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Enumerating score sequences and permutations by inversions and forbidden patterns

Atli Fannar Franklín

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FACULTY OF PHYSICAL SCIENCES

Enumerating score sequences and permutations by inversions and forbidden patterns

Atli Fannar Franklín

Dissertation submitted in partial fulfillment of a
Philosophiae Doctor degree in Mathematics

Ph.D. Committee
Anders Karl Claesson
Henning Ulfarsson
Sergey Kitaev
Sigurður Örn Stefánsson

Opponents
Luca Ferrari
Robert Brignall

Faculty of Physical Sciences
School of Engineering and Natural Sciences
University of Iceland
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Faculty of Physical Sciences
School of Engineering and Natural Sciences
University of Iceland
Dunhagi 5
107 Reykjavík
Iceland

Telephone: +354 525-4000

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Abstract

This thesis studies the enumeration of score sequences and permutations. The first paper settles a conjecture of Hanna on a recursion for the number of score sequences of a tournament, derives a closed formula, and gives a quadratic-time algorithm. The second presents generating functions for permutations with few inversions: those with as many inversions as elements, and those with a fixed number of inversions fewer than elements. The third continues on the theme of inversions, enumerating pattern-avoiding permutations by inversions for all patterns of length at most 3. The fourth and last paper explores how to obtain bounds on the number of 1324-avoiding permutations by encoding permutations as walks in a directed graph.

Ágrip

Þessi doktorsritgerð rýnir í talningu stigaruna og umraðana. Fyrsta rannsóknarritgerðin í henni sannar tilgátu Hanna um rakningu fjölda stigaruna, leiðir út lokaða formúlu og gefur kvaðratískt reiknirit fyrir útreikning þeirra. Næsta ritgerð setur fram framleiðandi föll fyrir umraðanir með fáar umhverfingar, fyrst fyrir umraðanir með jafn mörg stök og umhverfingar og svo fyrir þær með fastan fjölda umhverfinga færri en stök. Sú þriðja heldur sig við umhverfingar og telur allar mynstursforðandi umraðanir útfrá umhverfingum fyrir mynstur af lengd mest 3. Fjórða og síðasta rannsóknarritgerðin kannar hvernig megi leiða út efra mark á fjölda 1324-forðandi umraðanir með því að túlka umraðanir sem vegi í stefndu neti.

In memory of
Hjalti Þór Ísleifsson

Contents

List of papers

The thesis is composed of the following papers:

Paper I: Counting tournament score sequences.
A. Claesson, M. Dukes, A. F. Franklín, S. Ö. Stefánsson.
Proceedings of the American Mathematical Society, Vol. 151 (2023).

Paper II: Permutations with few inversions.
A. Claesson, A. F. Franklín, E. Steingrímsson.
The Electronic Journal of Combinatorics, Vol. 30, Issue 4 (2023).

Paper III: Pattern avoiding permutations enumerated by inversions.
A. F. Franklín.
Discrete Mathematics and Theoretical Computer Science, Vol. 27, Issue 1, Permutation Patterns 2024 #5 (2025).

Paper IV: Pattern avoiding permutations as walks.
A. F. Franklín.
Submitted to a journal.
Preprint in arXiv with identifier 2512.19462.

A paper with the title "The difficulty of beating the Taxman" was also published in Discrete Applied Mathematics, Volume 339 on the 15th of November 2023, written by A. F. Franklín and R. K. Moniot. It is not included in the thesis, but was written during the PhD.

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Introduction

This thesis lies comfortably within the realm of combinatorics, more specifically enumerative combinatorics. The underlying problem of enumerative combinatorics, as the name suggests, is enumerating discrete structures. Usually this means we are given some collection of finite sets and want to evaluate the size of each one. An example of this would be enumerating permutations. Permutations are ways to shuffle a finite set of objects. So as an example for a given deck with n cards, the permutations of that deck would be all the ways to shuffle that deck of cards. The enumeration of this would then involve finding out in how many ways you can shuffle the deck.

In fact, we will not go beyond the bounds of this one example for the most part within this thesis, as the topic will revolve entirely around permutations in all but the first paper. We will analyse, enumerate and study ways to shuffle a finite number of things. This may seem somewhat limited, but just this one type of object, permutations, has a rich theory with plenty of depth to explore.

Permutations connect to mathematical fields far and wide. Some of the most well-known objects of enumerative combinatorics arise from permutations, such as Stirling numbers of the first kind and Eulerian numbers [23]. Through the theory of generating functions, one finds connections to analysis, particularly complex analysis [11, 26]. Via the Robinson-Schensted correspondence, one gets connections to symmetric function theory and representation theory [22, 24]. Yet further, Cayley's theorem gives us a connection to group theory [13]. Permutations also arise in other disciplines; in computer science, for instance, they play a central role in the analysis of sorting algorithms [17]. The list goes on and on.

But one might wonder how shuffling a few things can connect to so many things; it might even sound a bit too simple. Let us consider some examples. First, how many ways are there to shuffle n different cards? One of those cards has to go on top of the deck, so we have n options for which card comes first. When choosing which card comes second, we only have $n-1$ options, and for the third $n-2$ options and so on. So our total number of options turns out to be $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$, which we usually denote $n!$. But the more complex examples start to arise when we limit which shuffles we allow, as we will see later.

Two common ways to write down permutations are the one-line notation and the plot diagram, though there are many others. Suppose we have a deck of cards where we mark the cards $1, 2, 3, 4, 5, 6$ and shuffle them such that afterwards the order is $4, 2, 5, 1, 6, 3$. Then we could simply write the permutation in one-line notation as 425163. The corresponding plot diagram for this permutation can also be seen in Figure 1.

But there are many more ways to view a permutation, and this variety of interpretation is a large part of the reason permutations are so widely applicable.

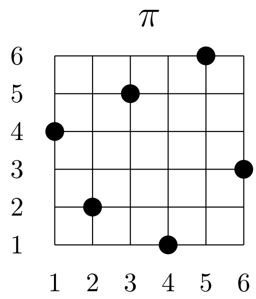

Figure 1: Plot diagram for $\pi = 425163$.

We can note that with $\pi = 425163$ the fourth card ends up first, and the first card ends up fourth. They swap, while 2 does not move at all. This kind of thinking leads us to view the permutation as a function rather than just the end result. This interpretation can be captured by the two-line notation, which for our example is

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 2 & 5 & 1 & 6 & 3 \end{pmatrix}$$

This can be read as 1 mapping to 4, 2 mapping to 2 and so on. But one can take this idea further and not just view it as a function, but see what cycles the function forms. We saw that 1 and 4 swap with each other while 2 does not move. But the last three elements (meaning cards) also form a cycle. 3 goes to 5, 5 to 6 and 6 back to 3. We could visualise this using a cycle diagram as seen in Figure 2.

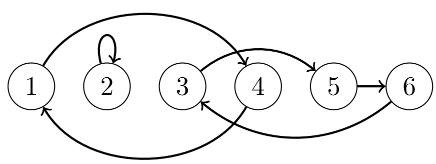

Figure 2: Cycle diagram for $\pi = 425163$.

This leads naturally to the cycle notation, letting us write π as (14)(2)(356). This is a valid way to write our permutation, but in some cases it can be helpful to write it such that the representation is unique. This is called canonical cycle notation. We write the biggest element first within each cycle, so we rewrite this as (41)(2)(635). Furthermore, we make sure to write the cycles in ascending order by their first element, so the canonical cycle notation would be (2)(41)(635).

Moving from objects to tools, a bijection f is a procedure for mapping a set of objects A to another set B, in such a way that they pair up one to one without collisions or any leftover elements, as shown in Figure 3.

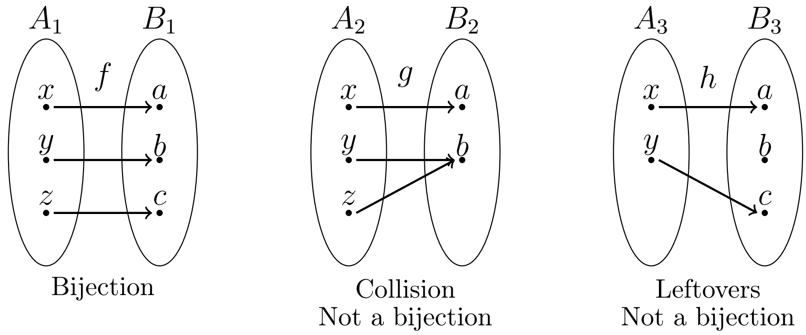

Figure 3: A bijection f, and two non-bijections g and h, between A_i and B_i.

Bijections are a useful tool for enumeration, because if you have some set of objects A and some other set of objects B, and there is a bijection f between them, there must be equally many objects in both sets. While this idea might be somewhat simple, the bijections are sometimes anything but simple.

A famous example of a bijection in the realm of permutations is Foata's fundamental bijection [12]. This bijection takes a permutation π to a new, possibly different, permutation π'. What we do is simply write π in canonical cycle notation, and then read it off as if it were a one-line notation. In our earlier example, we had $\pi = 425163$ and rewrote it as $(2)(41)(635)$. Then we just drop the parentheses to obtain $\pi' = 241635$. This is clearly well defined for any π, and it is not too hard to show that no two different π can lead to the same output. Furthermore, every possible permutation can be obtained as an output, so this is a bijection. We could even think of this bijection as a way to shuffle all the shufflings of a set of objects.

Just to give some more examples, there are many much simpler bijections on permutations. Taking the one-line notation and reversing it gives us a bijection, naturally called the reverse. Taking the plot diagram and flipping it vertically gives us a new permutation, called the complement permutation. We can also take the plot diagram and reflect it around the diagonal $x = y$, giving us the inverse permutation; see Figure 4. We can even combine some of these to first take the reverse, then the complement, which is predictably called the reverse complement. Bijections will be a fundamental and central tool in each of the four papers of this thesis.

Statistics

A prominent theme of the second and third paper of this thesis is permutation statistics. A statistic measures some property of the permutation which one

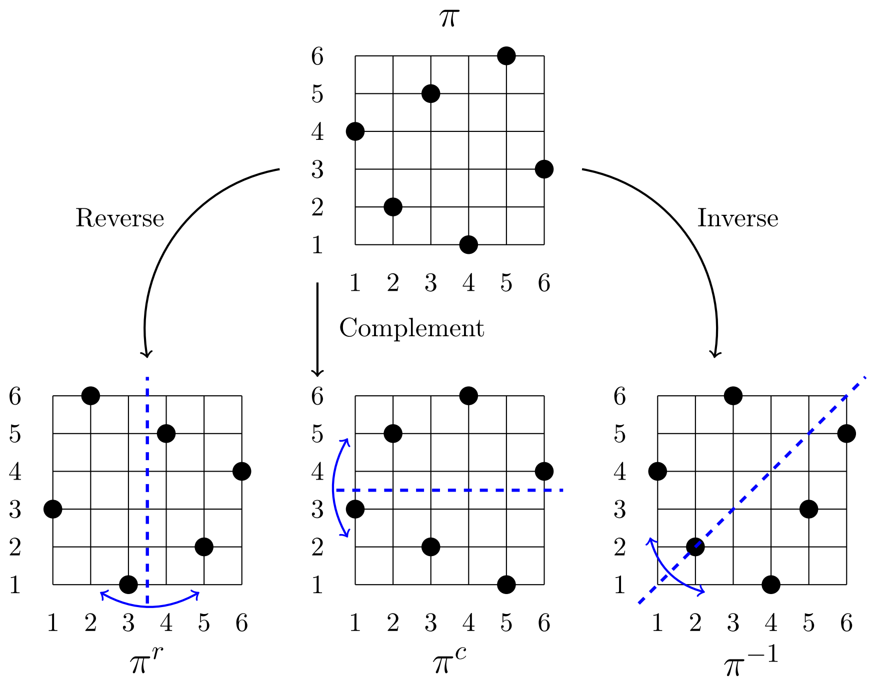

Figure 4: Some bijections applied to $\pi = 425163$.

might be interested in. For example, one could consider how many fixed points a permutation has. Calling it a statistic comes from these values often being considered on random permutations, so one could wonder for example if we shuffle a deck of cards, how many cards on average have not moved after the shuffle? The answer to this question turns out to be exactly 1, no matter the number of cards in the deck.

Other more common statistics of permutations that will be considered in the thesis include the left-to-right maxima or the number of cycles they have. The left-to-right maxima of a permutation are simply the numbers in its one-line notation that are the largest values encountered thus far when read left to right. So, for example, in our earlier permutation $\pi = 425163$, we have the left-to-right maxima $4, 5$ and 6 at positions $1, 3$ and 5. One can then often relate these statistics to one another via bijections or with other methods. We can for example see that Foata's fundamental bijection will map any permutation of n elements with k cycles to a permutation of n elements with k left-to-right maxima.

But why is this? When we apply Foata's fundamental bijection to a permutation π to get π', we start by collecting all the cycles in π, and denote the number of cycles we collect by k. Each cycle is written with its largest element first, so only the first element of each cycle has a chance to become a left-to-right maximum in π'. But then we order the cycles by their first element, so the first element of a cycle is always larger than the elements from all the cycles to the left of it in π'. But this means each first element of each cycle in π must become a left-to-right maximum in π'. Therefore we see that if the input has k cycles, the output must have k left-to-right maxima.

This means the number of permutations of n elements with k cycles must be the same as the number of permutations of n elements with k left-to-right maxima. In fact, both are equal to $\left[{n \atop k}\right]$, an unsigned Stirling number of the first kind. This means that the left-to-right maxima and the cycle count are equidistributed. This does not mean any particular permutation π must have the same number of cycles as it does left-to-right maxima; for example 623451 has 1 left-to-right maximum but 5 cycles. It simply means that across all permutations of a set of objects, the totals come out the same.

As another example, we can consider the inversion count of a permutation. This counts the number of pairs of elements in a permutation that are not in increasing order. If we once more look at our example permutation $\pi = 425163$, we can see that the element 4 comes before $1, 2$ and 3, which accounts for 3 such pairs. $(2, 1), (5, 1), (5, 3)$, and $(6, 3)$ are also not in increasing order, which gives us a total of 7 pairs, so the inversion count of π is 7. This is sometimes used as a measure of how ordered a sequence is, as for an ordered sequence the inversion count is 0 and for n elements in decreasing order it is $\binom{n}{2} = n(n-1)/2$. Thus sometimes the inversion count divided by $n(n-1)/2$ is used to measure sortedness, in which case it is known as the Kendall tau metric [15].

As with the left-to-right maxima and cycle count being equidistributed, there are statistics that are equidistributed with the inversion count as well. The most famous example of this is the major index, named after Major Percy Alexander MacMahon who first showed this equidistribution [18]. It is simply defined as the sum of the indices where the next element is smaller. In $\pi = \underline{4}2\underline{5}1\underline{6}3$, the corresponding indices have been underlined, which makes its major index $1+3+5 = 9$, which we see is different from its inversion count of 7. Statistics that are equidistributed with the inversion count statistic are known as Mahonian statistics.

Permutation patterns

When enumerating permutations, usually one ends up enumerating some subset of the whole, as we already know how many there are in total. We could for example imagine a rather bad shuffling technique. We take a sorted deck of cards, split it into two, not necessarily equally large, piles of cards, then interleave them in some way to produce a somewhat shuffled deck of cards. For example, we could take the sorted deck $1, 2, 3, 4, 5, 6, 7, 8$ and split it into two piles $2, 6, 8$ and $1, 3, 4, 5, 7$, then each pile is internally in ascending order and not necessarily contiguous. Then we interleave them, obtaining, for example $2, 6, 1, 3, 4, 5, 8, 7$. In how many ways could we perform such a shuffle? This kind of question will turn out to belong to the realm of permutation patterns.

Pattern containment defines a partial order on all permutations. We say that a permutation τ occurs in π if some subset of π has the same internal order as τ. Consider the bad shuffle described above on 8 cards, so our example becomes the permutation $\pi = 26134587$. Then consider a different permutation $\tau = 132$. We would say that τ occurs in π. This is because if we take the elements $1, 8, 7$ from π, they occur in an order such that the smallest element is first, largest second and middle third, the same as in 132. But $\tau = 321$ does not occur in π as there are no three elements in π such that the largest is first, middle second and smallest last. In fact, being a shuffle of this kind is equivalent to avoiding 321. If π does not have an occurrence of τ, we will say that π avoids τ. The normal riffle shuffle, where one cuts the deck into two contiguous piles and then interleaves them, can also be described by pattern avoiding permutations. These shuffles are exactly the ones avoiding the patterns 321, 2143 and 2413 [3].

The study of permutation patterns can be traced back to a few origins, but the usual prototypical example that most would point to is Knuth's study of stack-sortable permutations [16]. A case could be made for MacMahon being the first, having studied the permutations given by the shuffle example above, though in a different context [19, Sect. III, Ch. V]. Returning to Knuth's example, he investigated what sequences could be sorted using only a stack data structure. A way to visualise this is in terms of trains. We have a set of incoming double-headed trains, meaning they can drive both forward and backward, and need to reorder them in some way. But we only have a single T-junction to do so, as seen in Figure 5. The figure is inspired by a similar figure by Donald Knuth [16], which is in turn based on a suggestion by E. W. Dijkstra to interpret the problem in terms of railways.

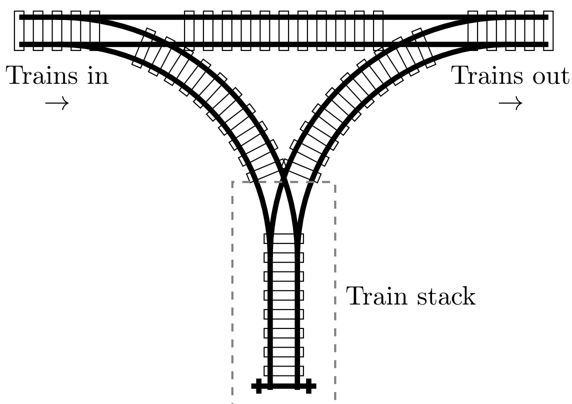

Figure 5: Diagram of stack behaviour.

The train stack area is arbitrarily large, so it can fit as many trains as we want. The diagram is simply drawn compactly. Suppose we get four trains coming in, which we number $4, 2, 1, 3$ because we want the first one to exit last, second one to exit second, third one to exit first and the last to exit third. This can be done by putting 4 on the stack, putting 2 on the stack, sending 1 through, sending 2 out from the stack, sending 3 through and finally sending 4 out from the stack.

But not all permutations can be sorted this way. Take 231 for example. The train numbered 2 has to go on the stack as it cannot exit right away. It would end up ahead of 1. But then once we look at the 3, we have a similar situation and must put it on the stack. But then 2 is stuck behind 3 on the stack, as only the train that most recently entered the stack can exit, so the trains will never be output in the right order. In fact, this same argument applies to any permutation such that the pattern 231 appears in it. That is to say, if π is such that there are three indices $i < j < k$ such that $\pi_k < \pi_i < \pi_j$, it cannot be sorted with a stack. For example, 13542 cannot be sorted with a stack as the values $3, 5, 2$ have the same relative order as 231. One can also show that if a permutation does avoid 231, a greedy approach to sorting it with a stack will work.

We denote the set of permutations on n elements which avoid the pattern τ by $\mathrm{Av}_n(\tau)$. Thus, the set of stack-sortable permutations on n elements is exactly $\mathrm{Av}_n(231)$. Similarly, the bad shuffle permutations on n cards will be $\mathrm{Av}_n(321)$. Seeing as there are 6 ways to shuffle 3 things, there should be 6 patterns of length 3 one could investigate. But using the bijections on permutations we saw earlier, many of these will turn out to be equinumerous. For example, reversing elements in $\mathrm{Av}_n(213)$ gets us elements in $\mathrm{Av}_n(312)$, and reversing again takes us back. So the number of elements in both sets is the same. If we then take an element of $\mathrm{Av}_n(312)$ and take the complement, we get elements in $\mathrm{Av}_n(132)$. Reverse again and we are in $\mathrm{Av}_n(231)$. This all goes to show that $132, 231, 213$ and 312 belong to the same Wilf-class, meaning they have equally many permutations avoiding them.

In the same manner we see that 123 and 321 are in the same Wilf-class. But are the bad shuffle permutations in a different Wilf-class than the stack-sortable ones? It turns out that they are in fact in the same Wilf-class. The bijections showing this are a bit more technical and will not be covered here; there are quite a few to choose from [9]. The first of these bijections was presented by Knuth [16].

As a note, what we refer to here as permutation patterns are often called classical permutation patterns. There exist many more types of permutation patterns not covered in this thesis. For example we have, vincular patterns, introduced by Eric Babson and Einar Steingrímsson to study Mahonian statistics [4], and mesh patterns, introduced by Petter Brändén and Anders Claesson to write permutation statistics as linear combinations of patterns [7].

Generating functions

But how many stack-sortable permutations are there? Or equivalently, how many permutations are there of length n that avoid any particular pattern τ of length three? The first few values, for $n = 0, 1, 2, 3, \ldots$ are $1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots$.

But to answer this, we must first consider what a satisfactory answer to this question even is. The best would be some kind of explicit closed formula, but this is often a bit much to hope for. A somewhat weaker answer might be to be satisfied with any procedure for efficiently computing arbitrarily many terms of the sequence, but this often leads to mathematically unhelpful formulations. What turns out to be a happy middle ground is an object known as a generating function, and most combinatorialists will be happy to call a sequence enumerated if a generating function is found. Even if a closed formula exists, the easiest path to that formula might be through the generating function.

> *A generating function is a device somewhat similar to a bag. Instead of carrying many little objects detachedly, which could be embarrassing, we put them all in a bag, and then we have only one object to carry, the bag.* - George Pólya [21]

But what is a generating function? It is simply a power series with our sequence as coefficients. So, for example, the famous Fibonacci numbers start with $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$, which means the corresponding generating function must start with

$$F(x) = 0 + 1x + 1x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + 21x^8 + 34x^9 + 55x^{10} + \ldots$$

But why would this F be more convenient than the original sequence if it consists of infinitely many terms as well? Well, it turns out that in many cases these generating functions can be expressed in finite terms. Denote the n-th Fibonacci number by f_n. It can be shown that

$$F(x) = \sum_{n=0}^{\infty} f_n x^n = \frac{x}{1 - x - x^2}$$

which is very handy. The original sequence can then be recovered by looking at the coefficients of the Taylor series of the generating function at $x = 0$. A diagram of the relation between a sequence and its generating function can be seen in Figure 6.

Generating functions let us bring in many powerful tools from analysis, both real and complex, to help us. How fast the terms of the sequence grow is often much more easily analysed using complex analysis and the generating function than by staring at the numbers from the sequence itself.

Generating functions also lend themselves to many operations that have useful combinatorial interpretations [5, 11]. For example, in the world of generating

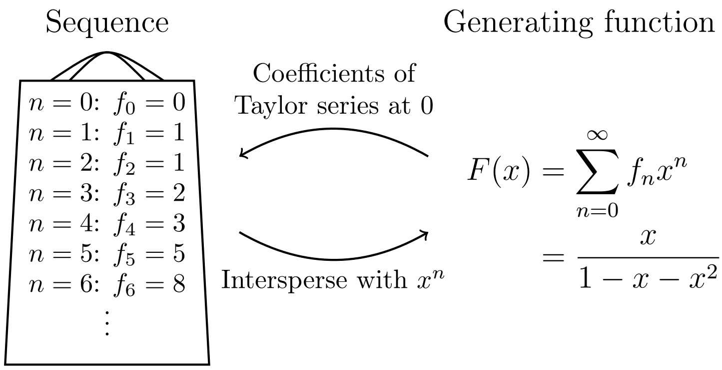

Figure 6: Relation between a sequence and its generating function.

functions, the structure of a singleton is described by the generating function x and the structure of the empty set by the generating function 1. Furthermore, addition becomes the equivalent of splitting into cases, and multiplication becomes the equivalent of building substructures.

Let us use the language of generating functions to enumerate the stack-sortable permutations, so we go back to our original sequence of numbers $1, 1, 2, 5, 14, \ldots$. We have to avoid the pattern 231, so we consider the maximum element m of some permutation π in the case that π is not empty. Let x be any element coming before m and y any element after m. If $x > y$, then x, m, y is an occurrence of 231, so we must have $x < y$ for any such x and y. But this means all the elements before m are smaller than all the elements after m, so we can factor our permutation π into two smaller 231-avoiding permutations π_L, π_R along with the singleton maximal element m; see Figure 7.

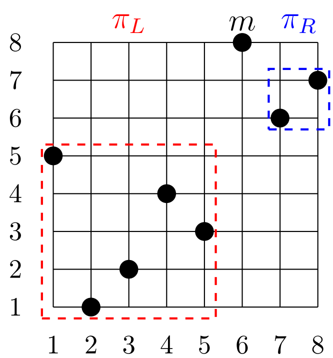

Figure 7: Factorisation of $\pi = 51243867$.

If C is the generating function for 231-avoiding permutations, this means that

$$C(x) = xC(x)^2 + 1$$

which we can simply solve to get

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x} = 1 + 1x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + \dots \quad (1)$$

Using a generalisation of the binomial theorem and omitting some calculation steps, we can even use this to derive that

$$|\mathrm{Av}_n(231)| = \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n+1} \cdot \frac{(2n)!}{n! \cdot n!}$$

This sequence is in fact a well-known sequence of numbers called the Catalan numbers. Often these generating functions have surprisingly simple forms even if the elements themselves satisfy no simple closed formula. An example of this is the zigzag numbers or Euler numbers, which count shuffles in which the first value is bigger than the second, the second smaller than the third, the third bigger than the fourth and so on. So, if the shuffled values are $a_1, a_2, a_3, \dots$ we have $a_1 > a_2 < a_3 > a_4 < a_5 > \cdots$. The number of such permutations on n values has no simple closed formula, but the (exponential) generating function E is described simply with

$$E(x) = \sec(x) + \tan(x)$$

where sec is the secant function and tan the tangent function [1]. It being an exponential generating function simply means that instead of getting out our desired sequence $1, 1, 1, 2, 5, 16, 61, 272, \dots$ we instead get 1/0!, 1/1!, 1/2!, 2/3!, 5/4!, 16/5!, 61/6!, 272/7!, ... In fact, there are many more types of generating functions. The ones we have looked at so far are usually called ordinary generating functions. There are ordinary, exponential, Dirichlet, Lambert, and many more types of generating functions.

As a last example, let us consider a case where more than one variable might be warranted in our generating function. If we want to study the distribution of a statistic on permutations, it might be useful to have two variables x and q, so that the coefficient of $x^n q^i$ is the number of permutations of n elements with some particular statistic equal to i. We consider as an example the inversion count statistic.

Let us consider constructing a permutation in its one-line notation by placing down the elements one at a time onto a line, going from 1 to n. As we place down the 1, no inversions are made. As we place down the 2, we put it in front of or after the 1, creating 1 or 0 inversions. As we place the 3, we have similar choices, getting 0, 1, or 2 inversions. And so on, getting $0, 1, \dots$ or $n-1$ inversions as we place the last element. These choices are all independent. This means the generating function we seek can be expressed as

$$I(x,q) = \sum_{n\geq 0} \left(1 \cdot (1+q) \cdot (1+q+q^2) \cdot \ldots \cdot (1+q+q^2+\cdots+q^{n-1})\right) x^n$$

This touches on a theory known as q-analogues. The product in the sum above is known as a q-factorial.

Stanley-Wilf limit

A common use case of generating functions is to determine asymptotics of sequences related to enumeration, that is to say how fast they grow as a function of the index n. Using standard methods from analysis, one can for example determine that the n-th coefficient c_n of the generating function C calculated earlier satisfies

$$c_n \sim \frac{4^n}{\sqrt{\pi n^3}}$$

This means the ratio between the left- and right-hand side approaches 1 as n approaches ∞. Compared to the total number of permutations, $n!$, this is very small. In fact, Stanley and Wilf conjectured that the number of pattern-avoiding permutations of length n for a given pattern τ always grows like α^n for some real number α. This means for any τ, almost all permutations have an occurrence of τ in some sense. More formally, the Stanley-Wilf limit of τ is defined as

$$L(\tau) = \lim_{n\to\infty} \sqrt[n]{|\mathrm{Av}_n(\tau)|}$$

The fact that $L(\tau)$ is always well-defined was then later proven by Arratia, Marcus and Tardos [2, 20]. We see then that the Stanley-Wilf limit is exactly 4 for any pattern of length 3. For length 4, all such limits have been found [8, 14] except one: up to symmetry, 1324 is the only length 4 pattern whose Stanley-Wilf limit remains unknown. The other limits for length 4 patterns are all 8 or 9. The best known bounds for 1324 are $10.271 \leq L(1324) \leq 13.5$, due to Bevan, Brignall, Price, and Pantone [6]. Numerical evidence suggests a value near 11.6 is likely [10].

Paper overview

The first paper in this thesis concerns score sequences of tournaments, and is the only paper not about permutations. A tournament is a directed complete graph, but this can be interpreted as n teams which play a round-robin tournament. So each pair of different teams play exactly one game and exactly one team wins each game. The score sequence is just the number of wins each team has at the end of the tournament, written in nondecreasing order. The question of enumerating score sequences then concerns finding the number of different score sequences one can end up with for a tournament of n teams. The problem of counting score sequences is more than 100 years old, considered by MacMahon in 1920. In 2013, Hanna [25] conjectured a surprising and elegant recursive formula for these numbers. In the paper we prove this conjecture of Hanna as a corollary of the main theorem of the paper. This main theorem factors the generating function of score sequences where one team has been marked. This main theorem also gives us a closed formula and a quadratic-time algorithm for counting score sequences.

The second paper, as the title suggests, is about permutations with few inversions. As a permutation of n elements may have up to $n(n-1)/2$ inversions, that paper investigates permutations on n elements with $\leq n$ inversions. A generating function S_0 is found that enumerates permutations on n elements with exactly n inversions. Then $(xC(x))^i S_0(x)$ is shown to be the generating function for permutations on n elements with exactly $n-i$ inversions, where C is the Catalan generating function shown in Equation 1.

The third paper is the first in the thesis to dive into permutation patterns. While $\mathrm{Av}_n(\tau)$ enumerates the permutations on n elements avoiding the pattern τ, the third paper looks at $I_k(\tau)$ which enumerates permutations with exactly k inversions avoiding the pattern τ. To make these sets finite, the I_k must be limited to indecomposable permutations, using a notion of factorisation not too dissimilar to what was shown before for 231-avoiding permutations in Figure 7. In the style of the seminal paper by Simion and Schmidt, all combinations of permutation patterns of length at most 3 are considered.

The fourth and last paper considers the open problem of determining the Stanley-Wilf limit of the pattern 1324. A possible avenue towards obtaining a lower bound is explored, involving encoding the permutations as ways to move around in a large graph. Each move in the graph corresponds to inserting a new maximum element into a permutation. This way computational methods can be used to determine the growth rate of the number of walks in the graph, which can be used to obtain lower bounds on the number of 1324-avoiding permutations. Since the graphs are much too large for direct computation, ways to compress the graph are considered, leading to an open conjecture which would yield a lower bound of 10.418.

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Paper I

Counting tournament score sequences

Anders Claesson Mark Dukes Atli Fannar Franklín
Sigurður Örn Stefánsson

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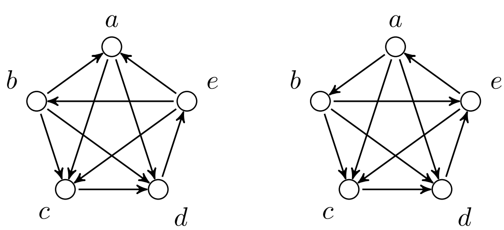
Abstract

The score sequence of a tournament is the sequence of the out-degrees of its vertices arranged in nondecreasing order. The problem of counting score sequences of a tournament with n vertices is more than 100 years old (MacMahon 1920). In 2013 Hanna conjectured a surprising and elegant recursion for these numbers. We settle this conjecture in the affirmative by showing that it is a corollary to our main theorem, which is a factorization of the generating function for score sequences with a distinguished index. We also derive a closed formula and a quadratic time algorithm for counting score sequences.

1 Introduction

In 1953 Landau [9] used oriented complete graphs—also called *tournaments*—to model pecking orders. If the vertices of the complete graph represent players (rather than chickens), then the initial vertex of a directed edge signifies the winner of a game between the two end-point players. The number of wins of a player is equal to the number of outgoing edges from that vertex. A *score sequence* is a sequence of these number of wins given in a nondecreasing order. For instance, with 3 players there are two possible score sequences, namely $(0, 1, 2)$ and $(1, 1, 1)$. Note that non-isomorphic tournaments may give rise to the same score sequence. With 5 players there are, up to isomorphism, 12 tournaments but only 9 score sequences. To be even more specific, here are two non-isomorphic[1] tournaments:



[1]The longest directed path between any pair of vertices in the left-hand graph is 3, while in the right-hand graph there is a directed path of length 4 from c to b.

The score sequence associated with both is $(1, 1, 2, 3, 3)$. The following characterization of score sequences is known as Landau's theorem.

Theorem 1 (Landau [9]). *A sequence of integers $s = (s_0, \dots, s_{n-1})$ is a score sequence if and only if*

(1) $0 \leq s_0 \leq s_1 \leq \cdots \leq s_{n-1} \leq n - 1$,

(2) $s_0 + \cdots + s_{k-1} \geq \binom{k}{2}$ *for* $1 \leq k < n$, *and*

(3) $s_0 + \cdots + s_{n-1} = \binom{n}{2}$.

Let S_n be the set of score sequences of length n. There is no known closed formula for the associated cardinalities (A000571 in the OEIS [7])

$$\big(|S_n|\big)_{n\geq 0} = (1, 1, 1, 2, 4, 9, 22, 59, 167, 490, 1486, 4639, 14805, \dots)$$

or their generating function.

It should be noted that Landau was not the first person to study score sequences, or attempt to count them. MacMahon [10] used symmetric functions and hand calculations to determine $|S_n|$ for $n \leq 9$ in 1920. Building on Landau's work, Narayana and Bent [11], in 1964, derived a multivariate recursive formula for determining $|S_n|$. They used it to give a table for $n \leq 36$. In 1968 Riordan [12] gave a simpler and more efficient recursion, but unfortunately it turned out to be incorrect [13].

Let $[a, b]$ denote the interval of integers $\{a, a + 1, \dots, b\}$. We may view a score sequence $s \in S_n$ as an endofunction $s : [0, n - 1] \to [0, n - 1]$. We now introduce the notion of a *pointed score sequence*. Define $S_n^\bullet$ as the Cartesian product $S_n^\bullet = S_n \times [0, n - 1]$. We call the members of $S_n^\bullet$ *pointed score sequences*; e.g. there are 6 pointed score sequences in $S_3^\bullet$:

$$((0, 1, 2), 0),\ ((0, 1, 2), 1),\ ((0, 1, 2), 2),$$
$$((1, 1, 1), 0),\ ((1, 1, 1), 1),\ ((1, 1, 1), 2).$$

Let $(s, i) \in S_n^\bullet$. Depending on the context, the element i will be interpreted as a position (element in the domain) or a value (element in the codomain) of s. If i is a value, then the cardinality of the fiber $s^{-1}(i)$ is the number of times i occurs in s; this number may be zero. Let

$$S_n^\bullet(t) = \sum_{(s,i)\in S_n^\bullet} t^{|s^{-1}(i)|}$$

be the polynomial recording the distribution of the statistic $(s, i) \mapsto |s^{-1}(i)|$ on $S_n^\bullet$. As an example, $S_3^\bullet(t) = 2 + 3t + t^3$. Let

$$S^\bullet(x, t) = \sum_{n\geq 1} S_n^\bullet(t) x^n.$$

To present the bijection that is the main result of this paper, we will first introduce a particular type of multiset that is an essential ingredient in our deconstruction of a pointed score sequence. At first glance it is not obvious what the relevance of these multisets to score sequences is.

We define EGZ_n as the set of multisets of size n with elements in the cyclic group $\mathbb{Z}_n$ whose sum is $\binom{n}{2}$ modulo n. To understand what the elements of EGZ_n look like it may be helpful to note that $\binom{n}{2}$, as an element of $\mathbb{Z}_n$, is 0 if n is odd and $n/2$ if n is even. For instance, EGZ_3 consists of the 4 multisets

$$\{0,0,0\},\ \{0,1,2\},\ \{1,1,1\},\ \text{and}\ \{2,2,2\}.$$

The notation EGZ_n refers to the Erdős-Ginzburg-Ziv Theorem [4], which is stated below. Following it we give a proposition motivating this terminology; its proof gives a simple one-to-one correspondence between EGZ_n and the sets considered by Erdős, Ginzburg, and Ziv.

Theorem 2 (Erdős, Ginzburg, and Ziv [4])**.** *Each set of $2n-1$ integers contains some subset of n elements the sum of which is a multiple of n.*

Proposition 3. *There is a one-to-one correspondence between EGZ_n and n-element subsets of $[1, 2n-1]$ whose sum is a multiple of n.*

Proof. Let $A = \{a_1, \ldots, a_n\}$ be a subset of $[1, 2n-1]$ such that $a_1 + \cdots + a_n$ is divisible by n. Without loss of generality we can further assume that $a_1 < a_2 < \cdots < a_n$. Let $b_i = a_i - i$. We claim that $A \mapsto \{b_1, \ldots, b_n\}$ is a bijection onto EGZ_n. Clearly, $i \leq a_i \leq n+i-1$ and hence $0 \leq b_i \leq n-1$. In this manner we can consider each b_i as an element of $\mathbb{Z}_n$. The sum $a_1 + \cdots + a_n$ is divisible by n, by assumption, and hence

$$b_1 + \cdots + b_n \equiv a_1 + \cdots + a_n - 1 - 2 - \cdots - n \equiv -\binom{n+1}{2} \pmod{n}.$$

That this is congruent to $\binom{n}{2}$ modulo n follows from $\binom{n}{2} + \binom{n+1}{2} = n^2$. Conversely, given a multiset $B = \{b_1, \ldots, b_n\}$, with $b_1 \leq \cdots \leq b_n$, in EGZ_n we define the set $A = \{a_1, \ldots, a_n\}$ by $a_i = b_i + i$. One can verify that A is a subset of $[1, 2n-1]$ whose sum is divisible by n, and that $A \mapsto B$ is the inverse of the map above, but we omit the details. □

The sequence of cardinalities

$$\big(|\mathrm{EGZ}_n|\big)_{n\geq 1} = (1, 1, 4, 9, 26, 76, 246, 809, 2704, 9226, 32066, \ldots)$$

is entry A145855 in the OEIS [7]. As recorded in that OEIS entry, Jovović conjectured and Alekseyev [1] proved in 2008 that

$$|\mathrm{EGZ}_n| = \frac{1}{2n} \sum_{d|n} (-1)^{n-d} \varphi(n/d) \binom{2d}{d}, \tag{4}$$

where the sum runs over all positive divisors of n and φ is Euler's totient function. A generalization of this result was given by Chern [3] in 2019.

The zeros in a multiset $M \in \mathrm{EGZ}_n$ play a prominent role in our construction. We now introduce a generating function to record their number. For a multiset $M \in \mathrm{EGZ}_n$ let $|M|_i$ be the number of occurrences of i in M. Furthermore, let

$$\mathrm{EGZ}_n(t) = \sum_{M \in \mathrm{EGZ}_n} t^{|M|_0}$$

be the polynomial recording the distribution of zeros in multisets belonging to EGZ_n. For instance, $\mathrm{EGZ}_3(t) = 2 + t + t^3$ (looking at the distribution of 1s or 2s in EGZ_3 would result in the same polynomial). Define the generating functions

$$\mathrm{EGZ}(x,t) = \sum_{n \geq 1} \mathrm{EGZ}_n(t) x^n \quad \text{and} \quad S(x) = \sum_{n \geq 0} |S_n| x^n.$$

Our main result (Theorem 4) is a factorization of the generating function for pointed score sequences:

$$S^{\bullet}(x,t) = \mathrm{EGZ}(x,t) S(x). \tag{5}$$

Let $(s,i) \in S_n^{\bullet}$. Viewing i as an element of the codomain of s we find that $S^{\bullet}(x,0)$ consists of terms stemming from pairs (s,i) such that $s^{-1}(i)$ is empty; i.e. i is outside the image of s. Thus, $S^{\bullet}(x,1) - S^{\bullet}(x,0)$ counts pairs (s,i) for which i is in the image of s. Let

$$\begin{aligned} S_n^{\circ} &= \{(s,i) \in S_n^{\bullet} : i \in \mathrm{Im}(s)\} \\ &= \{(s,i) \in S_n^{\bullet} : i = s_j \text{ for some } j \in [n]\} \end{aligned}$$

and let $S^{\circ}(x) = S^{\bullet}(x,1) - S^{\bullet}(x,0)$ be the corresponding generating function. For instance, S_3° consists of the 4 elements $((0,1,2),0)$, $((0,1,2),1)$, $((0,1,2),2)$, and $((1,1,1),1)$. We will show (in Corollary 11) that

$$S^{\circ}(x) = xC(x)S(x),$$

where $C(x) = (1 - \sqrt{1-4x})/(2x)$ is the generating function for the Catalan numbers $C_n = \binom{2n}{n}/(1+n)$. This striking occurrence of the Catalan numbers was in fact the original inspiration for our work. It was in the summer of 2019 that we experimented with score sequences and conjectured the identity. Despite ample attempts we were for the longest time unable to prove it.

By setting $t = 1$ in Equation 5 and noting that $S^{\bullet}(x,1) = xS'(x)$ it follows that

$$xS'(x) = \mathrm{EGZ}(x,1) S(x), \tag{6}$$

a fact conjectured by Paul D. Hanna as recorded in the OEIS entry A000571 in 2013. Equation 6 may alternatively be written $\big(\log S(x)\big)' = \mathrm{EGZ}(x,1)/x$ and so

$$S(x) = \exp\left(\sum_{n \geq 1} \frac{|\mathrm{EGZ}_n|}{n} x^n \right),$$

which arguably is the most elegant way of expressing the relation between $|S_n|$ and $|\mathrm{EGZ}_n|$. The most efficient way of computing the numbers $|S_n|$ is, however, to use the recursion underlying Equation 6. Namely, $|S_0| = 1$ and, for $n \geq 1$,

$$|S_n| = \frac{1}{n} \sum_{k=1}^{n} |S_{n-k}||\mathrm{EGZ}_k|.$$

See Corollary 13 and the discussion following it.

2 The main theorem and its bijection

Let the generating functions $S^{\bullet}(x,t)$, $\mathrm{EGZ}(x,t)$ and $S(x)$ be defined as in Section 1.

Theorem 4. *We have*

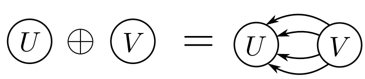
$$S^{\bullet}(x,t) = \mathrm{EGZ}(x,t)S(x).$$

We shall give a combinatorial proof of Theorem 4 using a bijection

$$\Phi : S_n^{\bullet} \to \bigcup_{k=1}^{n} \mathrm{EGZ}_k \times S_{n-k}$$

that maps a pointed score sequence to a pair consisting of a multiset and a score sequence. A property of this bijection is that, for $(M, v) = \Phi(s, i)$, the number of occurrences of i in s is equal to the multiplicity of zero in M. Before defining Φ we need to introduce several necessary concepts.

A nonempty directed graph is said to be *strongly connected* if there is a directed path between each pair of vertices of the graph. Note that we do not consider the empty graph to be strongly connected. A *strong score sequence* is one which stems from a strongly connected tournament. Equivalently (see Harary and Moser [6, Theorem 9]), $s = (s_0, \ldots, s_{n-1})$, with $n \geq 1$, is a strong score sequence if the inequality (2) of Theorem 1 is always strict; that is, $s_0 + \cdots + s_{k-1} > \binom{k}{2}$ for $1 \leq k < n$. Let us define the *direct sum* of two score sequences $u \in S_k$ and $v \in S_\ell$ by $u \oplus v = uv'$, where v' is obtained from v by adding k to each of its letters and juxtaposition indicates concatenation. For instance, $(0) \oplus (0) \oplus (1,1,1) = (0,1,3,3,3)$. If U and V are tournaments having score sequences u and v, one may view the direct sum $u \oplus v$ as the score sequence of the tournament where arrows are placed between the vertices of U and V such that they all point towards U:



This may easily be seen to be independent of the choice of tournaments.

Lemma 5. *Let $s \in S_n$. If $s_0 + \cdots + s_{k-1} = \binom{k}{2}$ for some $k < n$, then $u = (s_0, \ldots, s_{k-1})$ and $v = (s_k - k, \ldots, s_{n-1} - k)$ are both score sequences, and $s = u \oplus v$.*

Proof. That u is a score sequence is clear from Landau's theorem. By definition we have $s = u \oplus v$, so it only remains to show that v is a score sequence and we will use Landau's theorem to do so, proving each of the three parts separately:

(1) Since $s_0 + \cdots + s_{k-1} = \binom{k}{2}$ and $s_0 + \cdots + s_k \geq \binom{k+1}{2}$ we have $s_k \geq \binom{k+1}{2} - \binom{k}{2} = k$ and hence $v_0 = s_k - k \geq 0$. Since s is weakly increasing it is clear that v is weakly increasing as well. Moreover, the length of v is $n - k$ and $v_{n-k-1} = s_{n-1} - k \leq n - 1 - k$.

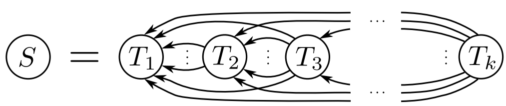
(2) For $1 \leq j < n - k$ we have

$$\begin{aligned} v_0 + v_1 + \cdots + v_{j-1} &= s_k + s_{k+1} + \cdots s_{k+j-1} - jk \\ &= s_0 + \cdots + s_{k+j-1} - (s_0 + \cdots + s_{k-1}) - jk \\ &\geq \binom{k+j}{2} - \binom{k}{2} - jk = \binom{j}{2}. \end{aligned}$$

(3) Similarly, $v_0 + v_1 + \cdots + v_{n-k-1} = \binom{n}{2} - \binom{k}{2} - (n-k)k = \binom{n-k}{2}$.

This concludes the proof. □

A direct consequence of Lemma 5, above, is that every score sequence s can be uniquely written as a direct sum $s = t_1 \oplus t_2 \oplus \cdots \oplus t_k$ of nonempty strong score sequences; in this context, the t_i will be called the *strong summands* of s. In terms of underlying tournaments we have the picture:



We are now almost in a position to define the promised map Φ, but first a couple of definitions. Assume that we are given a score sequence $s = (s_0, s_1, \ldots, s_{n-1}) \in S_n$.

- For any integer j, let $s + j$ denote the sequence obtained by adding j to each element of s, reducing modulo n, and sorting the outcome in nondecreasing order. Note that $s + j$ need not be a score sequence even though s is. E.g. $s = (1, 1, 1)$ is a score sequence, but $s + 1 = (2, 2, 2)$ is not. On the other hand, if $s = (0, 1, 2)$ then $s + 1 = s$ is a score sequence. A characterization of when $s + j$ is a score sequence will be given in Lemma 7.

- Let $\mu(s + j)$ denote the multiset $\{s_0 + j, s_1 + j, \ldots, s_{n-1} + j\}$ with elements in the cyclic group $\mathbb{Z}_n$.

Given a pointed score sequence $(s, i) \in S_n^\bullet$, write $s = t_1 \oplus t_2 \oplus \cdots \oplus t_k$ and let j be the smallest index such that $i < |t_1 \oplus \cdots \oplus t_j|$. Another way to define j is as the smallest prefix $t_1 \oplus \cdots \oplus t_j$ of strong summands of s that begins $s_0, s_1, \ldots, s_i$. Define the two score sequences u and v by

$$u = t_1 \oplus \cdots \oplus t_j \quad \text{and} \quad v = t_{j+1} \oplus \cdots \oplus t_k.$$

Finally, we let

$$\Phi(s, i) = \big(\mu(u - i), v\big).$$

As an example, consider the score sequence $s = (0, 2, 2, 3, 3, 5, 7, 7, 7)$; its decomposition into strong summands is $s = (0) \oplus (1, 1, 2, 2) \oplus (0) \oplus (1, 1, 1)$. With $i = 3$ we get $u = (0) \oplus (1, 1, 2, 2) = (0, 2, 2, 3, 3)$, $v = (0) \oplus (1, 1, 1) = (0, 2, 2, 2)$, $u - 3 = (0, 0, 2, 4, 4)$ and so $\Phi(s, 3) = \big(\{0, 0, 2, 4, 4\}, (0, 2, 2, 2)\big)$.

3 Proof of the main result

Our aim is to prove Theorem 4, but first we need to establish a number of lemmas. Let T_n be the set of strong score sequences of length n.

Lemma 6. *For any strong score sequence $s \in T_n$, the n multisets*

$$\mu(s + j) = \{s_0 + j, s_1 + j, \ldots, s_{n-1} + j\}, \quad \textit{for } j \in [0, n-1],$$

are all distinct.

Proof. Assume $j \in [0, n-1]$ is such that $\mu(s+j)$ and $\mu(s)$ are equal as multisets over $\mathbb{Z}_n$. Note that there is no loss of generality here: assuming that $\mu(s + j_1) = \mu(s + j_2)$ is equivalent to assuming that $\mu(s + j) = \mu(s)$ with $j = j_2 - j_1$. Let us write the values $s_0 + j, \ldots, s_{n-1} + j$ in nondecreasing order after reducing modulo n. Since $s_0 \leq \cdots \leq s_{n-1}$ the result must be a cyclic shift of the original order, say, $s_k + j, \ldots, s_{n-1} + j, s_0 + j, \ldots, s_{k-1} + j$ for some index k. As each s_i is less than n and $j < n$ we find that those elements must equal

$$s_k + j - n,\ \ldots,\ s_{n-1} + j - n,\ s_0 + j,\ \ldots,\ s_{k-1} + j. \tag{7}$$

Since we are assuming that $\mu(s + j)$ and $\mu(s)$ are equal as multisets over $\mathbb{Z}_n$, the sum of the elements listed in (7) must equal the sum of the elements $s_0 + s_1 + \cdots + s_{n-1} = \binom{n}{2}$. This implies $nj - (n-k)n = 0$, which gives $j = n - k$. Thus, the sequence (7) becomes

$$s_k - k,\ \ldots,\ s_{n-1} - k,\ s_0 + n - k,\ \ldots,\ s_{k-1} + n - k.$$

By assumption we have $s_0 = s_k - k$, $s_1 = s_{k+1} - k$, and so on; thus

$$\begin{aligned} s_0 + \cdots + s_{n-k-1} &= s_k - k + \cdots + s_{n-1} - k \\ &= \binom{n}{2} - (s_0 + \cdots + s_{k-1}) - (n-k)k \\ &= \binom{k}{2} + \binom{n-k}{2} - (s_0 + \cdots + s_{k-1}). \end{aligned}$$

Suppose $k \geq 1$. Since our score sequence s is strong we have $s_0 + \cdots + s_{k-1} > \binom{k}{2}$, but then $s_0 + \cdots + s_{n-k-1} < \binom{n-k}{2}$ contradicting that s is a score sequence. The only remaining possibility is that $k = 0$ and $j = 0$, which concludes the proof. □

Consider the following example concerning the strong summands of a score sequence $s + j$. Let $s = (1, 1, 2, 2, 4, 6, 7, 7, 7, 8)$ and $j = 5$. We add 5 to each element of s to get the list of numbers 6, 6, 7, 7, 9, 11, 12, 12, 12, 13, which when reduced modulo $|s| = 10$ reads 6, 6, 7, 7, 9, 1, 2, 2, 2, 3; finally we sort this list to arrive at $s + 5 = (1, 2, 2, 2, 3, 6, 6, 7, 7, 9)$. Note that s and $s + 5$ share the same strong summands only arranged differently:

$$s = (1, 1, 2, 2) \oplus (0) \oplus (1, 2, 2, 2, 3);$$
$$s + 5 = (1, 2, 2, 2, 3) \oplus (1, 1, 2, 2) \oplus (0).$$

As is detailed in the following lemma, this is not a coincidence. See also Fig. 1 where we show how this may be interpreted in terms of tournaments.

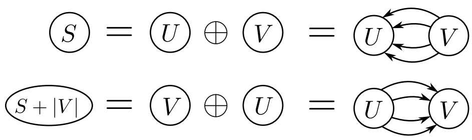


Figure 1: If $s = u \oplus v$, one may view $s + |v|$ as the score sequence of a tournament obtained by flipping the direction of all arrows pointing out of V towards U. Here, S, U, V are tournaments with score sequences s, u, v respectively.

Lemma 7. *Let s be any nonempty score sequence of length n and let $j \in [1, n-1]$. Then $s + j$ is a score sequence if and only if there are score sequences u and v with $|v| = j$, such that $s = u \oplus v$. In that case, $s + j = v \oplus u$.*

Proof. Let $s = (s_0, s_1, \ldots, s_{n-1})$. Define the sequences

$$u = (s_0, s_1, \ldots, s_{n-j-1});$$
$$v = (s_{n-j} + j - n, s_{n-j+1} + j - n, \ldots, s_{n-1} + j - n)$$

of length $n - j$ and j, respectively. Assume that u and v are score sequences. Clearly, $s = u \oplus v$. Since the length of v is j we find that

$$v \oplus u = (s_{n-j} + j - n, \ldots, s_{n-1} + j - n, s_0 + j, s_1 + j, \ldots, s_{n-j-1} + j).$$

If we consider the elements of this sequence modulo n we may add n to the first j of them. If we then sort the elements in nondecreasing order we obtain the sequence $(s_0 + j, \ldots, s_{n-1} + j)$. Thus, $s + j = v \oplus u$, which is a score sequence. For the other direction, assume that $s + j$ is a score sequence. Then by definition of $s + j$ there is an $\ell \in [0, n - 1]$ such that $s + j$ is

$$(s_\ell + j - n, s_{\ell+1} + j - n, \ldots, s_{n-1} + j - n, s_0 + j, s_1 + j, \ldots, s_{\ell-1} + j).$$

Since $s + j$ is a score sequence of length n, we find by item (3) in Landau's theorem that

$$\begin{aligned}\binom{n}{2} &= \sum_{i=\ell}^{n-1}(s_i + j - n) + \sum_{i=0}^{\ell-1}(s_i + j) \\ &= \sum_{i=0}^{n-1} s_i + (n-\ell)(j-n) + \ell j = \binom{n}{2} + n(\ell - n + j).\end{aligned}$$

Thus, $\ell = n - j$. Now consider the sum of the first j terms in $s + j$; let us call this sum K. By Landau's theorem we have

$$\begin{aligned}\binom{j}{2} \le K &= \sum_{i=n-j}^{n-1} (s_i + j - n) \\ &= \sum_{i=n-j}^{n-1} s_i + j(j-n) \\ &= \binom{n}{2} - \sum_{i=0}^{n-j-1} s_i + j(j-n) \\ &\le \binom{n}{2} - \binom{n-j}{2} + j(j-n) = \binom{j}{2}.\end{aligned}$$

In particular, the two inequalities above are in fact equalities and $K = \binom{j}{2}$. From Lemma 5 we deduce that u and v as previously defined are score sequences and that $s + j = v \oplus u$. As before, it is clear that $s = u \oplus v$. □

The previous lemma may be equivalently stated as follows:

Lemma 8. *Let $s = t_1 \oplus t_2 \oplus \cdots \oplus t_k$ be any nonempty score sequence decomposed into its strong summands, and let $j \in [1, n-1]$. Then $s + j$ is a score sequence if and only if there is an ℓ such that*

$$j = |t_\ell| + |t_{\ell+1}| + \cdots + |t_k|$$

in which case $s + j = t_\ell \oplus t_{\ell+1} \oplus \cdots \oplus t_k \oplus t_1 \oplus \cdots \oplus t_{\ell-1}$.

Our next lemma will be essential for proving that the mapping f defined in Lemma 10, below, is surjective.

Lemma 9. *For all multisets $M \in \mathrm{EGZ}_n$ there is a score sequence $s \in S_n$ and a constant $j \in [0, n-1]$ such that $\mu(s + j) = M$.*

Proof. Let $M \in \mathrm{EGZ}_n$. By definition of EGZ_n, the sum of the members of M is $\binom{n}{2}$ modulo n. We start by showing that for some integer j the members of the multiset $M + j = \{x + j : x \in M\}$ will have sum exactly $\binom{n}{2}$ (without reducing the sum modulo n). Suppose that the sum of the members of M is greater than $\binom{n}{2}$. We will show that we can always choose a j such that $M + j$ has a smaller sum. For $k \in [0, n-1]$, let y_k be the number of elements $x \in M$ such that $x \leq k$. Suppose that $y_k > k$ for all k. Then $y_0 \geq 1$ and there is at least one zero in M. Similarly, $y_1 \geq 2$ so in addition to that zero there is a value equal to at most one. Continuing like this we bound our values from above by $0, 1, \ldots, n-1$. The sum of these values is, however, at most $\binom{n}{2}$. Since we assumed that the sum is greater than $\binom{n}{2}$ we conclude that the assertion that $y_k > k$, for all k, is false. For the remainder of the argument, let k be the smallest index such that $y_k \leq k$.

If $k = 0$, then there are no zeroes in M, so subtracting one from all the elements in M simply causes the sum to drop by n since no modulo reductions occur. Thus, we can assume $k > 0$. Then $k - 1 < y_{k-1} \leq y_k \leq k$ and hence $y_k = k$. We now subtract $k + 1$ from all the elements in M. Exactly y_k of these values will be below zero. So we decrease the sum by $n(k + 1)$ but reducing modulo n adds kn to the sum. The combined effect is to decrease the sum by n. In this way we can always decrease the sum down to $\binom{n}{2}$. The proof that the sum be can be increased in the same way is nearly identical.

Let $M = \{x_1, \ldots, x_n\} \in \mathrm{EGZ}_n$ with $x_1 \leq \cdots \leq x_n$. In light of the last two paragraphs, we can assume that $x_1 + \cdots + x_n = \binom{n}{2}$. The sequence $(x_1, \ldots, x_n)$ satisfies items (1) and (3) of Landau's theorem. To prove that $(x_1, \ldots, x_n)$ is a score sequence it remains to prove item (2), namely that $x_1 + \cdots + x_k \geq \binom{k}{2}$ for $k \in [1, n-1]$. Suppose that $x_1 + \cdots + x_k < \binom{k}{2}$ for some k in $[1, n-1]$. For now, let k be the largest such k. Since k is maximal we have $x_1 + \cdots + x_{k+1} \geq \binom{k+1}{2}$, which gives us $x_{k+1} \geq \binom{k+1}{2} - x_1 - \cdots - x_k > \binom{k+1}{2} - \binom{k}{2} = k$. If $x_k \geq k$ then we can decrement k until this no longer holds true; note that $x_1 + \cdots + x_{k-1} < \binom{k-1}{2}$ will continue to hold while $x_k \geq k$ and $x_1 + \cdots + x_k < \binom{k}{2}$. We eventually arrive at a k such that $x_1 + \cdots + x_k < \binom{k}{2}$, $x_k < k$ and $x_{k+1} \geq k$. The values $x_1, \ldots, x_k$ are thus precisely the values among $x_1, \ldots, x_n$ that are smaller than k. Define y_i as before and let $w_i = y_i - i - 1$. The number of values equal to r among $x_1, \ldots, x_k$ is $y_r - y_{r-1}$ as long as $r < k$, and so

$$x_1 + \cdots + x_k = 0 \cdot y_0 + 1 \cdot (y_1 - y_0) + \cdots + (k-1)(y_{k-1} - y_{k-2}).$$

This implies $(k-1)y_{k-1} - y_{k-2} - \cdots - y_0 = x_1 + \cdots + x_k < \binom{k}{2}$, but we know that $y_{k-1} = k$, so we must have $k^2 - \binom{k}{2} < y_{k-1} + \cdots + y_0$. Rewriting in terms of w_k we get $\binom{k+1}{2} < w_0 + \cdots + w_{k-1} + 1 + 2 + \cdots + k$, which is equivalent to $w_0 + \cdots + w_{k-1} > 0$. Thus, if we can choose a shift (i.e. choose a constant to add to the elements of M) such that the prefix sums of the w_i are non-positive, then the original set must be a score sequence.

Consider the values k such that $w_k = 0$ or, equivalently, $y_k = k + 1$. If we subtract $k + 1$ from every value in our multiset, then exactly $k + 1$ of them will become negative. Thus, our sum decreases by $n(k+1)$ but the modulo reduction adds $n(k + 1)$ back to the sum, so it remains unchanged. Let us consider what this does to our sequences (y_i) and (w_i). Denote the new sequences after the subtraction by (y'_i) and (w'_i). Consider the value of y'_i. It counts the number of values from 0 to i after the subtraction, which are values from $k+1$ to $k+1+i$ modulo n before the subtraction. We consider two cases.

(a) If $k+1+i<n$, then $y'_i = y_{k+1+i} - y_k$. Since $y_k = k+1$ this gives us in terms of w_i that $w'_i + i + 1 = w_{k+i+1} + k + i + 1 + 1 - k - 1$, which simplifies to $w'_i = w_{k+i+1}$.

(b) If $k+1+i \geq n$, then $y'_i = y_{n-1} - y_k + y_{k+1+i-n}$. We already know that $y_k = k+1$ and $y_{n-1} = n$. In terms of the w_i we have $w'_i + i + 1 = n - k - 1 + w_{k+1+i-n} + k + 1 + i - n + 1$, which reduces to $w'_i = w_{k+1+i-n}$.

Hence, in both cases we have $w'_i = w_{k+i+1}$ when indices are considered modulo n. We can thus cyclically permute the sequence (w_i) as long as the final value continues to be 0 and still have it correspond to a shift of our original multiset M such that the shift has sum $\binom{n}{2}$.

By the same argument as above (when $k=n$) we have

$$x_1 + \cdots + x_n = (n-1)y_{n-1} - y_{n-1} - y_{n-2} - \cdots - y_0.$$

Furthermore, $y_{n-1} = n$ and $x_1 + \cdots + x_n = \binom{n}{2}$. Thus, $y_0 + \cdots + y_{n-1} = \binom{n}{2}$, which gives us $w_0 + \cdots + w_{n-1} = 0$. Hence, we can consider the sequence of sums of values between the zeroes that appear in (w_i). This new sequence (g_i) also has sum zero and we can cyclically permute the sequence (w_i) in the way we want if and only if we can permute the sequence (g_i) to satisfy the same desired property. This is because either all values between two zeroes in (w_i) are of the same sign, or the positive values all come after the negative ones. The reason for this is that one cannot go from positive values to negative without intercepting zero since we decrease by at most one at a time.

A sequence of integers with sum zero can be cyclically permuted to produce a sequence with non-positive prefix sums. To see this let $z_1, \ldots, z_n$ be integers that sum to zero. By applying Raney's lemma [5, §7.5, p. 345] to the sequence $-(z_1-1), -z_2, \ldots, -z_n$ there exists a cyclic permutation of this sequence having all positive prefix sums. Negate this new sequence and you have one with all negative prefix sums. Then add 1 to the element where $z_1 - 1$ ended up. This will increase any given prefix sum by at most 1, taking them from negative to at most non-positive, which proves the non-positive prefix sums claim. This allows us to conclude that the prefix sums, as discussed above, are non-positive and so the original set must be a score sequence, which completes our proof. □

Let $T(x)$ be the generating function for the number of strong score sequences according to length. Note that any nonempty score sequence s can be written $s = u \oplus t$, where u is a score sequence and t is the last strong summand of s, and thus $S(x) = 1 + S(x)T(x)$, or, equivalently, $S(x) = (1 - T(x))^{-1}$. Let

$$T_n^{\bullet} = T_n \times [0, n-1]$$

be the set of *pointed strong score sequences* of length n, and let

$$T_n^\bullet(t) = \sum_{(s,i)\in T_n^\bullet} t^{|s^{-1}(i)|} \quad \text{and} \quad T^\bullet(x,t) = \sum_{n\geq 1} T_n^\bullet(t)x^n.$$

Lemma 10. *Assume $n \geq 1$. Select those pointed score sequences (s,i) in $S_n^\bullet$ whose distinguished position is in the last strong summand, and denote the resulting set L_n. That is,*

$$L_n = \big\{(s,i) \in S_n^\bullet : s = t_1 \oplus \cdots \oplus t_k \text{ and } i \in [n - |t_k|, n-1]\big\}.$$

Then the mapping $f : L_n \to \mathrm{EGZ}_n$ defined by

$$f(s,i) = \mu(s-i)$$

is a bijection. Moreover, if $M = f(s,i)$, then $|M|_0 = |s^{-1}(i)|$. In terms of generating functions we have

$$\mathrm{EGZ}(x,t) = S(x)T^\bullet(x,t).$$

Proof. We start by proving that f is injective. To that end, assume that (u,i) and (v,j) in L_n are such that $f(u,i) = f(v,j)$; that is,

$$\mu(u-i) = \mu(v-j) = M, \quad \text{where } M \in \mathrm{EGZ}_n.$$

If $u = v$, then $i = j$ by Lemma 6, so we may assume that u and v are different score sequences. Without loss of generality we may further assume that $i \leq j$. Now, $\mu(u-i) = \mu(v-j)$ holds by assumption, and this multiset identity is true if and only if $\mu(u) = \mu(v + (j-i))$. Thus, $u = v + m$ with $m = j - i$. Assume that the decomposition of v into its strong summands is $v = t_1 \oplus \cdots \oplus t_k$. By Lemma 8, the score sequences u and v contain the same strong summands up to a cyclic permutation. To be more precise, there is an $\ell \geq 1$ such that

$$m = |t_\ell| + \cdots + |t_k| \quad \text{and} \quad u = t_{\ell+1} \oplus \cdots \oplus t_k \oplus t_1 \oplus \cdots \oplus t_\ell.$$

By definition of the set L_n we have $i \in [n - |t_\ell|, n-1]$ and $j \in [n - |t_k|, n-1]$. Therefore $m = j - i \leq n - 1 - (n - |t_\ell|) = |t_\ell| - 1$. This would however imply $|t_\ell| + \cdots + |t_k| < |t_\ell|$, which is impossible. Therefore no such i and j can exist and the mapping f is injective.

Next we prove surjectivity. We want to prove that every $M \in \mathrm{EGZ}_n$ is the image of some score sequence $s \in S_n$ with a distinguished element i in its last strong summand, i.e. $(s,i) \in L_n$. By Lemma 9 every $M \in \mathrm{EGZ}_n$ is the image of some such (s,j) where we place no restriction on j. If j is smaller than the size of the last strong summand of s, then f maps the pointed score sequence $(s, n-j)$ to our multiset M. If j is greater than the size of the last strong summand of s, then we can move the last summand to the front and decrease j by the corresponding size by Lemma 8. In this way we will eventually end up in the first case. Thus, we end up with a score sequence that maps to our multiset M.

We have shown that f is a bijection. Assume that $(s, i) \in L_n$ and $M = f(s, i)$. By definition of f it is clear that the number of zeros in M corresponds to the number of j for which $s_j = i$, and hence $|M|_0 = |s^{-1}(i)|$. Every $(s, i) \in L_n$ can be uniquely identified with a pair $(s', (t, j))$, where $s = s' \oplus t$, t is the last strong summand of s, and $j = i - |s'|$. In other words, s' is the score sequence s with its last strong summand t removed, and j is the distinguished position relative to t rather than s. Assuming $k = |s'|$, we have $s' \in S_k$ and $(t, j) \in T^{\bullet}_{n-k}$ with $|t^{-1}(j)| = |M|_0$. Consequently, $\mathrm{EGZ}(x, t) = S(x)T^{\bullet}(x, t)$ as claimed. □

We are now finally in a position to prove our main result.

Proof of Theorem 4. We will give a combinatorial proof of the power series identity $S^{\bullet}(x, t) = \mathrm{EGZ}(x, t)S(x)$ by showing that the mapping

$$\Phi : S^{\bullet}_n \to \bigcup_{k=1}^{n} \mathrm{EGZ}_k \times S_{n-k}$$

defined in Section is a bijection. Recall that $\Phi(s, i) = (\mu(u - i), v)$ is defined by writing s in terms of its strong summands, say $s = t_1 \oplus t_2 \oplus \cdots \oplus t_k$, and then splitting the score sequence $s = u \oplus v$, where $u = t_1 \oplus \cdots \oplus t_j$ and j is the smallest index such that $i < |t_1 \oplus \cdots \oplus t_j|$, and v is the remainder. Let L_n be the set defined in Lemma 10. Note that $(u, i) \in L_{|u|}$. In fact, it is easy to see that $(s, i) \mapsto \big((u, i), v\big)$ is a bijection from $S^{\bullet}_n$ to $\cup_{k=1}^{n} L_k \times S_{n-k}$. Moreover, by Lemma 10, $(u, i) \mapsto \mu(u - i)$ is a bijection from L_k to EGZ_k, and hence Φ is a bijection as well. Finally, $(u, i) \mapsto \mu(u - i)$ has the property that $|\mu(u - i)|_0 = |u^{-1}(i)|$ and so $S^{\bullet}(x, t) = \mathrm{EGZ}(x, t)S(x)$, as claimed. □

Corollary 11. *We have*

$$S^{\circ}(x) = xC(x)S(x),$$

where $C(x)$ is the generating function for the Catalan numbers.

Proof. From Theorem 4 we get

$$S^{\circ}(x) = S^{\bullet}(x, 1) - S^{\bullet}(x, 0) = \big(\mathrm{EGZ}(x, 1) - \mathrm{EGZ}(x, 0)\big)S(x).$$

It thus suffices to show that $\mathrm{EGZ}(x, 1) - \mathrm{EGZ}(x, 0) = xC(x)$. Note that the coefficient of x^n in $\mathrm{EGZ}(x, 1) - \mathrm{EGZ}(x, 0)$ is the number of multisets of EGZ_n that contain at least one zero. On removing a single zero from such a multiset we are left with a multiset of size $n - 1$ whose sum is $\binom{n}{2}$ modulo n. There are C_{n-1} such multisets [15, Exercise 6.19jjj]. □

Corollary 12. *We have*

$$S(x) \;=\; \exp\left(\sum_{n \geq 1} \frac{|\mathrm{EGZ}_n|}{n} x^n\right).$$

Proof. Since $S^\bullet(x,1) = xS'(x)$, by letting $t = 1$ in the main theorem we have $xS'(x) = \mathrm{EGZ}(x,1)S(x)$, or, equivalently, $x\bigl(\log S(x)\bigr)' = \mathrm{EGZ}(x,1)$. Thus

$$\log S(x) = \int_0^x \mathrm{EGZ}(u,1)\frac{1}{u}\,du = \sum_{n\geq 1} |\mathrm{EGZ}_n|\frac{x^n}{n},$$

as claimed. □

We end this section by comparing our result (Corollary 12) with earlier results on the enumeration of the *ordered* score sequences $(s_0, s_1, \ldots, s_{n-1})$, also called *score vectors*. That is, if G is a tournament on the vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$, then s_i is the out-degree of v_i in G. For instance, while there are only two score sequences of length 3, namely $(1,1,1)$ and $(1,2,3)$, there are 7 score vectors of length 3: the vector $(1,1,1)$ together with the 6 permutations of $(1,2,3)$.

Stanley and Zaslavsky [14] have shown that the number of score vectors of length n equals the number of (labeled) forests on n nodes. A combinatorial proof was subsequently given by Kleitman and Winston [8]. Cayley [2] famously gave the formula n^{n-2} for the number of trees on n nodes. From the theory of exponential generating functions it immediately follows that

$$\exp\left(\sum_{n\geq 1} n^{n-2}\frac{x^n}{n!}\right)$$

is the exponential generating function of forests, and thus also of score vectors.

4 The number of score sequences

If two power series $A(x) = 1 + \sum_{n\geq 1} a_n x^n$ and $B(x) = \sum_{n\geq 1} b_n x^n$ satisfy $xA'(x)/A(x) = B(x)$ and hence $\log A(x) = \sum_{n\geq 1} b_n x^n/n$, then one readily obtains a closed formula for a_n by expanding and identifying coefficients in $A(x) = \exp\bigl(b_1x^1/1\bigr)\exp\bigl(b_2x^2/2\bigr)\cdots$. Applying this to the equation in Corollary 12 we arrive at

$$|S_n| = \frac{1}{n!}\sum_{\pi\in\mathrm{Sym}(n)}\prod_{\ell\in C(\pi)} |\mathrm{EGZ}_\ell|, \tag{8}$$

where $\mathrm{Sym}(n)$ is the symmetric group of degree n and $C(\pi)$ encodes the cycle type of π; i.e. there is an $\ell \in C(\pi)$ for each ℓ-cycle of π. While having the virtue of being closed, this formula does not lend itself to quickly calculating $|S_n|$. For that purpose the following recursion is better suited.

Corollary 13. *For $n \geq 1$,*

$$|S_n| = \frac{1}{n}\sum_{k=1}^{n} |S_{n-k}||\mathrm{EGZ}_k|$$
$$= \frac{1}{n}\sum_{k=1}^{n} |S_{n-k}|\frac{1}{2k}\sum_{d|k}(-1)^{k-d}\varphi(k/d)\binom{2d}{d}.$$

Proof. Since $S_n^\bullet = S_n \times [0, n-1]$ there is a simple relation $|S_n^\bullet| = n|S_n|$ between the number of pointed score sequences and the number of (plain) score sequences. Thus the result immediately follows from identifying coefficients in the main theorem, with $t = 1$, and using Equation 4 in Section . ☐

This allows us to calculate all values of $|S_k|$ for $k \leq n$ in $\Theta(n^2)$ time, assuming constant time integer operations. This is an improvement on earlier results by Narayana and Bent [11]. Their recursive formula can be implemented to find $|S_n|$ in $\Theta(n^3)$ time, but no faster since their recursive function must always visit $\Theta(n^3)$ states to do so; to get all $|S_k|$ for $k \leq n$ takes $\Theta(n^4)$ time due to lack of overlap in the states recursively visited for different k.

Since $S(x) = (1 - T(x))^{-1}$ this recursive computation method can be extended to $|T_k|$. We can rewrite the equation as $1 + T(x) = S(x) - (S(x) - 1)T(x)$ which gives us the recursion

$$|T_n| = |S_n| - \sum_{i=1}^{n-1} |T_i||S_{n-i}|.$$

We first calculate the values $|S_k|$ and use this recursion to calculate all the values $|T_k|$ for $k \leq n$ in $\Theta(n^2)$ time. This is the same method as used by Stockmeyer [16], just calculating the underlying $|S_k|$ faster which brings the total time complexity down from $\Theta(n^4)$ to $\Theta(n^2)$.

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Paper II

Permutations with few inversions

Anders Claesson   Atli Fannar Franklín

Einar Steingrímsson

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Abstract

A curious generating function $S_0(x)$ for permutations of $[n]$ with exactly n inversions is presented. Moreover, $(xC(x))^i S_0(x)$ is shown to be the generating function for permutations of $[n]$ with exactly $n-i$ inversions, where $C(x)$ is the generating function for the Catalan numbers.

1 Introduction

The famous triangle of Mahonian numbers starts as follows:

1	0	0	0	0	0	0	0	0	0	⋯
1	0	0	0	0	0	0	0	0	0	⋯
1	1	0	0	0	0	0	0	0	0	⋯
1	2	2	1	0	0	0	0	0	0	⋯
1	3	5	6	5	3	1	0	0	0	⋯
1	4	9	15	20	22	20	15	9	4	⋯
1	5	14	29	49	71	90	101	101	90	⋯
1	6	20	49	98	169	259	359	455	531	⋯
1	7	27	76	174	343	602	961	1415	1940	⋯
1	8	35	111	285	628	1230	2191	3606	5545	⋯

Its n-th row records the distribution of inversions on permutations of $[n] = \{1, 2, \ldots, n\}$. The corresponding generating function is [6]

$$(1+x)(1+x+x^2)\cdots(1+x+\cdots+x^{n-1}) = \prod_{j=1}^{n} \frac{1-x^j}{1-x}. \tag{1}$$

We shall derive generating functions for the subdiagonals on or below the main diagonal of the table above. The first three of those are

$$\begin{aligned} S_0(x) &= 1 + x^3 + 5x^4 + 22x^5 + 90x^6 + 359x^7 + 1415x^8 + \cdots \\ S_1(x) &= x + x^2 + 2x^3 + 6x^4 + 20x^5 + 71x^6 + 259x^7 + 961x^8 + \cdots \\ S_2(x) &= x^2 + 2x^3 + 5x^4 + 15x^5 + 49x^6 + 169x^7 + 602x^8 + \cdots \end{aligned}$$

In general, if i is a non-negative integer, then $S_i(x)$ is the generating function for permutations of $[n]$ with exactly $n-i$ inversions. In other words, if we let $I_n(k)$ denote the number of permutations of $[n]$ with k inversions, then

$$S_i(x) = \sum_{n\geq 0} I_n(n-i)x^n.$$

It should be noted that there is a known closed expression for $I_n(k)$ when $k \leq n$, namely the Knuth-Netto formula [4, 7]:

$$I_n(k) = \binom{n+k-1}{k} + \sum_{j=1}^{\infty}(-1)^j\binom{n+k-u_j-j-1}{k-u_j-j} + \sum_{j=1}^{\infty}(-1)^j\binom{n+k-u_j-1}{k-u_j}$$

where $u_j = j(3j-1)/2$ is the j-th pentagonal number. This formula can be proved using (1) and Euler's pentagonal number theorem [1]. For instance, $u_1 = 1$, $u_2 = 5$, and the coefficient of x^6 in $S_0(x)$ is

$$I_6(6) = \binom{11}{6} - \binom{10-u_1}{5-u_1} - \binom{11-u_1}{6-u_1} + \binom{11-u_2}{6-u_2} = 90.$$

Let $C(x) = (1-\sqrt{1-4x})/(2x)$ be the generating function for the Catalan numbers, $C_n = \binom{2n}{n}/(n+1)$. We show (Theorem 3) that, for any non-negative integer i,

$$S_i(x) = \big(xC(x)\big)^i S_0(x),$$

thus reducing the problem of determining $S_i(x)$ to that of determining $S_0(x)$.

Denote by $\sigma(n)$ the sum of divisors of n, and denote by $p(n)$ the number of integer partitions of n. We show (Theorem 4) that

$$S_0(x) = R\big(xC(x)\big),$$

where the power series $R(x)$ can be expressed in any of the following three equivalent ways

$$\begin{aligned} R(x) &= \frac{1-x}{1-2x}\prod_{k\geq 1}(1-x^k); \\ \log R(x) &= \sum_{n\geq 1}(2^n - \sigma(n) - 1)\frac{x^n}{n}; \\ 1/R(x) &= 1 - \sum_{n\geq 1}\big(p(1)+p(2)+\cdots+p(n-1)-p(n)\big)x^n. \end{aligned}$$

See Equation (3), Proposition 12 and Proposition 13.

2 Factoring permutations with few inversions

Let $\mathcal{S}_n$ denote the set of permutations of $[n]$. The *inversion table* of $\pi = a_1a_2\cdots a_n$ in $\mathcal{S}_n$ is defined as $b_1b_2\cdots b_n$ where b_i is the number of elements to the left of and larger than a_i; in other words, b_i is the cardinality of the set $\{j \in [i-1] : a_j > a_i\}$. For instance, the inversion table of 3152746 is 0102021. The number of inversions in π, denoted $\mathrm{inv}(\pi)$, is simply the sum of the entries in the inversion table for π. We will work with an invertible transformation of the inversion table that we call the *cumulative inversion table.* It is obtained by taking partial sums of the inversion table: b_1, $b_1 + b_2$, $b_1 + b_2 + b_3$, etc. The cumulative inversion table of 3152746 is 0113356.

A *subdiagonal sequence* is a sequence of non-negative integers whose k-th entry is smaller than k. It is easy to see that the inversion table of a permutation is a subdiagonal sequence and that any such sequence is an inversion table, so the two concepts can be used interchangeably.

Lemma 1. *There are exactly $C_n = \binom{2n}{n}/(n+1)$ weakly increasing subdiagonal sequences of length n.*

Proof. Let a weakly increasing subdiagonal sequences $b_1b_2\cdots b_n$ be given, and form the sequence $a_1a_2\cdots a_n$ by setting $a_i = b_i + 1$. Then $a_i \leq i$ and $1 \leq a_1 \leq a_2 \leq \cdots \leq a_n$. By Exercise 6.19(s) in [9] there are exactly C_n such sequences. □

Let $\mathcal{S}_n^k = \{\pi \in \mathcal{S}_n : \mathrm{inv}(\pi) = k\}$ be the set of permutations of $[n]$ with k inversions, and let $\mathcal{C}_n$ be the subset of $\mathcal{S}_n^{n-1}$ consisting of those permutations whose every prefix of length $k \geq 1$ has fewer than k inversions. For $n = 0, 1, 2, 3, 4$ those are $\emptyset$, $\{1\}$, $\{21\}$, $\{231, 312\}$, and $\{1432, 2341, 2413, 3142, 4123\}$.

Lemma 2. *For $n \geq 1$ we have $|\mathcal{C}_n| = C_{n-1}$.*

Proof. Clearly, the cumulative inversion table $\gamma = c_1c_2\cdots c_n$ of any permutation $\pi \in \mathcal{S}_n$ is weakly increasing. Also, the last letter, c_n, of γ is the number of inversions in π. In particular, if $\pi \in \mathcal{C}_n$ then $c_n = n-1$ and π is uniquely determined by $\gamma = c_1c_2\cdots c_{n-1}$. Now, any k-prefix of γ is the cumulative inversion table of a permutation with fewer than k inversions. Moreover, since the only condition on π is that each k-prefix has fewer than k inversions, any weakly increasing subdiagonal sequence of length $n-1$ is the cumulative inversion table of such a permutation. As pointed out in Lemma 1, such sequences are counted by the Catalan numbers. □

Recall that $S_i(x)$ is the generating function for permutations of length n with $n - i$ inversions:

$$S_i(x) = \sum_{n\geq 0} |\mathcal{S}_n^{n-i}| x^n.$$

Also, let $C(x) = (1 - \sqrt{1-4x})/(2x)$ be the generating function for the Catalan numbers, $C_n = \binom{2n}{n}/(n+1)$.

Theorem 3. *For $i \geq 0$ we have*

$$\left|\mathcal{S}_n^{n-i-1}\right| = \left|\bigcup_{k=0}^{n} \mathcal{S}_k^{k-i} \times \mathcal{C}_{n-k}\right|$$

and thus the generating functions $S_{i+1}(x)$ and $S_i(x)$ satisfy the identity

$$S_{i+1}(x) = xC(x)S_i(x),$$

Equivalently,

$$S_i(x) = (xC(x))^i S_0(x).$$

Proof. Let $\pi = a_1a_2\cdots a_n \in \mathcal{S}_n^{n-i-1}$. We shall "factor" π into two parts σ and τ such that, for some k in $\{0,1,\ldots,n\}$, σ belongs to $\mathcal{S}_k^{k-i}$ and τ belongs to $\mathcal{C}_{n-k}$. For any permutation ρ, let $\Delta(\rho) = \mathrm{inv}(\rho) - |\rho|$. Consider what happens if we apply Δ to successive prefixes of π. In other words, consider the sequence

$$\Delta(\epsilon),\ \Delta(a_1),\ \Delta(a_1a_2),\ \ldots,\ \Delta(a_1a_2\cdots a_n), \tag{2}$$

where ϵ denotes the empty prefix. Clearly, this sequence can decrease by at most one at a time. Moreover, $\Delta(\epsilon) = 0$ and $\Delta(a_1a_2\cdots a_n) = \Delta(\pi) = -i-1$, and thus the value $-i$ occurs at least once. Note that the assumption $i \geq 0$ is crucial for this argument to work. Let $\sigma = a_1\cdots a_k$ be the longest prefix of π such that $\Delta(\sigma) = -i$. Additionally, let $\tau = a_{k+1}\cdots a_n$ be such that $\pi = \sigma\tau$. For instance, if $\pi = 4213675 \in \mathcal{S}_7^6$, then $i = 0$, the sequence (2) is $(0,-1,-1,0,0,-1,-2,-1)$, and π factors into $\sigma = 4213$ and $\tau = 675$.

By definition, $\mathrm{inv}(\sigma) = k - i$. We shall prove that σ is a permutation of $[k]$, and thus τ is a permutation of $\{k+1, k+2, \ldots, n\}$. Let

$$d = \#\big\{\,(i,j)\ :\ a_i > a_j,\ i \leq k,\ j > k\,\big\}.$$

That is, d is the number of inversions in π with one leg in σ $(i \leq k)$ and the other leg in τ $(j > k)$. Then $\mathrm{inv}(\pi) = \mathrm{inv}(\sigma) + \mathrm{inv}(\tau) + d$. We want to prove that $d = 0$. Suppose to the contrary that $d \geq 1$ and let τ' be the shortest prefix of τ such that $\pi' = \sigma\tau'$ has an inversion with one leg in σ and the other in τ'. Now, in any such inversion the "leg" in τ' must in fact be the last element of τ' due to the minimality of τ'. Thus there is an element of σ larger than the last element of τ', but smaller than all the other elements of τ', and hence the last element of τ' is its smallest. In particular, $\mathrm{inv}(\tau') \geq |\tau'| - 1$. Now, consider $\mathrm{inv}(\pi') = \mathrm{inv}(\sigma\tau')$. By definition, $\mathrm{inv}(\sigma) = |\sigma| - i$. We have just seen that $\mathrm{inv}(\tau') \geq |\tau'| - 1$ and, by assumption, there is also at least one inversion with one leg in σ and the other in τ'. Thus,

$$\mathrm{inv}(\pi') \geq (|\sigma| - i) + (|\tau'| - 1) + 1 = |\pi'| - i$$

and $\Delta(\pi') \geq -i$. Now, by the same intermediate value type argument as above, there is some prefix σ' of π containing π' that satisfies $\Delta(\sigma') = -i$, contradicting the maximality of σ. Thus there are no inversions with one leg in σ and the other in τ, and consequently σ is a permutation of $[k]$ and τ a permutation of $\{k+1, \dots, n\}$.

Having proved that $\sigma \in \mathcal{S}_k^{k-i}$ it immediately follows that $\mathrm{inv}(\tau) = n - i - 1 - (k - i) = n - k - 1$. It remains to prove that τ has no nonempty prefix with as many inversions as letters. Suppose that $\Delta(\tau') > 0$ for some nonempty prefix τ' of τ. Then some nonempty prefix τ'' of τ would satisfy $\Delta(\tau'') = 0$ by a similar argument as above, but then the prefix $\sigma\tau''$ of π would satisfy $\Delta(\sigma\tau'') = -i$, contradicting the maximality of σ. □

While the above theorem represents some progress in understanding permutations with few inversions one crucial piece of the puzzle is missing. Theorem 3 relates all the $S_i(x)$'s to $S_0(x)$, but we need a formula for $S_0(x)$, which is what we shall offer in the next section.

3 A formula for $S_0(x)$

Let us write $\lambda \vdash n$ to indicate that λ is an integer partition of n, and $\mu \vDash n$ to indicate that μ is an integer composition of n. Further, let

$$\mathrm{Par}(x) = \prod_{k \geq 1} \frac{1}{1 - x^k} \quad \text{and} \quad \mathrm{Comp}(x) = \frac{1 - x}{1 - 2x}$$

be the generating functions for integer partitions and compositions. With $\mathrm{Par}_+(x) = \mathrm{Par}(x) - 1$ denoting the generating function for nonempty integer partitions we have

$$\mathrm{Par}(x)^{-1} = \frac{1}{1 + \mathrm{Par}_+(x)} = \sum_{k \geq 0} (-1)^k (\mathrm{Par}_+(x))^k.$$

Thus $\mathrm{Par}(x)^{-1}$ counts signed tuples of nonempty integer partitions, where the sign of such a tuple $(\lambda^1, \dots, \lambda^k)$ is $(-1)^k$. Define

$$\begin{aligned} R(x) &= \mathrm{Comp}(x)\mathrm{Par}(x)^{-1} \\ &= 1 + x^3 + 2x^4 + 5x^5 + 9x^6 + 19x^7 + 37x^8 + \cdots \end{aligned} \tag{3}$$

Then $R(x)$ counts elements of the set

$$\mathcal{R}_n = \{ (\lambda^1, \dots, \lambda^k; \mu) : \lambda^i \vdash n_i,\, \mu \vDash m,\, n_1 + n_2 + \cdots + n_k + m = n \},$$

where the sign of the tuple $(\lambda^1, \dots, \lambda^k; \mu)$ is $(-1)^k$. Writing $(\lambda^1, \dots, \lambda^k; \mu) \vdash n$ when $(\lambda^1, \dots, \lambda^k; \mu)$ is in $\mathcal{R}_n$ (reusing $\vdash$ despite it not being a partition) we then have, by definition,

$$R(x) = \sum_{n \geq 0} \left(\sum_{(\lambda^1, \dots, \lambda^k;\, \mu) \vdash n} (-1)^k \right) x^n.$$

For illustration we list the elements of $\mathcal{R}_3$ below. Negative elements are found in the left column and positive elements in the right column:

$(1\,;11)$	$(\emptyset\,;111)$
$(1\,;2)$	$(\emptyset\,;12)$
$(1,1,1\,;\epsilon)$	$(\emptyset\,;21)$
$(11\,;1)$	$(\emptyset\,;3)$
$(111\,;\epsilon)$	$(1,1\,;1)$
$(21\,;\epsilon)$	$(1,11\,;\epsilon)$
$(2\,;1)$	$(1,2\,;\epsilon)$
$(3\,;\epsilon)$	$(11,1\,;\epsilon)$
	$(2,1\,;\epsilon)$

Here, ϵ denotes the (empty) integer composition of 0 and $\emptyset$ denotes an empty tuple (of integer partitions). The sequence $1, 0, 0, 1, 2, 5, 9, 19, 37, 74, \ldots$ of coefficients of $R(x)$ is recorded in entry A178841 of the OEIS [8]. There it is said to count the number of *pure inverting compositions* of n; see Propositions 2 and 3 in [5].

We are now in position to state our main result regarding $S_0(x)$.

Theorem 4. *We have $S_0(x) = R(xC(x))$, or, equivalently, $S_0\big(x(1-x)\big) = R(x)$, which, by Theorem 3, implies that $S_i(x) = \big(xC(x)\big)^i R(xC(x))$.*

Before proving this we need to better understand what combinatorial structures $R(x)$ enumerates, so we shall define a sign-reversing involution ϕ on $\mathcal{R}$ that singles out a positive subset $\mathrm{Fix}(\phi)$ of $\mathcal{R}$ for which

$$R(x) = \sum_{n\geq 0} |\,\mathrm{Fix}(\phi) \cap \mathcal{R}_n|\, x^n.$$

First, however, we define the auxiliary function

$$\mathrm{split} : \{\mu : \mu \vDash n\} \to \bigcup_{i=0}^{n} \{\lambda : \lambda \vdash i\} \times \{\mu : \mu \vDash n-i\}$$

by $\mathrm{split}(\mu) = (\lambda, \mu')$ where $\mu = \lambda\mu'$ and λ is the longest prefix of μ that is weakly decreasing, and thus defines a partition. For instance, $\mathrm{split}(311212) = (311, 212)$, $\mathrm{split}(21) = (21, \epsilon)$, $\mathrm{split}(12) = (1, 2)$ and $\mathrm{split}(\epsilon) = (\epsilon, \epsilon)$. Let $\mathrm{lir}(\mu)$ be the length of the longest strictly increasing prefix (also called leftmost increasing run) of μ. For instance, $\mathrm{lir}(121) = 2$, $\mathrm{lir}(213) = \mathrm{lir}(1122) = 1$ and $\mathrm{lir}(\epsilon) = 0$.

Lemma 5. *Let λ be a nonempty partition and μ a composition such that $\mathrm{lir}(\mu)$ is even. Then $\mathrm{lir}(\lambda\mu)$ is odd. Moreover, if a is the last element of λ, then*

$$\mathrm{split}(\lambda\mu) = \begin{cases} (\lambda, \epsilon) & \text{if } \mu = \epsilon \text{ is empty} \\ (\lambda, \mu) & \text{if } (b, \mu') = \mathrm{split}(\mu) \text{ and } a < b; \\ (\lambda b, \mu') & \text{if } (b, \mu') = \mathrm{split}(\mu) \text{ and } a \geq b. \end{cases}$$

Note that if μ is nonempty and $\text{lir}(\mu)$ is even, then the first element of μ must be smaller than the second, and hence the longest weakly decreasing prefix of μ is a singleton (the first letter of μ). Thus the lemma above covers all cases. We now define the promised involution ϕ on $\mathcal{R}_n$.

Definition 6. Let $(\lambda^1, \ldots, \lambda^k; \mu) \vdash n$. If $\text{lir}(\mu)$ is even then

$$\phi(\lambda^1, \ldots, \lambda^k; \mu) = \begin{cases} (\emptyset; \mu) & \text{if } k = 0; \\ (\lambda^1, \ldots, \lambda^{k-1}; \lambda^k \mu) & \text{if } k > 0. \end{cases}$$

If $\text{lir}(\mu)$ is odd and $(\rho x, \mu') = \text{split}(\mu)$ then

$$\phi(\lambda^1, \ldots, \lambda^k; \mu) = \begin{cases} (\lambda^1, \ldots, \lambda^k, \rho x; \mu') & \text{if } \text{lir}(\mu') \text{ is even}; \\ (\lambda^1, \ldots, \lambda^k, \rho; x\mu') & \text{if } \text{lir}(\mu') \text{ is odd}. \end{cases}$$

The idea behind the map is that we can create an involution by moving a partition λ back and forth between being considered as part of the list of partitions or as a prefix of our composition μ. The parity of $\text{lir}(\mu)$ allows us to know if we, so to speak, have already prepended a λ or not; indeed $\text{lir}(\lambda\mu)$ is odd if $\text{lir}(\mu)$ is even.

Let us look at a few cases illustrating Definition 6. A simple case is that of a fixed point: $\text{lir}(3644) = 2$ is even and

$$\phi(\emptyset; 3644) = (\emptyset; 3644).$$

Consider $(\lambda^1; \mu) = (6211; \epsilon) \vdash 10$. Then $\text{lir}(\mu) = 0$ is even, $k = 1$ and

$$\phi(6211; \epsilon) = (\emptyset; 6211).$$

Another example of when $\text{lir}(\mu)$ is even is

$$\phi(11, 62; 243352) = (11; 62243352).$$

Finally, three cases when $\text{lir}(\mu)$ is odd are

$$\begin{aligned} \phi(11, 62; 643452) &= (11, 62, 643; 452); \\ \phi(11, 62; 643425) &= (11, 62, 64; 3425); \\ \phi(\emptyset; 643425) &= (64; 3425). \end{aligned}$$

Lemma 7. *The map ϕ is a sign-reversing involution on $\mathcal{R}_n$ whose fixed points are of the form $(\emptyset; \mu)$ with $\mu \vDash n$ and $\text{lir}(\mu)$ even.*

Proof. Let $w = (\lambda^1, \ldots, \lambda^k; \mu) \vdash n$ be given. It is clear that $\phi(w) \vdash n$ and that the first case of the definition, namely $\text{lir}(\mu)$ is even and $k = 0$, covers all fixed points. Further, the second case shortens the list of partitions by one while the third and fourth cases lengthen the same list by one. In all three cases the sign of w is thus reversed. It remains to show that $\phi(\phi(w)) = w$ and we consider each of the last three cases of the definition of ϕ separately.

If $\mathrm{lir}(\mu)$ is even and $k > 0$, then $\phi(w) = (\lambda^1, \ldots, \lambda^{k-1}; \lambda^k\mu)$. To show that $\phi(\phi(w)) = w$ we consider the three cases of Lemma 5. If μ is empty then $\mathrm{split}(\lambda^k\mu) = (\lambda^k, \epsilon)$, $\mathrm{lir}(\epsilon) = 0$ is even and

$$\phi(\lambda^1, \ldots, \lambda^{k-1}; \lambda^k\mu) = (\lambda^1, \ldots, \lambda^{k-1}, \lambda^k; \mu) = w.$$

If μ is nonempty then let $(b, \mu') = \mathrm{split}(\mu)$. Also, let a be the last element of λ^k. If $a < b$ then $\mathrm{split}(\lambda^k\mu) = (\lambda^k, \mu)$, $\mathrm{lir}(\mu)$ is even (by assumption) and

$$\phi(\lambda^1, \ldots, \lambda^{k-1}; \lambda^k\mu) = (\lambda^1, \ldots, \lambda^{k-1}, \lambda^k; \mu) = w.$$

If $a \geq b$ then $\mathrm{split}(\lambda^k\mu) = (\lambda^k b, \mu')$, $\mathrm{lir}(\mu') = \mathrm{lir}(\mu) - 1$ is odd and

$$\phi(\lambda^1, \ldots, \lambda^{k-1}; \lambda^k\mu) = (\lambda^1, \ldots, \lambda^{k-1}, \lambda^k; b\mu') = (\lambda^1, \ldots, \lambda^k; \mu) = w.$$

If $\mathrm{lir}(\mu)$ is odd then let $(\rho x, \mu') = \mathrm{split}(\mu)$. If, in addition, $\mathrm{lir}(\mu')$ is even, then

$$\phi(\phi(w)) = \phi(\lambda^1, \ldots, \lambda^k, \rho x; \mu') = (\lambda^1, \ldots, \lambda^k; \rho x\mu') = w.$$

If $\mathrm{lir}(\mu')$ is odd, then $\phi(w) = (\lambda^1, \ldots, \lambda^k, \rho; x\mu')$ and, since $\mathrm{lir}(x\mu')$ is even,

$$\phi(\phi(w)) = \phi(\lambda^1, \ldots, \lambda^k, \rho; x\mu') = (\lambda^1, \ldots, \lambda^k; \rho x\mu') = w,$$

which concludes the last case and thus also the proof. □

Next we aim at proving Theorem 4. That is, we wish to prove that

$$S_0(x(1-x)) = R(x). \tag{4}$$

The proof is somewhat involved and we have divided it into three lemmas that we now outline:

- Lemma 7 above gives a convenient combinatorial interpretation of the right-hand side of (4). In Lemma 8 we provide a (signed) combinatorial interpretation of left-hand side of (4): We define a family of sets $\{T_n\}_{n\geq 0}$ such that the coefficient of x^n in $S_0(x(1-x))$ is

$$\sum_{(S,\beta)\,\in\, T_n} (-1)^{|S|}. \tag{5}$$

- The next step would ideally be to define a sign-reversing involution on T_n whose fixed-points are all positive and thus arrive at a result akin to Lemma 7. What we have found is a sign-reversing involution that does not quite fulfill this ideal, in that some fixed-points are negative: Lemma 10 shows that the sum (5) can be rewritten as

$$\sum_{(\lambda,\mu)\,\Vdash\, n} (-1)^{|\lambda|}, \tag{6}$$

where the meaning of $(\lambda, \mu) \Vdash n$ is given in Definition 9 below. This sum is preferable to (5) for two reasons. First, it has fewer terms. Second, the combinatorial structures being summed over are closer in spirit to the fixed points of ϕ (Lemma 7) than the members of T_n are.

- Finally, by means of a natural equivalence relation, Lemma 11 shows that the value of the sum (6) equals the number of fixed points of ϕ on $\mathcal{R}_n$ as desired.

Having presented an outline of the proof of (4) we now dive into the details. Let $T_{n,k}$ be the set of pairs (S, β), where $S \subseteq [n-k]$, $|S| = k$, and

$$\beta = (\beta_1, \beta_2, \ldots, \beta_{n-k})$$

is a subdiagonal sequence with sum $\beta_1 + \beta_2 + \cdots + \beta_{n-k} = n - k$. Also, let

$$T_n = \bigcup_{k=0}^{n} T_{n,k}.$$

For instance, $T_0 = \{(\emptyset, \epsilon)\}$, T_1 and T_2 are empty, $T_3 = \{(\emptyset, 012)\}$, and T_4 consists of the following 8 elements:

$$(\emptyset, 0121),\ (\emptyset, 0112),\ (\emptyset, 0103),\ (\emptyset, 0022),\ (\emptyset, 0013),$$
$$(\{1\}, 012),\ (\{2\}, 012),\ (\{3\}, 012).$$

Lemma 8. *We have*

$$S_0(x(1-x)) = \sum_{n \geq 0} \Bigg(\sum_{(S,\beta) \in T_n} (-1)^{|S|} \Bigg) x^n.$$

Proof. Let $(S, \beta) \in T_{n,k}$. View β as an inversion table and let π be the corresponding permutation on $[n-k]$. Note that π has exactly $n-k$ inversions and thus the cardinality of $T_{n,k}$ is $\left|\mathcal{S}_{n-k}^{n-k}\right| \binom{n-k}{k}$. The result now follows from a direct calculation:

$$\begin{aligned}
S_0(x(1-x)) &= \sum_{n \geq 0} |\mathcal{S}_n^n|\, x^n (1-x)^n \\
&= \sum_{n \geq 0} |\mathcal{S}_n^n|\, x^n \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k \\
&= \sum_{n \geq 0} \Bigg(\sum_{k=0}^{n} \left|\mathcal{S}_{n-k}^{n-k}\right| \binom{n-k}{k} (-1)^k \Bigg) x^n \\
&= \sum_{n \geq 0} \sum_{k=0}^{n} \Bigg(\sum_{(S,\beta) \in T_{n,k}} (-1)^{|S|} \Bigg) x^n \\
&= \sum_{n \geq 0} \Bigg(\sum_{(S,\beta) \in T_n} (-1)^{|S|} \Bigg) x^n.
\end{aligned}$$

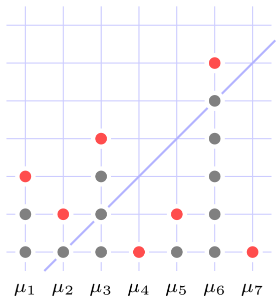
□

We shall show that the set T_n in the inner summation in Lemma 8 can be replaced with a smaller set, but first we give a few definitions.

For a composition $\mu = (\mu_1, \ldots, \mu_k)$ define $\operatorname{dmax}(\mu)$ as 0 if $k \leq 1$ and

$$\operatorname{dmax}(\mu) = \max\{\, \mu_j - j + 1 : 2 \leq j \leq k \,\}$$

otherwise. If we plot μ_j against j this is the largest distance it goes over the line $y = x - 1$, excluding μ_1 for technical reasons. For instance, if $\mu = 3241261$ then $\operatorname{dmax}(\mu) = \mu_3 - 3 + 1 = 2$ as depicted below:



Up until this point we have listed the parts of a partition λ in weakly decreasing order. In what follows, it will be convenient to instead list them in weakly increasing order. For instance, we may write $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (1, 3, 4) \vdash 8$.

Definition 9. Let λ be an integer partition and μ an integer composition. Let their total sum be n and let $d = \operatorname{dmax}(\mu)$. We shall write

$$(\lambda, \mu) \Vdash n$$

if the following three conditions hold:

- λ has distinct parts (and is hence strictly increasing);
- $\lambda \neq \epsilon \implies \lambda_{|\lambda|} < d$,
- $\mu \neq \epsilon \implies \mu_1 \leq d$.

For instance, (λ, μ) with $\mu = 3241261$ as in the example above does not satisfy Definition 9 regardless of what the partition λ is; the reason being that $3 = \mu_1 > \operatorname{dmax}(\mu) = 2$. Let us consider the sets of pairs $(\lambda, \mu) \Vdash n$ for small n. For $n = 0$ there is a single pair, (ϵ, ϵ); for $n = 1, 2$ there are none; for $n = 3$ there is a single pair, $(\epsilon, 12)$; for $n = 4$ there are two, $(\epsilon, 121)$ and $(\epsilon, 13)$; and for $n = 5$ there are seven:

$$(\epsilon, 113),\ (\epsilon, 1211),\ (\epsilon, 122),\ (\epsilon, 131),\ (\epsilon, 14),\ (\epsilon, 23),\ (1, 13).$$

As a larger example we offer $(134, 161121) \Vdash 20$.

Lemma 10. *We have*

$$\sum_{(S,\beta)\,\in\, T_n} (-1)^{|S|} = \sum_{(\lambda,\mu)\,\Vdash\, n} (-1)^{|\lambda|}.$$

Proof. We shall give a sign-reversing involution on T_n whose fixed points can be bijectively mapped to pairs $(\lambda, \mu) \Vdash n$.

Let $(S, \beta) \in T_{n,n-r}$ with $\beta = (\beta_1, \beta_2, \ldots, \beta_r)$. We will say that β_i is *marked* if $i \in S$. An index i such that $\beta_i = 0$ and $\beta_{i+1} > 0$ will be called a *0-ascent*. If $\beta_i = i - 1$, then we call i a *diagonal index* and β_i a *diagonal entry*. We shall now define an endofunction

$$\psi : T_n \to T_n$$

which we will later prove is a sign-reversing involution. Consider the entries β_i in descending order by index and define $\psi(S, \beta)$ according to which of the following four mutually exclusive cases is encountered first:

1. If β_i is marked and there is no 0-ascent $j > i$, then we replace β_i with an unmarked bigram xy whose first letter is zero, $x = 0$, and whose last letter is $y = \beta_i + 1$. In particular, S is mapped to $S \setminus \{i\}$.

2. If β_i is marked, i is not a diagonal index and there is a 0-ascent $j > i$, then we replace β_i with an unmarked $\beta_i + 1$ and append an unmarked zero to the end of β. Again, S is mapped to $S \setminus \{i\}$.

3. If β_i and β_{i+1} are both unmarked, i is a 0-ascent and there is no diagonal index $j > i + 1$, then we replace the bigram $\beta_i\beta_{i+1}$ by a single marked $\beta_{i+1} - 1$. Here, S is mapped to $S \cup \{i\}$.

4. If $\beta_i \neq 0$ is unmarked, $\beta_r = 0$ and there is some 0-ascent $j > i$, then we replace β_i by a marked $\beta_i - 1$ and remove β_r. Here, S is mapped to $S \cup \{i\}$.

If none of these cases are encountered we let $\psi(S, \beta) = (S, \beta)$ be a fixed point. It is easy to see that each case preserves subdiagonality. Cases 1 and 2 remove a mark, increase the sum by one and add an element; consequently the image $\psi(S, \beta)$ is in $T_{n,n-r-1}$. Cases 3 and 4 add a mark, decrease the sum by one and remove an element, so in these two cases $\psi(S, \beta)$ is in $T_{n,n-r+1}$. Thus ψ is well-defined and sign-reversing. Let us consider some examples:

- Case 1 at $i = 3$: $\psi(\{1, 3\}, 0103) = (\{1\}, 01013)$
- Case 2 at $i = 3$: $\psi(\{2, 3\}, 0010150) = (\{2\}, 00201500)$
- Case 3 at $i = 4$: $\psi(\{1\}, 002040) = (\{1, 4\}, 00230)$
- Case 4 at $i = 2$: $\psi(\{3\}, 0120250000) = (\{2, 3\}, 002025000)$
- A fixed point: $\psi(\{1, 3\}, 0020152000) = (\{1, 3\}, 0020152000)$.

Next we shall prove that ψ is an involution; that is, $\psi(\psi(S, \beta)) = (S, \beta)$. If (S, β) is a fixed point, then the claim is trivially true, so we can assume that we encounter one of the four cases above. Suppose $\psi(S,\beta) = (T,\gamma)$ after falling into one of the cases at β_i. We want to show that $\psi(T,\gamma) = (S,\beta)$. The map ψ leaves the suffix $\beta_{i+2}\beta_{i+3}\cdots\beta_r$ of β unchanged, aside from possibly appending an unmarked trailing zero; hence this suffix, with possibly an appended zero, will also be present in γ. If no zero was appended then clearly none of the cases apply to (T,γ) at $j > i$, or else that case would have applied to (S,β) as well. Suppose a zero was appended. Since this trailing zero is unmarked (T,γ) cannot fall into case 1 or 2 for any $j > i$. Adding a zero at the end cannot introduce a 0-ascent, so (T,γ) cannot fall into case 3 for $j > i$. Case 4 is also easy to exclude, so we conclude that (T,γ) cannot fall into any of the four cases at an index $j > i$.

Going through each of the cases at index i, we see that if β_i falls into case 1, then γ_i must fall into case 3, which undoes what case 1 just did. Similarly, case 2 is cancelled by case 4, 3 by 1, and 4 by 2; thus, ψ is an involution.

We now consider the fixed points of ψ. We wish to show that any nonempty fixed point (S,β) of ψ, when considered as a marked sequence, can be written

$$\sigma\tau\zeta$$

where each letter of σ is either a marked diagonal entry or an unmarked zero, ending in an unmarked zero; τ consists of unmarked positive entries at least one of which is a diagonal entry; and ζ is a (possibly empty) sequence of zeros. One instance of a fixed point is $(\emptyset, 0121)$ in which $\sigma = 0$, $\tau = 121$ and $\zeta = \epsilon$. Another instance is $(\{1,3\}, 0020152000)$ in which $\sigma = 0020$, $\tau = 152$ and $\zeta = 000$.

Since β is subdiagonal it starts with a zero. Moreover, its sum r is positive, and hence it must contain a 0-ascent. Let σ be the prefix of β consisting of every letter of β up to and including the rightmost 0-ascent. Define τ as the subsequent contiguous run of positive entries in β and let the remaining suffix be ζ. In particular, τ is nonempty. Also, ζ must consist entirely of zeros; otherwise, it would contain a 0-ascent, contradicting that ζ is right of the rightmost 0-ascent in β. Now, if β_i is marked and there is no 0-ascent $j > i$, then case 1 would apply at β_i. Thus every entry of $\sigma_{|\sigma|}\tau\zeta$ must be unmarked. There also has to be a diagonal entry in τ, otherwise the bigram $\sigma_{|\sigma|}\tau_1$ would make us fall into case 3. Thus, there is an $\ell > 1$ such that $\tau_\ell = \ell - 1 + |\sigma|$. If σ_i is marked, then i is a diagonal index, else we would fall into case 2 at σ_i because it is to the left of a 0-ascent. To show that σ is of the desired form we shall consider two cases.

Suppose ζ is empty. Every element to the right of τ_ℓ is ≥ 1, and $\tau_1 > 0$, so

$$\tau_1 + \tau_\ell + \cdots + \tau_{r-|\sigma|} \geq 1 + (\ell - 1 + |\sigma|) + (r - |\sigma| - \ell) \geq r.$$

The sum of entries in β is r (by definition of $T_{n,n-r}$) and consequently the sum above is exactly r. Thus, every entry of σ is zero. Aside from the first one, none of those zeros can be marked, or else we would have a contradiction with the earlier result that any marked element of σ is a diagonal entry. Thus σ is of the desired form.

Suppose ζ is nonempty. There cannot be any positive unmarked σ_i since then we would fall into case 4 at σ_i. Thus, σ is of the desired form by the same argument as above.

Let us now define a function θ mapping fixed points of ψ to pairs $(\lambda, \mu) \Vdash n$. Given a fixed point (S, β) factored as $\sigma\tau\zeta$ we let $\theta(S, \beta) = (\lambda, \mu)$, where λ consists of the marked indices of σ written in increasing order and $\mu = \tau$. In other words, the entries of λ are the elements of S, the reason being that τ and ζ contain only unmarked elements. For example $\theta(\emptyset, 0121) = (\epsilon, 121)$ and $\theta(\{1, 3\}, 0020152000) = (13, 152)$. It is clear that λ has distinct parts and that μ defines a composition. Furthermore, the sum of values in λ and μ is the sum of elements in β plus the number of marked elements, which is $r + n - r = n$. Note that the sign simply is $(-1)^{|\lambda|}$.

We wish to show that $(\lambda, \mu) \Vdash n$. Our diagonal index ℓ gives us

$$\begin{aligned} \operatorname{dmax}(\mu) &\geq \mu_\ell - \ell + 1 \\ &= \tau_\ell - \ell + 1 \\ &= (\ell - 1 + |\sigma|) - \ell + 1 = |\sigma| \,. \end{aligned}$$

Suppose $\operatorname{dmax}(\mu) = \mu_j - j + 1$. Then

$$\begin{aligned} \operatorname{dmax}(\mu) &= \beta_{j+|\sigma|} - j + 1 \\ &\leq j + |\sigma| - 1 - j + 1 = |\sigma| \,. \end{aligned}$$

in which the inequality is a consequence of subdiagonality. Thus $\operatorname{dmax}(\mu) = |\sigma|$. If λ is nonempty, then λ_1 corresponds to a marked diagonal index, which must then be in σ. Thus $\lambda < |\sigma| = \operatorname{dmax}(\mu)$ since σ ends on a zero. If μ is nonempty, then $\mu_1 = \tau_1 \leq |\sigma|$ by subdiagonality, and hence $\mu_1 \leq \operatorname{dmax}(\mu)$. Thus $(\lambda, \mu) \Vdash n$.

To complete our proof we have to show that θ is bijective, which we do by constructing its inverse. Assume that $(\lambda, \mu) \Vdash n$ and $k = \operatorname{dmax}(\mu)$. Let $\sigma = \sigma_1\sigma_2 \cdots \sigma_k$, where $\sigma_i = i - 1$ is a marked diagonal entry if $i = \lambda_j$ for some $j \in [|\lambda|]$, and $\sigma_i = 0$ is an unmarked zero otherwise. Also, let $\tau = \mu$ and let ζ be a segment consisting of $n - |\sigma| - |\tau|$ unmarked zeros. By the same argument as above we have $|\lambda|$ marked elements, and the sum of all elements in λ and μ is $n - |\lambda|$, so $r = |\lambda|$. Since $\lambda \neq \epsilon \Rightarrow \lambda_1 < \operatorname{dmax}(\mu)$ we have $\sigma_{|\sigma|} = 0$. Thus the image is in $T_{n,n-r}$ and θ maps $\sigma\tau\zeta$ back to (λ, μ), completing our proof. □

Lemma 11. *We have*

$$\sum_{(\lambda,\mu) \Vdash n} (-1)^{|\lambda|} \;=\; \big|\{\mu \vDash n : \operatorname{lir}(\mu) \text{ even}\}\big|.$$

Proof. Let $\sim$ be the equivalence relation generated by postulating that

$$(\lambda a, \mu) \sim (\lambda, a\mu)$$

whenever both $(\lambda a, \mu) \Vdash n$ and $(\lambda, a\mu) \Vdash n$ hold. For example, when $n = 5$ the equivalence classes are all singletons except the class $\{(\epsilon, 113), (1, 13)\}$. For $n = 6$ there are three non-singleton classes, namely

$$\{(\epsilon, 1131), (1, 131)\},\ \{(\epsilon, 114), (1, 14)\}\ \text{and}\ \{(\epsilon, 123), (1, 23)\}.$$

We wish to show that the inner sum in the expression for $R(x)$ above when restricted to a single equivalence class is 0 or 1. In other words, if C is an equivalence class, then

$$\sum_{(\lambda,\mu) \in C} (-1)^{|\lambda|} \in \{0, 1\}.$$

Assume $(\lambda a, \mu) \Vdash n$. Then $a < \operatorname{dmax}(\mu)$, but $\operatorname{dmax}(a\mu) \geq \operatorname{dmax}(\mu) - 1$, so $a \leq \operatorname{dmax}(a\mu)$. Furthermore, if λ is nonempty, then $\lambda_{|\lambda|} < a$ because λa is strictly increasing. Thus $\lambda_{|\lambda|} < a \leq \operatorname{dmax}(a\mu)$, and so $(\lambda, a\mu) \Vdash n$. By induction on the number of elements moved we see that (λ, μ) is in the same equivalence class as $(\epsilon, \lambda\mu)$. Clearly we cannot have two pairs of the form (ϵ, μ) in the same equivalence class, so we make them our representatives.

Let (ϵ, μ) be such a representative. We will call k *valid* if

$$(\mu_1 \cdots \mu_k,\ \mu_{k+1} \cdots \mu_{|\mu|}) \Vdash n.$$

Let $\lambda = \mu_1 \cdots \mu_k$ and $\nu = \mu_{k+1} \cdots \mu_{|\mu|}$. By the argument above, if some k is valid, then all smaller k are valid too. We want to find the largest valid k. The sign of $(\mu_1 \cdots \mu_k, \mu_{k+1} \cdots \mu_{|\mu|})$ is $(-1)^k$, so if the largest valid value is ℓ, then the sum of the equivalence class of (ϵ, μ) is $(-1)^0 + (-1)^1 + \cdots + (-1)^\ell$, which is zero if ℓ is odd and 1 if ℓ is even.

Let $s = \operatorname{lir}(\mu)$. We wish to show that ℓ and s have the same parity. Clearly, $\ell \leq s$ since otherwise $\lambda = \mu_1 \cdots \mu_\ell$ would not be strictly increasing. If $\ell = s$ we have nothing left to prove, so we can assume that $\ell < s$. Then ν is nonempty and $\nu_1 \leq \operatorname{dmax}(\nu)$, so $|\nu| \geq 2$ and $\ell \leq s-2$. Let $k = s-2$. Then $\lambda = \mu_1 \cdots \mu_k$ is strictly increasing and $\nu_1 < \nu_2$. Thus $\operatorname{dmax}(\nu) \geq \nu_2 - 1 \geq \nu_1$. If λ is nonempty, then $\lambda_k < \nu_1 \leq \operatorname{dmax}(\nu)$. Thus $k = s-2$ is valid, so $\ell = s-2$, which has the same parity as s. The representatives (ϵ, μ) whose equivalence classes have sum one are hence exactly those where $\operatorname{lir}(\mu)$ is even. □

Theorem 4 follows directly from Lemmas 7, 8, 10 and 11.

4 Structure of $R(x)$

By Lemma 7 the elements of $\operatorname{Fix}(\phi) \cap \mathcal{R}_n$ are of the form $(\emptyset; \mu)$ with $\mu \vDash n$ and $\operatorname{lir}(\mu)$ even. With this in mind let

$$\operatorname{Fix}_n(\phi) = \{\mu \vDash n : \operatorname{lir}(\mu)\ \text{even}\}.$$

Let $\mathcal{M}_n$ be the set of compositions of n that start with an ascent and are weakly decreasing after the initial ascent and let $M(x)$ be the corresponding generating function. That is, $(\mu_1, \dots, \mu_k) \in \mathcal{M}_n$ if and only if $k \geq 2$, $\mu_1 < \mu_2 \geq \mu_3 \geq \cdots \geq \mu_k$ and $\mu_1 + \cdots + \mu_k = n$. For instance, $\mathcal{M}_n = \emptyset$ for $n \leq 2$, $\mathcal{M}_3 = \{12\}$, $\mathcal{M}_4 = \{121, 13\}$, $\mathcal{M}_5 = \{1211, 122, 131, 14, 23\}$ and the first few terms of the power series $M(x)$ are

$$M(x) = x^3 + 2x^4 + 5x^5 + 8x^6 + 15x^7 + 23x^8 + 37x^9 + \cdots$$

Let $\mu^1, \mu^2, \dots, \mu^k$ be compositions with $\mu^i \in \mathcal{M}_{n_i}$. Their concatenation

$$\mu = \mu^1 \mu^2 \cdots \mu^k$$

is a composition of $n = n_1 + \cdots + n_k$ with $\mathrm{lir}(\mu)$ even, and so $\mu \in \mathrm{Fix}_n(\phi)$.

Conversely, given a composition $\mu \in \mathrm{Fix}_n(\phi)$, let μ^1 be the longest suffix of μ that belongs to $\mathcal{M}_{n_1}$, where n_1 is the length of μ^1. Writing $\mu = \nu\mu^1$ we can recursively do the same with ν, stopping if ν is empty. This works because μ starts with an ascent, so if ν is nonempty then ν starts with an ascent as well. This way we arrive at a factorisation $\mu = \mu^k \cdots \mu^2 \mu^1$ with $\mu^i \in \mathcal{M}_{n_i}$ and $n = n_1 + \cdots + n_k$. For instance, the factors of $123511211 \in \mathrm{Fix}_{17}(\phi)$ are 12, 351 and 1211.

In terms of generating functions the factorisation we have established translates to the functional equation $R(x) = (1 - M(x))^{-1}$. Now, by (3),

$$M(x) = 1 + \left(\frac{x}{1-x} - 1\right)\mathrm{Par}(x).$$

Thus, aside from the constant term, the coefficient of x^n in $M(x)$ equals

$$p(0) + p(1) + \cdots + p(n-1) - p(n) \tag{7}$$

and coincides with sequence A058884 in the OEIS [8]. Moreover, (7) is the number of compositions with exactly one inversion according to Theorem 4.1 of [3]. To summarise we have established the following proposition.

Proposition 12. *With $p(n)$ denoting the number of partitions of n,*

$$R(x) = \Big(1 - \sum_{n \geq 1} \big(p(1) + p(2) + \cdots + p(n-1) - p(n)\big)x^n\Big)^{-1}.$$

An alternative formula can be obtained from considering the logarithmic derivative of $R(x)$:

Proposition 13. *With $\sigma(n)$ denoting the sum of the divisors of n,*

$$R(x) = \exp\left(\sum_{n \geq 1} \left(2^n - \sigma(n) - 1\right) \frac{x^n}{n}\right).$$

Proof. Taking the logarithmic derivative of (3) we get

$$\begin{aligned} x\big(\log R(x)\big)' &= \frac{x\mathrm{Comp}'(x)}{\mathrm{Comp}(x)} - \frac{x\mathrm{Par}'(x)}{\mathrm{Par}(x)} \\ &= \frac{x}{(1-x)(1-2x)} - \sum_{k\geq 1} \frac{kx^k}{1-x^k} \end{aligned}$$

An expression of the form $F(x) = \sum_{k\geq 1} a_k x^k/(1-x^k)$ is called a Lambert series, and it is well known, and easy to see, that

$$F(x) = \sum_{n\geq 1} b_n x^n, \quad \text{where } b_n = \sum_{k|n} a_k.$$

In our case $a_k = k$ and hence $b_n = \sigma(n)$. Consequently,

$$x\big(\log R(x)\big)' = \sum_{n\geq 1}(2^n - 1 - \sigma(n))x^n \tag{8}$$

and it follows that

$$\begin{aligned} \log R(x) &= \int_0^x \sum_{n\geq 1}(2^n - 1 - \sigma(n))t^{n-1}dt \\ &= \sum_{n\geq 1}\big(2^n - 1 - \sigma(n)\big)\frac{x^n}{n}, \end{aligned}$$

which proves the claim. □

Corollary 14. *The cardinalities $r_n = |\mathcal{R}_n|$ can be computed recursively by $r_0 = 1$ and, for $n \geq 1$,*

$$r_n = \frac{1}{n}\sum_{k=1}^{n} r_{n-k}(2^k - \sigma(k) - 1).$$

Moreover, we have the closed formula

$$r_n = \frac{1}{n!}\sum_{\pi\in\mathrm{Sym}(n)}\prod_{\ell\in C(\pi)}(2^\ell - \sigma(\ell) - 1),$$

where $\mathrm{Sym}(n)$ *is the symmetric group of degree n and $C(\pi)$ is a multiset that encodes the cycle type of π, that is, there is an $\ell \in C(\pi)$ for each ℓ-cycle of π.*

Proof. Since $\big(\log R(x)\big)' = R'(x)/R(x)$ it follows from (8) that

$$xR'(x) = R(x)\sum_{n\geq 1}(2^n - 1 - \sigma(n))x^n$$

and on identifying coefficients we get the claimed recursion. For the closed formula we refer to Equation (8) in [2] and the paragraph preceding that formula. □

It easy to see that the coefficient of x^n in $xM'(x)/(1-M(x))$ is $2^n-\sigma(n)-1$. Thus, if we consider the factorisation $\mu = \mu^1\mu^2\cdots\mu^k$ of elements in $\mathcal{R}$, as above, together with a distinguished site of μ^1, then such structures should be counted by $2^n-\sigma(n)-1$. Finding a bijective proof of this remains an open problem.

5 Superdiagonals

Having derived formulas for the main diagonal and the subdiagonals of the Mahonian triangle one naturally wonders if similar formulas exist for the superdiagonals. In other words, does Theorem 3 generalize to negative i? If so then, in particular, $S_{-1}(x)x - S_0(x)C(x)^{-1}$ would be the zero power series. This is, however, not the case and experimentally we have found that this difference is the rational power series $(-1+2x)/(1-x)$. More generally, it appears that the coefficients of $S_{-i}(x)x^i - S_0(x)C(x)^{-i}$ are linearly recurrent with minimal polynomial $(1-x)^i$, and we make the following conjecture, which has been verified for $1 \le i \le 50$ and power series truncated to their initial 150 terms.

Conjecture 15. *For any $i \ge 1$, there is a polynomial $P_i(x) \in \mathbb{Z}[x]$ of degree $3i-2$ such that*

$$S_{-i}(x)x^i = S_0(x)C(x)^{-i} + \frac{P_i(x)}{(1-x)^i}.$$

In particular, the first four polynomials $P_1(x)$, $P_2(x)$, $P_3(x)$ and $P_4(x)$ are

$$-1+2x;$$
$$-1+4x-4x^2+x^3+x^4;$$
$$-1+6x-12x^2+10x^3-2x^4-x^5+2x^6-x^7;$$
$$-1+8x-24x^2+35x^3-25x^4+6x^5+4x^6-3x^7+3x^8-3x^9+x^{10}.$$

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Paper III

Pattern avoiding permutations enumerated by inversions

Atli Fannar Franklín

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Abstract

Permutations are usually enumerated by size, but new results can be found by enumerating them by inversions instead, in which case one must restrict one's attention to indecomposable permutations. In the style of the seminal paper by Simion and Schmidt, we investigate all combinations of permutation patterns of length at most 3.

1 Introduction

To enumerate a family of permutations by their number of inversions we need the family of permutations with a given number of inversions to be finite. This is generally not the case, so to get around this we use the notion of decomposability. We define the *sum* of two permutations $\rho_1, \ldots, \rho_n$ and $\tau_1, \ldots, \tau_m$ as a permutation on $n + m$ elements given by

$$\rho \oplus \tau = \rho_1 \ldots \rho_n (n + \tau_1) \ldots (n + \tau_m)$$

Following the definitions of Lentin [15] and Comtet [10] we will say that π is *decomposable* if $\pi = \rho \oplus \tau$ for some non-empty permutations ρ and τ. In the same vein we call permutations that are not decomposable *indecomposable*. With this definition every permutation can be factored into a set of indecomposable factors, we call these factors its *components*.

An *inversion* in a permutation $\pi = \pi_1 \ldots \pi_n$ is a pair of indices (i, j) such that $i < j$ and $\pi_i > \pi_j$. The number of inversions in a permutation π will be denoted $\text{inv}(\pi)$. The following lemma is due to Claesson, Jelínek and Steingrímsson and highlights a relation between these two concepts.

Lemma 1 ([9], Lemma 8)**.** *Let π be a permutation on n elements and c be the number of components of π. Then* $\text{inv}(\pi) \geq n - c$.

Crucially, if $c = 1$ then $\text{inv}(\pi) \geq n - 1$ and thus the set of indecomposable permutations with k inversions is finite since it is contained in the set of all permutations on $k+1$ or fewer elements. With this in mind we define I_k to be the set of all indecomposable permutations with exactly k inversions.

We recall some notions related to patterns in permutations, following the notation summarised by Bevan [6]. We will restrict ourselves to only consider classical permutation patterns. We say that two sequences $a_1, \dots, a_m$ and $b_1, \dots, b_m$ are *order-isomorphic* if $a_i < a_j$ holds if and only if $b_i < b_j$. For permutations $\pi = \pi_1 \dots \pi_n$ and $\tau = \tau_1 \dots \tau_m$ we say that π contains the pattern τ if there exists a set of indices $i_1, \dots, i_m$ such that $\pi_{i_1} \dots \pi_{i_m}$ is order-isomorphic to $\tau_1 \dots \tau_m$. We call such a set of indices an *occurrence* of the pattern τ. If π has no occurrence of τ we will say that π *avoids* the pattern τ. The subset of I_k consisting of the permutations avoiding τ by will be denoted $I_k(\tau)$. Similarly, we will denote the subset of I_k consisting of permutations avoiding several patterns $\tau_1, \tau_2, \dots, \tau_r$ by $I_k(\tau_1, \tau_2, \dots, \tau_r)$.

In this paper we will investigate all combinations of patterns of length ≤ 3 in the style of Simion and Schmidt's [19] paper. Something similar in the case of indecomposable permutations enumerated by size (rather than inversions) has been done before by Gao, Kitaev and Zhang [14], where they determine the number of indecomposable permutations for all patterns of length ≤ 4 in terms of the number of decomposable permutations.

Another paper in this area is due to Dokos, Dwyer, Johnson, Sagan and Selsor [11]. That paper finds q-polynomials for permutations enumerated jointly by size and inversions. They find many algebraic connections related to polynomials and q-analogues, covering analogues of the Catalan numbers, Fibonacci numbers, triangular numbers and powers of two. We deviate from this approach of joint enumeration for the two reasons outlined below. Our results do not follow directly from theirs either as they do not consider all the cases of this paper and in the cases where they do the indecomposability criterion prevents direct derivation.

The first reason is to get novel bijective maps to other combinatorial objects, rather than algebraic equivalences. The motivating example for this paper came from Claesson, Jelínek and Steingrímsson [9]. In Lemma 10 of their paper they note that the inversion tables of permutations are weakly decreasing if and only if the permutations avoids the pattern 132. Adding indecomposability as a condition turns this connection into a bijective map providing the first instance of the type of bijections studied in this paper.

The other reason was to possibly obtain bounds on the growth rates of pattern avoiding permutations. Let $S_n(\pi_1, \dots, \pi_r)$ be the set of permutations of size n avoiding the patterns $\pi_1, \dots, \pi_n$. Define $S_n^*(\pi_1, \dots, \pi_r)$ similarly for indecomposable permutations. The generating function for S_n^* can often be expressed in terms of the generating function of S_n, as is done for many different generating functions in Gao et. al. [14]. Thus if we can bound the growth rate of $|S_n^*|$ we can often get bounds for $|S_n|$ as well. Clearly we have

$$S_n^*(\pi_1, \ldots, \pi_r) \subseteq \bigcup_{k=0}^{\binom{n}{2}} I_k(\pi_1, \ldots, \pi_r)$$

Consider this bound for the pattern 132. The number of partitions of n grows like $\exp\left(\pi\sqrt{2n/3}\right)$; see for instance Apostol [2, p. 316–318]. The bound above would then give us that the asymptotic growth of $|S_n^*(132)|$ is at most $\exp\left(n\pi\sqrt{1/3}\right) \approx 6.13^n$. By Gao et. al. [14] we know $|S_n^*(132)|$ grows like the Catalan numbers, so the actual growth rate is $4^n n^{-5/2}$.

Some other cases seem to give tighter bounds, for example $I_k(132, 4321)$. We conjecture based on numerical evidence that this is enumerated by the number of partitions with no parts strictly between the smallest and the largest part. From Bloom and McNew [7] we get $\sum_{k=1}^{n} |I_k(132, 4321)|$ is asymptotically bounded by $n^2 \log(n)^2$, so $S_n^*(132, 4321)$ grows at most like $n^4 \log(n)^2$. From Albert, Bean, Claesson, Nadeau, Pantone and Úlfarsson [1] we get that $S_n(132, 4321)$ grows like n^4 and experimental results suggest $S_n^*(132, 4321)$ grows like n^3, so the bound is much closer in this case. Another case is $I_k(321, 1342)$, which we conjecture to have $k(k+1)/2 + 1$ elements. Thus would asymptotically bound $S_n^*(321, 1342)$ as $n^4/8$. From Albert et. al. [1] we know that $S_n(321, 1342)$ grows like 2^n, which shows that almost all permutations avoiding $321, 1342$ are decomposable. Experimental results suggest that $S_n^*(321, 1342)$ grows like $n^3/6$ which would make the bound quite good.

A summary of the results in this paper is given in the following table. We omit sets of patterns that give the same sequence as a subset of those patterns. We also omit sequences that decay to zero.

Pattern set	Result
123	No OEIS entry, recurrence in Theorem 9
132	`A000041`, Partitions
231	`A005169`, Fountains
321	`A006958`, Parallelogram polyominoes
123, 231	No OEIS entry, generating function in Theorem 12
132, 123	`A135278`, Pascal triangle with first column removed
132, 213	`A117629`, Gorenstein partitions
132, 231	`A000009`, Partitions into distinct parts
132, 321	`A000005`, Partitions into equal parts, i.e. divisors
231, 312	`A010054`, Characteristic of triangular numbers
231, 321	`A000012`, All ones
123, 132, 213	No OEIS entry, generating function in Theorem 21
123, 132, 231	`A000012`, All ones
132, 213, 231	`A001227`, Odd divisors
123, 132, 213, 231	`A103451`, Pascal triangle with values > 1 zeroed out

The next section will consider single patterns, and introduce many of the fundamental objects we will be mapping to, such as partitions, fountains of coins and parallelogram polyominoes. In section 3 we start tackling pairs of patterns, which introduces many refinements of previous maps. We obtain maps to many different types of partitions for example, almost triangular partitions and Gorenstein partitions among them. Lastly in section 4 all remaining sets of patterns are tackled.

2 Single patterns

The sets $I_k(1)$ and $I_k(21)$ have no non-empty elements. Furthermore $|I_k(12)|$ is the characteristic function of the triangular numbers, listed in the OEIS [18] as A010054. This is because to avoid 12 the permutation must be decreasing and the decreasing permutation of length n has $\binom{n}{2}$ inversions. The single patterns of length 3 are $123, 132, 213, 231, 312$ and 321. To reduce the number of cases we consider symmetries on I_k, one of them being the *reverse complement* of a permutation, written out explicitly this is

$$\pi_1 \dots \pi_{n-1}\pi_n \mapsto (n+1-\pi_n)(n+1-\pi_{n-1}) \dots (n+1-\pi_1)$$

The number of inversions remains constant under reverse complement. Consider an inversion $i < j$ in π. The values $\pi_i > \pi_j$ get mapped to $n+1-\pi_i, n+1-\pi_j$ at indices $n+1-i, n+1-j$ in the image, so it cancels out. We also see that π is decomposable if and only if its reverse complement is, so the reverse complement is an involution on I_k for every k. It maps the pattern 132 to the pattern 213 and from this we see that $|I_k(213)| = |I_k(132)|$ so we only have to consider one of these patterns. In our case the only symmetries are inverse, reverse complement and the composition of those two maps. This is in contrast to when enumerating by size where both reverse and complement are symmetries unto themselves.

Using these symmetries we see that it suffices to consider 123, 132, 231 and 321. We start with 132 and we will make use of a well known bijection. The *inversion table* of a permutation $\pi = \pi_1 \dots \pi_n$ is the sequence $b_1 b_2 \dots b_n$ where b_i is the number of values after π_i in π that are smaller than π_i. The image under the map $\pi \mapsto b_1 \dots b_n$ is the set of all sequences such that the first value is $\leq n-1$, the next $\leq n-2$ and so on. We call such sequences *subdiagonal*. Furthermore, we call elements in a subdiagonal sequence that are equal to their maximum possible value *diagonal* elements.

First we recall a lemma due to Claesson, Jelínek and Steingrímsson that will be used a few times going forward.

Lemma 2 (Lemma 10, [9]). *The inversion table of a permutation is weakly decreasing if and only if the permutation avoids* 132.

Now we simply have to formally prove that adding the indecomposability criterion turns this lemma into a bijective map.

Theorem 3. $|I_k(132)|$ *is equal to the number of partitions of k, listed as* `A000041` *in the OEIS.*

Proof. Let $\pi \in I_k(132)$ and consider its inversion table, it is weakly decreasing by Lemma 2. The inversion table may have trailing zeroes, so let $\rho_1, \ldots, \rho_r$ be the values in the inversion table once the trailing zeroes have been removed. Clearly, the sum of the ρ_i is equal to k and hence ρ is a partition of k. We claim that $\pi \mapsto \rho$ is a bijection between $I_k(132)$ and the set of partitions of k.

We start by showing that the map $\pi \mapsto \rho$ is injective. As the mapping from permutations to inversion tables is bijective, we only have to establish that dropping the trailing zeroes in the inversion table is an injective operation. Suppose that we have two permutations π and τ that have the same inversion table up to trailing zeroes. Suppose τ has ℓ extra trailing zeroes compared to π. Adding a trailing zero to the inversion table of π results in the inversion table of $\pi \oplus 1$, as this new element can not be smaller than any previous element in the permutation due to other entries of the inversion table not changing. Then we must have $\tau = \pi \oplus (1 \oplus 1 \oplus \cdots \oplus 1)$ where the 1 has been repeated ℓ times, . But then τ is decomposable, so it is not in our domain. Thus the map is injective on the domain.

Finally we show surjectivity. Let $\rho = (\rho_1, \ldots, \rho_r)$ be a partition. We can append zeroes to ρ until it becomes subdiagonal. Let the total number of entries after appending zeroes be n, so $\rho = (\rho_1, \ldots, \rho_n)$. Now let $d := \max_{i=1}^{r} \rho_i - (n-i)$, because ρ is subdiagonal we have $d \leq 0$. We have $\rho_r \geq 1$ so if $d < 0$ then $n - r \geq 2$. Therefore if $d < 0$ we could have omitted one of the trailing zeroes and still maintained subdiagonality, so this is not the case. Then there must be some index $i \leq r$ that achieves the maximum $d = 0$, so $\rho_i = n - i > 0$. Treating ρ as an inversion table, let the corresponding permutation be $\pi = \pi_1 \ldots \pi_n$. We prove that π is indecomposable by contradiction, so assume the first $j < n$ elements of π form a component. The last value in a component is smaller than every value after it, so we have $\rho_j = 0$. Furthermore, $(\rho_1, \ldots, \rho_j)$ is subdiagonal and ρ_j must be followed by only zeroes as ρ is weakly decreasing. But this contradicts the existence of the index i, so π must be indecomposable. □

To work with 231-avoiding permutations we define the *skew-sum* of two permutations $\pi_1 \ldots \pi_n$ and $\tau_1 \ldots \tau_m$ as a permutation on $n + m$ elements given by $\pi \ominus \tau = (\pi_1 + m) \ldots (\pi_n + m)\tau_1 \ldots \tau_m$. We will say that π is *skew-decomposable* if $\pi = \tau_1 \ominus \tau_2$ for non-empty permutations τ_1 and τ_2. If π is not skew-decomposable we call π *skew-indecomposable.*

A *fountain of coins* is an arrangement of coins in rows such that the bottom row is full (that is, there are no "holes"), and such that each coin in a higher row rests on two coins in the row below [17], see example in Figure 1. The connection between the inversions of 231-avoiding permutations and fountains is not new, it is due to Brändén, Claesson and Steingrímsson [8]. But while their result is very

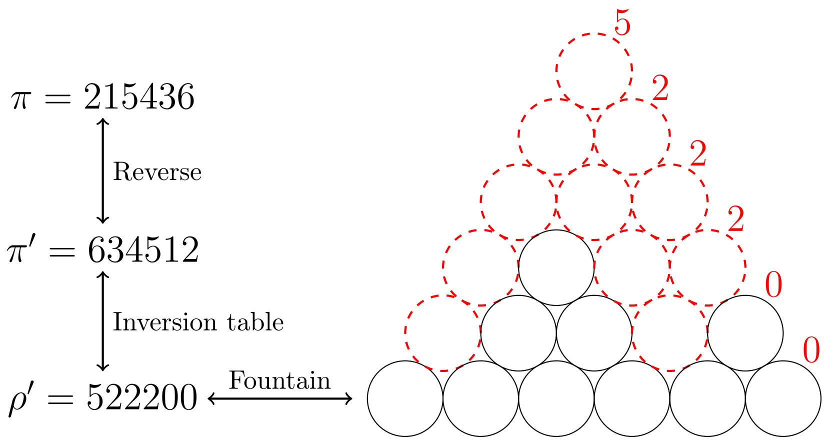

Figure 1: Example of bijective map in Theorem 4.

similar to ours, we need to incorporate indecomposability into our argument, so we need a separate proof.

Theorem 4. $|I_k(231)|$ *is equal to the number of fountains on* k *coins. The corresponding sequence is entry* **A005169** *in the OEIS.*

Proof. Let π' be the reverse of $\pi \in I_k(231)$ and n be the number of elements of π, this means every pair of indices in π is an inversion if and only if that pair is not an inversion in π'. Therefore π' has $\binom{n}{2} - k$ inversions. We note that the indecomposability of π is equivalent to the skew-indecomposability of π'. Since π' avoids 132 we get a partition ρ' of $\binom{n}{2} - k$ in the same manner as in Theorem 3. Note, however, that since π' might be decomposable we cannot recover the value of n from ρ' alone. We know $\rho'_i \leq n - i$ since ρ_i is given by the entries of an inversion table, which is subdiagonal. Suppose $\rho'_i = n - i$ for some i, since ρ' is weakly decreasing this means $\rho'_j \geq n - i$ for $j \leq i$. Then the first i elements in π' dominate the last $n - i$, meaning that π' is skew-decomposable. This contradicts the fact that π is indecomposable, so we must have $\rho'_i < n - i$ for all i.

We now define a bijective map from $I_k(132)$ to fountains with k coins. Consider a full fountain of coins, meaning it has $n - 1$ coins in the bottom row, $n - 2$ coins in the one above and so on, so $\binom{n}{2}$ coins in total. Let us consider diagonals going from a coin in the bottom row up and to the right. The i-th diagonal has $n - i$ coins. Since $\rho'_i < n - i$ we can remove the last ρ'_i coins from the i-th diagonal without altering the bottom row. This leaves us with an arrangement of $\binom{n}{2} - (\binom{n}{2} - k) = k$ coins. Let us show that this results in a fountain. The only way it could fail to is if we do not remove some coin c, but remove a coin out from under it. The coin below and to the left of c is in the same diagonal, and since we remove coins starting from the end, we never remove this coin before c. Thus we consider the coin below and to the right of c. It is in the

$$\ell = (0, 1, 1, 3, 3)$$

$$r = (3, 4, 4, 4, 6)$$

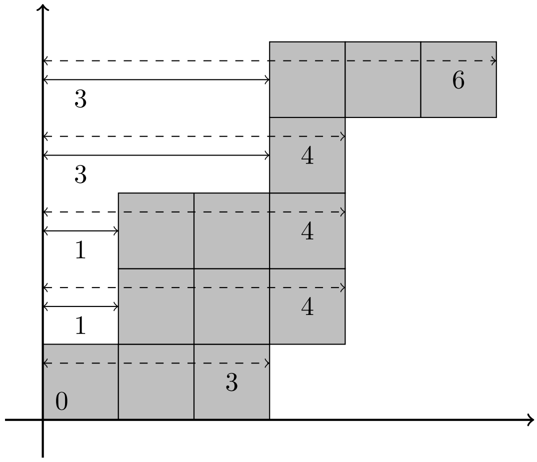

Figure 2: Example of parallelogram polyomino.

diagonal after the one c is in. For it to be removed we would have to remove strictly more elements from that diagonal than from the one c is in. But ρ' is a partition and is thus weakly decreasing, so this cannot happen. Thus a 231-avoiding permutation produces a fountain on k coins.

To show bijectivity we give the inverse. Let a fountain on k coins with $n-1$ coins in the bottom row be given. We can read out the number of coins missing from each diagonal to get a sequence of numbers. By the same argument as above this sequence is weakly decreasing. Therefore, this gives us a valid partition ρ' of $\binom{n}{2} - k$. Since we know n from the number of coins in the bottom row we can recover π' uniquely. Since ρ_i is the number of coins missing from diagonal i we must have $\rho'_i < n - i$. Thus the first i elements of π' can never dominate the last $n - i$, so π' is skew-indecomposable. Reversing π' to get π we then get an indecomposable permutation. Since π' is constructed from a partition it avoids 132 by Lemma 2, so π avoids 231, completing our proof. □

Enumerating 321-avoiding permutations by size and inversions (allowing decomposable permutations) has been investigated before by Barcucci, Del Lungo, Pergola and Pinzani [4]. They derive a generating function, but we will need a bijective proof for a later result, so we give a different proof. Our bijective map will map our permitations to *parallelogram polyominoes*. Following the definition given by Bender [5] we can define parallelogram polyominoes in terms of two weakly increasing non-negative sequences $(\ell_1, \ell_2, \ldots, \ell_n)$ and $(r_1, r_2, \ldots, r_n)$ satisfying $\ell_1 = 0$ and $\ell_i, \ell_{i+1} < r_i$ for all i. The polyomino itself can be viewed as consisting of cells in the segment $[\ell_i, r_i]$ at height i, see Figure 2.

Theorem 5. $|I_k(321)|$ *equals the number of parallelogram polyominoes with k cells, listed as* `A006958` *in the OEIS.*

Proof. We call π_k a *left-to-right maximum* of $\pi = \pi_1 \dots \pi_n$ if $\pi_i < \pi_k$ for all $i < k$. Suppose π has s left-to-right maxima at indices $i_1, \dots, i_s$. Consider the values in π that are not left-to-right maxima, let us call these values *short* going forward. If any two short values form an inversion, there must be a left-to-right maximum left of them that forms an occurrence of 321. Thus the short values of π are in increasing order, so π is uniquely determined by $i_1, \dots, i_s$ and $\pi_{i_1}, \dots, \pi_{i_s}$. We can also recover n since $\pi_{i_s} = n$.

We now show that taking $\ell \coloneqq (i_1-1, i_2-2, \dots, i_s-s)$ and $r \coloneqq (\pi_{i_1}-1, \dots, \pi_{i_s}-s)$ we get a parallelogram polyomino. The sequence $i_1, \dots, i_s$ is strictly increasing, so ℓ is weakly increasing. Similarly since $\pi_{i_1}, \dots, \pi_{i_s}$ are left-to-right maxima they are strictly increasing, so r is weakly increasing. The first value of a permutation must be a left-to-right maximum, so $i_1 = 1$ and so ℓ starts with a zero and is therefore also non-negative. Next we note that because π_{i_j} is a left-to-right maximum it must be greater than the $i_j - 1$ different values before it, so $\pi_{i_j} \geq i_j$. But if $\pi_{i_j} = i_j$ then the first i_j values of π are a permutation unto themselves, making π decomposable. Thus $i_j < \pi_{i_j}$ for every j, so we get $\ell_j < r_j$ by substracting j from both sides. This leaves only the condition $\ell_{j+1} < r_j$, which simplifies to $i_{j+1} \leq \pi_{i_j}$. This follows similarly since the π_{i_j} is the largest element out of the $i_{j+1} - 1$ elements preceding $\pi_{i_{j+1}}$ and they can't exactly be the elements 1 through $i_{j+1} - 1$ due to indecomposability.

Starting with a parallelogram polyomino we can trace the same steps backward to obtain a permutation in $I_k(321)$ by the same arguments.

All there is left to prove is that this map takes permutations with k inversions to polyominos with k cells. The number of cells in the polyomino defined by $(i_1 - 1, \dots, i_s - s)$ and $(\pi_{i_1} - 1, \dots, \pi_{i_s} - s)$ is

$$\pi_{i_1} - 1 + \dots + \pi_{i_s} - s - (i_1 - 1) - \dots - (i_s - s) = \pi_{i_1} - i_1 + \dots + \pi_{i_s} - i_s$$

Since the short values and left-to-right maxima are both in internally increasing order, the inversions are all between left-to-right maxima and short values. Because π_{i_j} is greater than the $i_j - 1$ elements before it and greater than $\pi_{i_j} - 1$ elements in total, it must be greater than $\pi_{i_j} - i_j$ elements that come after it. All inversions are of this form, so summing over these values gets us the total number of inversions, completing our proof. □

Interpreting parallelogram polyominoes as permutations yields a formula for efficiently computing new terms of `A006958`, taking $\mathcal{O}(k^2)$ time and space to compute $|I_k(321)|$. We need a small lemma first. We note that if a sequence is not subdiagonal, it can be made subdiagonal by appending some number of zeroes.

Lemma 6. *A sequence ρ of non-negative integers is the inversion table of a permutation in $I_k(321)$ (after adding the minimal required number of trailing zeroes) for some k if and only if the following conditions all hold. Whenever $\rho_i > 0$ and $\rho_j > 0$ with $i < j$ are separated by ℓ zeroes then $\rho_j \geq \rho_i - \ell$.*

Furthermore, $\rho_i > 0$ can be followed by at most $\rho_i - 1$ zeroes, aside from trailing zeroes, and $\rho_1 \neq 0$.

Proof. Let ρ be a sequence of non-negative integers. Let π be the corresponding permutation when ρ is considered as an inversion table after adding the minimum number of trailing zeroes, and assume $\pi \in I_k(321)$ for some k. By the same argument as in the proof of Theorem 5 above we can split $\pi \in I_k(321)$ into two subsequences, right-to-left minima and other values, such that both subsequences are increasing. We will call these other values *tall* values. If π_i is a right-to-left minimum it is smaller than every element after it, so the corresponding entry ρ_i in the inversion table must be 0. Since π is indecomposable this means $\rho_1 \neq 0$, so π_1 is tall. Next consider an index i such that π_i is tall, and so $\rho_i > 0$. Say it has ℓ zeroes following it. Those all correspond to left-to-right minima, and since π_i is tall they are all smaller than π_i. Thus π_i is greater than the next ℓ values, so $\rho_i \geq \ell$. But if π_i is greater than these values and no other values, π would be decomposable, so $\ell \leq \rho_i - 1$. By the same argument if $i < j$ with $\rho_i, \rho_j > 0$ are separated by ℓ zeroes then those zeroes all correspond to elements who form an inversion with π_i. Since the tall values are increasing $\pi_i < \pi_j$, so aside from these ℓ values j is in all the inversions i is in. Thus $\rho_j \geq \rho_i - \ell$, so ρ satisfies the required constraints.

For the other direction, let ρ be a sequence satisfying the above constraints. We define $\pi = \pi_1 \dots \pi_n$ as in the other case, which we want to show is in $I_k(321)$. It suffices to show that π is indecomposable and is the union of two increasing subsequences. We let those subsequences be the right-to-left minima and the tall values. The right-to-left minima are increasing by defintion, so we consider the tall values. The value π_i is a right-to-left minimum if and only if $\rho_i = 0$, so the tall values have $\rho_i > 0$. Let i be some index such that $\rho_i > 0$ and j be the next index after i such that $\rho_j > 0$. Suppose there are k zeroes between ρ_i and ρ_j, then our condition says that $\rho_j \geq \rho_i - k$. Every zero corresponds to a right-to-left minimum, and hence π_i is greater than all those elements. This means we cannot have $\pi_i \geq \pi_j$ since then $\rho_i \geq \rho_j + k + 1$, which is a contradiction. Thus $\pi_i < \pi_j$, so the tall values are in increasing order, which means π avoids 321.

Suppose π is decomposable with $\pi = \tau \oplus \sigma$. Since $\rho_1 \neq 0$ there must be some greatest index r such that $\rho_r > 0$ and π_r is in τ. Then τ_r must be followed by τ_r zeroes for τ to be a valid component, breaking the assumption on ρ since $\rho_r = \tau_r$ can have at most $\rho_r - 1$ zeroes after it. Therefore π is indecomposable which completes our proof. □

Theorem 7. *$|I_k(321)| = a_{k,1}$ where*

$$a_{n,m} = \begin{cases} 1 & \text{if } n = 0 \\ \sum_{i=1}^{n} a_{n-i,i} & \text{if } m = 1 \\ a_{n,m-1} + \sum_{i=m}^{n} a_{n-i,i} & \text{otherwise} \end{cases}$$

Proof. In this proof we will consider sequences equal if they differ only in their number of trailing zeroes, since by the same argument as in the proof of Theorem 3 we can uniquely recover the number of trailing zeroes due to the indecomposability condition. We show that $a_{n,1}$ enumerates sequences given by the description in Lemma 6. More specifically we show that $a_{n,m}$ enumerates prefixes of such sequences with sum n and whose next non-zero element must be at least m. Clearly if $n = 0$ there is only one such prefix, the empty prefix. For the other two cases we consider what element could be appended to our prefix to create a longer prefix. If $m = 1$ then our current prefix is either empty or ends with a 1, so we cannot append a zero. But we can append any positive integer $i \leq n$, which leaves $n - i$ of the sum left, and the next non-zero value must be at least i, giving the sum above. In the final case our prefix's last non-zero element is greater than 1, so we can append a zero. If we append a zero the next non-zero element can be allowed to be one less, so we get $a_{n,m-1}$ for this case. This also prevents us from having more than $x - 1$ zeroes after the value x as we create our prefix. The sum is the same as in the last case, completing our proof. □

Not only can this view help with computing new terms, but it also gives rise to new bijective correspondences. Consider fountains of coins where we only count coins in even rows. We can still place coins as we like, but when tallying the number we only count those in the bottom row, those 2 rows up, 4 rows up and so on. We will call such fountains with n counted coins *even fountains of size* n. Bala [3] has tackled the problem of enumerating such even fountains. He shows that they are equinumerous to parallelogram polyominoes by an algebraic argument, and leaves it open to find a bijective proof. Using $I_k(321)$, we give this bijective proof.

Theorem 8. *There exists a bijective map taking parallelogram polyominoes to even fountains of size* k*.*

Proof. Since Theorem 5 is proved by bijection, it suffices to map the inversion tables of elements in $I_k(321)$ bijectively to even fountains of size k.

Write out the fountain in the usual manner (see Figure 3), calling coins in even rows red and the ones in odd rows black for convenience. For each coin in the bottom row, going from left to right, we do the following procedure repeatedly:

- If we are on a red coin we remove it and move to the coin above and to the right. If there is no such coin we stop.
- If we are on a black coin we remove it and move to the coin above and to the right. If there is no such coin we move to the coin below and to the right instead.

Once done with a coin in the bottom row we write down the number of red coins removed during this procedure. This produces a sequence of numbers, which we then append a single zero to. Call the result ρ. We claim this procedure is our desired map.

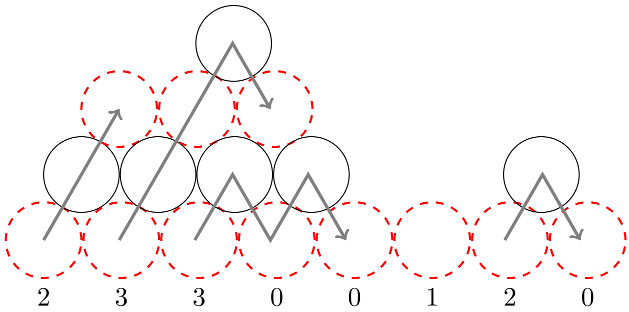

Figure 3: Example of bijective map in Theorem 8.

First we prove that ρ is the inversion table of an element in $I_k(321)$. It suffices to show that ρ fits the description given by Lemma 6. The path for the i-th coin from the right can move at most $i-1$ times upwards and at most $i-1$ times downwards. Since every other coin encountered is red, this means the resulting value can be at most i. Since we append a single zero afterwards ρ is subdiagonal.

For a value ρ_i to be zero, the path corresponding to the next non-zero value $\rho_j = x$ to the left must have touched the i-th coin in the bottom row. That path can touch at most $x-1$ coins in the bottom row aside from where it started, so a value of $x > 0$ can be followed by at most $x-1$ zeroes.

Next we prove that if the i-th coin and $(i+k+1)$-th coin are separated by k zeroes, where k can be zero, and $\rho_{i+k+1}, \rho_i > 0$ then $\rho_{i+k+1} \geq \rho_i - k$. We start with the case $k > 0$. In this case, the path corresponding to ρ_i must have touched k coins in the bottom row, then continued on from the coin corresponding to ρ_{i+k}. Since the path can never cross down below a red coin once above it, the path must have zig-zagged up and down alternatingly until reaching ρ_{i+k}. This means we can ignore what is to the left of ρ_{i+k} and consider ρ_{i+k} to have a value of $\rho_i - k$, and refer to the case where $k = 0$.

For $k = 0$ we have to prove $\rho_i \leq \rho_{i+1}$ for $\rho_i, \rho_{i+1} > 0$. Denote the path corresponding to ρ_i by P and the path corresponding to ρ_{i+1} by Q. We will show that for any red coin that P touches aside from the first, Q will touch the black coin below and to the right of it. This means Q touches at least $\rho_i - 1$ black coins, and thus $\rho_{i+1} \geq \rho_i - 1 + 1$, which is what we want to prove. We use induction along the number of steps of the path. For the base case we see that because $\rho_{i+1} \neq 0$, P must start with two upwards steps. Thus Q will start with an upward step as well, so the base case is true. Now for the inductive step P touches some red coin, and Q touches the black coin below and to the right. There are two cases, the first one being that P takes two upward steps next. Since P can never go below that red row again in this case, Q is free to take two upward steps as well, and thus the induction step is complete. In the other case P takes one upward step, then a downward step. Then P ends on a red coin not in the bottom row after these steps. This means Q is free to take a downward step, followed by an upward step since the coins make a valid fountain, completing the induction step. Thus both cases are done, and by induction $\rho_i \leq \rho_{i+1}$.

What remains is to show is that this map is bijective. We note that our map removes a path of coins from the fountain to construct the leftmost value, then applies itself to the remaining fountain to recursively create the remaining suffix. Since the empty fountain is uniquely mapped to the sequence of a single zero, we can use structural induction. We start with injectivity, in which case the problem reduces to showing that there is only one path of any given length you can add onto the left of a fountain. But this can be seen from the fact that as you lengthen the new path, the new coins have uniquely determined positions. A new black coin is always placed above and to the right, and the following red coin can be placed up and to the right if and only if the slot below and to the right is already occupied. Thus the new coins only have one place to go, so the path only has one valid shape for any given length, giving us injectivity.

This leaves only surjectivity. We note that a single path corresponds to a single value in the inversion table along with a possibly empty run of consecutive zeroes. By Lemma 6 we can see that removing such a sequence of values from the front of an inversion table of a permutation in $I_k(321)$ leaves us with an inversion table in $I_{k-\ell}(321)$ where ℓ is the non-zero value removed. Thus we can use the same structural induction as above. Let $\pi \in I_k(321)$, we will show that the lengths of the paths that can be prepended to the corresponding even fountain are exactly the non-zero values that can be prepended to its inversion table $\rho = (\rho_1, \dots, \rho_n)$, along with some number of zeroes. By Lemma 6 we never prepend a single zero, so our path is always non-empty. If we are prepending ℓ zeroes then our path will start by touching ℓ coins in the bottom row, so we can reduce to the case where our path is ℓ steps shorter and starts ℓ steps further to the right. Therefore by Lemma 6 we only have to show that any path of length at least 1 and at most equal to ρ_1 can be achieved. We can always remove two coins at a time from the end of a valid new path and have a smaller valid new path, so we only have to show the existence of a path of length ρ_1. Let P be the path corresponding to ρ_1. Above and to the left of each black coin in P place a red coin. If two adjacent such red coins are at different heights, a black coin will fit between them as that black coin lies to the left of P. Otherwise a black coin will fit above them as it lies above P. This way we construct a valid path of maximum length. Thus the map is surjective and therefore bijective, completing our proof. □

This leaves us only with $I_k(123)$, the only single pattern giving rise to a sequence not in the OEIS. In accordance with how we define decomposability for permutations, we will consider the sum of two subdiagonal sequences $\rho_1, \dots, \rho_k$ and $\tau_1, \dots, \tau_\ell$ to be $\rho_1, \dots, \rho_k, \tau_1 + k, \dots, \tau_\ell + k$. As a reminder we say that a sequence of integers of length n is *subdiagonal* if the first value is $\leq n-1$, the next $\leq n-2$ and so on. A subdiagonal sequence is then indecomposable if it cannot be written as the sum of two non-empty subdiagonal sequences.

Theorem 9. $|I_k(123)|$ *enumerates indecomposable subdiagonal sequences where non-diagonal elements are in decreasing order. Let*

$$c_{n,m,k} = \begin{cases} 0 & \text{if } n < 0 \text{ or } k < 0 \\ 1 & \text{if } n = k = 0 \\ c_{n-1,m,l-n+1} + \sum_{i=0}^{\min(n-2,m-1)} c_{n-1,i,k-i} & \text{otherwise} \end{cases}$$

Then the total number of permutations with k inversions (including decomposable ones) avoiding 123 *can be calculated as* $\sum_{n=0}^{k+1} c_{n,n,k}$.

To obtain the number of such permutations that are indecomposable, subtract the k-th coefficient of $\left(\sum_{i\geq 0} x^{i(i+1)/2}\right)^2$.

Proof. First we consider decomposable permutations avoiding 123. Let $\pi = \tau \oplus \sigma$ be such a permutation. Then $\tau_{|\tau|}\sigma_1$ is an ascent, so neither τ nor σ can contain any other ascents, and are thus decreasing. We also see that the sum of any two decreasing permutations is a decomposable permutation avoiding 123, so this is a complete description of such permutations. The generating function for decreasing permutations enumerated by inversions is $\sum_{i\geq 0} x^{i(i+1)/2}$ so the composition of two decreasing permutations is enumerated by $\left(\sum_{i\geq 0} x^{i(i+1)/2}\right)^2$.

We now consider permutations avoiding 123, including decomposable ones, and prove they are in bijection with subdiagonal sequences where non-diagonal elements are in decreasing order. Let $\pi = \pi_1 \dots \pi_n$ be a permutation with right-to-left maxima at all indices except $i_1, \dots, i_r$. By the same argument as above the values $\pi_{i_1}, \dots, \pi_{i_r}$ are decreasing. Let $\rho = (\rho_1, \dots, \rho_n)$ be the inversion table of π. Since a right-to-left maximum is greater than all the values that come afterwards, it corresponds to a diagonal value in ρ, so $i_1, \dots, i_r$ are the non-diagonal indices of ρ. Because $i_{j+1} > i_j$ and $\pi_{i_{j+1}} > \pi_{i_j}$ we get $\rho_{j_{i+1}} > \rho_{j_i}$. Thus the non-diagonal elements of the inversion table are in decreasing order.

For the other direction, let $\rho = (\rho_1, \dots, \rho_n)$ be a subdiagonal sequence with non-diagonal elements in decreasing order and let π be the permutation corresponding to ρ when it is considered as an inversion table. We consider the indices of the non-diagonal elements $j_1, \dots, j_s$. The other elements, the right-to-left maxima, are in decreasing order, so it suffices to prove that $\pi_{j_1}, \dots, \pi_{j_s}$ are also in decreasing order, since any occurrence of 123 would have to involve an ascent among these elements. By assumption we have $\rho_{j_k} > \rho_{j_{k+1}}$ and we also note that any elements between π_{j_k} and $\pi_{j_{k+1}}$ are right-to-left maxima. Since π_{j_k} is not a right-to-left maximum, it is smaller than all of the elements between π_{j_k} and $\pi_{j_{k+1}}$. Thus in order for $\rho_{j_k} > \rho_{j_{k+1}}$ to hold we must have $\pi_{j_k} > \pi_{j_{k+1}}$, so the sequence is decreasing.

All that is left to prove is the formula involving $c_{n,m,k}$. We see that $c_{n,m,k}$ enumerates subdiagonal sequences of n elements with sum k where the non-diagonal elements are decreasing and less than m. The three cases $n < 0, k < 0$

and $n = k = 0$ are straight-forward. The last case is covered by considering what the first element of our subdiagonal-sequence is, where $c_{n-1,m,l-n+1}$ is the case where we put a diagonal element first and the sum is when we put a non-diagonal value i in front. Since all elements of a subdiagonal sequence of n elements are less than n, we have that $c_{n,n,k}$ counts subdiagonal sequences of n elements with sum k where the non-diagonal elements are decreasing. Thus by Lemma 1 we only need to sum n from 0 to $k + 1$ to get our answer. □

3 Several patterns

We now investigate permutations avoiding several patterns. Some groups of patterns are restrictive enough to make all supersets of those patterns trivially determined. For example $|I_k(123, 321)|$ quickly decays to zero by the Erdős-Szekeres Theorem [13], making all supersets of $\{123, 321\}$ easy to determine. We have two more such trivial pairs of patterns.

Theorem 10. $|I_k(231, 321)| = 1$ *and the unique indecomposable permutation with* k *inversions avoiding these patterns is* $(k + 1)12 \dots k$*.*

Proof. Let $\pi \in I_k(321, 231)$. Since $\pi = \pi_1 \dots \pi_n$ avoids 321 it is composed of two increasing sequences, the left-to-right maxima and the short values. Suppose $\pi_1 \neq n$. This means we have some inversion (i, j) of π where $i \neq 1$. Thus π_i is a left-to-right maximum and π_j is short. Now we have $\pi_j > \pi_1$ since otherwise $1, i, j$ would be an occurrence of 231. Therefore π is decomposable into the permutation consisting of the first π_1 values and then the rest. This cannot be, so the assumption $\pi_1 \neq n$ is false, i.e. $\pi_1 = n$. But then since π avoids 321 there can be no inversion after π_1. Thus π is exactly $n123 \dots (n-1)$. We also see that this avoids 321 and 231, completing the proof. □

Theorem 11. $|I_k(231, 312)| = |I_k(12)|$.

Proof. Consider the maximum element π_m of $\pi \in I_k(231, 312)$. Since $\pi = \pi_1 \dots \pi_n$ is indecomposable, π_m cannot be the last element. Now any element π_i after π_{m+1} must be smaller than π_{m+1} since otherwise $m, m+1, i$ would be an occurrence of 312. Thus, the values after π_m must be in decreasing order. Suppose π_i comes before π_m, then it must be smaller than π_n since otherwise i, m, n would be an occurrence of 231. But this means all values before π_m are smaller than the decreasing sequence $\pi_m, \dots, \pi_n$, which contradicts the indecomposability of π. Thus there is no such π_i and π must be a decreasing sequence, which avoids 12. We also see that any decreasing sequence avoids 231 and 312, completing the proof. □

All supersets of $\{231, 321\}$ and $\{231, 312\}$ are now trivial to determine, as they all equal to one of the two cases above or decay to zero.

Up to symmetries the only pairs of patterns left to investigate are $\{123, 231\}$ and pairs containing 132.

Theorem 12. $|I_k(123, 231)|$ *enumerates fountains of k coins where the missing coins with respect to a full triangular fountain form a rectangle (removing no coins counts as a rectangle). The generating function is given by*

$$\sum_{i\geq 1} x^{\binom{i}{2}} + \sum_{i\geq 1}\sum_{j\geq 1}\sum_{\ell=0}^{\min(i,j)-1} x^{\binom{i+1}{2}+\binom{j+1}{2}-\binom{\ell+1}{2}}$$

Proof. Let $\pi \in I_k(123, 231)$. Like in the proof of Theorem 4 we can reverse π to get π' which corresponds to a partition ρ. This means π' avoids 321, so by the results for $I_k(123)$ we know that the non-zero entries in the inversion table of π' are in weakly increasing order. π' also avoids 132 so the inversion table of π' is weakly decreasing as well by Lemma 2. But therefore, aside from trailing zeroes, the entries of ρ are weakly increasing and weakly decreasing, and are thus fixed. Thus the rows where coins are removed are contiguous and the coins removed in those diagonals is fixed, forming a removed rectangle. For the other direction we see that if anything is removed, it has to include the top coin, so ρ is fixed aside from trailing zeroes. Therefore π' avoids 132 by Lemma 2 and the non-zero entries of ρ are weakly increasing, so π' avoids 321 as well. This means π avoids 123 and 231, completing our proof of the first claim.

We obtain the generating function by first splitting into cases based on whether the fountain is full or not. If we remove nothing, we are left with full fountains which are enumerated by the first sum in our result. If we remove a non-empty rectangle we are left with two peaks which each form a full fountain with $i > 0$ coins and $j > 0$ coins at the base respectively. When these fountains overlap the overlap takes the shape of another full fountain with ℓ coins at the base. The overlap cannot contain either fountain so $\ell < i, j$. Under these constraints any choice of i, j, ℓ will give a valid fountain corresponding to an element in $I_k(123, 231)$ with $\binom{i+1}{2} + \binom{j+1}{2} - \binom{\ell+1}{2}$ coins. □

This leaves us with four pairs, pairing 132 with any of the patterns 123, 213, 231 and 321. We now tackle them in that order. To deal with the first of them we have to introduce a new kind of partition.

Almost triangular partitions of k are partitions $\rho = (\rho_1, \ldots, \rho_r)$ of k such that each ρ_i is either i or $i - 1$, allowing $\rho_1 = 0$ for convenience. We note however that $0 + 1 + \cdots + r$ and $1 + \cdots + r$ are the same partition, so since we allow $\rho_1 = 0$ we must forbid $\rho_i = i - 1$ being true for all i to avoid duplicates.

Theorem 13. $|I_k(132, 123)|$ *enumerates the Pascal triangle with the first column removed, the corresponding sequence is entry* `A135278` *in the OEIS. Its generating function is*

$$\sum_{n\geq 1} x^{(n-2)(n+1)/2} \left((x + 1)^n - 1\right)$$

.

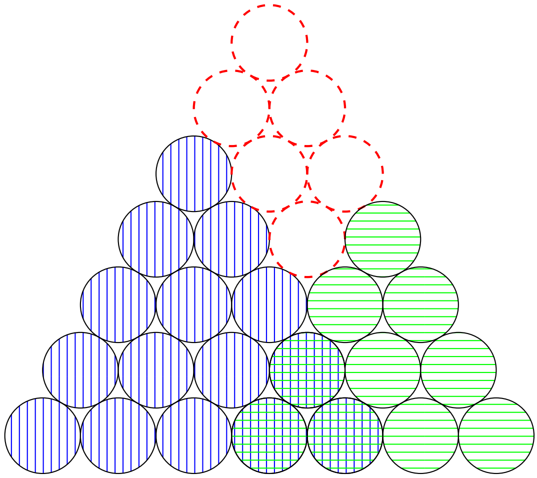

Figure 4: Example of decomposition into smaller fountains.

Proof. We first show that almost triangular partitions of k are enumerated by the Pascal triangle with the first column removed. Suppose we have an almost triangular partition $\rho = (\rho_1, \ldots, \rho_r)$ of k. The right hand side is at most $1 + \cdots + r = r(r+1)/2$ and at least $0 + 1 + \cdots + (r-1) = r(r-1)/2$. Therefore the sum is at least $r(r-1)/2 + 1$. This means that for any given k, the r is uniquely determined as the index of the next triangular number after or equal to k. To create an almost triangular partition we can then start with the least possible value for each of the r summands, then choose $k - r(r-1)/2$ of them to be increased by one. Since we must always increase at least one summand there will be $\binom{r}{k-r(r-1)/2}$ ways to do this, which walks through the Pascal triangle row by row without the first column as k increases.

Now we show that $I_k(132, 123)$ corresponds bijectively to almost triangular partitions. Let $\pi \in I_k(132, 123)$. For some index i suppose at least two values after π_i are not less than π_i, say π_j, π_k with $j < k$. Then either $\pi_j > \pi_k$ and i, j, k is an occurrence of 132 or $\pi_j < \pi_k$ and i, j, k is an occurrence of 123. Thus each π_i is greater than all of the values after it or all but one, proving that ρ is almost triangular. For the other direction we now assume we have $\pi \in I_k(132)$ such that ρ is almost triangular. Then each value in π must be greater than every value after it but one. Thus we can have no occurrence of 123 since that requires a value to have two greater values after it.

A row of the Pascal triangle has generating function $(x+1)^n$, so to delete the first column we simply subtract 1. Simple counting gets us that this row will start at $(n-2)(n+1)/2$, so by simply shifting over each row the appropriate amount and summing them together shows that this generating function corresponds to our description. ☐

Our next result involves a kind of partition called a Gorenstein partition. *Goren-*

$$24 = 7 + 4 + 4 + 4 + 2 + 2 + 1$$

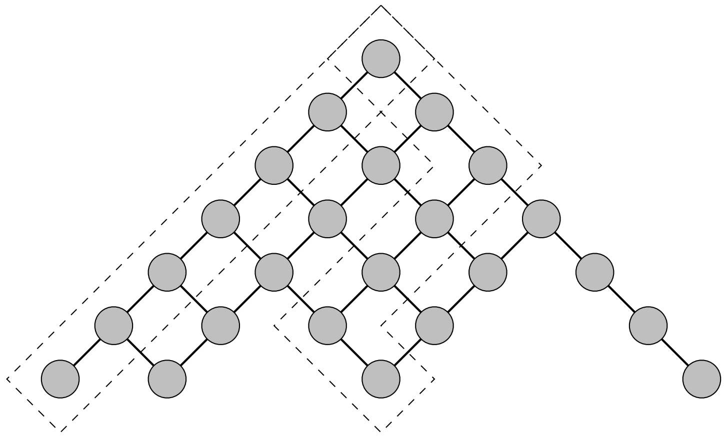

Figure 5: Hasse/Ferrers diagram of a Gorenstein partition.

stein partitions are partitions whose corresponding Schubert variety has a Gorenstein homogeneous coordinate ring, see Svanes [21] and Stanley [20]. The terminology appears to be due to Sloane [18] under A117629 however. By Theorem 5.4 in Stanley's work [20] the definition is equivalent to partitions whose maximal chains are all of the same size when regarded as order ideals of $\{1, 2, \dots\} \times \{1, 2, \dots\}$ under the product order. See Figure 5 which shows a combined Hasse and Ferrers diagram for such a partition and the corresponding partial order. The figure is rotated by 45° compared to a normal Ferrers diagram to conform to Hasse diagram conventions. The maximal chains all have size seven in this example, and a couple of them have been highlighted in the figure. This definition is rather unwieldy for our purposes however, so we first translate this condition.

Lemma 14. *A partition ρ is Gorenstein if and only if $\rho_i + i$ is constant across indices i that satisfy $\rho_i \neq \rho_{i+1}$.*

Proof. Let $\rho_1, \dots, \rho_r$ be a partition of n. Let us assume j is the maximal index such that $\rho_j = \rho_1$. Since $\rho_j \neq \rho_{j+1}$ the chain from $(1, 1)$ to (ρ_1, j) is maximal since neither the point right of it or below it are in the order ideal corresponding to the partition. Then this maximal chain length must be $\rho_1 + j - 1$. The same argument holds for any ρ_i such that $\rho_i \neq \rho_{i+1}$ and the chain length they give is $\rho_i + i - 1$. However, if $\rho_i = \rho_{i+1}$ then one can always go further, making the chain from $(1, 1)$ to (ρ_i, i) not maximal. Therefore all maximal chains are of the same size if and only if $\rho_i + i$ is constant for all i such that $\rho_i \neq \rho_{i+1}$. □

Theorem 15. *$|I_k(132, 213)|$ is the number of Gorenstein partitions of k. The corresponding sequence is entry* `A117629` *in the OEIS.*

Proof. Let $\pi \in I_k(132)$ and ρ be the partition corresponding to its inversion table as given by Theorem 3. We wish to show that if $\pi \in I_k(213)$ then ρ is Gorenstein. Suppose we have two indices $i < j$ such that $\rho_i > \rho_{i+1}$ and $\rho_j > \rho_{j+1}$. For $\rho_i > \rho_{i+1}$ to hold we must have $\pi_i > \pi_{i+1}$ and similarly $\pi_j > \pi_{j+1}$. Then there cannot be any index $i < k < j$ such that $\pi_i < \pi_k$ since then $i, i+1, k$ would be an occurrence of 213. Neither can there be an index $k > j$ such that $\pi_j < \pi_k$ since then $j, j+1, k$ would be an occurrence 213. Thus any values contributing to the inversion count ρ_i but not ρ_j must occur between the indices i and j, and conversely every value there between must contribute to that difference. Thus we arrive at $\rho_i - \rho_j = j - i$ or equivalently $\rho_i + i = \rho_j + j$. Thus ρ is Gorenstein by Lemma 14.

For the other direction assume ρ is Gorenstein. If ρ contains only one unique non-zero value then π must be of the form $k, k+1, \ldots, n, 1, 2, \ldots, k-1$ for some k, n. In this case π clearly has no occurrence of 213, so we can assume ρ has at least two different non-zero values. We proceed by contradiction so assume $\pi = \pi_1 \ldots \pi_n$ has an occurrence of 213 at indices a, b, c. Without loss of generality we can choose the values a, b, c such that a is maximal among all such choices. Next we show we can assume $b = a+1$ by showing that if $b > a+1$ there exists a smaller valid choice of b. Suppose $b \neq a+1$ so there is some $a < k < b$. We cannot have $\pi_k > \pi_c$ since then a, k, c would be an occurrence of 132. If $\pi_k < \pi_a$ then we can replace b with k. The only case left then is $\pi_a < \pi_k < \pi_c$, but in that case we can replace a with k which contradicts the maximality of a. Thus we can assume $b = a+1$ and still have a maximal. Now $\rho_a > \rho_{a+1}$. Since ρ has more than one non-zero value there must be some other index i such that $\rho_i > \rho_{i+1}$, where we define $\rho_j = 0$ for $j > |\rho|$. We now split into two cases.

We consider the case $a < i$ and can assume this i to be minimal among such i. This means $\rho_{a+1} = \rho_i$, so $\pi_{a+1}, \ldots, \pi_i$ is increasing. Suppose that $\pi_a < \pi_i$. Then the only inversions contributing to ρ_a but not ρ_i are of the form (a, j) with $j < i$. But there can be at most $i - a - 1$ such inversions and there are only $i - a - 1$ indices between i and a. But then $\rho_a - \rho_i < i - a$, which contradicts the fact that ρ is Gorenstein by Lemma 14. Thus we can assume $\pi_a > \pi_i$. Hence the values $\pi_{a+1}, \ldots, \pi_i$ are all less than π_a, so we must have $i < c$. But then $i, i+1, c$ is an occurrence of 213 with a larger value of a, which gives us a contradiction that completes this case.

Now we assume there is no such $i > a$, so ρ_a is the last non-zero value which makes $\rho_{a+1} = 0$. This means $\pi_{a+1}, \ldots, \pi_n$ is increasing, so we can assume $c = n$. Since π is indecomposable there is some index $j < a$ such that $\pi_j > \pi_n$. Then $\pi_j > \pi_a$ and $j < a$ so we must have $\rho_a < \rho_j$. We can now let j be the maximum index such that $\pi_j > \pi_n$ and $\rho_j > \rho_a$. Then $\rho_j > \rho_{j+1}$ since otherwise $j+1$ would be a valid greater index. Now since $j < a$ there is some minimum index ℓ such that $\rho_j > \rho_\ell > \rho_{\ell+1}$. Since ℓ was not a valid choice for j we must have $\pi_j > \pi_n > \pi_\ell$. Since ℓ is minimal we have that $\rho_{j+1} = \rho_\ell$, so $\pi_{j+1}, \ldots, \pi_\ell$ is increasing. Thus π_j is greater than all of these values. Therefore we get that $\rho_j - \rho_\ell > \ell - j$ which contradicts the fact that ρ is Gorenstein by Lemma 14.

This completes our proof. ☐

Theorem 16. *Let $\mu \vDash s$ denote that μ is a composition of s. $|I_k(132, 213)|$ has generating function*

$$\sum_{s\geq 0} \sum_{\mu \vDash s, |\mu| \neq 1} x^{\binom{s}{2} - \sum_{m\in\mu} \binom{m}{2}}$$

Thus, $|I_k(132, 213)|$ also enumerates finite sequences of positive integers of length > 1 such that k equals the second elementary symmetric function of the values of the sequence, as noted in the OEIS entry `A117629`.

Proof. For a Gorenstein parititon ρ let i be an index such that $\rho_i \neq \rho_{i+1}$, we then define $\rho_i + i$ as the *diagonal constant* of ρ, as it is the same for all such i by Lemma 14. We consider some fixed index s in the generating function sum. The case $s = 0$ is obvious, so we can assume $s > 0$. Let ρ be a Gorenstein partition with diagonal constant s. Let $i_1 < \cdots < i_r$ be the indices of ρ such that $\rho_{i_j} \neq \rho_{i_j+1}$. We then prepend $i_0 = 0$ and append $i_{r+1} = s$ to the sequence. Then $\sum_{j=1}^{r+1} i_j - i_{j-1} = s$, so $\mu = i_1 - i_0, i_2 - i_1, \ldots, i_{r+1} - i_r$ is a composition of s with at least two terms. We also see that any such composition with at least two terms can be transformed back into a valid Gorenstein partition by reversing the steps.

Next we consider what the sum of ρ is in terms of s and μ. We compare ρ to the triangular partition $\sigma = s-1, s-2, \ldots, 1$ which has sum $\binom{s}{2}$ and is Gorenstein. We see that $\sigma_1 + 1 = s$ so $\sigma_{i_j} + i_j = \rho_{i_j} + i_j$ for all j since $\sigma_\ell \neq \sigma_{\ell+1}$ for all ℓ. If $\mu_j = x$ this means ρ contains x equal consecutive values ending in $\rho_{i_j} = \sigma_{i_j}$. Thus the sum of those x values is $\binom{x}{2}$ lower than the sum of the corresponding values in σ. Taking the sum over all the values of μ we get that the sum of ρ is $\binom{s}{2} - \sum_{m\in\mu} \binom{m}{2}$, which completes our proof for the generating function.

The relation to symmetric functions can be seen by expanding the exponent

$$\binom{\sum_{m\in\mu} m}{2} - \sum_{m\in\mu} \binom{m}{2}$$

and seeing that the pure terms m, m^2 cancel and the mixed terms add to cancel the 2 in the denominator of the binomial. ☐

The generating function $\sum_{s\geq 0} \sum_{\mu \vDash s, |\mu| \neq 1} x^{\binom{s}{2} - \sum_{m\in\mu} \binom{m}{2}}$ is not useful for actually computing new terms in the sequence, but we can use the following recurrence instead. By ignoring all but the first and last $\sqrt{n}$ summands in the recurrence below, as they are zero, we can compute the n-th value in $\mathcal{O}(n^{2.5})$ time and $\mathcal{O}(n^2)$ space.

Theorem 17. *The number of Gorenstein partitions with sum n, and thus also* *the number of elements in* $|I_n(132, 213)|$, *is given by the sum* $\sum_{d=0}^{n} f(n, d)$ *where*

$$f(n, d) = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n = 0 \\ \sum_{k=1}^{d} f(n - k(d + 1 - k), d - k) & \text{otherwise} \end{cases}$$

Proof. We claim that $f(n, d)$ equals the number of Gorenstein partitions with sum n and diagonal constant $d + 1$, from which the result would follow.

The cases with $n \leq 0$ are trivial, so we focus on the last one. Say we have some Gorenstein partition $\rho_1, \ldots, \rho_r$ with sum n and diagonal constant $d + 1$. Then there is some maximal index k such that $\rho_1 = \rho_k$ and $\rho_k \neq \rho_{k+1}$. Then we must have $\rho_k + k = d + 1$. Thus $\rho_1 = \rho_2 = \cdots = \rho_k = d + 1 - k$, so the sum of these entries is $k(d + 1 - k)$. If we consider the remaining entries unto themselves, assuming there are any, they must form another Gorenstein partition with sum $n - k(d + 1 - k)$ and diagonal constant $d + 1 - k$, since all entries are shifted left by k. Summing over all such k we get the formula above. □

Theorem 18. $|I_k(132, 231)|$ *is equal to the number of partitions on* k *elements into distinct parts, the corresponding sequence is entry* `A000009` *in the OEIS.*

Proof. Let π be a permutation and ρ be the partition corresponding to its inversion table as given by Theorem 3. It suffices to show that an occurrence of 231 in π is equivalent to ρ containing two equal values. Since ρ is written in decreasing order this is equivalent to ρ containing two equal adjacent values, say $\rho_j = \rho_{j+1}$. We cannot have $\pi_j > \pi_{j+1}$ since then we would have $\rho_j > \rho_{j+1}$, so $\pi_j < \pi_{j+1}$. Since $\rho_{j+1} > 0$ we must have some ℓ such that $\pi_{j+1} > \pi_\ell$ and $j + 1 < \ell$. But then $j, j + 1, \ell$ is an occurrence of the pattern 231.

For the other direction suppose we have a 132-avoiding permutation $\pi = \pi_1 \ldots \pi_n$ that has an occurrence of the pattern 231 at the indices t, u, v. Without loss of generality take t to be the maximal valid index t for the given u, v. Suppose there is an index $t < i < u$ such that $\pi_t > \pi_i$. Since t is maximal this means $\pi_i < \pi_v$, otherwise i would be a valid choice for the index t. But then i, u, v is an occurrence of the pattern 132, which cannot be. Thus there is no such index i. Therefore any inversion (t, j) gives rise to an inversion (u, j). This means $\rho_u \leq \rho_t$. But since the inversion table must be weakly decreasing $\rho_u \geq \rho_t$, so $\rho_u = \rho_t$ which completes our proof. □

Theorem 19. $|I_k(132, 321)|$ *is equal to the number of partitions on* k *elements into equal values. This is in turn equal to the number of divisors of* k*. The corresponding sequence is entry* `A000005` *in the OEIS.*

Proof. In a similar manner to the previous proof, all we have to show is the equivalence of the partition containing two different values and the permutation having an occurrence of the pattern 321. Suppose first that our partition $\rho = (\rho_1, \ldots, \rho_r)$ contains two unequal values and let $\pi = \pi_1 \ldots \pi_n$ be the permutation that has ρ has an inversion table after appending the minimum number of required zeros to make it subdiagonal. Since the partition is written in decreasing order we can without loss of generality assume these values are adjacent, say $\rho_j \neq \rho_{j+1}$. Since the partition is weakly decreasing we must then have $\rho_j > \rho_{j+1}$. This means $\pi_j > \pi_{j+1}$. Since $\rho_{j+1} > 0$ we must then have some ℓ such that $\pi_{j+1} > \pi_\ell$ and $j+1 < \ell$. But then $j, j+1, \ell$ is an occurrence of the pattern 321.

For the other direction we assume we have a 132-avoiding permutation $\pi = \pi_1 \dots \pi_n$ that has an occurrence of the pattern 321 at t, u, v. We let $\rho = (\rho_1, \dots, \rho_r)$ be the non-zero entries of the inversion table of π. Then any inversion (u, w) gives us an inversion (t, w) so $\rho_t \geq \rho_u$. Furthermore $\pi_t > \pi_u$ so $\rho_t > \rho_u$. Since $\pi_v < \pi_u$ we also have $\rho_u > 0$. Thus the partition contains two different positive values, completing our proof. □

4 More than two patterns

Most of the remaining pattern combinations are trivially deduced as some subset of the patterns forces the sequence to die out or contain only very specific permutations, as noted before. We consider here the complement of those cases.

Theorem 20. $|I_k(123, 132, 231)| = 1$.

Proof. We know that $I_k(123, 132)$ gives us almost triangular partitions and that $I_k(132, 231)$ gives us partitions with distinct parts, so this gives us almost triangular partitions with distinct parts. The number of values in an almost triangular partition of k is uniquely determined as noted in Theorem 13. Suppose we start with the partition $0, 1, 2, \dots, r$. Incrementing any subset of $\ell > 0$ values by one produces a duplicate, except if we increment exactly the last ℓ values. This is because if we do not choose the last ℓ values there will be some value that is increased by one, followed by a value that is unchanged, producing a duplicate value. Thus if $1 + 2 + \cdots + r = k - \ell$ we have to increment the last ℓ values, producing a unique partition. □

Theorem 21. $|I_k(123, 132, 213)|$ *enumerates the Pascal triangle, read by diagonals, offset by two elements. In other words, it reads the binomials* $\binom{n}{m}$ *in increasing order by the sum* $n + m$*, with each set being read in increasing order by* n*, starting at* $\binom{1}{1}$ *for* $k = 0$*. Aside from* $\binom{1}{1}$ *it walks over all values with* $n \geq 2$*, including zeroes, see Figure 6. Furthermore its generating function can be written as*

$$1 + x + \sum_{d \geq 3} x^{1+\binom{d-1}{2}} \sum_{n=2}^{d} \binom{n}{d-n} x^{n-2}$$

Proof. We know that $I_k(123, 132)$ gives us almost triangular partitions and that $I_k(132, 213)$ gives us Gorenstein partitions, so the intersection gives us almost

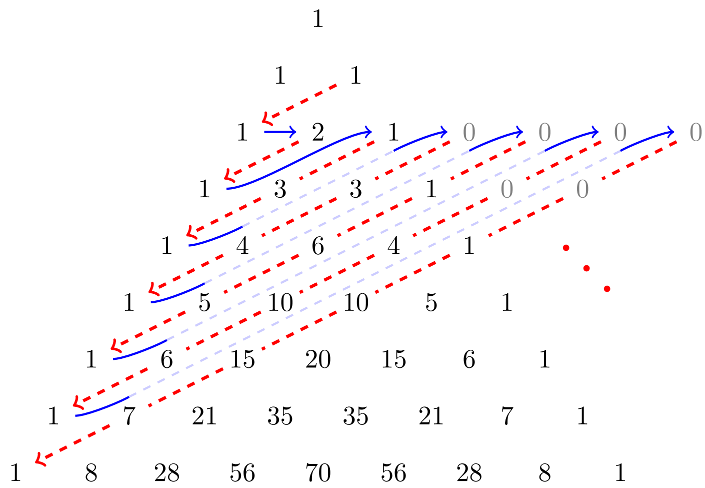

Figure 6: Enumeration order of Pascal Triangle in Theorem 21.

triangular Gorenstein partitions. We show that this is equivalent to considering all partitions given by the following construction. Starting with $r-1, \dots, 1, 0$ we construct our partition by selecting some subset of the partition that contains at least one out of every two adjacent elements, and then increment every selected value by 1.

This construction clearly gives us almost triangular partitions, so it suffices to show that failing to increment two adjacent values is equivalent to the partition not being Gorenstein. Assume then we have two indices $a, a+1$ that do not get incremented. There is always at least one value that gets incremented, so we have some index b that gets incremented. Now $\rho_b = r-b+1$ and depending on whether or not $b+1$ gets incremented we have either $\rho_{b+1} = r-b-1$ or $\rho_{b+1} = r-b$. In either case we have $\rho_b \neq \rho_{b+1}$. Furthermore $\rho_a = r-a \neq r-a-1 = \rho_{a+1}$. Since $\rho_a + a = r \neq r+1 = \rho_b + b$ we get that ρ is not Gorenstein by Lemma 14.

Now we assume ρ is not Gorenstein and wish to show there are two adjacent indices that fail to get incremented. By Lemma 14 we have indices $a < b$ such that $\rho_a \neq \rho_{a+1}$, $\rho_b \neq \rho_{b+1}$ and $\rho_a + a \neq \rho_b + b$. If we had incremented index $a+1$ but not a we would have $\rho_a = \rho_{a+1}$, so we get that if index a was not incremented, index $a+1$ was not either. But if neither of them was incremented, we already have two adjacent elements neither of which was incremented, which completes this case. Thus we can assume this is not the case, so index a was incremented. The same applies to b. But then $\rho_a + a = \rho_b + b$, which is a contradiction, so we are done.

Lastly we show that this construction gives us the claimed enumeration. Let r be the number of elements in the partition, which then also determines how many elements we have to increment, given by $s = k - r(r-1)/2$. We can now view the construction as choosing the sizes of the consecutive runs of elements we increment with one non-incremented element between each run. The first and last run can be empty, but the others must have at least one element, so this is given by

$$[x^s]\left(\frac{1}{1-x}\left(\frac{x}{1-x}\right)^{r-s-1}\frac{1}{1-x}\right) = [x^{2s+1-r}]\frac{1}{(1-x)^{r-s+1}} = \binom{s+1}{r-s+1}$$

We see now that for r fixed, the sum of the arguments in the binomial coefficient is fixed. Furthermore as s increases the upper argument increases as well, so this enumerates the binomial coefficients in the order claimed above. Furthermore for $k = 0$ we have $r = s = 0$, giving the coefficient $\binom{1}{1}$, so the offset is correct as well.

We take $k = 0, 1$ aside especially in the generating function with the $1 + x$ at the start. We let d be the sum of the arguments to the binomial, which is at least 3 for $k \geq 1$. Then we need to sum over all such binomials which appear in the generating function. For $k \geq 2$ we increment some element in the procedure above so $s \geq 1$, so the upper argument is always at least 2 and at most d. Thus the generating function for one fixed d is the inner summand given in the generating function above. Finally we just have to count how many entries appear before a given value of d starts, which comes out to $1 + \binom{d-1}{2}$, completing our proof. □

Theorem 22. *$|I_k(132, 213, 231)|$ is equal to the number of odd divisors of k, the corresponding sequence is entry `A001227` in the OEIS.*

Proof. The number of odd divisors of k is the same as the number of ways to write k as the difference of two triangular numbers, as noted by Mason [16]. We know that $I_k(132, 231)$ gives us partitions with distinct parts and $I_k(132, 213)$ gives us Gorenstein partitions, so the intersection gives us Gorenstein partitions with distinct parts.

Using our characterisation of Gorenstein partitions we note that since $\rho_i \neq \rho_{i+1}$ for all i, we must have that $\rho_i + i$ is constant across the entire partition. Thus if $\rho_i, \rho_{i+1} > 0$ we must have $\rho_i = \rho_{i+1} + 1$ since the partition is weakly decreasing. Therefore Gorenstein partitions with distinct parts are equivalent to writing k as a sum of a set of consecutive run of integers. This is equivalent to writing k as the difference of two triangular numbers by considering the maximum included integer in the sum and minimum excluded integer in the sum. □

Theorem 23. $|I_k(132, 213, 321)| = |I_k(132, 321)|$.

Proof. Inclusion in one direction is obvious, so we simply show that an occurrence of 213 in π implies an occurrence of either 132 or 231. Let a, b, c be the indices at which 213 occurs in π. If $\pi_k > \pi_c$ for all $k > c$ then π would be decomposable, so there exists some $k > c$ such that $\pi_k < \pi_c$. Now if $\pi_k > \pi_b$ then b, c, k would be an occurrence of 132, in which case we are done. Thus we can assume $\pi_k < \pi_b$. But then b, c, k is an occurrence of 231, which completes our proof. □

Theorem 24. $|I_k(123, 132, 213, 231)|$ *enumerates the Pascal triangle, except all values* > 1 *are replaced by* 0*. The corresponding sequence is entry* `A103451` *in the OEIS and has the generating function*

$$\sum_{i \geq 0} x^{i(i+1)/2} + x^{(i+1)(i+4)/2}$$

Proof. We already know the unique partition $I_k(123, 132, 231)$ gives us for a given k. Thus we only have to check for what k this is in $I_k(132, 213, 231)$, i.e. when it is a run of consecutive integers. The smallest non-zero value in an almost triangular partition is always 1 or 2, so this is a consecutive run of integers starting with 1 or 2. That always gets us a sum of the form $n(n+1)/2$ or $n(n+1)/2 - 1$. This is exactly the indices of ones in the Pascal triangle when read row by row, completing our proof. The generating function follows directly as the first summand enumerates the indices of the leftmost column in the Pascal triangle, including zero, and the other the rightmost column, excluding zero. ☐

Acknowledgements

The algorithm to generate permutations with a fixed number of inversions by Effler and Ruskey [12] was used to generate elements of all the sequences above, which helped tremendously in finding the formulas and other results in this paper.

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Paper IV

Pattern avoiding permutations as walks

Atli Fannar Franklín

23 December 2025

Abstract

The Stanley-Wilf limit of the pattern 1324 is known to lie between 10.271 and 13.5. We obtain lower bounds on this limit by encoding permutations as walks in directed graphs: building a permutation by successive insertion of maxima corresponds to traversing edges, and the growth rate of walks equals the spectral radius of the adjacency matrix. For 1324, this graph is too large for direct computation, so we pass to a quotient graph with weighted edges. Conditional on a natural conjecture, this yields a lower bound of 10.418.

1 Introduction

A permutation π contains a pattern τ if some subsequence of π has the same relative order as τ, otherwise π avoids τ. For example $\pi = 465213$ contains the pattern $\tau = 132$ as the relative order of the elements 465 is the same as 132. If we swap the first and third elements such that $\pi = 564213$ then π avoids τ. We write $\mathrm{Av}_n(\tau)$ for the set of τ-avoiding permutations of length n. The Stanley-Wilf limit of τ is

$$L(\tau) := \lim_{n\to\infty} \sqrt[n]{|\mathrm{Av}_n(\tau)|}$$

By Arratia, Marcus and Tardos [12, 2], this limit exists for every pattern τ.

For length 3 patterns, all Stanley-Wilf limits equal 4, the growth rate of the Catalan numbers. For length 4, all patterns have been exactly enumerated and their limits been found [5, 9] except one: up to symmetry, 1324 is the only length 4 pattern whose Stanley-Wilf limit remains unknown. The best current bounds are $10.271 \leq L(1324) \leq 13.5$ [4], with numerical evidence suggesting a value near 11.6 [7].

We develop a method for bounding Stanley-Wilf limits from below by encoding pattern-avoiding permutations as walks in directed graphs. Any permutation can be built by starting from a single element and repeatedly inserting a new maximum. Each insertion corresponds to an edge in a graph whose vertices are equivalence classes of intermediate permutations. The exponential growth rate of walks equals the spectral radius of the adjacency matrix, and the Perron-Frobenius theorem together with the Collatz-Wielandt formula provide tools to bound this from below. To some extent, our approach resembles that of Albert, Elder, Rechnitzer, Wextcott and Zabrocki [1]: we construct a sequence of increasing lower bounds through computational methods. Our method builds on that paper, but uses the simpler insertion scheme convered by Poh [16] which considers how one can make state machines encoding 1324-avoiding permutations via repeatedly inserting a new maxmimum element.

For 1324-avoiding permutations, the natural graph is too large for direct computation, so we pass to a quotient graph with weighted edges. We conjecture that the sum of weighted walks is at most the number of unweighted walks (Conjecture 8); conditional on this, we obtain a lower bound of 10.418 on $L(1324)$.

Section 2 develops the method through three examples: Av(213), Av(2134) and Av(3124). Section 3 treats 1324-avoiding permutations, establishing the combinatorial machinery and stating the conjecture. Section 4 discusses open questions.

2 Motivating examples

First we summarise notation and definitions for classical permutation patterns, following Bevan [3]. Two sequences $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$ are *order-isomorphic* if $a_i < a_j$ holds if and only if $b_i < b_j$ for all i, j. For permutations $\pi = \pi_1 \ldots \pi_n$ and $\tau = \tau_1 \ldots \tau_m$, π *contains the pattern* τ if there exists a set of indices $i_1, \ldots, i_m$ such that $\pi_{i_1}, \ldots, \pi_{i_m}$ is order-isomorphic to $\tau_1 \ldots \tau_m$. We call such a set of indices an *occurrence* of the pattern τ in π. If π has no occurrence of τ then π *avoids* τ.

In an effort to bound the growth rate of 1324-avoiding permutations we will look at encoding the permutations as walks in directed graphs. Traversing an edge in this graph will correspond to inserting a new maximum element in the permutation. Clearly we can construct any given nonempty permutation by inserting a new maximum element over and over, starting from the trivial permutation. The *trivial permutation* is the unique permutation of a single element. To illustrate the construction, we examine Av(213) as a motivating example.

As a naïve example we could imagine creating a directed graph with the elements of Av(213) as vertices, and placing an edge from π to τ if τ can be obtained by inserting a new maximum element into π. Our numerical methods will only work for finite graphs, so we will have to define a cutoff N, and only work with vertices corresponding to permutations with $\leq N$ elements. Clearly the number of walks in this smaller graph is a lower bound for the walks in the full graph. This reduced graph will still be much too large for our purposes, so we will try to reduce the size of the graph without losing any of the information contained

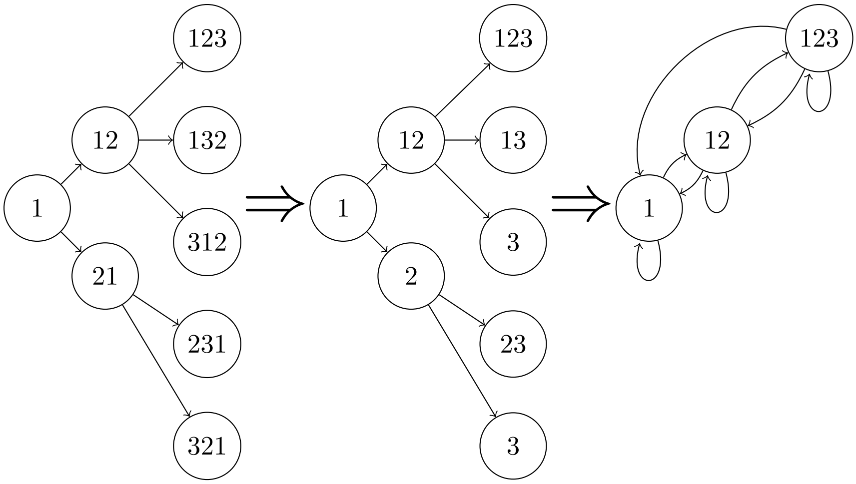

Figure 1: Construction of Av(213) *graph with cutoff* $N = 3$

in our walks.

We note that inserting a new maximum element to the right of an instance of the pattern 21 in a permutation necessarily creates an instance of the pattern 213. So if we want to avoid the pattern 213 we have to insert any new maximum elements to the left of all instances of the pattern 21. This means that for the purposes of our walks, any elements in a permutation after the first instance of 21 are entirely irrelevant. Thus we can consider only the maximal prefix that is in Av(21) and then standardise. Standardising a sequence of n distinct values means renaming the elements to obtain the order-isomorphic permutation on $1, 2, \ldots, n$. Finally we can equate vertices that correspond to the same permutation, reducing our graph to just having vertices in Av(21), see Figure 1.

As illustrated by the edge from 123 to 1, we also consider the edges that would have been present were the cutoff value one greater, in case that points back into the lower cutoff. That edge would have gone from 123 to 4123, which becomes an edge from 123 to 1. Assuming we have no cutoff, the walks in this new graph are in one-to-one correspondence with walks in the original graph, so it suffices to count walks in the new graph. Clearly the number of walks in the cutoff graph is a lower bound for the walks in the full graph. We also see that each vertex corresponds to an increasing permutation $12 \ldots n$. The outgoing edges from $12 \ldots n$ will point to all smaller permutations, the permutation itself and the permutation of one size greater, as inserting a new maximum element at index i will create a permutation of size i.

Now that we have our graph and a description of its edges, the next matter is to enumerate walks starting at the vertex 1. We can consider the adjacency matrix A of our graph. The entries of A^k give us the walks of length k in our graph, so we are interested in the growth rate of the entries of A^n as $n \to \infty$. To this end we use the Perron-Frobenius theorem [8, 15] . We need the more general form later stated by Frobenius as our later matrices will have many zero entries, and the original version by Perron covers only positive matrices. The portion of the theorem we need can be phrased as follows:

Theorem 1 (Perron-Frobenius Theorem)**.** *Let A be an aperiodic irreducible non-negative $N \times N$ matrix with spectral radius r (largest absolute value among all eigenvalues). Then $r \in \mathbb{R}_{>0}$ is an eigenvalue of A which we call the Perron-Frobenius eigenvalue. This eigenvalue is simple and both the left and right eigenspaces are one-dimensional. The corresponding left and right eigenvectors are all non-negative, and moreover these are the only eigenvectors of A whose components are all non-negative. If we denote these left and right eigenvectors by v, w and normalise them so $w^T v = 1$ then $\lim_{n\to\infty} A^n/r^n = vw^T$.*

Irreducibility is equivalent to every vertex being able to reach every other vertex, which is called *strong connectivity*, while for the aperiodicity it will suffice that some vertex has an edge pointing to itself, called a *loop*. Since all vertices can reach 1 in a single step and 1 has a loop, this is satisfied. The theorem also tells us that the number of walks starting at 1 of length n must grow like λ^n where λ is the Perron-Frobenius eigenvalue. Thus each cutoff N gives us some Perron-Frobenius eigenvalue λ_N that will be a lower bound for the Stanley-Wilf limit.

To determine the Perron-Frobenius eigenvalue of a graph we make use of the Collatz-Wielandt formula [6, 19]. The result can be stated as follows:

Theorem 2 (Collatz-Wielandt Formula)**.** *Let A be a matrix with Perron-Frobenius eigenvalue r. For a non-zero vector v with non-negative entries, the minimum value of $(Av)_i/v_i$ taken over i such that $v_i \neq 0$ is $\leq r$.*

Thus if we produce a vector v such that Av is componentwise $\geq \rho v$ for some real value ρ, then the Perron-Frobenius eigenvalue is at least ρ. In this case we will show that the limit is at least $\rho = 4$, which is of course also its true value.

If we take $v = (1, 1, 2/3, (2/3)^2, (2/3)^3, \dots)$, cutting the vector off to the right length, we claim we get our desired bound. The sum of the coordinates of v is then $\alpha := 4 - 2^{N-1}/3^{N-2}$ for cutoff N. By direct calculation we get that

$$Av = \left(\alpha, \alpha, \alpha - 1, \alpha - 1 - \frac{2}{3}, \alpha - 1 - \frac{2}{3} - \left(\frac{2}{3}\right)^2, \dots\right)$$

As $N \to \infty$, the ratios $(Av)_i/v_i$ all approach 4 from below. Therefore it will be a lower bound for the growth rate, so the growth rate must be at least 4. Thus in the case of Av(213) we recover the exact Stanley-Wilf limit.

Next we consider a bigger example, Av(2134). This is known to have a Stanley-Wilf limit of 9 due to Regev [17]. If we apply the same kind of construction as above we will have a graph with vertices in Av(213) up to some cutoff. There are however some small changes that have to be made, see Figure 2.

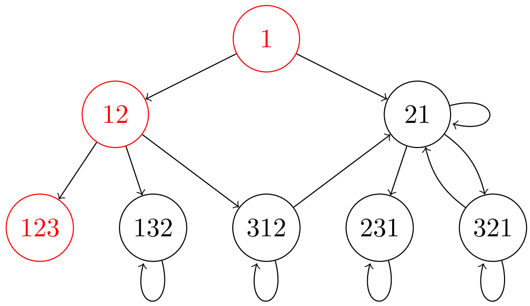

Figure 2: Graph of Av(2134) *with cutoff* $N = 3$*, version one*

The vertices corresponding to increasing permutations are not strongly connected to the rest of the graph, causing an issue. Suppose we have two permutations $\pi, \tau \in \mathrm{Av}(213)$, each with at least one inversion, and we want to find a path from τ to π in the graph. Then we can insert each element of π at the start of τ, and then a new maximum after that. For example if we started at $\tau = 1243$ and our target were $\pi = 132$ we would make the walk

$$1243 \to 51243 \to 561243 \to 5761243 \to 57681243$$

768 is an occurrence of 213, so the last vertex is actually 576, which is standardised as 132. Thus aside from the increasing permutations, any vertex can reach any other vertex. As we are calculating a lower bound, we can fix this problem by simply omitting the increasing permutations.

We make one more modification to our construction process that did not come up for Av(213). Suppose we have some walk starting at a vertex corresponding to the permutation π. If π is order-isomorphic to some prefix of another permutation τ, then all the insertions into π could be done on τ as well. This way we can map the walks starting at π injectively to the walks starting at τ. Therefore if π has an edge to τ in the full graph but τ is not contained in the cutoff graph, we could retain more information by having an edge from π to the maximum prefix of τ that is still in the graph. For example if our graph has cutoff $N = 3$ and we are looking at $\pi = 321$ and insert a maximum to get 3421, then this is not in our graph. So we instead put an edge from π to 342, which is standardised as 231. This modified graph is shown in Figure 3. Some edges that do not contribute to strong connectivity are omitted for clarity.

Two vertices are not strongly connected to the rest, the vertices corresponding to 1423 and 1243. But if we were to go up to cutoff $N = 5$ then they would become connected, so at any given point we can just omit the vertices that are not yet strongly connected. This is similar to how 132 would not be strongly connected to the other length 3 permutations without the length 4 permutations.

This construction yields an exponential number of vertices in the cutoff N, so we would like to reduce their number somewhat. To this end we consider

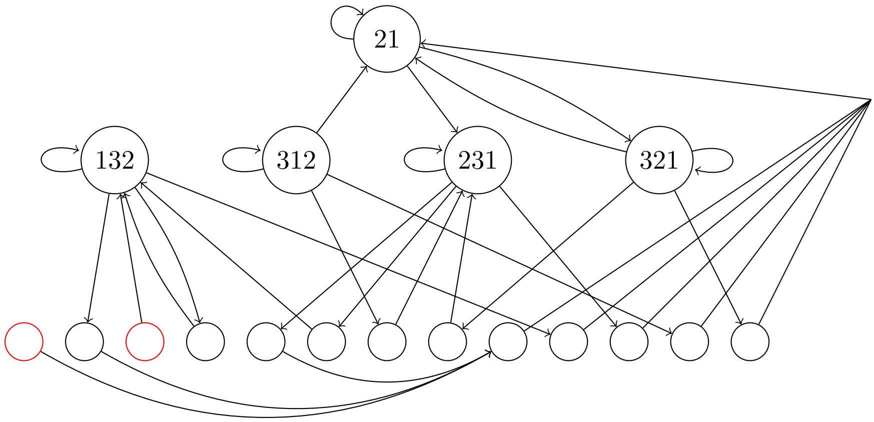

Figure 3: Filtered Av(2134), *cutoff* $N = 4$, *version two*

what happens when a new maximum element is inserted into a 213-avoiding permutation.

If a new maximum element is inserted into $\pi \in \text{Av}(213)$ one of two things happens. That insertion can happen in an initial increasing run, changing the length of that run and increasing the length of the permutation by 1. Otherwise the insertion happens after a descent so we will have to delete everything after the inserted element, and then the inserted element as well. This decreases the size of the permutation but leaves the initial increasing run unchanged. So if we know the length of the initial increasing run and the length of the permutation before and after the insertion, we can recover where the element was inserted. Applying this inductively, if we know the sequence of permutation sizes and initial run lengths, we can recover the walk in the graph. We define the partition of $\text{Av}_n(213)$ into sets with initial run of length r, which we denote $B(n, r)$. Then it suffices to consider the quotient graph with respect to this partition, where we collapse all vertices in the same set $B(n, r)$ into the same vertex. It will give us the same information about the paths, as we can map bijectively between the paths in the quotient graph and the original graph. This way, instead of working with a graph with about 4^N vertices for a cutoff N, we can have about N^2 vertices. Now numerical computations suddenly become much more feasible. In terms of the vector we try to find for the Collatz-Wielandt formula, this is equivalent to deciding that we will give all coordinates in the same part $B(n, r)$ get the same value. An example graph can be seen in Figure 4. The node for $B(5, 4)$ gets omitted because it would not be strongly connected to the rest of the graph.

To do the numerical computations we need three things. We need to know the sizes of the sets $B(n, r)$, the number of edges from one vertex $B(n, r)$ to another vertex $B(m, s)$ and we need some numerical method to determine the

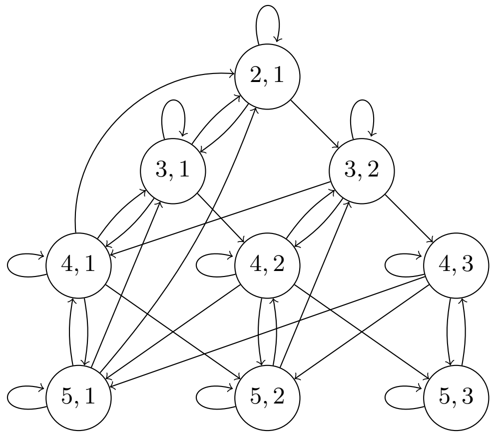

Figure 4: Quotient graph on $B(n, r)$ for cutoff $N = 5$

Perron-Frobenius eigenvalue. This eigenvalue can then be validated using the Collatz-Wielandt formula.

Using the standard bijection between Dyck paths and 213-avoiding permutations, we have due to Kreweras [11] that $|B(n, r)| = T_{n-1,n-r}$ where

$$T_{n,k} = \frac{n-k+1}{n+1}\binom{n+k}{n}$$

which will make an appearance again later. From earlier we also know that for any $\pi \in B(n, r)$ there will be $n+1$ edges out, each of which points to one of the sets $B(n+1, 1), B(n+1, 2), \ldots, B(n+1, r+1)$ or $B(r+1, r), B(r+2, r), \ldots, B(n, r)$. Thus if $B(m, s)$ is among these $n+1$ sets, there are $|B(n, r)|$ edges to it from $B(n, r)$ and otherwise zero edges.

Lastly we will use the power method [13] to calculate the Perron-Frobenius eigenvalue. We start with the vector v of all ones, then repeatedly replace v with Av, then normalise. This can be repeated until we get acceptably close to the true eigenvalue, and in the case of a Perron-Frobenius eigenvalue this will always converge to the right value. Now we can use a computer to calculate the growth rate for larger and larger cutoffs N, getting Figure 5.

This gives a lower bound of 8.90952 on the growth rate, within 1% of the true value. The computation was simply done on a personal computer in about a minute. A more serious computation could likely get a fair bit closer to the true value. We also note that this is seemingly not the only partition that will work, the same example can be worked through when partitioning on the value of the last element of the permutation rather than the length of the initial run.

As a last motivating example we can do something similar for Av(3124). Then we are looking at a graph with vertices in Av(312) and we want to find some

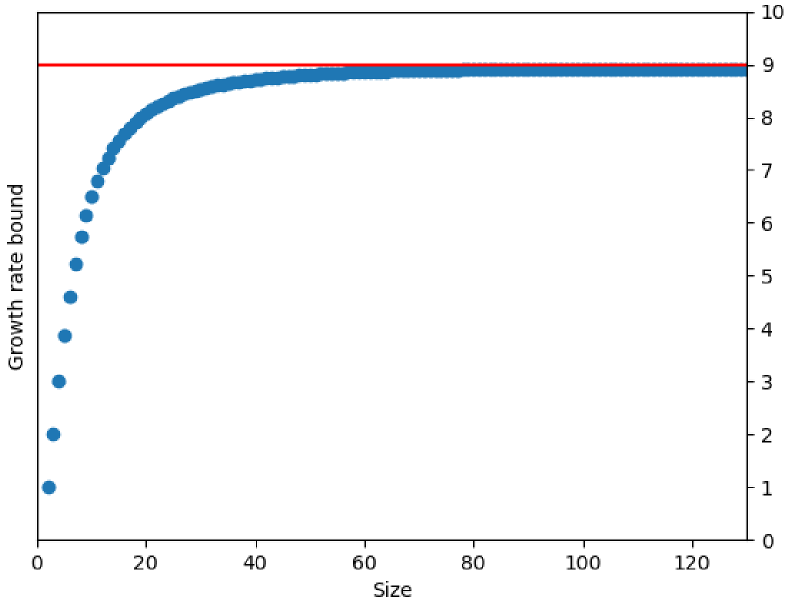

Figure 5: Stanley-Wilf estimate of 2134 *as a function of cutoff*

analogous partition that allows us to recover the walks. Many different statistics can be considered, but one that works to reduce the graph somewhat is to partition by the descent set. The descent set of a permutation π is the set of indices i such that $\pi_i > \pi_{i+1}$. Thus we denote the subset of $\mathrm{Av}_n(312)$ with descent set S by $C(n, S)$. For permutations of size n there are 2^n possible descent sets. Thus the size of the graph would grow like 2^N rather than 4^N as a function of the cutoff N. While this is still exponential, it is an improvement.

Again we have to determine the sizes $|C(n, S)|$ and the number of edges going from some $C(n, S)$ to some $C(m, T)$. Explicit formulas were not derived, but recurrences were sufficient for computing the values on a computer. For brevity we will omit some details here, but $|C(n, S)|$ can be calculated using a three-dimensional push-forward recurrence for a given size n and descent set S. Push-forward means that instead of calculating $C(n, S)$ based on smaller cases, each case $C(n, S)$ contributes something to later cases as we calculate them in some order. Then when we arrive at the case $C(n, S)$ it has the correct value stored. Similarly we omit details on the number of edges between sets, but it follows the same pattern as earlier cases. Finally, we must show the walks in the quotient graph are one-to-one with walks in the original graph. That is, given a sequence of sizes and descent sets, we can recover the original sequence of permutations in $\mathrm{Av}(312)$.

We can show we can do this recovery by induction on the length of the path. In the base case we know what permutation we have and can thus recover it, as it is the trivial permutation. So we just have to show that if we know π and its descent set and size after inserting a new maximum, we know what permutation we have after the insertion. If we insert the maximum before some ascent in π, then we will remove every element after the ascent and the ascent top after the insertion. So we know in what decreasing run of the permutation the insertion was made. But then we can tell where the insertion was made due to the new ascent, which we can read from the descent set. Therefore we can recover where the insertion was made, so the walks are one-to-one between the full graph and the quotient graph.

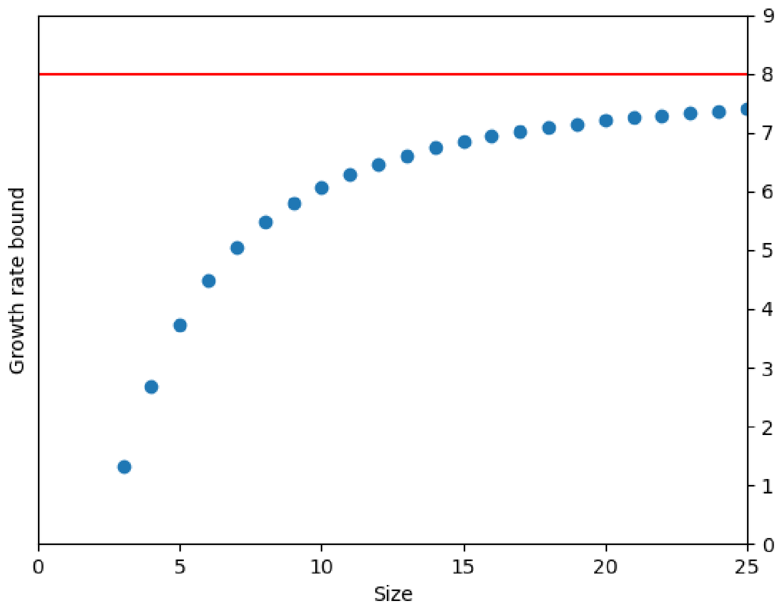

Figure 6: Stanley-Wilf estimate of 3124 *as a function of cutoff*

All in all this gets us Figure 6. The true Stanley-Wilf limit is known to be 8, due to Bóna [5]. We achieve the bound 7.40341, which again could be improved with more computation time and more memory.

3 1324-avoiding graph

We now formalise the same construction as in the motivating examples.

Definition 3. Let π be a permutation that ends with its maximum. Let π' be π with the last element removed. The avoider graph of π with cutoff N has as vertices all nonempty permutations of size at most N avoiding π'. For each permutation τ in the graph, there is a directed edge to every permutation obtained by inserting a new maximum element into τ , removing the minimal suffix to avoid π' and maintain size $\leq N$, then standardising. We denote the number of walks starting from the trivial permutation with $k-1$ steps with $W_{N,k}(\pi)$.

The shift in the definition of $W_{n,k}(\pi)$, making it correspond to $k-1$ steps, is so that $W_{n,n}(\pi) = |\mathrm{Av}_n(\pi)|$. The number of vertices of our graph will grow like 4^N when we look at the Stanley-Wilf limit of 1324, so our numerical methods can only go up to about a cutoff of $N = 17$ before this becomes infeasible due to computational limitations. This small cutoff only yields a lower bound of 8.18, which is not good enough. We want to do something similar to the grouping $B(n,r)$ and $C(n,S)$ but it seems we cannot get such a lossless partition in this case. For brevity we introduce the notation $\pi \to \tau$ for $\pi, \tau \in \mathrm{Av}(132)$, denoting that there is an edge from π to τ in the avoider graph with no cutoff. This means that we start with π and insert a new maximum to obtain π'. Next we remove the minimal suffix such that π' becomes 132-avoiding. Finally we standardise and if this results in τ, then $\pi \to \tau$.

Theorem 4. *The multiset* $\{|\tau| : \pi \to \tau\}$ *uniquely determines* π *for any* $\pi \in \mathrm{Av}(132)$.

Proof. Let S be the multiset $\{|\tau| : \pi \to \tau\}$ for some π. The multiset S already determines the size of the permutation π. Therefore we can instead work with $S' := \{|\pi| + 1 - x : x \in S\} \setminus \{0\}$, the multiset of how many elements get removed. Inserting on the left always yields the same size $|\pi| + 1$, so we remove a zero to ignore this case. This means all the elements of S' correspond to an insertion after an element in π.

We split 132-avoiding permutations on the maximum value using the classical decomposition. This lets us interpret the permutations as binary trees, letting the trivial permutation correspond to a single node and then using the decomposition. Then everything left of the maximum becomes the left subtree of the root of the binary tree, and everything to the right the other subtree.

Suppose π has a maximum at index m. Let π_L be the permutation on the elements before index m in π and π_R be the permutation on the elements after. If we insert a new maximum before m and $m \neq 1$ then the new permutation will have an occurrence of 132. To remove this occurrence by removing a suffix, everything after and including the old maximum must be removed. Therefore the values the elements left of index m contribute to S' are simply $\{|\pi_L| + 1 - |\tau| + |\pi_R| : \pi_L \to \tau\}$.

Consider next an insertion after index m. Since π avoids 132, every element of π_L is larger than every element of π_R. So even after inserting a new maximum we never have an occurrence of 132 that involves elements before the maximum. Therefore the values the elements right of the index m contribute to S' are $\{|\pi_R| + 1 - |\tau| : \pi_R \to \tau\}$.

With this information we can present the map taking π to its multiset S' in terms of the binary tree corresponding to π. We create a map g that assigns a non-negative integer to each node of a binary tree, and let S' be the image of the nodes of the whole binary tree under g. For a node v we consider the path P from v to the root, including both v and the root. Then we define $g(v)$ as the sum of size of the right subtree of w, plus one for w itself, over all w such that both w and its left child are on the path P. The multiset of these numbers will then give S', see Figure 7. A more imperative way to view g is to think of starting at the root and walking down to the vertex v. Any time you would move from a parent p to its left child, you add the size of the right subtree of p plus one to a running tally. The total sum you get at the end is then $g(v)$.

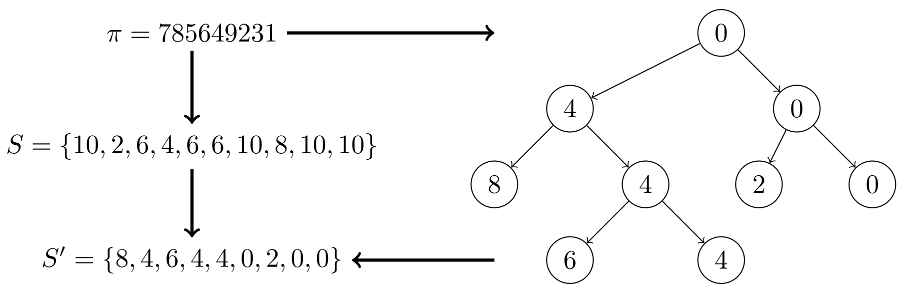

Figure 7: A permutation, its binary tree and multisets S, S'

This formulation will allow us to present an inverse, constructing the permutation π from the multiset S'. But first we need to prove three properties of g.

The first one is that g is injective on the leaves. Suppose this is not true and two leaves v, w are mapped to the same value. Consider their lowest common ancestor ℓ, if this is not the root, we can consider instead the subtree rooted at ℓ. We can do this because any node outside this subtree either contributes to both $g(v)$ and $g(w)$ or neither. Thus we may assume without loss of generality that v is in the left subtree of the root and w in the right. But $g(w)$ is at most the size of the right subtree, not including the root, and $g(v)$ is at least the size of the right subtree plus the root. This is clearly a contradiction, so all leaves are assigned different values.

The second property we need is that the reverse preorder will visit the nodes in increasing order of the value they map to. The reverse preorder is a recursive procedure for enumerating the vertices of a binary tree. Given a node it enumerates that node, then recursively enumerates the right subtree, then finally recursively enumerates the left subtree. So in Figure 7 it would enumerate the values $0, 0, 0, 2, 4, 4, 4, 4, 6, 8$ starting from the root. To prove this we consider some node v and its left and right subtrees. The path from the root to some node in either subtree will contain more nodes, so $g(v)$ cannot be greater than $g(w)$ for w in either subtree. Any $g(w)$ for w in the right subtree will only exceed $g(v)$ by at most the size of the right subtree, while $g(u)$ for u in the left one are at least $g(v)$ plus the size right of the subtree and one more. Thus all the values in the right subtree come before the values in the left subtree, giving us our desired result.

The third and last property we need is that if v is among the nodes that contributes to the sum $g(w)$ for some node w, then $g(v) < g(w)$. But if $g(v) < g(w)$, there is some node u on the path from w to the root such that w is in the left subtree of u and v in the right. And by the second property, this means $g(v) < g(w)$.

We consider the values of S in increasing order, denoting them $s_1 \leq s_2 \leq \cdots \leq s_k$, and construct the tree one node at a time. Denote the node corresponding to s_i by v_i, so we want to satisfy $g(v_i) = s_i$. Because of the third property we proved, we can do this one node at a time, since all the nodes v_j contributing to s_i must have $j < i$. To start we must have $s_1 = 0$, which will correspond to the root v_1. As we consider adding v_i to the tree, we know it comes next after v_{i-1} in the reverse preorder traversal of the tree. Thus it is either the right child of v_i, or the left child of some node on the path from v_{i-1} to the root.

Consider just the tree consisting of the path from v_{i-1} to the root, then add a left and right child to all nodes in that path where one is not already present. Then all the leaves of this tree are our possible insertion points for v_i. But we already know that all leaves correspond to different values, so only one of them can be s_i. This uniquely determines where we place v_i, before then moving onto v_{i+1} and so on. Finally the third property we proved ensures that $g(v_i) = s_i$ continues to be true as we insert $v_{i+1}, v_{i+2}, \ldots$.

Thus we have a binary tree which maps to our set S and it is unique as each node insertion is forced. The 132-avoiding permutation is uniquely determined as well then, completing our proof. □

This theorem shows we cannot get a partition like in the earlier two cases, where we could massively shrink the number of vertices while maintaining all walks. We will still partition the permutations, but we have to accept getting something approximate rather than exact. But this can still possibly provide a lower bound.

But first one can ask the question of how to find a good partition to try. Ultimately we want to compute some set of weights w_π on the permutations such that

$$\sum_{\tau \to \pi} \frac{1}{\rho} w_\tau \geq w_\pi$$

Then the Collatz-Wielandt formula would ensure that ρ is a lower bound for the Stanley-Wilf limit of 1324. Furthermore in the limit of the power method we'd have equality instead of inequality, as we converge to the Perron-Frobenius eigenvector. While these w_π cannot be calculated directly, as far as we know, a very similar set of weights w'_π can be found computationally. The set of weights satisfying

$$\sum_{\tau \to \pi} \frac{1}{|\tau|} w'_\tau = w'_\pi$$

is exactly the stationary distribution of the random walk on our graph.

A random walk is a random process, a discrete time Markov chain, in which each state is a vertex of our graph. At any given point the process chooses a neighbour of that vertex uniformly at random and transitions to that state, repeating this infinitely. The stationary distribution then tells us the proportion of time the process spends in any given vertex as time goes to infinity. Because our graph is strongly connected and has a loop a stationary distribution will exist, and can be computed in terms of recurrence times [10]. The recurrence time of a vertex v is the expected number of steps it will take to return to v during our random process if we start the process from v.

One can calculate that w'_π appears to only depend on the number of elements of π and the number of right-to-left maxima π. In fact after rescaling the weights w'_π it appears that they are given by $f(\pi)/|\pi|!$ where $f(\pi)$ is the number of non-right-to-left-maxima. As we will be working with such elements a fair bit going forward, we will call non-right-to-left-maxima *short* values. For 2134-avoiders, this approach seemingly yields the formula $(n - \ell)/n!$ after scaling, where ℓ denotes the last element. The 3124-avoiding permutations appear to have no such simple formula, though perhaps one can be found in terms of the descent set.

Theorem 5. *The set of* 132*-avoiding permutations of length* n *with* k *short values, which we denote* $A(n, k)$*, has* $T_{n-1,k}$ *elements where*

$$T_{n,k} = \frac{n-k+1}{n+1} \binom{n+k}{n}$$

These values are in the OEIS [14] under `A009766`*.*

Proof. Suppose we have $\pi \in A(n, k)$. We use the classical decomposition of 132-avoiding permutations, splitting on the maximum element. Let τ_1 denote everything to the left of the maximum of π, and τ_2 everything to the right. Every right-to-left maxima of τ_2 will continue to be a right-to-left maxima in π, but the ones in τ_1 will not because they are to the left of the maximum element of π. Thus,

$$|A(n, k)| = \sum_{m=1}^{n-1} C_m |A(n - m - 1, k - m + 1)|$$

where C_m is the m-th Catalan number. Using the formula $C_m = \frac{1}{m+1}\binom{2m}{m}$ the desired result follows easily by induction. □

As we saw in the motivating examples we still need one more thing before we can compute the bound with respect to the partition into the sets $A(n,r)$. We need to know the number of edges going from $A(n,r)$ to $A(m,s)$ in the graph for any given integers n,r,m,s. But to get there we first need a lemma.

Lemma 6. *For a permutation $\pi \in \mathrm{Av}_n(132)$, the $n+1$ permutations τ such that $\pi \to \tau$ all belong to different sets $A(m,s)$.*

Proof. Suppose inserting a new maximum at index i and at index j results in the same number of elements in the output permutations τ_1 and τ_2. Then it suffices to show that τ_1 and τ_2 have a different number of left-to-right maxima. Without loss of generality we assume $i < j$. This means the elements $\pi_1, \ldots, \pi_{i-1}$ are all greater than the elements $\pi_i, \pi_{i+1}, \ldots, \pi_{j-1}$, as otherwise an occurrence of 132 would form upon inserting a new maximum at index i and τ_1 would have less than j elements while τ_2 necessarily has at least j elements.

Now τ_1 and τ_2 must be equal after the first j elements. Suppose that this common tail contains an element greater than any of the elements $\pi_i, \pi_{i+1}, \ldots, \pi_{j-1}$. Then the insertion of a new maximum at index j must have created an occurrence of 132 that is still present in the common tail, which cannot be, so this is not the case. Suppose then this common tail contains s right-to-left maxima.

Now in τ_2 the right-to-left maxima are these s common right-to-left maxima and the newly inserted maximum element. In τ_1 we must have the s common right-to-left maxima, but also at least one right-to-left maxima among $\pi_i, \pi_{i+1}, \ldots, \pi_{j-1}$ since the s common right-to-left maxima are smaller than these elements. Furthermore τ_1 also has a right-to-left maxima at index i where the new maximum was inserted. Therefore τ_1 and τ_2 have different numbers of right-to-left maxima which completes our proof. □

Theorem 7. *Denote the number of edges in the 1324-avoider graph going from an element of $A(n,r)$ to an element of $A(m,s)$ by $E(n,r,m,s)$. That function satisfies*

$$E(n,r,m,s) = \begin{cases} 0 \textit{ if } r \geq n, s \geq m, n < 1, m < 1, r < 0, s < 0 \textit{ or } m > n+1 \\ 0 \textit{ if } n < m \textit{ and } s < r \\ T_{n-1,r} \textit{ if } n+1 = m \\ E(n-1,r,m,s) + E(n,r-1,m,s) \textit{ otherwise} \end{cases}$$

Proof. The first case says there is no permutation with more short elements than total elements, there is no way to have a negative number of short elements, there is no way to create the empty permutation from a non-empty one and there is no way gain more than one element by inserting a single element.

Next the second case. If $n < m$ we must have inserted a new maximum element and not removed anything due to occurrences of 132. If $s < r$ it means we have fewer short elements after insertion than before insertion. The newly inserted maximum cannot be short, and inserting a new maximum cannot make something go from being not a right-to-left maxima to being one. Therefore this case is impossible and the result is zero.

Next we move onto the third case. In this case we have inserted a new maximum and removed no elements afterwards. To remove no elements we must insert our maximum directly right of a right-to-left maxima. If we insert the new maximum to the right of what was previously a right-to-left maxima, it is a short value afterwards. So to land in $A(m,s)$ we know after which right-to-left maxima we have to insert, since we must create the correct number of short values. Thus each element of $A(n,r)$ has exactly one position where inserting lands us in $A(m,s)$, giving the base case.

Finally we just have one case left. Due to Lemma 6 each $\pi \in A(n,r)$ has at most one insertion landing in $A(m,s)$. Thus the value $E(n,r,m,s)$ is simply the number of $\pi \in A(n,r)$ that have such an insertion. Furthermore as $m \leq n$ we must remove at least one element from π after this insertion. We split into cases depending on whether π ends with a 1.

Any $\pi' \in A(n-1,r)$ can be made into a permutation in $A(n,r)$ by adding a new minimum element at the end and increasing every other element by one, as this introduces a new right-to-left maxima, leaving r unchanged. Furthermore since the insertion into π must remove at least element, the corresponding insertion in π' gives the same result. Thus if our $\pi \in A(n,r)$ ends with a 1 we can map it to a unique element in $A(n-1,r)$.

Now we move onto the case where π does not end with 1. Suppose the last element of π is $x > 1$. Then $x-1$ is in π, suppose that the smallest element preceding or equal to $x-1$ is y. Then we create π' by increasing every element equal to at least y and less than x by one and replacing the last element by y, see Figure 8.

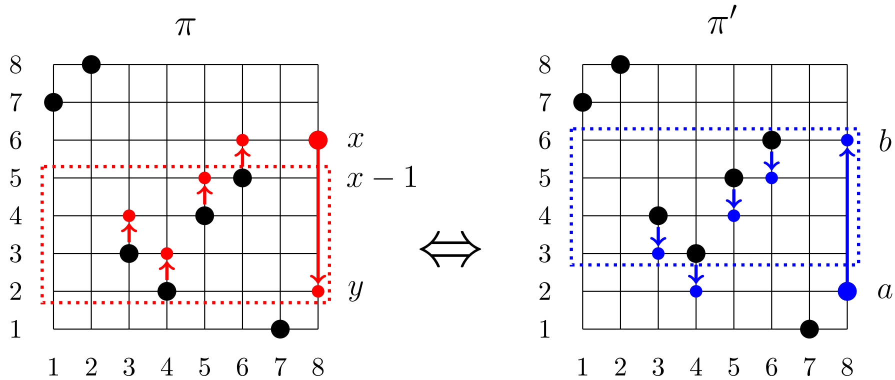

Figure 8: Map from π to π' when the last element is > 1

Clearly no value $z > x$ can exist between $x-1$ and x in π as $x-1, z, x$ would be an occurrence of 132. So $x-1$ becomes a right-to-left maxima in π' and was not one before in π, while x was and remains a right-to-left maxima. Furthermore there can be no element z satisfying $z > x - 1$ between $x - 1$ and x as otherwise y, z, x would be an occurrence of 132. So no other element between becomes a right-to-left maxima, so the number of short values decreases by exactly one. Thus $\pi' \in A(n, r - 1)$.

Going the other way we can start with $\pi \in A(n, r - 1)$. If π is the increasing permutation, any insertion results in no element being removed afterwards, so it cannot end in $A(m, s)$. Therefore we can discount the case where π is increasing, in the other case we have some last element a and previous right-to-left maxima b. Then we can get the inverse by decreasing every element with value between a and b by one and making the last element b, completing the proof. □

This could likely be reduced to a recurrence in less than four variables. But for our purposes we need all the values $E(n, r, m, s)$ anyway and this calculates each of them in constant time, so this will suffice.

If we repeat our earlier constructions to create the avoider graph for 1324 and take the quotient with respect to $A(n, r)$, checking some examples by hand reveals that the walks do not correspond one-to-one like before. In an attempt to still make the construction work, we introduce one new feature. The edge from $A(n, r)$ to $A(m, s)$ will have weight $E(n, r, m, s)/|A(n, r)|$, where we omit zero weight edges. A walk will then no longer contribute 1 to the total sum, but the product of the weights on the edges it travels along. This is done to replicate the earlier behaviour of forcing two coordinates belonging to the same $A(n, r)$ to have the same weight, just without the luxury of always having the same number of edges going out of a partition to each outneighbour. For this reason we choose the weight $E(n, r, m, s)/|A(n, r)|$, as it is the ratio of vertices in $A(n, r)$ having an edge pointing to $A(m, s)$. In previous cases, this ratio was always one. To distinguish these weighted walks from the normal ones, we denote the sum of the length k walks in the graph with cutoff n by $\widetilde{W}_{n,k}$ in the weighted case. We conjecture, but unfortunately do not prove, that these weighted walks provide a lower bound for the unweighted ones.

Figure 9 shows $\widetilde{W}_{n,k}/W_{n,k}$ for some values of n and k. We start at $n = 4$ and $k = 6$ as all values before that point are simply 1. All values for $k \leq 100$ and $n \leq 15$ have been confirmed to be ≤ 1. As all paths of length $\leq \ell$ steps are all the same in graphs with cutoff $\geq \ell$, this means the conjecture is true for $k \leq 15$ for all n.

Formally speaking, the conjecture to be proven is the following.

Conjecture 8. *The walks in the grouped 1324-avoiding graph provide a lower bound for the Stanley-Wilf limit of* 1324*-avoiding permutations. That is to say* $\widetilde{W}_{n,k}(1324) \leq W_{n,k}(1324)$ *for all* $n, k \geq 0$.

Now if this were true, we get the following for free based on the computations involved. The above conjecture is in fact quite strong, as the corollary only needs the case $n = k$. The case $n = k$ has been verified to be true for $n \leq 50$ using the

$n \backslash k$	6	7	8	9	10
4	0.999906	0.999893	0.999819	0.999747	0.999677
5	0.999971	0.999833	0.999605	0.999333	0.999053
6	0.999974	0.999864	0.999572	0.999111	0.998545
7	0.999975	0.999873	0.999622	0.999126	0.998388
8	0.999975	0.999875	0.999638	0.999192	0.998450
9	0.999975	0.999875	0.999641	0.999212	0.998523

Figure 9: Values of $\widetilde{W}_{n,k}/W_{n,k}$ *for various* n, k*.*

known values of $|\operatorname{Av}_n(1324)|$, provided by Conway, Guttmann and Zinn-Justin [7]. A plot of the true value compared to the esimate can be seen in Figure 10. The eagle-eyed viewer might see that the two values diverge very slightly as they get to index 50. On a non-logarithmic plot the difference somewhat clearer as the estimate is about 30% smaller than the true value at $n = 50$.

Corollary 9. *If Conjecture 8 is true, the growth rate of* 1324*-avoiding permutations is at least* 10.418*.*

The computed value for $n = 220$ is ≥ 10.418, though the true bound for $n = 220$ is likely a bit better. For the largest values of n, it was difficult to have the power method converge fully due to computation constraints. But thanks to the Collatz-Wielandt formula we can still check and be sure that the bound 10.418 is definitely correct. If we plot the three examples we have looked at, it seems like the difference from the final limit as a function of the cutoff N seems to be approximately $N^{-3/2}$ near the limit, see Figure 11.

It seems that this method is a genuine lower bound and does not tend to the conjectured growth rate of 11.6. But this gap could partially be a result of our limited floating-point precision method failing to find good solutions for larger cases. A guess based on how quickly the value seems to be converging for $n \leq 220$ is that the limit is about 10.65. So with more computation the value 10.418 could be improved, but maybe not by much. With much more computation time this could also be repeated using higher precision floating points, which could reveal how much of this gap is a result of computational constraints.

4 Open questions

- What properties do these sequences of increasing graphs need to satisfy for the growth rates of their walks to converge to the growth rate of walks in the limit graph? Since the graphs are directed there are some obvious counterexamples to the limit always being correct if we place no constraints.

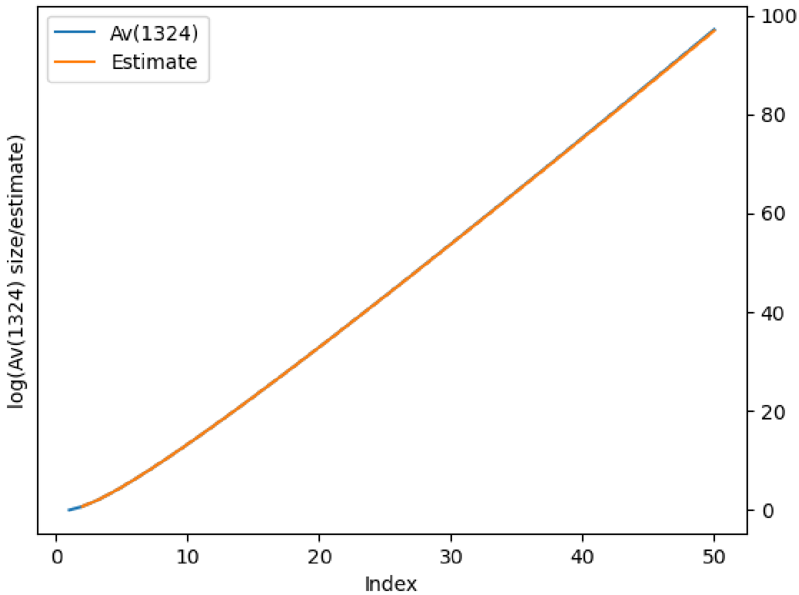

Figure 10: Logarithm of estimate of $|\operatorname{Av}_n(1324)|$ *compared to true value*

For example an infinite binary tree with edges oriented from parent to child would have growth rate 0 for any finite section, but growth rate 2 in the limit. Does it suffice that each graph in the sequence is strongly connected? This question could also be equivalently posed in terms of automata to fit with earlier work, like the paper by Albert, Elder, Rechnitzer, Wextcott and Zabrocki [1].

- For a stationary distribution of the random walks on one of these (non-truncated) graphs, does there always exist an $\alpha \in \mathbb{R}$ and integer valued function f on permutations so the coordinate for π is $\alpha f(\pi)/|\pi|!$ for all π?
- What is the theoretical connection between the stationary distributions of the random walks and the Perron-Frobenius eigenvector of the graph? They are both eigenvectors of a matrix related to the graph, but not of the same matrix. Yet they seem to share some structural properties, for example being constant on the same partition in some of the simpler examples considered.
- Is there some way to compute or approximate the Perron-Frobenius eigenvector in the infinite graph? Simply computing a coordinate for larger and larger cutoffs N does not even clearly reveal simple things like whether two coordinates are equal in the limit due to "noise" emanating from where

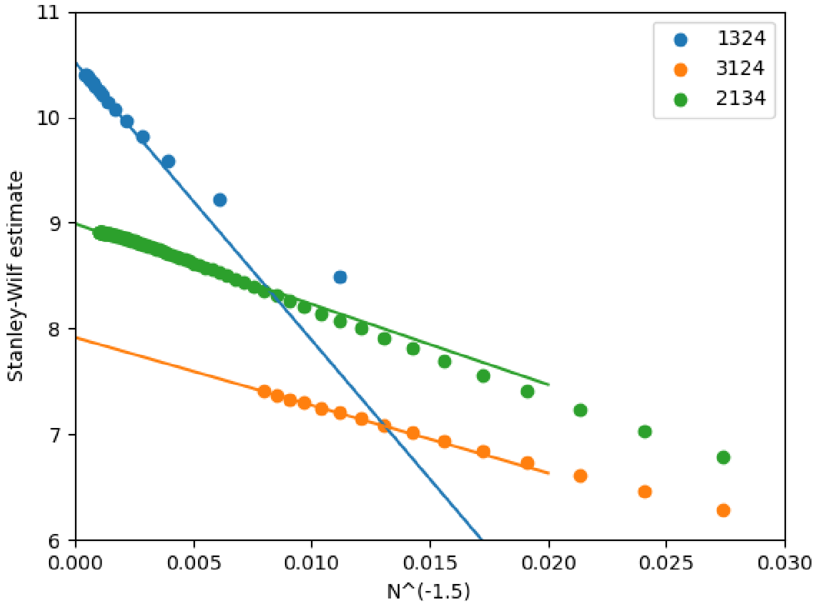

Figure 11: Plot of convergence of estimates to their limit

the graph was truncated.

- Is there some other partition that gives a better bound in the case of 1324-avoiding permutations, while still being computationally feasible?

Acknowledgements

Figuring out the formula for the weights for the random walks was done using the FindStat [18] tool.

Appendix

The calculated values of $\widetilde{W}_{n,n}$ for $n = 1, \ldots, 50$ follow in Figure 12. These can be compared against the known values of $|\operatorname{Av}_n(1324)|$ from Conway, Guttmann and Zinn-Justin [7].

n	$\widetilde{W}_{n,n}$	n	$\widetilde{W}_{n,n}$
1	0	26	4.706543085e+19
2	2	27	3.820046813e+20
3	6	28	3.121218687e+21
4	23	29	2.566262389e+22
5	103	30	2.122495423e+23
6	513	31	1.765314252e+24
7	2762	32	1.476043875e+25
8	15792.6	33	1.240398630e+26
9	94764.14143	34	1.047369958e+27
10	591737.5476	35	8.884154482e+27
11	3821110.811	36	7.568625284e+28
12	25394500.09	37	6.474664471e+29
13	173036190	38	5.560801289e+30
14	1205205579	39	4.794057984e+31
15	8559183937	40	4.148068322e+32
16	6.185211294e+10	41	3.601642999e+33
17	4.540198358e+11	42	3.137668938e+34
18	3.380289108e+12	43	2.742259729e+35
19	2.549431847e+13	44	2.404096157e+36
20	1.945671483e+14	45	2.113911500e+37
21	1.501133141e+15	46	1.864087697e+38
22	1.169850253e+16	47	1.648336933e+39
23	9.202016022e+16	48	1.461449357e+40
24	7.301188881e+17	49	1.299092264e+41
25	5.839947044e+18	50	1.157649550e+42

Figure 12: $\widetilde{W}_{n,n}$ *for* $n = 1, \ldots, 50$

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