



Robustness of Hilbert space-valued stochastic volatility models

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Abstract

In this paper, we show that Hilbert space-valued stochastic models are robust with respect to perturbations, due to measurement or approximation errors, in the underlying volatility process. Within the class of stochastic-volatility-modulated Ornstein–Uhlenbeck processes, we quantify the error induced by the volatility in terms of perturbations in the parameters of the volatility process. We moreover study the robustness of the volatility process itself with respect to finite-dimensional approximations of the driving compound Poisson process and semigroup generator, respectively, when considering operator-valued Barndorff-Nielsen and Shephard stochastic volatility models. We also give results on square root approximations. In all cases, we provide explicit bounds for the induced error in terms of the approximation of the underlying parameter. We discuss some applications to robustness of prices of options on forwards and volatility.

Keywords Hilbert space-valued stochastic volatility · Robustness under perturbation · Stochastic-volatility-modulated Ornstein–Uhlenbeck processes · Robustness of options on forwards and volatility

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1 Introduction

In mathematical finance, the dynamics of forward and futures prices may be modelled as an infinite-dimensional stochastic process taking values in some suitable Hilbert space. To capture probabilistic features of the price evolution, operator-valued stochastic volatility models have recently gained some attention in the literature. In this paper, we address the question of robustness of Hilbert-valued stochastic processes with respect to perturbations in the underlying stochastic volatility dynamics.

The HJM approach in fixed-income theory (see Heath et al. [33] and Filipović [29, Chap. 4]) has been adopted into commodity and energy futures price models (see Benth et al. [14, Chap. 6]). Using the so-called Musiela parametrisation, the forward price $f_t(x)$ at time $t \geq 0$ of a contract delivering a commodity (say oil or coffee beans) in time $x \geq 0$ defines a stochastic process $(f_t(\cdot))_{t \geq 0}$ which takes values in a Hilbert space of real-valued functions on $\mathbb{R}_+$. By no-arbitrage considerations, its risk-neutral dynamics may be modelled as the solution of a stochastic partial differential equation

$$df_t(x) = \partial_x f_t(x)dt + \sigma_t dB_t. \quad (1.1)$$

Here, ∂_x is the derivative operator with respect to x , $(B_t)_{t \geq 0}$ is a Wiener process in the Hilbert space and $(\sigma_t)_{t \geq 0}$ is some suitable operator-valued stochastic process acting on elements of the Hilbert space. We refer to Peszat and Zabczyk [40, Chap. 9] for details on such stochastic partial differential equations, and to Benth and Krühner [11] for a current account on the application to energy and commodity markets. We remark in passing that our analysis is also relevant in more exotic weather derivatives markets as well as fixed-income theory. Moreover, ambit fields as defined in Barndorff-Nielsen et al. [4, Chap. 5] can be re-cast as solutions of equations like (1.1) (see Benth and Eyjolfsson [9]), which ties our work to a rather general class of random field models where stochastic volatility plays an important role (e.g. turbulence, see [4, Chap. 9]). Ambit fields have been applied to futures price modelling in Di Persio and Perin [26] and Barndorff-Nielsen et al. [3].

In recent years, there has been a growing interest in developing and analysing infinite-dimensional stochastic volatility models, that is, stochastic dynamics for σ taking values in a suitable space of operators. We refer to Benth et al. [13], Benth and Simonsen [18], Cox et al. [23, 24], Friesen and Karbach [30], Cuchiero and Svaluto-Ferro [25] and Benth and Sgarra [17] for models of the variance $\mathcal{V}$ being a dynamics on the positive Hilbert–Schmidt operators, where one defines σ as the (positive) square root of $\mathcal{V}$. Rough volatility models have gained a lot of attraction, see e.g. El Euch and Rosenbaum [27] for rough Heston models. Benth and Harang [10] propose an infinite-dimensional version of such stochastic volatility models feasible for a forward term structure dynamics. A stochastic volatility dynamics in infinite dimensions gives a versatile model for describing the second-order risk variations across maturities.

The proposed infinite-dimensional volatility models can be split into two classes. The first extends the Barndorff-Nielsen and Shephard (BNS) stochastic volatility model. The BNS model, suggested by Barndorff-Nielsen and Shephard [5], was later

extended to a multivariate context by Barndorff-Nielsen and Stelzer [6]. The BNS model in infinite dimensions belongs to the affine class of models studied in Cox et al. [23, 24]. The second stream of operator-valued stochastic volatility models is the lifting of the Heston model (see Heston [34] for the original one-dimensional model). Benth and Simonsen [18] and Benth et al. [8] have studied the Heston model in infinite dimensions. The present paper takes the infinite-dimensional BNS class of models as its starting point to study robustness.

When estimating a particular volatility model, or doing numerical simulations of it, one inevitably introduces perturbations of the “real” volatility. One can view these perturbations as projections of the infinite-dimensional volatility dynamics onto a finite set of (principal) component processes. In this paper, we analyse the robustness of the volatility process itself along with the forward prices, and we show that the induced error can be controlled. Indeed, for the volatility, variance and forward price, we are able to show pathwise robustness.

There are some recent accounts on the numerical simulation of such volatility models together with the forward price, i.e., the solution of (1.1). In Benth et al. [7], a semi-discrete finite difference scheme for the Heston infinite-dimensional volatility process is combined with a finite element approximation of (1.1). Based on regularity results for the model, sharp convergence rates are proved and illustrated in numerical examples. In parallel to our studies, we learned that Karbach [35] recently has studied finite-rank approximations of affine stochastic volatility models. As mentioned, this class of volatility models includes the infinite-dimensional BNS dynamics that we consider here. Karbach [35] shows that there is a sequence of finite-rank volatility models which converges weakly in the path space equipped with the Skorohod topology. Moreover, a Galerkin approximation for the corresponding generalised Riccati equations is introduced and studied for the dynamics as in (1.1) with affine volatility. In this paper, we provide explicit bounds in $L^2(\mathbb{P})$ of the maximum pathwise error in terms of the various parameters in the volatility model. The pathwise error is measured in the norm of the Hilbert space where the dynamics take values.

In a large part of this paper, we analyse the difference between the stochastic volatility and its approximation in terms of three main parameters. These are the initial (current) variance, the drift of the BNS volatility model (which is a bounded operator on the space of Hilbert–Schmidt operators) and the driving Lévy process (which is a compound Poisson process in the space of Hilbert–Schmidt operators). We establish bounds on the pathwise error measured by the Hilbert–Schmidt and operator norms, both for the variance model and the square root of it (the volatility). The estimates for the volatility process rely on results by Birman et al. [19] on fractional powers of self-adjoint operators. Our main result, Theorem 2.3, assesses the pathwise error of the Hilbert-space-valued and volatility-modulated Ornstein–Uhlenbeck dynamics $t \mapsto f_t$ (as given by (1.1)). The result is shown using arguments involving the measurable selection theorem of Kuratowski and Ryll–Nardzewski [36].

Our results are directly applicable for a continuity and robustness analysis of option prices considered as functionals of the different parameters in the volatility model. We refer to Benth and Krühner [12] for an exposition of various options on infinite-dimensional forward price models in commodity markets, including flow forwards. Cuchiero and Svaluto-Ferro [25] analyse options on realised volatility in an

infinite-dimensional framework. We also want to mention the recent work by Benth et al. [15, 16] on limit theorems for realised variation of stochastic partial differential equations like (1.1), which can be utilised for non-parametric estimation of the infinite-dimensional stochastic variance.

The paper is structured as follows. In Sect. 2, we present the notation we employ in the paper and discuss the stochastic volatility processes as elements in a separable Hilbert space. The main result of this section is that the stochastic-volatility-modulated Ornstein–Uhlenbeck process is robust with respect to perturbations of the variance process, meaning that the error such a perturbation causes in the volatility-modulated process can be quantified in terms of the error induced by the variance process. In Sect. 3, we focus our attention on the variance process and present results on how to quantify the error induced by employing approximations. We focus on the case when the variance process is given by an Ornstein–Uhlenbeck-type infinite-dimensional process which involves an initial value, a bounded semigroup generator and a compound Poisson process. In particular, we elaborate on error bounds in the case of finite-dimensional approximations of the compound Poisson process and semigroup generator. In Sect. 4, we discuss square root approximations of positive definite variance processes and approximative processes.

2 Approximation of forwards

Fix a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Let H denote a separable real Hilbert space, with an inner product $(\cdot, \cdot)_H$, and norm $|\cdot|_H$, and denote by $\mathcal{H} := L_{\text{HS}}(H)$ the space of Hilbert–Schmidt operators, i.e., the Hilbert space of bounded operators $\mathcal{T} \in L(H) = L(H, H)$ with $\|\mathcal{T}\|_{\mathcal{H}} = \sqrt{\langle \mathcal{T}, \mathcal{T} \rangle_{\mathcal{H}}} < \infty$, where

$$\langle \mathcal{S}, \mathcal{T} \rangle_{\mathcal{H}} = \sum_{k=1}^{\infty} (\mathcal{S}e_k, \mathcal{T}e_k)_H$$

denotes the inner product of $\mathcal{H}$ for any $\mathcal{S}, \mathcal{T} \in \mathcal{H}$ and $(e_k)_{k \in \mathbb{N}}$ is an orthonormal basis (ONB) in H . A compact operator $\mathcal{T} \in L(H)$ is a *trace-class operator* if it has an absolutely convergent series of eigenvalues and the trace-norm is defined by (see Bogachev [21, Sect. 2.5])

$$\|\mathcal{T}\|_1 := \|(\mathcal{T}^* \mathcal{T})^{1/4}\|_{\mathcal{H}}^2, \quad (2.1)$$

where $\mathcal{T}^*$ is the adjoint of $\mathcal{T}$. Denote the Banach space of trace-class operators with norm (2.1) by $\mathcal{B}_1$. In what follows, we consider an H -valued stochastic-volatility-modulated Volterra process

$$X_t = \int_0^t \mathcal{S}(t-s) \sqrt{\mathcal{V}_s} dB_s, \quad (2.2)$$

where $(B_t)_{t \geq 0}$ is an H -valued Wiener process with the covariance operator Q being a positive definite trace-class operator on H , and $(\mathcal{V}_t)_{t \geq 0}$ is an $\mathcal{H}$ -valued nonnegative definite stochastic variance process. We assume $(\mathcal{V}_t)_{t \geq 0}$ to be predictable (with

respect to the strong topology on $\mathcal{H}$ and satisfy the integrability condition

$$\mathbb{E} \left[\int_0^t \|\mathcal{S}(t-s)\sqrt{\mathcal{V}_s} Q^{1/2}\|_{\mathcal{H}}^2 ds \right] < \infty \quad (2.3)$$

for all $t \leq T$ for a fixed time horizon $T < \infty$. We refer to Sect. 3 for a specification of the volatility process $\mathcal{V}$. Finally, $(\mathcal{S}(t))_{t \geq 0}$ is supposed to be a C_0 -semigroup on H . We recall from Peszat and Zabczyk [40, Chap. 9] that the stochastic process X in (2.2) is a mild solution of the SPDE

$$dX_t = \mathcal{A}X_t dt + \sqrt{\mathcal{V}_t} dB_t \quad (2.4)$$

with zero initial condition $X_0 = 0$. Here, $\mathcal{A}$ denotes the generator of the semigroup $\mathcal{S}$. Remark that we can easily include an initial condition $X_0 \in H$ by adding the term $\mathcal{S}(t)X_0$ to the dynamics in (2.2).

In the following example, we discuss the stochastic process in (2.2) in view of term structure modelling in commodity markets based on the analysis in Benth and Krühner [11, 12].

Example 2.1 Given a Hilbert space H of real-valued measurable functions on $\mathbb{R}_+$, we can define the forward price $f_t(x)$ of a commodity at time $t \geq 0$ and with time to delivery $x \geq 0$ as $f_t(x) := \delta_x(X_t)$. Here, X is given by (2.2) and δ_x is the evaluation operator, which is assumed to be a continuous linear functional, i.e., $\delta_x \in H^*$. Furthermore, in this commodity market context, $\mathcal{S}(t)$ is the shift semigroup, i.e., $\mathcal{S}(t)(g) = g(\cdot + t)$ for $g \in H$, which has the generator $\mathcal{A} = \partial/\partial x$. Letting H be the Filipović space (see Filipović [29, Chap. 5] as well as the end of this section for the original definition), the evaluation operator is an element in H^* and $\mathcal{S}$ is a C_0 -semigroup. This model of f defines the risk-neutral dynamics of the term structure dynamics of forward prices. Possible operator-valued stochastic models for the variance process $\mathcal{V}$ are provided later in this paper.

Here are some considerations on the approximation of (2.2). Let $\mathcal{V}^n$ be some approximation or perturbation of $\mathcal{V}$, which is supposed to satisfy the same predictability and integrability assumptions (2.3) as $\mathcal{V}$. We want to assess the error $X_t - X_t^n$, where

$$X_t^n = \int_0^t \mathcal{S}(t-s)\sqrt{\mathcal{V}_s^n} dB_s. \quad (2.5)$$

Clearly, it holds for $I_t^n := X_t - X_t^n$ that

$$I_t^n = \int_0^t \mathcal{S}(t-s)(\sqrt{\mathcal{V}_s} - \sqrt{\mathcal{V}_s^n}) dB_s. \quad (2.6)$$

In this section, we want to bound the random variable $\sup_{0 \leq t \leq T} |I_t^n|_H^2$ in $L^1(\mathbb{P})$ for the given time horizon $T > 0$. It is relatively straightforward to obtain a t -dependent $L^1(\mathbb{P})$ -bound for $|I_t^n|_H^2$. Our approach is then to demonstrate that the real-valued stochastic process $t \mapsto |I_t^n|_H^2$ is a.s. continuous and then to apply the measurable

selection theorem of Kuratowski and Ryll-Nardzewski (see e.g. Aliprantis and Border [1, Sect. 18.3] for an excellent exposition). Their selection theorem states that there exists a random variable ξ that takes values in $[0, T]$ such that

$$\sup_{0 \leq t \leq T} |I_t^n|^2_H = |I_\xi^n|^2_H.$$

To this end, denote by $C([0, T]; H)$ the space of all continuous H -valued functions on $[0, T]$ equipped with the supremum norm. Furthermore, for any $q \geq 1$, let $L^q(0, T; H) := L^q((0, T), \mathcal{B}((0, T)), \text{Leb}; H)$, where Leb denotes Lebesgue measure. The following result is inspired by Peszat and Zabczyk [40, Chap. 11], and we use it to prove that the real-valued stochastic process $t \mapsto |I_t^n|^2_H$ is a.s. continuous. This is needed in the proof of Theorem 2.3.

Lemma 2.2 *Suppose that $\mathcal{Z}$ is a predictable $\mathcal{H}$ -valued stochastic process such that*

$$\mathbb{E} \left[\int_0^T \|\mathcal{Z}_s Q^{1/2}\|_{\mathcal{H}}^q ds \right] < \infty$$

holds for some $q > 2$ and $T > 0$. If

$$X_t = \int_0^t \mathcal{S}(t-s) \mathcal{Z}_s dB_s,$$

then $X \in C([0, T]; H)$ holds a.s.

Proof Suppose $\alpha \in (0, 1/2)$ is a constant with $1/q < \alpha$. For each $t \in [0, T]$, let

$$Y_t := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \mathcal{S}(t-s) \mathcal{Z}_s dB_s.$$

Then by the Burkholder–Davis–Gundy inequality, see Marinelli and Röckner [37], Young’s inequality and the Hille–Yosida bound $\|\mathcal{S}(t)\|_{\text{op}} \leq ce^{kt}$, it follows that

$$\begin{aligned} & \int_0^T \mathbb{E}[|Y_t|_H^q] dt \\ & \leq C \int_0^T \mathbb{E} \left[\left(\int_0^t (t-s)^{-2\alpha} \|\mathcal{S}(t-s) \mathcal{Z}_s Q^{1/2}\|_{\mathcal{H}}^2 ds \right)^{q/2} \right] dt \\ & \leq C \sup_{0 \leq t \leq T} \|\mathcal{S}(t)\|_{\text{op}}^q \left(\int_0^T s^{-2\alpha} ds \right)^{q/2} \mathbb{E} \left[\int_0^T \|\mathcal{Z}_s Q^{1/2}\|_{\mathcal{H}}^q ds \right] < \infty, \end{aligned}$$

where $C > 0$ is a constant that depends only on q . Thus $|Y|_H^q \in L^1(\text{Leb} \times \mathbb{P})$, and Fubini’s theorem implies that $|Y|_H^q \in L^1(\text{Leb})$ a.s., i.e., $Y \in L^q(0, T; H)$ a.s. Now for each $\psi \in L^q(0, T; H)$, define the Liouville–Riemann operator I_α by

$$(I_\alpha \psi)(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mathcal{S}(t-s) \psi(s) ds.$$

By the stochastic Fubini theorem and the semigroup property of $\mathcal{S}$, it follows that

$$\begin{aligned}\Gamma(\alpha)\Gamma(1-\alpha)(I_\alpha Y)_t &= \int_0^t (t-s)^{\alpha-1} \mathcal{S}(t-s) \int_0^s (s-r)^{-\alpha} \mathcal{S}(s-r) \mathcal{Z}_r dB_r ds \\ &= \int_0^t \mathcal{S}(t-r) \left(\int_r^t (t-s)^{\alpha-1} (s-r)^{-\alpha} ds \right) \mathcal{Z}_r dB_r.\end{aligned}$$

Furthermore, by Peszat and Zabczyk [40, Lemma 11.2], we find that

$$\int_r^t (t-s)^{\alpha-1} (s-r)^{-\alpha} ds = \int_0^1 (1-z)^{\alpha-1} z^{-\alpha} dz = \frac{\Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1)}.$$

Therefore $X_t = (I_\alpha Y)_t$. According to Peszat and Zabczyk [40, Theorem 11.5], I_α is a bounded linear operator from $L^q(0, T; H)$ to $C([0, T]; H)$; so $X \in C([0, T]; H)$ holds almost surely. $\square$

We are now in a position to prove our main results. The following maximal-inequality-type result is related to the Burkholder-type inequalities in Gawarecki and Mandrekar [31, Lemma 3.3]. The result we present here has to the best of our knowledge not been given in the case when $\mathcal{S}$ is a general semigroup (not necessarily quasi-contractive), and our result is moreover stated with the operator norm on the right-hand side, rather than the Hilbert–Schmidt norm, which enables us to prove Corollary 2.4 concerning the robustness of the stochastic volatility process with respect to the variance process (2.6). In the proof, we employ the measurable selection theorem of Kuratowski and Ryll–Nardzewski [36] in the final step to obtain the inequality with the supremum over the compact interval $[0, T]$ inside the expected value. We observe that this is necessary since we cannot employ the steps we do at the beginning of the proof with the supremum inside the integral, nor could we have used the methodology in [31] under our assumptions.

Theorem 2.3 *Suppose that $\mathcal{Z}$ is a predictable $\mathcal{H}$ -valued stochastic process and that*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{Z}_t\|_{\text{op}}^2 \right] \vee \mathbb{E} \left[\int_0^T \|\mathcal{Z}_s Q^{1/2}\|_{\mathcal{H}}^q ds \right] < \infty$$

for some $q > 2$ and $T > 0$. Then

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \left\| \int_0^t \mathcal{S}(t-s) \mathcal{Z}_s dB_s \right\|_H^2 \right] \leq C(T) \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{Z}_t\|_{\text{op}}^2 \right],$$

where $C(T) = c^2 \text{Tr}(Q) \int_0^T e^{2ks} ds$ and $\|\mathcal{S}(t)\|_{\text{op}} \leq ce^{kt}$ for constants $k \in \mathbb{R}$, $c > 0$ and all $t \geq 0$.

Proof Let $X_t := \int_0^t \mathcal{S}(t-s) \mathcal{Z}_s dB_s$. By the Itô isometry, we have

$$\mathbb{E}[\|X_t\|_H^2] = \mathbb{E} \left[\int_0^t \|\mathcal{S}(t-s) \mathcal{Z}_s Q^{1/2}\|_{\mathcal{H}}^2 ds \right].$$

Since the Hilbert–Schmidt norm satisfies $\|\mathcal{L}\mathcal{K}\|_{\mathcal{H}} \leq \|\mathcal{L}\|_{\text{op}}\|\mathcal{K}\|_{\mathcal{H}}$ for a bounded linear operator $\mathcal{L}$ and a Hilbert–Schmidt operator $\mathcal{K}$, it follows that

$$\begin{aligned}\|\mathcal{S}(t-s)\mathcal{Z}_s Q^{1/2}\|_{\mathcal{H}} &\leq \|\mathcal{S}(t-s)\|_{\text{op}}\|\mathcal{Z}_s Q^{1/2}\|_{\mathcal{H}} \\ &\leq ce^{k(t-s)}\|\mathcal{Z}_s\|_{\text{op}}\|Q^{1/2}\|_{\mathcal{H}} \\ &= ce^{k(t-s)}\text{Tr}(Q)^{1/2}\|\mathcal{Z}_s\|_{\text{op}}.\end{aligned}$$

In the second inequality, we appealed to the general Hille–Yosida bound of the operator norm of a C_0 -semigroup. In the last step, we used the well-known identity $\text{Tr}(Q) = \|Q^{1/2}\|_{\mathcal{H}}^2$. Hence, if $t \leq T$,

$$\begin{aligned}\mathbb{E}[|X_t|_H^2] &\leq c^2\text{Tr}(Q)\mathbb{E}\left[\int_0^t e^{2k(t-s)}\|\mathcal{Z}_s\|_{\text{op}}^2 ds\right] \\ &\leq c^2\text{Tr}(Q)\int_0^t e^{2k(t-s)}ds\mathbb{E}\left[\sup_{0\leq t\leq T}\|\mathcal{Z}_t\|_{\text{op}}^2\right] \\ &\leq C(t)\mathbb{E}\left[\sup_{0\leq t\leq T}\|\mathcal{Z}_t\|_{\text{op}}^2\right],\end{aligned}$$

where $C(t) = c^2\text{Tr}(Q)\int_0^t e^{2ks}ds$ for all $t \in [0, T]$. Now we apply Lemma 2.2 to conclude that $X \in C([0, T]; H)$ and therefore $t \mapsto |X_t|_H^2 \in C([0, T]; \mathbb{R})$. By the measurable selection theorem of Kuratowski and Ryll–Nardzewski [36] (see also Peszat and Zabczyk [40, Theorem 8.8]), we conclude that there exists a random variable ξ which takes values in $[0, T]$ and such that $\sup_{0\leq t\leq T}|X_t|_H^2 = |X_\xi|_H^2$ holds. Hence, since $\xi \leq T$ and $C(\xi) \leq C(T)$, it follows that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t|_H^2\right] = \mathbb{E}\left[\mathbb{E}[|X_\xi|_H^2|\xi]\right] \leq C(T)\mathbb{E}\left[\sup_{0\leq t\leq T}\|\mathcal{Z}_t\|_{\text{op}}^2\right].$$

The proof is completed. $\square$

We remark that we may use the fact that $\|\mathcal{L}\|_{\text{op}} \leq \|\mathcal{L}\|_{\mathcal{H}}$ holds for a Hilbert–Schmidt operator $\mathcal{L}$ to obtain a version of the preceding result with the Hilbert–Schmidt norm on the right-hand side of the inequality. However, we do genuinely need the operator norm to obtain the following result. We furthermore remark that in Sect. 4, we discuss approximations of the square root process using the operator and the Hilbert–Schmidt norms.

Corollary 2.4 *Suppose that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\|\mathcal{V}_t - \mathcal{V}_t^n\|_{\mathcal{H}}^{q/2}\right] < \infty$$

for some $q > 2$. Then

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|I_t^n|_H^2\right] \leq C(T)\mathbb{E}\left[\sup_{0\leq t\leq T}\|\mathcal{V}_t - \mathcal{V}_t^n\|_{\mathcal{H}}\right],$$

where we recall the definition of I_t^n in (2.6). Here $C(T) = c^2 \text{Tr}(Q) \int_0^T e^{2ks} ds$, and $k \in \mathbb{R}$, $c > 0$ are constants such that $\|S(t)\|_{\text{op}} \leq ce^{kt}$.

Proof According to Bogachev [21, Lemma 2.5.1], it holds that

$$\|\sqrt{\mathcal{V}_s} - \sqrt{\mathcal{V}_s^n}\|_{\text{op}}^2 \leq \|\mathcal{V}_s - \mathcal{V}_s^n\|_{\text{op}}.$$

Hence by the assumption and the fact that $\|\mathcal{L}\|_{\text{op}} \leq \|\mathcal{L}\|_{\mathcal{H}}$ for a Hilbert–Schmidt operator $\mathcal{L}$, it follows that

$$\begin{aligned} \mathbb{E} \left[\int_0^T \|(\sqrt{\mathcal{V}_s} - \sqrt{\mathcal{V}_s^n}) Q^{1/2}\|_{\mathcal{H}}^q ds \right] &\leq C \mathbb{E} \left[\int_0^T \|\sqrt{\mathcal{V}_s} - \sqrt{\mathcal{V}_s^n}\|_{\text{op}}^q ds \right] \\ &\leq CT \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{V}_t - \mathcal{V}_t^n\|_{\mathcal{H}}^{q/2} \right] < \infty, \end{aligned}$$

where $C = (\text{Tr}(Q))^{q/2}$. Note furthermore that Hölder's inequality yields

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\sqrt{\mathcal{V}_t} - \sqrt{\mathcal{V}_t^n}\|_{\text{op}}^2 \right] &\leq \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{V}_t - \mathcal{V}_t^n\|_{\mathcal{H}} \right] \\ &\leq \mathbb{E} \left[\left(\sup_{0 \leq t \leq T} \|\mathcal{V}_t - \mathcal{V}_t^n\|_{\mathcal{H}} \right)^{q/2} \right] \\ &= \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{V}_t - \mathcal{V}_t^n\|_{\mathcal{H}}^{q/2} \right]. \end{aligned}$$

Thus the result follows from Theorem 2.3. $\square$

Consider a financial derivative written on a forward or futures contract on a commodity. Forwards are contracts delivering an underlying (oil, metals, soybeans, power or gas, say) at a predetermined (period of) time. Following the analysis in Benth and Krühner [12], we can express the price at time $t \geq 0$ of such a forward as $\mathcal{D}(X_t)$ for some $\mathcal{D} \in H^*$ and X_t in (2.2). The linear functional $\mathcal{D}$ models either fixed-delivery contracts or contracts where delivery takes place over a time period. Assuming that X is modelled directly under the risk-neutral probability measure, the current arbitrage-free price of a European option with exercise time τ paying $p(\mathcal{D}(X_\tau))$ is given by

$$P = \mathbb{E}[p(\mathcal{D}(X_\tau))].$$

Here, we have let the risk-free interest rate be equal to zero for simplicity. If we consider a payoff function $p : \mathbb{R} \rightarrow \mathbb{R}$ which is Lipschitz-continuous with Lipschitz constant $K > 0$, we find with X^n as in (2.5) that

$$\begin{aligned} |P - P^n| &= |\mathbb{E}[p(\mathcal{D}(X_\tau))] - \mathbb{E}[p(\mathcal{D}(X_\tau^n))]| \\ &\leq K \mathbb{E}[|\mathcal{D}(X_\tau - X_\tau^n)|] \\ &\leq K \|\mathcal{D}\|_{\text{op}} \mathbb{E}[|X_\tau - X_\tau^n|_H]. \end{aligned}$$

By Corollary 2.4, we can bound $|P - P^n|$ by the norm difference of the variance processes $\mathcal{V} - \mathcal{V}^n$. This tells us that option prices depend continuously on the variance process. In the next sections, we are going to establish precise estimates for the error induced by the difference in variance, giving robustness estimates for the option price.

In portfolio management and estimation of models, the forward dynamics are considered under the objective market probability rather than the risk-neutral probability. In such models, a *risk premium* enters into the general model (2.4) (or (1.1)) as an additional drift term. This drift term may depend on the variance. This would mean that we have a mild solution of the form (2.2) with an additional term

$$\int_0^t \mathcal{S}(t-s) \gamma(\mathcal{V}_s) ds.$$

A stability analysis would be possible by modifying the above arguments under appropriate conditions on γ . Furthermore, if we consider in such a context pricing of derivatives, the change of measure to the risk-neutral one involves a dependency on the variance process as well. To assess stability then is becoming a challenging task that we plan to investigate in future works. Notice also that in risk management, one may be interested in calculations of the so-called value-at-risk (VaR) of positions in forwards. The term structure of variance is the fundamental variable in such calculations, but the risk premium is also of importance.

Before proceeding, let us discuss a specific choice of a Hilbert space H particularly relevant for commodity forward market modelling. As already indicated in Example 2.1, the Filipović space is a convenient choice for H when modelling the term structure of commodity forwards. To this end, consider real-valued measurable functions g on $\mathbb{R}_+$ which are weakly differentiable and satisfy

$$|g|_w^2 := (g(0))^2 + \int_0^\infty w(x) (g'(x))^2 dx < \infty$$

for a nondecreasing continuously differentiable scaling function $w : \mathbb{R}_+ \rightarrow [1, \infty)$ with $w(0) = 1$ and $w^{-1} \in L^1(\mathbb{R}_+)$. Here, g' denotes the weak derivative of g . The space H equipped with the norm $|\cdot|_w$ is a separable Hilbert space (see Filipović [29, Chap. 5]) and furthermore, the evaluation operator is a continuous linear functional on H and the shift semigroup defines a C_0 -semigroup. Moreover, the Filipović space is a Banach algebra. From Benth and Krühner [11, Lemma 3.5], the shift semigroup is quasi-contractive on the Filipović space. A typical choice of scaling function is $w(x) := \exp(\alpha x)$ for $\alpha > 0$.

The models we consider in this paper are driven by noise processes that are elements in infinite-dimensional separable Hilbert spaces (i.e., the noise has infinite rank), which allows e.g. modelling *maturity-specific risk*. This is a realistic model characteristic from the perspective of hedging in these models; see for example Carmona and Tehranchi [22, Chap. 6]. In practical applications, one would project onto finitely many principal components (eigenfunctions). Theorem 2.3 and the results of the following sections quantify the expected (pathwise) error introduced by approximating the volatility in such models with finite-rank models, where the latter describe the behaviour of the forward curves under the n leading factors.

3 The variance process

In this section, we focus on the BNS stochastic volatility model for $\mathcal{V}$, as introduced in Benth et al. [13] and later studied in an affine extension by Cox et al. [23, 24]. We introduce various perturbations of $\mathcal{V}$ and quantify the induced distance to the original volatility process.

Given a square-integrable Lévy process $(\mathcal{L}_t)_{t \geq 0}$ with covariance operator $\mathcal{Q}_{\mathcal{L}}$, consider the variance process

$$d\mathcal{V}_t = \mathfrak{C}\mathcal{V}_t dt + d\mathcal{L}_t \quad (3.1)$$

with initial condition $\mathcal{V}_0 \in \mathcal{H}$ and where $\mathfrak{C} \in L(\mathcal{H})$ is a bounded operator. Then we have

$$\mathcal{V}_t = \mathfrak{S}(t)\mathcal{V}_0 + \int_0^t \mathfrak{S}(t-s)d\mathcal{L}_s, \quad (3.2)$$

where $\mathfrak{C}$ generates the uniformly continuous C_0 -semigroup $\mathfrak{S}(t) = \exp(\mathfrak{C}t)$ on $\mathcal{H}$. We remark that since $L(\mathcal{H})$ is a Banach algebra, it follows that $\mathfrak{S}(t) \in L(\mathcal{H})$. Note that the variance process $\mathcal{V}$ depends on the initial position $\mathcal{V}_0$, the semigroup generator $\mathfrak{C}$ and the Lévy process $\mathcal{L}$. We furthermore remark that one can let $\mathfrak{C}$ be an unbounded operator in the variance process. However, in what follows, we analyse a class of generators which is restricted to being bounded in Benth et al. [13] due to technical reasons. Therefore we only consider bounded generators in what follows, a property that we frequently make use of. Recall the definition of the mean of an $\mathcal{H}$ -valued Lévy process, cf. Peszat and Zabczyk [40, Definition 4.45]: If $\mathcal{L}$ is an $\mathcal{H}$ -valued Lévy process with a mean $\mathcal{M} \in \mathcal{H}$, then the process $(\tilde{\mathcal{L}}_t)_{t \geq 0}$ given by

$$\tilde{\mathcal{L}}_t := \mathcal{L}_t - \mathcal{M}t \quad (3.3)$$

is a zero-mean martingale.

Proposition 3.1 *It holds that*

$$\mathbb{E}[\|\mathcal{V}_t\|_{\mathcal{H}}^2] \leq C_0 \|\mathfrak{S}(t)\|_{\text{op}} + C_1 \int_0^t \|\mathfrak{S}(s)\|_{\text{op}} ds + C_2 \left(\int_0^t \|\mathfrak{S}(s)\|_{\text{op}} ds \right)^2,$$

where $C_0 = 3\mathbb{E}[\|\mathcal{V}_0\|_{\mathcal{H}}^2]$, $C_1 = 3\text{Tr}(\mathcal{Q}_{\mathcal{L}})$ and $C_2 = 3\|\mathcal{M}\|_{\mathcal{H}}^2$ and $\mathcal{M}$ and $\mathcal{Q}_{\mathcal{L}}$ denote the mean and covariance operator of $\mathcal{L}$, respectively.

Proof By (3.2) and (3.3), it holds that

$$\begin{aligned} \mathbb{E}[\|\mathcal{V}_t\|_{\mathcal{H}}^2] &\leq 3 \left(\mathbb{E}[\|\mathfrak{S}(t)\mathcal{V}_0\|_{\mathcal{H}}^2] + \left\| \int_0^t \mathfrak{S}(t-s)ds \mathcal{M} \right\|_{\mathcal{H}}^2 \right. \\ &\quad \left. + \mathbb{E} \left[\left\| \int_0^t \mathfrak{S}(t-s)d\tilde{\mathcal{L}}_s \right\|_{\mathcal{H}}^2 \right] \right). \end{aligned}$$

Observe that $\mathbb{E}[\|\mathfrak{S}(t)\mathcal{V}_0\|_{\mathcal{H}}^2] \leq \mathbb{E}[\|\mathcal{V}_0\|_{\mathcal{H}}^2]\|\mathfrak{S}(t)\|_{\text{op}}$, and similarly, it holds that

$$\left\| \int_0^t \mathfrak{S}(t-s)ds\mathcal{M} \right\|_{\mathcal{H}} \leq \|\mathcal{M}\|_{\mathcal{H}} \int_0^t \|\mathfrak{S}(t-s)\|_{\text{op}}ds.$$

By [40, Corollary 8.17], it moreover follows that

$$\begin{aligned} \mathbb{E} \left[\left\| \int_0^t \mathfrak{S}(t-s)d\tilde{\mathcal{L}}_s \right\|_{\mathcal{H}}^2 \right] &= \int_0^t \|\mathfrak{S}(t-s)\mathcal{Q}_{\mathcal{L}}^{1/2}\|_{L_{\text{HS}}(\mathcal{H})}^2 ds \\ &\leq \text{Tr}(\mathcal{Q}_{\mathcal{L}}) \int_0^t \|\mathfrak{S}(t-s)\|_{\text{op}} ds, \end{aligned}$$

where we used the trace equality $\text{Tr}(\mathcal{Q}_{\mathcal{L}}) = \|\mathcal{Q}_{\mathcal{L}}^{1/2}\|_{L_{\text{HS}}(\mathcal{H})}^2$. The result follows. $\square$

Given Proposition 3.1, it follows that if the semigroup $(\mathfrak{S}(s))_{s \geq 0}$ is uniformly exponentially stable, then $\limsup_{t \rightarrow \infty} \mathbb{E}[\|\mathcal{V}_t\|_{\mathcal{H}}^2] < \infty$. Note furthermore that since $(\mathfrak{S}(s))_{s \geq 0}$ is by definition a uniformly continuous semigroup, the uniform exponential stability is equivalent to $s(\mathfrak{C}) < 0$, where $s(\mathfrak{C}) := \sup\{\Re \lambda : \lambda \in \sigma(\mathfrak{C})\}$ is the spectral bound of $\mathfrak{C}$ and $\sigma(\mathfrak{C})$ denotes the spectrum of $\mathfrak{C}$ (see Engel and Nagel [28, Theorem V.3.7]). Thus if $\Re \lambda < 0$ for all $\lambda \in \sigma(\mathfrak{C})$ and $\partial\sigma(\mathfrak{C}) \cap \{z \in \mathbb{C} : \Re z = 0\} = \emptyset$, then $\limsup_{t \rightarrow \infty} \mathbb{E}[\|\mathcal{V}_t\|_{\mathcal{H}}^2] < \infty$. We give examples of specifications for the operator $\mathfrak{C}$ and relate them to these results in Sect. 3.2.

Introduce the process $\mathcal{V} = (\mathcal{V}_t^{\text{stat}})_{t \geq 0}$, where

$$\mathcal{V}_t^{\text{stat}} := \int_{-\infty}^t \mathfrak{S}(t-s)d\mathcal{L}_s. \quad (3.4)$$

It can be shown that $\mathcal{V}^{\text{stat}}$ is stationary with a time-independent distribution. One can show that when $s(\mathfrak{C}) < 0$, the variance process $\mathcal{V}$ is asymptotically stationary when $t \rightarrow \infty$ with a stationary distribution coinciding with that of $\mathcal{V}_0^{\text{stat}}$, meaning that the characteristic functional of $\mathcal{V}_t$ converges to the characteristic functional of $\mathcal{V}_0^{\text{stat}}$ as $t \rightarrow \infty$; see Benth and Eyjolfsson [9] for more details. As can be seen from the characteristic functional of $\mathcal{V}_t^{\text{stat}}$ (see [9]), the distribution of the scalar stochastic process $t \mapsto \langle \mathcal{V}_t^{\text{stat}}, \mathcal{T} \rangle_{\mathcal{H}}$ is invariant to shifts with respect to t for any $\mathcal{T} \in \mathcal{H}$, and we furthermore have $\mathbb{E}[|\langle \mathcal{V}_t^{\text{stat}}, \mathcal{T} \rangle_{\mathcal{H}}|^2] \leq \|\mathcal{T}\|_{\mathcal{H}}^2 \mathbb{E}[\|\mathcal{V}_t^{\text{stat}}\|_{\mathcal{H}}^2] < \infty$ by the Cauchy–Schwarz inequality.

We consider approximations of the variance process in (3.2) of the form

$$d\mathcal{V}_t^n = \mathfrak{C}^n \mathcal{V}_t^n dt + d\mathcal{L}_t^n, \quad (3.5)$$

where the initial value $\mathcal{V}_0^n$, semigroup generator $\mathfrak{C}^n$ and a Lévy process $\mathcal{L}^n$ approximate the initial value, generator and Lévy process of the variance process $\mathcal{V}$. We remark that in order to ensure that the square root of the volatility process is well defined, it must be self-adjoint and nonnegative definite. Conditions which ensure this are given in Sect. 4. In the following subsections, we study results which quantify the error induced in the variance process by approximating the corresponding initial value, the Lévy process and the semigroup generator, respectively.

3.1 Approximation in the compound Poisson process case

In this subsection, we suppose that $\mathcal{L}$ is an $\mathcal{H}$ -valued compound Poisson process

$$\mathcal{L}_t = \sum_{i=1}^{N_t} \mathcal{X}_i, \quad (3.6)$$

where $(N_t)_{t \geq 0}$ is a Poisson process with intensity $\lambda > 0$ and $(\mathcal{X}_i)_{i \in \mathbb{N}}$ is a sequence of independent and identically distributed (i.i.d.) $\mathcal{H}$ -valued random variables. We choose an RCLL version of N so that $\mathcal{L}$ becomes an RCLL Lévy process with values in $\mathcal{H}$. Suppose furthermore that

$$\mathcal{L}_t^n = \sum_{i=1}^{N_t} \mathcal{X}_i^n, \quad (3.7)$$

where the compound Poisson processes $\mathcal{L}$ and $\mathcal{L}^n$ are driven by the same Poisson process N for each $n \geq 1$. We study finite-dimensional approximations of the compound Poisson process (3.6) by a sequence of compound Poisson processes of the form (3.7).

Example 3.2 Suppose that $\mathcal{X}_i = (Y_i)^{\otimes 2} = (Y_i, \cdot)_H Y_i$, where $(Y_i)_{i \in \mathbb{N}}$ are i.i.d. and H -valued random variables. Given an ONB $(e_j)_{j \in \mathbb{N}}$ of H , define

$$Y_i^n = \sum_{j=1}^n (Y_i, e_j)_H e_j \quad (3.8)$$

for $n \geq 1$. In Proposition 3.4, we study the approximation of the i.i.d. $\mathcal{H}$ -valued random variables $\mathcal{X}_i$, i.e., the jumps of the compound Poisson process defined in (3.6), with $\mathcal{X}_i^n = (Y_i^n)^{\otimes 2}$, where $n \geq 1$.

The following result shows that the tensor product forms a locally Lipschitz-continuous operator.

Lemma 3.3 *If $f, g \in H$, then*

$$\|f^{\otimes 2} - g^{\otimes 2}\|_{\mathcal{H}} \leq (|f|_H + |g|_H) |f - g|_H.$$

Proof By the definition of the Hilbert–Schmidt norm and the triangle inequality, it holds that

$$\begin{aligned} \|f^{\otimes 2} - g^{\otimes 2}\|_{\mathcal{H}}^2 &= \sum_{j=1}^{\infty} |(f, e_j)_H f - (g, e_j)_H g|_H^2 \\ &\leq \sum_{j=1}^{\infty} (|(f, e_j)_H| |f - g|_H + |(f - g, e_j)_H| |g|_H)^2. \end{aligned}$$

By Parseval's identity, it follows that

$$\sum_{j=1}^{\infty} ((f, e_j)_H^2 |f - g|_H^2 + (f - g, e_j)_H^2 |g|_H^2) = (|f|_H^2 + |g|_H^2) |f - g|_H^2,$$

and an application of the Cauchy–Schwarz inequality together with Parseval's identity yields

$$\sum_{j=1}^{\infty} |(f, e_j)_H| |(f - g, e_j)_H| \leq |f|_H |f - g|_H.$$

Hence we conclude that $\|f^{\otimes 2} - g^{\otimes 2}\|_{\mathcal{H}}^2 \leq (|f|_H + |g|_H)^2 |f - g|_H^2$. $\square$

An application of Hölder's inequality now yields the following result.

Proposition 3.4 *Suppose that $\mathcal{X}_i = (Y_i)^{\otimes 2}$, where $(Y_i)_{i \in \mathbb{N}}$ is a sequence of i.i.d. H -valued random variables, and $\mathcal{X}_i^n = (Y_i^n)^{\otimes 2}$, where Y_i^n is defined by (3.8) for all $n, i \geq 1$. Then if $p, q \in [1, \infty]$ are conjugate exponents, it holds that*

$$\mathbb{E}[\|\mathcal{X}_i - \mathcal{X}_i^n\|_{\mathcal{H}}^2] \leq 2\mathbb{E}[|Y_i|_H^{2p}]^{1/p} \mathbb{E}[|Y_i - Y_i^n|_H^{2q}]^{1/q}. \quad (3.9)$$

In particular, we have that

$$\mathbb{E}[\|\mathcal{X}_i - \mathcal{X}_i^n\|_{\mathcal{H}}^2] \leq 2(\mathbb{E}[|Y_i|_H^4] \mathbb{E}[|Y_i - Y_i^n|_H^4])^{1/2}, \quad (3.10)$$

and if $|Y_i|_H \in L^4(\mathbb{P})$, the right-hand side of (3.10) converges to zero as $n \rightarrow \infty$.

Proof It follows by Lemma 3.3 and the inequality $|Y_i^n|_H \leq |Y_i|_H$ that

$$\|\mathcal{X}_i - \mathcal{X}_i^n\|_{\mathcal{H}}^2 \leq 2|Y_i|_H^2 |Y_i - Y_i^n|_H^2.$$

The bound in (3.9) now follows by Hölder's inequality. We obtain the particular case (3.10) by choosing $p = q = 2$. According to the triangle inequality,

$$|Y_i - Y_i^n|_H \leq |Y_i|_H + |Y_i^n|_H \leq 2|Y_i|_H.$$

Because $Y_i - Y_i^n \rightarrow 0$ a.s. due to (3.8), the dominated convergence theorem yields that the right-hand side of (3.10) converges to 0 as $n \rightarrow \infty$. $\square$

Let us state some useful lemmas; the first is known, but we include it here for the convenience of the reader and for future reference.

Lemma 3.5 *Suppose that U is a separable Hilbert space and $L_t = \sum_{i=1}^{N_t} J_i$ for $t \geq 0$ is a U -valued compound Poisson process, where the Poisson process N has intensity $\lambda > 0$ and $(J_i)_{i \in \mathbb{N}}$ are i.i.d. U -valued random variables. Then:*

1) *We have*

$$\mathbb{E}[|L_t|_U^2] = \lambda t \mathbb{E}[|J_1|_U^2] + \lambda^2 t^2 \mathbb{E}[J_1]_U^2.$$

2) We have

$$\mathbb{E}[|L_t|_U^2] \leq \lambda t(1 + \lambda t)\mathbb{E}[|J_1|_U^2].$$

Proof By conditioning, we note that

$$\mathbb{E}[|L_t|_U^2] = \mathbb{E}\left[\mathbb{E}\left[\left|\sum_{i=1}^{N_t} J_i\right|_U^2 \middle| N_t\right]\right] = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \mathbb{E}\left[\left|\sum_{i=1}^k J_i\right|_U^2\right].$$

Since $\langle J_i, J_i \rangle_U = |J_i|_U^2$ and $\mathbb{E}[\langle J_i, J_j \rangle_U] = \langle \mathbb{E}[J_i], \mathbb{E}[J_j] \rangle_U$ when $i \neq j$ and the $(J_i)_{i \in \mathbb{N}}$ are i.i.d., it follows that

$$\mathbb{E}\left[\left|\sum_{i=1}^k J_i\right|_U^2\right] = \sum_{i,j=1}^k \mathbb{E}[\langle J_i, J_j \rangle_U] = k\mathbb{E}[|J_1|_U^2] + k(k-1)\mathbb{E}[J_1]_U^2.$$

Putting these equations together, it follows that

$$\mathbb{E}[|L_t|_U^2] = \lambda t\mathbb{E}[|J_1|_U^2] + \lambda^2 t^2 |\mathbb{E}[J_1]|_U^2.$$

Finally, 2) follows from 1) and

$$|\mathbb{E}[J_1]|_U^2 \leq (\mathbb{E}[|J_1|_U])^2 \leq \mathbb{E}[|J_1|_U^2],$$

where we have used Jensen's inequality twice. $\square$

The following lemma will be used for the Hilbert space $\mathcal{H}$ and the Banach space of trace-class operators.

Lemma 3.6 Suppose that $\|\cdot\|_{\mathcal{B}}$ denotes the norm of a Banach space $\mathcal{B}$ which is a subspace of $L(H)$, and assume that $\mathfrak{C} \in L(\mathcal{B})$. Suppose moreover that $\mathcal{V}$ and $\mathcal{V}^n$ are the variance processes defined by (3.1) and (3.5), respectively, where $\mathfrak{C}^n = \mathfrak{C}$ and the compound Poisson processes $\mathcal{L}$ and $\mathcal{L}^n$ are driven by the same Poisson process N , and that $\mathcal{V}_t, \mathcal{V}_t^n \in \mathcal{B}$ for all $t \in [0, T]$, where $T > 0$. Then it holds that

$$\|\mathcal{V}_t - \mathcal{V}_t^n\|_{\mathcal{B}} \leq e^{\|\mathfrak{C}\|_{\text{op}} t} \left(\|\mathcal{V}_0 - \mathcal{V}_0^n\|_{\mathcal{B}} + \sum_{i=1}^{N_t} \|\mathcal{X}_i - \mathcal{X}_i^n\|_{\mathcal{B}} \right).$$

Proof Let $(T_i)_{i \in \mathbb{N}}$ be the (exponentially distributed) jump times of the Poisson process N which the compound Poisson processes $\mathcal{L}$ and $\mathcal{L}^n$ have in common. Then by

the triangle inequality and the assumption that $\mathfrak{C} \in L(\mathcal{B})$,

$$\begin{aligned} \|\mathcal{V}_t - \mathcal{V}_t^n\|_{\mathcal{B}} &= \|\mathfrak{S}(t)(\mathcal{V}_0 - \mathcal{V}_0^n) + \sum_{T_i \leq t} \mathfrak{S}(t - T_i)(\mathcal{X}_i - \mathcal{X}_i^n)\|_{\mathcal{B}} \\ &\leq \|\mathfrak{S}(t)\|_{\text{op}} \|\mathcal{V}_0 - \mathcal{V}_0^n\|_{\mathcal{B}} + \sum_{T_i \leq t} \|\mathfrak{S}(t - T_i)\|_{\text{op}} \|\mathcal{X}_i - \mathcal{X}_i^n\|_{\mathcal{B}} \\ &\leq e^{\|\mathfrak{C}\|_{\text{op}} t} \|\mathcal{V}_0 - \mathcal{V}_0^n\|_{\mathcal{B}} + \sum_{T_i \leq t} e^{\|\mathfrak{C}\|_{\text{op}}(t - T_i)} \|\mathcal{X}_i - \mathcal{X}_i^n\|_{\mathcal{B}} \\ &\leq e^{\|\mathfrak{C}\|_{\text{op}} t} \left(\|\mathcal{V}_0 - \mathcal{V}_0^n\|_{\mathcal{B}} + \sum_{i=1}^{N_t} \|\mathcal{X}_i - \mathcal{X}_i^n\|_{\mathcal{B}} \right), \end{aligned}$$

where the second inequality uses the exponential bound of the semigroup. $\square$

Now we investigate the error induced on the compound Poisson and variance processes by approximating $\mathcal{X}_i$ with $\mathcal{X}_i^n$, where we do not assume any specific form for the jump-size approximation $\mathcal{X}_i^n$.

Proposition 3.7 *Suppose for each $n \geq 1$ that $(\mathcal{X}_i)_{i \in \mathbb{N}}$ and $(\mathcal{X}_i^n)_{i \in \mathbb{N}}$ are sequences of $\mathcal{H}$ -valued i.i.d. random variables and that $\|\mathcal{X}_i\|_{\mathcal{H}}, \|\mathcal{X}_i^n\|_{\mathcal{H}} \in L^2(\mathbb{P})$ for all $n, i \geq 1$. Then $\mathcal{L}$ and $\mathcal{L}^n$ defined by (3.6) and (3.7), respectively, are square-integrable compound Poisson processes. Suppose furthermore that $\mathcal{V}$ and $\mathcal{V}^n$ are the variance processes defined by (3.1) and (3.5), respectively, where $\mathfrak{C}^n = \mathfrak{C} \in L(\mathcal{H})$ for all $n \geq 1$ and the compound Poisson processes $\mathcal{L}$ and $\mathcal{L}^n$ are driven by the same Poisson process N . Then for every $T > 0$,*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{V}_t - \mathcal{V}_t^n\|_{\mathcal{H}}^2 \right] \leq C_0(T) \mathbb{E}[\|\mathcal{V}_0 - \mathcal{V}_0^n\|_{\mathcal{H}}^2] + C_1(T) \mathbb{E}[\|\mathcal{X}_1 - \mathcal{X}_1^n\|_{\mathcal{H}}^2],$$

where $C_0(T) = 2e^{2T\|\mathfrak{C}\|_{\text{op}}}$ and $C_1(T) = 2e^{2T\|\mathfrak{C}\|_{\text{op}}} T\lambda(1 + \lambda T)e^{2T\|\mathfrak{C}\|_{\text{op}}}$.

Proof By part 2) of Lemma 3.5, it holds that

$$\mathbb{E}[\|\mathcal{L}_t\|_{\mathcal{H}}^2] \leq \lambda t(1 + \lambda t) \mathbb{E}[\|\mathcal{X}_1\|_{\mathcal{H}}^2] < \infty.$$

So $\mathcal{L}_t$ and $\mathcal{L}_t^n$ (by the same reasoning for $\mathcal{L}^n$) are square-integrable. According to Lemma 3.6 (using that $\mathcal{H}$ is Hilbert with $\mathfrak{C} \in L(\mathcal{H})$), it holds that

$$\|\mathcal{V}_t - \mathcal{V}_t^n\|_{\mathcal{H}} \leq e^{\|\mathfrak{C}\|_{\text{op}} t} \left(\|\mathcal{V}_0 - \mathcal{V}_0^n\|_{\mathcal{H}} + \sum_{i=1}^{N_t} \|\mathcal{X}_i - \mathcal{X}_i^n\|_{\mathcal{H}} \right).$$

It moreover follows by the triangle inequality and square-integrability assumption that

$$\mathbb{E}[\|\mathcal{X}_i - \mathcal{X}_i^n\|_{\mathcal{H}}^2] \leq 2(\mathbb{E}[\|\mathcal{X}_i\|_{\mathcal{H}}^2] + \mathbb{E}[\|\mathcal{X}_i^n\|_{\mathcal{H}}^2]) < \infty.$$

Hence by Lemma 3.5, we can conclude that $t \mapsto \sum_{i=1}^{N_t} \|\mathcal{X}_i - \mathcal{X}_i^n\|_{\mathcal{H}}$ is a square-integrable compound Poisson process with values in $\mathbb{R}_+$ (a subordinator). Thus by appealing to the inequality $(x + y)^2 \leq 2(x^2 + y^2)$ and Lemma 3.5, we get

$$\begin{aligned} & \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{V}_t - \mathcal{V}_t^n\|_{\mathcal{H}}^2 \right] \\ & \leq 2e^{2T\|\mathfrak{C}\|_{\text{op}}} \mathbb{E} \left[\|\mathcal{V}_0 - \mathcal{V}_0^n\|_{\mathcal{H}}^2 + \sup_{0 \leq t \leq T} \left(\sum_{i=1}^{N_t} \|\mathcal{X}_i - \mathcal{X}_i^n\|_{\mathcal{H}} \right)^2 \right] \\ & \leq 2e^{2T\|\mathfrak{C}\|_{\text{op}}} \left(\mathbb{E}[\|\mathcal{V}_0 - \mathcal{V}_0^n\|_{\mathcal{H}}^2] + \mathbb{E} \left[\left(\sum_{i=1}^{N_T} \|\mathcal{X}_i - \mathcal{X}_i^n\|_{\mathcal{H}} \right)^2 \right] \right) \\ & \leq 2e^{2T\|\mathfrak{C}\|_{\text{op}}} (\mathbb{E}[\|\mathcal{V}_0 - \mathcal{V}_0^n\|_{\mathcal{H}}^2] + \lambda T(1 + \lambda T) \mathbb{E}[\|\mathcal{X}_1 - \mathcal{X}_1^n\|_{\mathcal{H}}^2]). \end{aligned}$$

The result follows. $\square$

We can easily obtain the following corresponding result for the approximation of the compound Poisson process.

Corollary 3.8 *Suppose for each $n \geq 1$ that $(\mathcal{X}_i)_{i \in \mathbb{N}}$ and $(\mathcal{X}_i^n)_{i \in \mathbb{N}}$ are sequences of $\mathcal{H}$ -valued i.i.d. random variables and moreover that $\|\mathcal{X}_i\|_{\mathcal{H}}, \|\mathcal{X}_i^n\|_{\mathcal{H}} \in L^2(\mathbb{P})$ for all $n, i \geq 1$. If the compound Poisson processes $\mathcal{L}$ and $\mathcal{L}^n$ defined by (3.6) and (3.7), respectively, are driven by the same Poisson process N , then for every $T > 0$,*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{L}_t - \mathcal{L}_t^n\|_{\mathcal{H}}^2 \right] \leq C(T) \mathbb{E}[\|\mathcal{X}_1 - \mathcal{X}_1^n\|_{\mathcal{H}}^2],$$

where $C(T) = 2T\lambda(1 + \lambda T)$.

Proof This follows from Proposition 3.7 by setting $\mathcal{V}_0^n = \mathcal{V}_0$ for all n and $\mathfrak{C} = 0$. $\square$

Given an ONB $(e_j)_{j \in \mathbb{N}}$ of H , we introduce by tensorisation elements in $L(H)$ defined by

$$e_j \otimes e_k = (e_k, \cdot)_H e_j \quad (3.11)$$

for each $(j, k) \in \mathbb{N}^2$. It is well known (see e.g. Pedersen [39, Proposition 3.4.14]) that the family $(e_j \otimes e_k)_{(j,k) \in \mathbb{N}^2}$ defines an ONB of $\mathcal{H}$.

Recall that an operator $A \in L(U)$, where U is a separable Hilbert space, is said to be *compact* if it transforms any bounded subset of U into a set whose closure is compact (i.e., the set is relatively compact). Note that if $\mathcal{T} \in \mathcal{H}$, then $\mathcal{T}$ is compact; see Pedersen [39, Proposition 3.4.8]. Recall furthermore that an operator $A \in L(U)$ is said to be *diagonalisable* if there exists an ONB of U which consists of eigenvectors of A . If an operator $\mathcal{T}$ is compact and *normal*, meaning that $\mathcal{T}\mathcal{T}^* = \mathcal{T}^*\mathcal{T}$, where $\mathcal{T}^*$ is the adjoint of $\mathcal{T}$, then it is diagonalisable, i.e., there exists an ONB of eigenvectors $(e_k)_{k \in \mathbb{N}}$ of $\mathcal{T}$, and the corresponding eigenvalues $(\lambda_k)_{k \in \mathbb{N}}$ vanish at infinity, i.e., $\lambda_k \rightarrow 0$ when $k \rightarrow \infty$ (see [39, Theorem 3.3.8]). In what follows, we

use the notation $(E_j)_{j \in \mathbb{N}}$ for an ONB of $\mathcal{H}$. Note in particular that if $d : \mathbb{N} \rightarrow \mathbb{N}^2$ is a bijective correspondence, then we can let $E_j = \psi_{d(j)}$, where $\psi_{(j,k)} = e_j \otimes e_k$ is defined by (3.11) and $(e_j)_{j \in \mathbb{N}}$ is an ONB on H .

Suppose that $(\mathcal{U}_n)_{n \in \mathbb{N}}$ is a nested sequence of finite-dimensional subspaces of $\mathcal{H}$, meaning that $\mathcal{U}_n \subseteq \mathcal{U}_{n+1}$ for $n \geq 1$. Denote by $\Pi_n \in L(\mathcal{H})$ the orthogonal projection onto $\mathcal{U}_n$ and let

$$\mathcal{T}^n := \Pi_n \mathcal{T}. \quad (3.12)$$

Note in particular that $\|\mathcal{T}^n\|_{\mathcal{H}} \leq \|\mathcal{T}\|_{\mathcal{H}}$ for all $n \geq 1$ since $\|\Pi_n\|_{\text{op}} \leq 1$. So $\mathcal{T}^n \in \mathcal{H}$.

Proposition 3.9 *Suppose that $\mathcal{T} \in \mathcal{H}$ and $\mathcal{T}^n$ is defined in (3.12), where we define $\mathcal{U}_n = \text{span}\{e_j \otimes e_k : j + k \leq n\}$ and $(e_j)_{j \in \mathbb{N}}$ is an ONB of H . Then it holds that*

$$\|\mathcal{T} - \mathcal{T}^n\|_{\mathcal{H}}^2 = \sum_{j+k>n} (\mathcal{T}e_k, e_j)_H^2.$$

If furthermore the basis $(e_j)_{j \in \mathbb{N}}$ consists of the eigenfunctions of $\mathcal{T}$ and $(\lambda_j)_{j \in \mathbb{N}}$ are the corresponding eigenvalues, then $(\lambda_j)_{j \in \mathbb{N}} \in \ell^2$ and

$$\|\mathcal{T} - \mathcal{T}^n\|_{\mathcal{H}}^2 = \sum_{k>n/2} \lambda_k^2.$$

Proof Note that according to Parseval's identity on $\mathcal{H}$, it holds that

$$\begin{aligned} \|\mathcal{T} - \mathcal{T}^n\|_{\mathcal{H}}^2 &= \left\| \sum_{j+k>n} \langle \mathcal{T}, e_j \otimes e_k \rangle_{\mathcal{H}} (e_j \otimes e_k) \right\|_{\mathcal{H}}^2 \\ &= \sum_{j+k>n} \langle \mathcal{T}, e_j \otimes e_k \rangle_{\mathcal{H}}^2 \\ &= \sum_{j+k>n} (\mathcal{T}e_k, e_j)_H^2. \end{aligned}$$

Here we applied the identity

$$\langle \mathcal{T}, e_j \otimes e_k \rangle_{\mathcal{H}} = \sum_{m=1}^{\infty} (\mathcal{T}e_m, (e_k, e_m)_H e_j)_H = (\mathcal{T}e_k, e_j)_H$$

in the last step. Suppose now that $(e_j)_{j \in \mathbb{N}}$ is the basis of eigenvectors of $\mathcal{T}$. Then

$$\sum_{j=1}^{\infty} \lambda_j^2 = \sum_{j=1}^{\infty} |\mathcal{T}e_j|_H^2 = \|\mathcal{T}\|_{\mathcal{H}}^2 < \infty$$

and

$$\sum_{j+k>n} (\mathcal{T}e_k, e_j)_H^2 = \sum_{j+k>n} \lambda_k^2 (e_k, e_j)_H^2 = \sum_{k>n/2} \lambda_k^2.$$

Hence the proof is complete. $\square$

We remark that if we let $\mathcal{T} = \mathcal{X}$, a random variable, in Proposition 3.7, the eigenvalues are random variables. Furthermore, according to Proposition 3.9, if the operator $\mathcal{T} \in \mathcal{H}$ is diagonalisable with an ONB of $\mathcal{H}$ which consists of the eigenfunctions of $\mathcal{T}$, then the corresponding sequence of eigenvalues vanishes at infinity since it is in the sequence space ℓ^2 . Hence according to [39, Theorem 3.3.8.], it follows that $\mathcal{T}$ is normal and compact.

3.2 Finite-dimensional approximation of the semigroup generator

Now, before proceeding with the general results, let us consider specific examples of operators $\mathfrak{C} \in L(\mathcal{H})$ and their finite-dimensional approximations $\mathfrak{C}^n$. Motivated from Benth et al. [13], let

$$\mathfrak{C}_1 \mathcal{T} := \mathcal{C} \mathcal{T} \mathcal{C}^*, \quad (3.13)$$

$$\mathfrak{C}_2 \mathcal{T} := \mathcal{C} \mathcal{T} + \mathcal{T} \mathcal{C}^*, \quad (3.14)$$

where $\mathcal{C} \in L(H)$ and $\mathcal{C}^*$ is the adjoint of $\mathcal{C}$. We notice that the operator $\mathfrak{C}_2$ is an infinite-dimensional extension of the matrix-valued volatility model considered in Barndorff-Nielsen and Stelzer [6]. Note moreover that according to the identity $e^{\mathfrak{C}_2 t} \mathcal{T} = e^{\mathcal{C} t} \mathcal{T} e^{\mathcal{C}^* t}$ (see Benth et al. [13]), it holds that $\|e^{\mathfrak{C}_2 t}\|_{\text{op}} \leq \|e^{\mathcal{C} t}\|_{\text{op}}^2$. Hence if $s(\mathcal{C}) < 0$ (i.e., if $\Re \lambda < 0$ for all $\lambda \in \sigma(\mathcal{C})$ and the spectrum has a non-zero Hausdorff distance to $\{z \in \mathbb{C} : \Re z = 0\}$), then the semigroup $(\mathfrak{S}(t))_{t \geq 0}$ is uniformly exponentially stable (see the discussion following Proposition 3.1), so that there exist constants $M > 1$ and $\alpha > 0$ such that $\|e^{\mathfrak{C}_2 t}\|_{\text{op}} < M e^{-\alpha t}$ for all $t \geq 0$. If $\mathcal{C}$ is furthermore normal, then according to the spectral theorem (see Engel and Nagel [28, Theorem I.3.9]), there exist a σ -finite measure space (X, μ) and a measurable function $g : X \rightarrow \mathbb{C}$ such that $\mathcal{C}$ is unitarily equivalent to a multiplication operator $\mathcal{C}_g$ on $L^2(X, \mu)$, given by $\mathcal{C}_g f = g f$ for $f \in H$, where g is a fixed function. This means that $\mathcal{C} = \mathcal{U}^{-1} \mathcal{C}_g \mathcal{U}$, where $\mathcal{U} \in L(H, L^2(X, \mu))$ is a unitary operator. Thus the canonical example of an operator $\mathcal{C} \in L(H)$, where H is a function space, is the multiplication operator $\mathcal{C}_g$.

Note that if $\mathcal{C}_g$ is a multiplication operator on $L^2(X, \mu)$, then its spectrum is given by the essential range of g (see [28, Proposition I.3.10]), and since $\mathcal{C}_g - \lambda$ is invertible if and only if $\mathcal{U}^{-1} \mathcal{C}_g \mathcal{U} - \lambda = \mathcal{U}^{-1} (\mathcal{C}_g - \lambda) \mathcal{U}$ is invertible, it moreover holds that the operator $\mathcal{C} = \mathcal{U}^{-1} \mathcal{C}_g \mathcal{U}$ on H has the same spectrum as $\mathcal{C}_g$.

Example 3.10 Suppose that $H = H_w$ is the Filipović Hilbert space of Example 2.1. Let $\mathfrak{C} = \mathfrak{C}_2$ and $\mathcal{C} = \mathcal{U}^{-1} \mathcal{C}_g \mathcal{U}$, where $\mathcal{U}$ is a unitary operator, $\mathcal{C}_g$ is a multiplication operator on $L^2([0, 1], \text{Leb})$ such that $\mathcal{C}_g f = g f$ for $f \in L^2([0, 1], \text{Leb})$, and moreover that $g \in L^\infty([0, 1], \text{Leb})$ is real-valued with an essential range that is a subset of $(-\infty, -\delta]$ for some $\delta > 0$. If $(e_j)_{j \in \mathbb{N}}$ denotes an ONB in H_w and $(f_j)_{j \in \mathbb{N}}$ is an ONB in $L^2([0, 1], \text{Leb})$, then $\mathcal{U} \in L(H_w, L^2([0, 1], \text{Leb}))$ can be defined by

$$\mathcal{U} g := \sum_{j=1}^{\infty} (g, e_j)_w f_j$$

for $g \in H_w$. It is easy to check that $\mathcal{U}^{-1}$ is specified by $\mathcal{U}^{-1}g = \sum_{j=1}^{\infty} (g, f_j)_2 e_j$ for $g \in L^2([0, 1], \text{Leb})$, and that $(\mathcal{U}g, \mathcal{U}h)_2 = (g, h)_w$ for $g, h \in H_w$, i.e., $\mathcal{U}$ is unitary. Thus $\mathcal{C}$ has a negative spectrum which equals the essential range of g , and so $\mathcal{C}$ is uniformly exponentially stable, which is equivalent to $s(\mathcal{C}) < 0$, and $\mathcal{V}$ is asymptotically stationary, with a stationary distribution coinciding with that of $\mathcal{V}_0^{\text{stat}}$ as defined in (3.4).

In the next result, we state a general approximation estimate for $\mathcal{V} - \mathcal{V}^n$ in terms of $\mathcal{C} - \mathcal{C}^n$.

Proposition 3.11 *Suppose that $\mathcal{C}, \mathcal{C}^n \in L(\mathcal{H})$ and that $\mathcal{V}$ and $\mathcal{V}^n$ are the variance processes defined by (3.1) and (3.5), respectively, where $\mathcal{V}_0^n = \mathcal{V}_0$ and $\mathcal{L}^n = \mathcal{L}$ for all $n \geq 1$ and $\mathcal{L}$ is a square-integrable compound Poisson process. Then it holds that*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{V}_t - \mathcal{V}_t^n\|_{\mathcal{H}}^2 \right] \leq C(T) \|\mathcal{C} - \mathcal{C}^n\|_{\text{op}}^2,$$

where

$$C(T) = 2T^2 e^{2T(\|\mathcal{C}\|_{\text{op}} \vee \|\mathcal{C}^n\|_{\text{op}})} (\mathbb{E}[\|\mathcal{V}_0\|_{\mathcal{H}}^2] + \lambda T \mathbb{E}[\|\mathcal{X}_1\|_{\mathcal{H}}^2] + \lambda^2 T^2 (\mathbb{E}[\|\mathcal{X}_1\|_{\mathcal{H}}]^2)).$$

Proof Suppose $A, B \in L(U)$ are linear operators on a Banach space U . Then for any $k \geq 1$, it holds that

$$\|A^k - B^k\|_{\text{op}} \leq k(\|A\|_{\text{op}} \vee \|B\|_{\text{op}})^{k-1} \|A - B\|_{\text{op}}.$$

Assume $\|B\|_{\text{op}} = 0$. This implies $B = 0$ and we have by the algebra property of $L(U)$ that $\|A^k\|_{\text{op}} \leq \|A\|_{\text{op}}^k$, and thus the assertion holds in the case when $\|A\|_{\text{op}} = 0$ or $\|B\|_{\text{op}} = 0$. Suppose therefore without loss of generality that $\|A\|_{\text{op}}, \|B\|_{\text{op}} > 0$. By employing a telescope argument, it holds that

$$A^k - B^k = \sum_{m=0}^{k-1} A^{k-1-m} (A - B) B^m.$$

Since the space $L(U)$ of linear operators is a Banach algebra, it follows that

$$\begin{aligned} \|A^k - B^k\|_{\text{op}} &\leq \|A\|_{\text{op}}^{k-1} \sum_{m=0}^{k-1} \left(\frac{\|B\|_{\text{op}}}{\|A\|_{\text{op}}} \right)^m \|A - B\|_{\text{op}} \\ &\leq k(\|A\|_{\text{op}}^{k-1} \vee \|B\|_{\text{op}}^{k-1}) \|A - B\|_{\text{op}}. \end{aligned}$$

By employing the series representation of the exponential function, it follows that if $t \geq 0$,

$$\|e^{\mathcal{C}t} - e^{\mathcal{C}^n t}\|_{\text{op}} \leq \sum_{k=0}^{\infty} \frac{t^k}{k!} \|\mathcal{C}^k - (\mathcal{C}^n)^k\|_{\text{op}} \leq t e^{t(\|\mathcal{C}\|_{\text{op}} \vee \|\mathcal{C}^n\|_{\text{op}})} \|\mathcal{C} - \mathcal{C}^n\|_{\text{op}}.$$

Hence (3.2) and a repeated application of the triangle inequality yield

$$\begin{aligned}\|\mathcal{V}_t - \mathcal{V}_t^n\|_{\mathcal{H}} &\leq \|(\mathfrak{e}^{\mathfrak{C}_t} - \mathfrak{e}^{\mathfrak{C}_t^n})\mathcal{V}_0\|_{\mathcal{H}} + \left\| \int_0^t (\mathfrak{e}^{\mathfrak{C}(t-s)} - \mathfrak{e}^{\mathfrak{C}^n(t-s)}) d\mathcal{L}_s \right\|_{\mathcal{H}} \\ &\leq \|\mathfrak{e}^{\mathfrak{C}_t} - \mathfrak{e}^{\mathfrak{C}_t^n}\|_{\text{op}} \|\mathcal{V}_0\|_{\mathcal{H}} + \sum_{i=1}^{N_t} \|\mathfrak{e}^{\mathfrak{C}(t-\tau_i)} - \mathfrak{e}^{\mathfrak{C}^n(t-\tau_i)}\|_{\text{op}} \|\mathcal{X}_i\|_{\mathcal{H}} \\ &\leq t\mathfrak{e}^t(\|\mathfrak{C}\|_{\text{op}} \vee \|\mathfrak{C}^n\|_{\text{op}}) \left(\|\mathcal{V}_0\|_{\mathcal{H}} + \sum_{i=1}^{N_t} \|\mathcal{X}_i\|_{\mathcal{H}} \right) \|\mathfrak{C} - \mathfrak{C}^n\|_{\text{op}}.\end{aligned}$$

Note furthermore that by the properties of a real-valued compound Poisson process with intensity λ , it follows by Lemma 3.5 that

$$\begin{aligned}\mathbb{E} \left[\sup_{0 \leq t \leq T} \left(\|\mathcal{V}_0\|_{\mathcal{H}} + \sum_{i=1}^{N_t} \|\mathcal{X}_i\|_{\mathcal{H}} \right)^2 \right] \\ \leq 2 \left(\mathbb{E}[\|\mathcal{V}_0\|_{\mathcal{H}}^2] + \mathbb{E} \left[\left(\sup_{0 \leq t \leq T} \sum_{i=1}^{N_t} \|\mathcal{X}_i\|_{\mathcal{H}} \right)^2 \right] \right) \\ = 2(\mathbb{E}[\|\mathcal{V}_0\|_{\mathcal{H}}^2] + \lambda T \mathbb{E}[\|\mathcal{X}_1\|_{\mathcal{H}}^2] + \lambda^2 T^2 (\mathbb{E}[\|\mathcal{X}_1\|_{\mathcal{H}}])^2).\end{aligned}$$

Thus

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{V}_t - \mathcal{V}_t^n\|_{\mathcal{H}}^2 \right] \leq C(T) \|\mathfrak{C} - \mathfrak{C}^n\|_{\text{op}}^2.$$

The proof is complete. $\square$

Proposition 3.12 Suppose $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are specified by (3.13) and (3.14), respectively, for a given $\mathcal{C} \in L(H)$, and $\mathfrak{C}_1^n, \mathfrak{C}_2^n \in L(\mathcal{H})$ are given by $\mathfrak{C}_1^n \mathcal{T} = \mathcal{C}^n \mathcal{T} + \mathcal{T}(\mathcal{C}^n)^*$ and $\mathfrak{C}_2^n = \mathcal{C}^n \mathcal{T}(\mathcal{C}^n)^*$ for some $\mathcal{C}^n \in L(H)$. Then

$$\|\mathfrak{C}_1 - \mathfrak{C}_1^n\|_{\text{op}} \leq 2\|\mathcal{C} - \mathcal{C}^n\|_{\text{op}}$$

and

$$\|\mathfrak{C}_2 - \mathfrak{C}_2^n\|_{\text{op}} \leq (\|\mathcal{C}\|_{\text{op}} + \|\mathcal{C}^n\|_{\text{op}}) \|\mathcal{C} - \mathcal{C}^n\|_{\text{op}}.$$

Proof Suppose $\mathcal{T} \in L(\mathcal{H})$. Then using that $(\mathcal{A}\mathcal{B})^* = \mathcal{B}^* \mathcal{A}^*$ and $\|\mathcal{A}^*\| = \|\mathcal{A}\|$ for $\mathcal{A}, \mathcal{B} \in L(\mathcal{H})$, we get

$$\|(\mathfrak{C}_1 - \mathfrak{C}_1^n)\mathcal{T}\|_{\mathcal{H}} \leq \|(\mathcal{C} - \mathcal{C}^n)\mathcal{T}\|_{\mathcal{H}} + \|\mathcal{T}(\mathcal{C}^* - (\mathcal{C}^n)^*)\|_{\mathcal{H}} \leq 2\|\mathcal{C} - \mathcal{C}^n\|_{\text{op}} \|\mathcal{T}\|_{\mathcal{H}}.$$

Similarly, we obtain that

$$\|(\mathfrak{C}_2 - \mathfrak{C}_2^n)\mathcal{T}\|_{\text{op}} \leq (\|\mathcal{C}\|_{\text{op}} + \|\mathcal{C}^n\|_{\text{op}}) \|\mathcal{C} - \mathcal{C}^n\|_{\text{op}} \|\mathcal{T}\|_{\mathcal{H}}.$$

The result follows from the definition of the operator norm. $\square$

Example 3.13 Consider the setting of Example 3.10 with $H = H_w$, where we have $\mathcal{C} = \mathcal{U}^{-1}C_g\mathcal{U}$, C_g is a multiplication operator on $L^2([0, 1], \text{Leb})$ and $\mathcal{U}$ is unitary. Suppose that $\mathcal{C}^n := \mathcal{U}^{-1}C_g^n\mathcal{U}$ for some bounded operator C_g^n on $L^2([0, 1], \text{Leb})$ and $f \in H_w$. Then using that $\mathcal{U}$ is unitary, it holds that

$$|(\mathcal{C} - \mathcal{C}^n)f|_w = |\mathcal{U}^{-1}(C_g - C_g^n)\mathcal{U}f|_w = |(C_g - C_g^n)\mathcal{U}f|_2 \leq \|C_g - C_g^n\|_{\text{op}}|f|_w.$$

Hence we conclude that $\|\mathcal{C} - \mathcal{C}^n\|_{\text{op}} \leq \|C_g - C_g^n\|_{\text{op}}$, and similar arguments give $\|\mathcal{C} - \mathcal{C}^n\|_{\text{op}} \geq \|C_g - C_g^n\|_{\text{op}}$ so that $\|\mathcal{C} - \mathcal{C}^n\|_{\text{op}} = \|C_g - C_g^n\|_{\text{op}}$.

We refer to Morrison [38] for spectral approximation techniques for multiplication operators and Hansen [32] for approximation techniques of linear operators on separable Hilbert spaces.

We finish this subsection by exploring the case of finite-dimensional approximations of compact operators.

Proposition 3.14 Suppose $\mathfrak{C}_1, \mathfrak{C}_2$ are defined by (3.13) and (3.14), respectively, where $\mathcal{C}$ is a normal and compact operator. Then there exists an ONB $(e_j)_{j \in \mathbb{N}}$ of H consisting of eigenfunctions of $\mathcal{C}$ and such that the corresponding sequence $(\lambda_j)_{j \in \mathbb{N}}$ of eigenvalues vanishes at infinity. It furthermore holds that $(e_j \otimes e_k)_{(j,k) \in \mathbb{N}^2}$ is an ONB of $\mathcal{H}$ consisting of eigenfunctions of $\mathfrak{C}_i$, i.e., $\mathfrak{C}_i(e_j \otimes e_k) = \Lambda_{j,k}(e_j \otimes e_k)$ holds for all $j, k \geq 1$, where the eigenvalues are given by $\Lambda_{j,k} = \lambda_j \lambda_k$ when $i = 1$ and $\Lambda_{j,k} = \lambda_j + \lambda_k$ when $i = 2$. Moreover, $\mathfrak{C}_1$ is a compact operator, and $\mathfrak{C}_2$ is not a compact operator.

Proof The existence of the ONB with eigenvalues that vanish at infinity follows by Pedersen [39, Theorem 3.3.8]. We recall that according to [39, Proposition 3.4.14], $(e_j \otimes e_k)_{(j,k) \in \mathbb{N}^2}$ is a basis of $\mathcal{H}$. If $f \in H$, then

$$\begin{aligned} \mathfrak{C}_1(e_j \otimes e_k)f &= (e_k, C^*f)_H C e_j = (C e_k, f)_H C e_j \\ &= \lambda_j \lambda_k (e_k, f)_H e_j = \lambda_j \lambda_k (e_j \otimes e_k)f \end{aligned}$$

and

$$\begin{aligned} \mathfrak{C}_2(e_j \otimes e_k)f &= (e_k, f)_H C e_j + (e_k, C^*f)_H e_j = (e_k, f)_H C e_j + (C e_k, f)_H e_j \\ &= (\lambda_j + \lambda_k)(e_k, f)_H e_j = (\lambda_j + \lambda_k)(e_j \otimes e_k)f. \end{aligned}$$

Thus the family $(e_j \otimes e_k)_{(j,k) \in \mathbb{N}^2}$ consists of eigenfunctions of $\mathfrak{C}_i$, $i = 1, 2$, where the eigenvalues are given by $\Lambda_{j,k} = \lambda_j \lambda_k$ when $i = 1$ and $\Lambda_{j,k} = \lambda_j + \lambda_k$ when $i = 2$. In particular, $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are diagonalisable. Now for any $\epsilon > 0$, there are only finitely many λ_j , $j \geq 1$, which are larger than ϵ in absolute value. Thus for any $\epsilon > 0$, the set $\{(j, k) \in \mathbb{N}^2 : |\lambda_j \lambda_k| > \epsilon\}$ is finite. Hence according to [39, Lemma 3.3.5], $\mathfrak{C}_1$ is compact. On the other hand, if $|\lambda_1| > \epsilon > 0$, then $\{(j, k) \in \mathbb{N}^2 : |\lambda_j + \lambda_k| > \epsilon\}$ has infinitely many elements since we may select $N \geq 1$ such that $|\lambda_1 + \lambda_k| > \epsilon$ for all $k \geq N$. Thus according to [39, Lemma 3.3.5], $\mathfrak{C}_2$ is not compact. $\square$

Example 3.15 Let $H = L^2([0, 1], \text{Leb})$ and define

$$(\mathcal{C}f)(t) = \int_0^1 (s \wedge t) f(s) ds$$

for $f \in H$. Then as the kernel function $c(s, t) = s \wedge t$ is symmetric, the operator $\mathcal{C}$ is of trace-class, symmetric and nonnegative definite (see Peszat and Zabczyk [40, Theorem A.8]). The kernel function $c(t, s)$ is moreover the covariance function of a univariate Brownian motion, $c(t, s) = \mathbb{E}[B_t B_s]$, which is fixed throughout the example. Now given this construction, it follows by the Karhunen–Loève expansion that

$$B_t = \sum_j \sqrt{\lambda_j} \xi_j e_j(t),$$

where $(\xi_j)_{j \in \mathbb{N}}$ are i.i.d. (orthonormal) standard Gaussian random variables. In this case, the eigenvectors and eigenvalues of $\mathcal{C}$ are given by

$$e_j(t) = \sqrt{2} \sin\left(\frac{(2j+1)\pi t}{2}\right), \quad \lambda_j = \left(\frac{2}{(2j+1)\pi}\right)^2$$

for $j \geq 0$. If H_w is the Filipović Hilbert space of Example 2.1, we can define the operator $\mathcal{C}_w \in L(H_w)$ by $\mathcal{C}_w = \mathcal{U}^{-1} \mathcal{C} \mathcal{U}$, where $\mathcal{U}$ is a unitary operator, like we did in Example 3.10. Note that $\mathcal{C}_w$ is compact (since $\mathcal{C}$ is compact) and that the spectra of $\mathcal{C}_w$ and $\mathcal{C}$ coincide, i.e., the eigenvalues of $\mathcal{C}_w$ equal the eigenvalues of $\mathcal{C}$.

Suppose that $(E_j)_{j \in \mathbb{N}}$ is an ONB of $\mathcal{H}$, $\mathfrak{C} \in L(\mathcal{H})$ and for each $n \geq 1$, define $\mathcal{U}_n := \text{span}\{E_j : j \in J_n\}$, where $J_n \subseteq \mathbb{N}$ and $(J_n)_{n \in \mathbb{N}}$ is a nested sequence of finite subsets of $\mathbb{N}$ so that $J_n \subseteq J_{n+1}$ for all $n \geq 1$. Define $\mathfrak{C}^n$ by

$$\mathfrak{C}^n := \Pi_n \mathfrak{C} \Pi_n, \quad (3.15)$$

where $\Pi_n \in L(\mathcal{H})$ is the orthogonal projection onto $\mathcal{U}_n$. Note that since $L(\mathcal{H})$ is a Banach algebra and Π_n is a contraction operator, it follows that

$$\|\mathfrak{C}^n\|_{\text{op}} \leq \|\Pi_n\|_{\text{op}}^2 \|\mathfrak{C}\|_{\text{op}} \leq \|\mathfrak{C}\|_{\text{op}}, \quad (3.16)$$

so that clearly $\mathfrak{C}^n \in L(\mathcal{H})$ and $\mathfrak{C}^n \mathcal{T} \in \mathcal{U}_n$ for each $\mathcal{T} \in \mathcal{H}$. Our next result demonstrates that the approximation (3.15) converges in operator norm as $n \rightarrow \infty$ if $\mathfrak{C}$ is normal and compact.

Proposition 3.16 *Suppose that $\mathfrak{C}$ is a normal and compact operator. Then there exists an ONB $(E_j)_{j \in \mathbb{N}}$ of $\mathcal{H}$ consisting of eigenfunctions of $\mathfrak{C}$ and such that the corresponding sequence $(\Lambda_j)_{j \in \mathbb{N}}$ of eigenvalues vanishes at infinity. Assume furthermore that $\mathcal{U}_n = \text{span}\{E_j : j \in J_n\}$, where $J_n \subseteq \mathbb{N}$ and $(J_n)_{n \in \mathbb{N}}$ is a nested sequence of finite subsets of $\mathbb{N}$ so that $J_n \subseteq J_{n+1}$ for all $n \geq 1$. Then*

$$\|(\mathfrak{C} - \mathfrak{C}^n) \mathcal{T}\|_{\mathcal{H}}^2 = \sum_{j \in J_n^c} \Lambda_j^2 \langle \mathcal{T}, E_j \rangle_{\mathcal{H}}^2,$$

where $\mathfrak{C}^n$ is defined by (3.15) and $\mathcal{T} \in \mathcal{H}$. It furthermore holds that

$$\|\mathfrak{C} - \mathfrak{C}^n\|_{\text{op}}^2 \leq 2 \sup_{m \in J_n^c} \Lambda_m^2 \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof The existence of the ONB with eigenvalues that vanish at infinity follows by Pedersen [39, Theorem 3.3.8]. Take $\mathcal{T} \in \mathcal{H}$; then we may write

$$\mathcal{T} = \sum_j \langle \mathcal{T}, E_j \rangle_{\mathcal{H}} E_j.$$

Since $\mathfrak{C}E_j = \Lambda_j E_j$ and

$$\Lambda_j \langle \mathcal{T}, E_j \rangle_{\mathcal{H}} = \langle \mathcal{T}, \Lambda_j E_j \rangle_{\mathcal{H}} = \langle \mathcal{T}, \mathfrak{C}E_j \rangle_{\mathcal{H}} = \langle \mathfrak{C}^* \mathcal{T}, E_j \rangle_{\mathcal{H}},$$

it follows by Parseval's identity that

$$\begin{aligned} \|(\mathfrak{C} - \mathfrak{C}^n)\mathcal{T}\|_{\mathcal{H}}^2 &= \left\| \sum_{j \in J_n^c} \langle \mathcal{T}, E_j \rangle_{\mathcal{H}} \Lambda_j E_j \right\|_{\mathcal{H}}^2 \\ &= \left\| \sum_{j \in J_n^c} \langle \mathfrak{C}^* \mathcal{T}, E_j \rangle_{\mathcal{H}} E_j \right\|_{\mathcal{H}}^2 \\ &= \sum_{j \in J_n^c} \langle \mathfrak{C}^* \mathcal{T}, E_j \rangle_{\mathcal{H}}^2 = \sum_{j \in J_n^c} \Lambda_j^2 \langle \mathcal{T}, E_j \rangle_{\mathcal{H}}^2. \end{aligned}$$

Since the eigenvalues vanish at infinity, i.e., $\Lambda_j \rightarrow 0$ as $j \rightarrow \infty$, it follows that

$$\begin{aligned} \|\mathfrak{C} - \mathfrak{C}^n\|_{\text{op}}^2 &= \sup_{\|\mathcal{T}\|_{\mathcal{H}}=1} \sum_{j \in J_n^c} \Lambda_j^2 \langle \mathcal{T}, E_j \rangle_{\mathcal{H}}^2 \\ &\leq \sup_{\|\mathcal{T}\|_{\mathcal{H}}=1} \sum_{j \in J_n^c} \langle \mathcal{T}, E_j \rangle_{\mathcal{H}}^2 \sup_{m \in J_n^c} \Lambda_m^2 \\ &= \sup_{\|\mathcal{T}\|_{\mathcal{H}}=1} \|\mathcal{T} - \mathcal{T}^n\|_{\mathcal{H}}^2 \sup_{m \in J_n^c} \Lambda_m^2 \\ &\leq 2 \sup_{m \in J_n^c} \Lambda_m^2 \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad \square$$

Corollary 3.17 Suppose that $\mathfrak{C}$ is a normal and compact operator with eigenvectors $(e_j \otimes e_k)_{(j,k) \in \mathbb{N}^2}$ and eigenvalues $(\Lambda_{j,k})_{(j,k) \in \mathbb{N}^2}$, where $(e_j)_{j \in \mathbb{N}}$ is an ONB of H . Assume that $\mathcal{U}_n = \text{span}\{e_j \otimes e_k : j + k \leq n\}$. Then

$$\|(\mathfrak{C} - \mathfrak{C}^n)\mathcal{T}\|_{\mathcal{H}}^2 = \sum_{j+k > n} \Lambda_{j,k}^2 (\mathcal{T}e_k, e_j)_H,$$

where $\mathfrak{C}^n$ is defined by (3.15) and $\mathcal{T} \in \mathcal{H}$. It furthermore holds that

$$\|\mathfrak{C} - \mathfrak{C}^n\|_{\text{op}}^2 \leq 2 \sup_{j+k > n} \Lambda_{j,k}^2 \longrightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof Using the identity

$$\langle \mathcal{T}, e_j \otimes e_k \rangle_{\mathcal{H}} = \sum_{m=1}^{\infty} (\mathcal{T}e_m, (e_k, e_m)_H e_j)_H = (\mathcal{T}e_k, e_j)_H,$$

we establish the first claim, the expression for the norm. The second claim follows by Proposition 3.16. $\square$

Combining Propositions 3.11 and 3.16, we can control the error by the tail convergence of the eigenvalues when $\mathfrak{C}$ is normal and compact. This is made precise in the next result.

Corollary 3.18 *Suppose that $\mathfrak{C}$ is a normal and compact operator with eigenvectors $(E_j)_{j \in \mathbb{N}}$ and eigenvalues $(\Lambda_j)_{j \in \mathbb{N}}$ and that $\mathcal{U}_n = \text{span}\{E_j : j \in J_n\}$, where $J_n \subseteq \mathbb{N}$ and $(J_n)_{n \in \mathbb{N}}$ is a nested sequence of finite subsets of $\mathbb{N}$ so that $J_n \subseteq J_{n+1}$ for all $n \geq 1$. Suppose furthermore that $\mathcal{V}$ and $\mathcal{V}^n$ are the variance processes defined by (3.1) and (3.5), respectively, where $\mathfrak{C}^n$ is defined by (3.15), $\mathcal{V}_0^n = \mathcal{V}_0$ and $\mathcal{L}^n = \mathcal{L}$ for all $n \geq 1$ and $\mathcal{L}$ is a square-integrable compound Poisson process. Then it holds that*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{V}_t - \mathcal{V}_t^n\|_{\mathcal{H}}^2 \right] \leq C(T) \sup_{m \in J_n^c} \Lambda_m^2,$$

where $C(T) = 4T^2 e^{2T\|\mathfrak{C}\|_{\text{op}}} (\mathbb{E}[\|\mathcal{V}_0\|_{\mathcal{H}}^2] + \lambda T \mathbb{E}[\|\mathcal{X}_1\|_{\mathcal{H}}^2] + \lambda^2 T^2 (\mathbb{E}[\|\mathcal{X}_1\|_{\mathcal{H}}]^2))$, and the right-hand side of the above inequality converges to zero as $n \rightarrow \infty$.

Proof This follows from Propositions 3.16 and 3.11 and the inequality (3.16). $\square$

Recalling the option pricing example discussed at the end of Sect. 2, we can now apply for example Proposition 3.11 and Corollary 3.18 to assess the pricing error $|P - P^n|$ in terms of the approximation error induced by $\mathfrak{C}^n$, or Proposition 3.7 for the option price robustness towards errors in $\mathcal{L}$ and $\mathcal{V}_0$.

4 The square-root process

In this section, we study the robustness of the volatility process which is the square root of the variance process.

If an operator A is self-adjoint and nonnegative definite, there exists a unique nonnegative definite operator $\sqrt{A}$ such that $\sqrt{A}\sqrt{A} = A$. A sufficient set of conditions that together guarantee that $\mathcal{V}(t)$ in (3.1) is nonnegative definite, as given by Benth et al. [13], is that

- a) $(\mathfrak{C}\mathcal{T})^* = \mathfrak{C}\mathcal{T}^*$;
- b) $\mathfrak{C}\mathcal{H}_+ \subseteq \mathcal{H}_+$, where $\mathcal{H}_+$ denotes the convex cone of nonnegative definite operators on $\mathcal{H}$;
- c) $\mathcal{L}$ is a self-adjoint and nonnegative definite square-integrable Lévy process with values in $\mathcal{H}$;

d) $\mathcal{V}_0$ is self-adjoint and nonnegative definite.

In the following result, we employ the operator norm to investigate the error induced on the square root of the variance process.

Proposition 4.1 *Suppose that $\mathcal{V}$ and $\mathcal{V}^n$ are nonnegative definite. If $\mathcal{V}$ and $\mathcal{V}^n$ are the variance processes defined by (3.1) and (3.5), respectively, then for every $T > 0$, it holds that*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\sqrt{\mathcal{V}_t} - \sqrt{\mathcal{V}_t^n}\|_{\text{op}}^2 \right] \leq \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{V}_t - \mathcal{V}_t^n\|_{\text{op}} \right].$$

Proof This follows immediately according to Bogachev [21, Lemma 2.5.1]. $\square$

Remark 4.2 The inequality $|\sqrt{x} - \sqrt{y}|^2 \leq |x - y|$ is well known for real numbers x, y . However, extending this inequality to norms of (positive definite) operators is not immediate. Bogachev [21, Lemma 2.5.1] has shown this result we rely on in the proof of Proposition 4.1 above.

We remark that the inequality $\|\cdot\|_{\text{op}} \leq \|\cdot\|_{\mathcal{H}}$ can be employed to obtain the Hilbert–Schmidt norm on the right-hand side in the above result. In the following result, we obtain an analogue of Proposition 4.1 where the Hilbert–Schmidt norm is on the left-hand side and the trace-class norm appears on the right-hand side.

Proposition 4.3 *Suppose that $\mathcal{V}$ and $\mathcal{V}^n$ are nonnegative definite and of trace-class, where $\mathcal{V}$ and $\mathcal{V}^n$ are the variance processes defined by (3.1) and (3.5), respectively. Then for every $T > 0$, it holds that*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\sqrt{\mathcal{V}_t} - \sqrt{\mathcal{V}_t^n}\|_{\mathcal{H}}^2 \right] \leq \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{V}_t - \mathcal{V}_t^n\|_1 \right].$$

Proof According to Ando [2, Corollary 2], which holds for nonnegative and compact operators on $\mathcal{H}$ (see also Birman et al. [19] for the initial result and Birman and Solomyak [20, Sect. 8.4] for an overview), it holds that if $\mathcal{A}, \mathcal{B}$ are nonnegative definite and compact operators, then

$$\|\sqrt{\mathcal{A}} - \sqrt{\mathcal{B}}\|_{\mathcal{H}} \leq \| |\mathcal{A} - \mathcal{B}|^{1/2} \|_{\mathcal{H}},$$

where $|\mathcal{T}| = (\mathcal{T}^* \mathcal{T})^{1/2}$ denotes the modulus of an operator $\mathcal{T} \in \mathcal{H}$. Note that according to Pedersen [39, Proposition 3.4.8], all Hilbert–Schmidt operators are compact. Hence according to the definition of the trace-norm (2.1), it follows that

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\sqrt{\mathcal{V}_t} - \sqrt{\mathcal{V}_t^n}\|_{\mathcal{H}}^2 \right] \leq \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{V}_t - \mathcal{V}_t^n\|_1 \right].$$

The proof is complete. $\square$

We end this subsection with corollaries which detail how the above result with the trace-norm on the right-hand side can be employed when either the compound Poisson process or the generator $\mathfrak{C}$ are approximated.

Corollary 4.4 Suppose that $\mathcal{V}$ and $\mathcal{V}^n$ are nonnegative definite and of trace-class, where $\mathcal{V}$ and $\mathcal{V}^n$ are the variance processes defined by (3.1) and (3.5), respectively, with $\mathcal{V}_0^n = \mathcal{V}_0$ and $\mathfrak{C}^n = \mathfrak{C} \in L(\mathcal{B}_1)$ for all $n \geq 1$, where the compound Poisson processes $\mathcal{L}$ and $\mathcal{L}^n$ are driven by the same Poisson process N . Then for every $T > 0$, we have

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\sqrt{\mathcal{V}_t} - \sqrt{\mathcal{V}_t^n}\|_{\mathcal{H}}^2 \right] \leq C(T) \mathbb{E}[\|\mathcal{X}_1 - \mathcal{X}_1^n\|_1],$$

where $C(T) = ke^{\|\mathfrak{C}\|_{\text{op}} T} \lambda T$ and $k > 0$ is a constant.

Proof According to Lemma 3.6, it holds that

$$\begin{aligned} \mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{V}_t - \mathcal{V}_t^n\|_1 \right] &\leq \mathbb{E} \left[\sup_{0 \leq t \leq T} e^{\|\mathfrak{C}\|_{\text{op}} t} \sum_{i=1}^{N_t} \|\mathcal{X}_i - \mathcal{X}_i^n\|_1 \right] \\ &\leq e^{\|\mathfrak{C}\|_{\text{op}} T} \mathbb{E} \left[\sum_{i=1}^{N_T} \|\mathcal{X}_i - \mathcal{X}_i^n\|_1 \right] \\ &= e^{\|\mathfrak{C}\|_{\text{op}} T} \lambda T \mathbb{E}[\|\mathcal{X}_1 - \mathcal{X}_1^n\|_1]. \end{aligned}$$

The result follows by Proposition 4.3. $\square$

The following result reduces the trace-norm of a tensor product of Hilbert space elements to a product of their norms.

Lemma 4.5 If $f, g \in H$, and $\mathcal{T} = f \otimes g$ so that $\mathcal{T}h = (g, h)_H f$ for $h \in H$, then $\|\mathcal{T}\|_1 = |f|_H |g|_H$.

Proof It is easy to show that $\mathcal{T}^* = g \otimes f$ and $\mathcal{T}^* \mathcal{T} = |f|_H^2 g \otimes g$. A simple calculation then shows that $(\mathcal{T}^* \mathcal{T})^{1/4} = |f|_H^{1/2} |g|_H^{-3/2} g \otimes g$. The result follows from an application of Parseval's identity because

$$\|\mathcal{T}\|_1 = |f|_H |g|_H^{-3} \|g \otimes g\|_{\mathcal{H}}^2 = |f|_H |g|_H^{-1} \sum_{k=1}^{\infty} |(g, e_k)_H|^2 = |f|_H |g|_H. \quad \square$$

Example 4.6 Suppose that we have $\mathcal{X}_i = (Y_i)^{\otimes 2}$, where $(Y_i)_{i \in \mathbb{N}}$ is a sequence of i.i.d. H -valued random variables, and $\mathcal{X}_i^n = (Y_i^n)^{\otimes 2}$, where Y_i^n is defined by (3.8) for all $n, i \geq 1$. Then according to Lemma 4.5, it holds that

$$\begin{aligned} \|\mathcal{X}_i - \mathcal{X}_i^n\|_1 &\leq \|(Y_i - Y_i^n) \otimes Y_i\|_1 + \|Y_i^n \otimes (Y_i - Y_i^n)\|_1 \\ &= (|Y_i|_H + |Y_i^n|_H) |Y_i - Y_i^n|_H \\ &\leq 2|Y_i|_H |Y_i - Y_i^n|_H. \end{aligned}$$

Hence by appealing to the Cauchy–Schwarz inequality, we obtain that

$$\mathbb{E}[\|\mathcal{X}_i - \mathcal{X}_i^n\|_1] \leq 2(\mathbb{E}[|Y_i|_H^2] \mathbb{E}[|Y_i - Y_i^n|_H^2])^{1/2}.$$

Corollary 4.7 Suppose $\mathcal{V}$ and $\mathcal{V}^n$ are nonnegative definite and of trace-class, where $\mathcal{V}$ and $\mathcal{V}^n$ are the variance processes defined by (3.1) and (3.5), respectively, with $\mathcal{V}_0^n = \mathcal{V}_0$, $\mathcal{L} = \mathcal{L}^n$ and $\mathfrak{C}^n$, $\mathfrak{C} \in L(\mathcal{B}_1)$ for all $n \geq 1$. Then for every $T > 0$, we have

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\sqrt{\mathcal{V}_t} - \sqrt{\mathcal{V}_t^n}\|_{\mathcal{H}}^2 \right] \leq C(T) \|\mathfrak{C} - \mathfrak{C}^n\|_{\text{op}},$$

where $C(T) = kTe^{T(\|\mathfrak{C}\|_{\text{op}} \vee \|\mathfrak{C}^n\|_{\text{op}})} (\mathbb{E}[\|\mathcal{V}_0\|_1] + \lambda T \mathbb{E}[\|\mathcal{X}_1\|_1])$ and $k > 0$ is a constant.

Proof Using the same reasoning as in the proof of Proposition 3.11 with the Hilbert–Schmidt norm replaced by the trace-class norm, it follows that

$$\|\mathcal{V}_t - \mathcal{V}_t^n\|_1 \leq te^{t(\|\mathfrak{C}\|_{\text{op}} \vee \|\mathfrak{C}^n\|_{\text{op}})} \left(\|\mathcal{V}_0\|_1 + \sum_{i=1}^{N_t} \|\mathcal{X}_i\|_1 \right) \|\mathfrak{C} - \mathfrak{C}^n\|_{\text{op}}.$$

Hence we obtain

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \|\mathcal{V}_t - \mathcal{V}_t^n\|_1 \right] \leq Te^{T(\|\mathfrak{C}\|_{\text{op}} \vee \|\mathfrak{C}^n\|_{\text{op}})} (\mathbb{E}[\|\mathcal{V}_0\|_1] + \lambda T \mathbb{E}[\|\mathcal{X}_1\|_1]) \|\mathfrak{C} - \mathfrak{C}^n\|_{\text{op}}.$$

The result follows by Proposition 4.3. $\square$

Cuchiero and Svaluto-Ferro [25] analyse options on the realised volatility (so-called VIX options) in an infinite-dimensional framework. One could introduce options on the realised volatility of forward prices, with a payoff $p(\mathcal{D}\mathcal{V}_\tau^{1/2})$ at some exercise time τ . Here, $\mathcal{D} \in \mathcal{H}^*$ maps the volatility operator $\mathcal{V}_\tau^{1/2}$ into the real line by an integral and evaluation operator, say. If the payoff function p is Lipschitz-continuous, we can apply the results of this section to assess the robustness of such volatility options with respect to the parameters $\mathfrak{C}$ and $\mathcal{L}$, both in operator and in Hilbert–Schmidt norms.

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References

1. Aliprantis, C.D., Border, K.C.: *Infinite Dimensional Analysis: A Hitchhiker's Guide*, 3rd edn. Springer, New York (2006)
2. Ando, T.: Comparison of norms $|||f(A) - f(B)|||$ and $|||f(|A - B|)|||$. *Math. Z.* **197**, 403–409 (1988)
3. Barndorff-Nielsen, O.E., Benth, F.E., Veraart, A.: Modelling electricity futures by ambit fields. *Adv. Appl. Probab.* **46**, 719–745 (2014)
4. Barndorff-Nielsen, O.E., Benth, F.E., Veraart, A.: *Ambit Stochastics*. Springer, Cham (2018)
5. Barndorff-Nielsen, O.E., Shephard, N.: Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in economics. *J. R. Stat. Soc. Ser. B* **63**, 167–241 (2001)
6. Barndorff-Nielsen, O.E., Stelzer, R.: Multivariate supOU processes. *Ann. Appl. Probab.* **21**, 140–182 (2011)
7. Benth, F.E., Di Nunno, G., Lord, G., Petterson, A.: The heat modulated infinite dimensional Heston model and its numerical approximation (2022). Preprint. <https://arxiv.org/abs/2206.10166>
8. Benth, F.E., Di Nunno, G., Simonsen, I.C.: Sensitivity analysis in the infinite dimensional Heston model. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **24**, 2150014 (2021)
9. Benth, F.E., Eyjolfsson, H.: Representation and approximation of ambit fields in Hilbert space. *Stochastics* **89**, 311–347 (2017)
10. Benth, F.E., Harang, F.A.: Infinite dimensional pathwise Volterra processes driven by Gaussian noise – probabilistic properties and applications. *Electron. J. Probab.* **26**, 1–42 (2021)
11. Benth, F.E., Krühner, P.: Representation of infinite dimensional forward price models in commodity markets. *Commun. Math. Stat.* **2**, 47–106 (2014)
12. Benth, F.E., Krühner, P.: Derivatives pricing in energy markets: an infinite dimensional approach. *SIAM J. Financ. Math.* **6**, 825–869 (2015)
13. Benth, F.E., Rüdiger, B., Süß, A.: Ornstein–Uhlenbeck processes in Hilbert space with non-Gaussian stochastic volatility. *Stoch. Process. Appl.* **128**, 461–486 (2018)
14. Benth, F.E., Šaltytė Benth, J., Koekebakker, S.: *Stochastic Modelling of Electricity and Related Markets*. World Scientific, Singapore (2008)
15. Benth, F.E., Schroers, D., Veraart, A.: A weak law of large numbers for realised covariation in a Hilbert space setting. *Stoch. Process. Appl.* **145**, 241–268 (2022)
16. Benth, F.E., Schroers, D., Veraart, A.: A feasible central limit theorem for realised covariation of SPDEs in the context of functional data. *Ann. Appl. Probab.* **34**, 2208–2242 (2024)
17. Benth, F.E., Sgarra, C.: A Barndorff-Nielsen and Shephard model with leverage in Hilbert space for commodity forward markets. *Finance Stoch.* **28**, 1035–1076 (2024)
18. Benth, F.E., Simonsen, I.C.: The Heston stochastic volatility model in Hilbert space. *Stoch. Anal. Appl.* **36**, 733–750 (2018)
19. Birman, M.S., Koplienko, L.S., Solomyak, M.Z.: Estimates of the spectrum of a difference of fractional powers of selfadjoint operators. *Izv. Vysš. Učebn. Zaved., Mat.* **3**, 3–10 (1975)
20. Birman, M.S., Solomyak, M.: Double operator integrals in a Hilbert space. *Integral Equ. Oper. Theory* **47**, 131–168 (2003)
21. Bogachev, V.I.: *Weak Convergence of Measures*. Am. Math. Soc., Providence (2018)
22. Carmona, R., Tehranchi, M.: *Interest Rate Models: An Infinite Dimensional Stochastic Analysis Perspective*. Springer, Berlin (2006)
23. Cox, S., Karbach, S., Khedher, A.: Affine pure-jump processes on positive Hilbert–Schmidt operators. *Stoch. Process. Appl.* **151**, 191–229 (2022)
24. Cox, S., Karbach, S., Khedher, A.: An infinite-dimensional affine stochastic volatility model. *Math. Finance* **32**, 878–906 (2022)
25. Cuchiero, C., Svaluto-Ferro, S.: Infinite-dimensional polynomial processes. *Finance Stoch.* **25**, 383–426 (2021)
26. Di Persio, L., Perin, I.: An ambit stochastic approach to pricing electricity forward contracts: the case of the German energy market. *J. Probab. Stat.*, Article ID 626020-1–17 (2015)
27. El Euch, O., Rosenbaum, M.: The characteristic function of rough Heston models. *Math. Finance* **29**, 3–38 (2019)
28. Engel, K.-J., Nagel, R.: *A Short Course on Operator Semigroups*. Springer, New York (2006)
29. Filipović, D.: Consistency Problems for Heath–Jarrow–Morton Interest Rate Models. *Lecture Notes in Mathematics*, vol. 1760. Springer, Berlin (2001)
30. Friesen, M., Karbach, S.: Stationary covariance regime for affine stochastic covariance models in Hilbert spaces. *Finance Stoch.* **28**, 1077–1116 (2024)

31. Gawarecki, L., Mandrekar, V.: *Stochastic Differential Equations in Infinite Dimensions*. Springer, Berlin (2011)
32. Hansen, A.C.: On the approximation of spectra of linear operators on Hilbert spaces. *J. Funct. Anal.* **254**, 2092–2126 (2008)
33. Heath, D., Jarrow, R., Morton, A.: Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. *Econometrica* **60**, 77–105 (1992)
34. Heston, S.L.: A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Financ. Stud.* **6**, 327–343 (1993)
35. Karbach, S.: Finite-rank approximation of affine processes on positive Hilbert–Schmidt operators (2023). Preprint. <https://arxiv.org/abs/2301.06992>
36. Kuratowski, K., Ryll-Nardzewski, C.: A general theorem on selectors. *Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys.* **13**, 397–403 (1965)
37. Marinelli, C., Röckner, M.: On the maximal inequalities of Burkholder, Davis and Gundy. *Expo. Math.* **34**, 1–26 (2016)
38. Morrison, K.E.: Spectral approximation of multiplication operators. *New York J. Math.* **1**, 75–96 (1995)
39. Pedersen, G.K.: *Analysis Now*. Springer, New York (1989)
40. Peszat, S., Zabczyk, J.: *Stochastic Partial Differential Equations with Lévy Noise*. Cambridge University Press, Cambridge (2007)

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