alternately in Rainbow School and Sunshine School. Two days before the workshops, I reminded the teachers of our workshops, told them my plan for the next workshop and urged them to come up with ideas they would like to discuss. After each workshop I sent a

follow-up letter to summarise our work and clarify things that I felt that we did not have time to discuss in depth and reminded the teachers about collecting data from their work. In Appendix B, an overview of the workshops is presented.

8.9 Overview

The methodological approach in this research study was based on finding an approach that supported the process of tracing the way in which this developmental project itself proceeded. I began by revisiting the developmental aims of the study and related the discussion of my former experience of researching my own practice, discussed in chapters 5 and 6, to the process of finding a methodology that would support me in reaching my goals. As the purpose of this study was to develop an understanding of how collaboration between teachers in schools and a mathematics teacher educator could impact their work and researching their own practice, the methodology of developmental research was chosen.

The developmental research cycle, as introduced in Goodchild et al. (2013), supported the analyses the process, through the two interconnected cycles of development and research, and thus supporting growth of both practice and theory. This approach also supported the process of analysing the roles of the participants as insiders and outsiders in the different parts of the collaborative project.

The aims and methods used for collecting information and for analysing the developmental process were explained. Three research questions for Part II as well as the main question for the whole project will be answered in the three last chapters. The questions called for different approaches to the analysis of this collaborative project and the data were therefore analysed through three different lenses. In Chapter 9, I answer the first question for Part II by adopting the methods of grounded coding of the collaborative work with teachers. I answer the question about the teachers' interpretation of their learning by a narrative inquiry, presented in Chapter 10. Finally, I discuss my own learning throughout this process, in Chapter 11, by writing my own narrative, which adopts an autobiographical approach in presenting my self-study into the process of taking on this research.

9 The emergence of collaborative research

In this chapter, I will present the answer to the first research question in Part II:

- What learning processes emerge through long-term collaborative inquiry undertaken by classroom teachers and a mathematics teacher educator?

In order to provide a representative image of the developmental process of the project, I decided to report the findings in chronological order, i.e. in steps that ran parallel to the progression of the project. This means that the cyclic process of the study's development is made visible by describing the steps taken within the workshops, accounting for the continuous awareness of what was taking place in the ongoing process. The intention with reporting the outcomes in this detailed way is to give the readers a chance to involve themselves in the activities and interpret what happened, in the same order that events occurred. The excerpts are chosen to represent critical moments in the developmental process.

To identify the themes and interpret what was happening, I used the model of developmental research introduced in Section 8.2. I described the process of deliberating on the results from the developmental cycle, and how practical experiments nurtured thought experiments, that in turn, affected the choice of practical experiments we engaged with at each given time. The results from the developmental process then in turn informed the research process, which moved between local and global theories, with the revision of local theories guiding the developmental cycle. Finally, the results from the developmental process then facilitated the research process, moving between local and global theories where the revision of local theories guided the developmental cycle, as described in Section 8.3.

The main body data in this chapter is extracted from 17 workshops, held from September 2009 to May 2012. The approach taken in the workshops was based on sociocultural theories with the concept of learning communities forming the basis of the global theories. The local theories that guided the process were based on teachers' need for collaborative opportunities to investigate with mathematics and solve mathematical problems. The intention was that teachers, through their involvement in the workshops, interviews and observations, would develop knowledge

about teaching and learning mathematics in diverse classrooms. Additionally, it was expected that all the participants within the project, including me the teacher educator and researcher, would enhance their competence in reflecting on their practice, which in turn, would support them in researching their own practice for the benefit of their teaching.

Finally, the intention was also to learn about how collaborative inquiry in mathematics teaching and learning could support teachers and a teacher educator in developing their practice. As a researcher I wished to improve my competence in carrying out a research project with teachers where I was both an outsider and an insider to the research process. In particular, I wanted to learn about developmental research and the structure and application of the developmental research cycle.

The project was carried out in accordance with principles I have developed over time in collaboration with my colleagues. They are based on my extensive experience of working with prospective and in-service teachers and my belief that individual and cultural processes mutually constitute people's development.

- The protocol for each workshop was based on the teachers' expectations for what to attend to.
- The teachers were urged to come up with proposals for activities at the workshops.
- I offered the teachers tools to work with and entrusted them to decide what they found helpful.
- The teachers were challenged to rethink their teaching habits by participating in collaborative investigations into their practices.
- I did not confront their views but emphasised collaboration and mutual support.

In presenting the findings to this study, I do not differentiate between my role as a teacher educator leading a study with teachers and as a researcher who studies this practice. Researching into my practice is an inseparable part of my profession.

A sequence of six themes emerged and developed as our co-learning in the project progressed. These themes represent the nature and substance of the six core developmental research cycles that characterised the emerging collaboration between the teachers themselves and with me, as the researcher. The findings are built on data gathered from my field-notes, in transcripts of recordings of the seventeen workshops (each with its own protocol) and other data, drawn from several interludes, three observations

in each teacher's classroom and four interviews with each teacher. I then utilised the data to identify and process the six sequential themes, each involving its own developmental research cycle.

The six sequential themes are as follows:

- Theme 1: Initial steps to an investigative approach
- Theme 2: Reflective practice, hindrances and opportunities
- Theme 3: A focus on interactions in mathematics classrooms
- Theme 4: Focussing on pupils' learning in the classroom
- Theme 5: Teacher reflections lead our discussions
- Theme 6: Towards an investigative approach and inclusion

An additional two themes became evident and were constant factors throughout the process. These recurring themes concerned the learning of pupils in the teachers' classrooms, on the one hand, and the teachers' own learning on the other hand. Moreover, the themes had both pessimistic and optimistic elements with regard to the teachers' own capabilities and those of their students.

The notions of insider researcher and outsider researcher also became apparent, which have to do with identifying who carries out research and into which practices. I will comment on these two terms throughout this chapter.

A more detailed overview of the workshops is presented in Appendix B. The overview shows how conclusions of one workshop affected the structure of the next workshop. This is also described in connection to the protocol for each workshop in Sections 9.1 to 9.6.

In Workshop 1, we negotiated our collaboration and in Workshop 2, we tried out various ways of calculating. The first theme emerged from these two workshops and related activities, Theme 1, or initial steps to an investigative and collaborative approach. In Workshops 3-6, an awareness of the quality and value of reflective practice gradually gained ground through our work and discussions, which led to Theme 2, or hindrances and opportunities in reflective practice. Bit by bit, the next four themes appeared, for example, the teachers' influence on the workshops, Theme 5, and by the end of the period, the teachers were becoming competent at using an investigative approach, Theme 6, which contributed to the inclusion of all children in their mathematics classes.

Table 1. Overview of workshops

Year 1	Year 2	Year 3
Establishing common ground	Geometry and measurement	Teachers lead the workshops
Various ways of calculating	Dealing with children's dependence	Discussing curriculum guides
Teacher reflections	Patterns and algebra	Teachers reflect on their learning with their pupils
Supporting children's initiative	Exploring with the calculator	
Learning to listen	Conflicts about assessment	
Mutual visits to classrooms	Learning to listen	
Teaching and learning of fractions		
Focus on the learning of algebra		

I will now present the nature and substance of the six main themes, delineated in six sections. Each workshop is discussed in a sub-section under the relevant theme and with some minor exceptions, are structured as follows:

- Name and short description of the workshop and its protocol
- Particular topics, activities and discussion in the workshop
- Mathematics activities
- Stories from teachers' classrooms
- Conclusions and looking ahead

An analysis of the findings and conclusions are presented only after I have offered a description of all workshops contributing to a particular theme.

In Section 9.1, the findings from the emergence of Theme 1 are presented in 9.1.1 to 9.1.3 and the findings are then analysed in 9.1.4. Likewise, the five remaining sequential themes are described and analysed in sections 9.2 to 9.6. The recurring themes and the notions of an insider and an outsider are discussed in Section 9.7.

9.1 Theme 1: Initial steps to an investigative and collaborative approach

The initial steps to the project started when I observed each teacher's classroom and interviewed them in order to learn about the culture in their classrooms, their background and visions for the project. During the first two workshops, we took the first steps in the collaborative project as we became acquainted with each other's expectations.

9.1.1 The project begins: Learning about the teachers' practices

In early September, I joined each of the seven teachers in one mathematics class in order to get a sense of the culture in their classrooms and meet their pupils. In my visits I discovered that they did not discuss much with their pupils about the mathematics that they were learning. Most of the lessons began by reviewing homework and then the teacher introduced the topic of the lesson and asked the children a few questions in relation to the topic and described how to proceed with their work. During the lesson, the teachers talked to individual pupils or groups of pupils and there was no round up or whole-class discussion at the end of the lessons. Only one of the teachers urged her pupils to further investigate the topics they were studying.

After the observations, I interviewed the teachers to get a further sense of where support was needed to help me structure the workshops. The teachers all expressed their wish to learn about diverse ways to teach mathematics and particularly to lead discussions in their classrooms. None of the teachers had studied mathematics as their major in their teacher education program and they were all interested in being given the opportunity to engage with mathematical problems (see Appendix A). In Chapter 10, I will further discuss the findings from the observations and interviews in relation to the teachers' narratives.

9.1.2 Workshop 1: Establishing common ground

Following the initial interviews and observations in the teachers' classrooms, I concluded that the teachers would benefit from working with problems that would challenge them to rethink their way of approaching mathematical tasks and envisioning the ways their pupils would solve the problem.

The protocol for Workshop 1 was based on: Global sociocultural theories and local theories about teacher development in teaching mathematics. Findings from the interviews and observations about the teachers' interest in and need for learning about diverse ways in teaching mathematics and in particular to lead discussions in their classrooms.

9.1.2.1 Establishing collaboration within the project

I asked for permission to audiotape the workshop and videotaping our future workshops. The teachers expressed their consent and said that they were aware of the fact that participation in a research project would require data collection. Gróa though was sceptical about recording the workshops and was concerned about the possible uses of the information gathered. I confirmed that I would be the only one who would listen to and look at the recordings, and further, that the information would be destroyed after the project was finished. Our negotiations resulted in an agreement from all the teachers all agreed which allowed me to audiotape the first lesson and then videotape future workshops.

I presented the structure of the first workshop and told the teachers that I had based my choice of the topics for this workshop on both my observations in their classrooms, their expressed visions for the project and my former experience of working with teachers. I emphasised that we were in the beginning phases of a collaborative project and I encouraged them to actively engage in the development of the project, come up with proposals for what to focus on and bring in work from their own classrooms to discuss. I gave each teacher a folder to store materials from our workshops and also copies of children's work that they wanted to keep as data on the project.

9.1.2.2 Collaborative problem solving – diversity of solutions

Drawing on what I had learned from the visits to the teachers' classrooms and the interviews, I decided to start our first workshop with a problem that I had picked up in former work with teachers and student teachers, which called for discussions about diverse ways of solving problems. The problem was about Sigga and her 10 dogs, Labradors and Icelandic Sheepdogs. Together the dogs ate 56 biscuits at a feast. The Labradors ate 6 biscuits each and the Icelandic Sheepdogs ate 5. The question is: how many dogs are there were of each type?

I suggested that the teachers should split into two groups for discussing the problem. As Dóra started to think about the problem, she said that a drawing always aided her in solving problems. She needed to visualise what

the problem was about and her process of solving it. I urged her and the other teachers to draw or use manipulatives like counters, cubes and an abacus as means to support them in solving the problem. Gróa and Pála sat next to each other and discussed their thinking, I joined them and the other teachers listened to our discussion:

Gróa: Yes, I think so too, 6 times 6 make 36

Pála: Then there are 20 left, that is 4 times 5

Gróa: 6 Labradors and 4 Icelandic

Jónína: Could you think of different ways to solve the problem? Ways that your pupils might approach this problem?

Gróa: I think they would start by dividing by 6, the most capable ones ...

Many voices: Yes, yes they would.

Gróa: I mean 6 times 6 ...

Jónína: Try to write this down ...

Gróa: I think they would do it like this (points to her own calculations)

Jónína: But you also have pupils that have a less developed number sense ...

Gróa: They would just close.

Jónína: Would none of them use their fingers or counters?

Gróa: No. Or, maybe, some would do (starts counting cubes), what have I got, 11. They could do 56 minus 11 ...

Pála: Yes, proceed like this ...

Jónína: Could you show us their thinking through this process and write the steps you think they would take?

Gróa: (to Pála) do you think it will work, 56 minus 11?

We discussed further the diverse ways in which this problem could be approached and Gróa said that only a few of her pupils would make use of multiplication facts like she did and most of them would need to draw or count cubes. Inga showed us how she expected her pupils would use the abacus to count first 56 counters and then subtract by moving alternately 5 and 6 counters until 12 counters remained. She added that they might find it difficult to keep track of how many dogs ate 5 biscuits and 6 respectively.

Vala told us about two boys in her group who always write dots or lines as checkmarks or count cubes and other things at hand when they calculate. Dóra said that her pupils were not always willing to draw and they often said "I think this in my head".

Edda: But can they explain it? That is often difficult.

Dóra: I know. It is often difficult to tease it out.

Jónína: Why is it difficult? Why is it difficult to explain one's thinking?

Vala: Is it not just a lack of practice?

Pála: I think it is a lack of practice.

Gróa: This is mainly a question of practice. If you have the right answer, if you calculate differently, if you can explain your way of solving the problem even if it differs from mine and the textbooks' then it is equivalent, but you need to be able to justify it.

Jónína: If someone asks you about something that you know, can you always answer why you know it or how you can be sure about your answer?

Many voices: No, no.

Dóra: It is a bit like this, they say it is just there (points to her head), I just knew.

Jónína: Yes, and children at this age know many facts about numbers, like 5 times 11 equals 55. And when they work with numbers close to these numbers they relate to this knowledge and are quick to find a solution. But they cannot necessarily repeat to you the steps they took when calculating mentally. But it is important to explain ones thinking because then you need to think about your own thinking; mediate to others what you are thinking and this helps you focus on how you structured your thinking when you calculated in your head.

Dóra: Yes, it is important.

We continued discussing the teachers' diverse solutions to the problem as they explained what they were thinking. During this discussion, Dóra and Edda pointed out that they were debating the ways that they chose to solve the problem and at the same time they were supporting each other in this process. Pála said that she enjoyed discussing the problem with us, and learning about diverse ways of solving it. She felt that our collaboration helped her in thinking about her way of solving problems and that she would like to learn more about how to foster collaboration in her mathematics classes.

9.1.2.3 Approaches to researching one's own practice

In the latter part of the workshop, I invited the teachers into my own classroom as I taught children in 4th grade. I showed them some of the children's work with tessellations and then a video clip where one boy presented his work and discussed it with his classmates. The reasons for showing the teachers this video clip were threefold. First, I wanted to open up discussions about one's own teaching, so I offered them to discuss my teaching. Secondly, I wanted to show the teachers how the children in my class approached an investigative task and how their work with this task had paved the way for more in-depth discussions about geometry than I had originally planned. Third, I wanted to encourage them to record their lessons and discuss what could be learned from looking at recordings from one's own classrooms and from reflecting on what happened. I told the teachers that the recording of this lesson enabled me to analyse the in-

depth learning that took place in in my classroom, and the later observations allowed me to examine aspects that would have gone unnoticed.

9.1.2.4 *Stories from teacher's classrooms – algorithm for division*

Before we met, I had urged the teachers to think about events in their mathematics classes that they would like to discuss at the workshop. Dóra from Rainbow School was eager to talk about the use of the 'traditional' algorithm for division. She and Pála had discussed if it was important to learn the procedure for the algorithm.

Pála: What we have been reflecting on is, is it bad, does it spoil anything for them? Does it destroy their thinking process, does it stop anything?

Rúna: It is important for my pupils to be able to make use of the algorithm, to know the process because it makes them feel confident. They are proud of themselves when they can follow through with the algorithm step by step.

Edda: It does not help them to understand, it only helps them to be quick at finding the answer.

Rúna: It is important for my pupils to quickly solve division problems as required on standardised tests like in Grade 4.

To challenge Rúna, I wrote $360 \div 24$ on the whiteboard and asked the teachers how they would find the answer. They replied that they would use an algorithm that they had learned at school and none of them mentioned other ways of thinking about the solution.

Jónína: Can we solve it differently?

Rúna: You probably can but I do not understand why you cannot do it the old way.

Together we reflected on diverse ways to solve this problem and wrote our solutions on the whiteboard.

Gróa: This, they would just stare, be terrified, if you showed them something like this.

Dóra: We need to emphasise understanding.

Edda: I learned algorithms for calculating at school without understanding why to proceed like I was supposed to, just thoughtlessly.

Rúna: I do not agree that you cannot use these practical old procedures and understand what you are doing.

Jónína: Of course you can, but we must ask ourselves how much time we want to spend on practicing them.

Edda: When I started to learn algebra I lacked understanding, had no basis for learning algebra. That is why I want my pupils to develop an understanding of numbers and arithmetic rather than learning quick steps to find a solution to problems.

9.1.2.5 Conclusions and looking ahead

Before we completed this first workshop, I asked the teachers what they would like to work with during our next meeting. Dóra said that she would like to learn about how to use co-operative learning approaches in mathematics classes. The other teachers backed her suggestion and Edda repeated her former comment regarding their mutual learning in the problem solving process. Pála and Gróa had noted their experience with collaboration in their own classes, where their pupils had collaboratively worked on many projects and that the children had taken on different roles and responsibilities in their groups. None of the teachers though had emphasised collaboration in mathematics classes and they all expressed a wish to learn more about how to create supportive mathematical learning culture in their classrooms.

Through these discussions we concluded that we were all interested in continuing the discussions concerning the diverse ways of solving mathematical problems and co-operative learning in mathematics classes. I reminded the teachers to bring in stories from their classes and I told them that I had the intention of writing them down.

9.1.3 Workshop 2: Various ways of calculating

Drawing on experience gained from the first workshop and the discussion of teaching an algorithm, I decided to address diverse ways of calculating, particularly division problems, and respond to Dóra's wish of further addressing co-operative learning.

The protocol for Workshop 2 was based on: Conclusions from the first workshop where teachers expressed their wish to learn about co-operative learning in mathematics classes and the teachers' interest in discussing diverse ways of calculating.

9.1.3.1 Discussing calculations

The teachers solved four number problems, one for each of the four operations: addition, subtraction, multiplication and division. They were asked to find different ways to calculate to find solutions to the problems. The seven teachers were divided into two groups, and I assigned them roles such as discussion leader, writer, active listener, supporter and timekeeper. After they had discussed diverse ways to calculate, they were urged to focus on the issue of how their pupils would solve these problems.

Moreover, the teachers were asked to suggest possible contexts for the problems.

Gróa: [Was calculating $2387+8735+43$]. Put the numbers in the right position. Rúna, what is it called again, this structure with the places? Place value, base ten system?

Edda: What do we call the procedure when we connect units together and tens together? Is it the associative rule?

Gróa: Are we supposed to find a solution to the problems? Show our computations?

Jónína: You need to present your work to the other group later. What would you find helpful to do?

Rúna: Are we supposed to find an accurate solution?

Vala: Some of my pupils draw dots and lines to support their mental calculations.

Gróa: How do you think the children will respond if we 'gabble' all this, and draw dots and lines?

The division problem I gave the teachers was to find the solution to $408 \div 17$.

Dóra: We could use the traditional algorithm like we discussed last time and the calculator.

Pála: 17×10 and then double it, makes 340. Then we were going to split the rest.

Jónína: 340, how much is remaining?

Pála: It is 68, four times 17.

Jónína: What helps you find out ...

Pála: Two times 17 make 34, and then two times 34 make 68.

Jónína: Do your pupils see this connection? Are children in 5th and 6th grade capable of doing that?

Pála: Yes some of them, the clever ones.

Dóra: Those who know that double 17 is 34 and can visualise that 34 is half of 68.

The teachers shared their experience of taking on particular roles in their groups, like discussion leader, timekeeper and writer. Those who were familiar with using the co-operative learning approach in their teaching of other subjects, related to their former experience of teaching children become aware of their responsibilities in collaborative activities.

9.1.3.2 Looking into other teachers' classrooms

When we had finished our discussions about the problems, I showed the teachers a video-clip from a classroom. The clip is from Martin Dolk and Catherine Fosnot and is a part of the teachers' handbook titled *Young*

Mathematicians at Work: Multiplication and Division. Grades 3-5. (Fosnot & Dolk, 2005). A teacher discussed division with her pupils and started the lesson by asking the children to solve $340 \div 17$. She drew rectangles on the whiteboard to represent their explanation of their thinking. Similarly she asked the children for their solutions to $68 \div 17$ and $408 \div 17$. They concluded that $68 \div 17 = 4$, were quick to find that $340 + 68 = 408$ and concluded that $408 \div 17 = 24$.

I asked the teachers if their solution strategies had anything in common with this approach. Dóra and Pála referred to how they used their knowledge of multiplication and place value to solve this same problem. The teachers found that there was a lot to digest from what they had seen and claimed that this approach to division problems was new to them.

9.1.3.3 Stories from teachers' classrooms: Ratio and proportion

In the latter part of the workshop, the teachers brought in stories from their classrooms. Dóra and Pála in Rainbow School told us about a project that revolved around African animals that they had worked on with their 5th graders and how they solved a problem about the proportional weight of the animals. In order to solve the problem, they were given the following information:

A lion weighs the same as four cheetahs. The weight of three lions and three cheetahs equals the weight of one buffalo. One cheetah, two lions and one buffalo weigh together 1200 kg.

The children had worked in groups and their teachers tried to follow their discussions, wrote notes and reported their findings to us. Most of the children had started by guessing the weight of the lion, tried if it worked in relation to the other information given and then refined their guess. The children in Dóra's group were quick to find the weight of each type of animal. The children in Pála's group found the problem difficult because they did not know where to start. Dóra told us about two girls in her group:

They discussed a lot, started with guessing the weight of the lion, and then the cheetah and the bison. They started by guessing that the lion was 80 kg and found that it was too little, then they tried 105 then 150, the number was still too low so they tried 300 and found that it was too much. Next they tried 200 and then found the number that worked for all the clues.

In Pála's group, two children knew that a lion weighs about 200 kg and therefore could solve the problem quickly. Pála said that children at this age are interested in animals and know many facts about them.

Pála and Dóra both focused on discussing with two groups in their classes and said that it would have helped if they had recorded the discussions in other groups. We discussed what could be gained from recording lessons and the teachers said that they would not be able to record their lessons without permission from the children and their parents. I proposed that instead of trying to follow the discussions in all groups at every lesson they could focus on different groups each time.

9.1.3.4 Stories from teachers' classrooms: Length, width and area

Gróa, Rúna and Vala, from Sunshine School, proceeded to inform us about a project they had assigned pupils with, which entailed measuring the length and width of classrooms, the hallway and other areas in the school building. They then calculated the area of these surfaces. Their main focus was on the structure of the project and they did not provide any detailed information on how their pupils went about measuring these areas and calculating their solutions. Gróa added: "The children felt it was great fun to measure the hall but they soon gave up because it was much more complicated than they originally thought". They then gave me a folder with a copy of the children's work and their instructions for the project, as well as a short description of the process of the lesson.

9.1.3.5 Conclusions and looking ahead: Writing for development

Toward the end of this workshop, where the teachers had shared stories from their classrooms, I asked if they were willing to develop them further. I gave them a protocol for case and commentary writing, developed by Guðjónsdóttir et al. (2007). I then (Appendix D) described the 'four steps guidelines' to aid them in writing about their cases and I urged them to make use of the protocol. I also gave them copies of articles from the Icelandic mathematics teachers' journal "Flatarmál", in which teachers expound the various ways they facilitate their pupils' mathematical learning. I also referred them to a book chapter about co-operative learning (Guðjónsdóttir, Guðmundsdóttir, & Ívarsdóttir, 2005) and based in part on the writing of Johnson and Johnson (1994).

Two days later, I sent the teachers a mail in which I told them that I had learned much from listening to the stories from their classrooms and that it was good to hear about all the diverse tasks they were working on with their pupils. I then explained further the purpose of the guidelines I gave them:

The goal is to support teachers in reflecting critically on their teaching; aiming at learning from their own experience by continuously relating their process of learning to their prior knowledge of the profession. This can of course be achieved in different ways but these guidelines have been helpful to teachers.

I added that when we met next time, I would like to discuss their analysis of their stories with them and it was therefore important that they look at them and reflect on what they could learn from their experience and from their pupils' responses to the tasks with which they had worked.

In drawing conclusions from our discussions at the first two workshops, I decided to give more room for discussing the teachers' stories from their classrooms. The teachers had not mentioned much about their pupils' learning and I felt we needed to focus on analysing what was taking place, particularly on what their pupils were learning from participating in their classes. Bearing in mind our discussion from the first workshop about children's difficulties with explaining their thinking about problem solving, and the repeated comments on children's lack of knowledge, I also decided to focus on ways to support discussions about their own thinking and introduce the idea of concept cartoons (Dabell, Keogh, & Naylor, 2000).

9.1.4 Discussion of findings: Initial steps to investigative and collaborative approach

The theme that emerged during the first two workshops revolved around establishing common ground within our group. The categories that developed during this initial process were already traced while I participated in the workshops and listened to and looked at the recordings. During later steps of the data analysis process, they were refined and further established by tracing codes, as discussed in Section 8.6.1. The teachers were eager to solve the tasks that I assigned in our workshops and said that they enjoyed collaborating and sharing their thoughts about their way of solving them. They showed willingness to research their own practice by bringing in and discussing cases from their classes, however, some of them challenged the means for collecting data as proposed by me. In order to support the teachers in looking into alternative approaches to teaching, I suggested some texts to read which I hoped would broaden their perspectives on how to create supportive mathematics classrooms. I also urged them to start analysing the stories they had told at the second workshop as a means to focus on researching their own practice.

9.1.4.1 Negotiating collaboration

When I contacted the head of Sunshine School during the initial stage of planning the project, she proposed that we offered the neighbouring school, Rainbow School, the chance to participate, since she felt that her teachers would gain from collaborating with teachers from a neighbouring school, as discussed in Section 8.3. The head of Rainbow School saw this offer as a professional development opportunity for her teachers. The heads' response indicated that they expected their teachers to benefit professionally from participating in a learning community, acknowledging the position I adopted in planning the project. The position that learning can be understood as both a social process and a personal experience (Lave, 1988; Lave & Wenger, 1991; Rogoff, 2003; Wenger, 1998) was seemingly shared by the teachers who agreed to participate in the project and expressed their vision of furthering their learning about diverse ways of teaching mathematics and leading discussions.

At a preparation meeting with the teachers and heads of the schools, I explained the goal of this project and that I needed to collect information in order to be able to analyse the learning that was expected to take place, as discussed in 8.7. The teachers and heads of the schools also presented their visions for the project and teachers' interest in improving their practice for the benefit of their pupils was apparent. This enthusiasm was shared by Inga, who felt that her competences in teaching mathematics were weak. The teachers gave their permission that our workshops and my interviews with them would be recorded. And so, the emphasis I placed on negotiating collaboration and sharing information about our expectations seemed to have resulted in mutual understanding with regard to both the research process and the methods of data gathering.

9.1.4.2 Reluctance to record workshops and one's own teaching

This mutual understanding was opposed by Gróa at the first workshop who protested against record the session. She stressed the importance for secrecy in relation to what happens in school and underlined her duty to protect her pupils from external interference. Her alignment to the community of teachers within a school with a diverse population of children and where the anonymity of children is respected, conflicted with the interests of the researcher's community of practice that I represented, where collection of data is central. We needed to negotiate the boundary between our communities (Wenger-Trayner & Wenger-Trayner, 2015) and resolve the conflicts that arose between our diverse cultures. I thought that our conflict had been resolved when Gróa agreed to videotape our

workshops. I later proposed that the teachers should record their own lessons as a means for researching their own practice. Gróa was once again reluctant to do this and indicated that she was not ready to align with the co-learning culture that I hoped to establish within our community. It seemed that differences of power and authority might develop within our learning community, as discussed in Goodchild (2008).

9.1.4.3 Investigative approach to mathematics learning

The problem about Sigga and her dogs that I chose to work with at the first workshop opened up discussions for diverse ways of calculating, similar to what I had experienced earlier with other teachers when we explored with problems of the same structure (Guðjónsdóttir & Kristinsdóttir, 2011). To begin with, the teachers were hesitant to approach the problem and when Dóra said that she would like to draw, I urged them to use the tools they were confident with using. Their approach to solving the problem was similar to what I had experienced before with teachers in primary grades and I tried to support them in looking for diverse way to solve these problems by asking them challenging questions.

9.1.4.4 Conflicting views on capability of children

Pála and Gróa made use of their knowledge of multiplication facts when they solved the problem about the dogs. When I challenged them to think about alternative ways to solve the problem, Gróa responded by claiming that the most capable children in her group would solve it in a similar way as she did. Responding to the question regarding how the children with less developed number sense would solve the problem, Gróa expected them to “just close”, indicating that they would take on an accepting mode, as explained in Mason & Johnston-Wilder (2006). She thus anticipated that her most capable pupils would be able to think about the problem as she and Pála did, assuming that they were capable of what she might have interpreted as mathematical ways of thinking, in the words of Lerman (2006a). At the same time, she was indicating that the less able pupils were not capable of such thinking, even though she could envision that they could use counters to support their thinking about the problem. When we discussed ways of explaining one’s thinking about mathematics, she said that the ability to explain one’s way of solving problems was just a matter of practice, voicing the assumption that if given access to the pedagogical site of mathematics in school (Bernstein 2000; Solomon, 2009) they would be empowered to learn mathematics in a meaningful way.

9.1.4.5 Learning mathematics with understanding

The notion of learning mathematics with relational understanding as opposed to instrumental understanding, as described by Skemp (1976), was salient from the outset. Dóra's request during the first workshop for discussing the teaching of traditional algorithms (Fosnot & Dolk, 2005) revealed that she had been questioning the norms at her school, which emphasised learning by mechanical procedures for calculations. When discussing these norms with Pála, Dóra was concerned that if children were forced to follow a predetermined procedure, their way of making their own sense of why these procedures work would be compromised. By returning to their discussion at the workshop, she was bringing debates from the teachers' community at her school into our community. Pála's question about, whether it destroys their thinking path to follow a predetermined algorithm, indicated that her beliefs had been questioned. Dóra's felt a need to voice her concerns and her worries signalled that she was afraid that this approach would lead to instrumental understanding and prevent them from the thoughtful and connected learning of principles that, according to Skemp (1976), may lead to relational understanding. Rúna emphasised that instrumental understanding was consistent with her view that her pupils, who were sent to the support centre, should be familiar with the traditional algorithm in order to be hasten their abilities of calculating. Edda, on the other hand, maintained that this process, which she herself had experienced in her own schooling, does not strengthen children's understanding of the traditional algorithm.

9.1.4.6 Disturbing practice initiates debate and reflection

The conflicting views about teaching the traditional algorithm, voiced at this first workshop, were familiar to me. I had discussed traditional algorithms and rote learning with teachers, student teachers and parents, both in my role as a teacher in primary grades and as a teacher educator, and knew well how persistent the norms for rote learning have been in traditional school cultures. At the second workshop I therefore decided to respond to Dóra's initiative by challenging the teachers to think about diverse ways of calculating. Our discussions opened up for further discussions of their conflicting views of approaches to calculations and for taking initial steps to inquiring into their own practices (Cochran-Smith & Lytle, 1999; Jaworski, 2006a). Through this process, we were disturbing practice on the inside, which served to challenge the status quo of accepted ways of teaching in schools (Goodchild et al., 2013). Diverse views of mathematical learning were also apparent at our second workshop as the teachers shared further

stories from their classrooms. Dóra and Pála told us how the children solved a problem about African animals, and how different their approaches were, while Gróa, Rúna and Vala explained how they structured their teaching and monitored the children's work. In the words of Niss & Højgaard (2011), Dóra and Pála sought to reveal learning, while the other three teachers attended to planning lessons.

Dóra and Edda initiated a discussion about collaboration at the first workshop and pointed out that they were supporting each other in debating their way of solving the problem during our first workshop, bringing the notion of scaffolding into our discussions (Bruner, 1985). Dóra wanted to learn more about collaboration in mathematics classrooms, indicating that she believed that individual learners draw on knowledge in communities of learning in which shared knowledge can lead to improved learning (Askew, 2015; Vygotsky, 1978; Wenger, 1998).

9.1.4.7 Reflecting on practice is a challenge

The purpose of looking into other teachers' classrooms and reading journal articles was to inspire them to seek alternative ways of teaching mathematics with the intention of supporting them in developing their pedagogical content knowledge (Schulman, 1986; Ball et al., 2008). When we observed an instructional video about division during the second workshop, Dóra and Pála could relate their own solutions to the instructions in the video. I chose this problem deliberately to stimulate such discussion and the fact that Dóra and Pála had responded to my challenge in finding alternative ways to solve it helped them make sense of the discussion in the video. They were thus empowered to relate their way of solving the problem to how children can be supported in sharing their thinking about division. It was therefore crucial that this discussion precluded the watching of the video.

In urging the teachers to write about important instances from their teaching and share them with us at the workshops, I challenged them to reflect on their practice (Schön, 1983; 1987). Inspiring them to further describe and explain these cases (Kruger & Cherednichenko, 2006) was meant to support them in attending to and accounting for their experiences (Mason, 2002). Furthermore, by reflecting on what took place in their classrooms they were engaging in core reflection, and relating to their own beliefs, identities and mission for teaching (Korthagen, 2004; 2013). The first steps toward reflecting on their classroom experiences were taken at the first two workshops. In Section 9.3 I will further discuss the teachers'

initial steps toward reflecting on their own teaching, and for the remainder of this chapter, I will elaborate their further steps along this process.

The initial steps to an investigative approach and negotiating collaboration were shaped by a willingness to collaborate, on one hand, and tensions about approaches to researching one's own practice, on the other. Into this figured also the teachers' conflicting views of the ways children learn mathematics. The global theories about learning as social participation helped make sense in the process of analysing these first steps, particularly in relation to the boundaries between the communities with which we engaged. The local theories about teachers' need for reflecting on their own way of solving problems as well as the learning in their classrooms guided our work and this need was confirmed by the teachers' responses. The conclusions of this first cycle affected the next cycle where the emphasis was placed on reflective practices.

9.2 Theme 2: Reflective practice, hindrances and opportunities

The teachers' willingness to share with us their work in their classrooms was promising and I advised them to develop their stories further and analyse what they had noticed about their pupils' learning. In the four ensuing workshops, our capability of collectively reflecting on their work was progressing. To begin with, however, some of the teachers questioned the means I advised for explaining and analysing their practice.

9.2.1 Interlude between workshops

While preparing for the third workshop, I received an email from three of the teachers. They wrote that they were sitting together and reflecting on how they could better explain their work. They referred to the protocol for case and commentary writing that I gave them at the second workshop and wrote: "We worked with measurements and gave you a folder that we believe answers all your questions in step 2".

I replied to their email and emphasised that they needed to attend in more detail to what they had learned from the lesson they referred to, since they had only written a summary of how they presented the task and of their pupils' findings. I added that I would like to see them discuss how they attended to individual children and also what they would learn about themselves in this process.

The next day I received another email from these same teachers. They wrote that they felt that they, as teachers with wide-ranging experience, were capable of assessing their work without writing in detail about how

they succeed and what they can improve. They also questioned the goal with this report, in whose favour it was, and where the data would be stored. They added that a long time had passed since they worked with their pupils on this project and it was difficult to write in detail about things that happened more than a month ago.

In my reply to them, I said that I was aware of the fact that in their work they always reflect on and attend to the needs of individual children. I then explained that the goal of the task was to urge them to reflect on individual cases, supporting them in analysing what they have learned about their work with their pupils. I added that if we even wrote only a few notes, then we could read them later and discover what has changed concerning ourselves, and our pupils.

I further underscored that I probably did not make it clear enough that the writings are intended for them and they did not need to give me a copy. My purpose was to encourage them to reflect on their teaching and I offered them the tools in order to support them in achieving that end. Of course, when reflecting on our work we often go through these steps even if we do not write anything. What we gain from writing about things that happen is that it provides an opportunity to return to them, which helps us remember what happened.

In this second mail I apologised for responding to them like a teacher who is guiding her students and I underlined that it was not my intention to criticise their work. I urged them to write their own notes and keep them for themselves. We would continue discussing our teaching at the workshops.

This instance was the first confrontation in our collaborative work and I decided to start the following workshop by discussing the goal for writing cases from our classrooms and the process of analysing them.

9.2.2 Workshop 3: Teacher reflections

The protocol for Workshop 3 was based on: Conclusions from the first two workshops where teachers expressed their wish to learn more about diverse ways to encourage discussions in their classrooms. Also on the recurrent comments on pupils' lack of knowledge. The email from the three teachers indicated that we needed to negotiate further our collaborative approaches.

9.2.2.1 The goal of writing about and analysing own work

I opened the discussion in the workshop by clarifying the purpose of writing about one's experience and stressed the point that when the teachers write about their work they write for themselves:

Write what you want to write yourself. Do not write for me. ... There is so much we reflect on, we talk about it and we remember some of the instances because they are memorable. Other things we forget, often something that would have been good to examine further.

Gróa said that she would have liked to know beforehand that she was supposed to write about her work and "have all the instructions beforehand and follow them throughout the process". She said that it was a part of teachers' daily routine to think about what they had done and what they might do differently next time even though they did not write about it. The other teachers did not comment on her response and the discussion moved to their pupils work.

9.2.2.2 Stories from teachers' classrooms – collaboration

The teachers were eager to talk about their experiences from their classrooms and responded enthusiastically to my invitation to discuss and reflect together on their teaching. I will only account briefly for some of their stories here and refer to the teacher's narratives in chapter 10 where they are discussed in more detail.

Pála from Rainbow School wanted to add to her story from our earlier workshop. She read the book chapter I had given them at the second workshop about co-operative learning, and she discovered that the idea of co-operative learning, as well as control theory and restorative philosophy (Gossen, 1998) underpinned the program used at her school for teaching pupils to solve conflicts, also based on. As a way to teach their pupils how to tackle disagreement, the teachers assigned group discussion where the children adopted certain roles and assumed positions of responsibility in relation to each group. We discussed why it was important for their pupils to learn to take responsibility for their work and support each other to reach a conclusion and how this approach could support them in learning mathematics. Teachers in both schools had tried group work with their pupils and could share their experience about this practice, even though they had not yet emphasised collaboration in their mathematics lessons.

9.2.2.3 Stories from teachers' classrooms – hands-on materials

Inga, from Rainbow School, was keen to tell us about her experience of working with children who she had been identified as having difficulties with mathematics, on the problem about the proportional weight of African animals, as we discussed at the second workshop. Inga's pupils found it difficult to understand what the problem was about and she was concerned about their lack of initiative in solving problems. She had urged them to use manipulatives like counters to help them solve the problem. We discussed whether hands-on materials could be helpful when solving mathematical problems. The teachers found it important to have access to such models and Dóra said that she had made sure that manipulatives were accessible at Rainbow School. Edda, who was new at Sunshine School, said that she had asked for more access to such materials.

Edda told us about her experience of posing the problem, about dogs and biscuits, to her class, that we had solved at the first workshop. She had encouraged her pupils to draw the problem in order to help them visualise the situation but they were hesitant to draw. Rúna and Vala had worked with this same problem with the children in Edda's class who came to the support centre. They, in contrast to their classmates in Edda's group, were eager to draw.

Rúna: They do not know the multiplication table, not completely, they therefore needed to use something else, they drew.

Vala: Therefore they made the drawings ...

Rúna: Made sets ...

Vala: It needed to add up, 6 and 5, and then finally there were 12 left. They could not make sets of 6 and 5 from 12 so there had to be two sets of 6.

Rúna showed us one picture with ten dogs eating from a bowl. There were four similar dogs that could easily be identified as Icelandic Sheepdogs and six larger dogs that looked like Labradors. She added: "What this boy dislikes most of all is calculating. He likes to show his findings by drawing".

9.2.2.4 Conclusions and looking ahead – children's initiative

My experience from exchanging emails with three of the teachers who were concerned about the goal with writing about their work, urged me to start the workshop with a discussion on the goal of the writing. I did not mention our email exchange to the group because I wanted to respect their privacy and I expected them to take it up if they wanted to discuss it with the other teachers.

The teachers had looked at what they wrote before and started to analyse their writing, and we proceeded by discussing these initial steps at the workshop. Some of them had read the articles I gave them at the second workshop and were inspired by what they read. The teachers in Rainbow School saw similarities between the ways they teach their pupils to take responsibility for their actions and the co-operative learning approach I had introduced to them. All the teachers seemed concerned about how to inspire their pupils to take initiative in solving problems.

Children's different mathematical knowledge and ways of calculating was current ongoing theme in our discussions. Children identified with difficulties in learning mathematics like to use manipulatives or draw pictures and other pupils want to be able to calculate quickly and are not interested in discussing their thinking.

The time passed quickly when we discussed the stories from the teachers' classrooms and I decided to offer them enough space for their discussions. My response to the three teachers' reluctance to write about their reflections was to give time at the workshops to discuss their mathematics teaching so we could proceed in supporting each other in analysing their work. We also needed to focus on ways to support children in taking initiative in solving problems. Additionally, I wanted to challenge them to bolster their learning by visiting each other's classrooms and reflect together on their experiences.

9.2.3 Workshop 4: Supporting children's initiative

The protocol for Workshop 4 was based on: Conclusions from the first three workshops where teachers expressed their wish to learn more about diverse ways to encourage discussions in their classrooms. The teachers' interest in discussing their own teaching is also attended to.

9.2.3.1 Mutual visits to the teachers' classrooms

I initiated discussion about mutual visits to each other's classrooms. The teachers said that it was difficult to find time to for the visits due to a lack of finance. The schools do not have the financial resources for an extra teacher to take over their teaching when they visit other teachers and they would therefore need to find time when there was an empty slot in their timetable. I talked about the purpose of such visits and mentioned that teachers plan such visits in different ways. I told them about lesson-study approaches and how teachers plan together beforehand and watch one teacher conduct the lesson according to their plan and then discuss afterwards what they learned and then improve the plan.

Edda was keen to visit the other teachers and thought that she could learn from them but was doubtful that the other teachers would learn anything from observing her. Gróa did not see the process as meaningful and said that she would rather like to see teachers' plans and the product of their pupils' work. Inga agreed that she would probably learn more from discussing with teachers rather than observing them teach, and Vala added that it would be helpful to get ideas about projects to work with. Pála said that following the class the first 10 minutes could be helpful, but she did not see the significance of following a whole session.

9.2.3.2 Introducing Concept Cartoons

At our former workshops, the teachers often mentioned that their pupils found it difficult to describe their problem solving processes. To open up discussions about problems, I decided to introduce concept cartoons, developed by Dabell, Keogh & Naylor (2000), which are designed to provoke discussion and stimulate thinking. The cartoons show pictures of children who have solved a problem and each child's solution to a problem is shown in a speech bubble. The goal with showing the cartoons to children is to encourage them to justify their own ideas and defend their points against other ideas and interpretations.

We discussed a problem about a tower of cans. There was a picture of four rows of cans with four cans on the bottom row and subsequently one less at each row. The question was how many cans were needed to build a tower of 10 rows.

The five children on the picture each explained how many cans they thought were needed, 34, 40, 55 and 100 respectively, and one child had no answer. We discussed how we solved the problem.

Pála: There would be 10 here [points to the bottom row of the 10 storey tower she drew]. Then I would count 9 and 1, 8 and 2, 7 and 3, 6 and 4. Then I have 10, 20, 30, 40, 50 and then add these 5 [points to her drawing for each step] and have got 55. I do this to be quick at counting.

Edda: They [refers to her pupils] often pair like this when they count.

Inga: My pupils would count.

Gróa: They would draw. I have two children who would do this without difficulties. The others would count cubes or draw, otherwise they would get lost.

We discussed further the solutions the children on the picture had found and the teachers reflected further on how their pupils would solve these problems. I then decided to direct the discussion towards the structure of the task.

Jónína: What is different with posing the problem this way than asking the children how many cans are needed to build a 10 storey tower?

Dóra: Well, the discussion is then missing, yes ...

Inga: They work out the different solutions. One could try 40 cans, another 55 and so on. They need to try ...

Pála: It is probably different, what they will do, how reasonable they are. Will they work out each solution, try the first proposal first and so on? Or will they try to solve it themselves and then say what is the right answer?

Inga: I think many would search for the right answer, work towards the right answer ...

Gróa: Yes, I think so. They verify the answer. My pupils would start with 40.

Inga: There are so many that doubt that they can solve problems. And here they get some clues to start with.

Inga said that her pupils would gain from working with the concept cartoons “because they are so afraid of discussing their own ideas”. Here they can read what other children proposed and then try to find out if their answer is right and discuss what those children were thinking. It supports you in explaining your own thinking if you can start with trying to interpret what others thought.

9.2.3.3 Stories from teachers’ classrooms – children write problems

Edda told us about a project where the teaching of mathematics and Icelandic was integrated. She showed us a book that she had made of her pupils’ story problems. She copied their stories and then each of them got a copy to take home in order to solve each other’s problems. We discussed their work and what could be gained from creating your own problems and then solving the problems your classmates made.

9.2.3.4 Conclusions and looking ahead – listen to each other

I gave the teachers a copy of two new articles from “Flatarmál” (journal for the Icelandic mathematics association). Dóra is a member of the association and therefore receives a copy of their journal. I urged the other teachers to join the association and suggested they check whether the journal is accessible at their school library.

The conclusions I drew from discussing mutual visits to each other’s classes were that the teachers had difficulties in seeing what they could learn from observing each other’s classrooms and discussing their visits. They were more focused on searching for ideas about what to work with and how to structure their teaching than observing the learning that was taking place in their classrooms. After the workshop, I wrote guidelines that

I thought might be helpful for the teachers and emailed them to them (Appendix D).

To support the teachers in focusing on how they themselves and other people approach mathematical tasks, I decided that at the next workshop we would pay attention to how we listen to each other when we solve problems.

9.2.4 Workshop 5: Learning to listen

The protocol for Workshop 5 was based on: Conclusions from the fourth workshop about supporting children's initiative were that the teachers could not envisage beforehand what they could learn from observing each other's classrooms and discussing their visits. The focus was therefore on how they themselves and other people approach mathematical tasks. I decided that at the next workshop we would pay attention to how to listen to each other when solving problems.

9.2.4.1 Teachers solve problem and discuss their solutions

To begin with, the teachers worked in two groups in two separate rooms. Each group solved one problem and the teachers discussed their solution strategies. Dóra, Edda and Inga worked together and discussed how to find all possible pairs of two digit numbers, whose product is 360 and 265 respectively.

Dóra: We can start with 10 times 36.

Edda: Two digit numbers, yes ...

Dóra: Twelve ...

Edda: Times this number makes 360, some number times this number.

Dóra: Three times 12 make 36, it is 30.

Edda: Can 13 be ...

Inga: No, it does not work ...

Dóra: All possible pairs?

Jónína: Yes, you have found two pairs, are there more?

Inga: I find this very difficult.

Edda: If we turn this around.

Inga: Do we proceed multiplying until we find all ...?

Edda: I found 20x18 or 18x20, can we find more?

Dóra: 10, 12, 15, 18, 20, 24, 26 no, 28 no, 30, 32 no, 34 no, 36

Edda: These are the pairs [points to her writing: 10x36, 12x30, 15x24, 18x20].

They then tried to find two digit numbers that give the product of 265 and found that 5x53 equals 265. Dóra said that 53 is a prime number and

therefore the only numbers that worked with 5 and 53 and said: “Five is not a two digit number so there is no solution to this problem”.

In another room, Gróa, Pála, Rúna and Vala explored relationships among factors, divisors, and remainders, with the goal of discovering patterns. Initially, they had difficulties with understanding the problem and criticised how the text was written. It took them a long time to solve it and they then reflected on their way of finding the target number. They knew that they were going to support their colleagues in solving this problem afterwards and wanted to be sure that they themselves had understood the process of finding the target number of 138.

Rúna: We decided to use the number 7 ...

Pála: Yes we decided to use the number 7 and always add 5.

Rúna: Maybe we could have found a quicker way.

Vala: Maybe there is an easier way.

Pála: The thing is, girls, that it needed to be a number with the last digit either 3 or 8, because there are 3 left when you divide by 5. It couldn't be 3 because when you divide by 4 there are 2 left, the last digit needed to be 8.

Gróa: Yes, so we realise when we have found it. Then ...

Pála: We realise it when we have gone through all the steps ...

Gróa: We needed to have started with, what would be the last digit in the target number, when we think of what is left ...

Pála: Yes, 2, 3 and 5. It must be a number, 5, 10, 15, you always add 3, therefore it is 8, 13, 18, 23. And then it is always an even number. It must be 8. If we had thought of that before we had solved the problem more quickly.

9.2.4.2 *Teachers work in pairs*

The two groups then met and worked in pairs. One teacher from each group joined one teacher from the other group. They solved each other's problems and supported their peer by asking questions. In this process they took on a role as either a learner or a teacher. When in the role of a teacher they urged the learner to talk aloud so they could follow their thinking, asked probing questions and were careful not to give any clues or describe how to solve the problem. I gave the teachers a copy of guidelines they could follow in this process (Appendix D).

Dóra posed the problem to Rúna of finding all possible pairs of two digit numbers whose product is 360 and they proceeded with the discussion until Rúna had found all the pairs. Finally Dóra asked her if she found that her initial approach to solving the problem was helpful and Rúna said that in fact it was and she felt confident in using this approach.

9.2.4.3 Collective discussion about the experience of this workshop

The teachers felt that this experience of solving numerical problems was fruitful, particularly the opportunity to discuss their own way of solving these problems, before supporting another person in solving it. They found the guidelines useful and reminded them not to be too quick to interfere with the learner's process.

Gróa said that these problems would be too difficult for her 6th graders, but she believed that the problems might be better suited for the following year's work when the pupils learned about factorising numbers. Inga said that she would like to learn more about factorising and we discussed how the approach could be useful in finding the pairs of two digit numbers that give the product of 360 and 265 respectively. Similarly, we discussed the steps the teachers took in finding the target number 138 and how the different clues given supported them in finding the number. The teachers said that they were not familiar with such problems and therefore not focused on how each of the clues could aid them in finding the target number. In sum, they found this workshop challenging and a valuable learning opportunity.

9.2.4.4 Conclusions and looking ahead – observe teaching

The teachers were making arrangements for visiting each other. I urged them to make a plan for their visits and decide beforehand what to focus on, and reminded them about the guidelines I had sent to them.

From the experience gained at this workshop I concluded that the teachers had learned from discussing their own thinking about the problems. They found it important to reflect on their learning, as emphasised by Pála when she summarised how she and her colleagues had worked when finding the target number of 138. Edda also pointed this out as she described the pairs of two digit numbers that give the product of 360.

While the teachers were solving the problems in the initial groups, they initiated a discussion about their own teaching. They were working in two different rooms and I moved between these rooms and did not hear these stories until I looked at the videos the day after the workshop. Their initiative to discuss the learning in their classrooms indicated that they were taking a step towards closer collaboration and that we were building trust in our learning community. These stories are discussed in Chapter 10.

9.2.5 Workshop 6: Mutual visits to classrooms

In my email to the teachers two days before Workshop 6 reminded them that we were going to discuss their observations in each other's classrooms and that the workshop was devoted to discussions about their visits.

The protocol for workshop 5 was based on: Conclusions from the fourth workshop where the teachers struggled with envisioning what they could learn from observing each other's classrooms and discuss their visits. The experience gained from Workshop 5 gave promising signs for the teachers' growing focus on mathematics learning and ways to support it. I expected that we would need to devote the whole workshop to discuss the teachers' mutual visits.

9.2.5.1 Visits to classrooms – learning to prepare for visits

In Rainbow School, all the teachers had observed each other's classrooms. Dóra teaches the blue group (children who had been identified as being 'strong' at learning mathematics). She had chosen a difficult problem to solve and the visiting teachers, Inga and Pála, had not seen the problem before. Pála said:

We had not seen the problem before and therefore could not reflect on where they could possibly be stuck. Where they needed support.

The children worked in groups and Dóra had told them that they needed to describe to their group mates how they solved the problem. Pála added that the children reminded each other of this fact and that helped her and Inga learn about how they were thinking when they solved the problem. After this first visit to Dóra's class, the three teachers had discussed what they learned from this visit, and they drew on this experience when Pála and Inga prepared for their lessons.

Inga teaches the red group (children who had been identified as having difficulties with mathematics). She decided to pair what she calls more able children together and in turn less able together. Inga was concerned that children with low expectations of their capabilities to learn mathematics would be passive when working with classmates that are quick to find solutions to problems. Inga reflected on her experience:

I think it is good when they are beginning to learn to work in groups. There they were two at the same level. They then come stronger into the group when they have found out that they are capable of doing something, instead of always being the one who is inactive. It is so difficult to take the first step.

Inga had chosen an easier problem to work with than the children in the blue group had solved and Dóra and Pála said that the children had tried to do their best and that they enjoyed themselves. Inga was satisfied with this lesson, she had seen a positive change, the children all worked hard and she had never heard them talk so much about mathematics before. We discussed whether the presence of Dóra and Pála might have had a positive influence on their engagement. They are their classroom teachers, and had time to discuss with individual groups. The teachers agreed that their presence might have affected their willingness to work with the problems, and Inga added: “Óli adores Dóra, says that she is the best teacher in the world”. Dóra responded by saying that she had not seen him so confident in his work with mathematics before.

Pála teaches the green group (the children who had been identified as less able than those in the blue group but more able than the children in the red group). She had benefited from visiting the other classes and decided to tell the children that one person in the pair would need to read the problem aloud, which they would then have to discuss. The children were active and the teachers could discuss with individual groups and expressed more about their ways of solving the problems than they did when they discussed their former visits. Inga and Pála told us how one pair approached the problem they were working on:

Inga: They went back and forth. She looked into the air, was thinking, and he started to explain to her. He then asked her what she was thinking and then they started to make progress.

Pála: Yes, and when he started to explain to her she could add to his explanation. She was so happy with their solution and she has discussed it with me every day since.

Pála said that the children were not used to solving problems together and they were learning to collaborate. Dóra said: “I do this more often, they are not negative, they start right away to try”. Inga added that she does this step by step: “First, when I started with the problems, it was just, they never came up with anything”.

The teachers had discussed many things after their visits to each other’s classrooms and Pála said:

Yes, I felt we discussed this, how we grouped the pupils and how we are reflecting on each and everyone’s learning. How we can maximise their work, activate them. And how we have succeeded. We have seen before that some of them cannot work together and now have found which individuals can work together.

The teachers in Rainbow School were all satisfied with their mutual visits and claimed that they would like to visit each other again and prepare themselves better for these visits.

9.2.5.2 Visits to classrooms – opportunity to discuss with pupils

In Sunshine School, the teachers also had paid mutual visits to each other's classrooms. Edda observed Gróa's classroom the day before we met and told us about her experience. Gróa discussed results from an assignment with her pupils.

Edda: She asked them to take turns in telling the answer, and it worked out well.

Gróa: They have difficulties with reading numbers and need to practice.

Edda: They read the numbers and then you reviewed how to solve these problems.

Gróa: Yes, they asked me to, wanted me to repeat. I have been emphasising multiplication and division now in January and February.

Gróa asked the children to help her with the processes of carrying out the algorithms as she wrote on the whiteboard and tell her what they would do. What the children had found most difficult was to carry out the procedure of finding the product of two digit numbers and she asked them questions to help them clarify their thinking. Edda added that she would have learned more from observing the lesson if the children were working with problems in groups since she would then have been able to discuss with them.

Edda commented on Gróa's visit to her classroom and said: "She mentioned a lot of things that I did not notice". We discussed how useful it was to have a visitor who notices things that the teacher misses. The children were making fraction models out of large cardboard circles to put on display in their classrooms. They then worked with fractional problems in their textbooks. Edda found the structure of the fraction chapter confusing and we discussed what might have been the reason behind the structure of the chapter and what the children would learn from solving these tasks. I pointed out that to solve the problems the children were urged to use different kinds of fractional models and not only area models like the ones Edda's children had made.

In Sunshine School, Rúna and Vala worked together in the support centre. They did not have time to visit each other's classrooms but wanted to discuss a lesson where they observed each other's teaching. They worked with fractions with a group of six children from Edda's class. Vala described how Rúna had played with fractions with the children and how

they had found that it supported them in understanding the fraction concept. Rúna asked how many children in their group had blue eyes, long hair, and so on, and how large a proportion they were of the whole. She drew a circular cake on the whiteboard, divided it into 12 parts and asked how many parts made on third of the cake. Rúna and Vala urged the children to look around in their classrooms, some counted the windows and calculated the proportion of a whole two windows in relation to the wall and then three windows. Others counted the lights on the ceiling and found how large a proportion the lights in each row represented to the whole.

I pointed out that it was interesting to learn how Vala and Rúna had worked with both set models and area models. I used this opportunity to discuss different kinds of fractional models and how children develop understanding of the fraction concept when they explore with fractions in diverse ways. Edda found the textbook she had referred to in her discussion about fractions earlier and we explored together how the children were urged to use three types of fractional models, length, area, and set models to make sense of relational size of fractions. The teachers had not looked at the teacher guides that come with the textbook and the proposals for fraction models that could be used when teaching about fractions.

9.2.5.3 Conclusions and looking ahead

The teachers in Rainbow School were all eager to visit each other again. I encouraged them to continue using means for professional practice such as recording their lessons. They could start with recording audio and the next step could then be to video-record their lessons. I asked the teachers if they would like to discuss fractions next time and mentioned that we had neither worked with fractions nor algebra. Pála wanted to learn more about both but we decided to focus on fractions at our next workshop.

9.2.6 Discussion of findings: Reflective practice, hindrances and opportunities

The teachers had shown interest in discussing their own teaching and shared some stories with us at the workshops. When I urged them to write their stories, and analyse what they had noticed about their teaching, some of them were reluctant to do so. I responded by giving them space at the workshops to reflect on and discuss their learning in their own classrooms. The conflicting views of the means to support reflection into one's own practices that had occurred at the first workshop thus recurred, and this time, were dominant. My response was to create space at the workshops for the teachers to reflect on and discuss their classroom learning.

9.2.6.1 Hindrances to reflective practices

The emphasis I placed on reflective practice in relation to one's own teaching and the means for engaging in such practices, challenged the community I had envisioned. The boundaries between our communities, the one I offered them to partake and those that the teachers engage in at their schools seemed to be restricting the possibility for our community to grow (Wenger-Trayner & Wenger-Trayner, 2015). For instance, when the three teachers sent me an email (see Section 9.2.1) and told me that they had already answered the questions I asked them to respond to. My enthusiasm to engage the teachers in practices that I believed were vital in developing their own practice resulted in an authoritative response, which then led to a certain dissonance between our communities. I was too quick to step into the role as a teacher who responds to students' concerns, forgetting that these teachers were not my students and their alignment to their own community of teachers might be stronger than their willingness to engage with what I offered them.

When we met at the following workshop (see Section 9.3.2), I was still in the teachers' role and emphasised the purpose of the task I had given them. I did not mention our exchange of emails at the workshop, as I did not know if these teachers were confident in discussing the issue with the other teachers. They did not initiate a discussion in relation to them, and Gróa only confirmed that she would have wanted the detailed information that I gave them at the second workshop when they wrote their first version of their case. My response to this confrontation was to suggest that the teachers would keep their writings for themselves as well as all other data they collected. In the words of Wenger (1998) and Wenger-Trayner & Wenger-Trayner (2015), I was aligning myself to these teachers' perspectives and their exposed reluctance to aligning to the collaborative milieu I wished to create. The confusion that arose at the boundaries between our communities can also be interpreted as constraints within our zones of free movement, as theorised by Valsiner (1997). The teachers' zone of free movement was constrained by the requirements to write about their analysis of their case and my zone in turn was restricted by their response to my request.

9.2.6.2 Reflecting on tools for supporting learning

The teachers' enthusiasm to discuss their teaching was apparent at the following workshops and they took the initiative and brought in stories from their classrooms for us to reflect on together. My determination to respond to the confrontation with three of the teachers by giving them

enough space to feel confident resulted in them sharing their concerns with us. I set aside the work that I had planned for this workshop and postponed it to the next one.

To respond to children's difficulties with solving problems, Inga emphasised using hand-on material in order to appropriately model the situation represented in the problem and to clarify our understanding of the number system (see 9.3.2.3) Edda was concerned that she did not succeed in motivating her pupils' interest in solving problems and Dóra wanted her pupils to become better at explaining their thinking when solving problems. These concerns reflected their awareness of the diverse tools needed to support learning, as theorised by Vygotsky (1999), and of their role in scaffolding (Bruner, 1985) their pupils' mathematics learning.

The nature of the mathematical tasks that the teachers and their pupils worked with, allowed them to direct the focus in their mathematics classrooms in more meaningful ways. Pála addressed this point when she said that the reason as to why the African animals problem was difficult for her pupils was because they did not know where to start, which emphasises the investigative nature (Jaworski, 1994) of the task.

When we explored tasks at the workshops, the teachers were eager to discuss their thinking about them. Inga often expressed her wish to learn more about the mathematics we were exploring with, voicing that she felt she was badly prepared for teaching mathematics. We were all willing to support her and in that way our learning community was gradually developing as we shared our thinking about our understanding of the problems and ways to solve them, which Askew (2005) claims characterises an inclusive learning community.

9.2.6.3 Reflecting on mutual visits to classrooms

When the teachers observed each other's classrooms (see 9.2.5) they found that if they prepared themselves for these visits they gained more from participating in the classes than if they did not. At Rainbow School they learned from their first observation in Dóra's class and focused more on the pupils collaboration in their next observations. They met after each visit and reflected on what they had noticed during these visits, which is an activity that, according to teachers need to make a habit of (Mason, 2002; 2011). In Sunshine School, the teachers neither prepared themselves for the visits nor met to discuss them afterward. During the observations, their focus was more on how they organised the classroom experiences than on the children's learning. Edda said that she wished that they had prepared her visit to Gróa's classroom better and so that she could have followed the

children's work rather than observing Gróa explain procedures for solving multiplication and division tasks. The teachers in Rainbow School felt that they had learned from these visits and wanted to pay visits to each other classrooms again. Their response was a sign of that they experienced that their competences in teaching mathematics developed through such collaboration (Niss & Højgaard-Jensen, 2002).

Before the classroom visits, some of the teachers found it hard to imagine that they would learn much from observing each other teaching. In case of the teachers in Sunshine School, their assumptions might have been confirmed, since they did not dedicate themselves seriously to engaging with the activity proposed for improving their practice. This though was not the case with Edda, who had expected that she would learn from these visits and was disappointed that they did not prepare them. There were also other constraints to the teachers' possibilities for paying these visits, for instance, financial issues preventing the school from hiring a replacement teacher. Their zone of free movement (Goos, 2005; 2008) was thus restricted.

9.2.6.4 Constraints and opportunities

The boundaries between our communities of practice were disrupted, as some of the teachers did not find it meaningful to use the tools for inquiring into their own practice, in the ways proposed by myself, the teacher educator. In their discussions about their teaching, their focus more on organising pupils' learning than their learning of mathematics. The teachers' need to focus on their own understanding of mathematics, as a means for understanding children's diverse ways of thinking about mathematics, was apparent. An opportunity presented itself when Pála expressed that she would like to learn more about fractions and algebra, and the attention was therefore directed toward these areas at the two following workshops. The developmental cycle thus affected the local theories and the need to centre on one's own exploration with mathematics and discussing mathematics learning.

9.3 Theme 3: A focus on interactions in mathematics classrooms

The original plan for the project was coming to an end and the teachers were gradually beginning to focus on their own understanding of the mathematical content they were teaching and to take on an investigative approach to learning. The conflicts that arose at the first workshops

concerning means for researching into one's own practice were settled. The teachers had paid visits to each other's classrooms and were content with their experience. The next step was to challenge them to record their lessons as well as to explore more using tasks and problems. The findings from the two remaining workshop of the first year are presented in this section. The categories that were traced at this time centred around classroom interactions such as shaping the cultures in the classroom, focusing on understanding, children's possibilities to participate in discussions and the shaping of our community.

9.3.1 Workshop 7: Teaching and learning of fractions

At Workshop 6, the teachers' confusion about teaching fractions and tools for supporting the understanding of fractions became apparent. Pála expressed a wish to focus on the teaching of fractions.

The protocol for Workshop 7 was based on: Conclusions from Workshop 6 where teachers showed interest in learning more about how to support children in understanding of fractions and proportions. The teachers' interest in discussing their own teaching was also attended to.

9.3.1.1 Stories from teachers' classrooms – using teacher guides

Dóra wanted to talk about her experience of a measurement project with her pupils. She had always found it difficult to teach the measurement of area but when she read the teacher guide that came with the textbook she discovered suggestions for how she could support her pupils in developing understanding of area. Dóra's reflections on her experience of this project are discussed further in Section 10.3.

9.3.1.2 Fractions and number lines

At Workshop 6, Edda raised concerns about the teaching of fractions and through our discussions I learned that the teachers were not familiar with different models for fractions. They used circular area models to represent fractional parts and did not seem to be aware that the number line could be helpful in making sense of the relational size of fractions. I proposed that the teachers wrote fractional parts on 'sticky labels', ordering them from the lowest number to the highest, and I made a number line to put on display at the wall.

Edda: $\frac{8}{9}$ is larger than $\frac{7}{8}$.

Vala: We need to think about the nominator and denominator. The difference between $\frac{8}{9}$ and $\frac{9}{8}$.

Edda: Where do we put $\frac{2}{3}$?

Jónína: [hands Edda a sheet with number lines divided into equal parts, halves, third, fourths, etc. respectively] Would you like to use fraction strips?

Edda: This is helpful.

Gróa: Only half of the pupils in my group would be able to order these fractions.

Vala: This is smart, to experiment with ordering the fractions this way.

Edda: [to Vala] We can do this next week.

Gróa: Why did we not think of this when we taught fractions in the fall? Then we made the cake. It is used a lot.

Vala: This is so plain and visual.

The teachers all engaged with ordering the fractions and discussed problems that their pupils had encountered when using the circular area model. We reflected on what could be gained from ordering fractions on a number line and how it could support the children's understanding of the fraction concept to discern between numbers like $\frac{3}{4}$ and $\frac{2}{3}$. Gróa asked in an ironical tone if I was going to test them on their knowledge of fractions. She added that one could detect how quick they were at ordering fractions with a minute test like they use when assessing their pupils' knowledge of multiplication facts. Edda said that she does not find the minute tests helpful.

9.3.1.3 Solving fractional problems

During the latter part of the workshops, the teachers solved a problem about sharing sandwiches.

At lunchtime Gunnar, Gylfi and Siggí shared 8 sandwiches equally between them. Gylfi contributed 5 sandwiches and Gunnar 3. Siggí had not brought any sandwich. After lunch Siggí wanted to pay 80 kr. for his share. Gylfi said that it was fair that he got 50 kr. and Gunnar 30 kr. Gunnar wanted to share the money equally between them. Siggí proposed that Gylfi would get 70 kr. and Gunnar 10 kr. Who proposed a fair solution?

Pála: I think it is Gylfi.

Jónína: [to the group] Do you agree?

Vala: At first sight.

Gróa: One contributes 3 and the other 5.

Dóra: They all eat equally much.

Pála: Each of them eats $2\frac{2}{3}$ of a sandwich

Jónína: How much does Gylfi offer, and how much does Gunnar offer?

Pála: It is just $5 - 2\frac{2}{3}$. Gylfi gives $2\frac{1}{3}$

Dóra: Gunnar gives $\frac{1}{3}$. This is smart. Siggi was right. Gylfi gives 7 times as much as Gunnar.

Gróa was sceptical about teaching fractions through problem solving and said that her pupils would not understand these problems. Edda replied that children needed to learn to solve problems and not always depend on their teacher to explain how to approach problems. Dóra and Pála both voiced their belief that if children worked together they could support each other in this process like they had been doing at our workshops.

9.3.1.4 Discussing fractional models and the classrooms cultures

Drawing on the experience gained from Workshop 6, where I learned that the teachers were not accustomed to use different kinds of fractional models and did not understand the way fractions were introduced in the textbook they were using, I decided to discuss the learning of fractions in more depth. I brought with me some slides about the development of understanding of fractions that I had used with my student teachers. We discussed different kinds of fractional models such as length or measurement models like the number line, set models such a counters, and area or region models like the circular models.

We also discussed how the teacher shapes the culture in the mathematics classroom and develops norms for collaboration and sharing discussions and findings. During this discussion, the teachers reflected on and shared their thinking about their teaching with us as we discussed the core features of the mathematics classroom.

Jónína: The teacher shapes the culture in the classroom. We model how we communicate in the classroom. How we discuss our findings.

Dóra: It is important to send the message to them that we learn from doing mistakes and they must not be afraid of telling others about their thinking.

Vala: It is better to be able to give some kind of an explanation, than not dare to try.

Jónína: It is a matter of building trust and confidence in the classroom, to urge the children to share their thinking.

Gróa: I am often too impulsive. Too quick to respond negatively when they humiliate each other and then I have bad conscience.

Jónína: We are different, how we talk to our pupils and tackle such problems. But we are responsible for guiding the discussion, shaping the culture. Sending the message that we are all here to learn and that we learn together. Some are quicker than others to solve problems and we need to respect that we do not all go through the same learning path.

9.3.1.5 *Conclusions and looking ahead – reflection on our learning*

The teachers concluded that they found this workshop valuable and they had learned about aspects of fractional learning that they had not been aware of before. Pála looked forward to learning more about the teaching of algebra at our next workshop as she had proposed at our previous workshop. I concluded that we needed to keep on focusing on ways to support children in learning mathematics, particularly in response to Pála's wish to discuss algebra.

I urged the teachers to audiotape their mathematics lessons and tell us about their experience next time, and Pála wanted to give it a try. I emphasised that they would listen to their recordings themselves and then tell us about their experience. I also mentioned that the following workshop would be the last one and as such we needed to reflect on our collaborative experience.

9.3.2 Workshop 8: Focus on the learning of algebra

The protocol for Workshop 8 was based on: Conclusions from Workshop 6 and 7 where teachers showed interest in learning about algebra and children's development in learning mathematics. Focus on diversity and ways to support all children in learning mathematics.

9.3.2.1 *Stories from teachers' classrooms*

Pála had audiotaped one mathematics lesson and told us about her experience. She found it difficult to listen to herself and was critical of her own performance. She discovered that she talked most of the time and whereas the children listened. We discussed the purpose of recording one's own teaching and what one can learn from listening to the recordings. I pointed out that the teacher could learn much from her recordings of herself and how she communicates with her pupils. Of equal importance is what you can learn about individual pupil's learning, things that you do not notice because you are so occupied with conducting the teaching. Rúna commented: "You need to have eyes all around". Pála added that she would like to record her lessons later and we decided to focus on recordings next year.

Edda, Rúna and Vala gave me a copy of a book with story-problems that the 5th graders in Sunshine School had made and Vala said that the wording was not always perfect because many of the children are newcomers in Iceland and were yet to learn the language. I asked if discussing mathematics and writing about it could be an effective way to

learn a new language. Dóra said that she uses the language of mathematics to communicate with children who do not speak Icelandic and added:

One girl did not understand anything and started crying. Then I talked to her with mathematical symbols. That is how we made contact and developed mutual trust.

9.3.2.2 *Teachers solve a problem about a growing pattern*

The teachers explored the process of developing a growing pattern with black and white tiles. First they were asked to extend the pattern and then to find a general formula that could be used to predict the number of black and white tiles in a pattern of any size. I brought with me squares of two colours and Inga and Rúna used them to make the pattern. The other teachers started to draw the pattern on a square paper.

The first steps were easy and the teachers could count the numbers of tiles in each step but found it hard to develop a formula. I urged them to draw a table to aid them in finding relationships between the number of tiles in every step, both the total number and for each of the two colours. We then discussed how the formula could be developed through looking at the relationship between the numbers.

Pála: It grows by, 1, 2, 3 ... we have found that there are 55 black tiles in step 10.

Jónína: How many would there be in step 11?

Dóra: They would be 66.

Edda: Yes, because we knew how many were in the previous step. But I do not know, I only know how many are added in each step. We do not know the total.

Jónína: Pála, you said there were $1+2+3$...

Pála: Did I say that?

Jónína: You said that the pattern grows by 1, 2, 3 ...

Pála: Yes, $+3+4$...

Edda: Can we not simplify this?

Jónína: If you look at the numbers you have found, how many there are in each step [writes on the whiteboard] 1, 3, 6, 10, 15, how many would there be in step n ?

Edda: Tell us the formula! [ironical tone]

Jónína: [writes on the whiteboard] $1+2+3+ \dots +(n-1)+n$.

I reminded the teachers of how Pála had calculated when she solved the problem about a tower of cans at Workshop 4 and told us that her pupils

would count first 10, and then add 9 and 1, 8 and 2, and so on. Dóra replied that she knew there was a formula that could be used to calculate these numbers and she had heard a story of Gauss. By attending to Pála's way of solving the task of the tower of cans and Dóra's former experience of calculating these numbers, we found that the formula that could be used to find the number of black tiles in every step of the pattern, $\frac{n(n+1)}{2}$.

I asked the teachers if their pupils could solve the first steps of this problem and if they thought that any of them could find the general rule that we had developed together. My intention was to draw their attention to how people can work together at solving the same problem even though they were not all able to work through all the steps. In other words, everyone will learn something from approaching the problem even though the level of learning may differ with each pupil. Dóra noted that there was one girl in her class who could probably work through all these steps by herself and she liked to grapple with challenging problems.

We also discussed how using hand-on materials and drawings could support children in solving the problem and how making tables and writing information that one collects can help finding relationships between the numbers of tiles in each row. Inga believed that her pupils would like to use squares like those she and Rúna had used which might help them structure their thinking about this process.

The last part of the task was to come up with one's own tiling pattern. We discussed what could be gained from making one's own patterns.

Jónína: What is the difference between this last part of the problem and the first part?

Pála: Is it not just the same?

Edda: They need to make a pattern themselves.

Jónína: They are urged to make a growing pattern that is different from this one. What do we need to do then? How is it different from investigating a pattern that someone else has made?

Edda: Their own imagination.

Inga: Keep on with the rule.

Edda: Find yourself how the pattern grows.

Jónína: You need to use your imagination and think logically. What do I need to do to let the pattern grow?

We concluded by discussing algebraic symbolism and the importance of developing understanding of how it is used to express generalisations and patterns. We discussed the equal sign and how children develop understanding of its meaning.

9.3.2.3 Looking into other teachers' classrooms

The next step was to watch a video of a 4th grade bilingual classroom exploring the meaning of the equal sign (from a CD-disk with video recordings accompanying the book *Thinking Mathematically: Integrating Arithmetic and Algebra in Elementary School*) (Carpenter, Franke, & Levi, 2003). We discussed how the teacher supported the children in developing their understanding of the equal sign when they discussed the equation $5+7=_+4$. The teachers were familiar with that notion that children believe that the sum always follows the equal sign, as did most of the children we observed. We discussed benchmarks for children's understanding of the equal sign that Carpenter and his co-authors (2003) present in their book and reflected on how the teacher we had observed used her knowledge of these benchmarks to guide her teaching.

The teachers noticed that this teacher was consistent in her use of mathematical concepts and urged her pupils to talk together in their mother tongue. We discussed how two languages, English and Spanish, were spoken in the classroom, how the teacher led the discussion in English and when the children responded in Spanish she replied in Spanish. Vala told us about her experience of working with children who did not have a good grasp of Icelandic language and how she and Rúna approach the mathematics with their pupils by focusing on symbols and the language of mathematics. The teachers are not familiar with all the languages spoken in their classes and we reflected on how parents could be supported in discussing mathematics with their children in their mother tongue.

9.3.2.4 Conclusions and looking ahead

We concluded this workshop by discussing the future. This was meant to be our last workshop, however, all the teachers wanted to continue the workshops in the next year. Pála said: "I wasn't aware that this was our last meeting. I will miss them and would like to continue if possible". We decided that we would meet at three workshops in the fall. I asked for their permission to observe each of their classrooms before school finished and to interview them about their experience of our research collaboration.

This was the first workshop in which we had focused on the diverse languages spoken in these two schools and the fact that the teachers had difficulties with communicating with some of their pupils. Vala pointed out that some of her pupils had difficulties with writing in Icelandic and when we watched the video from the bilingual classroom, the discussion proceeded on that topic. We were also gradually focusing more on how we

could create conditions in which all children can learn from participating in mathematical activities, where everyone learns something new but they do not necessarily learn the same.

In addition to focussing on the learning of algebra, my intention at this workshop was to direct the teachers' attention to ways for supporting children with different backgrounds and learning needs. The multi-layered task gives space for all pupils in the group to work together with the same problem even though they are not all capable of carrying out all the steps. By observing a teacher discuss the equal sign in two languages, I addressed how language barriers can be resolved.

In late May I visited all the teachers in their mathematics classes, except for Gróa, who had a leave at the time. The findings from these observations and interviews are discussed in Chapter 10.

9.3.3 Discussion of findings: A focus on interactions in mathematics classrooms

By the end of the first year, the teachers were starting to take the lead in discussing inquiry approaches, initiated by Dóra, and the other teachers, added to the discussion by bringing in examples from their classrooms. Diversity was also addressed and ways to include children with diverse backgrounds in the classroom community. The culture in our learning community was developing and we all added to it, and this was confirmed in the interviews with the teachers who clearly expressed that they felt they belonged to it. The focus on inquiring into own mathematics learning and diverse ways of learning mathematics had a positive effect on the teachers, which they found challenging and a benefit to their mathematical thinking. We therefore concluded that that the teachers would continue concentrating on their inquiry into mathematics.

I was content with the improvements we were making. When the teachers proposed that we should continue to meet in the workshops, I was pleased that they had initiated further collaboration. I knew that it takes time to embrace new norms and hoped that I would see more effects from our co-learning as time passed.

9.3.3.1 Fostering learning cultures in mathematics classrooms

During the two last workshops of the first year, the focus was steadily moving to discussions on how to create cultures in mathematics classrooms that foster meaningful ways of learning mathematics. Dóra initiated this discussion by telling us how she had explored the topic of measuring

different areas at the schoolyard with her pupils. She had read about an investigative approach in a teacher guide that comes with the textbook she used. She looked for information that supported her in developing her curriculum and teaching competency (Niss & Højgaard, 2011) and used them as a source for learning, as described in Ahl et al. (2015).

When we discussed the cultures we create in our classrooms, Dóra emphasised how we learn from making mistakes and telling others about our thinking. Vala affirmed this point and added that she finds it important to build trust and confidence in the classroom. They were both emphasising the features that Hiebert et al. (1997) and Askew (2015) claim characterised classroom communities, where sharing thinking is seen to promote mathematics learning. During this discussion, we shared our visions for meaningful learning in mathematics classrooms and the ways in which the teacher shapes the culture in our community (Wenger, 1998; Wenger-Trayner & Wenger-Trayner, 2015).

9.3.3.2 Instrumental learning and relational understanding

When Gróa expressed her scepticism towards teaching about fractions through problem solving, and asserting that her pupils would be dependent on her explanations, Edda responded by saying that she believes that children need to learn to be independent, and Dóra and Pála both supported her opinion. The discussion centred on pupils being assuming an accepting mode as opposed to taking on an asserting mode, as articulated by Mason and Johnston-Wilder (2006), who emphasise instrumental learning, on the one hand, and relational understanding (Skemp, 1976), on the other. Gróa's insistence on rote learning was also confirmed when she raised the question about testing and mentioned the minute test on multiplication tables. Edda critiqued her suggestion of the minute test, and voiced her conflict with criticisms of the established norms at her school on testing children on their capability to memorise facts (Jaworski, 2006a).

9.3.3.3 Multicultural education

Gróa expressed the view that her pupils had difficulties with understanding word problems because they do not understand the words. She was voicing concerns for her bilingual pupils not being able to participate effectively in the school mathematics, an issue addressed by Valero and her co-researchers (2008) had challenged, and who sought to frame language and school mathematics practices in terms of socio-cultural-political perspectives. When we discussed a book with story problems that the 5th graders in Sunshine School had written, Vala mentioned that newcomers in

Iceland had difficulties with wording their problems. She later added that she and Rúna had focused on mathematical language and symbols when communicating with these children. Dóra added that she uses mathematical symbols to talk to children with whom she cannot communicate with through spoken language, indicating that she makes an effort to find ways to include these children in the school culture (Valero, et al., 2008). Mathematical symbolism could be therefore figures as a potentially effective tool for communicating thought (Lerman, 2006b; Vygotsky, 1999).

9.3.3.4 Development of a community of co-learning

At Workshop 8, when we discussed the teachers' approach to a growing pattern problem, they were occupied with counting the tiles and did not start to look at relationships until I proposed that they should draw a table to help them look at relationships between the numbers. Still, they did not find how they could put forth a general rule for calculating the number of tiles in each row, despite them having adopted an asserting mode. This was apparent when I asked a probing question in relation to Pála's explorations, Dóra replied, and Edda followed their thinking by adding that even though they knew how many were added in each step they did not know the total. When I reminded them that Pála counted, 10, 9+1, 8+2, and so on, as they solved the problem about a 10 storey tower of cans in Workshop 4, Dóra related to her former knowledge about adding consecutive numbers, and together we all supported each other in developing the formula. Our community of co-learning was developing as we brought to bear the knowledge that was growing within our community, in conjunction to our personal knowledge (Askew, 2015; Lerman, 2000a; Vygotsky, 1978; Wenger, 1998). As we built on each other's input, the signs of collective learning arising through a mutual reflexive process of knowledge made apparent the solidifying bond between individuals and community (Jaworski, 2003).

9.4 Theme 4: Focussing on pupils' learning in the classroom

The teachers expressed that they would like to continue our workshops as the original plan for the project approached its end they were starting to focus on interactions within their classrooms and paying increased attention to their pupils learning. This was apparent during my visits to their classroom by the end of the first year of the project, and I planned to continue urging them to maintain their attention on the learning that took place in their classrooms. This applied both to their pupils learning and their own learning from communicating with their pupils and noticing critical

learning moments. The codes that reoccurred from our second year discussions manifested specific focus on pupils' learning and the teachers' own learning, both the negative and positive aspects. The emerging categories centred on pupils' learning and our community building in which solidarity and co-learning was fostered.

9.4.1 Workshop 9: Geometry and measurement

The second year of the program, Vala was not teaching mathematics and she did therefore not participate in the project. Karl, who had taught at the school for almost 40 years, was now collaborating with Edda and Rúna in teaching mathematics in 6th grade. I invited him to participate in the project and he joined us for this first workshop of the second year. Gróa's pupils were now in 7th grade and the pupils from Rainbow School were in 6th grade.

The protocol for Workshop 9 was based on: Conclusions from the first year of the project and the teachers' interest in learning about diverse ways of solving mathematical tasks. Growing interest in focusing on diversity and ways to support all children in learning mathematics.

9.4.1.1 Negotiating how to proceed with our collaboration

We started the workshop by discussing our program the next months and what the teachers would like to focus on. The teachers found it important to continue experimenting with problem solving and Pála added that she would like to record her lesson again in order to be able to learn about her own teaching.

The teachers at Sunshine School told us that the head of the school had decided that the pupils would not be grouped into 'ability groups' in their mathematics classes this year was to be the custom at the school before. Karl said that he disagreed with this arrangement and believed that it was not a wise decision. Rúna found it more rewarding to work with mixed ability groups than working with the 'low achievers' that she used to work with. Edda said that she is aware that grouping pupils into 'ability groups' does not favour the children's mathematical learning though she finds it difficult to work with pupils in 'mixed ability groups'. She is concerned for pupils in her group who she feels do not get enough support from her.

9.4.1.2 Designing a water container and reflecting on volume

The teachers worked in two groups and the task was to design a ½ litre container for a company that wanted to sell water in containers made of waterproof paper. There were three conditions the designers had to bear in

mind: the containers had to be easy to hold and drink out of, as little as possible would be used of paper, and they had to look presentable so people will want to buy them.

Dóra took a piece of paper and rolled around Inga's $\frac{1}{2}$ a litre water bottle and asked Inga and Pála if they could start from there. She said that she liked the cylinder formed better than those that were more like boxes. The teachers discussed what information was needed to make the cylinder.

Karl: What is the formula for a cylinder?

Jónína: What information is needed if you want to make a cylinder?

Karl: We need to calculate for half a litre.

Jónína: [Holds a cup in her hand and touches the bottom and sides]. What do we need to know?

Karl: Yes, it is times something.

Pála: Is it not radius times pi, something?

Karl: This is too complicated.

Jónína: We can help each other in working this out.

Karl: [To Edda and Rúna] Can we not just choose another form? This is too complicated.

Meanwhile, Pála, Dóra and Inga had been exploring with using centi-cubes to make rectangular prisms.

Pála: Is it like this? [shows 5 plates of 10x10 centi-cubes]

Jónína: What do you think?

Inga: We think so.

Jónína: Why do you think so?

Pála: Because, is it not true that one like this [shows one centi-cube] 1 centimetre each edge, holds 1 centilitre?

Jónína: How many do we need ...

Pála: One litre is 1000.

Karl: Yes.

Pála: We need 500.

Both groups explored with designing a container according to the conditions given and made rectangular prisms out of paper. They then measured the sides to find how much paper was needed. For one of the containers 408 cm² of paper were needed and for the other 450 cm². We discussed why more paper was needed to make the long and narrow prism

than the other that was shorter and wider. We also compared their results to the other conditions that the company required.

Finally, we discussed what one learns from carrying out this project and how the hands-on material had supported them in designing the containers. Edda said that she was afraid that very few of her pupils would be able to carry out this project unaided by the teachers and Karl agreed. However, the teachers thought that the children would enjoy working on such a project, and selling their idea would be of special interest.

We discussed further the importance of hands-on activities when learning geometry and Edda commented that she would like to have access to more hands-on material to use in her own mathematics classes.

9.4.1.3 Teachers' stories from their own classrooms

Pála wanted to tell us about her work. She was surprised how good her pupils were at solving the problems in the textbook and they were not as tired of listening to her as last year. We discussed what might be the reason for this, and I wondered if the problems were structured in a way that motivated the children or if she had changed the way she structured her mathematics classes. Pála said that she was at least glad to have succeeded in awakening their interest, because her former experience was that they did not listen to her explanations. In this case, she had proposed that the children worked in groups and discussed their ideas. Pála's experience is further discussed in Chapter 10.

9.4.1.4 Conclusions and looking ahead

At the end of the workshop, I urged the teachers to record their lessons. Pála reminded us of her experience of recording her lesson and added that she had learned that the children had little opportunities to discuss their ideas. I asked her if this experience could have affected her teaching, as she seemed to pay more attention now to the children's ideas than before. Pála responded that it could have affected her teaching even though she had been unaware of it.

The teachers' enthusiasm to discuss their investigations when designing the water cartons indicated that they found it helpful to reflect on their own understanding of mathematics. They did not refer much to their pupils thinking about volume as they were deeply caught up in their own thinking about the project. Pála's reflection on her pupils' enthusiasm to engage with the mathematical tasks in their textbook opened up a discussion concerning what kinds of tasks prompt them to investigate with the mathematics. Pála was uncertain in analysing her pupils' rising interest and

confidence in working with the mathematics tasks. Their reflections called for a more in-depth discussion on the role of the teacher in the classroom and how one can lead a discussion with children about their thinking of mathematics. At the following workshop I therefore decided to direct the attention to the issue of the learner.

9.4.2 Workshop 10: Dealing with children's dependence

The protocol for Workshop 10 was based on: Conclusions from the previous workshop about the nature of mathematical tasks and the effects the teacher has on the mathematical learning in the classroom. Growing interest in focusing on diversity and ways to support all children in learning mathematics.

After participating in Workshop 9 Karl decided not to join the group so only six teachers took part in the project in the second year. Edda and Gróa were sick at this time and Rúna came late so we started with only the three teachers from Rainbow School. We agreed to have a short workshop and devote it to discussing their own teaching and wait with the boarder problem until next time.

9.4.2.1 Stories from teachers' classrooms

Pála, who told us how satisfied she was with her pupils' work, at our last workshop was concerned about their performance this time. She wondered if she demanded too much of them as most of them did not seem to be able to solve the problems she assigned them. Despite her pupils having made progress, she claimed that they were not able to think critically and reason about their work.

Pála: Last year Dóra and I stuck together. She of course went deeper than I did, gave them extra work. And now I really want them to be more skilful than they are. Is it possible that I am teaching them at a stage above their abilities?

Dóra: Do you think so?

Pála: I imagine that it could be. The thing is that with some logical problems... I try to talk to the group and challenge them a bit. I get so little response and feel that they need to sharpen themselves. I feel that these are kids who could think more, be capable of more critical thinking than they show in class.

Inga: You are getting into the same package as I am in.

Pála: Am I demanding too much of them? But I do not think so. Like yesterday I took some problems from the inter-net. The first was about 12 kids. Six of them were wearing boots, 4 of them had socks on and 2 had both socks and boots on. The question was how many were barefoot.

Jónína: Yes.

Pála: None of them could do this. Then I showed them, drew a diagram. I expected that this problem would be difficult but not the other problems.

She then told us that the next problem was about currency and the question was how many dollars one could get for 50.000 ISK. She had looked for the latest currency rate and found that 1\$ costs 110.58 ISK.

Only few of her pupils could solve the problem and when she asked them to explain how they calculated, the others did not seem to understand their explanations. We discussed the content of the problem and the teachers agreed that eleven-year-old children might not be familiar with currency. Pála told us that she had made a simpler problem of the same structure about the cost of apples and everyone could solve that problem. Pála had tried to help her pupils to visualise and see that the problems were of the same type, and relate their solution of the apple problem to the currency problem. Only the children who had been able to solve the currency problem could see the relationship between the two problems. The others could not solve the currency problem even if they had calculators at hand and Pála said: “they did not know if they needed to multiply or divide”.

Jónína: Can I ask you, Pála? When they talked about their way of solving the problem, those who found a solution, do you think that those, who did not, understood their explanations?

Pála: I am not sure. Even though I used all my, everything I could, to try to explain this. What I feel is, feel that they are too restless to listen.

Dóra: But did you explain, or did you let them explain, those who could?

Pála: Those who could explain. But I also explained. They could not always express themselves in a way the others understood.

We reflected on what children could gain from explaining their solution strategies and that it requires them to think about their own thinking. Pála said:

How can I move them onwards? That is what I am struggling with. ...
What I feel is missing is the happiness and joy. Have I destroyed it by demanding too much of them and what can I do to awaken it?

Dóra talked about the work with her pupils in the blue group. They enjoy reasoning about their work and even continued to reason about things they found interesting for several days after. I asked Pála if she thought that her pupils would gain from working with children from Dóra’s group who are more willing to reason about their solution strategies. Inga added that her

pupils would gain from working with pupils in Pála's group since they also needed to be challenged to discuss their thinking. I urged the teachers to try to work with mixed ability groups and they decided to plan at least one such lesson to begin with.

9.4.2.2 *Conclusions and looking ahead*

Pála's story about her struggling with her teaching and her reflections on her mathematics teaching confirmed the need for discussing the role of the teacher in the classroom in more depth and how one could lead a discussion with children about their mathematical thinking as concluded from the previous workshop.

9.4.3 Workshop 11: Patterns and algebra

The protocol for Workshop 11 was based on: Conclusions from previous workshops about the nature of mathematical tasks and the effects the teacher has on the mathematical learning in the classroom. Growing interest in focusing on diversity and ways to support all children in learning mathematics.

9.4.3.1 *The Border Problem*

Drawing on my experience from Workshop 8 and our discussions at Workshop 9, I looked for problems to work with that might help the teachers experience how we can learn from each other. The teachers had expressed that working with the algebra and algebraic notations had contributed significantly to their knowledge of the topics. I decided to use a problem, 'the Border Problem' that I adopted from the book *Connecting Mathematical Ideas* (Boaler & Humphreys, 2005, p. 15) (Appendix D) and show the teachers a video clip from Cathy Humphreys' class, which accompanies the book on a CD disc. Originally, I had planned to work with this problem at the 10th workshop.

We first had a quick look at a 10 by 10 grid with its border coloured and I asked the teachers to calculate the number of the coloured squares of the border without counting one by one.

Pála: 18, no 36.

Edda: Yes, 36.

Jónína: How did you think? Write it down.

Inga: $10+10+10+10-4$.

Dóra: Yes, I did the same.

Edda: It is 10, 10, 8, 8.

We then looked at the other questions asked. How many coloured squares would there be in a 6x6 grid and then if they were 9x9 or 8x8. The final question was to find a general rule that could be applied to such a grid of any size.

Rúna: It is 22.

Jónína: Why do you say 22?

Rúna: 6+6 and 4+4.

Pála: It is 20. Is it not just always minus 4.

Rúna: No, what am I saying? It is 20

Pála: 32 minus 4. It is 28.

Jónína: Pála can you explain why you said minus 4

Pála: Because Dóra said minus 4.

Dóra: The corners of the box. We do not count them.

Edda: We always need to subtract two from each length.

Pála: But if we want to write this as an equation?

Jónína: Yes, it was the last question.

Edda: [ironical tone] The most difficult task on the paper.

Dóra: It is 4 times x minus 4?

Pála: Yes, 4 times n minus 4. If n is 25 how many would the coloured squares be?

Dóra: 4 times 25 minus 4. It is always like this.

Jónína: Edda you said 10, 10, 8, 8 and Rúna you counted the same way for the 6x6 grid. How could an equation for your thinking be written?

Rúna: 2 times x plus 2 times x-2.

9.4.3.2 *Look into classrooms*

We then looked at the video and observed how Cathy worked with her pupils and led discussions about their solutions of the task. We discussed what kinds of questions she asked and how she responded to her pupils' ideas. The teachers related to their thinking of the problem and reflected on Cathy's communication with her pupils.

Jónína: Did you notice what she did when she had written all the different explanations on the blackboard?

Edda: Compare their strategies.

Jónína: Why does she do that?

Edda: Yes, from one of the boy's and the girl.

Jónína: How is her solution different from his, what does he do differently from her. Did you notice it?

Gróa: That is very good.

Jónína: Why does one do that?

Edda: They might both be equally good.

Gróa: And it calls for more reasoning.

Jónína: Yes, you can approach it differently but reach the same conclusion. One is not necessarily better than the other.

Rúna: No, no.

Jónína: She is directing their attention towards what is similar and what is different. Then they need to think about what it is that is different. You need to look closer at the task, what is in it, what is one doing there, immerse oneself deeper into the task.

We also discussed the purpose with the task and the goal with spending such a long time on discussing it. Pála said:

The children must get a solid understanding if they spend such a long time on it. ... When they get another kind of task, then they have this solid understanding.

We paid special attention to the pupils' diverse approaches to the task and how Cathy supported them in explaining their thinking about it. Rúna added: "It must add to their independence and independent thinking". To these discussions about reasoning and arguing for one's own way of finding solutions, Edda commented:

I think one must learn from it, I mean if one does not understand, then you cannot reinforce your thinking with arguments. But if you have understood then you can reason about your findings. ... If you go quickly through this, do not go into the depth, some will fall out but others might keep on in a rapid tempo.

9.4.3.3 Conclusions and looking ahead

By the end of the workshop, the teachers in Rainbow School told us that they had decided to audio record their lessons the following week and wanted me to meet them individually to discuss their recordings. The teachers at Sunshine School were reluctant to record their lessons, except for Edda, who was planning to record her teaching in January. I reminded them that we had planned to meet three times this year and now we had just finished our third workshop. Inga said that they could not stop now when they were starting to audiotape their lessons and wanted to be able to discuss the recordings with us. We agreed to meet again for three workshops during the spring and then we could discuss their experience of recording their own teaching.

Our discussions about the border problem and the video from Cathy's classroom strengthened my mission of encouraging the teachers to look into their way of teaching mathematics, and to apply diverse approaches when working with their pupils.

9.4.4 Workshop 12: Exploring with the calculator

The protocol for Workshop 12 was based on: Conclusions from previous workshops about the nature of mathematical tasks and the effects the teacher has on the mathematical learning in the classroom. Growing interest in focusing on diversity and ways to support all children in learning mathematics.

9.4.4.1 Relationships between numbers and operations

From our discussions at former workshops I learned that the teachers were hesitant to use diverse ways to explore with numbers. I therefore decided to investigate with the teachers how the calculator could be a helpful tool in exploring relationships between numbers and the properties of the operations. I brought with me one of the booklets that are a part of the 5th – 7th grade textbooks the teachers were using. The teachers had not used it before and were not aware of what kinds of tasks are to be found in the booklet. The tasks were designed with the goal of exploring relationships between numbers and the properties of the operations, with the help of calculators and spread sheets. We discussed what could be gained from playing with the calculator; exploring connections and researching what happens when you push keys for functions such as square roots and power exponents. I challenged the teachers to reflect on what children in their groups might learn from approaching these tasks and we discussed if these tasks were suitable for children in “mixed ability groups”.

Gróa: When you have mixed ability groups. One boy in my group solved a task in two minutes and then helped another one who asked: “What does this key do?” And he explained it to him.

Jónína: Yes, and we need to adapt our teaching to respond to everyone's needs. It does not necessarily mean that they all solve different tasks, rather that you bring in tasks that allow them to explore; everyone is not learning the same even though they are dealing with the same task. Therefore it is so important to think of what kinds of tasks you choose to work with. These were examples of such tasks.

9.4.4.2 Stories from teachers' classrooms

Edda told us about her recording. She was content with this experience and found that listening to her recording helped her in reflecting on what happens in the classroom.

I learned that I was too monotonous in my presentation and need to be more flexible in my teaching. ... I often raise my voice and as a result I have lost my voice several times. ... I need to get the pupils interested and more focused and they need to have the possibility to concentrate on their work.

She had thought much about how to improve her communication with her pupils, find ways to motivate and support them in their work. I pointed out that recording one's lessons is one way to grasp what we would like to do differently because things tend to pass us by.

Inga talked about her recording and said: "I talked all the time and never gave them a chance to respond to my questions". She was also concerned that she did not use mathematical concepts when talking to her pupils and felt that she was underestimating them by using everyday language. Inga added that listening to her recordings supported her in reflecting on her teaching as well as the opportunity to discuss this experience with others.

Pála was satisfied with what she heard when she listened to her recording this time. Her pupils worked with the border problem that we had dealt with at Workshop 11. Pála said:

The children are often unfocused, but with this project they were focused all the time. I put them in three groups and they were thinking about it the whole time. I found it incredibly remarkable. ... I think that everyone in the class, they all understood this. ... Before I gave them this problem I was certain that it was too difficult for them, had decided beforehand but it was not the case.

Dóra had destroyed the recording of her lesson because she found it difficult to listen to herself. She experienced, as Inga did, that she did not use mathematical concepts and felt that she needed to sharpen herself and plan her instruction better. I added that it is important that children learn the mathematical language but if they do not understand what the teacher is talking about, she may need to explain it in everyday language.

The four teachers who recorded their lessons were all pleased with this experience and urged Gróa and Rúna to also record their mathematics classes. Both of them were reluctant to give it a try.

9.4.4.3 Conclusions and looking ahead

At this workshop I wanted to encourage the teachers to explore relationships and properties of numbers and operations and to discuss with them how such an investigative approach can support inclusive mathe-

mathematics teaching. The teachers liked exploring with the calculator and discussed what their pupils could learn from such explorations. I concluded this workshop by urging the teachers to reflect on their teaching and using the means for professional development we had been discussing; writing cases from their classrooms and analysing them, observing each other teaching and collaboratively reflecting on their experience as well as recording their own lessons.

9.4.5 Discussion of findings: Focussing on pupils' learning in the classroom

The focus on children's learning in the classroom was developing as the teachers changed their way of working with their pupils and their approaches progressed, from explaining procedures for solving problems, towards a more investigative approach. The teachers found the inquiry tasks we dealt with at the workshops demanding but at the same time they added to their confidence of using such approaches in their teaching. The culture in our learning community was flourishing as the teachers proceeded to shape it. Their experiences of solving inquiry based problems at the workshops and discussing with us their experiences from their classroom seemed to have affected their teaching.

9.4.5.1 Pupils' learning

During our second year of collaboration, the focus of our discussion was on pupils' learning in the teachers' classroom. This was an indication that they were gradually paying more attention than before to how their pupils' learning was reflected in their classrooms. Pála was amazed to observe her pupils' growing interest and competences in solving mathematical problems and wanted to share her reflection on this experience with us (9.4.1.3). She was interested in discussing what the reason might be for their sudden enthusiasm in learning mathematics. When I asked if the nature of the tasks she was using and her focus on listening to her pupils, instead of talking herself, might have affected their willingness to engage with these tasks, she replied that it might be possible but she had not considered the option. On another occasion (9.4.2.1), she expressed how disappointed she was with their declining motivation, she was still hesitant in analysing what might be the reason for why the pupils had adopted an accepting mode again (Mason & Johnston-Wilder, 2006). When Dóra asked her if the children had explained their way of solving a problem, or if she explained how to find a solution, she replied that she had needed to add to their explanations because they did not always express themselves in a way

the others understood. As we discussed Pála's concerns, our co-learning partnership was cultivated by her openness to involve us in her reflections on her own learning (Jaworski, 2003). We were inspired to discuss the nature of the tasks she was using and how tasks that are constructed in such a way that the learner can encounter important ideas in curriculum topics and whose solution require active engagement, can further mathematics learning (Mason & Johnston-Wilder, 2006).

9.4.5.2 Teacher investigations direct our lens towards collaboration

When we collaborated on inquiring into mathematics (Jaworski, 2006a), and solving the tasks I brought into the workshops, we listened to each other's solutions, shared our thinking about them and refined our methods, which also respected our diverse experiences (Askew, 2015). Looking into a classroom, where a teacher led a discussion about the border problem, further helped us focus on what individual learners add to the collective learning in the classroom. This was revealed when Pála told us about her experience of working with this problem in her class where she had learned that her pupils were capable of more advanced thinking than she had expected.

Even though I decided what problems to work with at our workshops, the teachers were progressively taking the lead in our discussions, both in terms of the problems and by bringing in cases from their classrooms to reflect on collectively. The teachers found it challenging to design a container for water at Workshop 9 and asked me to confirm their conceptions. I responded by asking them questions to help them further their thinking. When the teachers explored with algebra at Workshop 11 (9.4.3.1), I confronted them by asking them to justify their thinking, thus challenging their contingency ability (Rowland, 2014). In the case when Pála was confused about her pupils' bewilderment in face of the currency task at Workshop 10, Dóra and I asked her challenging questions to confront her analysis of her pupils' incapability to show initiative in their work, in order to motivate her competency to evaluate her pupils' learning (Niss & Højgaard, 2011).

9.4.5.3 Shaping the culture of our community

We were all shaping the culture in our community through our active engagement in all our activities. Each one of us supported the learning of the others as we asked probing questions and scaffolded each other in making sense of the issues we were dealing with. We made assumptions about each other as we recalled the past and talked about our future and

imagined how we would like to proceed in future workshops (Wenger-Trayner & Wenger-Trayner, 2015).

The teachers proposed that we would proceed with our project by the end of the first year and again in the middle of the second year. Inga insisted that it was too early to end the workshops because they were just getting started with recording their lessons and needed the opportunity to discuss their experience with us. They all decided to keep on meeting at the workshops, except for one teacher who transferred to a different position within her school. Karl, who substituted her, was invited to join us and participated in one workshop. He then decided to leave the project and due to time constraints. I was not surprised that he did not want to join the project since we had established a culture within our community that he might have found difficult to align with.

In responding to the teachers' enthusiasm to collectively reflect on their teaching, I decided to give them leeway in shaping the following workshops and was prepared to discuss inquiry problems with them to further challenge their approach to mathematics teaching.

9.5 Theme 5: Teacher reflections lead our discussions

The second year of our collaboration was coming to an end and the teachers still wanted to proceed. They had been gradually developing their own teaching where they increasingly emphasised interactions and explorative approaches, and where they paid more attention to the learning in their classrooms. Their enthusiasm in sharing their experiences and reflecting on them with us at the workshops resulted in them taking the lead in structuring the workshops. The categories emerging this second year centred on concerns for their pupils' learning and finding meaningful ways to support them all. The culture in our community was being cultivated by the teachers' willingness to share their experiences. Boundaries between our community and the one's in their schools were also in focus.

9.5.1 Workshop 13: Conflicts about assessment

The protocol for Workshop 13 was based on: Conclusions from previous workshops where the teachers wanted to discuss their experience from paying attention to their pupils learning. They also wanted to improve their understanding of proportions and I brought with me some tasks to look at with them and to discuss what could be learned from solving them.

9.5.1.1 *Stories from teachers' classrooms - assessment*

As soon as the workshop began, Pála said that she wanted to discuss an event from her class the same morning. The teachers in Rainbow School had put together an end-of-term test and one girl in Pála's group interpreted one of the problems differently from what she was expected to do. She argued that her response was right even though Pála had deemed it wrong. Pála wanted to share this experience with us.

Pála: It was not the case that she did not understand how to find $\frac{1}{3}$ of 12. She just interpreted that she was supposed to subtract $\frac{1}{3}$ from 12.

Edda: She subtracted?

Jónína: What do you think, is it unreasonable? If she had interpreted it this way, subtract $\frac{1}{3}$ from 12?

Pála: No, it is not unreasonable, but I marked her answer to this task as wrong.

Edda: Yes, if she misinterprets it!

Pála: It is not the same to subtract $\frac{1}{3}$ from 12 and find $\frac{1}{3}$ of 12. Maybe both answers are right. When we talked about this task. I know that she understands how to find $\frac{1}{3}$ of 12, but she was consistent in that she was supposed to subtract $\frac{1}{3}$ from 12. I probably need to change her grade.

Inga: The tasks before this one were about subtracting fractions.

Pála: It is unfair to mark this as wrong.

Pála realised that the wording of the problem was misleading and the girls' interpretation of what she was supposed to do was sensible. She and Dóra had picked some tasks from an old end-of-term test and did not concentrate on how the questions were phrased. They agreed that the problem was badly phrased even though they had not discerned it before we discussed this event. Together we reflected on this case, both on what was misleading in the wording of the problem and how we assess our pupils' knowledge of mathematics.

Jónína: You know that this girl can find $\frac{1}{3}$ of 12 because you have seen her do it earlier. She understands it and knows how to do it. Now she gets this same example at a test and evidently relates it to something else. So she does not do what she is required to do but does what she thinks she is supposed to do. Then the question is, how we respond in such a situation?

Pála: If it had been an oral examination I could have learned what she understands and what not.

Jónína: You would have asked her?

Pála: Yes, I would have asked her and could have rated her response right.

Jónína: So you are reflecting on ...

Pála: I showed them their solutions to the test and they saw how I rated it. She got tears in her eyes and said: "This is true, Pála, I must tell you that this is true".

Jónína: I subtract $\frac{1}{3}$ from 12?

Pála: The instructions are not good enough. Of course we learn from this.

I asked Pála if she was willing to reflect deeper on this case or until we would meet at our next workshop where we could then discuss it further. I urged her to write first a description of her discussion with the girl and then her further reflections and interpretation of the incidence:

What could be the reason for the girls 'misinterpretation' of the task and what do you find reasonable to do in this situation? Because now you are thinking about what to do, this is an end-of-term test, and you are reflecting on whether you should change her grades. This is an ethical question.

9.5.1.2 Making sense of pupils' response

When we discussed Pála's story about this girl's misunderstanding, Inga was prompted to talk about her experience of trying to understand how one of her pupils interpreted a task he was solving.

Inga: Then I thought, wait, what is he thinking? They were supposed to write an answer to a problem in centimetres. The solution in metres was 4 and he said it is 4000. I tried to explain to him that it is 400 but he was consistent in saying that it was 4000.

Jónína: What was the task about?

Inga: It was a story problem. When he did not understand my explanation I started to think, what is he thinking, to turn this around, and asked him to explain it to me.

Dóra: Rather than telling him?

Inga: Yes, he did not understand when I tried to explain to him.

Jónína: Then he has had some idea about this problem that you did not understand?

Inga: Yes, I wanted to know what he based it on.

Inga was unable to learn more about what he was thinking as the lesson came to an end but she planned to ask discuss this the next time they met.

9.5.1.3 Conclusions and looking ahead

By the end of the workshop we looked at two problems I had brought with me and discussed what their pupils would gain from solving them, but we

did not have time to solve them together as we had spent such a long time on discussing Pála's case.

At this workshop we mainly discussed the teachers' stories and I decided to remind them of the protocol for case and commentary writing I gave them at the beginning of our collaboration. I urged them to keep telling us stories from their classrooms. I said that if they did not have time to write the stories they could just write some notes to remind themselves of what had happened. We would then discuss their stories and reflect on them together. Since we did not have time to solve the problems on proportional reasoning I decided to put them on the agenda at our next workshop.

9.5.2 Workshop 14: Learning to listen

The protocol for Workshop 14 was based on: Drawing from our experience from the last two workshops I expected the teachers to want to discuss their teaching and did not anticipate that we had much time to explore with problem solving. I was though prepared to work with them on solving problems about proportions in rectangles.

9.5.2.1 Teachers' stories from their own classrooms – cultural days

The teachers at Sunshine School wanted to tell us about the cultural days in their school. The pupils could choose to work with different projects and the teachers were each responsible for a project at one station. Gróa and Rúna told us a story from their station, which was a café where the pupils could have juice and cookies. They knew that one of the boys who came to this station had been identified as having problems with learning mathematics. They therefore decided to put on a play and pretend that they were at a birthday party. The participants had to decide how much money to spend on gifts, food and drink and how to share the costs. Gróa told the story in a lively manner, imitating the pupils' contribution and told us how she supported these children in calculating how much each person was supposed to pay.

Gróa: This was our contribution to our mathematical discussions this time.

Jónína: And this is not about number problems to calculate, these are real problems, and when they solve them we learn what they know and understand.

Rúna: The one who could not give the change from 100 ISK, he is good at grammar.

Edda: Cannot tell the time, so many things he is not capable of doing.

Jónína: You have told us beautiful stories of what happened when the boy solved real life problems. We need to ask ourselves, how we as teachers, can document this so we can look at it again. You already asked yourselves how you could support this particular boy.

Rúna: Yes, in the future.

Jónína: Yes, he has his strong sides, as you said Rúna, but he seems to have problems with numbers and cannot tell the time. What can we do for this boy, what kinds of solutions can we find?

Gróa: This applies to so many.

Edda: But this is our least able pupil.

Jónína: Concerning number sense?

Edda: Yes, and then there are others who have other problems.

We discussed their story further and how important it is for teachers to be responsive to their pupils' contribution. When the teachers know their pupils and have reflected on their work with them they are more capable of reacting to diverse situations that come up. Gróa told us how she often tells her pupils stories to help them relate the contents she is working with in her classes to the environment the children are familiar with.

Jónína: It is important to relate to different situations, we have diverse tasks and we adapt them to our conditions. You tell these stories and try to relate what you think will work to help them understand. When we as teachers think as you do, reflect on our conditions and then respond to the situations, what are we then doing?

Gróa: What am I doing?

Jónína: Yes, when you decided to talk about sharing raisins when solving fractional problems.

Gróa: You do not think when you are in the action, you just, you see that something needs to happen.

Jónína: Yes, and why do you do that?

Dóra: So, the pupils will understand.

Gróa: To try to make the pupil understand. Particularly when you see that one pupil understands, and the other does not. What can I do?

Jónína: This is what we are looking into, how professionals work, your response did not come out of the blue. It is so important to consider what is happening there in your teaching. Your experience and your knowledge of the subject helped you reach this conclusion. It seemed to have worked out there. It is so important to think about what one can learn from such an experience and how it can help in future work.

Rúna: We are connecting to their real world.

Jónína: Yes, and this is what teachers do. What I am asking you to do is look into how you do this. What is it that you reflect on and how is it represented in

what you do? How you come to these conclusions, because when you are in the classroom you are not thinking about how you reached the conclusion.

Dóra: No, no.

Jónína: One just acts.

Edda: Yes, yes.

Jónína: Then the next day arrives and we proceed as usual. Of course we get better at this. The reason for that I asked you to work with me was that I wanted to learn about how you do this. Have you look at it with me, the way you do it, so we can together make meaning of how this happens.

Gróa: We just feel that we are doing routine work. There is nothing to talk about.

Jónína: But why is that ...

Gróa: And then, it is not until one starts to talk about it, you have been doing something that is interesting. I have experienced this.

Jónína: What is routine work, and what is professional teaching? I believe that all teachers base their work on their professionalism, but their background is different and it also varies how much attention they pay to improving their practice.

Gróa: Of course they do.

Jónína: That is why, to meet like we do and talk about your mathematics teaching, what happens in your classrooms, you reflect on your work, what you can learn from your own experience and the others. This is what I wanted you to help me learn about.

9.5.2.2 Teachers' stories from own classrooms – assessment

Pála said that she had talked to the girl that she told us about at our last workshop and had told her that she would grade her answer right as she supported it with such good arguments. We then discussed how we base our evaluation of the pupils work. Pála referred to a workshop I had led earlier with all the teachers at Rainbow School, where we focused on alternative assessment, and the teachers had discussed how they would like to develop their methods for assessing mathematical competences.

Dóra: We will do this differently next time.

Edda: This is an example to learn from about tests, how to phrase the questions.

Jónína: And what kinds of tasks to choose. What are we assessing and how do we pose these tasks to our pupils? It is important to ask oneself, what the purpose with the assessment is and the goals with the tasks. Do the results tell us anything about the pupils' knowledge?

Pála: It is different. Sometimes when one rates the results from assignments, one gets astonished. Then you see that some children are capable of doing much more than you expected and others less. I always ask myself after every test, am I assessing in the right way. Am I stuck with this, old tradition of testing?

Edda: Like we have been doing in Sunshine School.

Pála: And I always ask myself, is it fair or unfair, you know.

9.5.2.3 *Teachers' stories – understanding formulas*

Dóra and Pála then shared their experience of asking their pupils how many handshakes there would be in their class if they all shook hands with each other. The children decided to try to shake hands with each other and were quick to realise that they would only shake hands once with each person. They developed a rule that could be used to calculate the handshakes $14+13+12+ \dots +2+1$. They then split into smaller groups to test if their rule could be applied to a group of any size. Pála had not thought about the solution of this problem before she posed it to her pupils and therefore took an active part in the solution process. By comparing the total handshakes for different number of pupils, they then had developed a formula together. Pála was keen to discuss with us whether the formula $\frac{n(n-1)}{2}$ could be applied to calculate the handshakes for a group of any size.

Pála: If we are 13, then it is 13×12 divided by 2.

Dóra: Yes it is like this.

Pála: I do not understand why this equation works, why this connection. I know it works, we have tried it for many cases. Can you help me to understand why it works? I would like to proceed to work with the children in this way.

Jónína: This is a wonderful example. You gave your pupils a problem that neither you nor they knew beforehand how to approach. Then you all started to investigate and look for patterns and then developed a rule. ... When a teacher works in this way, she takes an active part in the process. Not only does she learn about the children's thinking but also about her own thinking about the problem.

We then discussed the formula they had developed together and why it could be used to calculate the handshakes. Pála told us that some of her pupils understood why the formula could be applied to solve this problem and others did not. They though all understood that they could calculate the total number of handshakes by adding $(n-1)+ \dots +1$. I reminded the teachers of our earlier discussions about tasks that could be solved at many levels and were therefore suitable to work with in diverse classrooms.

Jónína: When we work with problems like this, so much happens. Therefore it is important to allow the pupils to deal with such problems and give them the time they need to develop their own thinking about them.

Dóra: Yes, it is so interesting, we learn so much.

Jónína: It is good that you told us about your thinking Pála. We do not necessarily need to know everything beforehand. We can discuss with the children and ask them about their ideas. We then decide if we tell them that we do not know the answer.

Pála: I feel now that I can tell them. Once I felt I could not.

I then reminded the teachers that we had twice before discussed problems where we added consecutive numbers. Pála said that she remembered that we had done it before but she still could not understand why the formula she had developed with her pupils worked.

9.5.2.4 Conclusions and looking ahead

The teachers' enthusiasm to discuss their stories from their classrooms was growing and when I mentioned that this was meant to be our last workshop they said that they would like to proceed. The teachers discussed what they had learned from participating in the workshops and all agreed that they were more conscious of what they were doing with their pupils, the steps every child was taking and what kinds of tasks were worth working on.

I told the teachers that I would like to meet each of them individually and they proposed that we would all meet for a few workshops next year, because they felt that they gained so much from our meetings. So the project continued for a third year.

9.5.3 Workshop 15: Teachers lead the workshop

The protocol for Workshop 15 was based on: During the workshops the second year of the study the teachers were eager to discuss their work and I expected that at this workshop we would devote our time to reflect on their stories. I brought the problem about proportions in rectangles with me in case we would have time for solving it.

9.5.3.1 Teachers' stories from their own classrooms

We were starting the third year of our collaboration and nine months had passed since we met last time. Rúna had retired and Gróa was ill so there were only four teachers present. The teachers all needed time to talk about their situation within their schools, what classes they were teaching, with whom they were working this year, and what textbooks were used at their schools. It was obvious that they needed time to talk about their situation

and exchange stories about their work. I decided not to interfere with their discussions because I found it important to give time for such unstructured reflections, in order to re-establish the culture within the community we had managed to build the previous two years. The concerns each of them raised are attended to in Chapter 10, in connection to their narratives.

Dóra told us that during the annual mathematics day at Rainbow School she had proposed that all pupils in grades 5-10 would solve a puzzle she had discovered at the annual conference of the mathematics teachers' association (Flötur) in the fall. The teachers had trusted her to bring in a project that was worth working with and this was a sign that she was respected as a mathematics teacher in her school. To be able to solve the puzzle, the pupils needed to find equivalent fractions. The purpose was to work collaboratively at finding equivalent fractions that were represented as fractions, decimals, proportions and percentages. The teachers in Rainbow school wanted to share their experience from working with their pupils in solving the puzzle.

Inga: There were different representations of the same number.

Dóra: Like 0.08, 8/100, 8%. Difficult for them to find.

Inga: Their discussions were fruitful. When they looked at the numbers. What number is equivalent to $\frac{1}{8}$?

Edda: We need to work more with such problems.

Inga told us that her pupils were eager to discuss their work and she was surprised to see how excited they were to find all the solutions and clues needed to solve the puzzle. Dóra and Pála were also satisfied with their 7th graders' work and how willing they were to collaborate when finding the clues to the puzzle. The other teachers in the school had also expressed that they liked to work with this puzzle in their classes.

Dóra and Pála told us that our discussion about assessment at Workshop 13 had inspired them to try some of the proposals for assessment that come with the teachers' manuals connected to the textbooks they were using. They had initiated negotiations at their school about reviewing their way of assessing their pupils' mathematical competences. They referred to the workshop on assessment that I had led with all the teachers at their school during the first year of our project and felt that it had paved the way for looking at alternative assessment in their school.

Pála brought with her a fraction problem that she had been trying to solve without success and wanted us to help her solve it. The structure of the problem was similar to a problem we had solved the year before and we discussed the similarities and differences of these two problems. We

also discussed what children learn from solving such problems and in what way they are structured differently from the task that confused a girl in Pála's class, as was discussed in Workshop 13.

9.5.3.2 Conclusions and looking ahead

At this workshop, the teachers led the discussion and brought with them problems they had been using in their schools and they wanted to share their experience with us. Dóra had taken a lead in her school by proposing that all pupils in grades 5-10 would solve a problem that she had worked with earlier in collaboration with colleagues, and she believed would be interesting and fruitful in the context of her school. The teachers reflected on their experiences and discussed their concerns for their pupils' learning and how their pupils could be supported in gaining confidence in learning mathematics. Their enthusiasm impelled me to ask them if they would like to discuss the new national curriculum guidelines at the next workshop, and in doing so, help me in my role as the editor of the mathematics chapter.

9.5.4 Discussion of findings: Teacher reflections lead our discussions

The teachers were taking the lead in deciding what to discuss at our workshops, and were always prepared to share their reflections on their work with us. I reminded them of the workshops two days in advance where I summarised briefly what we had done at the previous workshop and what we had decided to the focus on the next time. I also urged them to share their experiences with us but they never gave me information beforehand about what they wanted to discuss. The plans for the workshops therefore gradually became preliminary plans.

9.5.4.1 Teachers share their concerns

As the collaborative project progressed, Pála repeatedly initiated discussions at our workshops, which often lead the group discussions down unforeseen routes. She was always willing to discuss her concerns about her mathematics teaching and her collaboration with her pupils. This was a clear indication that she was aligning herself to our community (Wenger, 1998; Wenger-Trayner & Wenger-Trayner, 2015) as she responded to my encouragement to share her reflections, which at the same time contributed to the shaping of our culture.

One turning point was when Pála told us about her conflict with a girl who solved a task differently than expected, on an end-of-term test. Pála knew that the girl was capable of calculating the way she was supposed to do but she had misinterpreted the instructions. By bringing in this case, Pála

turned our attention to how we assess learning and the mathematical competences children are expected to develop at school (Niss & Højgaard, 2011). Pála had together with colleagues put together a test based on established norms at her school and now discovered that using this tool did not meet her expectations. She asked for our support in resolving this conflict, and reflected on her own response, which in turn resulted in her discussing the matter with the girl and they came to a conclusion which satisfied them both. Pála engaged with her ideals and resolved the conflict between her beliefs and expectations from her environment by engaging in what Korthagen (2004; 2013) describes as core reflection. At the same time, her competencies of revealing and assessing learning were developing (Niss & Højgaard, 2011).

9.5.4.2 Rethinking assessment

The assessment incident introduced by Pála, inspired her and Dóra to rethink their way of assessing their pupils' learning and they looked for alternative methods as their awareness of means for assessment grew (Mason, 1998; 2008). In our discussions about assessment, Pála referred to a workshop on assessment with all the teachers at Rainbow School (see Appendix C) that had opened up discussions at her school about diverse ways to assess mathematical competences. The interest of other teachers in reviewing their assessment methods supported them in improving their practice and negotiating the boundaries between the communities in which they engaged with their colleagues, on the one hand, and the communities we had established through the workshops, on the other (Wenger-Trayner & Wenger-Trayner, 2015).

9.5.4.3 Shared ownership of workshops

Pála also shared with us her experiences of posing a problem to her pupils that she had not solved herself beforehand (Workshop 14). She was puzzled about the conclusion she and her pupils arrived at and wanted us to help her understand why the formula which they had developed worked. She also brought a problem into our workshop that she had been trying to solve and wanted to discuss with us (Workshop 15). In doing so she was accepting my invitation to take responsibility for shaping our community, thus partly taking on my role as an initiator of where to focus our mathematical learning at the workshops, and in taking the control of the discourse she was sharing the ownership (Goodchild, 2008) with me, in the words of Bernstein (2000).

9.5.4.4 Shaping the culture in the community at school

Dóra was taking the lead in her school with regard to what kind of mathematics would be the focus by proposing that all pupils in the middle and upper grades would engage with a puzzle that she had encountered at a mathematics workshop. The other teachers' positive response was a sign that they respected her as a skilful mathematics teacher and that they were willing to follow her advice, accepting that she was actively shaping the culture in their community.

9.5.4.5 Inclusive practices

In Workshop 14, Gróa and Rúna addressed an important aspect of diversity in mathematics classrooms by sharing with us a story from an informal setting at their school. When they were joined by a boy, who they knew had difficulties with learning mathematics, they decided to awaken his interest by means of mathematical activities, thus involving him in mathematics learning with his schoolmates. Inga was also consistently brought stories from her work with children identified as having difficulties with learning mathematics. She focused on what her pupils were thinking and drew our attention to their competences instead of addressing only their deficiencies. The teachers' focus on ways to include all learners in meaningful mathematics learning was apparent, envisioning inclusive practices (Ainscow, 1995; 2007; Askew, 2015) as opposed to the integrative approach that had previously been practiced at their schools.

9.5.4.6 Constraints in boundaries of communities

When we discussed the stories that the teachers shared with us, Edda would often comment that she felt that the culture at Sunshine School prevented her from developing her practice as she had envisioned. Pála said that she felt that she was stuck with old traditions concerning assessment and Edda responded by saying that this also applied to her school. She noted, during a discussion regarding diverse types of problems, that at her school they needed to work more with such problems. She was thus sharing her internal mediation with us as she reflected on the external mediation (Wertsch, 2007) of the other teachers shared their concerns with us at the workshops.

9.5.4.7 Developing co-learning partnership

The conclusions drawn from the two last workshops of the second year of the project, and the first workshop of the third year, indicated that a co-learning partnership was developing within our community even though

the teachers did not all identify strongly with it. We were developing collective learning and we had established a flourishing community, through mutually reflexive processes of knowledge growth, as individuals and a group (Jaworski, 2003).

When the teachers shared their reflections with us, I tried to bring up a discussion about professionalism and how teachers can work at developing as professionals (Dalmau & Guðjónsdóttir, 2002). Our collaboration competency was strengthening and we were progressively engaging in more in-depth discussions than we had in the first year. The teachers were not experienced in discussing their professional practice, as was reflected in Gróa's response in Workshop 14 to my question about her reflective practice, when she said that she was just doing routine work. Even though Gróa was not willing to record her lessons or write about her reflections, she was willing to discuss them with us. Her frankness about her beliefs and ways of working with her pupils provoked discussions that often led us down unforeseen routes. Her implicit mediation (Wertsch, 2007) in Workshop 14, when she shared her story in a dramatic way from the cultural days is an example of how she was engaging with our community and finding her way of aligning to it.

The teachers proceeded to develop their interactions with pupils and shape the culture in our learning community, and I challenged them to reflect on the purpose of this study. The findings from our 15 workshops suggested that the teachers' process of reflecting on their practices was gradually progressing and they were increasingly challenging the established norms of mathematics teaching in their schools. They were not only shaping the cultures of our community, they were taking additional steps into shaping the communities within their schools. I therefore decided to invite them into my world as a teacher educator and asked them to help me in my work of editing and writing of the mathematics chapter in the new national curriculum guidelines, that was at the time still at an embryonic stage.

9.6 Theme 6: Towards investigative approach and inclusion

The project had been running for almost three years and this third year we only met in the spring as we all were occupied with other things during the fall. The teachers had taken the lead in deciding what to focus on at the workshops, thus shaping the culture within the community we had established. I invited the teacher to discuss my work as a teacher educator at Workshop 16, and in that way, I showed them respect as professionals

that are capable of offering advice about what is valuable for children to learn at school. At our last workshop, we engaged in discussions about the teacher as a professional and summarised our collaborative work over the three years. The following year, I met each of the four teachers that had participated in all phases of the project and observed their lessons with the intention of learning about their work with their pupils.

9.6.1 Workshop 16: Discussing curriculum guides

The protocol for Workshop 16 was based on: The teachers expressed visions for their mathematics teaching at our last workshop and raised concerns about what kinds of tasks to work with and how to communicate with their pupils about the mathematics they were studying. Drawing conclusions from our reflections I decided to propose that we discuss what points they would like to see addressed in the mathematics chapter in new national curriculum guidelines.

9.6.1.1 *Discuss the mathematics chapter in curriculum guidelines*

I asked the teachers if they were willing to discuss the new national curriculum guidelines. The general section had already been published and at this time I was leading the writing of the mathematics section. I wanted to discuss with the teachers their opinions of what was important to include in the mathematics section. In both schools, the teachers had discussed the general section of the guidelines at staff meetings and Dóra and Pála had been asked to lead the discussion about two chapters in the general section at a staff meeting the following week.

We discussed the fundamental pillars on which the curriculum guidelines are based: literacy, sustainability, democracy and human rights, equity, health and welfare, and creativity. The concepts that the fundamental pillars are based on should be evident in all educational activities and in the content of school subjects and fields of study, both regarding the knowledge and the skills that children and youth are expected to acquire (Mennta- og menningarmálaráðuneytið, 2011).

I asked the teachers to share with me their thoughts about how the pillars could be reflected in mathematics teaching. They felt it was necessary to aim at mathematical literacy, and added that they wished that their pupils would become confident in using mathematical language.

Jónína: How can we support our pupils in becoming mathematically literate?

Dóra: The concepts, teach them the concepts, to read ...

Pála: They need to be able to use the calculator, know the symbols on the buttons and all that.

Jónína: What about the mathematics in their environment?

Pála: Yes, their environment, the tasks need to have connections to their daily life.

Edda: But they do not often need to use mathematics in their daily life.

Pála asked if the teachers thought the new curriculum guidelines would bring much change in the schools and Edda responded by saying that there would probably be some change over time. We also discussed the difference between intended curriculum and implemented curriculum and Inga added: “We are doing all this today but we do not use these fancy words when we talk about our work”.

The emphasis on inclusive education in the guidelines was discussed and the need for more financial support if the schools are to be able to fulfil the requirements of including all children in the mainstream schools. The teachers offered many examples of the diverse challenges that schools are faced with in trying to include children with special needs in the work at school, due to both a lack of resources and people to support them.

9.6.1.2 Conclusions and looking ahead

The teachers were positive towards the new national curriculum guidelines. The schools are expected to develop their own curriculum guidelines based on the national guidelines and the teachers felt that they had freedom to attend to the needs of their own schools. The decrease of financial support as a result of the financial crisis three years ago worried them and they felt that it intruded on their possibilities for professional teaching. The teachers talked enthusiastically about the circumstances in their schools and the focus shifted to general concerns about teaching and learning in schools, away from the mathematics section of the curriculum guidelines.

The project was coming to an end and I decided that I would not extend the project even though the teachers had expressed the desire to proceed with the workshops. I expected that we would discuss our learning from the project at the final workshop.

9.6.2 Workshop 17: Teachers reflect on their learning with their pupils

The protocol for Workshop 17 was based on: This was our final workshop and I expected the teachers to want to discuss their work. We had worked together for three years and we also needed to have time for discussing our collaborative learning.

9.6.2.1 *Stories from teachers' classrooms*

Pála told us about a mathematics lesson in which her pupils worked with different kinds of word-problems in their textbook. They were required to write their solutions to the problems with algebraic expressions. She gave examples of the pupils' discussions about the problems and how they wrote the expressions. She had recorded these examples in her notebook and now wrote on the whiteboard to show us how the pupils calculated and how she interpreted their thinking about the problems.

We discussed two of the problems:

- Klara is 4 years younger than her brother Kári. Their total age is 18 years. How old is Kári?
- A large apple costs 11 ISK more than a small apple. The total price of a small apple and a large apple is 59 ISK. What is the price of a large apple?

Pála had solved the problems herself and her thinking was different from her pupils' but they all came to the same conclusions. She wanted to discuss this experience with us and hear my interpretation of the different ways they solved the problems. She was particularly keen to hear my opinion with regard to the way she had accepted her pupils' way of solving a problem instead of telling them to think about it in the same terms she did.

Jónína: Did any of them solve the problem before they wrote the equation?

Pála: Yes, yes, they did it first. But they were asked to write an equation, they started ...

Jónína: So, when they start to record, they record in accordance with their thinking?

Pála: Yes, they did.

Jónína: Pála, you said that the children wrote $x+x+4=18$ and you wrote $x+x-4=18$.

Pála: Yes. And for the apples they wrote $x+x+11=59$ and I wrote $x+x-11=59$.

Dóra: Yes, yes.

We discussed how the value of the unknown variable in Pála's equation was different from the value in the children's equation. Still in both cases they came to the same conclusion about the age of the siblings and the price of the apples.

Pála: And then there was this problem: Tell the story that could lie behind the problem $x+x+25=65$. And this they all managed to do.

Jónína: And what kind of stories did they make?

Pála: They used the context of the other problems, did the same: Nonni is 25 years older than his sister Dóra. Their total age is 65. How old is Dóra?

Jónína: Used the same context? And this step requires more of them.

We discussed the different approaches to solving a word problem; the difference of writing the answer with algebraic equation and writing the story behind the equation. All the children in Pála's class were able to solve the word problems by first trying some numbers and then adjusting them until they found the right numbers. Most of them could then write the equations and they then supported each other in doing so. Many could write stories for the equations given but some had difficulties with visualising what stories the equations could represent.

Pála: These were just my thoughts. I found it interesting to see how they understood and thought about this.

Jónína: Yes, and their discussions about what they did.

Pála: Yes, they discussed a lot. They all enjoyed this and found it easy.

Jónína: And still this is algebra. Actually they were solving simultaneous equations.

Dóra: Yes.

Jónína: Like when you said $x+x-4$, and they did $x+x+4$, you have written the y in terms of the x . In your equation the x represents the age of Kári, and in there, the x is the age of Klara.

Dóra: Why did they do it this way?

Pála: I do not know.

Dóra: They were asked to find the age of Kári.

Pála: But they did find the age of both. They knew that Klara was younger.

Jónína: Many people find this difficult, the algebraic notations.

Pála: Yes.

Jónína: This problem was in the form of a story that the children could visualise. When we teach this in abstract form without context many pupils have difficulties with this abstract form.

Pála: It is important that the problems are about something, something they know.

After our fruitful discussions about Pála's reflections on what she had learned through communicating with her pupils about the mathematics problems, it came as a surprise when she said:

Pála: But I am quite sure that if they would see a problem like this in one a month from now they would be confused. Then there is such a long time gone and they have worked with other things in between, they would need to refresh.

Jónína: Would they not be able to calculate the age or the price of the apples even though they could not write the equations?

Pála: Yes, maybe they could, yes.

Jónína: You have worked this out together and they have discussed their ideas. Do you think they could recall their thinking during this process and relate to it?

Pála: Maybe they could.

In relation to Pála's story, I added that we had been discussing inclusive practices and that the design of this chapter in the textbook helped teachers in supporting all children in their learning. Every child could work through the first steps of these problems and participate in the discussions about their solutions and some of them could develop their work further. Even though they were not all capable of writing the equations or making their own stories for the equations, they would learn from the pupils who did.

9.6.2.2 Conclusions

The time passed quickly as the teachers came up with stories that they were felt impelled to talk about in connection to things we discussed, like what kinds of words we use when we express our thinking. The teachers agreed that they felt they had benefited from our meetings and claimed that they were more aware of what kinds of mathematical tasks were worthwhile. They also felt more confident in leading discussions with their pupils and listening to them instead of explaining to them how to calculate as they used to do. Pála added that she believes that there is a need to talk more about the mathematics with the children than has been the custom at school. Edda said: "We have learned to emphasise hands-on activities and collaboration and lead discussions". Inga told us how she had realised already at the first workshop that mathematical problems could be solved in diverse ways and this was a new experience for her. She felt that her competency in solving mathematics problems grew as she participated in further workshops and that she learned more fruitful ways to discuss with her pupils. She added that she wanted to follow the children that she, Dóra and Pála had taught these last three years and work with them in their mathematics classes when they reach 8th grade in the next year.

After three years of collaboration, I decided to visit all the teachers in their classrooms and interview them individually. Gróa had a leave at the time of the observations and interviews, so I only visited Dóra, Edda, Inga and Pála. I noticed a considerable change in the way they worked with their pupils and they all had progressed in terms of planning their teaching in line

with an investigative approach. I will discuss the interviews and observations further in Chapter 10, where I present the narratives of these four teachers.

9.6.3 Discussion of findings: Towards investigative approach and inclusion

I was inviting the teachers into my practice as a teacher educator when I asked them to discuss what they would like to see addressed in the mathematics chapter in new national curriculum guidelines. They all presented their opinion of what they would like to see in the guidelines where their main focus was on general issues and less on the mathematics.

At our final workshop, the teachers were still eager to share stories from their classrooms. In their accounts of their experiences, they described conflicting understandings of their communications with their pupils. They presented an image of them as skilled mathematics learners yet at the same time they voiced worries about their lack of capabilities to learn mathematics. They all agreed that they had learned much from participating in the project, and this is also reflected in my observations in their classrooms and interviews with them, as discussed further in Chapter 10.

9.6.3.1 Inviting teachers into my professional community

I was respecting the teachers as professionals (Dalmau & Guðjónsdóttir, 2002) by offering them to join me in discussing the mathematics chapter of the national curriculum guidelines. I invited them to discuss questions in which I was engaged with experienced mathematics teachers and teacher educators, within my professional community, allowing them to influence our work and explore boundaries between communities (Wenger-Trayner & Wenger-Trayner, 2015). Their enthusiasm was apparent but their main concerns centred on the lack of financial support with regard to information technology and in responding to children with special needs. Even though I asked them to focus on the six fundamental pillars on which the guidelines are based, I did not structure the workshop to discuss a special topic at a time, as I wanted to give them space to voice their thoughts. The discussions revolved around issues that had emerged in our discussions at the workshops, as was reflected in their suggestions concerning mathematical literacy. It was apparent that the teachers wanted to be able to respond to diverse needs and include all children in their school culture (Ainscow, 1995, 2007; Askew, 2015). The fact that they did not come up with any new suggestions for what to attend to in the mathematics chapter

indicated that they as professionals identify as classroom teachers rather than mathematics teachers (Guðjónsdóttir, 2000).

9.6.3.2 Mismatch in interpretations

When Pála shared her experience of working with word problems and algebraic symbolism at our last workshop, she expressed clearly how her pupils, had thought differently about these problems than she did and that she had respected their descriptions. Once again she wanted to reflect on her learning from interacting with her pupils and gave us detailed information about what she had observed the children do and how their interactions had affected her thinking, thus inquiring into her mathematics teaching (Jaworski, 2006a). She was happy to have experienced that her pupils were capable of solving these problems without her telling them what to do. This was a sign of her growing strength in supporting her pupils inquire into mathematics (Jaworski, 2006a), and moreover, she was gaining relational understanding (Skemp, 1997). It therefore surprising when she added that she thought her pupils soon would forget how to solve these kinds of problems, referring to her former experience in which she had emphasised instrumental understanding.

It was apparent that the culture in the teachers' community at school was colliding with the culture we had been establishing within our community, and that our discussion was on two levels. When Pála talked about her interactions with her pupils she was clearly describing that she had adopted an investigative stance into her work and aligning to our culture. When adding that her pupils would forget how to solve the same kinds of problems, she was aligning with the culture in her school, the one that she had represented in the beginning when she accentuated that her pupils were slow learners and dependent on her. She was clearly developing her knowledge for teaching and her persistence in improving her way of scaffolding her pupils in learning mathematics (Bruner, 1985) reflected her investigative approach (Ball et al., 2008). It was apparent that Pála's beliefs about her teaching were changing and even though she referred to her former way of thoughts about their learning, she could now envision that her pupils would be capable of solving similar problems later.

9.6.3.3 Inquiry learning in teachers' classrooms

When the project was coming to an end, I visited the four teachers that were still working at their schools, observed their lessons and interviewed them, where I noticed a shift towards an investigative approach to mathematics teaching (Jaworski, 1994) and inclusive practices (Askew,

2015). This is further described in Chapter 10. Their visions for their teaching had also changed, as appeared in Inga’s description of what she had learned from solving mathematical problems with us and how she learned to discuss mathematics with her pupils. Dóra and Pála had decided not to divide their classes into ability groups, even though it was still a custom at their school, thus challenging dominant culture. In the discussion of the teachers’ narratives this will be considered further.

9.7 Discussions of findings from recurring themes

Four themes repeatedly manifested themselves in our discussions as the project developed, which I will address in the next two sections. The notion of insider and outsider researchers, i.e. identifying who was carrying out research and into which practices, also emerged, and will be discussed in Section 9.8.

Table 2. Two recurrent themes that occurred throughout the project

Themes	Challenges	Possibilities
Pupils’ learning	Concerns for pupils’ incapability to solve mathematical tasks and understand teachers’ explanations.	Focus on pupils’ competences in solving mathematical problems and on discussing approaches with them.
Teacher learning	Reluctance to use means that could support reflection on one’s own practice.	Focus on how collaboration and reflective practices support the development of one’s own mathematics teaching.

Table 3. Different roles of outsiders and insiders into the research process

Activity	Insider roles	Outsider roles
The collaborative research project	Teachers and teacher educator	
Teachers’ learning	Teachers	Teacher educator
Teacher educators’ learning	Teacher educator	Teachers

9.7.1 Pupils’ learning, challenges and possibilities

During our collaboration at 17 workshops there were repeated instances of conflicting views about children’s capability to learn mathematics. This was

apparent in the process of coding from the outset of the study and confirmed when in later steps of the process of analysing the data. At our first workshops, Gróa mentioned several times that her pupils would not be capable of solving problems or understanding teacher explanations. At the same time, Gróa claimed that if her pupils could explain their way of solving problems, even if it differed from hers or the textbook instructions, she would accept their responses. This was indicated that she believed they might be capable of more advanced thinking than she had originally thought. She also often said that only a few of her pupils were capable of solving the problems we discussed, and added that other pupils required much more work, or in her words, had to be “shovelled with work”. Throughout the project, Gróa frequently addressed her disbelief about the children’s competences in mathematics, yet at the same time telling stories about her pupils’ capabilities in solving mathematical tasks.

These conflicting views were to some extent apparent in Pála’s reflections on her collaborations with her pupils. She did not express her concern about her pupils’ competences as clearly as Gróa, but she often reflected on how disappointed she was about how dependent they were on her, placing themselves in an accepting mode, and she wished that they would become more independent, that they would more often adopt an asserting mode, in the words of Mason and Johnston-Wilder (2006). When the teachers were solving problems at the workshops they sometimes assumed an accessing mode, like Gróa and Rúna did at the second workshop, when they asked me what they were supposed to do. More often though they entered an asserting mode by asking themselves probing questions and using their initiatives to make conjectures, and experiment with and reconstruct their ideas, as they did at Workshop 5.

The other teachers did not share this disbelief in their pupil’s mathematical competences to the same extent as Gróa and Pála did. Inga emphasised that the pupils identified as having special education needs were afraid of expressing themselves about their work and taking initiative, because they felt they had often received the message that they lacked competences in mathematics.

Rúna often mentioned that she found it important to explain algorithms for calculations and the steps needed to carry them out to her pupils, displaying an emphasis on instrumental understanding. Edda, on the other hand, said that she found it more important that her pupils understood what they were doing rather than learning by heart how to carry out the algorithms, indicating relational understanding (Skemp, 1976). Dóra

manifested her conflicts with her school's dominant culture at our first workshop when she wanted to discuss the traditional algorithm for division. She had been debating with her colleague, Pála, who felt it was important for their pupils to practice the steps needed to carry out the algorithm, and Dóra had questioned her view.

As our collaborative project progressed and the teachers experienced the process of discussing their own thinking in relation to solving mathematical problems, they started to listen to their pupils and discuss with them in proportion to the growth of their mathematical knowledge for teaching (Ball et al., 2008) and their competences in teaching mathematics (Niss & Højgaard-Jensen, 2002). Inga and Pála often gave detailed information about things that they had noticed and not been aware of before. For example, Inga's story of the size of the mouth of the blue whale, that she told us at Workshop 14 and further discussed in Section 10.4 and Pála's detailed description of her pupil's thinking, which she shared at Workshop 17. Their development echoes the process of shared thinking that Fennema et al. (1993; 1996) and Wood et al. (1991) (see Chapter 2) discovered in their research, namely, that when teachers began to increasingly emphasise on problem solving and collaborative learning, the teachers learned to learn from their pupils' thinking. Similarly, Boaler (2002) revealed that in focusing on the mathematical practices of their pupils, teachers learn about their pupils' learning of mathematics. This was also evident in my own research with the four teachers, and in my own development of teaching mathematics in primary grades (Kristinsdóttir, 2007; 2010a) (see also Chapter 3).

The negative comments about children's learning became less prominent as our co-learning developed and the teachers began to talk more about the positive things they had noticed.

9.7.2 Teacher learning, challenges and possibilities

The teachers' responses to my emphasis on researching into their own practices were contradictory. The three teachers in Rainbow School were all interested in using the tools I proposed for reflecting on their practices, such as writing reflective journals, observing and discussing each other's lessons, as well as recording their own teaching. At Sunshine School, Edda was the only teacher who showed interest in using these tools, although, during the first year, Rúna and Vala often noted the various ways in which they had focused on their communication with their pupils, and Gróa also shared her reflections on her teaching throughout the project.

At our first workshop, Edda and Dóra commented that when the teachers solved a problem about dogs and biscuits they had been defending their own way of solving problems and in supporting each other (9.1.2.2), thus acknowledging that we were creating a learning community in which the sharing of thoughts was respected (Jaworski, 2006a; 2006b). As the project developed, the teachers mentioned what they had learned from our collaborative explorations with mathematics. Dóra began seeking ideas for investigative approaches in the educative guides for teachers (Ahl et al., 2015) that accompanied the textbooks she was using.

Gróa was concerned about where the data collected from their work would be stored and who would read them. In response, I proposed that the teachers kept all the written material they collected themselves. They told their stories at our workshops, showed us examples of pupils' work, and we discussed their contributions. By responding in this way I gained access to their data and they could then use the information to support their further reflections. When the teachers told stories from their classrooms, we reflected together on what could be learned from their accounts of their teaching, thus elaborating on their experiences and reflecting on what happened in order to learn from it, focusing on the process of accounting for their experiences (Mason, 2002). The reflections regarding Pála's concerns about a pupil's misinterpretation of a task at an end-of-term test prompted her and Dóra to rethink their way of assessing their pupils and instead seek alternative ways of pupil assessment.

The teachers had some difficulties with finding time for writing cases from their classrooms and using the guidelines I had given them for analysing their accounts (Guðjónsdóttir et al., 2007). Responding to their concerns, I decided to allot specific time for them to elaborate on their stories at our workshops, and as the project developed more and more time was devoted to our mutual reflections and the teachers increasingly took the lead in deciding the focus. I sought to give the teachers room to find their own ways to approach their teaching through their stories of their pupils' learning. In doing so, I respected their underlying beliefs on teaching and learning, as addressed in West and Staub (2003).

In our discussions about the teachers' classroom experiences, I often challenged them to think about what they were learning from reflecting on their practices. During Workshop 14, Gróa responded that she was just doing routine work and there was nothing to talk about. When further challenged she added that the more she talked about her work, the more she realised that she had been doing something interesting indicating that

she had been reflecting on her behaviour, the outer layers of Korthagen's onion model and not necessarily on her beliefs, identity or mission, or the inner layers (Korthagen, 2004; 2013). When I first challenged the teachers to write about their experiences, Gróa said that it was a part of teachers' daily routine to think about what they had done and what they might do differently next time, even though they kept not written documentation of it. Her responses suggested conflicts between established norms within her school culture and the culture I was offering her to align with. The different 'Discourses' (Gee, 2004) within the communities we belonged to, the teachers' school communities and the teacher educators' research project community, might explain the dissonance in our understanding of reflective practices, especially if Gróa was not accustomed to discussing her core reflections. It was evident that she had a mission for her teaching, as she constantly referred to her pupil's dependency on her leadership and she felt responsible for guiding them.

At the outset of the project, Gróa's expressed her reluctance to recording our communication at the workshops, and she and Rúna did refused to record their own lessons. This underlines their role as safeguards for their vulnerable children and the latter's right of being anonymous. I respected their attitude and did not press them to record their lessons. However, the other teachers urged them to record their lessons and explained that it would help them learn more about their communications and interactions with their pupils. It took the teachers a long time to make the necessary arrangements to be able to record their lessons, and even though they found it helpful, none of them recorded their lessons again. This was a sign that the culture in their school communities did not support them in taking advantage of these tools and their zone of free movement was thus restricted (Goos, 2005; Valsiner, 1997).

When I proposed that the teachers should observe each other's lessons, their first response was that they would not learn much from observing each other teach and would rather prefer receiving good ideas from discussions with the other teachers and reviewing pupils' work. I emphasised that the goal was to learn about both their communication with their pupils and the latter's learning, and eventually they were all willing to give it a try. All the teachers then proceeded to observe other's lessons. In Rainbow School, the three teachers paid visits to each other's classrooms. They had not prepared themselves sufficiently for their first visit. From this experience they learned that it was important to be prepared for discussing with the children. As a result, they prepared together for their next visits and also reflected collectively on what they

had learned. In Sunshine School, the teachers did not prepare their visits and neither discussed them afterwards. Edda was dissatisfied with her visit to Gróa's classroom, as she did not have a chance to discuss with the pupils and wished that they had planned her observation beforehand. In her report, Edda was signalling that her zone of free movement was being restricted (Goos, 2005; Valsiner, 1997).

9.7.3 Researching from inside and outside

During the three years of collaboration within the research project we were all researching into our practices. The teachers were reflecting on their practices and gradually taking on the role as researchers into their practices as they started to pay more attention to the learning that took place in their classrooms, both their pupil's learning and their own learning. I was reflecting on my practice as a teacher educator, working with teachers, and as a researcher into our collaborative project. In these roles we were insiders into our research but also took on the roles as outsiders when we looked into each other's practices and collectively reflected on them, thus supporting each other in researching into our own practices (Jaworski, 2003).

The teachers expressed in different ways how they reflected on their practices and it was not always evident that they had been researching into their practices in a systematic way. However, the cases that Inga and Pála brought from their classrooms into our workshops were often a clear indication of reflective thinking. At Workshop 13, Inga told us about her discussion with a boy about metres and centimetres and her response reflected her beliefs and mission for her teaching (Korthagen, 2004; 2013) where she avoided reacting according to established norms (Mason, 2002). In addition, she had been reflecting on how children are labelled as incapable of mathematical learning based on a diagnosis of their responses in test situations instead of giving them opportunities to express themselves in a way that makes sense to them (Dalvang & Lunde, 2006). When Pála recounted her communication with her pupils about problems they were solving at workshops 14 and 17, and asked us to help her resolve her confusion regarding how to write equations, she was attending to her own awareness (Mason, 1998; 2008). In inviting us to reflect with her on these experiences, she was asking us to be her critical friends (Schuck, 2011) and in doing so, inviting us to adopt in the role of outsider researchers in relation to her practice.

The teachers were in dual positions when they observed each other's classrooms, both researching their own learning through the communication with the children and as they studied the pupils' work, they were learning about the children's learning. The teachers placed themselves in the role of an outsider researcher and a critical friend to their colleagues as they observed each other's teaching and collectively reflected on each other's way of teaching mathematics. The teachers, who were observed, then had a chance to reflect on their experiences with their colleagues who in turn supported them in researching into their own practices.

When we discussed the teachers' reflections on their teaching, we were all researching into our own learning within the project. This was particularly evident at Workshop 14, when we discussed the nature of reflection and why we were reflecting together on their teaching. This was a sign that we were all researching into our practices within the project.

When I offered the teachers to look into my primary school classroom during the first workshop and into my practice as a teacher educator at Workshop 16, they were offered the chance to take on the role as outsider researchers with respect to my work, which helped me in reflecting on my practice. They were also placing themselves in the roles of outsider researchers when they told me what they had learned from participating in the project, as expressed by Dóra, Edda and Pála at the first workshop where they pointed out how they had been supporting each other in the problem solving process.

We all brought different experiences into the research project and collectively framed it. The teachers were not experienced researchers but they all had knowledge about the children they were teaching and the environment within their schools I brought in knowledge about mathematics teaching and reflective practices. Our co-learning was dependent on the participation and learning of others, but what was learned was not the same for all, of the same form or at the same level (Jaworski, 2003; 2008b).

In Chapter 10, I will discuss further the findings from my outsider research into the teachers' practices, and in Chapter 11, I will present our narratives and my analysis of them and in that way reveal my insider research into my own practice of carrying out this research project.

9.7.4 Summary and conclusions

I have discussed themes that repeatedly occurred throughout the project and concerned the learning of pupils in mathematics classrooms and the teachers' ability to improve their practices as mathematics teachers. The

dominant view that some children are less capable of learning than other children because they do not carry out their work in the way the teachers expect them to do was apparent in our discussions. As the project developed, the teachers started to pay attention to their pupils' different pathways to learning. They found that if children get the chance to share their thinking with others and express themselves in a way they find meaningful, they are capable of more in-depth learning beyond what is normally expected of them.

The teachers' willingness to situate themselves as researchers within our community and research their own practices differed. They all showed interest in participating in our reflective discussions but the degree to which they were willing to use the tools for researching practice I offered them varied. When researching their own practices, they were insiders to their investigation and I was an outsider to their research. I was an insider to the developmental research process of this project and to my own development as a teacher and researcher participating in this process.

9.8 Conclusions of findings from the emerging collaborative research

The learning community we succeeded in building when inquiring into our practices was built on mutual trust that developed as the project progressed and was reflected in the teachers enthusiasm to carry on with the project two years longer than originally planned. There were though many hindrances on our way as we travelled through the landscapes of the professional communities we participated in (Wenger-Trayner & Wenger-Trayner, 2015). Our different perspectives on mathematics learning and diverse experiences as researchers into our practices, as well as the extent to which we aligned with the culture in the other communities we belonged to, shaped the development of our own community (Jaworski, 2008a).

During the three years of our collaborative inquiry into mathematics teaching and learning, the teachers made progress in inquiring into their practices. The degree to which they participated in the inquiry research varied (Jaworski, 2006a) and their engagement with the research process depended on their possibilities for such a venture, which was sometimes disrupted due to time constraints and lack of support. They all actively engaged with the mathematical tasks we explored with at the workshops and shared their experiences from their classrooms as we collaboratively reflected on them and supported each other in learning from their stories. We addressed many aspects of mathematics learning relating to our

awareness as learners. The teachers learned to position themselves in an asserting mode, asking probing questions and using their initiative in exploring into the world of mathematics (Mason and Johnston-Wilder, 2006). Gradually, they also took the first steps towards inquiry approaches in their classrooms. This will be discussed further in Chapter 10.

We discussed diverse approaches to mathematics teaching and the teachers' concerns for facilitating their pupils' mathematics learning. We debated to what extent the learning of algorithms (Fosnot & Dolk) supports mathematics learning and whether to emphasise learning through the memorisation of facts and instrumental understanding or relational understanding (Skemp, 1976), by making conjectures and trying out ideas. We also addressed inclusive approaches (Ainscow, 1995; 2007) in mathematics teaching and whether to separate pupils into groups based on a diagnosis of their weaknesses or to embrace all children in the school culture (Askew, 2015) by attending to individual needs through collaborative activities.

The teachers in Rainbow School all engaged actively in shaping the inquiry culture (Jaworski, 2006a; 2006b) of our community by bringing in resources they shared with us. Dóra had recently participated in a problem-solving course where she had the opportunity to reflect on her own learning habits and relating to her pupil's ways of learning mathematics. This experience empowered her to share perspectives with us that often challenged the views of the other teachers. Inga, who had a different background from the others, shared with us her skills of working with children who had special learning needs, and helped us focus on what we could learn from attending to their strengths instead of focusing on their weaknesses (Dalvang & Lunde, 2006). Pála helped develop the co-learning partnership (Jaworski, 2003) within our community by constantly bringing her concerns about her teaching into our workshops, asking for our support in resolving her conflicts. The boundaries between our community and the community at their school did not seem to create many oppositions even though the teachers started to question some of the traditions at their school as their alignment with our community developed (Wenger-Trayner & Wenger-Trayner, 2015).

The teachers at Sunshine School did not engage with shaping our community to the same extent, except for Edda, who always showed an interest in improving her teaching. She was new at her school but had extensive experiences from other schools and this was reflected in her open-minded attitude to our collaborative experiences. She often

questioned established norms at Sunshine School, indicating her critical alignment (Goodchild, Fuglestad, & Jaworski, 2013; Jaworski, 2006a) to the teacher community at her new school. Gróa, Rúna and Vala did not participate in all the activities within the project due to personal reasons and transfer to other duties. Gróa often questioned the means for reflective practices I proposed the teachers used (Korthagen, 2004; 2013; Kruger & Cherednichenko, 2006; Mason, 2002) and her alignment with established norms at her school was apparent. Her engagement in our community was mainly reflected in her willingness to investigate in mathematics (Jaworski, 2006a) at the workshops and in telling us about her experiences of working with children of diverse background. Rúna and Vala worked closely together at Sunshine School and were therefore in the position of observing each other teaching, and reflecting on their experiences together. They often shared experiences that added to our understanding of collaborative practices and ways to awaken children's interest in learning mathematics.

Edda often questioned the established norms at her school and she lacked support in fulfilling the goals with her teaching. In Rainbow School, the teachers never expressed that they lacked support even though they questioned the norms at their school. Dóra and Pála decided to ignore the established practice at their school to arrange pupils in ability groups and found that the culture in mixed ability groups favoured all learners. Their close collaboration throughout the project helped them gain confidence in their work, and Inga, who also met them when preparing their teaching, found that she benefited from their collaboration.

The results from this three-year developmental project indicate that the teachers gained confidence in reflecting on their practices and in researching into their work. Their approach to teaching changed as they emphasised inquiry based approaches by the end of the project as opposed to highlighting instrumental understanding at the outset. The support from their communities at school had effect on their experiences and was vital in sustaining their confidence in fulfilling their goals of improving their practices. This is further addressed in Chapter 10.

I expected that conflicts of interest might arise since I was one of the authors of the textbooks that were used in the schools I worked with. At in-service courses, I had experienced that some teachers expected to be told how they should teach their pupils to solve the problems in the textbooks. At the outset of the study, I therefore emphasised that my goal was to support the teachers in developing their practice and to avoid telling them how to teach. I never initiated discussions about the textbooks but the

teachers often discussed how they worked with problems in the books and asked for my advice. To begin with, they admitted that they had not often looked for advice in the teacher guides that accompany the books, but they gradually started to use them as the project developed, and I pointed out what kind of assistance they could find in the guides. These guides were written with the goal of supporting teachers make informed decisions about their teaching and were classified as ‘educative guides’ in the findings from Ahl et al. (2015). Dóra told us several times how she had taken advantage of these guides and her positive experience of extracting good advice from them urged her to use them more often. Pála and Dóra told us at Workshop 14 that they had found help in these guides with reviewing their ways of assessing their pupils’ mathematics learning, as they began to question their former ways of making tests.

As a teacher educator I learned to respect the teachers as professionals who are capable of developing their practice. In our community a co-learning culture was cultivated as we all participated in shaping the culture of our community. The degree to which we aligned to this culture differed as will be further addressed in Chapters 10 and 11.

10 Narratives of teachers

The second question of Part II will be answered in this chapter, by presenting narratives of the teachers and then discussing the analysis of their participation within the project. The question that will be answered is:

- When teachers are participants in a project based on collaborative inquiry how do they perceive this as being reflected in their mathematics teaching?

Narratives of the four teachers who participated in all phases of the study, Dóra, Edda, Inga and Pála, from August 2009 to February 2013 are presented and discussed in sections 10.1 to 10.4. At the outset of the project, seven teachers participated in the project, however, the three teachers, Gróa, Rúna and Vala were not able to join us in all phases for different reasons. Their participation will be attended to in section 10.5, in relation to common threads traced in individual teachers' development. The narratives are based on an analysis of videotapes from 17 workshops, audio recordings from four interviews with each teacher, notes from three classroom observations in their classes, samples of children's work, and my own reflective notes. In Section 10.5 the narratives are summarised and common threads discussed.

Through the analysis of the teachers' participation in the project, different themes developed for each teacher. Some themes were common to two or three teachers but the participation in the project led to differing developmental trajectories for each teacher. The themes that developed during the coding process were not the same as developed within our collaborative community. The teachers' backgrounds were diverse and their experience of teaching mathematics differed. Their learning path through this project depended on their earlier experiences, the support within their communities at their schools as well as their personal characteristics. All the quotes in this chapter are taken from my own field data, including classroom activities and interviews.

10.1 Edda

In her teacher education studies, Edda specialised in teaching Icelandic and social sciences. When she joined the project she was new at Sunshine School, but had taught for a few years in grades one to four in two other

schools. She had lived for seven years in a foreign country where her own children went to two international schools and she found it fruitful to learn about the work in those schools and how she could support her own children's learning.

10.1.1 Developing her own teaching in a new school

When the research project began Edda was the classroom teacher for 27 pupils in 5th grade in Sunshine School. In Icelandic and mathematics lessons 8-10 children went to the support centre where Rúna and Vala taught pupils who needed special support. Edda found her work challenging as she had never taught 5th grade before and she was new at her school. Therefore, she depended on the support of her colleagues.

In the first visit to Edda's class:

There were 19 children present and a class assistant who supported one boy. Edda used a smart board (a digital board connected to the teacher's computer) to show the pages in the textbook she was attending to. She asked the children to help her fill in the answers to the first questions and wrote down their responses and then told them to proceed with their work. During the lesson she walked around and discussed with individual children. She called for the attention of the whole class several times to discuss questions from individuals. There was no time for a review or summary at the end of the lesson as the children who had been to the support centre came back and another lesson started. (Notes from observation September 15th 2009)

We discussed the lesson later the same day and Edda was concerned about her pupils' dependence on her telling them what to do. She felt that many of her pupils lacked a sense for numbers, especially for large numbers, and they needed to practice mental calculation. She responded to this by teaching them to play card games that required certain skill from the participants at mental calculation and found that they benefited from this.

At our 3rd workshop, Edda talked about her experience of a problem-solving lesson in her 5th grade class where the children solved a problem from our first workshop. She found that many of her pupils were hesitant to approach the problem and wanted her to tell them what to do. Edda said:

There were at least five pupils who were, could not begin. Just looked at the page, would not write anything on this page. It might all be wrong. I said: "Just try to write something, try to draw".

Edda was concerned about how reluctant they were to approach their work independently and always required her to explain to them what to do.

She felt that it was difficult to teach them to be self-reliant and tried to find ways to motivate them.

Later, Edda told us how surprised she was to see how the passive children became interested in geometry and work with Tangram patterns:

Yesterday, it was so much fun when we worked with the Tangram in Geisli [the textbook for 5th grade]. And then, it was so nice to see two girls that never have confidence to initiate anything they just did it. Did not need to ask any questions. They were more confident than the others and somehow seemed to be more able to visualise the patterns.

Edda was happy to learn that these girls enjoyed solving the puzzle and showed initiative in their work.

By the end of the first year, Edda was satisfied with how her work had developed during her first year at Sunshine School. She still found her pupils dependent on her and wanted them to become more self-reliant. She was though happy to see them make progress.

The pupils are more willing to discuss their thinking. They are starting to think about if their answers make sense. This is new and I feel that they are making progress. I want to keep on with problem solving and urge them to reflect on their work.

At Sunshine School, children were regularly tested on multiplication facts and Edda did not oppose this tradition even though she did not believe that it helped her pupils:

I am always thinking about if I should give up on trying to teach them the multiplication tables or not. But anyway, I feel quite confident in teaching the way I do. Working with you at the workshops also gives one something, opens your mind; one is more creative and thus more open to new methods. I do not feel as pedantic or stuck, like this has always been done this way.

She was concerned that at her school's old tests were being used to assess the children's learning where the emphasis is placed on assessing instrumental understanding and knowledge of facts. In the new mathematics curriculum and textbooks, the focus is more on inquiry and relational understanding, and Edda felt that the teachers at her school had not adapted their teaching in line with this approach. She felt that they needed to review the way they taught and assessed their pupils learning

and referred to some of her colleagues who emphasised rote learning. She was concerned that her pupils were not quick at recalling facts and might not do as well as pupils in other classes on the end-of-term tests.

At the workshops we often discussed what was achieved by working with the same problem over a long period of time, even for many lessons. When we at Workshop 11 observed a teacher discuss “the Border Problem”, Edda said:

Somebody would say that it was a waste of time to use a whole lesson for this small task that one could solve in two minutes. But if you solve it in two minutes, and then proceed with other work, then the time comes when many fall behind and do not understand well enough.

Edda’s concerns for her pupils’ independence in their mathematics learning was apparent and she sought to develop her own teaching to be able to more fruitfully support them in this process.

10.1.2 Efforts to improve her own practice

Edda was eager to improve her practice and learn from collaborating with others. When Gróa observed Edda’s teaching she discovered how helpful it was to have a chance to discuss what she had noticed in the classroom:

We are a bit stuck in our ways and do not notice what they are really doing and discussing; this is what happened today. What Gróa told me, what she noticed, I had not noticed that myself.

When Edda visited Gróa’s class, the latter led a whole-class discussion during most of the time. Edda realised that she would have gained more from her observation if they had planned her visit better and if the children had engaged more in exploring on their own and in groups- discussions:

But I like to look into other classrooms and observe other teachers, because you always learn and see what they do, how they explain, how she does this and this, that one can learn from and maybe take in and assimilate. We did not plan this beforehand but we might plan another visit with a problem-solving approach.

At the end of the first year of our collaboration, Edda discussed her participation in the project, and said that she found it useful to meet with us at the workshops:

One often gets stuck in a rut and you get somehow uninventive when you are alone; ... everything we have done makes us reflect on our own work. And then I use it in my teaching, even though I do not always make use of everything we have done. But these kinds of tasks, I take notice of them.

When Edda recorded her own lesson, she discovered that she had problems with motivating the pupils' interest and was too monotonous in the way she talked. The children paid little attention to her and she sometimes had to raise her voice. She also found that instead of asking the children questions, as she intended to do, she ended up explaining to them what to do and then she even answered her own questions. She often misspoke and believed that it might have confused the children. Edda found this experience helpful and decided to record her lessons again. She often referred to this experience and her lack of getting the message through to the children, whom she felt were not always listening to her at all.

10.1.3 Attending to learners' diverse needs

Edda taught the same children in the second year of the research project, now in 6th grade. She was still the classroom teacher for all the children in 6th grade and in mathematics classes the children were randomly grouped into three groups. The school principal had decided that the children would not be arranged into ability groups in mathematics and Icelandic lessons, as was previously the custom at the school. Many of the teachers opposed this new arrangement but agreed to try it for one term and then review the decision. Edda felt that testing this arrangement was interesting and looked forward to her second year in Sunshine School. She later told us that she found this arrangement demanding and often talked about the children of who she felt worked independently and said: "It is difficult for clever kids who always need to study on their own."

At my second visit to Edda's classroom, she brought in a fraction task we had worked with at one of our workshops.

The children wrote fractional parts on the "sticky labels" and explored with arranging them from the lowest to the highest. The children asked for fraction models and Edda told them to use the area models (circular models) that were at hand. Two girls did not get any models to work with and seemed confused. I asked them which one of three fractional parts was the highest and they were quick to answer. They kept on arranging the labels and when Edda later gave the children a length model with fractions written on a number line they used them to confirm their arrangement. (Notes from observation May 19th 2010)

Edda emphasised that her pupils need to have access to mathematical models and believed that hands-on activities could help them in developing an understanding of mathematical concepts. However, she lacked access to manipulatives at the school, as most of them are stored in the classrooms for younger children.

In my last visit to Edda, she taught a 5th grade class. This year the pupils were again arranged into ability groups in mathematics classes and Edda taught those who were identified as slow learners. Edda was satisfied with this arrangement and felt that she could appropriately attend to the children's diverse needs when she had a group with only eight children.

On the agenda was division and Edda showed the number line on the smart board and handed the children plastic strips with number lines written on them. She showed them how to find what 40:5 equals by starting at 0 and counting by increments of 5 until reaching 40 and then counting the steps taken. She showed more examples and sometimes asked for the children's response. Some of the children solved the problems on their own but others followed Edda's lead. She then asked them to make their own problems and they came up with numbers that did not divide by the divisor chosen. They then discussed what to do with the remainder.

Edda asked the children to solve some division problems with the support of counters and paper money that they used as a tool for calculating. At the end of class, Edda asked the children to recall what they had learned and what methods they liked to use when solving division problems. Some mentioned the number line, others the counters and still others the paper money, and at least one mentioned that it was good to use all of them. Edda added that it was useful to be able to approach problems in diverse ways. (Notes from observation February 5th 2013)

In my last interview with Edda, we discussed her work with the slow learners. Her pupils had not been arranged into ability groups the two previous years and they had often worked individually with their textbooks. Edda found that when she had a homogeneous group in mathematics classes she could work more with the whole group than when she had a heterogeneous group. She often brought in manipulatives like base ten blocks, number lines or fraction models for the children in order for them to visualise what they were dealing with. She found it important to discuss different approaches to solving problems and was happy to learn that her pupils were willing to discuss their thinking. She felt though that the children in her group required much support:

We work this out together. I do lead them, use the smart board, I must admit that. It is only when they are working with something they are more experienced with that I allow them to work at their own pace.

10.1.4 Participation in the research project and in class

Edda tried to approach her teaching in a variety of ways to make her mathematics classes enjoyable and to encourage her pupils' understanding. She sought to develop her teaching by drawing on what she learned at the workshops where she was always active, engaging with the other teachers on how she solved different problems. When we discussed their work, she often told us that her pupils did not like to explain how they calculated and that she found it important to teach them how to share their thinking and collaborate. Edda often underlined the significance of teaching for understanding and that she believed it took time to change one's approach. Rote learning might strengthen children's abilities to recall basic facts but they forget them quickly:

When I went to school I learned all kinds of procedures, calculated something but never understood why this worked, just totally thoughtlessly ... I had no basis for learning algebra. That is why I think it is better that they deepen their understanding before they jump into using procedures for quick calculations.

Later, Edda told us that she was happy to see how good her pupils were at writing their own word problems. She had collected their work and made a copy for everyone and the children were eager to solve each other's problems. Edda told us that the content of their problems varied and the complexity reflected their mathematical knowledge.

During my final interview with Edda we discussed her participation in the project and once again she mentioned that she felt she needed support to be able to develop her teaching:

I found it illuminating and enjoyed tackling the tasks that you brought in and discussing with the others. One often gets stuck in something, in one's own world, how one approaches things. To open up to diverse approaches you need to practice ... and the discussions about our teaching were fruitful.

Edda had not participated in mathematics education courses but said that she would like to attend some. She told us that she reads the mathematics teachers' journal, *Flatarmál*, and that she notices what teachers have written there about their own mathematics teaching.

10.1.5 Edda – an active learner

10.1.5.1 Visions for teaching and learning

Edda had a strong vision for her mathematics teaching and often mentioned that she wanted her pupils to deepen their understanding in mathematics. She referred to her own learning at school, where she had learned mechanical procedures without understanding what she was doing, and therefore had no basis for learning algebra. In her teaching she wanted to emphasise relational understanding as opposed to instrumental understanding (Skemp, 1976), indicated by her reflection on her own learning at school and her emphasis on her pupils sharing their thinking about their problem solving. Her beliefs conflicted with the views of many of her colleagues and she often mentioned that they overemphasised rote learning and procedural knowledge (Hiebert & Carpenter, 1992), and testing pupils' knowledge of facts. She was concerned that her pupils would not do as well as pupils in other classes when taking these tests because she did not strongly emphasise rote learning. However, she was reluctant to oppose the tradition of using these tests to assess pupils' learning. This was a clear sign of her critical alignment to the community at her new workplace despite the difficulties she had with aligning to it (Jaworski, 2006a; Wenger-Tayner & Wenger-Tayner, 2015). She indicated several times that she felt her zone of free movement was restricted (Goos, 2005; 2008; Valsiner, 1997) and mentioned the shortage of manipulatives at her school and the culture where rote learning was emphasised. Additionally, she mentioned several times that her pupils wanted her to explain to them how to solve tasks and they were dependent on her telling them what to do. They were thus placing themselves in an attending mode instead of showing initiative and taking on an asserting mode (Mason & Johnston-Wilder, 2006) as she wanted them to do.

10.1.5.2 Participating in professional development

Edda always participated actively in our collaborative work at the workshops. She enjoyed exploring with the problems we dealt with and freely discussed her thinking about them with us. She was worried that she did not succeed in awakening her pupils' interest in exploring with mathematics and asked for our advice, voicing her transformation concerns (Rowland, 2014; Rowland et al., 2005). When I observed her teaching, her awareness of the cognitive demands of different tasks was emerging, reflected in her discussions with her pupils about fractions and later about division. The way she rounded up with her pupils in my final visit to her was

an indication that her competency as mathematics teacher was developing (Niss & Højgaard-Jensen, 2002).

Edda was consistent in applying the opportunities acquired through participating in the project in order to develop her professional competency as a mathematics teacher (Niss & Højgaard-Jensen, 2002), both by engaging in discussions with us and finding ways to improve her way of working with her pupils. Edda utilised the means for professional development that I proposed to the teachers. She shared with us what she learned from paying attention to the way she noticed her pupils' learning and lucidly accounted for her interpretation of what she had noticed, such as her pupils' dependency in relation to problem solving (Mason, 2002). She also recorded her lessons and participated in mutual visits to classrooms.

As our project developed, Edda reflected more often with us on positive things she noticed in her classroom like when she saw girls who lacked confidence in approaching mathematical tasks engage in their work with a Tangram puzzle in an independence way, and thus Edda shifted her focus to their strengths instead of their weaknesses (Dalvang & Lunde). She was satisfied with her participation in the project and felt that it had supported her in developing her teaching, acknowledging that her zone of promoted action was developing (Goos, 2005; Valsiner, 1997).

It had been a tradition in Sunshine School to send children identified as having problems with learning mathematics to the support centre for mathematics classes. Edda said that she was aware that this arrangement did not benefit them and therefore miss participating in discussions with their classmates. Still, she felt that it was difficult to combine children in the same group who were quick to solve problems and those that were slow, voicing her conflicts about inclusion and integration, as described in Ainscow (1995; 2007). When she solved problems at our workshops, she did experience how people with diverse backgrounds could support each other in learning mathematics, as emphasised by Askew (2015). Edda made an effort to ensure to provide her pupils with a chance to explore with mathematics in different ways and access to manipulatives like counters and fraction models to support them in detecting patterns, expressing these in diverse ways and making conjectures, as Mason (1999) and Mason and Johnston-Wilder (2006) have emphasised. She often mentioned that there was a shortage of manipulatives at her school, thus indicating that her zone of free movement was restricted (Goos, 2005).

In my visits to Edda's classroom, I saw a steady progress in how she approached her teaching, always trying to lead discussions with her pupils

and urging them to discuss their work. In my last visit to her she was teaching a group of children that had earlier been sent to the support centre and she felt that they needed her support and guidance since they were dependent on her. Yet, she strove to encourage them to show initiative in their work.

Edda helped in shaping our community within the project by participating in our reflections, voicing questions, giving advice, asking critical questions and responding to the concerns of other participants. In the words of Wenger (1998) and Wenger-Trainer and Wenger-Trainer (2015), she was aligning herself to our community by a two-way process of following our plans and coordinating our work.

10.2 Pála

At the time our collaboration began Pála had taught in Rainbow School for almost thirty years. In her teacher education she specialised in teaching Icelandic and textiles and has mainly been a classroom teacher in grades five to seven. Teaching Icelandic has figured as her main interest and she has participated in developmental projects on the subject. Pála is enthusiastic about teaching and learning in general as well as the welfare of all her pupils.

10.2.1 Ability grouping

At our first meeting we discussed the ability grouping in Rainbow School. The school leaders had made a decision some years ago to arrange children into ability groups in mathematics classes as well as in Icelandic classes. In mathematics classes Dóra teaches those that received the highest grades at an end-of-term test, Pála teaches the mid-level group and Inga teaches the children in the lowest grade category. The idea behind this system is to individualise the teaching of these subjects, however, as Pála put it:

The idea is of course to meet everyone's needs, but this is probably discrimination, this is classification. ... But I think that ability grouping in mathematics classes suits the children better than working in mixed ability groups.

In my final interview with Pála she told me that she and Dóra no longer grouped the children according to ability in mathematics classes. Pála and Dóra teach the pupils in their classes and two from each class attend Inga's special needs education classes, where she now teaches children from grades five and six together. Pála was satisfied with this arrangement even

though she thought she was more capable of meeting children's individual needs when she taught children in the mid-level group:

This works well ... most of them are good at mathematics and they like to have extra work to do. I do not demand as much of those who are not as capable but I do not individualise the teaching, they just do less than the others.

Pála was concerned about some of her pupils who needed a lot of support, both in learning mathematics and other subjects. She would like more access to professional support for those children and finds that the resources at her school are too limited.

10.2.2 Children need to be guided

At our first meeting, we discussed the use of new mathematics textbooks at the school in which an emphasis is placed on problem solving and explorative work, and there is more text in the new books than in previous editions. Pála said that she liked the books and claimed they were suitable for children with good reading comprehension, but many of her pupils find them difficult. She feels that the pupils in her group need a lot of support and that they require explanation with regard to how to proceed with their work. "But they get tired when you are always explaining. They want to work individually like they could with the old textbooks."

My first visit to Pála's classroom:

Pála started the lesson by reviewing homework and then discussed the content of the lesson. She described carefully to her pupils how to work through the problems in their textbook that she wanted them to solve. She then talked to individual students as they went on with their work. Most of the children used calculators, even though the problems could easily be solved with calculating mentally. One boy talked loudly and often shared his thinking about his mental calculations but Pála did not encourage discussions about his utterings. When the children had finished solving these problems they fetched another mathematics textbook and worked individually at their own pace through the problems in that book. (Notes from observation September 8th 2009)

When I observed Pála's class at the end of the first year of our collaboration, the children were working in groups with fractions.

When the lesson started Pála needed to talk to two girls and help them resolve a conflict between them. She told her pupils to work with fractional problems she had given them a few days earlier. I discussed with individual pupils about their work and found that they were confident in solving these problems. When Pála came back she gave the children a task from our workshops that included ordering fractional numbers on a number line. She referred to what the children had already learned about fractions when she introduced the task. The children then worked in pairs and ordered

the numbers from the lowest to the highest. Pála gave them fraction strips (length model with fractions written on a number line) they could use to help them find the relative size of the fractional parts. She discussed with the children as they worked with the problem and then told them that they would discuss their work collectively next time. (Notes from observation May 11th 2010).

We discussed this lesson a few days later and Pála said that she liked teaching about fractions.

I discuss the fractions from different points of view with the children. I ask the class and we talk about it, find this out together. ... I have a sense of what children remember. I take examples from the children's environment to further develop the concepts. The children remember better the fraction concepts than other mathematical concepts. I do not know if it is because they understand it from the beginning. ... The children tend to forget about other things, like the processes for the algorithms, such as borrowing and carrying in addition and subtraction.

Pála often told us how dependent her pupils were on her support and that she needed to lead them to find ways to solve problems. This was evident when she was dealing with new things, like when we worked with concept cartoons at Workshop 4: "I think it would take them a long time to figure this out, I would need to help them. I think the form ... I need to reflect on this myself."

10.2.3 Listening in class

Pála recorded a lesson of hers towards the end of the first year. She learned that she talked most of the time and wanted to become better at listening to her pupils. She recorded her lesson again in the fall and discovered that she still talked a lot and repeated again and again what she felt was important to explain: "But I think it might have been important for my message to come through to everyone."

When Pála recorded her lesson for the second time the children worked with the border problem' that we had solved together at Workshop 11. Pála emphasised collaboration and that they needed to explain their thinking about the problem to the others in the group. She felt that it took the children a long time to figure out a rule to find the number of squares for a grid of any size. The children were though focused throughout and enjoyed their work, but she was still concerned about the lengthy time for just one problem:

But then when I listened to the recording at home I learned that the time was not too long because they were not tired at all. One often thinks; I cannot spend too long on this, I must not make them tired. But we just went through this until all of them understood.

Pála was surprised of her pupils' focus on the task and that they could find the rule. She was satisfied to learn how quickly one pupil, who often has difficulties with concentrating in class, was to solve this problem.

Pála is concerned about her pupils' learning and is constantly trying to find ways to support them. At our first workshop in the second year of our collaboration, she told us how happy she was with her pupils' progress. The pupils were working with problems in their textbooks and Pála had expected them to be too difficult:

The problems require that you have good number skills and developed mathematical thinking. They are really good. Children who I did not believe could think so independently, it was so rewarding to follow their reasoning, how they figured things out and the ways they solved the problems and how they thought. All the time they were focused on their work. This took me by surprise, it is not often one experiences this.

However, during the next workshop, she had once again become concerned about her pupils' development. She wondered if she demanded too much of them, as they did not seem to be able to solve the problems she posed, and she worried that they were unable to think critically and reason about their work.

In my last visit to Pála's classroom, she had decided to let her pupils explore with the measurement of an area. They had been learning how to measure the area and the circumference of shapes and she wanted to see if they had a feeling for how big one square metre is.

Pála asked the children if they knew how many square meters the apartments they lived in were. They also discussed how big their classroom might be. There were two tasks on the agenda for this class; one was to draw one square meter on the floor of the classroom with the help of a metric stick and the other was to draw a sketch of the apartment they live in. The children worked in groups and Pála talked to them as they were working on their projects. Some of them seemed to have a good sense for the relative size of rooms in an apartment but others did not bother much about the project and seemed to be more interested in drawing all kinds of rooms for leisure and sports. Pála talked with the children while they were working with the tasks and asked them questions about their work. At the end of the lesson, she told the children that they would proceed with this work later. (Notes from observation February 8th 2013)

In the final interview with Pála we discussed this lesson. She was satisfied with how interested her pupils were in this project and how focused they were on their work. She told them to ask at home about the size of their apartments where they would need to revise their drawings in accordance with this new information. She hoped that the drawings of the squares on the floor would help them visualise the area of one square metre and also help them connect to the size of their classroom. She showed me some of the drawings. There was a two-storey house on one of them and a garden around it in the scale 1:100 that was clearly depicted. Pála said that the girl who drew the picture enjoyed working on projects where she could express herself visually and create something new. Her native tongue is not Icelandic and she has had difficulties with expressing herself in Icelandic. She therefore benefited from being able to use the language of mathematics to communicate and to develop her mathematical thinking. We also discussed the drawings of two boys who have concentration problems and are reluctant to write or draw. They participate in discussions in mathematics classes and feel confident in learning mathematics by communicating their thinking orally.

In Workshop 13, Pála wanted to discuss with us an event from her class the same day. The teachers had put together an end-of-term test and used tasks from older tests without reflecting on the wording of the tasks. One girl misinterpreted the instructions and our collective reflections on this incident resulted in further discussions about the goal with tests and how to structure them. Pála was concerned about the way pupils at her school were assessed. She was satisfied that self-evaluation and formative assessment were part of the end-of-term grades, yet she claimed that these tests need to be reviewed, even though she was not generally opposed to the structure of the evaluation program per se. Later, she and Dóra decided to use tests that came with the textbooks they were using where an emphasis was placed on explorative work as opposed to the types of tests they had used, where instrumental understanding was assessed.

10.2.4 Participation in the research project

Pála participated in all 17 workshops and was always active in the discussions, both about the problems they dealt with and our discussions about children's mathematical learning. She often asked questions that provoked discussions about mathematical thinking and learning and was eager to experiment in her teaching with ideas that we had discussed at the workshops.

At the first workshop, Pála pointed out that children would benefit from solving problems in groups and initiated the discussion about cooperative learning that we developed further in the second workshop. She often asked for more information about things I presented at the workshops, like access to videotapes, links to websites, and copies of written information.

She commenced a discussion at the first workshop about rote learning and memorisation, such as learning the multiplication tables by heart or using a traditional algorithm without understanding what they are based on. She wanted to discuss with us if these methods might undermine actual learning. Her questions stimulated discussions about traditions in mathematics teaching and what we can learn from new research on mathematics teaching and learning.

When I urged the teachers to visit each other's classrooms, Pála thought it might be worth a try. She shared Gróa's scepticism with regard to the benefit of observing a full class, but Pála was still interested in observing another teachers style of teaching as how the pupils begin their work. She visited both Dóra's and Inga's classrooms and felt that sitting with the groups and discussing their work with them was rewarding. These three teachers then met and discussed their observations and all of them found it helpful to have visitors in their classes who noticed aspects that have otherwise gone unnoticed and on which they could reflect on together.

Pála was pleased with her participation in the workshops. She felt that she gained a lot from talking with the other teachers about how they solved mathematical problems and also from watching the videotapes of mathematical classrooms where teachers discussed mathematics with their pupils:

It is good to work in such a small group and people are able to express themselves ... you pointed out many things that have been useful. I could do more of talking to the whole class and urging them to think. I often do talk to the whole class, but could do more of it.

Pála's stories from her classroom, changed as our project was approaching an end. At former workshops she had often raised concerns that her pupils were dependent on her explaining solutions to tasks. Now she told us how she had explored together with her pupils and wanted us to reflect together with her on her experience. Examples of this are in workshops 14 and 17.

When we discussed our collaboration within the project, Pála said:

I feel that this has opened new dimensions for me in many ways and just how much one small task can include so many factors that can deepen the pupils' understanding in many ways. I feel that I am more aware what kinds of tasks train this, this and this, and what kinds of tasks deepen this kind of understanding and this kind of understanding. Then something new always generates new considerations. I am not sure that I would have thought as much about my instructions and explanations as I have done lately. I have become more aware of this than before.

We discussed the development of mathematical concepts at our last workshop and Pála said that we needed to talk more about mathematics than we have done. She tries to make time for discussions but her schedule is tight and she feels that she needs to follow her plan. When we worked with problems at our workshops, Pála often mentioned that she has seen a similar problem and that she likes to solve problems that are published in the newspapers. Moreover, she commonly brought problems into our workshops that she wanted to discuss with us, and some of these problems she had posed to her pupils.

10.2.5 Pála – a careful and conscientious teacher

10.2.5.1 Care for children

In my initial visit to Pála, she described to me how the teachers at her school had come to the conclusion that in mathematics and Icelandic classes they would group the children into ability groups. The special education teachers then taught those who needed extra support. They were therefore making minimal arrangements in responding to diversity by integrating children with special learning needs instead of focusing on approaches that enable the inclusion of all children in the learning at school (Ainscow, 1995; 2007). Throughout the project, Pála often reflected on this arrangement. When she became more confident in inquiring into mathematics at the workshops and adapting a similar approach in her teaching, she found that it favoured all children and they benefited from the contributions of children with diverse background (Askew, 2015). In my final visit to her to class, she confirmed this view when she talked about how she responded to the children's needs in a mixed ability group by allowing them space in which to manifest their capabilities, thus focusing on their strengths (Dalvang & Lunde, 2006).

What characterised Pála's teaching to begin with was how determined she was in describing to her pupils how to carry out their work. She was

careful to ensure -that they knew how to solve mathematics tasks and this was reflected in her stories on her pupils' dependency on her. This was a clear indicator that she expected her pupils to enter an accepting mode (Mason & Johnston-Wilder, 2006). Nevertheless, she often mentioned that her pupils were often tired of her continuous explanations and instructions. After listening to the recording of her lessons, she became aware that she occupied most of the time and asserted that she wanted to become better at listening to the children.

10.2.5.2 Children's understanding

It was obvious that Pála wanted her pupils to understand what they were doing and she often related to their former work when she introduced new concepts into her discussions, voicing her beliefs about relational understanding (Skemp, 1976). When we worked with problems at our workshops, she was eager to discuss her understanding of the problems to empower herself to reflect on them together with her pupils. Her decisiveness in learning to reflect on how she taught her pupils was an indication of her awareness of the importance of teacher pedagogical content knowledge (Ball et al., 2008; Schulman, 1986). The foundation dimension of 'The Knowledge Quartet' (Rowland, 2014; Rowland et al., 2005) was clearly reflected in her work as she wanted to gain a deeper understanding of mathematics and her beliefs were evident concerning the purposes of mathematics education and the conditions under which students will best learn mathematics.

Pála often asked for our support in making sense of her own work in the classroom, for example, when she brought in problems that she had been thinking about or when she was content with her level of understanding of certain issues. She was clearly focused on conceptual knowledge or rich understanding, which Hiebert and Carpenter (1992) refer to as 'webs' of interrelated ideas. She wanted to understand what she was teaching and to begin with she indicated that she could transfer her knowledge to her pupils. Later workshops confirmed her growing awareness of the importance of discussing her thinking with her pupils and listening to them in deciding what to focus. Her competencies of teaching mathematics and of revealing learning were thus developing (Niss & Højgaard-Jensen, 2002) as she planned and carried out her teaching and related to her interpretation of her pupils' learning.

In my second interview with, Pála she told me that she was good at teaching about fractions, and her pupils remember the fraction concepts better than other mathematical concepts. She said that they tend to forget

the steps of algorithms such as carrying and borrowing of the addition and subtraction algorithms. The reason for this, she said, might be that they did not understand the process from the beginning. She was thus indicating that certain integral aspect had not been explained well enough to the children in earlier. It was interesting to learn about her consistency in explaining the mathematics to her pupils and her belief that her understanding of problems was enough for her to make them understand it too. When I planned our workshops, I sought to challenge this view by motivating the teachers to share their thinking about the problems they solved, as opposed to telling them what to do, with the notion in mind that teachers can neither make their pupils learn nor do their learning for them (Mason, 2008).

10.2.5.3 Sharing experiences

Pála often told us about her interactions with her pupils where she reflected together with them on the problems they were solving. This was evident at Workshop 14, when she told us about her explorations with her pupils on the total number of handshakes in their group, and at Workshop 17, when they explored and reflected together on how to write equations for information written in text. Her involvement with the other teachers in solving problems at the workshops seemed to have enabled her to transfer this experience to her classroom and engage in co-learning with her pupils (Jaworski, 2003, Watson & Mason, 2007). It therefore came as a surprise when, at Workshop 17, she claimed that her pupils' impressive abilities in dealing with algebra would probably soon be forgotten. However, I asked whether she believed that they could relate to their collective experience of problem solving and she answered affirmatively. Her aligning with the culture in her school's community, where emphasis had been placed on rote learning and repetition, was visible as she was negotiating the boundaries between our learning community and the one she belonged to at her school (Wenger-Trainer & Wenger-Trainer, 2015). The way we discuss our work and our tendency to focus on pupils' weaknesses instead of strengths is all too prevalent in the cultures within our schools (Dalvang & Lunde, 2006) and it takes time to change these notions on learning.

10.2.5.4 Developing as a professional teacher

Pála participated in all our workshops and took active part in all our activities. She was keen to try out all the means for professional development and ways for researching one's own practice that I proposed - and asked for my advice as well as guidance from the other participants.

She gradually learnt to take the lead in what to work with as the project developed by bringing in problems to look at with her and sharing her concerns and successes with us. She therefore took an active part in shaping our learning community and aligning to our culture by coordinating and engaging with our ventures (Wenger-Trainer & Wenger-Trainer, 2015).

In my visits to Pála's classroom, I saw much progress in how she worked with her pupils. During the first visit, she carefully explained to her pupils the steps to carry out when solving problems in the textbook. When one boy shared his thinking about his mental calculations she did not encourage discussions about his. In my final visit to her, the children explored with measurement and to draw a plan of their homes. She urged them to collaborate and discuss their findings and then to gather information at home to further explore with their work. She also pointed out to me a girl who did not speak Icelandic and how these explorations enabled her to participate in the collective work. This experience supported this girl's identity construction as she was empowered to her use her strengths in geometry in collaborating with her classmates (Dalvang & Lunde, 2006; Valero, et al., 2008). Pála has developed in her mathematics teaching, now emphasising exploration in tasks and problems (Jaworski, 2006a) as opposed to instructing her pupils how to solve them.

10.3 Dóra

When our collaboration started, Dóra had taught for 15 years in grades one to four. She specialised in teaching textiles in her teacher education but had mainly been a classroom teacher in grades 1-4. She decided to take a leave from teaching after 15 years of experience but decided to return to studies and attended an evening class on mathematical problem-solving in order to improve her mathematics knowledge. Her experience of being a support teacher in mathematics for a group of low-achieving 7th graders impelled her to improve her competence in teaching mathematics. Dóra was now back to teaching and taught 5th grade for the first time. A recurring theme through our collaboration was Dóra's desire to improve her mathematics teaching and in that relation she referred to the diverse opportunities she has engaged with in order to develop her teaching.

10.3.1 Working with others towards professional development

Dóra found that participating in a problem-solving course changed her beliefs about mathematics and mathematics teaching, and that this experience has affected all her teaching. The teacher of the course had

been consistent in asking the students probing questions and encouraged them to collaborate when solving the problems. Dóra said: “He never gave us the answers and we were equally lost to begin with”. She had experienced how important it was to have time to reflect on her work and wanted her pupils to experience the same. When she returned to teaching she felt that her pupils required her to respond quickly to their questions and tell them precisely how to calculate and proceed with their work. She said:

When they ask: “Am I supposed to add”, I would have answered: “Yes just add the numbers together.” The teacher said something important that I always keep in mind: “We never give them a chance to answer, we ask them questions and then we answer them ourselves. They don’t get time to think.”

She reminded herself of what this teacher said and tried to give her pupils time to reflect on their responses. She joined the mathematics teachers’ association, Flötur, and attended their conferences. At one conference she joined a workshop taught by teacher from the problem-solving course and integrated problems into her school that she had worked on there. She proposed that all the children from grades 5-10 at her school should engage with the problems at a ‘math-day’ in the school.

Later, Dóra participated in a course on creative mathematics, which she found helpful in keeping up with the process of reflecting on her teaching. At the course, they discussed what makes mathematics interesting and the blame for many children’s dislike of mathematics is not the mathematics itself but rather the way it is presented by teachers and in textbooks. Dóra said that this discussion opened her eyes to why some of her pupils say that mathematics is boring even though they enjoy mathematics lessons when they get a chance to explore with the mathematics. She herself always liked mathematics at school and enjoyed textbook work. She often referred to her own mathematics learning at school and wished that she had gotten a chance to investigate more deeply with mathematics.

The agenda at the start of 5th grade in Rainbow School concerned the tradition of learning the multiplication tables by heart. Dóra does not believe in rote learning and finds it more important to understand the base-ten-system and the additive property of the tables. She decided not to oppose this tradition and agreed to test her pupils on the tables.

I use my own ways to make their learning more meaningful. So, when you start talking about it, you understand what I mean, then it just adds to ... they do not find it the same as learning by heart.

Dóra was sceptical of the evaluation program in Rainbow School. She was not satisfied with the end-of-term tests they use for assessing their pupils, and wanted to find ways to reform the approach.

Dóra, Pála and Inga (the special education teacher) collaborated in mathematics classes. As mentioned in Section 10.2 Dóra teaches pupils who had the best outcomes from the end-of-term test at her school. She has never taught such a homogenous group before and she likes to reason with her pupils about mathematics. :

I think that it gives these powerful pupils the opportunity to work at a high speed and to discuss things with their teacher; because you do not discuss such things if you have kids that just sit and wait all the time, because you are hurrying to talk about something, things they do not understand. You can do it with like maybe a strong, homogenous group. I just, maybe because I am so quick myself somehow, I feel so good when I get the group with me.

Dóra, Pála and Inga get together for planning, and then each of them teaches their groups as they find this arrangement most fruitful for them. Dóra and Pála work closely together and often discuss what happens in their classrooms. They would like to be able to discuss their teaching more with Inga but she is occupied with teaching other children and collaborating with other teachers. When the project started, Dóra had never taught 5th grade pupils before and relied on Pála, although Dóra hoped that she would be able to add something to their collaboration. She looked for interesting projects to use with her group and was aware that her pupils were able to finish more projects than the other groups. She also tried to relate the work at school to the children's experiences and found that problems in the textbook could be the catalyst for making her own problems. The problems she had found meaningful to work with, like the one about the proportional weight of African animals and the handshake problem, discussed in workshops 2 and 14, she shared with Inga and Pála, who also used them in their teaching.

Dóra was always willing to try the methods for professional development that I introduced to the group. After discussing classrooms observations, Dóra proposed that she and her colleagues visited each other's classrooms. She wanted to plan her teaching well before the visits. "I would

like to look closer at how I ask questions, ask them to notice that.” The mutual visits to Inga and Pála were fruitful and they discussed group work and co-operative learning when they met after the visits.

Dóra took advantage of the teaching instructions that accompany the textbooks and extracted ideas about how to work with her pupils. She felt that she was trapped in a traditional way of teaching and wanted to learn more about how to investigate mathematics with her pupils:

This is just the beginning and we need to find the rhythm. Of course one has certain ideas, you know, as you yourself understand this; and try to get them to tell you how they understand this.

In my last interview with Dóra, we discussed her work with the new group of pupils. Together with Pála she decided not to arrange the children into ability groups as they had done before and she was happy with their decision. She felt that she had successfully managed to respond to all her pupils’ needs. When she prepares her teaching, she uses all kinds of material that is available about the content area she is working with. She found it important to have hands-on material available and mathematical tools like base ten blocks, fraction models, rulers, scales and cubes to build three- dimensional shapes. She got permission to buy more supplies for the school and urged other teachers in Rainbow School to make use of these materials.

10.3.2 Listening to children

When I visited Dóra’s classroom for the first time she started the lesson by handing out two sheets with homework.

Dóra reminded the children to write down their thinking step by step so she could follow their thinking when reviewing their homework. She asked the children to look at some of the problems and tell her how they would calculate in solving them and then she wrote what they told her on the whiteboard.

The children then solved problems in their textbook about the ‘Olympic Games’. Dóra asked the children what they knew about these games and they discussed how long it was since they were revived in 1896. The children referred to what they had discussed earlier about their own age. They were born in 1999 and had discussed how easy it was for them to find their age at any time by adding one year to the year that had passed since 2000. They saw that they could apply the same rule for calculating how long it was since 1896, by adding 4 to the years that had passed since the year 1900. They discussed several other ways to discover this and Dóra wrote all their proposals on the whiteboard. (Notes from observation September 3rd 2009)

The atmosphere in Dóra's classes was relaxed and there seemed to be mutual trust between her and the pupils. In one of my visits to her class three boys asked if they could measure the distance between two light posts outdoors and she urged them to take the measuring wheel and measure the distance. Dóra told me that she trusted them to do this on their own and the boys soon came back and reported proudly on their findings. Dóra said:

They like the mathematics classes. They would not lie to me. I care about all these children, you know, and they are not too many. You can easily keep track of everything. Then one can more easily follow each of them.

Dóra's colleagues have noticed the strong relationship she has with her pupils and when we discussed their mutual visits to each other's classrooms, Inga told us how much one of Dóra's pupils admires her and even said that she was the best teacher in the world.

Dóra often referred to her pupils' work in our discussions and the progress they were making in their mathematics learning:

They are more willing to discuss their thinking than before and not afraid of making mistakes. There are three boys who work differently though. They find it difficult to explain how they solve problems. I need to support them; they hardly ever participate in our discussions. I am not happy about that.

She also talked about a girl who found it difficult to socialise with the other children and disliked group work because she felt that nobody listened to what she had to offer. Her approach to the problems often differed from the others. She looked for patterns and tried to visualise the problems by drawing or working with manipulatives:

She is a bit eccentric but it is okay. She is smart and I hope that she achieves the most out of this. If she were not content with me this would not work.

When Dóra recorded her own teaching for the first time she destroyed it immediately after she had listened to it because she was struck by what she heard. She wanted to be able to teach like the teachers in the videos we had observed together but felt that she was insufficiently prepared, talked too fast and was unprofessional. In this lesson, she posed a problem to her pupils that we had solved together at one of our workshops and we then

had looked at a video from class where we saw how a teacher discussed this problem with her pupils. Dóra said:

What I found that I learned from the video you showed us was that I would like to teach in this way. One needs to prepare. I just jumped into this. Felt I was so good at it. I forgot to give them a chance to come up to the whiteboard to explain, like this teacher did.

From listening to her recording she also learned that she did not use mathematical concepts, for instance, she said box instead of square, and felt that this experience had helped her be more careful in relation to what concepts she uses when she talks about mathematics and said:

I do not want to talk like this. I realised that I need to practice using the right concepts. ... Now we are talking about triangles and I have been practicing the concepts, like isosceles and equilateral.

She regretted having destroyed the recording and wanted to record her lesson again to be able to listen to it several times and also so she would be able to listen to what her pupils said.

10.3.3 Contribution to the research project

Dóra liked to participate in our workshops and was always willing to discuss her work. She discussed her thinking about the problems we were solving in all workshops and often asked the other participants demanding questions and probed for their explanations. She also initiated discussions about issues that we examined further at our workshops, for example, when she raised concerns about teaching a ‘traditional algorithm’ and rote learning.

Dóra claimed that she was satisfied with having participated in the project in our discussion regarding the experiences from the first year of the project and felt that it had affected her teaching. Working together with the tasks and discussing their solutions had helped her structure her own teaching: “I also learn from the questions you ask us and how you discuss with us.” The articles we read and discussed together opened her eyes for new ways of working with her pupils on mathematics, and she said:

I need to sharpen myself and think about what and how to ask the children questions so they do not get confused. Why is she asking me, did I say something wrong? I want to be constructive, it is vital, that they enjoy themselves. I think I have succeeded in that.

Dóra was eager to improve her way of discussing mathematics with her pupils, their different approaches to the problems and the multiple ways to present one's thinking. "But is this not something that takes a long time, some of them do not dare to jot something down on paper, it might be wrong." She also found it important to discuss mathematical concepts and the children's understanding of them, but noted that it takes them time to get used to discussing them.

When we used investigative approaches at our workshops and worked with problems that were structured to aid people discover new patterns and connections when solving them, she often referred to pupils in her class who would benefit from being able to develop their thinking further. She also told us about how she used ideas, both from teacher manuals and articles in teacher journals, to obtain various ideas about approaching her teaching. One such project revolved around measuring the area of different parts of the playground in the schoolyard and then reporting their findings by drawing in scale 1:100 in their logbooks.

In my final visit to Dóra's classroom, her pupils were working on this type of project and Dóra started the lesson by introducing the concept 'flatarmál', the Icelandic word for 'area'.

Dóra asked the children for the meaning of the two parts of this composite word 'flatar' (e. area) and 'mál' (e. measurement). They pointed out different areas that could be measured such as the area of the classroom floor. Dóra asked for the support of two pupils to draw one square metre on the floor using a 1 m long ruler. One girl said that she had figured out that the area of the classroom would be about 48 m². Dóra asked how she had found that and the girl replied that by looking at the square metre on the floor she could see that there would be six such across the width of the floor and eight on the length. Dóra asked the tallest girl in the class to take long steps along the classroom, measuring both the length and the width of the floor. The results were that the length was approximately eight steps and the width six. By comparing her step to the sides of the square metre on the floor they could confirm that their guess of the area of their classroom was close.

The children then went outdoors equipped with rulers for measuring and logbooks to write their findings. When they came back they compared and discussed their findings. (Notes from observation February 7th 2013)

Dóra has found that the participation in the project has helped her improve her teaching of mathematics. The participation in a problem-solving course had opened her eyes to different approaches to mathematics teaching and our collaboration supported her in finding ways to improve her teaching. She was more consistent than before in urging her pupils to collaborate and discuss their thinking and to use more

investigative approaches. She referred to the articles she had read and the videotapes we had looked at and said that she still had a long way to go. Dóra was granted study leave and decided to take some courses about mathematics teaching and learning within the teacher education program to further improve her practice as a mathematics teacher.

10.3.4 Dóra – diligent and professional

Dóra took an active part in our workshops and she was always interested in making use of what we had addressed at the workshops in her teaching, both problems we had solved and means for researching one's own practice. She often said that she learned much from observing teachers we saw on videos and also from how I discussed with them at the workshops, indicating that she was aware that her zone of promoted action had expanded (Goos, 2005; 2008; Valsiner, 1997) as she collaborated with us.

10.3.4.1 Professional practice

Dóra was the only one of the teachers within this project who had actively participated in professional developmental courses on mathematics teaching. When she was granted study leave she decided to participate in courses about mathematics education. The fact that she joined the mathematics teacher association, reads their journals and uses other means for developing her teaching shows her willingness for professional development in teaching mathematics. She clearly expressed how she had been encouraged to rethink her understanding of mathematics teaching and learning when participating in courses (Niss & Højgaard-Jensen, 2002; Niss & Højgaard, 2011). Dóra often challenged the teachers at the workshops to discuss established norms within their schools, such as when she raised the question of teaching a traditional algorithm (Fosnot & Dolk, 2005) at our first workshop. Her close collaboration with Pála at her school helped them resolve the different beliefs they had on mathematics teaching and Dóra was consistent in the emphasis she placed on relational understanding (Skemp, 1976) and conceptual knowledge (Hiebert & Carpenter, 1992) in her teaching.

10.3.4.2 Collaboration

Dóra strove to create a culture in her classroom in which her pupils felt entrusted to take responsibility for their learning and her pupils all seemed to enjoy learning mathematics. She emphasised collaboration and discussions and felt that her pupils' interest in discussing their thinking grew with the progression of our project and as her competency in

revealing learning developed (Niss & Højgaard-Jensen, 2002; Niss & Højgaard, 2011). This was reflected in my last visit to her class in the way Dóra responded to a girls' guess of the classroom's area. She was though concerned that there were boys in her class that hardly ever participated in their discussions and she wanted to find ways to include them in the learning community within her group. Her concerns were an indication of that she was aware that pupils signal their way of interpreting and understanding the mathematics in diverse ways, as described in Gee (2004). Their reluctance to communicate their thinking thus does not necessarily mean that they do not benefit from the discussions.

Dóra often criticised established norms at her school like the habit of memorising facts and grouping children in mathematics classes according to their performances on tests, and thus, she was critically aligning with the community at her school (Jaworski, 2006a). She participated actively in shaping the culture of her own school's community and took initiative in suggesting what to work with at 'math-day' in the school (see Section 9.5.3) and in that way she aligned with the community by negotiating what to attend to at school. She also aligned with our collaborative learning community, both in suggesting what to discuss and by engaging with the activities on the agenda (Wenger-Tayner & Wenger-Tayner, 2015).

10.4 Inga

Inga's educational background is in social pedagogy and she originally worked as a social pedagogue for a community diagnostic and counselling centre. She later added to her education as a special educator and was in her second year as a special education teacher when she joined the collaborative research project. She had specialised in working with children with behavioural problems, and assisted the teachers at Rainbow School in diagnosing behavioural problems and planning intervention programs. She was concerned about how little knowledge of mathematics teaching and learning she had. In her background and education as a social pedagogue, this was not addressed and neither was teaching and learning emphasised in her special education program mathematics. At the outset of our project she expressed the view that she was happy to have obtained the opportunity to join this project and attend the workshops.

10.4.1 Collaboration

The first year of our collaboration, Inga joined Dóra and Pála who were the classroom teachers for the two 5th grade classes, for mathematics and

Icelandic lessons. The children were divided into three ability groups and Inga taught the children who struggled with mathematics. These children also had other learning problems as well as behaviour problems.

When I visited Inga's classroom at the beginning of our collaboration she emphasised co-operative learning and supported the children in taking responsibility for their learning. There were nine children present. Inga grouped them into three groups and gave each group three problems to work on. While the children were solving the problems, Inga took turns talking with each group. Her main focus was on the role of the students as participants in a group and less on the mathematics they were working with (Notes from observation September 7th 2009)

At our first workshop, Inga was quiet and did not take initiative in the problem solving process. When the teachers discussed their solution strategies she often referred to her pupils and how they might have solved the problem with support of manipulatives. "They would use the abacus, count by ten to 50 and then 1, 2, 3, 4, 5, 6 until they reach 56."

In our work with concept cartoons, Inga was quick to realise that her pupils would benefit from working with the cartoons: "There are so many of them that doubt that they can, and here they get some clues to start with".

Later, she told us about her experience of using the concept cartoons. Her pupils liked to work with the cartoons and she felt that it gave them confidence to talk about their own ideas with regard to the problems.

Those who have persistence to work through the problems, they like this ... and then there are others who do not have the patience.

At the third workshop, Inga told us how her pupils solved a problem about the proportional weight of the African animals that Dóra and Pála had discussed at Workshop 2. The children worked in groups to find the weight of each of the animals and Inga urged them to collaborate. She told us that she had expected the problem to be too difficult for the children in her group, but she wanted to see if they could find ways to approach it:

I was most curious to see what they did. They did not need to necessarily solve the problem. I wanted to see if they had the imagination to try, and it turned out that they did not.

The children were passive in their groups and had little idea of how to address the problem. Finally, Inga decided to go through the problem, step by step, with the whole group. She insisted that she needed to put more effort into teaching them collaboration and the responsibility of each person within a group:

I did not assign them any roles in the group-work and that was a mistake. I noticed a lot of things I need to do differently next time. This is a group with special needs children and they need much more preparation. They need to learn about their roles in the groups.

Inga did not give up on supporting the children in collaborating on mathematics problems. A few months later, in Workshop 6, she discussed her work with us. She had grouped the children into pairs to avoid some of them becoming passive. Often, only one child in each pair would be active and she realised that the 'weaker' child tended to let the 'stronger' child do the work. Instead of allowing them to choose a partner she decided to pair children who she estimated as being at the same ability level and said:

I saw a lot, I saw a positive change. I have never seen all of them so active. I have never seen them talk so much about the mathematics. It was great. And also those that are inactive. One of them said: "Wow I solved it". He just needed to tell us all about it. This lesson was such a victory at this time.

Inga then told us how the children discussed in pairs and supported each other in understanding their solution strategies: "He described his thinking, did this systematically and wrote down for every number, and then the next one." She was satisfied with the progress the children were making and that they were gradually becoming more independent in working on mathematical problems.

In an interview after the first year of the project, Inga said that she was beginning to see 'the big picture'. To begin with, she felt that she was going back and forth but now the children were becoming more independent and did not rely on her as much as they did before:

When we start a new project and I see that they have a light-bulb moment, and are capable of relating to what we have worked on before, then we can start from there. This does not always happen, but quite often.

10.4.2 Listening in classrooms

At the beginning of the second year of the project, Inga was concerned about her pupils when school started. She felt that the enthusiasm she had seen by the end of the first year had vanished and the children had lost interest. "They find everything we do boring and I'm continuously going

back to where we were last year. I see a huge difference from when they finished school last year.”

Inga wanted to improve her mathematics teaching and decided to audiotape her lessons to be able to better understand what was happening in her classroom. When she listened to the first recording she was struck by the fact that she did not use mathematical concepts when talking about mathematics. “We were working with fractions and I talked about the numbers above and below.” She was concerned that she was overly simplifying the mathematics for her special education group by using everyday language. Her experience was that if she used mathematical terms her pupils did not understand her and then she tried to make it simpler by explaining with everyday language. “But they need to learn the concepts. How can I intertwine them in my teaching?” Later in our discussions about her recording she asked herself: “Am I underestimating them by not using the concepts? I need to think about that too.”

Inga was pleased to hear on the recording how much she discussed with individual children while they were working with the problems and how she made an effort to understand their thinking when supporting them in their work. However, she was concerned that she did not give the children enough time to think, and found that she was too quick to answer her own questions in the whole-class discussions.

In the following workshops, Inga expressed how she was becoming more aware of the ways in which her pupils were thinking when solving mathematical problems. When we discussed a story that Pála told us about one of her pupil’s misinterpretation of a fraction problem, Inga was prompted to recount a similar situation in her class. In Inga’s case, she did not understand one boy’s description of his solution to a problem, where he was supposed to present the answer in metres, in terms of centimetres. Inga told us how worried she was about not understanding the boy’s justification and how she understood that her explanation that 4 m equals 400 cm made no sense to him (9.5.1.2).

Her interest in understanding her pupils’ development in mathematical thinking was growing, and also in how they discussed their work. In Workshop 14, she told us about something she had noticed the same day in her mathematics class:

I enjoyed this morning’s lesson. ...We were working on a whale project. The boys got the task to draw the Blue Whale. They went outside with a measuring wheel to visualise how big he is. Atli started to draw on the pavement, had done the mouth and every-

thing. Then Hilmar said: "This mouth is a bit small for all this whale". Atli just started to draw and did not think that he needed to draw in scale with the total length. Hilmar is the slowest in this group.

Inga described in more detail how this instance helped her look closer at what the children are capable of doing instead of always searching for what they are incapable of doing.

One is always thinking, do they find a way? ...We are more aware of the small steps we are taking. One is more aware of the development.

In the third year of our collaboration, Inga was no longer working with Dóra and Pála but still wanted to participate in our project. She was then a support teacher in grades eight to ten and felt that her participation in the workshops also helped her in teaching those children. She used tasks from our workshops and found that they supported her in awakening her pupils' interest and in making sense of the mathematics. She told us that her pupils' self-esteem in mathematics was low and said: "These kids show such little initiative and they are so uncertain of their ability to learn".

In the final interview with Inga, she proudly told me that she had been consistent in asking her pupils' to explain their thinking in class. She was collaborating with a mathematics teacher in the 10th grade and when this teacher interrupted a boy who calculated differently from what he was expected to do, "I said, tell me what you are doing, and he could explain it, and I said, this is great." Inga was worried that the children who have problems with learning mathematics are often told that they do things the wrong way and said:

Instead of getting the chance to explain their thinking, the teachers tend to explain to them again and again in a way that they do not understand. This makes them uncertain about themselves and they want the teacher to tell them what to do. But when the teacher listens to them they feel that they are capable of explaining their thinking like other children.

In my final visit to Inga's classroom she was teaching a group of 11 children in 5th grade and 6th grade.

The subject was geometry with focus on reflection. Inga referred to their former work with flipping shapes and then the children worked on three different kinds of mirroring projects. All the children were interested in these projects and tried to do the best they could. They then put their work on display on the classroom walls and

were satisfied with their work. Many of them initiated discussions with me about their work and proudly told me what they were thinking. The atmosphere in this classroom was different from the first one I visited. The children were more confident with their work and were not afraid of expressing themselves. Inga was also more secure in her role as a mathematics teacher and required the children to discuss their thinking when working with mathematics. (Notes from observation February 14th 2013)

10.4.3 Inga – eager to learn and listen

10.4.3.1 Developing mathematical competence

When the research project started, Inga was confident in teaching children with learning difficulties but felt that her own knowledge of mathematics was vague. She wanted to learn more about mathematics teaching and learning in order to be able to support her pupils' mathematics learning, expressing that she would like to improve her pedagogical content knowledge (Schulman, 1986). In my first visit to her classroom, she emphasised learning habits that would support the children in their problem solving techniques but she did not encourage them to discuss their calculations. As her own mathematical competence developed through participating in the workshops she became more confident in discussing mathematics with her pupils, as emphasised by Niss and Højgaard (2011), and this was evident in my final visit to her classroom. The atmosphere in her special education class with children in 5th and 6th grade differed from the one in my first visit to her. The children were more confident in their work and eagerly expressed their thoughts about their explorations with geometry. Inga seemed to have succeeded in empowering these children to participate in discussions about geometry in the mathematics classrooms to a degree that they could relate to in their everyday experiences with mirroring and reflections (Bernstein, 2000; Lerman, 2000a; Solomon 2009). Inga was now more secure in her role as a mathematics teacher than at the outset of the project and she actively encouraged the children to discuss their own thinking when working with the mathematics, suggesting that her mathematical knowledge for teaching had developed (Ball et al., 2008). She urged them to explore with reflections and make their own pictures with different kinds of techniques.

10.4.3.2 Learning to listen

When Inga recorded her own teaching she was stunned to learn that she did not use mathematical concepts in her discussions with the children and moreover felt that she was underestimating them by using everyday

language in her discussions with them. She sought to emphasise conceptual knowledge (Hiebert & Carpenter, 1992) and gradually learnt ways to encourage her pupils to explain their thinking about mathematics, as she explained to us at Workshop 13. In my final interview with her, she also told me how she had interrupted one of her colleagues who interfered with one boy's explanation of his thinking. Inga asked this boy to explain how he understood the problem he was solving and describe his thinking about it. He explained it in different terms from what his teacher had expected and was satisfied that Inga listened to his explanation. Inga added that children who struggle with mathematics are used to their thinking being second-guessed but when their teacher listens to them they find that they are indeed capable of explaining their thinking. Inga was supporting and empowering this boy in communicating his thinking, feelings and beliefs, and simultaneously facilitating the diverse ways (Gee, 2004) in which mathematics can be approached, interpreted and understood.

To begin with, Inga was hesitant to express her thinking about the mathematics we were working on at the workshops but gradually she became more confident in sharing her ideas, improving her cooperation competency (Niss & Højgaard, 2011). She often referred to her pupils and how they might solve the problems we explored with at the workshops. Her background as a social pedagogue had afforded her a strong ability in listening to her pupils, and as our collaboration developed, her confidence grew as a mathematical problem solver and a mathematics teacher, and she felt that working with us at the workshops sharpened her capability to discuss mathematics with her pupils. However, she was concerned that she underestimated their capability to learn mathematics. This issue appeared in Workshop 14 when she told us about a conflict that arose between two boys who were drawing a Blue Whale. In noticing boys' responses and focusing on their strength, Inga displayed increased abilities to attend to her pupils' diverse way of expressing their knowledge was growing as she was (Dalvang & Lunde, 2006).

Inga was satisfied with her participation in the research project. She found it valuable to reflect on her teaching together with us. Learning more about mathematics teaching and how children develop their understanding of mathematics had also been of aid, especially given that these aspects were not addressed in her education. She found the collaboration with Dóra and Pála helpful and they were able to support each other both by planning together and visiting each other's classrooms, thus developing a co-learning agreement (Wagner, 1997). She had not experienced such close collaboration before with classroom teachers.

Inga engaged actively in our discussions at the workshops and supported us in focusing on children's strengths instead of their weaknesses. At first, she was aloof at the workshops but with her increasing confidence in solving the mathematical tasks at the workshops, she steadily began sharing her thinking with us. She often referred to what she thought her pupils would do in the same situation. She took an active part in shaping the culture in our community in sharing her experience with us and also used the means for reflecting on her practice that I advised the teachers to use (Wenger-Tayner & Wenger-Tayner, 2015).

10.5 Conclusions to teacher participation in the project

Three other teachers also participated in the project. However, even though they were not able to participate in all activities they made progress and contributed significantly to the project. I add selected examples from their participation as I conclude this chapter. These examples contributed to the development of the project as a whole. I also mention the culture of the schools and the roles of the heads. This section thus extends the perspectives presented above. I will identify key points of the narratives of the four teachers supplementing these examples from my own observations in other three teachers' classrooms and interviews with them.

10.5.1 Starting out

I based my work at the first workshops on the preliminary observations and interviews with the teachers. What I learned from these observations and ensuing interviews with the teachers did not come as a surprise to me as I had similar experiences from former work with teachers. If teachers are used to lecturing and emphasising instrumental understanding (Skemp, 1976) then their pupils are likely to adopt an attending mode and wait to be told what to do. If they have taken an investigative approach to their teaching, their pupils are more likely to show initiative in their approach to learning, entering into an asserting mode (Mason & Johnston-Wilder, 2006). During my first visits to the teachers' classrooms, I experienced much of the former, and as the project developed, I began to see more investigation and communication than before.

An example of an emphasis on instrumental understanding is an incident that occurred in the support centre with a group of six 6th graders. Vala had written number problems on the whiteboard and called the children to come and show how they had carried out the procedures of the algorithms. The children were confused about the carrying and borrowing of digits and

Vala was accurate in showing them the steps they needed to take. She cared about whether her pupils would learn these procedures but she did not discuss with them why they needed to take these steps.

I observed a similar approach to repetition of homework in Gróa's 6th grade class. She asked the children to explain how they solved the problems and wrote their response on the whiteboard, carefully attending to each step. At our workshops, Rúna often said that she believed that learning the steps of the algorithm was vital for the children who came to the support centre and they were proud of themselves when they had mastered them. It was evident that these three teachers did what they believed would help their pupils in gaining confidence in learning mathematics. Their focus on procedural knowledge and instrumental understanding was apparent and I felt that it was important to challenge their beliefs.

10.5.2 Increasing collaboration

As our collaboration developed and the teachers talked more about their work, I learned that the approach they took in their teaching changed in proportion as well as their beliefs about teaching, and it was therefore important to address this change. Their developing focus on co-learning and investigations was apparent during Workshop 6 when Rúna and Vala talked about their observation of each other's teaching. Their focus was more on collaboration with their pupils than before as they urged their pupils to explore while finding out proportions of a whole by searching for things in their surroundings. During my visit to their classroom, at the end of our first year of the project, they also highlighted collaboration and explorations to a greater extent than earlier. In my interview with them the following day, it was evident that they had started to pay more attention to the fact that their pupils might think differently about calculations than they had themselves tried to teach them. Rúna said:

I have noticed one thing; it seems to be a certain talent at least with one pupil here, it is mental calculation. It is just that he likes to, or it suits him better, to calculate mentally than write down, I think I have never encountered that before.

We discussed why this pupil might be more willing to calculate when he was allowed to explain his thinking orally than when he needed to write down his calculations. I asked them if the reason could be that they now accepted that their pupils could explain orally how they calculated and Rúna admitted that they are in fact more aware of that than before. They

both felt that their approach to teaching had changed as they learned to inquire into mathematics themselves.

The interrelated discussions at our workshops, interviews with the teachers and observations in their classrooms helped me in developing my work with them and deciding on what to focus in the workshop sessions.

10.5.3 Established norms within Rainbow and Sunshine School

In both schools, children were grouped into ‘ability groups’ in mathematics lessons and the teachers did not seem to question these arrangements. I was careful not to give my opinion of such arrangements but emphasised that the teachers would experience inclusive practices themselves at our workshops. In our discussion about the problems we solved, we explored how solving them together might support cooperative learning in diverse groups. As our collaboration developed, the teachers often mentioned that their pupils would gain from working in more diverse groups. In my final visit to Dóra and Pála, I learned that they had decided not to group their pupils in ability groups, thus taking a stance against established norms at their school, and they were content in their conclusions. Pála had often expressed that she felt her pupils lacked initiative and was concerned that she might be requiring too much of them. At Workshop 10, we discussed if they might gain from working with children in Dóra’s group who showed more initiative in their work. They had now taken this step and felt that all their pupils benefited from working with children with diverse backgrounds.

When the teachers at Sunshine School told us at that the head of school had decided that they would not arrange children into ability groups, I was surprised to learn she had not consulted with the teachers about this decision. Rúna was content with this arrangement as she felt that it was more rewarding to work with children who were willing to show initiative in their work than those who used to come to the support centre. Edda was aware that ability grouping does not favour the pupils’ learning but she found it difficult to attend to the diverse needs of all the children in her class. She said that many teachers at their school were sceptical about this arrangement. It therefore did not come as a surprise when she later told us that the head of school had decided to go back to the former arrangement of mixed ability groups due to the teachers’ discontent with her decision.

10.5.4 Summary

The atmosphere in the two schools was reflected in the different decisions taken about these arrangements of dividing children into ability groups. The

teachers in Rainbow School were supported in their decision not to group their pupils into these groups and were trusted with a responsible decision. At Sunshine School, however, the teachers had nothing to say about the head's decision no matter if they agreed with the decision taken or not.

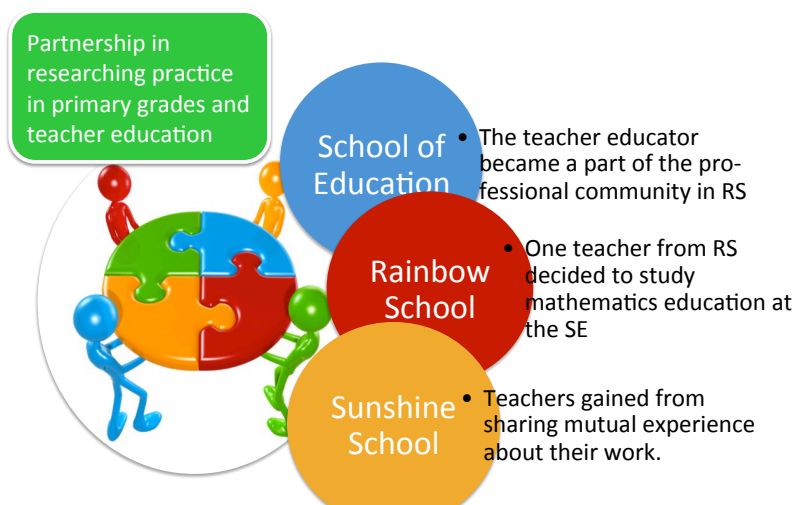
Another aspect that influenced our collaboration and reflected the culture in the schools was how I was received in these two schools. In Rainbow School my offer to work with other teachers (see section 8.8) was readily accepted and I led five workshops with other mathematics teachers at the school during the first year of the project (see Appendix C). At Sunshine School, I only met once with a few teachers during the second year of the project in order to discuss their teaching. When I went to the workshops at Rainbow School, the teachers welcomed me and often discussed their concerns for their teaching. In Sunshine School, very few people knew who I was and they were not informed about this project.

The four teachers whose narrative I wrote about earlier in this chapter actively took part in shaping the culture within our learning community, by sharing their work with us and collaboratively reflecting on the learning that developed within the project. Even though they did not record their findings on what they noticed in their classrooms in a systematic way or write an analysis of them, they reported on them and discussed them at our workshops. This information was recorded and analysed by me and is therefore accessible to other people who might be interested in learning about our work. They read their narratives in the first draft and commented on them. All the teachers read chapters 9 and 10 in the final version and met to discuss and comment on them. Inga and I have already published one article about our collaboration in "Glæður", a journal published by the association for special education teachers in Iceland. She took an active part in writing the text and her knowledge about inclusive practices was valuable in this process (Kristinsdóttir & Sveinbjörnsdóttir, 2015). My plan is to report further about our work with the teachers in "Flatarmál", the journal of the mathematics association in Iceland.

The other three teachers were also active in shaping our culture by their engagement within the workshops during the time they participated in the project. Two of them were reluctant to record their lessons but they were always willing to discuss their experiences from their classrooms. Since they did not participate in all activities of the study, due to other obligations, their participation did not shape our culture within our community to the same extent. As the process of analysing the learning of each of the teachers developed, so did my interpretation of how the collaborative

project was shaped by each of the teachers' participation. Writing the findings of the cyclic process of analysing the data helped me in writing their narratives, being loyal to their expressed views and beliefs reflected in their activities and discussions throughout the project.

In figure 3 I summarise how the different communities we belonged to, and the one we created together, affected each other.



All participants aligned with the learning community that was established. Conflicts were resolved by negotiating the boundaries between the different communities they belonged to.

Figure 3. Relationships between communities

In Chapter 11, I will discuss further how my own learning developed through participating in this project and how it has affected my professional practice as a teacher educator and researcher, as well as the implication it may have for teacher education practices.

11 Researching with teachers

Now that I approach the end of my journey through the landscape of diverse learning communities in which I have participated, I will summarise my findings and also assess how they have affected my vision for working with teachers. In doing this I will also answer the following two research questions:

- In what way has my understanding of teacher development in mathematics teaching and learning changed through working with teachers at improving their own practice?
- In what way do I interpret that my own learning from carrying out a collaborative inquiry project with teachers has influenced my practice as a teacher educator and a researcher?

These are the main research question for both parts of this study and the third question in Part II. I intend to reach conclusions about the learning derived from my self-study into my teacher education practices, Part I, and into the collaborative inquiry project, Part II.

Furthermore, I will consider the value of the findings in relation to teacher education practices, in particular the education of mathematics teachers and the provision of in-service education. Urgent questions are related to the opportunities for teachers to develop as mathematics teachers, and how teacher educators can work at improving their own practice? In a section on quality and value, I discuss ethical aspects and the validity of the study, as I revisit my writings about educational research in section 2.3 and its ethics in section 8.7.

To understand the future I start by going back to the beginning of my work as a mathematics teacher and teasing out the value, origin and aim of the study. In finding the value of this work for mathematics education and for teacher education, I ascertain for myself what I have learnt as well as some implications for teacher development.

11.1 Origins and nature of the study

The study had its origin in my passion that all children in schools should be able to experience meaningful mathematics learning in ways where everyone is respected and is seen as capable of learning mathematics. My

experience as a teacher early on in my career, in the dual role of teaching in primary grades and within teacher education, raised my interest in working with teachers at improving learning conditions in mathematics classrooms (see Chapters 1 and 3). I also learned from my parents about teacher professionalism. My mother was a textile teacher and my father a mathematics teacher. They were both active in improving their teaching and my home was a place where their colleagues could to meet and plan improvements in their practice, which gave me the opportunity to learn from their enthusiasm. Their interest in mathematics learning and their way of supporting me in solving everyday tasks with the help of mathematics fuelled my curiosity to focus on mathematics in my further studies.

11.1.1 Beginnings of the study

Throughout my teaching career, I have focused on and learned about how pupils in primary grades and students within teacher education can be facilitated in learning mathematics. Central to the process of developing my own practice were the opportunities for participating in learning communities with colleagues, both in courses and in meetings where we planned our work. When the opportunity arose to undertake doctoral studies, I decided to focus on how I could improve my practice as a mathematics teacher educator, and in particular, on how I could support teachers in schools improve their mathematics teaching.

I had built connections with teachers in schools all over the country and who would be able and willing to support me in this process. I had worked for one year with teachers in rural areas and several times we were forced to postpone our meetings due to weather conditions. For the purposes of this project and to ensure more continuity I decided that the participant school would have to be located in the urban southwest of Iceland. As discussed in Section 8.4, I contacted one school in Reykjavík and the head of school proposed that we invite a neighbouring school to collaborate in the project. I welcomed her proposition as it correlated with my intention of building a learning community of teachers with diverse experiences.

11.1.2 Decisions identifying the research problem

In conducting this study I wished to involve teachers as active research participants, respect their values and give them a voice to report about their work. I wanted to make a contribution to the field of educational research, one that focused on the possibilities of developing as mathematics teachers. My intention was to learn how teacher educators could

support teachers in this process and include them in educational research. In order to respond to teachers' concerns about the growing diversity in schools, I wanted to address issues on inclusive practices and how learning communities can be created to welcome diversity and educate all their pupils in successful ways (Ainscow & Miles, 2008; Askew, 2015).

As I developed my own ideas, I realised that I wanted to explore with teachers their capability to develop as professional mathematics teachers, participating in a learning community and focussing on how things work rather than on what works (Sfard, 2005). The community should offer teachers a say in determining what is educationally desirable (Biesta, 2007). In wanting to work with and carry out research with teachers I was responding to the growing interest in collaboration between teachers in the field and researchers within mathematics education, in which they learn about their work inside and outside classrooms for the benefit of both teachers and researchers (Artigue, 2009; Goos & Geiger, 2010). In framing the study, I emphasised that all participants should add knowledge to each other's practice. The teachers knew their pupils and schools, while I as a teacher educator had knowledge of theory, research and educational systems. I envisioned that together we would produce shared knowledge (Jaworski, 2008b).

11.1.3 The nature of the study

From the beginning it was clear that I wanted a developmental study where the teachers were active participants in the research process, researching their own practice while collaborating with me in learning about their development. My experience of practitioner research helped me in framing a cyclic study where learning from former steps guided actions taken in succeeding steps. As a result, I looked for a research approach that would support me in developing a cyclic process where teachers in schools and a teacher educator could research their own development while participating in a collaborative study. In Chapters 5 and 8, I accounted for how I arrived at the methodology of developmental research, and particularly, the developmental research cycle, as the appropriate methodology for the structure of this research process.

When I began to work with the teachers in interviews and the initial workshops the process of interpreting that data and analysing the outcomes started immediately. It was not until I was writing up my findings that I realised that I needed to return to the theories that underpinned the study. In Part I, I wrote about my trajectory to self-study, where I

investigated the process of defining my understanding of theories based on constructivism, on one hand, and sociocultural perspectives, on the other hand. As I have presented my findings and reflected on what I have learned from them, I have continued to reflect on my understanding of learning theories and how my understanding of these theories could support teachers, teacher educators and researchers in their work. In the following sections, I will discuss what I learned from my self-study of the collaborative research process with teachers.

11.2 Self-study to underpin the project

When I reflected on my former experience of working with teachers and student teachers I realised that they had not experienced discussion on their own approaches to solving problems, as the teacher often is the one who talks and the children are encouraged to work quietly through the problems in textbooks. This was also reflected in observations in mathematics classrooms in Iceland, discussed in section 2.1 (Gunnarsdóttir & Pálsdóttir, 2015; Savola, 2010).

11.2.1 Planning the workshops

In this project and in the workshops I wanted to focus on investigating in mathematics (Jaworski, 2006a) at the workshops. I felt that the project teachers should have an opportunity to realise how important it is for all mathematics learners to actively engage in exploring together with mathematics and discussing their work. I believed that this experience would support the teachers in this study in building space for their pupils to experience similar approaches.

When I planned the workshops, I felt that I needed to become aware of the extent to which I could place myself in the role of a teacher educator and when to rely on the teachers as professionals who can make informed decisions about their work (Dalmau & Guðjónsdóttir, 2002; Guðjónsdóttir, 2002; Handal & Lauvås, 1982). We were all bringing our knowledge into the project (Jaworski, 2006a) and my vision was that we would collectively create a learning community where we could learn together about improving our work.

In Chapter 9, I stated the position I took in working with the teachers and the principles I followed, namely, the respect for teachers as professionals and my role as a facilitator who challenges their beliefs by urging them to look critically into their practices. To ensure that our collaboration would be fruitful, I carefully attended to their expressed

beliefs about teaching and learning, never criticised them but carefully supported them in rethinking their beliefs. In adopting this position, I was remaining true to my belief that we come to understand our experience as we study it, in the words of Pinnegar and Hamilton (2009).

11.2.2 Alignment of communities

The boundaries between the communities we belonged to within our professions, and the one I wished to create for the purpose of carrying out this study, needed to be negotiated (Wenger, 1998; Wenger-Trayner & Wenger-Trayner, 2015) from the outset. I, as the teacher educator, had to decide what to focus on at the workshops to begin with based on my first observations and interviews with the teachers, so I decided to bring an inquiry problem. I was not surprised about the teachers' wish to discuss rote learning and their concerns that, if they did not emphasise instrumental learning, they would be depriving their pupils of the opportunity to become fluent in calculating mechanically. I realised that it was necessary to attend to these beliefs about learning mathematics even though I intended to enter into discussions about inquiry tasks and collective reflections on solving them.

Instead of criticising their beliefs and arguing about what is gained from different approaches to learning to calculate, I challenged them to rethink their own way of calculating. On one hand I expected that by participating in discussions about their different ways of calculating and reflecting on the ways their pupils might calculate, they would be empowered to enter into such discussions with their pupils. By responding in this way, I offered the teachers the chance to shape the culture within our community by bringing in concerns from their daily work and supporting them in looking into them from a different perspective to which they were accustomed. On the other hand I needed to align myself with the community they expected to create and at the same time empower them to critically align with the communities they belonged to at their schools (Jaworski, 2006a).

My reaction to the alignment issues resulted in the teachers developing focus on the pupils' way of learning and led to increasingly stronger collaboration and discussions in their classrooms. This became more and more visible in the cases they brought into our workshops such as Pála's story from her reflections with her pupils at our final workshop and Dóra's discussion with her pupils in my final visit to her classroom.

11.2.3 Continuity and progression

I relied first and foremost on what had happened in the most recent workshop when I made decisions about what to attend to at the workshops, and then developed my next protocol from that. I also used my knowledge and experience of teaching about mathematics teaching and learning within the university in interpreting and responding to the teachers' requirements. Their response to investigating in mathematics reflected what I had experienced earlier and my former experience of communicating about the mathematics helped me further their discussions and probe for alternative ways of solving tasks (Guðjónsdóttir & Kristinsdóttir, 2011; Gunnarsdóttir et al., 2013). The same applied to discussions about other material I brought into the workshops such as video-clips from classrooms, readings about mathematics teaching and to the discussions about mathematics teaching and learning, such as questions about instrumental or relational understanding.

As the project developed, the teachers gradually took the lead in what to focus on both with regard to the mathematical content they wished to address at the workshops and concerning their experiences in their classrooms. The topics they chose to discuss reflected their evolving awareness of their pupils' strengths in mathematics. At first, the emphasis was often placed on rote learning and their pupils' incapability of learning mathematics, as figured into many of Gróa's responses during the first year. Progressively, they began to bring in stories that reflected their attention more positively to what their pupils were expressing, as was evident in Pála's story at Workshop 13 of the assessment dilemma, along with Gróa and Rúna's discussion about cultural days at Workshop 14.

11.2.4 Modelling

The teachers all welcomed the participation in the process of mathematics investigations and engaged in exploring problems and discussing their thoughts about them in negotiating their understanding of the mathematics with which we were dealing. They often mentioned that they learned from the way I worked with them, in terms of introducing tasks, asking questions and leading discussions. They also articulated that watching teachers on video-clips informed them how to lead discussions with children about mathematics, acknowledging that their zone of promoted action could be affected (Goos, 2005; Valsiner, 1997).

I encouraged the teachers to reflect on their practices and use means for professional development such as writing reflective journals, visit each

other's classrooms and recording their lessons. They were hesitant to begin with and some were reluctant in using these tools, as reflected in my early conflict with three teachers about writing and analysing cases from their classrooms. I responded to their reluctance by suggesting that instead of giving me a copy of their writings they would take care of their writings themselves and we would discuss their concerns at the workshops. I was aware that they might not write about what they noticed in their classrooms if they were not required to give me a copy of their writings (Kruger & Cherednichenko, 2006; Mason, 2002). However, I had to respect that they were not my students and their commitment to the study needed to be renegotiated, respecting the boundaries between our communities of practice (Wenger, 1998; Wenger-Trainer & Wenger-Trainer, 2014).

In my original plan for the project, writing about one's own reflections was to be one of the main strands of development. I was aligning myself to the community we were creating together in responding to the teachers' reluctance to write and respecting that they were also aligning themselves to this same community (Wenger-Trainer & Wenger-Trainer, 2014). In Goos's (2008) terms, my zone of promoted action was restricted because of these teachers' critical alignment to the community I had envisioned would develop (Jaworski, 2006a).

11.2.5 Individual needs

Despite initial reluctance on part of three teachers to write about their classroom noticing, it was evident that the other teachers did write notes, which they then brought with them into the workshops and used as scaffolding in telling their stories. An example of this is Pála's contribution in Workshop 17 where she copied notes onto the whiteboard from her reflective journal about her discussions with her pupils in relation to algebraic notations.

If I had been consistent in requiring the teachers to write about their work and handing in copies of their writings to me, some of them might have been reluctant to proceed in the participation. By respecting them as professionals and offering them opportunities to shape the study, they were empowered to take the lead and participate in structuring the workshops. They were also challenged to question established norms at their schools and align critically to the cultures established in their schools (Jaworski, 2006a), as was confirmed by Dóra and Pála's decision to no longer group their pupils in ability groups.

11.2.6 Empowerment

On the outset, I was concerned about how uncritical the teachers were of the arrangements within their schools in grouping children in ability groups in mathematics classes. When I first observed Rúna and Vala's teaching in 5th grade in the support centre, I was struck by what I observed. In my notes I wrote:

What I found most astonishing was that these two competent teachers were working with so few pupils and I felt it was a waste of time and labour both concerning the children and their teachers. The teachers had managed to build a good relationship with their pupils and both parts seemed to be content. Still I felt that the children were not challenged enough, and nothing was done to urge them to show initiative in their studies. (Reflective notes September 4th, 2009)

When I observed them teaching 6th grade a few days later I was also concerned for what I observed. The children had worked with measurement and were supposed to work individually and not disturb each other:

The belief that the children who are sent to the support centre need to work individually is strong. I found the children's measurement skills more developed than I had expected, both knowledge of concepts and techniques. Their communication also went smoothly contrary to their teachers' concerns. (Reflective notes, September 9th, 2009)

As our project progressed, and the teachers increasingly experienced how their collaboration in solving problems inspired them to look for alternative solutions, they gradually started to focus on how their pupils could be empowered to learn mathematics through sharing methods and experimenting with particular approaches (Askew, 2015). When the teachers in Sunshine School told us, at the beginning of the second year (Workshop 9), that the head of school had decided to cease arranging their pupils into ability groups, Rúna was satisfied with this decision and looked forward to working with children in mixed ability groups. Through our discussions at the workshops, and my later observations in the teachers' classrooms, I learned that their approach to teaching mathematics changed. They acknowledged that the learning community in classrooms could be enriched through the diversity of learners' contributions (Askew, 2015).

The teachers' awareness of both their pupils' learning and their own learning advanced through participating in our learning community, and as our discussions developed and we could reflect in more depth on what we had noticed, as was apparent in Workshops 13 and 14, I was content to experience this and found that we had been able to create a learning community where we all were empowered to improve our practices.

In my notes after Workshop 14 I wrote:

It came as a surprise to hear Gróa's story from the cultural days at her school. It was obvious that she had been consistent in making this day a learning experience for a boy who had been identified as having problems with learning mathematics. I was happy to hear how focused she was on including him in calculating the cost of what was needed for a birthday party.

When I challenged her to reflect on her learning from this experience she was surprised and at first she did not seem to understand my question. The other teachers participated in our discussions and together I felt that we managed to reflect on important aspects about professional development. I am content with our decision to meet again next year and look forward to following up what seems to be developing within our community. (Reflective notes, May 4th, 2011)

The teachers had taken the initiative once more to extend the project and I was content to have been able to respond to their wish to proceed. I knew from earlier experience and from reading about developmental projects that it takes time to improve one's practice as a teacher and consequently felt that we were gaining from our persistence in proceeding with our collaboration (Berg, 2012; Breiteig & Goodchild, 2010; Fennema et al., 1993; 1996; Jaworski, 2007a; 2007b). During the third year, we were not able to meet in the fall and our 15th workshop was not held until February 2012. It took some time to revive the level the co-learning had developed and it was not until our 17th and final workshop, in May the same year, I felt that we were back to our former approach. When Pála discussed her work with her pupils at our final workshop, I was content to see how much progress she had made and I wrote the following:

I was glad to hear how much attention Pála paid to the children's thinking. She was happy to have experienced that her pupils were capable of solving these problems on their own. She is gradually stepping out of her role of guiding her pupils through the solution process. She has started to pay attention to their discussions and

support them in this process by discussing their ideas with them. The discussion in our group was fruitful and all the teachers participated in discussing how her pupils' thinking was developing. (Reflective notes, May 26th, 2012)

As a teacher educator, I have added to my former experience of working with teachers in the process of carrying out this study. As discussed in Chapter 1, I had studied my own teaching within the teacher education program but I had not been able to study my work with in-service teachers. I have learned that it is important to build mutual trust between all partners in developmental projects and it is vital to give teachers a sense of ownership and independence in shaping the culture within such projects. In my practice within teacher education, I have been consistent in offering my students the opportunity to influence what to attend to in our courses, and as addressed in Chapter 6. I have not planned and carried out any in-service courses during the time of my doctoral studies and I have therefore not been able to draw on my findings from this study yet within that context. In the final section of this chapter, I will discuss my plans for using findings from this project in my work with teachers in schools.

11.3 Developing as a researcher

The long journey I have undertaken in planning and carrying out this project has helped me develop as a researcher. From the initial steps of planning this research I was determined to utilise a developmental research approach where each step taken would guide further steps. The methodology of developmental research served my purpose, particularly in following the developmental research cycle with the interconnected cycles of research and development, as discussed in Section 8.2.

I based my decision on researching with teachers both on my own experience and findings from research in Iceland and international studies on teachers' development as discussed in Chapter 2. My focus was directed toward understanding how things work, since I wanted to learn about teacher development in the process of critically inquiring into their practices. The developmental nature of the project involved the cyclic process, which passes between the developmental cycle and the research cycle (Goodchild, 2008). The local theories that guided the project were based on my findings from research on teacher development and supported me in creating coherent foundations on which our work on findings from former cycles was then based. As the project developed, the extent to which we attended to the different aspects of our co-learning

varied, as the teachers gradually started to shape the process of the workshops.

Attending to the global sociocultural theories and constantly reminding myself that the teachers were my co-learners (Jaworski, 2003; Wagner, 1997), helped in being true to the goal of the study, which aimed at enriching my understanding of how learning develops, individually and collectively. The global theories about the ways people change their mode of understanding, perceiving, noticing and thinking in shared efforts with other people (Lave, 1988; Lave & Wenger, 1991; Rogoff, 2003; Wenger, 1998), informed my interpretation of what I experienced and thus shaped the local theories and the developmental cycle (Goodchild, 2008). This was materialised in my response to the teachers' critical alignment to the community I wished to develop when making assumptions about their needs and imagining how I could support their engagement within the community (Wenger, 1998; Wenger-Trayner & Wenger-Trayner, 2015).

Throughout the research study, I learned to be precise in collecting data and in writing my own reflective notes. From the outset, I learned to be meticulous in analysing the data that I extracted from the study (Creswell, 2007) by being diligent about keeping notes and reflections up to date. I decided to use a spread sheet computer program to store my data. In the course of analysing the data, I found the tools for organising the data helpful, and what I learned from this process has also helped me in analysing data from other research projects that I carried out alongside this process. The codes and categories that I already noticed at the outset of the study, in my reflective notes and when watching and listening to the tapes from the workshops, were supplemented by the codes that I identified at later steps of the analysis process. These codes in turn helped detect new categories and the themes that gradually developed. This is further described in Section 8.6. In this coding I looked for flow in the process of our collaboration and in tracing how the project developed by the contribution of each of the participants in accordance with the purpose for the study, as stated in Chapter 1.

Undertaking a study in which I was both an outsider and an insider (Jaworski, 2003) to the research process, required me to pay attention to my own learning as well as the teachers' learning. Writing reflective notes, along with careful and precise data collection and data analysis throughout this process was challenging and often overwhelming. I had to be precise in focussing on critical moments in the process and the small steps we were taking. Often, I was unaware of the fact that they were vital for the process

until I could relate to what I later experienced was developing. The teachers were my critical friends (Schuck, 2011) in this process in the role of outsider researchers into my own development, asking probing questions and supporting me in carrying out the research. My colleagues at the School of Education UI and my supervisors also served as my critical friends, always willing to listen and discuss.

Adopting 'The Developmental Research Cycle' (Goodchild, 2008) in carrying out the research project was challenging and I had to develop my understanding of what each of the cycles within in the main cycle represented in relation to my own research. As mentioned above, I realised early on that the study I wished to carry out aligned with the developmental research cycle. I built on local theories derived through my studies with colleagues (Guðjónsdóttir & Kristinsdóttir, 2006; 2007b; 2011; Gunnarsdóttir et al., 2008; Kristinsdóttir, 2010a). Our theories were in turn grounded on global theories of learning, and as described in Part I, I needed to revisit my understanding of them during this study. This process was challenging and there were times when I felt that my capability to understand the global theories on which I based my work was threatened. In writing the thesis and analysing my data, this thought-provoking process was rewarding and helped me in making sense of my findings.

Another challenge, no less striking, was writing in a foreign language. Not only did I need to translate all quotes I presented from my data, but I also had to formulate my thinking in a language in which I have always found it difficult to communicate. My native language, Icelandic, is a transparent language where most words reflect the meaning of the concept they represent. Since I know neither Latin nor Greek, I find the English words originating from these languages extremely difficult to understand and tend to forget them since they do not make sense to me. Struggling with reading in English about complex descriptions of theoretical terms was therefore often overwhelming and left me bewildered.

On my journey into the world of diverse theoretical foundations and interpretations, as well as methodological approaches, I often came across obstacles that I found difficult to overcome, as was discussed in Chapters 4, 5 and 6. Nevertheless, these challenges have strengthened me in my role as a researcher and a teacher educator. I found the work with the teachers encouraging and I enjoyed participating in the workshops with them. No less pleasant were my visits to their classrooms, meeting their pupils and having the chance to discuss with them. The cyclic process of moving

through the developmental cycle and the research cycle echoed my vision for the development of the project.

11.4 Qualities and value

The heads of both schools were positive about taking part when I contacted the schools and offered them an opportunity to participate in the study. They consulted with teachers in primary grades within their schools with the result that three teachers in Rainbow School and four teachers in Sunshine School accepted my invitation to participate. To ensure that they obtained sufficient information (Norton, 2009) on the nature of the study, I gave them a written summary of the purpose of the study, stated the research questions, informed them about the methodology and presented a time frame for the study (Appendix C). These teachers attended a preparation meeting for the study where they had an opportunity to discuss the purpose of the study and their reasons for taking part. There, we could negotiate our entry into the study, the teachers and mine, our involvement, data gathering and the time we would devote to our participation, as described by Creswell (2007) as the fundamental ethical concerns in qualitative inquiries. I also tried to make myself aware of power imbalances and respect the participants individually by inviting them to actively shape the culture of our community. The teachers accepted my offer, and as discussed in Chapter 9, they gradually took the lead in shaping the culture of our community.

Issues about confidentiality arose when I urged the teachers to record their lessons and observe each other in their classrooms. The teachers who were reluctant to record their lessons claimed that they had to respect their vulnerable pupils' anonymity. We needed to negotiate our understanding of confidentiality and anonymity and who would have access to the information gathered (Norton, 2009). We concluded that the teachers kept track of the information they gathered, I stored the data I collected, and ensured that the data would be destroyed when the analysis and writing of the thesis was finished. Teachers are gatekeepers for their pupils' rights, who should be protected from outside interventions to their life in schools. In my role as a researcher I had to respect the teachers' stand on these issues.

To strengthen the validity of the research, I have written a detailed and thick description of the process of the research, providing the reader the opportunity to personally interpret the findings. My intention with presenting the story of our collaboration in a narrative form, and not

interrupting the flow in the process of my analysis of what was happening in each workshop, was to ensure that the reader had the chance to deduce whether the findings could be trusted (Feldman, 2003). I spent extensive time with the teachers and our prolonged engagement with each other in the project resulted in a community built on trust, as presented and discussed in chapters 9-11 (Creswell, 2007).

I used multiple methods for gathering data, including observations, interviews and recordings from collective workshops, thus making a triangulation of data possible (Bryman, 2004; Bullough & Pinnegar, 2009; Creswell, 2007). I made explicit what counted as data both in the text itself and as an appendix (Feldman, 2003). In the process of analysing the data, I used a variety of methods by drawing on grounded coding and a narrative approach. The plan for each workshop was based on the analysis of findings from former steps, so it was vital that the process of analysing the data would begin at the same time as the project began.

I have been careful in clarifying the origin and accuracy of the data to improve reliability, particularly by presenting excerpts from our discussions and direct quotes from all participants. The reader is then given the opportunity “to check what is a statement of the subject on one hand, and where the researcher’s interpretation begins on the other hand” (Flick, 2002, p. 221). To further support this approach, I presented the findings from each workshop in the form of a narrative so that readers could themselves experience the flow of each workshop. As a further step in ensuring the reliability and validity of the research, four participants read and commented on their narratives in a preliminary draft and they all read the final version of my findings (Bryman, 2004; Creswell, 2007). In the workshops, the verbal contributions from the participants varied and what may appear to be an imbalance in Chapter 9 is a true reflection of their contributions to their discussions.

I discussed my findings with my supervisors and asked for support in analysing them. The data gathered throughout the study were recorded in Icelandic and the findings are presented and discussed in English. I myself translated all quotes presented and used dictionaries and textbooks in English about mathematics teaching and learning to find appropriate concepts to represent the content of our discussions. I also asked my colleagues and supervisors for advice in this process.

11.5 Reformed understanding of teacher development

The collaborative research with teachers was originally planned to last for one year. I knew from former experience of working with teachers and from studying findings from research about teacher development, that I might not see much change over only one year. I did not expect that the teachers would be willing to offer time to work with me over a longer period, particularly because I had no means for rewarding them, other than with giving them the opportunity to develop their teaching with my support. Iceland was in the midst of an economic crisis when the project started and the schools had no means for rewarding their teachers for participating in professional development. It therefore came as a surprise to me when the teachers proposed several times that we prolong our collaboration. We met at workshops for three years and my final observations in their classrooms and interviews with them took place the fourth year. Thus, I was in the position to follow their development over a period of almost four years.

My intentions with the project were clear from the beginning and I felt confident working with the teachers. As the project developed and I continued my doctoral studies, I began to steadily reflect more on the theories that I based my work on. I was challenged to revisit my understanding of these theories and try to make sense of how my understanding of individual and collective learning had developed by relating to my former teaching. In Chapter 6, I discussed my journey into this process and gave an example of how it has affected my teaching within the teacher education program today. This journey helped me in making sense of how my understanding of these theories as well as my beliefs about teaching and learning shaped the project.

In drawing conclusions about how my understanding of teacher development in mathematics teaching and learning has changed through working with teachers at improving their practice, I have traced a trajectory of gradual change in the way I focus on and look for teachers' professional strengths. My notion that it is not sufficient to tell teachers how to teach developed from early on in my practice as a teacher educator. The ways in which I, as a teacher educator, have chosen to support teachers in developing their teaching has also progressively undergone changes. Step by step I have begun to interweave teachers' own explorations into my talks about mathematics learning and now I provide increased space for collective reflection. The experience of providing the teachers in this project the opportunity to take part in shaping our learning community has

strengthen my beliefs that teachers are professionals who can make informed decisions about how to improve their practice. My responsibility is to offer them insights into alternative ways of teaching and challenge them to rethink established norms within their schools.

It is important to note that challenging teachers' beliefs is not the same as criticising them, and I have constantly worked at reminding myself of the fact that attacking peoples' core beliefs only results in them digging in their heels. I have tried to model good teaching by the way I teach and support teachers on their route toward improving their practice by working at changing my own.

By choosing teachers that I knew had little experience in developing their mathematics teaching, I challenged myself to work with teachers who might be reluctant in using the means for professional development I offered them to use. I had met teachers at in-service courses who were reluctant in reviewing their practices and claimed that the advice we, the teacher educators, gave them was not in harmony with their conditions in schools. I wanted to know more about these teachers' concerns and therefore looked for schools where I might meet them. In addition, I wanted to work with teachers in schools with children from diverse background in order to learn about their challenges in attending to their pupils' differing needs. My long and broad experience within the field of education strengthened my trusting with regard to working with these teachers. This experience also provided me with insight into what I might expect and I was prepared for negative response to my proposals aimed at professional development. To my surprise, the teachers welcomed most of what I proposed and we managed to resolve the conflicts that came up.

11.6 Implications and further work

My journey through the process of preparing, planning and implementing the study has been enjoyable and, even though there were some obstacles along the way, it ran smoothly most of the time. From the outset, my vision for a collaborative study that would lead to a co-learning agreement was clear. I expected all the participants within the project to support each other in learning about their practice and add to the learning that would emerge within our community. My understanding of how learning develops as our inner thoughts interrelate with what we learn through participation with the outer world helped me shape the study. In this context, I had to negotiate my constructive view that knowledge is not passively received with my understanding of learning as social participation. Throughout the

project, I also had to remind myself that, while it was important that I brought my knowledge as a teacher educator and a researcher into the project, I also had to respect the teachers' professional knowledge and their willingness to align to what I would like them to learn. I trusted the teachers to take the lead in the development of our project as it progressed, as well as in relation to what we attended to at the workshops and what tools they decided to use for researching into their practices.

By inviting the teachers to shape the study with me we were empowered to build a community where sharing knowledge was the key factor and a guiding tool for the project. The developmental research cycle, which moved between thought experiments and practical experiments, with both elements mutually informing each other, had a major impact on the way the project developed. Being consistent in structuring every workshop on former findings and always relating to the teachers' visions for our work, a full developmental cycle was carried out in each workshop, as presented and discussed in Chapter 9.

Within each theme, a full developmental research cycle was completed. The teachers' developing notion of their professional learning and where they needed support, affected what we focused on at the workshops, thus affecting the local theories. Their self-confidence developed through this experience and they took lead in shaping the culture within our community. Their developing confidence in turn led to the refinement of the global theories as they were empowered to take the lead in shaping the culture of a the co-learning community that to begin with was shaped by my visions for the project.

The initial local theories revolved around the idea that teachers can be empowered to inquire into their mathematics teaching by participating in learning communities. Here, they shared thinking and negotiated their understanding of important features of mathematical activities in classrooms. My findings showed that they progressed in this sense, as was discussed in Sections 9.8 and 10.5. Time is an essential factor in all developmental work and the teachers' initiative, to proceed with the project, helped trace their progress for a period of three years. I have also developed my competency as a teacher educator and a researcher, as discussed in Sections 11.2–11.4, and intend to continue my work with teachers, framed in terms of the model for teacher developmental projects I have started to develop and based on the findings from this study.

The extent to which the teachers saw themselves as researchers differed, but by the end of the three-year study, they had all actively

contributed to shaping the study and researching into their own practices by reflecting on their experiences and analysing what they had noticed in their classrooms, and by recognising choices and looking for alternative actions (Mason, 2002). In challenging them to focus on critical moments in their classrooms and discuss at our workshops I was offering them means by which they could research into their practices.

During this process, I was empowered in rethinking my practice. One turning point was three teachers' reluctance to write about their own reflections, where I realised how important it is to respect in-service teachers as professionals and not expect them to devote their time to engage in writings about their learning. When I decided to respond to their critique by offering the teachers the chance to discuss their reflections of what they had noticed in their classrooms, they gradually became comfortable with addressing what to attend to at our workshops. I felt relieved when I discovered how the project benefited from the decision to encourage them in taking the initiative in shaping the culture within our community. My reflections on my understanding of educational theories and the different methodologies of practitioner research, discussed in Part I, strengthened my resolve with regard to encouraging the teachers to shape the trajectory of the project, and by extension, the culture that emerged, always being prepared to respond to their contribution. In my original plan for the project, the developmental aspect was apparent and resulted in my decision to adopt the developmental research cycle. My decision to urge the teachers to take initiative in shaping the project was to assure that they were inspired to develop their practice, which provided me the opportunity to trace their learning path throughout the project.

The teachers I worked with all learned to focus on the learning in their classrooms and were strengthened in their ability to discuss mathematics with their pupils. I did not attempt to measure how their learning was reflected in their pupils' learning and I am therefore not in the position to make claims about the effect of the study on pupils' mathematics learning. However, I can confirm from my discussions with the teachers at the workshops, and in interviews as well as in my visits to their classrooms, that they felt empowered as professionals in teaching mathematics in primary grades.

We succeeded in developing a learning community, which we all aligned with by preparing collaboration and negotiating our plans and visions for the project (Wenger-Trayner & Wenger-Trayner, 2015), as has been discussed in Chapters 9 and 10 and further in this chapter. I aligned myself

to the teachers' needs by creating the environment in which the teachers could take the lead in what to attend to and discuss at our workshops. If I had been consistent in following the protocols, prepared for each workshop, they would not have had the same opportunity to shape the culture within the community that we were developing together. The interrelations between the local theories and the global theories, became manifest through the constant revising of our collaborative work.

The co-learning partnership I envisioned that we could develop was cultivated as we shared knowledge about our practice and mutually reflected on our work (Jaworski, 2003). The teachers invited me to take part in their professional practice by sharing their experiences at the workshops and by inviting me into their classrooms. Our collaborative research produced valuable knowledge that will be shared with other teachers and education professionals in the form of collective writings in teacher journals and my contribution to the research community by presenting the findings at conferences and in research journals.

Interweaving the methodologies of developmental research and self-study of teacher education practices has empowered me in improving my practice as a teacher educator and a researcher. By constantly reflecting on every step that was taken within the project, I adopted a role where I was both following the developmental process of the study and my own learning. Through the process of self-study into my own practice, I became aware of my relationship with the teachers I was working with and found that building trust through dialogue was essential, as well as respecting each other's knowledge and strengths (Loughran, 2007; Pereira, 2011; Russell, 2007).

As discussed in Chapter 2, research within the field of mathematics education in Iceland is meagre. Educational studies are a growing field in Iceland, and with my contribution, I seek to influence the way educational research is carried out in my country. I do not know of any study in Iceland where the methodology of developmental research has been applied. A few articles have been published about self-study into own practices in Iceland. Within the School of Education, however, there is a growing interest amongst teacher educators in studying their own practices. The combination of these two approaches is new within the field of educational research in Iceland and I have not come across any research within the field of mathematics education elsewhere where these two methodologies are intertwined in one study. I found it essential to adopt these two approaches in my study and plan to develop further my findings from this process.

My understanding of teacher development in mathematics teaching and learning has undergone significant changes.

- Through the careful use of the developmental process I have realised that it is essential for teachers to be aligned with and within their school communities and this necessity impacts their potential for professional learning.
- Conflicts exist between inquiry approaches and instrumental learning. The dominant belief that mathematics is best learned through instrumental learning impacted the teachers' potentials for adapting to inquiry approaches in teaching mathematics, a practice, which I wished to cultivate within our community.
- Tensions arising from teacher beliefs concerning inclusive teaching as opposed to individualised learning can hinder active engagement in a learning community. Individualised learning does not promote approaches to teaching that cultivates the collective construction of mathematical knowledge.
- Requirements within schools that teachers adopt the position of protecting the interest of vulnerable children can function as boundary objects between the communities in schools and the one that researchers wish to create when researching with teachers.
- In developmental projects collaboration with school authorities and professional communities within schools is vital for individual teachers to grow in their professional practice.
- Teachers are professionals who, if provided with classroom-related support, can work at developing their mathematics teaching with the goal of cultivating inquiry in mathematics within their classrooms.

The local theories that inspired the collaborative study, namely, that teachers need the opportunity to a) collaboratively investigate with mathematics in order to develop knowledge about mathematics teaching and learning in diverse classrooms, and b) research their own practice, were constantly informed by the teachers' response to the activities with which we engaged. The global theories were reinterpreted with regard to the extent to which the boundaries between our professional communities restricted or cultivated our community building.

Together with colleagues within the teacher education program at the University of Iceland, I plan to apply for a research grant, which would allow me to carry out a research project with teachers in compulsory schools in Iceland where the goal is to build on the emerging model I developed for this study. The purpose of the study will be to collectively develop ways to attend to all learners in schools where collaboration and mutual support is emphasised. By building on the strengths of individual learners and working

towards the goal of creating learning communities, where participants develop understanding through co-learning and shared reflections and experiences, they may be empowered to work towards the goal of inclusive practice in schools.

My study is the first collaborative research with teachers in Iceland. It is also the first study where diversity and inclusive practices are addressed in research about the teaching and learning of mathematics, and my findings will therefore be a valuable asset in planning the new study. The structure of the study will be informed by these findings and our guiding principles for the moment are:

- Teachers need to be respected as professionals in their practice and offered the opportunity to become active participants in deciding what to focus on together with the teacher educators. The contribution of both parts is equally important.
- Teacher educators must be responsive to the steps the teachers take during the process and support them in reflecting on their beliefs and practices in a constructive way. A cyclic process is essential for the project to develop.
- The teacher educators must attend to their own learning throughout the project, constantly being prepared to revise their plans as new steps are taken. In this process, their knowledge of theories of learning is vital and needs to be continuously revisited.
- Development in teaching takes time and the effects become apparent through gradual steps. Therefore, we must make special effort to seek what is invisible or what is at an embryonic stage at each time.
- Mutual trust between teachers and teacher educators and respect for diverse opinions is vital in building a learning community.
- Dialogue with colleagues the within the teachers' professional communities in their schools needs to be fostered.
- The experience the teachers gain from inquiring into their own practice will empower them to bolster such inquiry community with their pupils.

Teacher educators from different disciplines will collaborate in this research with teachers in schools and we will focus on diverse subject areas. The learning developed from my collaborative study into mathematics teaching and learning is equally applicable in other subject areas.

Working with the teachers who participated in this study has supported me in my belief that teachers can neither make their students learn nor do their learning for them, as Mason (2002; 2008) has addressed. I cannot change the way teachers in schools teach, but I can work at improving my competency in supporting teachers in changing their practices.

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Appendix A: Tables from Chapter 8

Table 8.1: Time schedule and overview of data generation

Time	Project activities	Data collected	Data processing
Spring 2009	Meeting with the school principals.	Minutes from the meeting.	
Autumn 2009	<p>First meeting with the teachers together with principals.</p> <p>Individual interviews with seven teachers and two focus group interviews with four school principals.</p> <p>Classroom observations in seven teachers' classrooms.</p> <p>Workshops 1-4 (90 min) with seven teachers.</p>	<p>Minutes from the meeting.</p> <p>Audio recordings from interviews.</p> <p>Notes from classroom observations.</p> <p>Video recordings from workshops.</p> <p>Teachers keep samples of children's work and own reflections.</p> <p>Reflective notes (teacher educator/researcher).</p>	<p>Transcription of data from interviews and observations.</p> <p>Analysis of the process started by listening to and reading transcriptions from interviews and looking at videos from workshops.</p> <p>Themes that emerged through this process guided what to attend to at the following workshops.</p>
Spring 2010	<p>Workshops 5-8 (90 min) with seven teachers.</p> <p>Teachers observed each other's classrooms.</p> <p>Individual interviews with six teachers and a focus group interview with two school principals.</p> <p>Classroom observations in six teachers' classrooms.</p>	<p>Video recordings from workshops.</p> <p>Teachers keep samples of children's work and own reflections.</p> <p>Reflective notes (teacher educator/researcher).</p> <p>Audio recordings from interviews.</p> <p>Notes from classroom observations.</p>	<p>Analysis of the process followed by looking at videos and tracing emerging themes.</p> <p>Transcription of data from interviews and observations.</p> <p>Themes from observations and interviews coded.</p> <p>Themes that emerged through this process guided what to attend to at the following workshops.</p>

Autumn 2010	Workshops 9-11 (90 min) with six teachers. One of seven teachers is not teaching mathematics this year and therefore no longer participates in the project. Two teachers audiotaped their teaching. Discussion with them about their reflections.	Video recordings from workshops. Teachers keep samples of children's work and own reflections/recordings. Reflective notes (teacher educator/researcher). Audio recordings from discussions with two teachers about their recordings of their teaching.	Analysis of the process proceeded by looking at videos and tracing emerging themes. Transcription of data from discussions about teachers recording. Themes that emerged through this process guided what to attend to at the following workshops.
Spring 2011	Workshops 12-14 (90 min) with six teachers. Two teachers audiotaped their teaching. Discussions with them about their reflections. Analysis of the process proceeded.	Video recordings from workshops. Teachers keep samples of children's work and own reflections/recordings. Reflective notes (teacher educator/researcher). Audio recordings from discussions with two teachers about their recordings of their teaching.	Analysis of the process followed by looking at videos and tracing emerging themes. Transcription of data from discussions about teachers recording. Themes that emerged through this process guided what to attend to at the following workshops.
Fall 2011	No workshops due to lack of time on both sides.	Reflective notes (teacher educator/researcher).	Transcriptions of videotapes from workshops started. Themes coded.
Spring 2012	Workshops 15-17 (90 min) with five teachers. Last workshop in May	Video recordings from workshops. Teachers keep samples of children's work and their own reflections. Reflective notes (teacher educator/researcher).	Transcriptions of videotapes from workshops. Themes coded. Themes that emerged through this process guided what to attend to at the following workshops.
Fall 2012		Reflective notes (teacher educator/researcher).	Transcriptions of videotapes from workshops. Themes coded.

Spring 2013	Interviews with four teachers and observations in their classrooms.	Audio recordings from interviews. Notes from classroom observations. Reflective notes (teacher educator/researcher).	Transcriptions of videotapes from workshops finished. Themes coded.
Fall 2013 to Spring 2016			Further analysis of data, wrapping up themes and writing results.

Table 8.2: List of participants and their education and experience

Participant	Education	Experience
Dóra, homeroom teacher in grades 5-7 in Rainbow School	B. Ed. with specialisation in teaching textile. Had added to her studies and participated in a problem-solving course for teachers.	15 years of experience, mainly as a classroom teacher in Grades 1-4
Edda, homeroom teacher in grades 4-6 in Sunshine School	B. Ed. with specialisation in Icelandic and social science	Was new at Sunshine School. Four years of experience as a classroom teacher in Grades 1-4
Gróa, homeroom teacher in grades 5-7 in Sunshine School	B. Ed. with specialisation in teaching textile.	15 years of experience, mainly as a classroom teacher in Grades 5-7
Inga, support teacher in grades 5-7 in Rainbow School	Educational background in social pedagogy and later in special education.	Worked for several years as a social pedagogue for a communal diagnostic and counselling centre. 2 years of experience as a special education teacher.
Pála, homeroom teacher in grades 5-7 in Rainbow School	B. Ed. with specialisation in teaching textile and Icelandic.	29 years of experience, mainly as a classroom teacher in Grades 5-7
Rúna, support teacher in grades 5-7 in Sunshine School	Teacher certificate before university degree was required. Specialised in teaching English and Icelandic.	23 years of experience, both as a classroom teacher in Grades 1-7 and support teacher in mathematics and Icelandic in Grades 5-7
Vala, support teacher in grades 5-7 in Sunshine School	B. Ed. with specialisation in Icelandic and Social science.	11 years of experience, mainly as a support teacher in mathematics and Icelandic in Grades 5-7
Jónína, mathematics teacher educator and a doctoral student	M. Ed. in mathematics education.	18 years of experience as a classroom teacher in Grades 1-7. 20 years of experience in teacher education as a practice teacher and a mathematics teacher educator.

Table 8.3: Interviews with teachers

Time	Focus	Conclusions
September 2009	<p>Teachers' education, years of teaching.</p> <p>Experience of teaching mathematics.</p> <p>How teachers structure their mathematics teaching, use of curriculum, assessment.</p> <p>Attitudes to mathematics teaching and learning, support from school, parents, authorities.</p> <p>Visions for the project.</p>	<p>Years of experience of teaching ranged from 2-29 years.</p> <p>None of the teachers had specialised in teaching mathematics.</p> <p>Teachers wished to learn about diverse ways of teaching mathematics and lead discussions.</p>
May/June 2010	<p>Teachers experience of participating in the project.</p> <p>What is memorable from our workshops and was interesting to discuss? Do the teachers feel that our collaboration has affected their teaching? Is there anything the teachers would like to discuss further?</p>	<p>The teachers were confident with their experience and felt that they had benefited from participating in discussions regarding their work and solving mathematical problems together. Pála added that the workshops with all the teachers in her school were helpful.</p> <p>Vala and Rúna focused more on their pupils' strength than before and listened more to how they explained their thinking.</p>
Dec 2010/Jan 2011	<p>The four teachers who audiotaped their lessons wanted to discuss their experience with me. The focus was on what they learned about their communication with their pupils and about their pupils learning.</p>	<p>The teachers found it difficult to listen to the recording and felt that they did things differently from what they intended. Pála had recorded her teaching earlier (language lesson) and she was more confident in listening to the recording. The teachers discussed their recordings at several workshops.</p>
February 2013	<p>Teachers' experience of participating in the project.</p> <p>What is memorable from our workshops and was interesting</p>	<p>Dóra expressed that her participation in the project has supported her in being open to investigative approaches to</p>

	<p>to discuss? Do the teachers feel that our collaboration has affected their teaching? Is there anything the teachers would like to discuss further?</p>	<p>mathematics teaching. Edda said that she had enjoyed solving the tasks and discussing them with us. The experience she gained at the workshops opened her eyes for diverse ways of teaching mathematics.</p> <p>Inga emphasised that she now was consistent in asking her pupils to explain their thinking about problems.</p> <p>Pála felt that she is now conscious of what kinds of problems open up for mathematics learning. Her confidence in leading mathematical discussions has also grown.</p>
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Table 8.4: Observations in classrooms

Time	Focus	Conclusions
September 2009	Classroom culture: Teachers' way of organising work and choice of topics, interactions: teacher/pupils, pupils/pupils.	Review of homework, introduction of classroom topic and explanation of how to approach tasks on the agenda. Pupils work individually with textbook tasks. No organised collaboration, some pupils discuss their work with persons sitting next to them, teachers interact with individual pupils. No summarising or discussion of work at end of lesson.
May/June 2010	Classroom culture: Teachers' way of organising work and choice of topics, interactions: teacher/pupils, pupils/pupils.	Teachers worked with problems that we had discussed together at our workshops. They introduced the task and asked for children's ideas on how to approach it. The children worked in groups and the teachers discussed with them. Finally the pupils shared their work with the class.
February 2013	Classroom culture: Teachers' way of organising work and choice of topics, interactions: teacher/pupils, pupils/pupils.	The teachers were more focused on investigation and collaboration than before. They started with probing for the children's opinions about the topic of the lesson. The children then worked in groups of 2-4 and teachers discussed with them. Emphasis was placed on talking about explorations and their ways of finding solutions to tasks. At the end of lesson the children shared their work with the whole class.

Appendix B: Tables from Chapter 9

Negotiating collaboration

Table 9.1: Theme 1 - Initial steps to an investigative approach

Dates	Theme 1	Conclusions and looking ahead
Initial steps to an investigative and collaborative approach		
Observations and interviews. The project begins	One lesson in each teacher's classroom was observed and followed up with an interview.	Teachers wished to learn about diverse ways to teach mathematics and lead discussions.
Workshop 1 Establishing common ground 19.09.09	<p>Protocol for workshop 1: Establishing common ground</p> <p>Establish a common ground and negotiate means for our collaboration. Solve a task about dogs and biscuits and discuss diverse ways to approach it. Discuss approaches to researching one's own practice. Discuss teachers' stories from their classrooms. Look ahead and discuss the teachers' vision for the collaborative project.</p>	<p>The teachers expressed that they liked to discuss their way of solving the problem and wanted to learn more about collaboration in mathematics classes and diverse ways of calculating. Focus on co-operative learning and calculation of number problems.</p>
Workshop 2 Various ways of calculating 29.10.09	<p>Protocol for workshop 2: Various ways of calculating.</p> <p>Teachers solve number problems and discuss various ways to calculate and discuss how they think their pupils would approach these problems. Focus on co-operative learning and the roles of participants. Look into other teachers' classrooms. Video clip from a classroom where the solution of a division problem was discussed. Stories from the teachers' classrooms. Look ahead.</p>	<p>Give more space for discussing the teachers' stories from their classrooms. Focus on ways to support discussions about own thinking.</p>

Table 9.2: Theme 2 - Reflective practice, hindrances and opportunities

Dates	Theme 2	Conclusions and looking ahead
Reflective practice, hindrances and opportunities		
Interlude between workshops	Three teachers were concerned about writing their reflections and where their writings would be stored.	Teachers will keep their writings for themselves. Give space for reflective discussions at workshops.
Workshop 3 Teacher reflections 03.12.09	<p>Protocol for workshop 3: Teacher reflections</p> <p>Discuss the goal of writing about and analysing cases from classrooms. Introduce concept cartoons and this approach to solving problems. Discuss cases from the teachers' classrooms. Look ahead. Step 2 was not implemented as step 3 took over.</p>	<p>Urge the teachers to discuss their mathematics teaching at our workshops with the goal of supporting each other in analysing their work. Focus on ways to support children in taking initiative in solving mathematical problems.</p>
Workshop 4 Supporting children's initiative 21.01.10	<p>Protocol for workshop 4: Supporting children's initiative</p> <p>Discuss mutual visits to the teachers' classrooms. Introduce concept cartoons and this approach to solving problems. Discuss cases from the teachers' classrooms. Look ahead. .</p>	<p>Support the teachers in focusing on how they themselves and other people approach mathematical tasks. Teachers urged to observe each other teaching and plan their visits with the goal of paying attention to the learning that takes place in the classroom.</p>
Workshop 5 Learning to listen 25.02.10	<p>Protocol for workshop 5: Learning to listen</p> <p>Teachers divided into two groups and each group works with one problem. Teachers in each group discuss diverse ways of solving their problem. The groups meet and one teacher in each group pairs with one from the</p>	<p>Teachers learned from the discussions concerning their own thinking about the problems. Their initiative to discuss their own teaching was a sign that we were building</p>

	<p>other group. One member of the pair solves the problem she had not solved in the first round and the other acts as a teacher who listens and asks probing questions but does not suggest how to solve it. The teachers then change roles.</p> <p>Discuss what we learned from this experience.</p> <p>Look ahead.</p> <p>The teachers added to the protocol by starting a discussion on stories from their own classrooms.</p>	<p>trust in our learning community.</p> <p>They were starting to focus on what kinds of problems and approaches to mathematical learning are worth paying attention to.</p>
<p>Workshop 6 Mutual visits to classrooms 18.03.10</p>	<p><i>Protocol for workshop 6: Mutual visits to classrooms</i></p> <p>Teacher stories from mutual visits to each other's classrooms.</p> <p>Look ahead.</p>	<p>Teachers felt that they had learned much from observing each other's teaching. Teachers wanted to focus on fractions.</p>

Awareness of learning

Table 9.3: Theme 3 – A focus on interactions in mathematics classrooms

Dates	Theme 3	Conclusions and looking ahead
A focus on interactions in mathematics classrooms		
Workshop 7 Teaching and learning of fractions 19.09.09	<p>Protocol for workshop 7: Teaching and learning of fractions</p> <p>Stories from teachers' classrooms. Explore with fractions and solve fractional problems. Discuss fractional models and the culture in mathematics classrooms. Look ahead.</p>	<p>Teachers showed interest in learning about algebra and children's development in learning mathematics with focus on diversity. They were urged to audiotape their teaching as means for learning more about their own teaching.</p>
Workshop 8 Focus on the learning of algebra 29.04.10	<p>Protocol for workshop 8: Focus on the learning of algebra</p> <p>Stories from teachers' classrooms. Teachers solve problem about a growing pattern. Look into other teachers' classrooms. Video clip from a bilingual classroom where the meaning of the equal sign is discussed. Look ahead.</p>	<p>Focus on diverse approaches to mathematics teaching was emerging. Teachers wanted to proceed with our collaboration and we decided to meet three times next year.</p>
Observations and interviews May/June 2010	<p>The teachers were confident with their experience and felt that they had benefited from participating in the workshops. Their approach to their teaching had changed. They discussed with their pupils and explored with problems.</p>	<p>Teachers were making progress towards investigative approach and focusing on the learning in their classrooms.</p>

Table 9.4: Theme 4 – Focusing on pupils’ learning in the classrooms

Theme 4: Focusing on pupils’ learning in the classroom		
Workshop 9 Geometry and measurement 07.10.10	<p>Protocol for workshop 9: Geometry and measurement Negotiate how to proceed with our collaboration the next months. Teachers design a container for ½ litre of water and reflect on their own understanding of volume. Teachers’ stories from own classrooms. Look ahead.</p>	Need to focus on the nature of mathematical tasks and the effects the teacher has on the mathematical learning in the classroom.
Workshop 10 Dealing with children’s dependence 11.11.10	<p>Protocol for workshop 10: Dealing with children’s dependence Teachers solve a task about the border problem. Look into other teachers’ classroom. Observe a discussion about the border problem. Teachers’ stories from own classrooms. Look ahead. Steps 1 and 2 were postponed until the next workshop.</p>	Need to focus on the nature of mathematical tasks and the effects the teacher has on the mathematical learning in the classroom.
Workshop 11 Patterns and algebra 25.11.10	<p>Protocol for workshop 11: Patterns and algebra Teachers solve the border problem. Look into other teachers’ classroom. Observe a discussion about the border problem. Look ahead.</p>	Teachers need to look into their own way of teaching mathematics. They were urged to record their lessons.
Workshop 12 Exploring with the calculator 03.03.11	<p>Protocol for workshop 12: Exploring with the calculator Explore with relationships between numbers and the properties of operations. Discuss reflections on the benefit of audiotaping one’s own classrooms. Look ahead.</p>	Teachers were satisfied with what they learned from exploring with the use of the calculator and envisioned how they could focus on such explorations with their pupils. Those who had recorded their lessons were content with their experience and urged the others to do the same.

Teachers' influence on workshops

Table 9.5: Theme 5 – Teacher reflections lead our discussions

Dates	Theme 5	Conclusions
Teacher reflections lead our discussions		
Workshop 13 Conflicts about assessment 31.03.11	<p><i>Protocol for workshop 13:</i> <i>Conflicts about assessment</i></p> <p>Stories from teachers' classrooms. Solve proportional problems. Look ahead. Step 2 was not implemented as step 1 took over</p>	The focus of the discussions shifted from how to monitor work to children's understanding.
Workshop 14 Learning to listen 04.05.11	<p><i>Protocol for workshop 14:</i> <i>Focus on listening to children</i></p> <p>Teachers' stories from their own classrooms. Solve proportional problems (if time allows). Look ahead. Step 2 was not implemented as step 1 took over.</p>	The focus was shifting to teacher professional development and the means for improving one's own practice.
Workshop 15 Teachers lead the workshops 09.02.12	<p><i>Protocol for workshop 15:</i> <i>Teachers lead the workshop</i></p> <p>Stories from teachers' classrooms. Solve proportional problems (if time allows). Look ahead. Step 2 was not implemented as step 1 took over.</p>	The teachers reflected on their experiences and discussed their concerns for their pupils' learning and how they could be supported to gain confidence in learning mathematics.

Table 9.6: Theme 6 – Towards an investigative approach and inclusion

Dates	Theme 6	Conclusions
Towards an investigative approach and inclusion		
Workshop 16 Discussing curriculum guides 14.03.12	<p>Protocol for workshop 16: Discussing curriculum guides Discuss our visions for the mathematics chapter in new curriculum guidelines. Look ahead.</p>	The decrease of financial support worried the teachers and they felt that it disrupted their possibilities for professional teaching.
Workshop 17 Teachers reflect on their learning with their pupils 26.05.12	<p>Protocol for workshop 17: Teachers reflect on their learning with their pupils Stories from teachers' classrooms. Conclusions about our three years of collaboration.</p>	The stories the teachers told were focused on the mathematics learning in their classrooms. The teachers were satisfied with their participation in the project. They had gained confidence in leading discussions about mathematics and in choosing tasks to work with.
Interviews and observations	Focus on investigative approach and inclusion.	Decision taken not to continue the project despite the teachers' wish to proceed.

Appendix C: Information about the project

Agreement with regard to a collaborative research project of teachers and researchers into mathematics teaching in a school for all

Rainbow School and Sunshine School and the University of Iceland, School of Education enter into the following agreement collaborative research project on teaching of mathematics. Some maths teachers in the middle grades of the two schools will undertake research into their own teaching of mathematics in cooperation with a PhD student, who is also an assistant professor in math education at the School of Education of the University of Iceland. The objective of the research is to analyse the development of teachers who work in a structured manner towards improving their mathematics teaching.

If a recording of persons does take place the teacher must inform the principal and the parents.

The doctoral student conducts interviews with teachers and school leaders and will occasionally be present in the classroom. Part of staff development is to observe one's own teaching. Participants in the research work could be, for example, videotaped otherwise recorded. If a recording is to be made it is necessary to inform the school administration ahead of time and inform the parents.

Teachers will be examining their own knowledge of learning and teaching of mathematics. They have met once a month during the school year 2009-2010 and will continue their cooperation for the coming year. The PhD student will conduct interviews with teachers and school managers a few times over the period. Part of a teacher professional development can entail an study into their own teaching, for instance, by using video or sound recordings. Teachers must inform managers and parents about intended recordings and receive their permission. The recordings are only intended for the teachers and will be destroyed once they have been examined. Part of the professional development is to take part in teaching with colleagues.

The PhD student will offer to work with other teachers of the school on specific projects related to learning and teaching of maths in consultation with school heads and will not receive any payment for that work.

Reykjavik, 1st July 2010

XXXX XXXX
Head of Sunshine School

XXXX XXXX
Head of Rainbow School

Jónína Vala Kristinsdóttir,
Menntavísindasviði Háskóla Íslands
(School of Education, University of Iceland)

Workshops with all teachers at Rainbow School

During the workshops in Rainbow School, I responded to the teachers' interest in problem solving, classroom discussions and assessment. At the first two workshops, we focused on problem solving and discussions in mathematics classrooms. The teachers solved problems in groups of 3-4 and then shared their thinking with the whole group about the problems group. Following their reports from the groups, we discussed what kinds of tasks open up explorative work in the mathematics classroom and how teachers can motivate discussions in their classrooms. At the third workshop, we focused on assessment, diverse approaches to assessing mathematics learning and the goals with these approaches. The two remaining workshops were devoted to the teachers' discussions of their approaches to teaching mathematics, where they shared their way of working in their mathematics classrooms, and we then reflected collectively on their stories. I first met the teachers in grades 1-4 and then the teachers in grades 5-10, in order to provide teachers space for discussing their work in-depth.

In Sunshine School, I met two groups of teachers in August, in the second year of the project,. They were planning their mathematics teaching for the fall and raised some concerns about how to organise their teaching and what was important to attend to. I responded to the teachers' concerns by probing for more information and descriptions of their worries, thus supporting them in finding their own solutions to the problems.

Appendix D: Information to teachers

Points sent to teachers before interviews in early September 2009

In the interviews I would like to discuss with you the following:

- Your background, i.e. your education and work experience. I would like to hear from you about everything you think has influenced your teaching and your views on teaching and the work of the school.
- I am especially interested in learning about your mathematics teaching, i.e. what age groups you have taught, how you like teaching it and how you have been getting on with your teaching.
- It is also interesting to learn about how you organise your teaching, both how you construct each lesson (although it is of course not always the same) and also the overall organisation of your work with regard to diverse pupils and the overall work plan for the whole winter.
- I am also interested in knowing what materials you have used in your teaching and what you think of it. Also how you use the materials. Do you like some materials better than others? Have you prepared your own materials or accessed material for instance from a repository in the school?
- I would also like to listen to your ideas on assessment and how evaluation is carried out in your school.
- It would like to hear about your views on learning and teaching of mathematics and also your opinions on how society (e.g. parents and education authorities) supports your teaching.
- I will not ask direct questions, but would like you to tell me about your teaching. I will follow up the interview by asking further questions about what I would like to understand better.

Information to teachers about the project August 27th 2009

Aims and objectives and research questions

The objective of the research is to analyse the professional development of teachers who work in a structured and systematic way towards improving their teaching

- How do their teaching and attitudes towards learning and teaching of mathematics change?
- How do the teachers perceive that their participation in the research affects their teaching?
- How does the research process develop?

Research approach

Collaborative research between teachers and the researcher on learning and teaching mathematics. Efforts will be made to create a learning community where participants use a research approach to analyse the learning that takes place in the classroom and how their teaching evolves. In such a learning community, participants are offered the opportunity to ask questions about how the condition of pupils for engaging in meaningful learning can be improved. At the same time, they learn about their own learning. Research based approach is used to develop one's own work and as a teaching method.

Implementation plan – Fall 2009

- Preparation and consultation with school leaders/managers
- Applying for grants
- First meeting with teachers to introduce the project and modalities of work (end of August)
- All teachers and school managers/leaders interviewed
- Field observations in all classes
- Three workshops (Sept. Oct. Nov.). Work together on mathematics research to strengthen own knowledge and experience in applying maths in solving problems.
- Discuss examples of learning and teaching from classrooms of teachers.
 - Teachers make note of stories from their teaching
 - Teachers observe teaching of colleagues in the group
 - Reading of articles on learning and teaching of mathematics

Spring 2010

Three workshops (Jan. Feb. March).

Solve mathematics problems together to strengthen one's own knowledge in mathematics and competency in exploring, discussing and reasoning together.

Discuss examples of learning and teaching from classrooms of teachers (videos, field observations, teachers' stories).

Also:

- Teachers observe colleagues in the classroom
- Recordings of lessons
- Teachers make note of stories from their teaching
- Reading of articles on learning and teaching mathematics
- Final workshop where the experiences from the winter are summarised

Autumn 2010

All teachers and school managers interviewed

Field observations in all classes

Data analysed

Spring 2011

Continue to analyse data and possibly follow-up with teachers

Results presented to teachers.

Developing a Case Commentary: Reflecting, questioning and acting in response to a practice example (case)

Your **Case Commentary** records what you can learn through reflecting on and questioning your experience. The following questions are designed to assist you in thinking about your commentary. The numbered questions (1-4) lead into the four dimensions of reflecting on practice and planning action. The bulleted questions are simply examples to get the reflection going, in thinking about your case you may use some of these questions or simply develop your own.

<p>1. Practice Described: What are my questions?</p> <p><i>Initiating questions:</i></p> <ul style="list-style-type: none"> • What happened for the students in this story? Were they all included in the learning, the relationships, and the activity? etc. • What did students contribute? How did they relate to one another? • What looks to be the everyday routine? What methods/approaches to teaching/learning/assessment did I use? • How did I respond to students? Did I treat some students differently? • Who else was involved? What was their contribution? • Who benefits from this situation? Who is disadvantaged? • Were there any significant “turning points” in the story? • What are my impressions and emotional response? • What confuses or worries me? • What pleases me? 	<p>2. Practice Explained: How can I understand and explain this event/issue/dilemma?</p> <p><i>Initiating questions:</i></p> <ul style="list-style-type: none"> • Why do students learn as they do? Why do some students achieve better than others? • How can I understand and explain this event/issue/dilemma based on my teaching experience? • What other explanations are there (e.g., from talking to teachers, talking with students, reviewing the literature, what I have learnt in classes)? • How is it that schooling treats some particular students and groups differently? • Who decides? Who is powerful? Who is less powerful? Who benefits from the decisions? • How could I look at this experience from a fresh viewpoint? What theories of learning and teaching could inform these perspectives? • What levels of relationships and/or controls affected (or could change) the outcomes in this case, e.g. <ul style="list-style-type: none"> ○ <u>Close/local</u>: intrapersonal responses and interpersonal relationships in the classroom, in the school, between school, home and community? ○ <u>Medium distance</u>: System requirements, state and national politics? ○ <u>Broad societal</u>: the socioeconomic and sociocultural structures and their impacts?
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<p>3. Practice Theorised: What is my personal theory of action (Why do I do what I do?)</p> <p><i>Initiating questions:</i></p> <ul style="list-style-type: none"> • What have I decided to keep doing? Why? • What have I decided not to keep doing? Why? • How will I explain what I do to myself (and my friends), ... to students, ... to other teachers/professionals and administrators, ... to parents and community members? • What personal ethics and values guide my commitments to all students? How can I apply these personal ethics and value stance in developing curriculum, pedagogical and assessment approaches that support all 	<p>4. Practice Changed: What have I learned & what could I do?</p> <p><i>Initiating questions:</i></p> <ul style="list-style-type: none"> • What can I do to make my practice more inclusive and more responsive to the learning of all students?? • What is supporting/getting in the way of my putting my insights into practice? • How can I know if what I do is good for students? • What can I do to improve and change my practice continuously?
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(Guðjónsdóttir et al., 2007)

Guidelines for observing lessons together

Before you visit each other, it is important to view the lesson topic together. What is the aim of the lesson and what teaching materials do you intend to use?

This is a good opportunity to try something in your teaching you have not done before, but would like to try out. You can also ask your colleague to look at something you have done before and you would like to have her view on.

It is important that the observer mingles with the pupils. The one who is teaching may ask you to pay particular attention to some pupils.

It is good for the observer to keep the following in mind when preparing how she is going to conduct the observation.

- What is the aim of the exercise?
- How do the pupils listen when the exercise is introduced?
- How do they start the exercise?
- What do they discuss among themselves?
- How do they work together?
- How do they approach the solution to the tasks?
- What do I do when a pupil runs into a problem?
- How do pupils take part in common discussions?

It is important that the observer is actively engaged during the lesson, walk around and talk to the pupils.

Try to find time, preferably on the same day to discuss the lesson.

- The one observing will discuss her observations and the one teaching will certainly also have much to discuss, both about how the pupils reacted and how they worked, and also about her own role in the lesson
- If the one teaching feels she has done something differently than intended, then it is important to discuss it and how the lesson could be improved.
- If the teacher is satisfied with the lesson it is also important to discuss what went well and discuss the ways the pupils solved the problems.

Next when we meet (18th of March), we will take good time to discuss the visits. Then it would be good if you bring notes from your observations and discussions.

Appendix E: A sample of problems the teachers worked with

Workshop 5:

Guidelines for listening to problem solvers

The group will be divided into two groups. The groups work in two different rooms to begin with. Members in each group solve one problem together (different problem for each group).

- Think of different ways to approach the problem and discuss how the clues given help you find a solution.

When both groups have come to a conclusion, they meet and one member from each group pairs with a member from the other group. The members of each pair now solve each other's problem. During this process, one takes on the role of a teacher and the other is the learner.

The teacher gives the learner a copy of the problem she has solved with her group. When the learner has solved the problem they reverse roles. The teacher is now in the role of a learner and the learner in the role of a teacher.

Guidelines for the teacher:

1. Give the learner a copy of the problem you just solved with your group
2. Ask the learner to read the problem out loud.
3. Then ask the learner the following questions:
 - a. Do you understand the problem?
 - b. Do you think the problem is hard or easy?
 - c. What strategies do you think you will use?
 - d. How do you think you will do?
4. After the learner replies to each question, she begins solving the problem by thinking out loud for the teacher.
5. The teacher may only ask questions to clarify her understanding of the process the learner is using. She should not describe, lead or tell the learner how to solve the problem.
6. When the learner has solved the problem or feels she is not able to find a solution the teacher asks the original questions
 - a. Did you understand the problem?
 - b. Do you still think the problem is hard/easy?

- c. Did your original strategies work well for you?
 - d. How do you think you did?
7. Reverse roles and problems.
8. When the pairs are done, the whole group will reconvene and discuss both problems as well as the roles of the teacher and learner.

Problem 1:

Part A. Find all possible pairs of two digit numbers whose product is 360.

Part B. Find all possible pairs of two digit numbers whose product is 265.

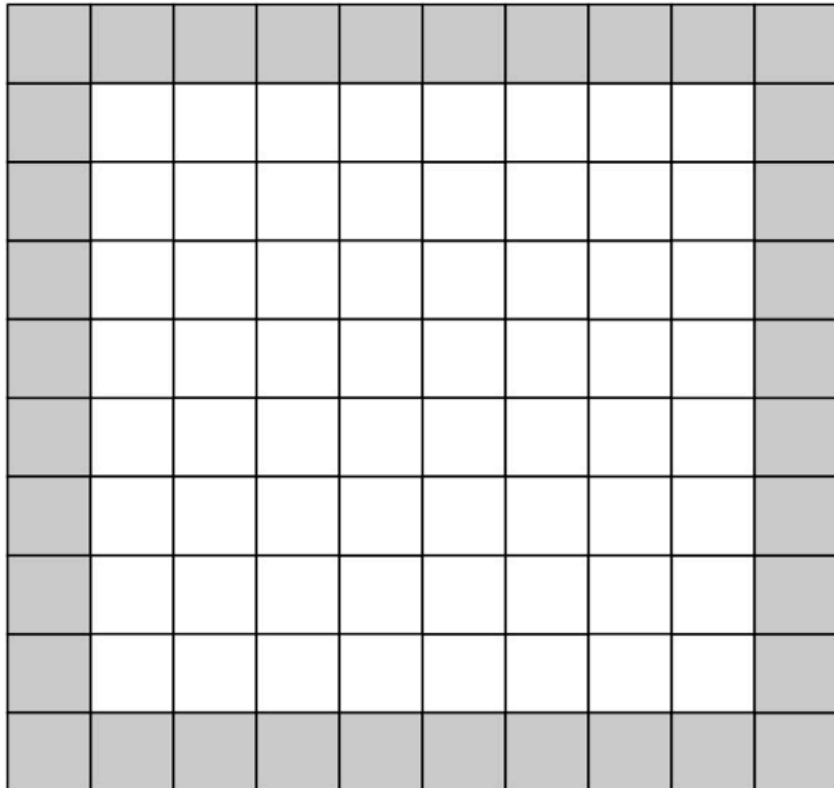
Problem 2:

If a collection of counters is placed in rows of 4, there are 2 counters left; if placed in rows of 5, there are 3 left; and if placed in rows of 7, there are 5 left. What is the smallest possible number of counters in the collection?

Adopted from Hart and Schultz, 2004

The Border Problem

- Calculate the number of coloured squares in the border, without counting one by one.



- How many coloured squares would there be in the border of a 6 by 6 grid?
- How many coloured squares would there be in the border of a 9 by 9 grid?
Or 8 by 8?
- Can you put forth a general rule that can be applied in finding the number of coloured squares in the border of a grid of any size?

Adopted from Boaler and Humphreys, 2005

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Appendix F: A list of papers and presentations of results from the research

Articles published in conference proceedings and journals

- Kristinsdóttir, J.V. (2009). Elementary school teachers' professional identity and competence as mathematics teachers. In C. Winsløw (Ed.), *Nordic Research in Mathematics Education*. Proceedings from NORMA08 (pp.199–204). Rotterdam: Sense Publishers.
- Kristinsdóttir, J.V. (2010a). Teacher development in the light of design-based research. Í C. Winsløw, & R. Evans (Eds.) *Didactics as design science* (pp. 105–115). København: Universitet, København.
- Kristinsdóttir, J.V. (2010b). Rethinking mathematics teaching in a community of inquiry. In L. B. Erickson, J. R. Young, & S. Pinnegar (Eds.), *Navigating the Public and Private: Negotiating the Diverse Landscape of Teacher Education*. Proceedings from the 8th International Conference on Self-Study of Teacher Education Practices (pp. 125–128). Brigham Young University: Provo UT.
- Kristinsdóttir, J.V. (2010c). Teacher development in a community of inquiry: primary school teachers rethink their mathematics teaching in diverse classrooms. In H. Guðjónsdóttir, J.V. Kristinsdóttir, & E. Óskarsdóttir (Eds.), *Challenges in teaching mathematics – Becoming special for all*. Proceedings of the 5th Nordic Research Conference on Special Needs Education in Mathematics (pp. 119–134). Reykjavík: Háskóli Íslands, Menntavísindasvið.
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Paper presentations at conferences

- Kristinsdóttir, J. V. (2008, March). *Teachers co-learning in mathematics teaching*. Paper presented at The 36th Conference of Nordic Educational Research Association (NERA), Copenhagen.

- Kristinsdóttir, J. V. (2008, April, 2008). *Elementary school teachers' professional identity and competence as mathematics teachers*. Paper presented at The fifth Nordic Conference on Research in Mathematics Education. NORMA08, Copenhagen.
- Kristinsdóttir, J. V. (2009, February). *Samvinnurannsókn kennara og rannsakanda um stærðfræðikennslu*. [Research with teachers into their mathematics teaching]. Paper presented at Ráðstefna Rannsóknarstofu í stærðfræðimenntun [Conference The Research Centre for Mathematics Education at University of Iceland], Reykjavík.
- Kristinsdóttir, J. V. (2009, October). *Samvinnurannsókn um stærðfræðikennslu í skóla fyrir alla*. [Collaborative research into mathematics teaching in a school for all]. Paper presented at Málþing Menntavísindasviðs [Annual Conference of The School of Education, University of Iceland], Reykjavík.
- Kristinsdóttir, J. V. (2010, August). *Rethinking mathematics teaching in a community of inquiry*. Paper presented at The 8th International Conference on Self-Study of Teacher Education Practices, East Sussex, England.
- Kristinsdóttir, J. V. (2011, May). *Teacher development in a community of inquiry*. Paper presented at The Sixth Nordic Conference on Research in Mathematics Education, NORMA11, Reykjavík.
- Kristinsdóttir, J. V. (2012, March). *Teacher professional development through reflective practice*. Paper presented at The 40th Conference of Nordic Educational Research Association (NERA), Copenhagen.
- Kristinsdóttir, J. V. (2012, March). *Að rannsaka með kennurum*. [Researching with teachers]. Paper presented at The málþing FUM – Félags um menntarannsóknir [Confrence of IERA – Icelandic Educational Research Association], Reykjavík.
- Kristinsdóttir, J. V. (2012, October). *Samvinnurannsókn grunnskólakennara og háskólakennara*. [Collaborative research of teachers in primary grades and teacher education]. Paper presented at Menntakvika [Annual Conference of The School of Education, University of Iceland], Reykjavík.
- Kristinsdóttir, J. V. (2013, March). *Primary school teachers develop their mathematics teaching*. Paper presented at the BSRLM – The British Society for Research into Learning Mathematics Conference, University of Bristol.
- Kristinsdóttir, J.V. (2013, October). *“Ég hef tekið eftir hæfileika nemenda til að reikna í huganum sem ég held að ég hafi aldrei rekist á fyrr”*. [“I have noticed a certain talent of one pupil, it is mental calculation. I think I have never encountered that before”]. Paper presented at Menntakvika [Annual Conference of The School of Education, University of Iceland], Reykjavík.
- Kristinsdóttir, J. V. (2014, October). *Að læra saman um stærðfræðinám- og kennslu*. [Learning together about mathematics learning and teaching]. Paper presented at Menntakvika [Annual Conference of The School of Education, University of Iceland], Reykjavík.
- Kristinsdóttir, J. V. (2014, June). *Supportive Mathematics Learning Community*. Paper presented at The Seventh Nordic Conference on Mathematics Education, NORMA14, University of Turku.
- Kristinsdóttir, J. V. (2015, November). *Collaborative research into mathematics teaching and learning in diverse classrooms*. Paper presented at The eighth Conference of

The Nordic Research Network on Special Needs Education in Mathematics (NORSMA 8), Kristianstad.

Kristinsdóttir, J. V. (2015, October). *Fagmennska stærðfræðikennara*. [Mathematics teacher professionalism]. Paper presented at Menntakvika [Annual Conference of The School of Education, University of Iceland], Reykjavík.

Kristinsdóttir, J. V. (2016, May). *Researching with teachers*. Paper presented at The málþing FUM – Félags um menntarannsóknir [Conference of IERA – Icelandic Educational Research Association], Reykjavík.

Kristinsdóttir, J. V. (2016, July). *Co-learning partnership in mathematics teacher in-service education*. Paper presented at The 13th International Congress on Mathematical Education, ICME13, Hamburg.

Poster presentatios at conferences

Kristinsdóttir, J. V. (2009, October). *Teacher development in a co-learning community: Primary school teachers research their mathematics teaching*. Poster presented presented at The fifth Conference of The Nordic Research Network on Special Needs Education in Mathematics, NORSMA5, Universty of Iceland, Reykjavik

Kristinsdóttir, J. V. (2012, April). *Building a community of practice by reflective discussions on mathematics teaching and learning*. Poster presented presented at the AERA – American Educational Research Association Annual conference, Vancouver.

Kristinsdóttir, J. V. (2014, April). *Collaborative inquiry into mathematics teaching. How Self- study made a difference*. Poster presented at the structured poster session at the AERA – American Educational Research Association Annual conference, Philadelphia.