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Restricted and weighted polynomial approximations

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Restricted and weighted polynomial approximations

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Dissertation submitted in partial fulfillment of a
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To Sandra

Abstract

The thesis contains material from papers studying polynomial approximations of holomorphic functions in several variables. These approximations are weighted, in the sense that the norm used to assess the approximation is weighted, and polynomials used for the approximation are restricted to a certain subring of polynomials. The notion of the degree of a polynomial is also different from the classical setting. The main aim is to develop an analog of the Bernstein-Walsh-Siciak theorem, a generalization of the Runge-Oka-Weil theorem.

To aid this study the Siciak-Zakharyuta functions, sometimes called global extremal functions or pluricomplex Green functions, from pluripotential theory are generalized. They are then used to construct weights for Hörmander's L^2-estimates of solutions to $\bar{\partial}$-equations and the solutions to those equations are then used to construct the desired polynomials.

Ágrip

Ritgerðin er samansett af efni úr greinum sem fjalla um margliðunálganir á fáguðum föll í mörgum breytistærðum. Þessar nálganir eru metnar með vegnum stöðlum og margliðurnar sem eru notaðar koma úr hlutbaugum margliðna. Hugtakið um stig margliðanna er einnig frábrugðið í hefðbundna tilfellinu. Megin markið rannsóknanna er að setja fram og sanna útgáfu af Bernstein-Walsh-Siciak setningunni, sem er alhæfing á Runge-Oka-Weil setningunni.

Til að sanna þessa setningu er notast við alhæfða útgáfu af Siciak-Zakharyuta fallinu úr fjölmættisfræði. Það er svo notað sem vigt í L^2-mati Hörmanders á lausnum á $\bar{\partial}$-afleiðujöfnum. Lausnir slíkra jafna eru síðan notaðar til að smíða margliðurnar sem á að nota í nálgununum.

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List of Papers

The thesis is based on the following papers:

I: Polynomials with exponents in compact convex sets and associated weighted extremal functions – Fundamental results
B. S. Magnússon, Á. E. Sigurðardóttir, R. Sigurðsson, **B. Snorrason**
To appear in Ann. Polon. Math.

II: Polynomials with exponents in compact convex sets and associated weighted extremal functions – Approximations and regularity
B. Snorrason
Manuscript

III: Polynomials with exponents in compact convex sets and associated weighted extremal functions – Generalized product property
B. Snorrason
To appear in Math. Scand.

IV: Polynomials with exponents in compact convex sets and associated weighted extremal functions – The Bernstein-Walsh-Siciak theorem
B. S. Magnússon, R. Sigurðsson, **B. Snorrason**
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Introduction

Our studies will focus on different forms of polynomial approximations. A lot is known about approximations of holomorphic functions by polynomials, both in one and several variables. This includes results such as Runge's well known theorem, Theorem 3.2. One way of studying polynomial approximations is through the application of potential theory, or pluripotential theory when working in several variables. This is the approach taken, for example, in Section 6.3 in Ransford [37].

The central aim of this thesis to study weighted polynomial approximations in certain subfamilies of polynomials. Additionally, the polynomials may have different degrees than in the classical setting. This becomes relevant when studying quantitative results such as the Bernstein-Walsh-Siciak theorem, Theorem 3.4. Much of the thesis is devoted to developing and studying the pluripotential theory related to these polynomial families.

The first three sections of this introduction provide definitions and classical results that inform the content of the thesis. Section 1 gives definitions and basic results from complex analysis in several variables and Section 2 does the same for pluripotential theory. Section 3 goes over some quantitative results in polynomial approximations along with giving an outline of a proof of the Bernstein-Walsh-Siciak theorem based only on Hörmander's L^2-estimates for solutions to $\bar{\partial}$-equations. Section 4 describes the contents of Papers I-III, where the pluripotential theory of Lelong classes is generalized to allow more precise control over the growth of the functions in the class. These Lelong classes are connected to gradings of subfamilies of polynomials. Section 5 applies this theory to generalize the Bernstein-Walsh-Siciak theorem.

1 Holomorphic functions

We denote by $\mathbb{C}^n$ the n-fold cartesian product of the complex plane. The real euclidean space $\mathbb{R}^{2n}$ can be identified with $\mathbb{C}^n$ by

$$(x_1, y_1, \dots, x_n, y_n) \mapsto (x_1 + iy_1, \dots, x_n + iy_n) = (z_1, \dots, z_n).$$

With this identification, we view $\mathbb{R}^{2n}$ as vector space over $\mathbb{C}$. For $z = (z_1, \dots, z_n)$ in $\mathbb{C}^n$ we define the euclidean norm by $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$ and the supremum norm by $\|z\|_\infty = \max\{|z_1|, \dots, |z_n|\}$. We will denote the punctured complex plane by $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and to simplify notation we let $\mathbb{C}^{*n} = (\mathbb{C}^*)^n$. We similarly let $\mathbb{C}^{n*} = \mathbb{C}^n \setminus \{0\}$, where we abuse notation by letting 0 denote the additive identity in $\mathbb{C}^n$. The open unit disc and unit circle in $\mathbb{C}$ will be denoted by $\mathbb{D} = \{z \in \mathbb{C}\,;\, |z| < 1\}$ and $\mathbb{T} = \{z \in \mathbb{C}\,;\, |z| = 1\}$.

Let $\Omega \subset \mathbb{C}^n$ be open and $f\colon \Omega \to \mathbb{C}$ be continuous. We say that f is *holomorphic on* Ω if it is holomorphic in each variable separately, and denote the class of all such functions by $\mathcal{O}(\Omega)$. This is equivalent to the mapping $\zeta \mapsto f(\zeta a + b)$ being holomorphic for all $b \in \Omega$, $a \in \mathbb{C}^{n*}$, and $\zeta \in \mathbb{C}$ such that

$\zeta a + b \in \Omega$. A result of Hartogs tells us that the assumption that f is continuous is superfluous.

Theorem 1.1 (Hartogs's theorem)**.** *Let $\Omega \subset \mathbb{C}^n$ be open and $f\colon \Omega \to \mathbb{C}$ be holomorphic in each variable. Then f is smooth.*

As in one dimension, we say that f is *entire* if it is holomorphic on $\mathbb{C}^n$.

By the definition of holomorphic functions, one might imagine that many properties of holomorphic functions in $\mathbb{C}$ generalize to several variables. While this is true, there are still many differences between $\mathbb{C}$ and $\mathbb{C}^n$, where $n > 1$. Another theorem of Hartogs shows such a difference.

Theorem 1.2 (Hartogs's extension theorem)**.** *Let $n > 1$, $\Omega \subset \mathbb{C}^n$ be an open set, $K \subset \Omega$ compact such that $\Omega \setminus K$ is connected, and $f \in \mathcal{O}(\Omega \setminus K)$. Then there exists $F \in \mathcal{O}(\Omega)$ such that $F = f$ on $\Omega \setminus K$.*

By taking K as a singleton, we see that holomorphic functions in several variables can not have isolated singularities and the level sets of entire functions are unbounded. In contrast, the level sets of a non-constant holomorphic function in one variable can not contain accumulation points. One way to justify the differences between one and several variables is to consider holomorphic functions as solutions to partial differential equations.

As in one dimension, we define the *homogeneous Cauchy-Riemann equations* by

$$\frac{\partial g}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial g}{\partial x_j} + \frac{i\partial g}{\partial y_j}\right) = 0, \quad j = 1, \dots, n.$$

So a smooth function is holomorphic if and only if it satisfies the homogeneous Cauchy-Riemann equations. We also have the *non-homogeneous Cauchy-Riemann equations* given by

$$\frac{\partial u}{\partial \bar{z}_j} = f_j, \quad j = 1, \dots, n,$$

where $f_1, \dots, f_n$ are smooth functions. We simplify our notation to $\bar{\partial} u = f$, where $f = (f_1, \dots, f_n)$. This notation can be formalized by considering u and f as differential forms, but we will omit this since it is not needed. We refer to the equation $\bar{\partial} u = f$ as a *$\bar{\partial}$-equation.* Solving $\bar{\partial}$-equations has many applications and will be our central tool later, where the solutions will be used to construct polynomials that approximate a given function. Note that the equation $\bar{\partial} u = f$ has a large solution space, since the kernel of the linear operator

$$u \mapsto \left(\frac{\partial u}{\partial \bar{z}_1}, \dots, \frac{\partial u}{\partial \bar{z}_n}\right),$$

is the space of holomorphic functions. Hörmander proved that a solution could be found that satisfies an L^2-estimate.

Theorem 1.3 (Hörmander's L^2-estimate)**.** *Let $\psi \in \mathcal{PSH}(\mathbb{C}^n)$, and $a > 0$. If $f_j \in L^2_{loc}(\mathbb{C}^n)$ and $\partial f_j/\partial \bar{z}_k = \partial f_k/\partial \bar{z}_j$, for $j, k = 1, \dots, n$ then there exists $u \in L^2_{loc}(\mathbb{C}^n)$ such that $\bar{\partial} u = f$ and*

$$\int_{\mathbb{C}^n} |u|^2 e^{-\psi_a}\, d\lambda \leq \frac{1}{a} \sum_{j=1}^{n} \int_{\mathbb{C}^n} |f_j|^2 e^{-\psi_{a-2}}\, d\lambda, \tag{1.1}$$

where $\psi_a = \psi + a \log(1 + |\cdot|^2)$.

Here $\mathcal{PSH}(\mathbb{C}^n)$ denotes the class of plurisubharmonic function on $\mathbb{C}^n$, which are defined and discussed in the next section. The stated version of Hörmander's estimate for the $\bar{\partial}$-equations is a global version. Usually, $\mathbb{C}^n$ is replaced with a more general Ω, but this is not needed for our studies. See Theorem 4.2.6 in Hörmander [23]. In the case of a more general Ω, and when $n > 1$, some geometric restrictions need to be imposed for the equations to be solvable. The classical restriction is that Ω is a *domain of holomorphy* or *pseudoconvex*. Domains of holomorphy are natural settings when studying holomorphic functions and pseudoconvex sets are their counterpart when studying plurisubharmonic functions. These two assumptions turn out to be equivalent, demonstrating the connection between complex analysis and pluripotential theory. See Theorem 4.2.8 in Hörmander [23]. The case when $n = 1$ is simpler since all open sets are domains of holomorphy.

To apply Hörmander's estimates we commonly have to convert this L^2-estimate into a pointwise estimate. In Section 7 of Paper I this is done using the Cauchy formula and in Section 2 of Paper IV this is done using the mean value property of holomorphic functions.

The Cauchy formula in one variable generalizes naturally to several variables, since we define a function f on $\Omega \subset \mathbb{C}^n$ to be holomorphic if it is holomorphic in each variable separately. To state a Cauchy formula in $\mathbb{C}^n$ we use *multi-index notation*, that is, for $z \in \mathbb{C}^n$ and $\alpha \in \mathbb{N}^n$, we let $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. If we assume $\overline{\mathbb{D}}^n \subset \Omega$ and set

$$c_\alpha = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \frac{f(\zeta_1, \dots, \zeta_n)}{\zeta^\alpha} \frac{d\zeta_1}{\zeta_1} \cdots \frac{d\zeta_n}{\zeta_n}, \quad \alpha \in \mathbb{N}^n,$$

then f is given by

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha, \quad z \in \mathbb{D}^n.$$

We refer to this identity as the *Cauchy formula*, since it follows by applying the classical Cauchy formula in each variable of f. In one variable we use the Cauchy formula to prove various Liouville theorems, for example that if $p \in \mathcal{O}(\mathbb{C})$ such that $|p(z)| \leq A + B|z|^m$, for constant A and B, and integer m, then p is a polynomial of degree no greater than m. We get the same result in several variables, but with a slight twist. The notion of the degree of a polynomial is more complicated in several variables. In fact, a flexible way of assigning polynomials degrees is at the heart of the research this thesis is based on. We circumvent this discussion, for the time being, by being direct. Let $p \in \mathcal{O}(\mathbb{C}^n)$, given by the Taylor series $p(z) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha$, be such that

$$|p(z)| \leq A + B|z|^m, \quad z \in \mathbb{C}^n,$$

for some constants A and B, and integer m. Then $c_\alpha = 0$ for all $\alpha \in \mathbb{N}^n$ such that $\alpha_1 + \cdots + \alpha_n > m$. This is the standard way of defining an m-th degree polynomial in several variables.

2 Plurisubharmonic functions

Solutions $u \colon U \to \mathbb{R}$, where U is an open subset of $\mathbb{R}^n$, to the differential equation $\Delta u = 0$, where $\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ denotes the *Laplace operator*, are called *harmonic*. An upper semicontinuous $u \colon U \to \mathbb{R} \cup \{-\infty\}$, that satisfies $\Delta u \geq 0$, in the weak sense of distributions, is said to be *subharmonic* on U and the family of subharmonic functions on U that are not identically $-\infty$ on any connected component of U is denoted by $\mathcal{SH}(U)$. The study of harmonic and subharmonic functions is referred to as *potential theory.* Subharmonic functions relate to their mean values, similarly to harmonic functions. In fact, an upper semicontinuous u on an open set $U \subset \mathbb{R}^n$ is subharmonic if and only if

$$u(x) \leq \frac{1}{S(x,r)} \int_{\partial B(x,r)} u(y)\, d\sigma(y), \quad x \in U,$$

for all $r > 0$ such that $\overline{B(x,r)} \subset U$, where $B(x,r)$ is the open euclidean ball with center x and radius r, $S(x,r)$ denotes the surface area of the ball, and σ is the surface area measure in $\mathbb{R}^n$. Potential theory in two (real) dimensions is commonly applied in the study of complex analysis in one variable. An explicit connection between complex analysis and potential theory is that the pullbacks of subharmonic functions by holomorphic maps are subharmonic. Namely, if U and V are open sets in $\mathbb{C}$, $f \colon U \to V$ holomorphic, and $u \in \mathcal{SH}(V)$, then $u \circ f \in \mathcal{SH}(U)$. In several complex variables, potential theory is not restrictive enough, since the subharmonic functions do not always respect the complex structure. For an example take $u(z_1, z_2) = -1/(|z_1|^2 + |z_2|^2)$, $z = (z_1, z_2) \in \mathbb{C}^{2*}$. We have that

$$\frac{\Delta u}{4} = \frac{\partial^2 u}{\partial z_1 \partial \bar{z}_1} = (|z_1|^2 + |z_2|^2)^{-2} - \frac{2|z_1|^2}{(|z_1|^2 + |z_2|^2)^3},$$

so by the symmetry $u(z_1, z_2) = u(z_2, z_1)$, we have that

$$\Delta u = \frac{4\partial^2 u}{\partial z_1 \partial \bar{z}_1} + \frac{4\partial^2 u}{\partial z_2 \partial \bar{z}_2} = 0.$$

So u is harmonic on $\mathbb{C}^{2*}$, and thus subharmonic as well. Let us now consider the univariate function $v(\zeta) = u(\zeta, 0)$, $\zeta \in \mathbb{C}^*$. If v were subharmonic on $\mathbb{C}^*$ then by

Theorem 3.6.1 in Ransford [37] it could be extended uniquely to a subharmonic function on $\mathbb{C}$. But no such extension exists since v is not integrable on any neighborhood of 0, which it would need to be by Theorem 2.5.1 in [37].

To remedy this we consider a subfamily of the subharmonic functions. We say a function $u\colon \Omega \to \mathbb{R} \cup \{-\infty\}$, where Ω is an open subset of $\mathbb{C}^n$, is *plurisubharmonic* if it is upper semicontinuous and subharmonic on all complex lines intersected with Ω. In terms of integral averages, an upper semicontinuous u is plurisubharmonic if and only if

$$u(z) \leq \frac{1}{2\pi} \int_{\mathbb{T}} u(z + \zeta w)\, d\zeta,$$

for all $z \in \Omega$ and $w \in \mathbb{C}^n$ such that $w\overline{\mathbb{D}} + z \subset \Omega$. We denote by $\mathcal{PSH}(\Omega)$ all plurisubharmonic functions on Ω that are not identically $-\infty$ on any connected component of Ω. Omitting functions that are $-\infty$ on an open set ensures that all functions in $\mathcal{PSH}(\Omega)$ are locally integrable, so

$$\mathcal{PSH}(\Omega) \subset L^1_{\text{loc}}(\Omega) \subset \mathcal{D}'(\Omega)$$

where $L^1_{\text{loc}}(\Omega)$ is the class of locally integrable functions in Ω and $\mathcal{D}'(\Omega)$ is the space of distributions on Ω. As a consequence we may endow $\mathcal{PSH}(\Omega)$ with the subspace topology of either $L^1_{\text{loc}}(\Omega)$ or $\mathcal{D}'(\Omega)$. These two topologies turn out to be the same topology. A discussion and application of this topology can be found in Section 3 of Paper II.

In potential theory *polar sets* are of importance as the insignificant sets, similar to the null sets of measure theory. Pluripotential theory has its analog. We say a set $E \subset \mathbb{C}^n$ is *pluripolar* if for all $z \in E$ there exists an open neighborhood U of z and $u \in \mathcal{PSH}(U)$ such that $u|_{U \cap E} = -\infty$. A theorem of Josefson says that this local definition is equivalent to a global one. That is, if E is pluripolar there exists $u \in \mathcal{PSH}(\mathbb{C}^n)$ such that $u|_E = -\infty$. Pluripolar sets are in a sense smaller than Lebesgue-null sets, since all pluripolar sets have Lebesgue measure zero but some Lebesgue-null sets are not pluripolar sets.

One source of plurisubharmonic functions are the holomorphic functions, since if f is holomorphic then $\log|f|$ is plurisubharmonic. Consequently, the zero sets of non-constant holomorphic functions are pluripolar sets.

One reason to study plurisubharmonic functions over holomorphic functions is because they are more flexible. For instance, if u and v are plurisubharmonic, then so is $\max\{u, v\}$. In the extremal case $v = \sup_{u \in \mathcal{F}} u$, for some family $\mathcal{F} \subset \mathcal{PSH}(\Omega)$ of plurisubharmonic functions which is locally bounded above, can fail to be plurisubharmonic because it is not generally upper semicontinuous. However, when v is upper semicontinuous it is also plurisubharmonic. In the case where v is not upper semicontinuous, we can take the *upper semicontinuous regularization*, given by

$$v^*(z) = \varlimsup_{w \to z} v(w), \quad z \in \Omega,$$

which is plurisubharmonic. Plurisubharmonic functions constructed in this way are referred to as *extremal.*

One common family of plurisubharmonic function to study is the *Lelong class*, given by

$$\mathcal{L}(\mathbb{C}^n) = \{u \in \mathcal{PSH}(\mathbb{C}^n)\,;\, u \leq \log^+ \|\cdot\|_\infty + c_u\}$$

where c_u is some constant depending on u and $\log^+ x = \max\{0, \log x\}$, $x \in \mathbb{R}$. The specific norm chosen does not matter since all norms on $\mathbb{C}^n$ are equivalent. Using the supremum norm is a natural choice for upcoming generalizations. Functions belonging to $\mathcal{L}(\mathbb{C}^n)$ are said to have *logarithmic growth* or *minimal growth*.

We can not take the supremum of $\mathcal{L}(\mathbb{C}^n)$ directly and expect it to be plurisubharmonic, since the Lelong class is not locally bounded above. Instead, we take supremum over subfamilies of $\mathcal{L}(\mathbb{C}^n)$.

For a compact $K \subset \mathbb{C}^n$ we define the *Siciak-Zakharyuta function of* K by

$$V_K(z) = \sup\{u(z)\,;\, u \in \mathcal{L}(\mathbb{C}^n), u|_K \leq 0\}, \quad z \in \mathbb{C}^n.$$

The family $\{u \in \mathcal{L}(\mathbb{C}^n)\,;\, u|_K \leq 0\}$ is locally bounded above if and only if K is not pluripolar. See Klimek [26], Corollary 5.2.2. If K is pluripolar then $V_K(z) = 0$ for $z \in K$ and $V_K(z) = +\infty$ for $z \notin K$.

The Siciak-Zakharyuta functions are also called *global extremal functions* or *pluricomplex Green's functions*. To elaborate on the latter term, we note that when $n = 1$ and $K \subset \mathbb{C}$ is compact with non-polar boundary, we have that V_K is the Green's function of $\mathbb{C} \setminus K$ with a pole at infinity, commonly denoted by $g_{\mathbb{C}\setminus K}$. That is V_K is the unique function in $\mathcal{L}(\mathbb{C})$ that is harmonic on $\mathbb{C} \setminus K$ and $V_K(w) \to 0$ as $w \to z$ for all $z \in \partial K \setminus F$, where F is a polar set.

If $K = \{z \in \mathbb{C}^n\,;\, \|z - a\| \leq r\}$, for $a \in \mathbb{C}^n$ and some norm $\|\cdot\|$, then

$$V_K(z) = \log^+ \frac{\|z - a\|}{r}, \quad z \in \mathbb{C}^n.$$

The basic properties of V_K, for more general K, can be derived from this example. It can be shown that V_K is lower semicontinuous if K is compact. This will be discussed in more detail later. Furthermore, if $V_K^* = 0$ on K then V_K is continuous. We will later use the fact that $V_K^* \in \mathcal{L}(\mathbb{C}^n)$ for all non-pluripolar $K \subset \mathbb{C}^n$. See Corollary 5.2.2 in Klimek [26].

3 Polynomial approximations

An early result in polynomial approximations is Weierstrass's theorem.

Theorem 3.1 (Weierstrass's theorem). *Let* $f\colon [-1, 1] \to \mathbb{R}$ *be continuous and* $\varepsilon > 0$. *Then there exists a polynomial* p *such that* $\|f - p\|_{[-1,1]} < \varepsilon$.

In other words, every continuous function on the interval $[-1, 1]$ can be approximated uniformly by polynomials. An analogous result in the complex plane is Runge's theorem. To state it we recall that a compact set $K \subset \mathbb{C}^n$ is said to be *polynomially convex* if it coincides with its *polynomial hull*, which is given by $\widehat{K} = \{z \in \mathbb{C}^n\,;\, |p(z)| < \|p\|_K,\ \text{for all}\ p \in \mathcal{P}(\mathbb{C}^n)\}$, where $\mathcal{P}(\mathbb{C}^n)$ denotes the space of all polynomials. For a compact $K \subset \mathbb{C}$ we can describe $\widehat{K}$ in terms of its complement, that is the complement of $\widehat{K}$ is the unbounded complement of K. So, when $n = 1$, the polynomial hull has the effect of filling in the holes in K. This nice description of polynomial convexity does not work when $n > 1$. Generally, if $K \subset \mathbb{R}^n \subset \mathbb{C}^n$ is compact then it is polynomially convex.

Theorem 3.2 (Runge's theorem)**.** *Let $K \subset \mathbb{C}$ be compact and polynomially convex, f be holomorphic in a neighborhood of K, and $\varepsilon > 0$. Then there exists a polynomial p such that $\|f - p\|_K < \varepsilon$.*

Weierstrass's theorem and Runge's theorem are examples of *qualitative results*, since they state when approximations are possible. Our focus will be on *quantitative results*. This includes theorems about how well one may approximate when approximations are possible. Our studies will center around the Bernstein-Walsh-Siciak theorem and related theorems.

To this end we define, for a function f on a set $K \subset \mathbb{C}^n$ and integer m, *the m-th approximation coefficient of f on K* by

$$d_{K,m}(f) = \inf\{\|f - p\|_K\,;\, p \in \mathcal{P}_m(\mathbb{C}^n)\},$$

where $\mathcal{P}_m(\mathbb{C}^n)$ is the class of polynomials of degree not greater than m. These coefficients describe how well f can be approximated by polynomials of bounded degree. Runge's theorem can be rephrased to say that $d_{K,m}(f) \to 0$, as $m \to +\infty$, if K is compact and polynomially convex and f is holomorphic in a neighborhood X of K. The rest of this section will study how the rate of decay of $d_{K,m}(f)$ relates to X.

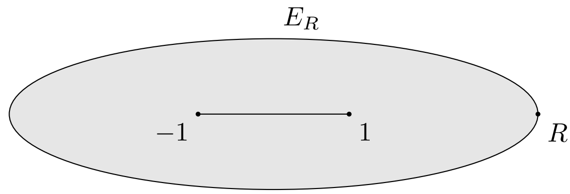
We first consider an initial result of Bernstein [9], which serves as a quantitative counterpart to Weierstrass's theorem.

Theorem 3.3 (Bernstein's theorem)**.** *Let $R > 1$, f be a continuous function on the interval $K = [-1, 1]$, and E_R be the open set bounded by an ellipse with foci -1 and 1 and semi major axis R. Then f can be extended to a holomorphic function on E_R if and only if*

$$\varlimsup_{m\to\infty} d_{K,m}(f)^{1/m} \le \frac{1}{R}. \tag{3.1}$$

The inequality in Bernstein's theorem describes how quickly $d_{K,m}(f)$ converges to 0 as m goes to infinity. In fact, it tells us that for all $0 < \gamma < R$ the inequality $d_{K,m}(f) \le (R-\gamma)^{-m}$ holds for all but finitely many m. Walsh [56, §4.6] calls this *overconvergence*.

To generalize Bernstein's theorem for a larger class of K we need to find some replacement for E_R. This was first done by Walsh [55]. We will go over an overview of Walsh's theorem in lieu of a direct statement as its generalization to several variables is nearly identical. Walsh's theorem replaces the interval $[-1, 1]$, in Bernstein's theorem, with a compact $K \subset \mathbb{C}$ such that the Green's



An illustration of Bernstein's theorem.

function $g_{\mathbb{C}\setminus K}$ is continuous and the ellipse E_R is replaced with the sublevel set $X_R = \{z \in \mathbb{C}\,;\, g_{\mathbb{C}\setminus K}(z) < \log R\}$. It can be shown that

$$g_{\mathbb{C}\setminus[-1,1]}(z) = \log|z + (z^2-1)^{1/2}|, \quad z \in \mathbb{C}\setminus[-1,1],$$

where the branch of the square root is chosen such $t + (t^2-1)^{1/2} > 1$ for $t > 1$. Then $X_R = E_R$, so Walsh's result matches Bernstein's in this case. See Lemma 5.4.2 in Klimek [26].

We have already discussed how the Siciak-Zakharyuta functions are a generalization of Green's functions to several variables. Siciak used this idea to generalize Walsh's theorem to several variables.

Theorem 3.4 (The Bernstein-Walsh-Siciak theorem)**.** *Let $R > 1$, $K \subset \mathbb{C}^n$ be non-pluripolar, compact and polynomially convex, $f\colon K \to \mathbb{C}$ be a function on K, and $X_R = \{z \in \mathbb{C}^n\,;\, V_K(z) < \log R\}$. Then there exists $F \in \mathcal{O}(X_R)$ such that $F|_K = f$ if and only if*

$$\varlimsup_{m\to\infty} d_{K,m}(f)^{1/m} \le \frac{1}{R}. \tag{3.2}$$

Both directions in the Bernstein-Walsh-Siciak theorem can be proven by constructing satisfactory polynomials, although one direction is simpler. If we assume (3.2) holds we can pick polynomials p_m that are close to achieving the infimum in $d_{K,m}(f)$. If this is done with care one can show that the sequence $(p_m)_{m\in\mathbb{N}}$ converges uniformly on compact subsets of X_R, and its limit, therefore, defines a holomorphic function F that agrees with f on K. Theorem 1.1 in Paper IV is a generalization of the "if" part of Theorem 3.4 and is proven using this same process.

Before considering the other half of the proof in full generality it is instructive to consider specifically the case when $n = 1$ and $K = \overline{\mathbb{D}}$. Then $V_K(z) = \log^+|z|$ so $X_R = R\mathbb{D}$. We assume we have some $F \in \mathcal{O}(R\mathbb{D})$ and want to show that (3.2) holds. We may write $F(z) = \sum_{j=0}^{\infty} c_j z^j$, $z \in R\mathbb{D}$, and we set $p_m(z) = \sum_{j=0}^{m} c_j z^j$. If we fix $0 < \gamma < R$ we have by a Cauchy estimate $|a_j| \le (R-\gamma)^{-j} C_\gamma$, where $C_\gamma = \sup\limits_{|z|=R-\gamma} F(z)$. So

$$d_{\overline{\mathbb{D}},m}(f) \le \|f - p_m\|_{\overline{\mathbb{D}}} \le \sum_{j=m+1}^{\infty} |a_j| \le C_\gamma \sum_{j=m+1}^{\infty} (R-\gamma)^{-j} = \frac{C_\gamma}{(R-\gamma)^m},$$

since the final series is geometric. The constant C_γ does not depend on m, so we have that

$$\varlimsup_{m\to\infty} d_{\overline{\mathbb{D}},m}(f)^{1/m} \le \frac{1}{R-\gamma}.$$

This holds for all $0<\gamma<R$, so (3.2) follows.

The case of $K=\overline{\mathbb{D}}$ is easy because we have a canonical polynomial approximation of functions in $\mathcal{O}(R\mathbb{D})$, $R>1$, in their Taylor series. Proving the general case is structurally similar to this special case, but we need a sufficiently flexible manner of constructing polynomials.

We will now show how Hörmander's estimates for $\bar\partial$-equations can be used to construct polynomials to prove the "only if" part of the Bernstein-Walsh-Siciak theorem. This is done in detail and more generality in Paper IV. To start we will give a general outline of the $\bar\partial$-equation we will be considering. We will then see how specific restrictions to the parameters will allow us to construct polynomials that approximate a given function.

To recall the setting of the theorem we assume that f is a holomorphic function on the open set X_R and that K is a compact set in X_R. We then let χ be a smooth function with compact support in X_R. Then χf is a smooth function on X_R that can be extended to a smooth function on $\mathbb{C}^n$ by assigning it the value 0 outside of X_R. For $m\in\mathbb{N}$, $0<a<1$, and $\psi_m\in\mathcal{PSH}(\mathbb{C}^n)$ we let u_m be a solution to $\bar\partial u_m = f\bar\partial\chi$ satisfying (1.1), and $p_m=\chi f-u_m$. Without specifying χ, ψ_m, or a we have that

$$\bar\partial p_m = \bar\partial(\chi f) - \bar\partial u_m = \chi\bar\partial f + f\bar\partial\chi - f\bar\partial\chi = 0,$$

since $\bar\partial f=0$, so $p_m\in\mathcal{O}(\mathbb{C}^n)$.

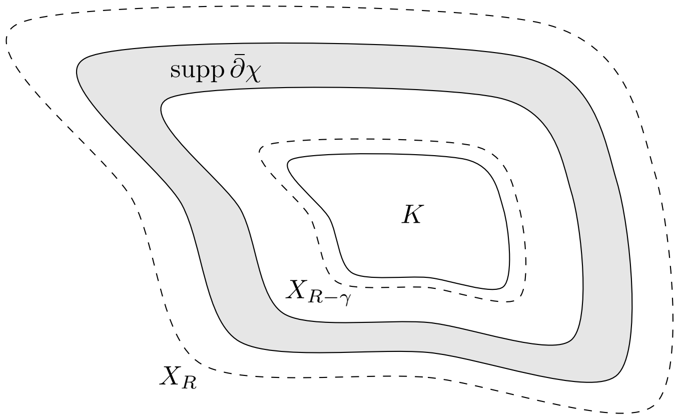
If we assume our weight ψ_m satisfies $\psi_m\le 2m\log^+|\cdot|+C$ for some constant C, then we can use the mean value property of holomorphic functions, along with (1.1) and Lipschitz continuity properties of the logarithm, to show that $|p(z)|\le A+B|z|^{m+a}$ for all $z\in\mathbb{C}^n$. The Liouville theorem then tells us that $p_m\in\mathcal{P}_m(\mathbb{C}^n)$, since $a<1$.

Let us now see how we can pick χ and ψ_m such that $\|f-p_m\|_K$ is close to being minimized. Fixing $\varepsilon,\gamma>0$, we set $\psi_m=2mV_K$ and χ such that $\{z\in X_R\,;\,\chi(z)=1\}$ is a neighborhood of $X_{R-\gamma}$. Note that $V_K\in\mathcal{L}(\mathbb{C}^n)$ so $p_m\in\mathcal{P}_m(\mathbb{C}^n)$. Using the mean value property for holomorphic functions and (1.1) we get that

$$|f(z)-p_m(z)| \le C'_{\varepsilon,\gamma}e^{m\varepsilon}\sum_{j=1}^n\int_{\mathbb{C}^n}|f\partial\chi/\partial\bar z_j|^2e^{-\psi_{m,a-2}}\,d\lambda, \quad z\in K,$$

for some constant $C'_{\varepsilon,\gamma}$ that does not depend on m. Now since χ is constant on $X_{R-\gamma}$ and outside of X_R we have that the union of the supports of the partial derivatives of χ lie in $X_R\setminus X_{R-\gamma}$. So $V_K>R-\gamma$ holds on the support of the integrands in the right hand side of the previous inequality, and consequently

$$|f(z)-p_m(z)| \le \frac{C_{\varepsilon,\gamma}e^{m\varepsilon}}{(R-\gamma)^m}, \quad z\in K,$$



Illustrated is a potential placement of supp $\bar{\partial}\chi$ *due to the construction of* χ. *As pictured the complement of the support has two components. We have that* $\chi = 1$ *on the bounded component and* $\chi = 0$ *on the unbounded component.*

where $C_{\varepsilon,\gamma}$ is another constant that does not depend on m. So

$$\overline{\lim_{m\to\infty}}\, d_{m,K}(f)^{1/m} \leq \overline{\lim_{m\to\infty}}\, \frac{C_{\varepsilon,\gamma}^{1/m} e^{\varepsilon}}{R-\gamma} = \frac{e^{\varepsilon}}{R-\gamma},$$

and letting γ and ε go to zero yields (3.2).

4 Generalized Lelong classes

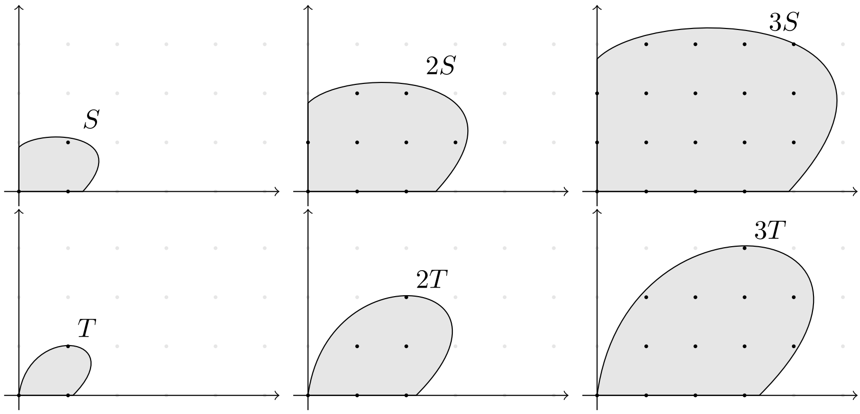
In this section, we will lay the groundwork for our generalization of the Bernstein-Walsh-Siciak theorem. First,we will consider how we restrict which polynomials we allow, and then we will look at the related pluripotential theory.

We call $\alpha \in \mathbb{N}^n$ a *multi-index*. The exponentiation by multi-indices is given by

$$z^{\alpha} = z_1^{\alpha_1} \cdots z_n^{\alpha_n}, \quad z = (z_1, \ldots, z_n) \in \mathbb{C}^n,$$

and the length of multi-indices is denoted by $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Note that we define $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. So $p \in \mathcal{O}(\mathbb{C}^n)$, given by the Taylor series $p(z) = \sum_{\alpha\in\mathbb{N}^n} c_\alpha z^\alpha$, is in $\mathcal{P}_m(\mathbb{C}^n)$ if $c_\alpha = 0$ for $\alpha \in \mathbb{N}^n$ such that $|\alpha| > m$. The choice to use the ℓ^1-norm to describe the size of the multi-indices is standard, but arbitrary. We are interested in a more flexible way to describe the degrees of polynomials. The method we develop will sometimes lead to us working with polynomial rings that are proper subfamilies of $\mathcal{P}(\mathbb{C}^n)$. This can be seen as certain polynomials having degree $+\infty$.

Let S be a compact convex subset of $\mathbb{R}^n_+$ that contains 0. From this set we will generate our polynomials. We say that a function p is an *m-th degree*



Although $\mathcal{P}_1^S(\mathbb{C}^2) = \mathcal{P}_1^T(\mathbb{C}^2)$ *we immediately have that* $\mathcal{P}_2^S(\mathbb{C}^2) \neq \mathcal{P}_2^T(\mathbb{C}^2)$. *Additionally* $\mathcal{P}^S(\mathbb{C}^2) \neq \mathcal{P}^T(\mathbb{C}^2)$, *since* $\mathcal{P}^S(\mathbb{C}^2) = \mathcal{P}(\mathbb{C}^2)$ *but* $\mathcal{P}^T(\mathbb{C}^2)$ *does not contain* $z \mapsto z_2^j$ *if* $j \geq 1$. *These are the only monomials that are not in* $\mathcal{P}^T(\mathbb{C}^n)$.

polynomial with respect to S if it can be written by

$$p(z) = \sum_{\alpha \in mS \cap \mathbb{N}^n} c_\alpha z^\alpha, \quad z \in \mathbb{C}^n.$$

The space of all such polynomials is denoted by $\mathcal{P}_m^S(\mathbb{C}^n)$ and we set $\mathcal{P}^S(\mathbb{C}^n) = \bigcup_{m=0}^\infty \mathcal{P}_m^S(\mathbb{C}^n)$. Let us consider some concrete examples.

First we note that if $S = \mathrm{ch}\{0, e_1, \ldots, e_n\}$, where ch denotes the closed convex hull and $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$, then $\mathcal{P}_m(\mathbb{C}^n) = \mathcal{P}_m^S(\mathbb{C}^n)$.

Now let $S = \mathrm{ch}\{0, \alpha\} \subset \mathbb{R}_+^n$, for some $\alpha \in \mathbb{N}^n$. Then $p \in \mathcal{P}_m^S(\mathbb{C}^n)$ if there exists $q \in \mathcal{P}_m(\mathbb{C}^1)$ such that $p(z) = q(z^\alpha)$, $z \in \mathbb{C}^n$. Generally, not all polynomials in $\mathcal{P}^S(\mathbb{C}^n)$ will be given in this way. To see this consider the case where $n = 2$, $p(z_1, z_2) = z_1 z_2$, and $\alpha = (2, 2)$. However, if $\gcd\{\alpha_1, \ldots, \alpha_n\} = 1$ this describes all polynomials in $\mathcal{P}_m^S(\mathbb{C}^n)$.

For another example, we set $S = \mathrm{ch}\{(0,0), (1,0), (1,1)\} \subset \mathbb{R}_+^2$. The monomial z^α, for $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, is then in $\mathcal{P}^S(\mathbb{C}^2)$ if and only if $\alpha_1 \leq \alpha_2$. Note that $z \mapsto z_2^j$, for an integer $j \geq 1$, is not contained in $\mathcal{P}^S(\mathbb{C}^2)$.

Throughout this thesis we will assume that $S \subset \mathbb{R}_+^n$ is a compact convex set containing 0. To justify these assumptions we note that the convexity of S implies that $\mathcal{P}_m^S(\mathbb{C}^n)\mathcal{P}_\ell^S(\mathbb{C}^n) \subset \mathcal{P}_{m+\ell}^S(\mathbb{C}^n)$, the compactness of S implies that $\mathcal{P}_m^S(\mathbb{C}^n)$ is finite dimensional, and the fact that S contains 0 implies that $\mathcal{P}_\ell^S(\mathbb{C}^n) \subset \mathcal{P}_m^S(\mathbb{C}^n)$, for $\ell \leq m$, and that $\mathcal{P}^S(\mathbb{C}^n)$ contains the constant polynomials.

Our next goal is to generalize the Lelong class to give a more precise description of its growth properties, giving us a class of plurisubharmonic functions related to $\mathcal{P}^S(\mathbb{C}^n)$. To this end we recall the *supporting function of* S, given by

$$\varphi_S(\xi) = \sup_{x \in S} \langle x, \xi \rangle, \quad \xi \in \mathbb{R}^n.$$

The supporting function is convex and positively homogeneous, that is $\varphi_S(t\xi) = t\varphi_S(\xi)$ and $\varphi_S(\xi+\eta) \leq \varphi_S(\xi) + \varphi_S(\eta)$, for all $t \geq 0$ and $\xi, \eta \in \mathbb{R}^n$. In fact, if $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is convex and positively homogeneous then it is the supporting function of

$$A = \{x \in \mathbb{R}^n\,;\, \langle x, \xi \rangle \leq \varphi(\xi), \text{ for all } \xi \in \mathbb{R}^n\},$$

that is $\varphi = \varphi_A$. The *logarithmic supporting function of* S can now be defined by

$$H_S(z) = \varphi_S(\log|z_1|, \dots, \log|z_n|), \quad z \in \mathbb{C}^{*n}.$$

This function can be extended to all of $\mathbb{C}^n$ by $\varlimsup_{\mathbb{C}^{*n} \ni w \to z} H_S(w)$, with its values on $\mathbb{C}^n \setminus \mathbb{C}^{*n}$ given by logarithmic supporting functions of lower dimensional sets. We then have that $H_S \in \mathcal{PSH}(\mathbb{C}^n) \cap \mathcal{C}(\mathbb{C}^n)$. The details of these properties of the logarithmic supporting functions can be found in Propositions 3.3 and 3.4 of Paper I. Our generalization of the Lelong class will be the plurisubharmonic functions that do not grow faster than H_S, that is $\mathcal{L}^S(\mathbb{C}^n)$ will denote the class of $u \in \mathcal{PSH}(\mathbb{C}^n)$ such that $u \leq H_S + c_u$ for some constant c_u, depending on u.

The logarithmic supporting functions had already appeared in the literature in Rashkovskii [39], which is related to Rashkovskii [38] and Lelong and Rashkovskii [27], where they are called *indicators*. There they are introduced to study the total Monge-Ampère masses of globally plurisubharmonic functions of logarithmic growth.

Central to our studies will be *the Siciak-Zakharyuta function of* E*, with respect to* S*, and with weight* q which we define by

$$V^S_{E,q} = \sup\{u\,;\, u \in \mathcal{L}^S(\mathbb{C}^n), u|_K \leq q\},$$

where E is a subset of $\mathbb{C}^n$ and $q\colon E \to \mathbb{R} \cup \{+\infty\}$. The weight q is said to be *admissible* if

(i) it is lower semicontinuous,

(ii) the set $\{z \in E\,;\, q(z) < +\infty\}$ is non-pluripolar,

(iii) $\varlimsup_{\substack{|z| \to +\infty \\ z \in E}} H_S(z) - q(z) = -\infty$.

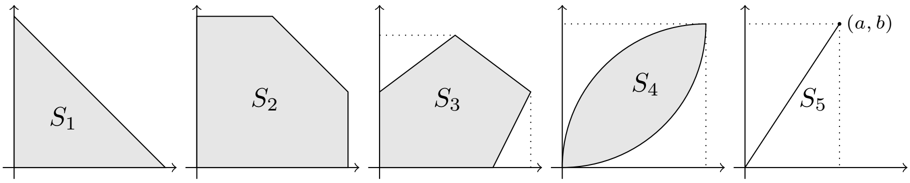
Note that **(iii)** is not relevant if E is bounded, which is often the case.

In the study of the Lelong classes $\mathcal{L}^S(\mathbb{C}^n)$, the following assumptions on S appear:

(i) $S = \Sigma = \operatorname{ch}\{0, e_1, e_2, \dots, e_n\}$, where $e_1, \dots, e_2$ is the standard basis in $\mathbb{R}^n$. We call Σ the *standard simplex in* $\mathbb{R}^n$.

(ii) $S = \Sigma_x = \operatorname{ch}\{0, x_1e_1, x_2e_2, \dots, x_ne_n\}$, where $x \in \mathbb{R}^n_+$.

(iii) S is such that for all $x \in S$ the box $[0, x_1] \times \cdots \times [0, x_n] \subset S$. In such a case, we say that S is a *lower set*.



Some examples in $\mathbb{R}^2$. From left to right, S_1 is the standard simplex, S_2 is a lower set, S_3 is not a lower set but contains a neighborhood of 0, S_4 does not contain a neighborhood of 0 but is a convex body, and S_5 is not a convex body. If $b = 0$ or a/b is rational then the rational points are dense in S_5. Otherwise they are not. In the case of S_3, S_4, and S_5 dotted lines have been added to denote their respective lower hulls.

(iv) S is such that exists $a > 0$ such that $a\Sigma \subset S$. In this case, we say that S contains a neighborhood of 0 in $\mathbb{R}^n_+$.

(v) S has a non-empty interior. In this case, we call S a *convex body*. This implies that S is the closure of its interior.

(vi) $\overline{S \cap \mathbb{Q}^n} = S$. In other words, the rational points are dense in S.

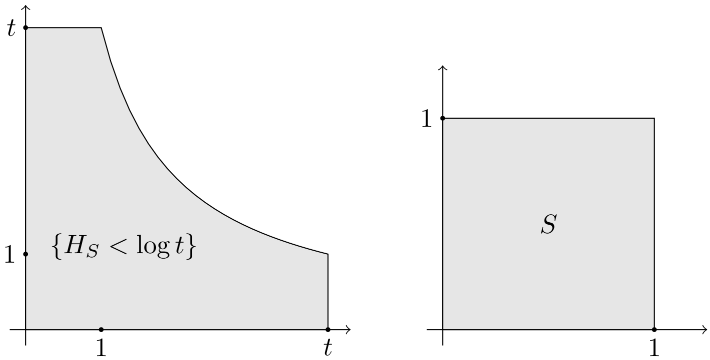
(vii) No additional assumptions on S are made.

Recall that in all cases we assume that $0 \in S \subset \mathbb{R}^n_+$ and that S is compact and convex. Assumption **(i)** encompasses the classical setting, that is $\mathcal{L}^\Sigma(\mathbb{C}^n) = \mathcal{L}(\mathbb{C}^n)$. Assumption **(ii)** is relevant when considering the convexity of the sublevel sets of V_K^S for compact K. Assumption **(iii)** is the setting of Bos and Levenberg [14]. Note that this assumption is restrictive, since the intersection of lower sets is a lower set. Notable examples of lower sets are the intersection of an ℓ^p unit ball and $\mathbb{R}^n_+$ for $1 \leq p \leq +\infty$, that is

$$S = \{x \in \mathbb{R}^n_+ \,;\, x_1^p + \cdots + x_n^p \leq 1\}, \quad 1 \leq p < +\infty,$$

and $S = \{x \in \mathbb{R}^n_+ \,;\, \max\{x_1, \ldots, x_n\} \leq 1\}$, corresponding to the case of $p = +\infty$. Assumptions **(iv)**, **(v)**, and **(vi)** are more malleable than assumption **(iii)**, since if no assumption is made on S then $S_j = \mathrm{ch}(S \cup (1/j)\Sigma)$ satisfies these assumptions and $S = \cap_j S_j$. This is relevant in the proof of Theorem 1.1 in Paper III. Assumption **(iv)** is the setting of [8] and [29], while assumption **(v)** is used in [30] and [34]. Assumption **(vi)** is relevant in [30]. The goal of Paper I was to start with no additional assumptions and add them only when necessary.

Most results in the classical setting of $S = \Sigma$ generalize to lower sets. An example that does not generalize can be found in Section 5 of Paper III. Lempert proved in discussions with Momm [33] that if $K \subset \mathbb{C}^n$ is compact and convex and $t > 0$ then the sublevel set $\{z \in \mathbb{C}^n \,;\, V_K(z) < t\}$ is convex. Dieu and Long [34] generalized to sublevel sets of V_K^S when S is a convex body. In Paper III it is shown that this is not true in general. In fact, if S is not a simplex of the form Σ_x, for $x \in \mathbb{R}^n_+$, then the sublevel sets $\{z \in \mathbb{C}^n \,;\, H_S(z) < t\}$ are not convex, for t large enough.



The left image shows the sublevel set $\{z \in \mathbb{C}^2\,; H_S(z) < \log t\}$*, for* $t > 1$ *and* $S = [0,1] \times [0,1] \subset \mathbb{R}^2_+$*, as pictured on the right. Note that the sublevel set is not convex. The sublevel set is drawn in the coordinates* $|z_1|$ *and* $|z_2|$*, which is justified by the fact that logarithmic supporting functions are rotationally symmetric in each variable, that is* $H_S(z_1, z_2) = H_S(|z_1|, |z_2|)$*. For our choice of* S *we have that* $H_S(z_1, z_2) = \log^+ |z_1| + \log^+ |z_2|$*. So when* $|z_1|, |z_2| \geq 1$ *the boundary of the sublevel set is parameterized by* $x \mapsto (x, t/x)$*,* $x \in [1, t]$*.*

In the case when S is not a lower set we are sometimes interested in the smallest lower set that contains S, called *the lower hull of* S and denoted by $\widehat{S}_\ell$, or sometimes $\widehat{S}_{\mathbb{R}^n_+}$. This notation makes sense when considering the more general Γ*-hull of* S, where Γ is a cone in $(\mathbb{R}^n \setminus \mathbb{R}^n_-) \cup \{0\}$, given by

$$\widehat{S}_\Gamma = \{x \in \mathbb{R}^n_+\,; \langle x, \xi \rangle \leq \varphi_S(\xi), \text{ for all } \xi \in \Gamma\}.$$

These hulls are introduced and studied in Sections 5 and 7 of Paper I. Setting $\Gamma = \mathbb{R}^n_+$ gives us the lower hull of S.

To justify our definition of $\mathcal{L}^S(\mathbb{C}^n)$ we look at a connection between it and $\mathcal{P}^S(\mathbb{C}^n)$. In the classical setting of $S = \Sigma$, we have for $p \in \mathcal{O}(\mathbb{C}^n)$ that $p \in \mathcal{P}_m(\mathbb{C}^n)$ if and only if $\log |p|^{1/m} \in \mathcal{L}(\mathbb{C}^n)$. This is a variant of the Liouville theorem discussed in Section 1. Theorem 3.6 in Paper I is a Liouville theorem for $\mathcal{P}^S(\mathbb{C}^n)$. It states that, for $p \in \mathcal{O}(\mathbb{C}^n)$ and $m \in \mathbb{N}$, the following are equivalent:

(i) $p \in \mathcal{P}^S_m(\mathbb{C}^n)$,

(ii) $\log |p|^{1/m} \in \mathcal{L}^S(\mathbb{C}^n)$,

(iii) $|p(z)| < C(1 + |z|)^a e^{mH_S(z)}$, for all $z \in \mathbb{C}^n$, where $a < d^1_m$,

where d^1_m denotes the distance from mS to $\mathbb{N}^n \setminus mS$ in the L^1-norm. We also have an L^2 version of this Liouville theorem. Theorem 7.2 in Paper I states that if

$$\int_{\mathbb{C}^n} |p|^2 e^{-2mH_S} (1 + |\cdot|^2)^{-a}\, d\lambda < +\infty,$$

for some $0 \leq a < d_m^2$, where d_m^2 denotes the euclidean distance from mS to $\mathbb{N}^n \setminus mS$, then $p \in \mathcal{P}_m^{\widehat{S}_{\Gamma_m}}(\mathbb{C}^n)$, where $\Gamma_m = \{\xi \in \mathbb{R}^n\,;\, \langle 1_n, \xi\rangle \geq (a - d_m)|\xi|\}$ and $1_n = (1, \dots, 1) \in \mathbb{N}^n$. Example 7.4 in Paper I shows that the Γ_m-hull is necessary.

In light of these Liouville theorems, studying the distance from mS to $\mathbb{N}^n \setminus mS$ is interesting. Proposition 4.1 in Paper IV gives a lower bound for these distances, independent of m, when S is a convex polytope with rational vertices, that is when $S = \operatorname{ch}\{v_1, \dots, v_n\}$, for $v_1, \dots, v_n \in \mathbb{Q}_+^n$. It states that

$$d(mS, \mathbb{N}^n \setminus mS) \geq \frac{1}{nq} d(nqS, \mathbb{N}^n \setminus nqS) > 0, \quad m \in \mathbb{N}^*,$$

where q is a common denominator for all coordinates of $v_1, \dots, v_\ell$. The lower bound follows from an interesting formula for convex sets that relates dilations with translations using the extremal points of the set. Recall that a point $a \in A$ is called *extremal in* A if for all $x, y \in A$ and $t \in [0,1]$ such that $tx + (1-t)y = a$, then either $x = a$ or $y = a$, and the set of all extremal points of A is denoted by $\operatorname{ext} A$. Then, for a convex set $A \subset \mathbb{R}^n$, we have

$$mA = A + \frac{1}{n} \sum_{j=1}^{mn-n} \operatorname{ext} A, \quad m \in \mathbb{N}^*,$$

which means that any $x \in mA$ can be written as $x = a + b_1 + \cdots + b_{mn-n}$, where $a \in A$ and $b_1, \dots, b_{mn-n} \in \frac{1}{n} \operatorname{ext} A$.

We will now turn our attention to the regularity of $V_{K,q}^S$. In the classical setting we have that $V_{K,q}$ is lower semicontinuous if K is compact and q is admissible. This can be shown by regularizing functions in $\mathcal{L}(\mathbb{C}^n)$. Central to the proof is a variant of Dini's theorem.

Theorem 4.1 (Dini's theorem)**.** *Let $K \subset \mathbb{C}^n$ be compact, f_j be a decreasing sequence of upper semicontinuous function on K with limit f and g be a lower semicontinuous function on K such that $f \leq g$. Then for every $\varepsilon > 0$ there exists j_ε such that $f_j \leq g + \varepsilon$ for $j > j_\varepsilon$.*

The proof does not differ greatly from the proof of the classical version of Dini's theorem. For every $z \in K$ we can choose j_z such that $f_j(z) \leq g(z) + \varepsilon$ for all $j > j_z$. Since $f_j - g$ is upper semicontinuous the set $U_z = \{z \in K\,;\, f_j(z) - g(z) \leq \varepsilon\}$ is an open neighborhood of z. The family $\{U_z\,;\, z \in K\}$ is an open cover of K, so by the compactness of K we can take $z_1, \dots z_\ell$ such that $\{U_{z_j}\,;\, j = 1, \dots, \ell\}$ is also an open cover of K. We can then conclude by taking $j_\varepsilon = \max\{j_1, \dots, j_\ell\}$.

Let us fix an open $U \subset \mathbb{C}^n$, and assume that for all $u \in \mathcal{L}^S(\mathbb{C}^n)$ there exists a sequence u_j in $\mathcal{L}^S(\mathbb{C}^n)$ such that all u_j are continuous on U and $u_j \searrow u$. If we fix $\varepsilon > 0$ and $u \in \mathcal{L}^S(\mathbb{C}^n)$ such that $u|_K \leq q$ and take such a sequence we have by Dini's theorem that there exists j such that $u_j - \varepsilon \leq q$ on K. Since $u_j \searrow u$ we have that $V_{K,q}^S$ can be taken as the supremum over a family of functions that are continuous on U. This implies that $V_{K,q}^S$ is lower semicontinuous on U. See Lemma 2.3.2 in Klimek [26].

So showing the lower semicontinuity of $V^S_{K,q}$ can be done by finding a method of regularization that preserves the Lelong class $\mathcal{L}^S(\mathbb{C}^n)$. When $S = \Sigma$ this is done using integral convolutions with standard smoothing kernels. For details see Theorem 2.9.2 in Klimek [26]. Theorem 5.8 in Paper I shows that this method works if and only if S is a lower set. So in the case where S is not a lower set, other methods must be used. Paper II is devoted to studying various other methods of regularizations, which we will now describe. Throughout the discussion, we will assume that $u \in \mathcal{L}^S(\mathbb{C}^n)$ is bounded below. This restriction is sometimes stronger than those in Paper II but it is sufficient when studying the Siciak-Zakharyuta functions, since $V^S_{K,q} \geq \inf_{w\in K} q(w) > -\infty$, when q is admissible. The first method considered is given by

$$R^a_{\delta,\mu}u(z) = -\log \inf_{w\in\mathbb{C}^n} \{e^{-u(w)} + \delta^{-1}\mu(z-w)\},$$

where $\delta > 0$ and μ is a function on $\mathbb{C}^n$. This is a generalization of a method of Siciak [44] and was studied because in [8] it is claimed that if $\mu = |\cdot|$ and δ is taken small enough then $R^a_{\delta,\mu}u \in \mathcal{L}^S(\mathbb{C}^n)$ if S contains a neighborhood of 0. Example 2.2 in Paper II gives a counterexample to this assertion. Theorem 2.1 in Paper II shows that $R^a_{\delta,\mu}u$ is continuous on $\mathbb{C}^n$, $R^a_{\delta,\mu}u \in \mathcal{L}^S(\mathbb{C}^n)$, and $R^a_{\delta,\mu}u \searrow u$, as $\delta \searrow 0$, if S is a lower set, μ is a distance function, and $\delta > 0$ is taken small enough. A continuous positive function μ is called a *distance function* if $\mu(z) = 0$ if and only if $z = 0$ and $\mu(tz) = |t|\mu(z)$, for all $z \in \mathbb{C}^n$ and $t \in \mathbb{C}$. It is also shown precisely how small δ needs to be taken.

We will allow us a slight abuse of notation by identifying a lower case vector in $\mathbb{C}^n$ with a diagonal matrix denoted by the corresponding upper case letter. So if $z \in \mathbb{C}^n$ then $Z \in \mathbb{C}^{n\times n}$ will denote the diagonal matrix with diagonal z. This notation is useful since the subadditivity of the supporting function φ_S implies that

$$H_S(Zw) = H_S(z_1w_1, \dots, z_nw_n) \leq H_S(z) + H_S(w), \quad z, w \in \mathbb{C}^n.$$

We don't a have nice upper bound for $H_S(z + w)$ unless S is a lower set. Proposition 3.2 in Paper I gives two upper bounds, both of which behave poorly near the coordinate hyperplanes $\mathbb{C}^n \setminus \mathbb{C}^{*n}$.

The infimum in $R^a_{\delta,\mu}u$ is sometimes called an *infimal convolution.* The second regularization operator considered in Paper II is given by the *supremal convolution*

$$R^b_\delta u(z) = \sup_{w\in\mathbb{C}^n} \{u(Zw) - \delta^{-1}\log(\|w - 1_n\|_\infty + 1)\},$$

where $1_n = (1, \dots, 1) \in \mathbb{N}^n$ and $\delta > 0$. Theorem 3.2 in Paper II states that $R^b_\delta u \in \mathcal{L}^S(\mathbb{C}^n)$ if $\delta < \sigma_S^{-1}$, where $\sigma_S = \varphi_S(1_n)$. Additionally, it is showed that $R^b_\delta u$ is continuous on $\mathbb{C}^{*n}$ and $R^b_\delta u \searrow u$ as $\delta \searrow 0$. This shows that $V^S_{K,q}$ is lower semicontinuous on $\mathbb{C}^{*n}$.

Lastly, two related integral operators are defined by

$$R^c_\delta u(z) = \int_{\mathbb{C}^n} u(Az)\psi_\delta(A)\, d\lambda(A),$$

$$R^d_\delta u(z) = \log R^c_\delta e^u(z) = \log \int_{\mathbb{C}^n} e^{u(Az)}\psi_\delta(A)\, d\lambda(A), \quad z \in \mathbb{C}^n$$

where ψ is a smooth function on $\mathbb{C}^n$ with compact support, is rotationally symmetric in each variable, and $\int_{\mathbb{C}} \psi\, d\lambda = 1$, and $\psi_\delta(z) = \delta^{-2n}\psi((z - 1_n)/\delta)$. Theorems 4.1 and 4.3 in Paper II tell us that $R^c_\delta u, R^d_\delta u \in \mathcal{L}^S(\mathbb{C}^n)$, $R^c_\delta u$ and $R^d_\delta u$ are both smooth on $\mathbb{C}^{*n}$, and $R^c_\delta u \searrow u$ and $R^d_\delta u \searrow u$ as $\delta \searrow 0$.

Siciak [44] proved a formula for V_K when K can be written as a cartesian product. He showed that, for non-pluripolar compact $K_1 \subset \mathbb{C}^{n_1}$ and $K_2 \subset \mathbb{C}^{n_2}$,

$$V_{K_1 \times K_2}(z_1, z_2) = \max\{V_{K_1}(z_1), V_{K_2}(z_2)\}, \quad z_1 \in \mathbb{C}^{n_1}, z_2 \in \mathbb{C}^{n_2}.$$

This formula is called *Siciak's product formula.* Bos and Levenberg [14] proved a related formula. They showed that if S is a lower set, $K_1, \ldots, K_n \subset \mathbb{C}$ are compact and non-polar, and $K = K_1 \times \cdots \times K_n$, then

$$V^S_K(z) = \varphi_S(V_{K_1}(z_1), \ldots, V_{K_n}(z_n)), \quad z \in \mathbb{C}^n. \tag{4.1}$$

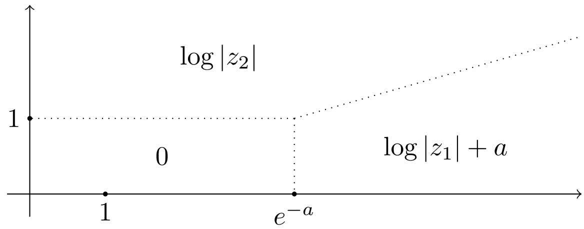
Levenberg and Perera [29], and then Dieu and Long [34], attempted to generalize this for S which are not lower sets. However, these results are false. The purpose of Paper III is to point out these errors and find a correct generalization. There it is shown that equation (4.1) can only hold if S is a lower set. The main result of Paper III is that if

- $S_1, \ldots, S_\ell$ are compact convex subsets of $\mathbb{R}^{n_1}_+, \ldots, \mathbb{R}^{n_\ell}_+$, respectively, all containing 0,
- $K_1, \ldots, K_\ell$ are compact subsets of $\mathbb{C}^{n_1}, \ldots, \mathbb{C}^{n_\ell}$, respectively,
- $n = n_1 + \cdots + n_\ell$,
- $T \subset \mathbb{R}^\ell_+$ is compact convex, and
- $S \subset \mathbb{R}^n_+$ given by
$$S = \bigcup_{x \in T} (x_1 S_1) \times \cdots \times (x_\ell S_\ell),$$

then S is compact convex, contains 0, and

$$V^S_K(z) = \varphi_T(V^{S_1}_{K_1}(z_1), \ldots, V^{S_\ell}_{K_\ell}(z_\ell)), \quad z = (z_1, \ldots, z_\ell), z_j \in \mathbb{C}^{n_j}.$$

We refer to this equation as the *generalized product formula*. In the case where $\ell = n$ then S is the lower hull of T, and this formula reduces to a slight generalization of (4.1). By setting $T = \mathrm{ch}\{(0,1),(1,0)\}$ we get a generalization



The zones correspond to the values taken by $V(z_1, z_2) = \max\{\log^+|z_1| + a, \log^+|z_2|\}$*, with* $a < 0$*. The picture is drawn in the coordinates* $|z_1|$ *and* $|z_2|$*, where the x-axis and y-axis correspond to* $|z_1|$ *and* $|z_2|$*, respectively. Note that generally* $\log^+|z_1| + a \neq \log^+|e^a z_1|$*, but since* $\log^+|z_2| \geq 0$*, we have that* $V(z_1, z_2) = \max\{\log^+|e^a z_1|, \log^+|z_2|\}$*.*

of Siciak's original formula. In this case $S = \mathrm{ch}\{(S_1 \times \{0\}^{n_2}) \cup (\{0\}^{n_1} \times S_2)\}$, and $\varphi_T(\xi_1, \xi_2) = \max\{\xi_1, \xi_2\}$, and thus

$$V_K^S(z) = \max\{V_{K_1}^{S_1}(z_1), V_{K_2}^{S_2}(z_2)\}, \quad z = (z_1, z_2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}.$$

This formula appears in [34] with the assumption that S_1 and S_2 are convex bodies. Their proof, however, only works if S_1 and S_2 contain a neighborhood of 0.

A natural next step is to add weights to the generalized product formula. However, this can not be done. In Proposition 4.2 in Paper III it is shown that

$$V(z) = \varphi_T(V_{K_1,q_1}^{S_1}(z_1), \dots, V_{K_\ell,q_\ell}^{S_\ell}(z_\ell)), \quad z \in \mathbb{C}^n,$$

may fail to be maximal outside of $K_1 \times \cdots \times K_\ell$, which is a necessary condition for $V_{K,q}^S$, for all admissible weights q. The maximality of $V_{K,q}^S$ outside of K follows from Theorem 6.1 in Paper I and [26, Corollary 3.7.6].

Proposition 4.2 in Paper III serves as a general counterexample. A simpler, more specific, counterexample can also be given. Let $n = \ell = 2$, $S_1 = S_2 = \Sigma$, $T = \mathrm{ch}\{(1,0),(0,1)\}$, $K_1 = K_2 = \overline{\mathbb{D}}$, $q_2 = 0$, and $q_1 = a < 0$. By Example 5.1.1 in Klimek [26] we have that

$$\begin{aligned} V(z_1, z_2) &= \max\{\log^+|z_1| + a, \log^+|z_2|\} \\ &= \max\{\log^+|e^a z_1|, \log^+|z_2|\} \\ &= V_{\overline{\mathbb{D}}^2}(e^a z_1, z_2), \qquad z_1, z_2 \in \mathbb{C}, \end{aligned}$$

since $V_{K,q+a}^S = V_{K,q}^S + a$ for all $K \subset \mathbb{C}^n$, weights q on K, and $a \in \mathbb{R}$. By Theorem 5.3.1 in [26] we have that $V_{\overline{\mathbb{D}}^2}(e^a z_1, z_2) = V_{\overline{D}}(z_1, z_2)$, where $D = (e^{-a}\mathbb{D}) \times \mathbb{D}$. So the support of $(dd^cV)^n$, given by $\mathrm{supp}(dd^cV)^n = (e^{-a}\mathbb{T}) \times \mathbb{T}$, intersects $\mathbb{C}^2 \setminus \overline{\mathbb{D}}^2$.

5 A generalized Bernstein-Walsh-Siciak theorem

When generalizing the Bernstein-Walsh-Siciak theorem to the weighted setting we must reconsider how the approximations are evaluated. In the classical setting we consider uniform approximations of f by a polynomial p of degree $\leq m$, so the error is quantified using the supremum norm $\|f - p\|_K$. In our weighted approximations, we will use a weighted supremum norm, that is $\|(f-p)e^{-mq}\|_K$, where $q\colon K \to \mathbb{R} \cup \{+\infty\}$ is our weight on K. To this end we define

$$d_{K,q,m}(f) = \inf\{\|(f-p)e^{-mq}\|_K \,;\, p \in \mathcal{P}_m(\mathbb{C}^n)\}.$$

A Bernstein-Walsh-Siciak theorem in this setting is novel and appears as a corollary to the main result in Paper IV. We are also interested in polynomial approximations using polynomials in $\mathcal{P}^S(\mathbb{C}^n)$, for compact convex $S \subset \mathbb{R}^n_+$ containing 0, so we define

$$d^S_{K,q,m}(f) = \inf\{\|(f-p)e^{-mq}\|_K \,;\, p \in \mathcal{P}^S_m(\mathbb{C}^n)\}.$$

In this setting Bos and Levenberg [14] proved a Bernstein-Walsh-Siciak theorem when S is a lower set and $q = 0$.

Before stating the main result of Paper IV we should recall that the Γ-hull of S, where Γ is a cone in $(\mathbb{R}^n \setminus \mathbb{R}^n_-) \cup \{0\}$, is given by

$$\widehat{S}_\Gamma = \{x \in \mathbb{R}^n_+ \,;\, \langle x, \xi\rangle \leq \varphi_S(\xi), \text{ for all } \xi \in \Gamma\},$$

and satisfies $\widehat{S}_{\Gamma_2} \subset \widehat{S}_{\Gamma_1}$ if $\Gamma_1 \subset \Gamma_2$. More basic properties of these hulls are found in Proposition 5.6 in Paper I. The main result of Paper IV is split into two theorems.

Theorem 5.1. *Let $S \subset \mathbb{R}^n_+$ be a compact convex set that contains 0, q be an admissible weight on a compact $K \subset \mathbb{C}^n$, such that $V^{S*}_{K,q} \leq q$, and for every $r > 0$ define $X_r = \{z \in \mathbb{C}^n \,;\, V^S_{K,q}(z) < \log r\}$. Let $f\colon K \to \mathbb{C}$ be bounded, assume that*

$$L = \{z \in K \,;\, \lim_{m\to\infty} d^S_{K,q,m}(f)e^{mq(z)} = 0\} \neq \varnothing$$

and that, for some $R > 0$, $K \subset X_R$, and

$$\varlimsup_{m\to\infty} d^S_{K,q,m}(f)^{1/m} \leq \frac{1}{R}.$$

Then the following hold:

- **(i)** *For every $\gamma > 0$, such that $K \subset X_{R-\gamma}$, the function $f|_L$ extends to a holomorphic function $F_\gamma \in \mathcal{O}(X_{R-\gamma})$.*
- **(ii)** *If X is an open component of X_R and $L_X = L \cap X$ is non-pluripolar, then $f|_{L_X}$ extends to a unique holomorphic function on X.*
- **(iii)** *If $q < \log R$ then $L = K$ and, consequently, there exists $F \in \mathcal{O}(X_R)$ such that $F|_K = f$.*

The set L denotes where we may assume our polynomial approximation tends to f, pointwise. Points **(i)-(iii)** illustrate how a larger L gives us stronger results.

Theorem 5.2. *Let $S \subset \mathbb{R}^n_+$ be a compact convex set that contains 0, such that $S = \overline{S \cap \mathbb{Q}^n}$, q an admissible weight on a compact subset K of $\mathbb{C}^n$, and $R > 0$. Assume that $V^S_{K,q}$ is continuous, X_R is bounded, and $K \subset X_R$. Let $d_m = d(mS, \mathbb{N}^n \setminus mS)$ be the euclidean distance between mS and $\mathbb{N}^n \setminus mS$. If $f \in \mathcal{O}(X_R)$ then*

$$\varlimsup_{m\to\infty} d^{\widehat{S}_{\Gamma_m}}_{K,q,m}(f)^{1/m} \leq \frac{1}{R},$$

where $\Gamma_m = \{\xi \in \mathbb{R}^n\,;\, \langle \mathbf{1}, \xi\rangle \geq -\frac{1}{2} d_m |\xi|\}$.

The proof of Theorem 5.1 is essentially the same as in the classical setting. To prove Theorem 5.2 the approach described at the end of Section 3 is adapted. The biggest hurdle is analyzing what type of polynomials are constructed. This is where the Γ_m-hulls come in. With the weight ψ_m chosen in the proof, these hulls can not be omitted. See Example 7.4 in Paper I.

When S is a lower set, we have $\widehat{S}_{\mathbb{R}^n_+} = S$, by Theorem 5.8 in Paper I. So $S \subset \widehat{S}_{\Gamma_m} \subset \widehat{S}_{\mathbb{R}^n_+} = S$, and hence $\widehat{S}_{\Gamma_m} = S$, since $\mathbb{R}^n_+ \subset \Gamma_m$. This gives us our first corollary.

Corollary 5.3. *Let S be a lower set, q be an admissible weight on a compact subset K of $\mathbb{C}^n$, such that $V^S_{K,q}$ is continuous, and $K \subset X_R$, for some $R > 0$. If $f \in \mathcal{O}(X_R)$ then*

$$\varlimsup_{m\to\infty} d^{S}_{K,q,m}(f)^{1/m} \leq \frac{1}{R}.$$

When $q = 0$ we have that $K \subset X_R$ if and only if $R > 1$. So, setting $q = 0$ yields the main result of Bos and Levenberg [14, Theorem 3.1].

Corollary 5.4. *Let S be a lower set, K be a compact subset of $\mathbb{C}^n$, such that V^S_K is continuous, and $R > 1$. Then $f \in \mathcal{O}(X_R)$ if and only if*

$$\varlimsup_{m\to\infty} d^{S}_{K,m}(f)^{1/m} \leq \frac{1}{R}.$$

Our final corollary is a weighted version of the classical Bernstein-Walsh-Siciak theorem. It is obtained by setting $S = \Sigma$, in Corollary 5.3.

Corollary 5.5. *Let q be an admissible weight on a compact $K \subset \mathbb{C}^n$, such that $V_{K,q}$ is continuous, and assume $K \subset X_R$, for some $R > 0$. If $f \in \mathcal{O}(X_R)$ then*

$$\varlimsup_{m\to\infty} d_{K,q,m}(f)^{1/m} \leq \frac{1}{R}.$$

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Paper I

Polynomials with exponents in compact convex sets and associated weighted extremal functions - Fundamental results

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Abstract

This paper is a collection of fundamental results for the study of polynomial rings $\mathcal{P}^S(\mathbb{C}^n)$ where the m-th degree polynomials have exponents restricted to mS, where $S \subseteq \mathbb{R}^n_+$ is compact, convex and $0 \in S$. We study the relationship between $\mathcal{P}^S(\mathbb{C}^n)$ and the class $\mathcal{L}^S(\mathbb{C}^n)$ of global plurisubharmonic functions where the growth is determined by the logarithmic supporting function of S. We present properties of their respective weighted extremal functions $\Phi^S_{K,q}$ and $V^S_{K,q}$ in connection with properties of S. Our ambition is to give detailed proofs with minimal assumptions of all results, thus creating a self contained exposition.

1 Introduction

Approximation theory deals with problems of determining whether a given function in some prescribed function space can be approximated by functions in a certain subspace. The theorems of Runge and Mergelyan are prototypes of results from approximation theory. The Runge theorem states that every function f holomorphic in some neighborhood of a simply connected compact subset K of $\mathbb{C}$ can be approximated in the uniform norm $\|\cdot\|_K$ on K by polynomials and the Mergelyan theorem states that it is enough to assume that f is continuous on K and holomorphic in the interior of K.

Quantitative approximation theory deals with problems of relating properties of the given function f to the error in approximation which is usually measured as the distance from f to a certain finite dimensional subspaces of the approximating subspace. The Bernstein-Walsh theorem is a prototype of a result from quantitative approximation theory. It states that a holomorphic function f defined in some neighborhood of a compact simply connected subset $K \subset \mathbb{C}$ extends as a holomorphic function to the sublevel set $\{z \in \mathbb{C}\,;\, g_{\mathbb{C}\setminus K}(z,\infty) < \log R\}$ if and only if $\overline{\lim}_{m\to+\infty} d_m(f,K)^{1/m} \leq 1/R$, where $R > 1$, $g_{\mathbb{C}\setminus K}(\cdot,\infty)$ is the Green function of $\mathbb{C} \setminus K$ with logarithmic pole at ∞, $d_m(f,K)$ is the distance from f to the space $\mathcal{P}_m(\mathbb{C})$ of all polynomials of degree $\leq m$ with respect to the uniform norm on K, and it is assumed that the domain $\mathbb{C} \setminus K$ is regular for the Dirichlet problem in the sense that $g_{\mathbb{C}\setminus K}(\cdot,\infty)$ vanishes at the boundary of K.

The Runge theorem is generalized to several variables where simple connectedness of K generalizes as polynomial convexity. This generalization is usually called the Oka-Weil theorem. There only exist fragmentary, but interesting, results generalizing the Mergelyan theorem to several complex variables. See Levenberg [20].

The origin of the subject of the present paper is the generalization by Siciak [30] of the Bernstein-Walsh theorem. There he introduced the extremal function $\Phi_K = \overline{\lim}_{m\to\infty} \Phi_{K,m}$, where $\Phi_{K,m} = \sup\{|p|^{1/m}\,;\, p \in \mathcal{P}_m(\mathbb{C}^n), \|p\|_K \leq 1\}$ and $\mathcal{P}_m(\mathbb{C}^n)$ is the space of all polynomials of degree $\leq m$. In his work $\log \Phi_K$ plays the role of $g_{\mathbb{C}\setminus K}(\cdot,\infty)$ and the regularity condition at ∂K is that $\Phi_K^*(z) = \overline{\lim}_{\zeta\to z} \Phi_K(\zeta) \leq 1$ for every $z \in \partial K$. Siciak's paper is a seminal work on the understanding of the interrelation between quantitative approximation theory in several complex variables and pluripotential theory. He studied this subject further in many of his works, for example [31, 32, 33, 34]. See also Bloom's Appendix B in the monograph by Saff and Totik [28].

We will systematically work with subspaces of the polynomial space $\mathcal{P}(\mathbb{C}^n)$ in n variables. For every non-empty bounded subset S of $\mathbb{R}^n_+$ we let $\mathcal{P}^S_m(\mathbb{C}^n)$ denote the space of all polynomials p which can be written on the form $p(z) = \sum_{\alpha\in(mS)\cap\mathbb{N}^n} a_\alpha z^\alpha$, $z \in \mathbb{C}^n$, and let $\mathcal{P}^S(\mathbb{C}^n) = \cup_{m\in\mathbb{N}}\mathcal{P}^S_m(\mathbb{C}^n)$. The standard simplex $\Sigma = \operatorname{ch}\{0, e_1, \dots, e_n\}$, where $\operatorname{ch} A$ denotes the convex hull of a set A and $(e_1, \dots, e_n)$ is the standard basis in $\mathbb{R}^n$, yields the standard grading of polynomials of degree $\leq m$, that is $\mathcal{P}^\Sigma_m(\mathbb{C}^n) = \mathcal{P}_m(\mathbb{C}^n)$. If S is a compact convex set in $\mathbb{R}^n_+$ with $0 \in S$ then the space $\mathcal{P}^S(\mathbb{C}^n)$ forms a polynomial ring where the degree of a polynomial p is the minimal $m \in \mathbb{N}$ such that $p \in \mathcal{P}^S_m(\mathbb{C}^n)$.

The polynomial spaces $\mathcal{P}^S_m(\mathbb{C}^n)$ appear in Shiffman-Zelditch [29] with S as an integral polytope and later in Bayraktar [4] as sparse polynomials. These polynomial classes and their relation to pluripotential theory have been studied by Bloom, Levenberg and their collaborators [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 21, 25]. Unfortunately, in some of these papers there are false results stated that we have not seen corrected. We point them out and correct them as far as we can.

In Section 2 we define the Siciak function with respect to S, E, and q for every function $q\colon E \to \mathbb{R}\cup\{+\infty\}$ on a subset E of $\mathbb{C}^n$ by $\Phi^S_{E,q} = \overline{\lim}_{m\to\infty} \Phi^S_{E,q,m}$, where

$$\Phi^S_{E,q,m} = \sup\{|p|^{1/m}\,;\, p \in \mathcal{P}^S_m(\mathbb{C}^n), \|pe^{-mq}\|_E \leq 1\}, \quad m \in \mathbb{N}^*.$$

We drop S in the superscript if $S = \Sigma$ and q in the subscript if $q = 0$. The grading of the polynomial classes $\mathcal{P}^S_m(\mathbb{C}^n)$ implies that for every $z \in \mathbb{C}^n$ the sequence $(-\log\big(\Phi^S_{E,q,m}(z)\big)^m)_{m\in\mathbb{N}^*}$ is subadditive and a lemma by Fekete implies that

$$\Phi^S_{E,q} = \lim_{m\to\infty} \Phi^S_{E,q,m} = \sup_{m\in\mathbb{N}^*} \Phi^S_{E,q,m},$$

without any restriction on S or q. This was first proved by Siciak [30, Theorem 6.1] for $S = \Sigma$. For the reader's convenience we prove the Fekete Lemma 2.3, because ingredients from the proof are needed in the proof of Proposition 2.2, where it is shown that the convergence is uniform on a compact subset X, if q is bounded below on E and $\Phi^S_{E,q}$ is continuous on X. This statement on uniform convergence appears in many arguments in pluripotential theory. See for example Bloom and Shiffman [12, Lemma 3.2], Bayraktar [1, 4, Theorem 2.10], and Bayraktar, Hussung, Levenberg, and Perera [8, Theorem 1.1]. The statements on uniform convergence in these papers all follow directly from Proposition 2.2 only with the conditions that S is compact and convex with $0 \in S$ and q is bounded below.

In Section 3 we introduce the *logarithmic supporting function* H_S of S and the *Lelong class with respect to* S. The function H_S is defined on $\mathbb{C}^{*n}$ as the supporting function φ_S of S, $\varphi_S(\xi) = \sup_{s\in S} \langle s, \xi\rangle$, $\xi \in \mathbb{R}^n$, in logarithmic coordinates extended to $\mathbb{C}^n \setminus \mathbb{C}^{*n}$ as an upper semicontinuous function and the Lelong class $\mathcal{L}^S(\mathbb{C}^n)$ with respect to S is defined as the set of all $u \in \mathcal{PSH}(\mathbb{C}^n)$ satisfying $u \leq c_u + H_S$, where c_u is a constant only depending on u and $\mathcal{PSH}(X)$ denotes the set of plurisubharmonic functions on an open set $X \subset \mathbb{C}^n$. We have $H_\Sigma(z) = \log^+ \|z\|_\infty$, which implies that $\mathcal{L}^\Sigma(\mathbb{C}^n)$ is equal to the Lelong class $\mathcal{L}(\mathbb{C}^n)$.

What values H_S takes at points on the coordinate hyperplanes $\mathbb{C}^n \setminus \mathbb{C}^{*n}$ is opaque from its definition itself, but Proposition 3.3 provides an explicit formula. Additionally, this formula shows that the zero set $\mathcal{N}(H_S)$ of H_S may very well be unbounded. We prove in Proposition 3.4 that H_S extends from $\mathbb{C}^{*n}$ as a continuous plurisubharmonic function on $\mathbb{C}^n$.

The usual grading of the polynomials can be characterized by growth, and so is also the case for $\mathcal{P}^S(\mathbb{C}^n)$. It is clear that if $p \in \mathcal{P}^S_m(\mathbb{C}^n)$ then $|p| \leq C_p e^{H_S}$ for some constant $C_p > 0$. In Theorem 3.6 we prove a Liouville type theorem which states that if $f \in \mathcal{O}(\mathbb{C}^n)$ satisfies a growth estimate $|f(z)| \leq C(1+|z|)^a e^{mH_S(z)}$ for some $C > 0$ and some $a > 0$ strictly less than the distance from mS to $\mathbb{N}^n \setminus mS$ in L^1-norm, then $f \in \mathcal{P}^S_m(\mathbb{C}^n)$.

In Section 4 we define the *Siciak-Zakharyuta function*

$$V^S_{E,q}(z) = \sup\{u(z)\,;\, u \in \mathcal{L}^S(\mathbb{C}^n),\ u|_E \leq q\}, \quad z \in \mathbb{C}^n,$$

with respect to S and any $q\colon E \to \mathbb{R} \cup \{+\infty\}$ on a subset E of $\mathbb{C}^n$. We drop S in the superscript if $S = \Sigma$ and q in the subscript if $q = 0$. By Klimek [19, Example 5.1.1], we have $V_K(z) = \log^+(\|z - a\|/r)$ for any norm $\|\cdot\|$ and $K = \{z \in \mathbb{C}^n\,;\, \|z - a\| \leq r\}$, $r > 0$, the closed ball in $\|\cdot\|$ with center a and radius r. These classical examples can not be generalized for V^S_E for the simple reason that the Lelong class $\mathcal{L}^S(\mathbb{C}^n)$ with respect to S does not need to be translation invariant. We prove in Proposition 4.3 that $V^S_E = H_S$ for any subset E of $\mathbb{C}^n$ such that $\mathbb{T}^n \subseteq E \subseteq \mathcal{N}(H_S)$, where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$ and $\mathcal{N}(H_S)$ the zero set of H_S. This is a generalization of Bayraktar [4, Example 2.3].

We introduce *admissible weights* in Definition 4.4, where we follow Bloom [28, Appendix B], with the natural generalization for the Lelong classes with respect to S given in Bayraktar, Bloom, and Levenberg [6]. In Proposition 4.5 we prove that for every compact convex $S \subset \mathbb{R}^n_+$ with $0 \in S$ and admissible weight q on a closed subset E the upper regularization $V^{S*}_{E,q}$ of $V^S_{E,q}$ is in $\mathcal{L}^S_+(\mathbb{C}^n)$, the set of $u \in \mathcal{L}^S(\mathbb{C}^n)$ such that $H_S - c_u \le u$ for some constant c_u.

It is a fundamental problem to characterize those S and admissible weights q for which $V^S_{E,q} = \log \Phi^S_{E,q}$. The classical Siciak-Zakharyuta theorem states that $V_K = \log \Phi_K$, for every compact subset K of $\mathbb{C}^n$, see [19, Theorem 5.1.7]. It was first proved by Zakharyuta [38] and generalized to $V_{K,q} = \log \Phi_{K,q}$ for continuous q on a compact K by Siciak [31, Theorem 4.12]. Magnússon, Sigurðardóttir, and Sigurðsson [22] prove that for every compact and convex $S \subset \mathbb{R}^n_+$ with $0 \in S$ and every admissible weight q on a closed $E \subset \mathbb{C}^n$, the equation $V^S_{E,q} = \log \Phi^S_{E,q}$ holds on $\mathbb{C}^{*n}$ if and only if $S \cap \mathbb{Q}^n$ is dense in S. In the case when $V^S_{E,q}$ is lower semicontinuous equality holds on $\mathbb{C}^n$. In Proposition 4.7 we show that if $S \cap \mathbb{Q}^n$ is not dense in S, then $V^S_{E,q} \ne \log \Phi^S_{E,q}$ for any admissible weight q on a closed set $E \subseteq \mathbb{C}^n$. Magnússon, Sigurðsson, and Snorrason [23] prove a generalization of the Bernstein-Walsh-Siciak theorem to the weighted setting with approximation by polynomials from the polynomial ring $\mathcal{P}^S(\mathbb{C}^n)$.

In Section 5 we study regularity of $V^S_{E,q}$. We introduce a regularization operator R_δ that preserves the class $\mathcal{L}^S(\mathbb{C}^n)$ and has the property that $R_\delta u \in \mathcal{C}^\infty(\mathbb{C}^{*n}) \cap \mathcal{L}^S(\mathbb{C}^n)$ for every $u \in \mathcal{L}^S(\mathbb{C}^n)$ and $R_\delta u \searrow u$ as $\delta \searrow 0$. This property enables us to prove in Proposition 5.4 that $V^S_{K,q}$ is lower semicontinuous on $\mathbb{C}^{*n}$ for every S and every admissible weight q on a compact set K. It is crucial to know if the upper regularization $V^{S*}_{E,q}$ of $V^S_{E,q}$ satisfies $V^{S*}_{E,q} \le q$ on K, for then $V^{S*}_{E,q} = V^S_{E,q}$ and $V^S_{E,q}$ is continuous at every point where it is lower semicontinuous.

It is a natural question to ask under which conditions on S the class $\mathcal{L}^S(\mathbb{C}^n)$ is preserved under the standard method for regularization of plurisubharmonic functions, that is convolution $u \mapsto u * \psi$, where $\psi \in \mathcal{C}^\infty_0(\mathbb{C}^n)$ with $\psi \ge 0$, and $\int_{\mathbb{C}^n} \psi \, d\lambda = 1$. In Theorem 5.8 we prove that $\mathcal{L}^S(\mathbb{C}^n)$ is preserved under convolution if and only if S is a lower set, which says that the cube $C_s = [0, s_1] \times \cdots \times [0, s_n]$ is contained in S for every $s \in S$. This is a complete answer to a question raised by Bayraktar, Hussung, Levenberg, and Perera [8, Section 2].

In Section 6 we consider an equilibrium measure $\mu^S_{E,q} = \left(dd^c V^{S*}_{E,q}\right)^n$ for $V^{S*}_{E,q}$. We prove in Theorem 6.1 that its support is located where $V^{S*}_{K,q} \ge q$ and in Theorem 6.2 we prove that its total mass is $\mu^S_{E,q}(\mathbb{C}^n) = (2\pi)^n n! \operatorname{vol}(S)$ where vol denotes the euclidean volume. See also Bayraktar [4, Proposition 2.7] and Rashkovskii [26, Section 3].

In Section 7 we return to the problem of characterizing the classes $\mathcal{P}^S_m(\mathbb{C}^n)$ by growth properties we started in Theorem 3.6. Our main result, Theorem 7.2, is that every entire function f in the weighted L^2-space $L^2(\mathbb{C}^n, \psi)$ consisting of all measurable functions which are square integrable with respect to the Lebesgue measure λ on $\mathbb{C}^n$ with weight $e^{-\psi}$ and $\psi = 2mH_S + a\log(1+|\cdot|^2)$ is a polynomial in $\mathcal{P}^{\widehat{S}_\Gamma}_m(\mathbb{C}^n)$, where $\widehat{S}_\Gamma$ is the Γ-hull of S defined by $\widehat{S}_\Gamma = \{x \in \mathbb{R}^n_+ \,;\, \langle x, \xi \rangle \le \varphi_S(\xi), \forall \xi \in \Gamma\}$ for a certain cone $\Gamma \subset \mathbb{R}^n$. Bayraktar, Hussung, Levenberg, and Perera [8, Proposition 4.3]. claim that $f \in \mathcal{P}^S_m(\mathbb{C}^n)$ for any polytope S and

a sufficiently small. Example 7.4 shows that their claim is false even if S is a polytope containing a neighborhood of 0 in $\mathbb{R}^n_+$. The Siciak-Zakharyuta type theorem [8, Theorem 1.1] does not have a sound proof as it is based on their Proposition 4.3. As far as we can see the proof is only valid if S is a lower set.

In Section 8 we continue the discussion in Perera [25] and show that for any polynomial map $f\colon \mathbb{C}^n \to \mathbb{C}^\ell$ and any compact convex $0 \in S \subseteq \mathbb{C}^n$, there is a canonical minimal choice of S' such that $f^*\colon \mathcal{P}^S_m(\mathbb{C}^n) \to \mathcal{P}^{S'}_m(\mathbb{C}^\ell)$ is well-defined for all $m \in \mathbb{N}$ and $f^*\colon \mathcal{L}^S(\mathbb{C}^n) \to \mathcal{L}^{S'}(\mathbb{C}^\ell)$ is well-defined. In the case when $\ell = n$ we show when such pullbacks are bijective.

Snorrason [36] generalized the Siciak product formula, Siciak [30, 31] and Klimek [19, Theorem 5.1.8]. He shows in Corollary 1.3, that the generalization of Bos and Levenberg [13] of Siciak's product formula only holds for lower sets. This shows that neither Levenberg and Perera [21, Proposition 1.3] and Nguyen Quang Dieu and Tang Van Long [24, Theorem 1.3] hold. The error in both the papers is the same, the authors implicitly assume that $\varphi_S(\xi) = \varphi_S(\xi^+)$ holds for every $\xi \in \mathbb{R}^n$, where $\xi^+ = \max\{0, \xi\}$, but in Theorem 5.8 we prove that this identity holds if and only if S is a lower set. In [36, Section 5] Snorrason shows that the sublevel sets of V^S_K are not convex in general. This contradicts Nguyen Quang Dieu and Tang Van Long [24, Theorem 1.2], where they claim that for every convex body S and every compact convex $K \subset \mathbb{C}^n$ the sublevel sets of V^S_K are convex. All these mistakes showed us the importance of a careful study of the values of H_S near points on the union of the coordinate hyperplanes as we have done in Propositions 3.3 and 3.4.

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2 Weighted polynomial classes and Siciak functions

Let S be a bounded subset of $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n; x_j \geq 0 \text{ for all } j = 1, \dots, n\}$. For every $m \in \mathbb{N}$ we associate to S the space $\mathcal{P}^S_m(\mathbb{C}^n)$ of all polynomials in n

complex variables of the form

$$p(z) = \sum_{\alpha \in (mS) \cap \mathbb{N}^n} a_\alpha z^\alpha, \quad z \in \mathbb{C}^n, \tag{2.1}$$

with the standard multi-index notation and let $\mathcal{P}^S(\mathbb{C}^n) = \cup_{m \in \mathbb{N}} \mathcal{P}^S_m(\mathbb{C}^n)$. If S is the standard simplex $\Sigma = \text{ch}\{0, e_1, \dots, e_n\}$, then the space $\mathcal{P}^\Sigma_m(\mathbb{C}^n)$ consists of all polynomials of degree $\leq m$, which we denote by $\mathcal{P}_m(\mathbb{C}^n)$. We let $\mathcal{P}(\mathbb{C}^n) = \cup_{m \in \mathbb{N}} \mathcal{P}_m(\mathbb{C}^n)$ denote the space of all polynomials in n complex variables.

Assume now that S is a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$. If $\alpha \in jS$ and $\beta \in kS$ for some $j, k \in \mathbb{N}^*$, say $\alpha = ja$ and $\beta = kb$ with $a, b \in S$, then convexity of S gives $\alpha + \beta = (j+k)\big((1-\lambda)a + \lambda b\big) \in (j+k)S$, where $\lambda = k/(j+k) \in [0,1]$. Thus, $z^\alpha z^\beta \in \mathcal{P}^S_{j+k}(\mathbb{C}^n)$ and by taking linear combinations of products of monomials we get

$$\mathcal{P}^S_j(\mathbb{C}^n)\mathcal{P}^S_k(\mathbb{C}^n) \subseteq \mathcal{P}^S_{j+k}(\mathbb{C}^n). \tag{2.2}$$

This tells us that $\mathcal{P}^S(\mathbb{C}^n)$ is a subring of $\mathcal{P}(\mathbb{C}^n)$. For every $p \in \mathcal{P}^S(\mathbb{C}^n)$ we define the *S-degree* $\deg^S(p)$ of p as the infimum over m for which $p \in \mathcal{P}^S_m(\mathbb{C}^n)$. We have

$$\deg^S(p_1 + p_2) \leq \max\{\deg^S(p_1), \deg^S(p_2)\}, \tag{2.3}$$

$$\deg^S(p_1 p_2) \leq \deg^S(p_1) + \deg^S(p_2). \tag{2.4}$$

Equality does not hold in general in either of these inequalities.

Definition 2.1. Let $S \subset \mathbb{R}^n_+$ be a compact convex set, $0 \in S$, and $q \colon E \to \mathbb{R} \cup \{+\infty\}$ be a function on $E \subset \mathbb{C}^n$. For $m \in \mathbb{N}^* = \{1, 2, 3, \dots\}$ the *m-th Siciak extremal function with respect to S, E, and q* is defined by

$$\Phi^S_{E,q,m}(z) = \sup\{|p(z)|^{1/m} \,;\, p \in \mathcal{P}^S_m(\mathbb{C}^n), \|pe^{-mq}\|_E \leq 1\}, \quad z \in \mathbb{C}^n,$$

the *Siciak extremal function with respect to S, E, and q* is defined by

$$\Phi^S_{E,q}(z) = \varlimsup_{m \to \infty} \Phi^S_{E,q,m}(z), \quad z \in \mathbb{C}^n.$$

We drop S in the superscripts if $S = \Sigma$ and q in the subscripts if $q = 0$. Note that the family $\{p \in \mathcal{P}^S_m(\mathbb{C}^n); \|pe^{-mq}\|_E \leq 1\}$ is never empty since it always contains the zero polynomial. Furthermore, we define $\Phi^S_{E,q,0} = 1$.

Observe that our definition of $\Phi^S_{E,q,m}$ deviates from the original definition of Siciak [30], which is $\big(\Phi^S_{E,q,m}\big)^m$ in our notation. We have that $\Phi^S_{E,q}, \Phi^S_{E,q,m}$ are lower semicontinuous on $\mathbb{C}^n$ for $m = 1, 2, 3, \dots$, for all these functions are suprema of continuous functions.

If q is bounded below, say by the real number q_0, then the constant polynomial $p(z) = e^{mq_0}$ is in $\mathcal{P}^S_m(\mathbb{C}^n)$ and $\|pe^{-mq}\|_E = \|e^{-m(q-q_0)}\|_E \leq 1$. Hence, it follows that $\Phi^S_{E,q,m}(z) \geq e^{q_0}$ for every $z \in \mathbb{C}^n$ and $m \in \mathbb{N}^*$.

Proposition 2.2. *Let $S \subset \mathbb{R}^n_+$ be a compact convex set with $0 \in S$, $E \subset \mathbb{C}^n$, and $q\colon E \to \mathbb{R} \cup \{+\infty\}$ be a function. Then for $j, k = 1, 2, 3, \dots$*

$$\big(\Phi^S_{E,q,j}(z)\big)^j \big(\Phi^S_{E,q,k}(z)\big)^k \leq \big(\Phi^S_{E,q,j+k}(z)\big)^{j+k}, \quad z \in \mathbb{C}^n, \tag{2.5}$$

and

$$\Phi^S_{E,q}(z) = \lim_{m\to\infty} \Phi^S_{E,q,m}(z) = \sup_{m\geq 1} \Phi^S_{E,q,m}(z), \quad z \in \mathbb{C}^n. \tag{2.6}$$

If q is bounded below and $\Phi^S_{E,q}$ is continuous on some compact subset X of $\mathbb{C}^n$, then the convergence is uniform on X.

We need ingredients from the proof from Tsuji [37, Lemma after Theorem III.25 on page 73]:

Lemma 2.3. (Fekete lemma) *Let $(a_m)_{m\in\mathbb{N}^*}$ be a subadditive real sequence, that is $a_{j+k} \leq a_j + a_k$ for $j, k = 1, 2, 3, \dots$. Then $\lim\limits_{m\to\infty} a_m/m = \inf\limits_{m\geq 1} a_m/m$.*

Proof. Denote the infimum of a_m/m by α and take $\beta \in \mathbb{R}$ such that $-\infty \leq \alpha < \beta$. Then there exists $j \in \mathbb{N}^*$ such that $a_j/j < \beta$. Every $m > j$ can be written as $m = sj + r$ for some $s, r \in \mathbb{N}$ with $s \geq 1$ and $0 \leq r < j$. By assumption, we have $a_m \leq a_{sj} + a_r \leq sa_j + a_r$, so

$$\frac{a_m}{m} \leq \frac{sa_j + a_r}{m} < \beta\big(1 - r/m\big) + \frac{\max_{\ell<j} a_\ell}{m}, \tag{2.7}$$

which implies $\overline{\lim\limits_{m\to\infty}}\, a_m/m \leq \beta$. Since β is arbitrary we have

$$\overline{\lim_{m\to\infty}}\, a_m/m \leq \inf_{m\geq 1} a_m/m \leq \underline{\lim_{m\to\infty}}\, a_m/m.$$

Hence, the limit (possibly $-\infty$) exists and the equality holds. □

Proof of Proposition 2.2. Take $p_j \in \mathcal{P}^S_j(\mathbb{C}^n)$ with $\|p_j e^{-jq}\|_E \leq 1$ and $p_k \in \mathcal{P}^S_k(\mathbb{C}^n)$ with $\|p_k e^{-kq}\|_E \leq 1$. Then $\|p_j p_k e^{-(j+k)q}\|_E \leq 1$ and (2.2) implies $p_j p_k \in \mathcal{P}^S_{j+k}(\mathbb{C}^n)$. Hence, $|p_j(z)p_k(z)| \leq \big(\Phi^S_{E,q,j+k}(z)\big)^{j+k}$. By taking supremum over p_j and p_k (2.5) follows. By (2.5) the sequence defined by $a_m = -\log\big(\Phi^S_{E,q,m}(z)\big)^m$ is subadditive for every z, so (2.6) follows from Lemma 2.3.

Assume now that $q \geq q_0$ for some $q_0 \in \mathbb{R}$ and that $\Phi^S_{E,q}$ is continuous on X. By the discussion after Definition 2.1 we have $\Phi^S_{E,q,m} \geq e^{q_0}$. Since $\Phi^S_{E,q} = \sup_{m\geq 1} \Phi^S_{E,q,m}$, it follows by a simple compactness argument that it is sufficient to show that for every $z_0 \in X$ and every $\varepsilon > 0$ there exists $\delta > 0$ and $k \in \mathbb{N}^*$ both depending on z_0 and ε such that

$$\Phi^S_{E,q}(z) - \Phi^S_{E,q,m}(z) < \varepsilon, \quad z \in B(z_0, \delta) \cap X,\ m \geq k. \tag{2.8}$$

Let $c = \sup_X \Phi^S_{E,q}$ and choose $\gamma > 0$ such that $c(1 - e^{-\gamma}) < \frac{1}{4}\varepsilon$ and $j \in \mathbb{N}^*$ so large that $\Phi^S_{E,q}(z_0) - \Phi^S_{E,q,j}(z_0) < \frac{1}{4}\varepsilon$. Since $\Phi^S_{E,q}$ is continuous on X and

$\Phi^S_{E,q,j}$ is lower semicontinuous on $\mathbb{C}^n$, there exists $\delta > 0$ such that for all $z \in B(z_0, \delta) \cap X$ we have

$$\Phi^S_{E,q}(z) - \Phi^S_{E,q}(z_0) < \tfrac{1}{4}\varepsilon \quad \text{and} \quad \Phi^S_{E,q,j}(z_0) - \Phi^S_{E,q,j}(z) < \tfrac{1}{4}\varepsilon.$$

The three estimates imply

$$\Phi^S_{E,q}(z) - \Phi^S_{E,q,j}(z) < \tfrac{3}{4}\varepsilon, \quad z \in B(z_0, \delta) \cap X, \tag{2.9}$$

and (2.8) follows from (2.9) if we can prove that there exists $k > j$ such that

$$\Phi^S_{E,q}(z) - \Phi^S_{E,q,m}(z) \le \Phi^S_{E,q}(z) - \Phi^S_{E,q,j}(z) + \tfrac{1}{4}\varepsilon, \quad z \in X,\ m \ge k. \tag{2.10}$$

For every $z \in \mathbb{C}^n$ the sequence $a_m = -\log\big(\Phi^S_{E,q,m}(z)\big)^m$ is subadditive. By (2.7) we have for every $m > j$ written as $m = sj + r$ with $s \in \mathbb{N}^*$ and $r \in \mathbb{N}$ with $0 \le r < j$ that

$$-\log \Phi^S_{E,q,m}(z) \le \frac{-sj \log \Phi^S_{E,q,j}(z) - r \log \Phi^S_{E,q,r}(z)}{m}, \quad z \in \mathbb{C}^n.$$

We have $sj/m = 1 - r/m$, $-\log \Phi^S_{E,q,j}(z) \ge -\log c$ for every $z \in X$, and $\log \Phi^S_{E,q,r}(z) \ge q_0$ for every $z \in \mathbb{C}^n$, so

$$\begin{aligned} \log \Phi^S_{E,q,m}(z) &\ge \log \Phi^S_{E,q,j}(z) + \frac{-r \log \Phi^S_{E,q,j}(z) + r \log \Phi^S_{E,q,r}(z)}{m} \\ &\ge \log \Phi^S_{E,q,j}(z) - \frac{j \log(c/e^{q_0})}{m}, \quad z \in X. \end{aligned}$$

We choose $k > j$ so large that $j \log(c/e^{q_0})/k < \gamma$. Then $\Phi^S_{E,q,m}(z) \ge \Phi^S_{E,q,j}(z)e^{-\gamma}$ and

$$\Phi^S_{E,q}(z) - \Phi^S_{E,q,m}(z) \le \Phi^S_{E,q}(z) - \Phi^S_{E,q,j}(z) + \Phi^S_{E,q,j}(z)\big(1 - e^{-\gamma}\big)$$

for every $z \in X$ and $m \ge k$. Since $\Phi^S_{E,q,j}(z) \le c$ for every $z \in X$ and $c(1-e^{-\gamma}) < \frac{1}{4}\varepsilon$ the estimate (2.10) holds. $\square$

3 The Lelong class with respect to a convex set

Let us begin by setting some notation for the sequel. We denote by $\mathcal{H}(X)$, $\mathcal{SH}(X)$, $\mathcal{PSH}(X)$, $\mathcal{O}(X)$, $\mathcal{LSC}(X)$, and $\mathcal{USC}(X)$ the classes of harmonic and subharmonic functions on a domain X in $\mathbb{C}$, plurisubharmonic and holomorphic functions on a complex manifold X, and lower and upper semicontinuous functions on a topological space X, respectively. We define the coordinate-wise logarithm of the modulus, exponential function, and positive part, by

$$\begin{gathered} \operatorname{Log}\colon \mathbb{C}^{*n} \to \mathbb{R}^n, \quad \operatorname{Log}(z) = (\log|z_1|, \dots, \log|z_n|), \quad z \in \mathbb{C}^{*n}, \\ \operatorname{Exp}\colon \mathbb{R}^n \to \mathbb{R}^n_+, \quad \operatorname{Exp}(\xi) = e^{\xi} = (e^{\xi_1}, \dots, e^{\xi_n}), \quad \xi \in \mathbb{R}^n, \\ {}^+\colon \mathbb{R}^n \to \mathbb{R}^n_+, \quad \xi^+ = (\xi_1^+, \dots, \xi_n^+), \quad \xi_j^+ = \max\{\xi_j, 0\}, \quad \xi \in \mathbb{R}^n. \end{gathered}$$

We let $\mathbb{D}$ denote the unit disc and $\mathbb{T}$ the unit circle in $\mathbb{C}$. The Lelong class $\mathcal{L}(\mathbb{C}^n)$ is the set of all $u \in \mathcal{PSH}(\mathbb{C}^n)$ such that for some constant c_u depending on u we have

$$u(z) \leq c_u + \log^+ \|z\|_\infty, \quad z \in \mathbb{C}^n.$$

It is clear that $\log^+ \|\cdot\|_\infty$ can be replaced by $\log^+ \|\cdot\|$ or $\log(1 + \|\cdot\|)$ for any norm $\|\cdot\|$ on $\mathbb{C}^n$.

Definition 3.1. For every compact subset of $\mathbb{R}^n_+$ with $0 \in S$ we define the *supporting function of* S as

$$\varphi(x) = \sup_{s \in S} \langle s, x \rangle, \quad x \in \mathbb{R}^n,$$

and the *logarithmic supporting function of* S as the function $H_S \colon \mathbb{C}^n \to \mathbb{R}_+$ defined on $\mathbb{C}^{*n}$ by

$$H_S(z) = (\varphi_S \circ \mathrm{Log})(z) = \max_{s \in S} \big(s_1 \log |z_1| + \cdots + s_n \log |z_n|\big), \quad z \in \mathbb{C}^{*n},$$

and extended to $\mathbb{C}^n \setminus \mathbb{C}^{*n}$ by

$$H_S(z) = \varlimsup_{\mathbb{C}^{*n} \ni w \to z} H_S(w), \quad z \in \mathbb{C}^n \setminus \mathbb{C}^{*n}.$$

The real number $\sigma_S = \varphi_S(\mathbf{1})$, where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^n_+$, is called the *logarithmic type* of H_S.

Since φ_S is positively homogeneous of degree 1 and convex, that is $\varphi_S(t\xi) = t\varphi_S(\xi)$ and $\varphi_S(\xi + \eta) \leq \varphi_S(\xi) + \varphi_S(\eta)$ for every $t \in \mathbb{R}_+$ and $\xi, \eta \in \mathbb{R}^n$, we have

$$H_S(z) = \tfrac{1}{\lambda} H_S(|z_1|^\lambda, \ldots, |z_n|^\lambda), \quad \lambda \in \mathbb{R}^*_+,\ z \in \mathbb{C}^{*n}, \tag{3.1}$$

$$H_S(z_1 w_1, \ldots, z_n w_n) \leq H_S(z) + H_S(w), \quad z, w \in \mathbb{C}^{*n}. \tag{3.2}$$

Observe that $\varphi_S(-\mathbf{1}) = 0$ and

$$H_S(\lambda z) \leq H_S(z) + \sigma_S \log^+ |\lambda|, \quad z \in \mathbb{C}^{*n},\ \lambda \in \mathbb{C}^*. \tag{3.3}$$

If we write $z = \|z\|_\infty w$ with $\|w\|_\infty = 1$, then this formula implies $H_S/\sigma_S \in \mathcal{L}(\mathbb{C}^n)$,

$$H_S(z) \leq \sigma_S \log^+ \|z\|_\infty, \quad z \in \mathbb{C}^n. \tag{3.4}$$

Directly from the definition we see that $H_S \in \mathcal{PSH}(\mathbb{C}^{*n}) \cap \mathcal{C}(\mathbb{C}^{*n})$. Since $\mathbb{C}^n \setminus \mathbb{C}^{*n}$ is pluripolar and H_S is locally bounded above we have $H_S \in \mathcal{PSH}(\mathbb{C}^n)$.

Proposition 3.2. *Let $S \subset \mathbb{R}^n_+$ be compact convex and with $0 \in S$. Then for every $z \in \mathbb{C}^{*n}$ and $w \in \mathbb{C}^n$ we have*

$$H_S(z + w) \leq H_S(z) + \varphi_S(|w_1|/|z_1|, \ldots, |w_n|/|z_n|),$$

and in particular, for every $w \in \overline{\mathbb{D}}^n$ and $\delta \in]0, 1[$ we have

$$H_S(\mathbf{1} + \delta w) \leq \delta \sigma_S.$$

Furthermore,

$$H_S(z + w) \leq H_S(z) + \varphi_S(\log^+(|w_1|/|z_1|), \ldots, \log^+(|w_n|/|z_n|)) + \sigma_S \log 2.$$

Proof. By plurisubharmonicity of H_S and convexity of the functions given by $w \mapsto \varphi_S(|w_1/|z_1|, \dots, |w_n|/|z_n|)$ we may assume that $w, z+w \in \mathbb{C}^{*n}$. For some $t \in S$ we have

$$H_S(z+w) = \log(|z_1+w_1|^{t_1} \cdots |z_n+w_n|^{t_n})$$
$$\leq \langle t, \operatorname{Log} z\rangle + \sum_{j=1}^{n} t_j \log(1+|w_j|/|z_j|).$$

Since $\log(1+x) \leq x$, for $x \geq 0$ the first estimate follows. Since $\mathbf{1}+\delta w \in \mathbb{C}^{*n}$ for $w \in \overline{\mathbb{D}}^n$, $\delta \in]0,1[$, and $H_S(\mathbf{1}) = 0$, the second estimate follows. For the third estimate we use the fact that $\log(1+x) \leq \log 2 + \log^+ x$ for $x \geq 0$. □

The zero set $\mathcal{N}(H_S)$ of H_S can be understood in terms of the zero set of φ_S which is a cone. Since $\mathcal{N}(H_S) \cap \mathbb{C}^{*n} = \operatorname{Log}^{-1}(\mathcal{N}(\varphi_S))$ and $\mathbb{R}^n_- \subseteq \mathcal{N}(\varphi_S)$, the closed unit polydisc $\overline{\mathbb{D}}^n$ is contained in $\mathcal{N}(H_S)$. Furthermore, $\mathcal{N}(H_S)$ is equal to $\overline{\mathbb{D}}^n$ if and only if $\mathbb{R}_+S = \mathbb{R}^n_+$. We have a complete description of the values of H_S at every point in $\mathbb{C}^n \setminus \mathbb{C}^{*n}$, the union of the coordinate hyperplanes.

Proposition 3.3. *Let S be a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$. For every $a \neq 0$ in some coordinate hyperplane we have*

$$H_S(a) = H_{S_J}(a_{j_1}, \dots, a_{j_\ell}),$$

where $J \subset \{1, \dots, n\}$ consists of the indices $j_1 < \cdots < j_\ell$ of the non-zero coordinates $a_{j_1}, \dots, a_{j_\ell}$ of a and $S_J \subseteq \mathbb{R}^\ell$ consists of all $t \in \mathbb{R}^\ell$ such that if $s \in \mathbb{R}^n_+$ is defined by $s_{j_k} = t_k$ for $j_k \in J$ and $s_j = 0$ for $j \notin J$, then $s \in S$.

Proof. After renumbering the variables we may assume that $J = \{1, \dots, \ell\}$ and $a_{\ell+1} = \cdots = a_n = 0$. We write $z = (z', z'') \in \mathbb{C}^n$, where $z' \in \mathbb{C}^\ell$ and $z'' \in \mathbb{C}^{n-\ell}$. Then $a' \in \mathbb{C}^{*\ell}$, so H_{S_J} is continuous at a' and we have

$$H_{S_J}(a') = \lim_{\mathbb{C}^{*\ell} \ni w' \to a'} H_{S_J}(w') = \lim_{\mathbb{C}^{*m} \ni w' \to a'} \sup_{t \in S_J} \langle t, \operatorname{Log} w'\rangle$$
$$= \lim_{\mathbb{C}^{*m} \ni w' \to a'} \sup_{s=(t,0)\in S} \langle s, (\operatorname{Log} w', 0, \dots, 0)\rangle$$
$$\leq \varlimsup_{\mathbb{C}^{*n} \ni w \to a} \sup_{s \in S} \langle s, \operatorname{Log} w\rangle = H_S(a).$$

To prove the converse inequality we take $\mathbb{C}^{*n} \ni w_j = (w_{j,1}, \dots, w_{j,n}) \to a$ such that $\lim_{j\to\infty} H_S(w_j) = H_S(a)$. There exists $s_j \in S$ such that $H_S(w_j) = \langle s_j, \operatorname{Log} w_j\rangle$ for every j. Since $H_S \geq 0$ and $\log|w_{j,\kappa}| \to -\infty$ as $j \to \infty$ for $\kappa = \ell+1, \dots, n$, it follows that $s_{j,\kappa} \to 0$ as $j \to \infty$ for $\kappa = \ell+1, \dots, n$. By compactness of S there exists a subsequence s_{j_k} converging to $(t, 0) \in S$. We have $t \in S_J$ and conclude that

$$H_S(a) = \lim_{k\to\infty} H_S(w_{j_k}) = \lim_{k\to\infty} \langle s_{j_k}, \operatorname{Log} w_{j_k}\rangle = \langle t, \operatorname{Log} a'\rangle \leq H_{S_J}(a').$$

□

With a proof similar to Rashkovskii [27, Proposition 2.2], we are able to show that $H_S \in \mathcal{C}(\mathbb{C}^n)$:

Proposition 3.4. *Let S be a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$. Then H_S is plurisubharmonic and continuous on $\mathbb{C}^n$.*

Proof. Let $0 \neq a \in \mathbb{C}^n \setminus \mathbb{C}^{*n}$. Since $H_S \in \mathcal{PSH}(\mathbb{C}^n) \cap \mathcal{C}(\mathbb{C}^{*n})$, it suffices to prove that H_S is lower semicontinuous at a. After renumbering the variables we may assume $a_{\ell+1} = \cdots = a_n = 0$ and $a_j \neq 0$ for $j \leq \ell$. We also write $z = (z', z'') \in \mathbb{C}^n$, where $z' \in \mathbb{C}^\ell$ and $z'' \in \mathbb{C}^{n-\ell}$. Let $z_j \in \mathbb{C}^n$ be such that $z_j \to a$. Since H_S is rotationally invariant in each variable it takes the constant value $H_S(z_j)$ on the distinguished boundary of the $n-\ell$ dimensional polydisc $\{(z', \zeta'')\,;\, |\zeta''_k| \leq |z''_{j,k}|, k = 1, \dots, n-\ell\}$, so by the maximum principle $H_S(z'_j, 0) \leq H_S(z_j)$. By Proposition 3.3 we have that $H_S(z'_j, 0) = H_{S_J}(z'_j)$, for $J = \{1, \dots, \ell\}$. By the continuity of H_{S_J} at a' we have

$$\varliminf_{j\to\infty} H_S(z_j) \geq \varliminf_{j\to\infty} H_S(z'_j, 0) = \varliminf_{j\to\infty} H_{S_J}(z'_j) = H_{S_J}(a') = H_S(a).$$

This proves the lower semicontinuity of H_S at a. □

Definition 3.5. For every compact convex subset S of $\mathbb{R}^n_+$ with $0 \in S$ we define the *S-Lelong class* $\mathcal{L}^S(\mathbb{C}^n)$ as the set of all $u \in \mathcal{PSH}(\mathbb{C}^n)$ such that

$$u(z) \leq c_u + H_S(z), \quad z \in \mathbb{C}^n,$$

for some constant c_u depending on u, and define $\mathcal{L}^S_+(\mathbb{C}^n)$ as the subclass of functions u, that have the same asymptotic behavior at infinity as the function H_S, that is

$$-c_u + H_S(z) \leq u(z) \leq c_u + H_S(z), \quad z \in \mathbb{C}^n. \tag{3.5}$$

The Liouville theorem tells us that an entire function $f \in \mathcal{O}(\mathbb{C}^n)$ which satisfies a growth estimate $|f(z)| \leq C(1+|z|)^{a+m}$, $z \in \mathbb{C}^n$, for some $m \in \mathbb{N}$ and $a \in [0, 1[$, is a polynomial of degree $\leq m$, that is $f \in \mathcal{P}_m(\mathbb{C}^n)$. The following is a Liouville type theorem for the polynomial classes $\mathcal{P}^S_m(\mathbb{C}^n)$:

Theorem 3.6. *Let d_m denote the distance between mS and $\mathbb{N}^n \setminus mS$ in the L^1-norm. Then for every $f \in \mathcal{O}(\mathbb{C}^n)$ the following are equivalent:*

- **(i)** $f \in \mathcal{P}^S_m(\mathbb{C}^n)$.
- **(ii)** $\log|f|^{1/m} \in \mathcal{L}^S(\mathbb{C}^n)$.
- **(iii)** *there exists $a \in [0, d_m[$ such that* $|f|e^{-mH_S - a\log^+\|\cdot\|_\infty} \in L^\infty(\mathbb{C}^n)$.
- **(iv)** *there exists $a \in [0, d_m[$ and a constant $C > 0$ such that*

$$|f(z)| \leq C(1+|z|)^a e^{mH_S(z)}, \quad z \in \mathbb{C}^n.$$

Proof. **(i)⇒(ii):** If $\alpha \in mS$, then $|z^\alpha| \leq e^{mH_S(z)}$, so for $f \in \mathcal{P}_m^S(\mathbb{C}^n)$, $f(z) = \sum_{\alpha\in(mS)\cap\mathbb{N}^n} a_\alpha z^\alpha$, we have $\log|f(z)|^{1/m} \leq c_f/m + H_S(z)$ with $c_f = \log\sum_{\alpha\in(mS)\cap\mathbb{N}^n}|a_\alpha|$.

(ii)⇒(iii): We have $|f|e^{-mH_S - a\log^+\|\cdot\|_\infty} \leq |f|e^{-mH_S} \in L^\infty(\mathbb{C}^n)$.

(iii)⇒(i): Observe first that $m\varphi_S(\xi) + a\|\xi^+\|_\infty = \varphi_{mS+a\Sigma}(\xi)$ for every $\xi \in \mathbb{R}^n$. Let $f(z) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha$ be the power series expansion of f at 0. We need to show that $a_\alpha = 0$ for all $\alpha \in \mathbb{N}^n \setminus mS$. Since $a < d_m$ we have $(mS + a\Sigma)\cap\mathbb{N}^n = (mS)\cap\mathbb{N}^n$, so $\alpha \in \mathbb{N}^n \setminus (mS + a\Sigma)$. Hence, there exists $\xi \in \mathbb{R}^n$ such that $\langle\alpha,\xi\rangle > \varphi_{mS+a\Sigma}(\xi)$. We let C_t denote the polycircle with center 0 and polyradius $(e^{t\xi_1}, \dots, e^{t\xi_n})$ and observe that by the Cauchy formula for derivatives we have

$$a_\alpha = \frac{1}{(2\pi i)^n}\int_{C_t}\frac{f(\zeta)}{\zeta^\alpha}\,\frac{d\zeta_1\cdots d\zeta_n}{\zeta_1\cdots\zeta_n}.$$

For $\zeta = (e^{t\xi_1+i\theta_1}, \dots, e^{t\xi_n+i\theta_n})$ on C_t we have

$$|f(\zeta)|/|\zeta^\alpha| \leq Ce^{-t(\langle\alpha,\xi\rangle - \varphi_{mS+a\Sigma}(\xi))}$$

, so the right hand side tends to 0 as $t \to +\infty$, and we conclude that $a_\alpha = 0$.

(iii)⇔(iv): Follows from the equivalence of the euclidean norm $|\cdot|$ and $\|\cdot\|_\infty$. □

Recall, from Klimek [19, p. 87], that a real valued $u \in \mathcal{PSH}(X)$ on an open subset X of $\mathbb{C}^n$ is said to be *maximal*, if for every relatively compact open subset G of X and every $v \in \mathcal{USC}(\overline{G})\cap\mathcal{PSH}(G)$ satisfying $v \leq u$ on ∂G we have $v \leq u$ on G. For the reader's convenience we prove the following well known result.

Lemma 3.7. *Let X be an open subset of $\mathbb{C}^n$ and $u \in \mathcal{PSH}(X)$ be real valued. Assume that for every relatively compact open subset G of X there exists a family $(g_z)_{z\in G}$ of holomorphic maps $g_z\colon D_z \to \mathbb{C}^n$ defined on open subsets D_z of $\mathbb{C}$, such that $K_z = g_z^{-1}(\overline{G})$ is compact, $z = g_z(\tau_z)$ for some $\tau_z \in K_z$, and $u \circ g_z$ is harmonic on D_z. Then u is maximal on X.*

Proof. By Klimek [19, Proposition 3.1.1], we may take $v \in \mathcal{PSH}(X)$ in the definition of maximality. We assume that $v \leq u$ on ∂G and need to prove that $v(z) \leq u(z)$ for every $z \in G$. The set $K_z = g_z^{-1}(\overline{G})$ is compact, the function $s_z = v\circ g_z \in \mathcal{SH}(D_z)$ is less than or equal to $h_z = u \circ g_z \in \mathcal{H}(D_z)$ on the boundary of K_z. By the maximum principle for harmonic functions $v(z) = s_z(\tau_z) \leq h_z(\tau_z) = u(z)$. □

For every $z \in \mathbb{C}^{n*}$ we define a parametric curve $f_z\colon \mathbb{C}\to\mathbb{C}^n$, $f_z = (f_{z,1}, \dots, f_{z,n})$ by

$$f_{z,j}(\tau) = \begin{cases} e^{-i\tau\log|z_j|}(z_j/|z_j|), & z_j \neq 0, \\ 0, & z_j = 0, \end{cases} \qquad \tau \in \mathbb{C}. \tag{3.6}$$

We have $f_z(i) = z$ and $\|f_z(\tau)\|_\infty = 1$ for every $\tau \in \mathbb{R}$. If $\|z\|_\infty > 1$, then for j with $|z_j| = \|z\|_\infty$ we have

$$f'_{z,j}(\tau) = -ie^{-i\tau\log|z_j|}(\log|z_j|)(z_j/|z_j|) \neq 0, \quad \tau \in \mathbb{C}.$$

Hence, f_z parametrizes an open Riemann surface in $\mathbb{C}^n$ through the point z. Furthermore, we have $H_S(f_z(\tau)) = \mathrm{Im}\tau\, H_S(z)$ for $\mathrm{Im}\tau \geq 0$. The function $\mathbb{C} \ni \tau \mapsto H_S(f_z(\tau))$ is subharmonic, harmonic in the upper half plane, equal to 0 on the real axis, and takes the value $H_S(z)$ at i. By Lemma 3.7 with f_z in the role of g_z and H_S in the role of u we get:

Proposition 3.8. *Let S be a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$. Then H_S is maximal on $\mathbb{C}^n \setminus \partial\mathcal{N}(H_S)$, where $\mathcal{N}(H_S)$ is the zero set of H_S.*

4 Weighted Siciak-Zakharyuta functions

Definition 4.1. Let $S \subset \mathbb{R}^n_+$ be a compact convex set, $0 \in S$, and $q\colon E \to \mathbb{R} \cup \{+\infty\}$ be a function on $E \subset \mathbb{C}^n$. The *Siciak-Zakharyuta function with respect to S, E, and q* is defined by

$$V^S_{E,q}(z) = \sup\{u(z)\,;\, u \in \mathcal{L}^S(\mathbb{C}^n),\ u|_E \leq q\}, \quad z \in \mathbb{C}^n.$$

We drop S in superscripts if $S = \Sigma$ and q in subscripts if $q = 0$.

From Theorem 3.6 it follows that $\log \Phi^S_{E,q} \leq V^S_{E,q}$ for every compact convex $S \subset \mathbb{R}^n_+$ and every $q\colon E \to \mathbb{R} \cup \{+\infty\}$ on $E \subset \mathbb{C}^n$. We will need a variant of the Phragmén-Lindelöf principle, see [18, Lemma 2.1].

Lemma 4.2. *Let v be subharmonic in the upper half plane $\mathbb{C}_+ = \{z \in \mathbb{C}\,;\, \mathrm{Im}z > 0\}$ such that for some real constants C and A we have $v(z) \leq C + A|z|$ for all $z \in \mathbb{C}_+$, and $\overline{\lim\limits_{\mathbb{C}_+ \ni z \to x}}\, v(z) \leq 0$ for all $x \in \mathbb{R}$. Then $v(z) \leq A\,\mathrm{Im}z$ for all $z \in \mathbb{C}_+$.*

By Klimek [19, Example 5.1.1], we have $V_K(z) = \log^+(\|z-a\|/r)$ if $\|\cdot\|$ is any norm and $K = \{z \in \mathbb{C}^n\,;\, \|z-a\| \leq r\}$, $r > 0$, is the closed ball in this norm with center a and radius r. The polynomial classes $\mathcal{P}^S_m(\mathbb{C}^n)$ are in general not translation invariant, so we can not expect to have a generalization of this example. The following is proved in special cases by Bos and Levenberg [13]:

Proposition 4.3. *Let S be a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$ and let E be a subset of $\mathbb{C}^n$ such that $\mathbb{T}^n \subseteq E \subseteq \mathcal{N}(H_S)$. Then $V^S_E = H_S$.*

Proof. Since $H_S \in \mathcal{L}^S(\mathbb{C}^n)$ and $H_S|_E = 0$ we have $H_S \leq V^S_E$, so it is sufficient to prove that if $u \in \mathcal{L}^S(\mathbb{C}^n)$ with $u|_E \leq 0$ we have $u(z) \leq H_S(z)$ for every $z \in \mathbb{C}^{*n}$ such that $H_S(z) > 0$. Define f_z by (3.6) and $v \in \mathcal{SH}(\mathbb{C})$ by $v(\tau) = u(f_z(\tau))$, $\tau \in \mathbb{C}$. Since $u \in \mathcal{L}^S(\mathbb{C}^n)$ we have $v(\tau) = u(f_z(\tau)) \leq c_u + H_S(f_z(\tau)) = c_u + \mathrm{Im}\tau\, H_S(z)$, $\mathrm{Im}\tau \geq 0$, and since $f_z(\mathbb{R}) \subseteq \overline{\mathbb{D}}^n$ we have $v \leq 0$ on $\mathbb{R}$. Lemma 4.2 gives $u(z) = v(i) \leq H_S(z)$. □

Since $\mathbb{T}^n \subset \mathcal{N}(H_S)$ for every S the maximum principle implies that $\overline{\mathbb{D}}^n \subset \mathcal{N}(H_S)$, where $\mathbb{D}$ denotes the unit disc in $\mathbb{C}$.

Definition 4.4. Let $0 \in S \subset \mathbb{R}^n_+$ be compact and convex and $q\colon E \to \mathbb{R} \cup \{+\infty\}$ be a function on $E \subset \mathbb{C}^n$. We say that q is an *admissible weight with respect to S on E* if

(i) q is lower semicontinuous,

(ii) the set $\{z \in E\,;\, q(z) < +\infty\}$ is non-pluripolar, and

(iii) if E is unbounded, then $\lim\limits_{E\ni z, |z|\to\infty}(H_S(z) - q(z)) = -\infty$.

This definition is taken from Bloom [28, Appendix B: Definition 2.1]. Some authors use the term *admissible external field* for q rather than *weight* in this situation and then they refer to e^{-q} as a weight. Observe that if q is an admissible weight, then E is non-pluripolar and that if E is unbounded then $q = 0$ does not need to be an admissible weight. In the case when E is unbounded and S is a neighborhood of 0 in $\mathbb{R}^n_+$, then $q = 0$ can not be an admissible weight.

Proposition 4.5. *Let S be a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$ and q be an admissible weight with respect to S on a compact subset K of $\mathbb{C}^n$. Then $V^{S*}_{K,q} \in \mathcal{L}^S_+(\mathbb{C}^n)$.*

Proof. The upper regularization $V^{S*}_{E,q}$ of $V^S_{E,q}$ is plurisubharmonic in $\mathbb{C}^n$ if q is an admissible weight with respect to S on E. Since $V^S_{K,q} \le q$ on K, q is admissible on K and $\{z \in K\,;\, q(z) < +\infty\} \subseteq \{z \in \mathbb{C}^n\,;\, V^S_{K,q}(z) < +\infty\}$, the set on the right is non-pluripolar. By Klimek [19, Proposition 5.2.1], it follows that the family $\mathcal{U} = \{u \in \mathcal{L}^S(\mathbb{C}^n)\,;\, u|_K \le q\} \subset \sigma_S\mathcal{L}(\mathbb{C}^n)$ is locally uniformly bounded above, where $\sigma_S = \varphi_S(\mathbf{1})$ is the logarithmic type of H_S. Let $c > 0$ be such that $u|_{\overline{\mathbb{D}}^n} \le c$ for all $u \in \mathcal{U}$. Then by Proposition 4.3, we have $V^S_{K,q} \le V^S_{\overline{\mathbb{D}}^n} + c = H_S + c$. Hence, $V^{S*}_{K,q} \in \mathcal{L}^S(\mathbb{C}^n)$. If $c = \max_{w\in K} H_S(w) - \min_{w\in K} q(w)$ then $H_S - c \le V^S_{K,q}$, so $V^{S*}_{K,q} \in \mathcal{L}^S_+(\mathbb{C}^n)$. □

Admissible weights on unbounded closed sets yield the same Siciak and Siciak-Zakharyuta functions as some compact subsets. Admissible weights do not obstruct the growth of Siciak-Zakharyuta functions. See Bloom [28, Appendix B].

Proposition 4.6. *Let S be a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$ and q an admissible weight on a closed subset E of $\mathbb{C}^n$, and set $E_R = E \cap \overline{B}(0,R)$ for every $R > 0$. Then $V^S_{E,q} = V^S_{E_R,q}$ and $\Phi^S_{E,q} = \Phi^S_{E_R,q}$ for R sufficiently large. Furthermore, $V^{S*}_{E,q} \in \mathcal{L}^S_+(\mathbb{C}^n)$.*

Proof. Since $E_R \subseteq E$ for every $R > 0$ we have $\Phi^S_{E,q} \le \Phi^S_{E_R,q}$ and $V^S_{E,q} \le V^S_{E_R,q}$, so we need to prove the reverse inequalities. By Definition 4.4, E is non-pluripolar and by [19, Corollary 4.7.7], a countable union of pluripolar sets is pluripolar. It follows that E_{R_0} is non-pluripolar for some $R_0 > 0$ and consequently E_R is pluripolar for every $R \ge R_0$. By Proposition 4.5, $V^{S*}_{E_R,q} \in \mathcal{L}^S(\mathbb{C}^n)$ and it follows that for some constant $c > 0$

$$V^{S*}_{E_R,q}(z) \le c + H_S(z), \quad z \in \mathbb{C}^n. \tag{4.1}$$

Condition (iii) in Definition 4.4 implies that we can choose $R_1 \geq R_0$ such that

$$q(z) - H_S(z) \geq c, \quad z \in E \setminus E_{R_1}. \tag{4.2}$$

Now we take $u \in \mathcal{L}^S(\mathbb{C}^n)$ and assume that $u \leq q$ on E_{R_1}. By (4.1) we have $u \leq c + H_S(z)$ for all $z \in \mathbb{C}^n$ and by (4.2) we have $u \leq q$ on $E \setminus E_{R_1}$. Hence, $u \leq V^S_{E,q}$.

Let $p \in \mathcal{P}^S_m(\mathbb{C}^n)$ for some $m \in \mathbb{N}$ be such that $\|pe^{-mq}\|_{E_{R_1}} \leq 1$. By Theorem 3.6 we have $u = \log|p|^{1/m} \in \mathcal{L}^S(\mathbb{C}^n)$ and $u \leq q$ on E_{R_1}. Again by (4.1) and (4.2) we have $u \leq q$ on $E \setminus E_{R_1}$ as well. Then $\|pe^{-mq}\|_E \leq 1$ and $|p|^{1/m} \leq \Phi^S_{E,q}$. The last statement follows directly from Proposition 4.5. □

For a general compact convex S with $0 \in S$, let $S' = \overline{S \cap \mathbb{Q}^n}$ be the closure of the set of rational points in S. Then $\mathcal{P}^S_m(\mathbb{C}^n) = \mathcal{P}^{S'}_m(\mathbb{C}^n)$ for every $m \in \mathbb{N}^*$ and consequently $\log\Phi^S_{E,q} = \log\Phi^{S'}_{E,q} \leq V^{S'}_{E,q} \leq V^S_{E,q}$. Observe that $S = S'$ if S is a convex body but $S \neq S'$ for example if S is a line segment in $\mathbb{R}^2$ with irrational slope.

Proposition 4.7. *Let $S \subset \mathbb{R}^n_+$ be compact and convex with $0 \in S$. If $S \cap \mathbb{Q}^n$ is not dense in S, then for every admissible weight q on a closed $E \subset \mathbb{C}^n$ we have $\log\Phi^S_{E,q} \neq V^S_{E,q}$.*

Proof. Since $S' = \overline{S \cap \mathbb{Q}^n} \subsetneq S$, there exists a $\xi \in \mathbb{R}^n$ with $\varphi_{S'}(\xi) < \varphi_S(\xi)$. By Proposition 4.5 there exist constants c and c' such that

$$H_S(z) - c \leq V^S_{E,q} \quad \text{and} \quad V^{S'}_{E,q}(z) \leq H_{S'}(z) + c'.$$

For $r > 0$ sufficiently large, we have $\varphi_{S'}(r\xi) + c' < \varphi_S(r\xi) - c$, so for $z = (e^{r\xi_1}, \dots, e^{r\xi_n})$

$$\begin{aligned}\log\Phi^S_{E,q}(z) = \log\Phi^{S'}_{E,q}(z) \leq V^{S'}_{E,q}(z) \leq H_{S'}(z) + c' < H_S(z) - c \\ \leq V^S_{E,q}(z),\end{aligned}$$

concluding the proof. □

The next result regards the Siciak-Zakharyuta functions with respect to S, E and q when we have decreasing sequences of sets $S_j \searrow S$ or increasing sequences of weights $q_j \nearrow q$.

Proposition 4.8. *Let S_j, $j \in \mathbb{N}$ and S be compact convex subsets of $\mathbb{R}^n_+$ with $0 \in S$ and $S_j \searrow S$, q be an admissible weight on a compact subset K of $\mathbb{C}^n$, $(K_j)_{j\in\mathbb{N}}$ be a decreasing sequence of compact sets with $\cap_j K_j = K$, and $(q_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{LSC}(K_j)$ such that $q_j \nearrow q$. Then:*

- **(i)** $\mathcal{L}^S(\mathbb{C}^n) = \cap_{j\in\mathbb{N}}\mathcal{L}^{S_j}(\mathbb{C}^n)$.
- **(ii)** *If $V^{S_j *}_{K,q} \leq q$ on K for some j, then $V^{S_j}_{K,q} \searrow V^S_{K,q}$ as $j \to \infty$.*

(iii) *Every q_j is an admissible weight on K_j, $V^S_{K_j,q_j} \nearrow V^S_{K,q}$ and $\Phi^S_{K_j,q_j} \nearrow \Phi^S_{K,q}$ as $j \to \infty$.*

Proof. **(i)** Obviously $\mathcal{L}^S(\mathbb{C}^n) \subseteq \cap_j \mathcal{L}^{S_j}(\mathbb{C}^n)$. Let $u \in \cap_j \mathcal{L}^{S_j}(\mathbb{C}^n)$ and set $c_u = \sup_{\overline{\mathbb{D}}^n} u$. Then by Proposition 4.3 we have $u - c_u \leq V^{S_j}_{\overline{\mathbb{D}}^n} = H_{S_j}$ for every j. We have $H_{S_j} \searrow H_S$, so $u \leq c_u + H_S$ and $u \in \mathcal{L}^S(\mathbb{C}^n)$.

(ii) We have $V^S_{K,q} \leq V^{S_j}_{K,q}$ and since $V^{S_j *}_{K,q} \leq q$ the equation $V^{S_j *}_{K,q} = V^{S_j}_{K,q}$ holds. By Proposition 4.5 we have $V^{S_j}_{K,q} \in \mathcal{L}^{S_j}(\mathbb{C}^n)$. Since the sequence is decreasing $V = \lim_{j\to\infty} V^{S_j *}_{K,q} \in \cap_j \mathcal{L}^{S_j}(\mathbb{C}^n) = \mathcal{L}^S(\mathbb{C}^n)$ and $V \leq q$ on K. Hence,

$$V^S_{K,q} \leq \lim_{j\to\infty} V^{S_j}_{K,q} \leq V \leq V^S_{K,q}.$$

(iii) Since $F = \{z \in K\,;\, q(z) < +\infty\} \subseteq F_j = \{z \in K_j\,;\, q_j(z) < +\infty\}$ for every j and F is non-pluripolar, the set F_j in non-pluripolar and every q_j is an admissible weight on K_j. Since the sequence $(V^S_{K_j,q_j})_{j\in\mathbb{N}}$ is increasing and bounded above by $V^S_{K,q}$ we need to show that $V^S_{K,q} \leq V = \lim_{j\to\infty} V^S_{K_j,q_j}$. For that purpose we take $u \in \mathcal{L}^S(\mathbb{C}^n)$ with $u \leq q$ on K and $\varepsilon > 0$. For every $z_0 \in K$ there exists j_ε such that $u(z_0) - \varepsilon < q_{j_\varepsilon}(z_0)$. Since $u \in \mathcal{USC}(\mathbb{C}^n)$ and $q_{j_\varepsilon} \in \mathcal{LSC}(K_j)$ it follows that we have $u(z) - \varepsilon < q_{j_\varepsilon}(z)$ for all $z \in K_j \cap U_0$ for some neighborhood U_0 of z_0 in $\mathbb{C}^n$. A simple compactness argument gives that there exists an open neighborhood U of K such that $u - \varepsilon < q_j$ on $K_j \subset U$ for $j \geq j_\varepsilon$, possibly with j_ε replaced by a larger number. Hence, $u - \varepsilon \leq V$, and since ε is arbitrary we conclude that $u \leq V$. By taking supremum over u we get $V^S_{K,q} \leq V$. The same argument for $\log|p|^{1/m}$, $p \in \mathcal{P}^S_m(\mathbb{C}^n)$, in the role of u implies that $\Phi^S_{K,q} \leq \lim_{j\to\infty} \Phi^S_{K_j,q_j}$ by Proposition 2.2. □

5 Regularity of Siciak-Zakharyuta functions

In this section we study regularity of the functions $\Phi^S_{E,q}$ and $V^S_{E,q}$, where we assume that S is a convex subset of $\mathbb{R}^n_+$ with $0 \in S$ and that $q\colon E \to \mathbb{R}\cup\{+\infty\}$ is a function on a subset E of $\mathbb{C}^n$. Since the Siciak function $\Phi^S_{E,q}$ is the supremum of a subclass of $\mathcal{C}(\mathbb{C}^n)$ we have $\Phi^S_{E,q} \in \mathcal{LSC}(\mathbb{C}^n)$. If q is an admissible weight on a closed set E, then by Proposition 4.6 we have $V^{S*}_{E,q} \in \mathcal{L}^S_+(\mathbb{C}^n)$.

If $V^{S*}_{E,q} \leq q$ on E then $V^S_{E,q} = V^{S*}_{E,q}$ and we conclude that $V^S_{E,q}$ is continuous at every point where it is lower semicontinuous. In this section we prove that $V^S_{E,q} \in \mathcal{LSC}(\mathbb{C}^{*n})$ for every admissible weight on a compact set K. For that purpose we need to discuss regularization of plurisubharmonic functions, but we begin with local $\mathcal{L}$-regularity of sets.

Definition 5.1. A subset E of $\mathbb{C}^n$ is said to be *$\mathcal{L}$-regular* at a point $a \in \overline{E}$ if V_E is continuous at a and E is said to be *locally $\mathcal{L}$-regular* at a if $E \cap U$ is $\mathcal{L}$-regular at a for every open neighborhood U of a. We say that E is (*locally*) *$\mathcal{L}$-regular* if E is (locally) $\mathcal{L}$-regular at every $a \in \overline{E}$.

The function V_E is continuous at every interior point of E. Moreover, V_E is continuous at $a \in \overline{E}$ if and only if $V_E^*(a) = 0$. Thus, it is sufficient to check the condition for local $\mathcal{L}$-regularity at boundary points only.

Lemma 5.2. *Let E be a closed subset of $\mathbb{C}^n$, $a \in E$ and assume that there exists a norm with closed unit ball $\mathbb{B}$ such that for some $\delta > 0$ and $b \in E$ we have $a \in B = b + \delta\mathbb{B} \subset E$. Then E is locally $\mathcal{L}$-regular at a.*

Proof. Let $\|\cdot\|$ denote the norm and U be open with $a \in U$. Choose $\tau > 0$ so small that $C = c + \tau\delta\mathbb{B} \subset U$, where $c = (1-\tau)a + \tau b$. We have $C \subset B \subset E$, so

$$0 \le V_E^*(a) \le V_C^*(a) = \log^+(\|a-c\|/\tau\delta) = \log^+(\|a-b\|/\delta) = 0.$$

□

Observe that the lemma implies that every set of the form $E = A + \delta\mathbb{B}$ is locally $\mathcal{L}$-regular, where $A \subset \mathbb{C}^n$ is closed and $\mathbb{B}$ is the closed unit ball with respect to some norm. The following result is a generalization of Siciak [31, Proposition 2.16].

Proposition 5.3. *For every continuous function q on a locally $\mathcal{L}$-regular closed subset E of $\mathbb{C}^n$ we have $V_{E,q}^{S*} \le q$ and consequently $V_{E,q}^{S} = V_{E,q}^{S*}$.*

Proof. Let $a \in E$ and take $\varepsilon > 0$. Since q is continuous on E there exists an open neighborhood U of a such that $q(z) \le q(a) + \varepsilon$ for every $z \in E \cap U$. Since E is locally $\mathcal{L}$-regular we have $V_{E\cap U}^*(a) = 0$ and

$$V_{E,q}^{S*}(a) \le V_{E\cap U,q(a)+\varepsilon}^{S*}(a) \le \sigma_S V_{E\cap U}^*(a) + q(a) + \varepsilon = q(a) + \varepsilon.$$

Since a and ε are arbitrary the inequality holds. □

Convolution is a standard tool for approximating functions $u \in \mathcal{PSH}(\mathbb{C}^n)$. We define a convolution operator $L^1_{\text{loc}}(\mathbb{C}^n) \to L^1_{\text{loc}}(\mathbb{C}^n)$, $u \mapsto u * \mu$, for a given positive Borel measure μ with compact support,

$$u * \mu(z) = \int_{\mathbb{C}^n} u(z-w)\, d\mu(w), \quad z \in \mathbb{C}^n. \tag{5.1}$$

If $u \in \mathcal{PSH}(\mathbb{C}^n)$ then $u * \mu \in \mathcal{PSH}(\mathbb{C}^n)$ and if μ is a probability measure then $u \mapsto u * \mu$ preserves $\mathcal{L}(\mathbb{C}^n)$. In the case that μ is presented by a $\mathcal{C}^\infty$ function, $\mu = \psi\, d\lambda$, then $u * \mu = u * \psi$ is a $\mathcal{C}^\infty$ function. If we define $\psi_\delta \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ by $\psi_\delta(z) = \delta^{-2n}\psi(z/\delta)$, $\delta > 0$, with $\psi \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ radially symmetric, $\psi \ge 0$, and $\int_{\mathbb{C}^n} \psi\, d\lambda = 1$, then $u * \psi_\delta \searrow u$ as $\delta \searrow 0$. See Klimek [19] or Hörmander [15, 16]. Siciak [31, Proposition 2.12] used convolution to prove that $V_{E,q} \in \mathcal{LSC}(\mathbb{C}^n)$ for compact E and $q \in \mathcal{LSC}(E)$. In general, the class $\mathcal{L}^S(\mathbb{C}^n)$ is not preserved under convolution (cf. Theorem 5.8).

In order to preserve a particular subclass of $\mathcal{PSH}(\mathbb{C}^n)$ under regularization, special methods are sometimes needed. For example homogeneity is preserved under

$$R_\delta u(z) = \int_G u(Az)\psi_\delta(A)\, d\mu(A), \quad z \in \mathbb{C}^n, \tag{5.2}$$

where G is some group of $n \times n$ matrices with real or complex entries and μ is a positive measure on the matrix space $\mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$. The smoothing kernel ψ_δ is chosen so that it converges to the Dirac measure δ_I at the identity I as $\delta \to 0$. This method only gives a $\mathcal{C}^\infty$ function on $\mathbb{C}^n \setminus \{0\}$, when we integrate over the group of *complex* invertible matrices, and it gives a $\mathcal{C}^\infty$ function on $\mathbb{C}^n \setminus \mathbb{C}\mathbb{R}^n$, where $\mathbb{C}\mathbb{R}^n = \{\lambda x; \lambda \in \mathbb{C}, x \in \mathbb{R}^n\}$, when we integrate over the group of *real* invertible matrices. See Sigurdsson [35] and Hörmander and Sigurdsson [18].

In order to preserve the classes $\mathcal{L}^S(\mathbb{C}^n)$ for a compact convex $S \subset \mathbb{R}^n_+$ with $0 \in S$ it is natural to choose G as the group of invertible diagonal matrices. This group can be identified with $\mathbb{C}^{*n}$ with coordinate wise multiplication, so it is natural to choose μ as the Lebesgue measure λ on $\mathbb{C}^n$.

In the following text we allow us a slight abuse of notation by identifying a vector denoted by a lower case letter with a diagonal matrix denoted by the corresponding upper case letter. Thus,, we identify the vector $a \in \mathbb{C}^n$ with the diagonal matrix A with diagonal a and in particular the vector $\mathbf{1}$ with the identity matrix I. We define

$$\begin{aligned} R_\delta u(z) &= \int_{\mathbb{C}^n} u(Az)\psi_\delta(A)\, d\lambda(A) = \int_{\mathbb{C}^n} u((I+\delta B)z)\psi(B)\, d\lambda(B) \qquad (5.3)\\ &= \int_{\mathbb{C}^n} u((1+\delta w_1)z_1, \dots, (1+\delta w_n)z_n)\psi(w)\, d\lambda(w), \quad z \in \mathbb{C}^n, \end{aligned}$$

where we choose a function $0 \le \psi \in \mathcal{C}_0^\infty(\mathbb{C}^n)$, rotationally symmetric in each variable with $\int_{\mathbb{C}^n} \psi\, d\lambda = 1$, and set $\psi_\delta(z) = \delta^{-2n}\psi((z-\mathbf{1})/\delta)$. By the Fubini-Tonelli theorem $R_\delta\colon L^1_{\text{loc}}(\mathbb{C}^n) \to L^1_{\text{loc}}(\mathbb{C}^n)$ and $R_\delta u \to u$ in the L^1_{loc} topology as $\delta \to 0$ and with local uniform convergence for $u \in \mathcal{C}(\mathbb{C}^n)$. Furthermore, $R_\delta\colon \mathcal{PSH}(\mathbb{C}^n) \to \mathcal{PSH}(\mathbb{C}^n)$. This implies that if $u \in \mathcal{L}^S(\mathbb{C}^n)$ then $R_\delta u \in \mathcal{L}^S(\mathbb{C}^n)$, more precisely, if $u \le c_u + H_S$, Proposition 3.2 implies

$$\begin{aligned} R_\delta u(z) &\le \int_{\mathbb{C}^n} \big(c_u + H_S(\mathbf{1}+\delta w) + H_S(z)\big)\psi(w)\, d\lambda(w)\\ &\le c_u + C\sigma_S\delta + H_S(z), \qquad z \in \mathbb{C}^n, \end{aligned}$$

where $C = \sup_{w\in \operatorname{supp}\psi} \|w\|_\infty$. The linear map $\mathbb{C}^n \to \mathbb{C}^n$, $A \mapsto Az$, has the Jacobi determinant $|z_1 \cdots z_n|^2$, so for every $z \in \mathbb{C}^{*n}$ and corresponding matrix Z with z on the diagonal we have

$$\begin{aligned} R_\delta u(z) &= \int_{\mathbb{C}^n} u(w)\psi_\delta(Z^{-1}w)\, |z_1 \cdots z_n|^{-2}\, d\lambda(w)\\ &= \int_{\mathbb{C}^n} u(w)\psi_\delta(w_1/z_1, \dots, w_n/z_n)\, |z_1 \cdots z_n|^{-2}\, d\lambda(w). \end{aligned}$$

By applying the Lebesgue dominated convergence theorem we may differentiate with respect to z_j under the integral sign infinitely often, so this shows that for every $u \in L^1_{\text{loc}}(\mathbb{C}^n)$ we have $R_\delta u \in \mathcal{C}^\infty(\mathbb{C}^{*n})$.

Proposition 5.4. *Let S be a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$ and q be an admissible weight on a compact subset K of $\mathbb{C}^n$. Then*

(i) *$V^S_{K,q} \in \mathcal{LSC}(\mathbb{C}^{*n})$, and if $\mathcal{L}^S(\mathbb{C}^n)$ is preserved under convolution (see Theorem 5.8) then $V^S_{K,q} \in \mathcal{LSC}(\mathbb{C}^n)$.*

(ii) *If $V^{S*}_{K,q} \le q$ on K then $V^S_{K,q} \in \mathcal{L}^S(\mathbb{C}^n) \cap \mathcal{C}(\mathbb{C}^{*n})$ and if $\log \Phi^S_{K,q} = V^S_{K,q}$ then $V^S_{K,q} \in \mathcal{C}(\mathbb{C}^n)$.*

Proof. **(i)** It is sufficient to show that there exists an increasing sequence $(u_j)_{j\in\mathbb{N}}$ in $\mathcal{L}^S(\mathbb{C}^n) \cap \mathcal{C}(\mathbb{C}^{*n})$ such that $u_j \le q$ on K for every j and such that $\lim_{j\to\infty} u_j(z) = V^S_{K,q}(z)$ for every $z \in \mathbb{C}^{*n}$. (See Klimek [19, Section 2.3].) We take $u \in \mathcal{L}^S(\mathbb{C}^n)$, $\varepsilon > 0$, and let R_δ be the regularization operator (5.3). Since $R_\delta u \searrow u$ and $q \in \mathcal{LSC}(K)$ there exists δ_ε such that $u - \varepsilon \le R_\delta u - \varepsilon < q$ on K for every $\delta \le \delta_\varepsilon$. These estimates tell us that there exists a family $\mathcal{F} \subseteq \mathcal{L}^S(\mathbb{C}^n) \cap \mathcal{C}^\infty(\mathbb{C}^{*n})$ such that $V^S_{K,q} = \sup \mathcal{F}$. By the Choquet lemma $V^S_{K,q} = \sup \mathcal{G}$ for some countable subfamily $\mathcal{G}$. By arranging the elements of $\mathcal{G}$ into a sequence $(v_j)_{j\in\mathbb{N}}$ and then setting $u_j = \max_{k\le j} v_k$, we have $u_j \nearrow V^S_{K,q}$ as $j \to \infty$. Hence, $V^S_{K,q} \in \mathcal{LSC}(\mathbb{C}^{*n})$. This is a modification the proof of [31, Proposition 2.12], where the regularization operator is given by (5.1), and that is the second statement.

(ii) We have $V^{S*}_{K,q} \ge V^S_{K,q}$ and by the definition of $V^S_{K,q}$ we have $V^{S*}_{K,q} \le V^S_{K,q}$. Hence, $V^S_{K,q} = V^{S*}_{K,q} \in \mathcal{USC}(\mathbb{C}^n)$ and from **(i)** it follows that $V^S_{K,q} \in \mathcal{C}(\mathbb{C}^{*n})$. The last statement follows from the fact that $\Phi^S_{E,q} \in \mathcal{LSC}(\mathbb{C}^n)$. □

Our next task is to characterize the $\mathcal{L}^S(\mathbb{C}^n)$ classes that are invariant under the convolution operator $u \mapsto u * \psi$ given by (5.1). For that purpose we need to review a few facts from convexity theory and define the hull of a convex set in $\mathbb{R}^n_+$ with respect to a cone. Recall that a point x in a convex set S is said to be an *extreme point of S* if the only representation of x as a convex combination $x = (1-t)a + tb$ with $a, b \in S$ and $t \in]0,1[$ is the case $x = a = b$. We let $\operatorname{ext} S$ denote the set of all extreme points in the convex set S. Note that by the Minkowski theorem [17, Theorem 2.1.9], every non-empty compact convex set S is the closed convex hull of its extreme points and

$$\varphi_S(\xi) = \max_{x\in\operatorname{ext} S} \langle x, \xi\rangle, \quad \xi \in \mathbb{R}^n. \tag{5.4}$$

Every affine hyperplane $\{x \in \mathbb{R}^n\,;\, \langle x, \xi\rangle = \varphi_S(\xi)\}$ with $\xi \in \mathbb{R}^{n*}$ is called a *supporting hyperplane of the set S*. For every $s \in \partial S$ the set $N^S_s = \{\xi \in \mathbb{R}^n\,;\, \langle s, \xi\rangle = \varphi_S(\xi)\}$ is a convex cone which is called the *normal cone of S at the point s*. For every ξ the upper bound in (5.4) is attained at some $s \in \operatorname{ext} S$, so $\mathbb{R}^n = \cup_{s\in\operatorname{ext} S} N^S_s$.

Every normal cone N^S_s is an intersection of closed half spaces in $\mathbb{R}^n$ with the origin in the boundary hyperplanes, that is $N^S_s = \bigcap_{t\in\partial S}\{\xi \in \mathbb{R}^n\,;\, \langle s - t, \xi\rangle \ge 0\}$. Observe that ∂S can be replaced by the set of extreme points $\operatorname{ext} S$. In the case S is a convex polytope and s is an extreme point, then N^S_s is an intersection of finitely many half spaces in $\mathbb{R}^n$ with boundary hyperplanes containing the origin. See Figure 2(b).

If $\xi \in \mathbb{R}^{n*}$ and $\xi_0 \in \mathbb{R}$, then $|\langle x, \xi\rangle + \xi_0|/|\xi|$ is the distance from the point $x \in \mathbb{R}^n$ to the hyperplane $\{y \in \mathbb{R}^n\,;\, \langle y, \xi\rangle + \xi_0 = 0\}$. For supporting hyperplanes with normal ξ we have $\xi_0 = -\varphi_S(\xi)$. Hence, $|\varphi_S(\xi)|/|\xi|$ is the euclidean distance from the origin in $\mathbb{R}^n$ to the supporting hyperplane $\{x \in \mathbb{R}^n\,;\, \langle x, \xi\rangle = \varphi_S(\xi)\}$.

Recall that a subset Γ of $\mathbb{R}^n$ is said to be a cone if $t\xi \in \Gamma$ for every $\xi \in \Gamma$ and $t \in \mathbb{R}_+$. The *dual cone* $\Gamma^\circ = \{x \in \mathbb{R}^n\,;\, \langle x, \xi\rangle \geq 0\ \forall \xi \in \Gamma\}$ of Γ is a closed convex cone and if $\Gamma \neq \mathbb{R}^n$ is closed and convex, then $\Gamma^{\circ\circ} = \Gamma$.

Definition 5.5. For every cone $\Gamma \subset (\mathbb{R}^n \setminus \mathbb{R}^n_-) \cup \{0\}$ with $\Gamma \neq \{0\}$ and every subset S of $\mathbb{R}^n_+$ we define the Γ-*hull* of S by

$$\widehat{S}_\Gamma = \{x \in \mathbb{R}^n_+\,;\, \langle x, \xi\rangle \leq \varphi_S(\xi)\ \forall \xi \in \Gamma\},$$

and we say that S is Γ-*convex* if $S = \widehat{S}_\Gamma$.

Note that if $\Gamma_1 \subseteq \Gamma_2$ then $\widehat{S}_{\Gamma_2} \subseteq \widehat{S}_{\Gamma_1}$. For every compact and convex S we have $S = \{x \in \mathbb{R}^n\,;\, \langle x, \xi\rangle \leq \varphi_S(\xi)\ \forall \xi \in \mathbb{R}^n\} = \widehat{S}_{\mathbb{R}^n}$ which implies that $S \subseteq \widehat{S}_\Gamma$ for every cone $\Gamma \subseteq \mathbb{R}^n$.

Proposition 5.6. *Let S be a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$ and Γ be a proper closed convex cone containing at least one point in $\mathbb{R}^{*n}_+$. Then*

$$\widehat{S}_\Gamma = (S - \Gamma^\circ) \cap \mathbb{R}^n_+,$$

and $\varphi_{\widehat{S}_\Gamma}(\xi) = \varphi_S(\xi)$ holds for every $\xi \in \Gamma$. If for every $\xi \in \mathbb{R}^n$ and every extreme point x of $\widehat{S}_\Gamma$ there exists $\eta \in \Gamma$ such that $\langle x, \xi\rangle \leq \langle x, \eta\rangle$ and $\varphi_S(\eta) = \varphi_S(\xi)$, then S is Γ-convex.

Proof. Take $a = s - t \in S' = (S - \Gamma^\circ) \cap \mathbb{R}^n_+$ with $s \in S$ and $t \in \Gamma^\circ$. For every $\xi \in \Gamma$ we have $\langle t, \xi\rangle \geq 0$ which implies $\langle a, \xi\rangle = \langle s, \xi\rangle - \langle t, \xi\rangle \leq \varphi_S(\xi)$ and $a \in \widehat{S}_\Gamma$.

Conversely, we take $a \notin S'$ and prove that $a \notin \widehat{S}_\Gamma$. Without restriction, we may assume that $a \in \mathbb{R}^n_+$. Since $S - \Gamma^\circ$ is convex the Hahn-Banach theorem implies that $\{a\}$ and $S - \Gamma^\circ$ can be separated by an affine hyperplane. Hence, there exists $\xi \in \mathbb{R}^n \setminus \{0\}$ and $c \in \mathbb{R}$ such that $\langle a, \xi\rangle > c$ and $\langle x, \xi\rangle \leq c$ for every $x \in S - \Gamma^\circ$. By replacing c by $\sup_{x \in S - \Gamma^\circ} \langle x, \xi\rangle$ we may assume that there exists $s \in S$ and $t \in \Gamma^\circ$ such that $\langle s - t, \xi\rangle = c$. Now we need to prove that $\xi \in \Gamma = \Gamma^{\circ\circ}$ by showing that $\langle y, \xi\rangle \geq 0$ for every $y \in \Gamma^\circ$. Since Γ° is a convex cone, we have $t + y \in \Gamma^\circ$ and $c - \langle y, \xi\rangle = \langle s - t - y, \xi\rangle \leq c$. Hence, $\langle y, \xi\rangle \geq 0$. This implies that $\langle a, \xi\rangle > c \geq \varphi_S(\xi)$ and we conclude that $a \notin \widehat{S}_\Gamma$.

Let $\xi \in \Gamma$ and take $a = s - t \in \widehat{S}_\Gamma$, such that $s \in S$, $t \in \Gamma^\circ$, and $\varphi_{\widehat{S}_\Gamma}(\xi) = \langle s - t, \xi\rangle$. Since $\langle s, \xi\rangle \leq \varphi_S(\xi) \leq \varphi_{\widehat{S}_\Gamma}(\xi) = \langle s, \xi\rangle - \langle t, \xi\rangle$ and $\langle t, \xi\rangle \geq 0$, we conclude that $t = 0$ and $\varphi_S(\xi) = \varphi_{\widehat{S}_\Gamma}(\xi)$. If for every $\xi \in \mathbb{R}^n$ and every extreme point x of $\widehat{S}_\Gamma$ there exists $\eta \in \Gamma$ such that $\langle x, \xi\rangle \leq \langle x, \eta\rangle$ and $\varphi_S(\eta) = \varphi_S(\xi)$, then $\widehat{S}_\Gamma = \widehat{S}_{\mathbb{R}^n} = S$. □

Definition 5.7. We say that a subset S of $\mathbb{R}^n_+$ is a *lower set* if for every $s \in S$ the cube $C_s = [0, s_1] \times \cdots \times [0, s_n]$ is a subset of S and we call

$$\widehat{S}_{\mathbb{R}^n_+} = \cup_{s\in S} C_s = \big(S - \mathbb{R}^n_+\big) \cap \mathbb{R}^n_+$$

the *lower hull of* S.

We have a characterization of lower sets:

Theorem 5.8. *Let $S \subset \mathbb{R}^n_+$ be compact and convex with $0 \in S$ and set $\sigma_S = \varphi_S(\mathbf{1})$. Then the following are equivalent:*

(i) *$S \subset \mathbb{R}^n_+$ is a lower set,*

(ii) *$S = \widehat{S}_{\mathbb{R}^n_+}$,*

(iii) *$\varphi_S(\xi) = \varphi_S(\xi^+)$ for every $\xi \in \mathbb{R}^n$,*

(iv) *$H_S(z - w) \leq \sigma_S \|w\|_\infty + H_S(z)$ for every $z, w \in \mathbb{C}^n$,*

(v) *$\mathcal{L}^S(\mathbb{C}^n)$ is translation invariant, and*

(vi) *if $u \in \mathcal{L}^S(\mathbb{C}^n)$ then $u * \psi \in \mathcal{L}^S(\mathbb{C}^n)$ for every $\psi \in \mathcal{C}^\infty_0(\mathbb{C}^n)$ with $\psi \geq 0$ and $\int_{\mathbb{C}^n} \psi \, d\lambda = 1$.*

Proof. **(i)⇔(ii)⇔(iii):** Observe that the extreme points of C_s are $t = (t_1, \ldots, t_n)$ with $t_j = 0$ or $t_j = s_j$, so $\varphi_{C_s}(\xi) = \sup_{t\in \mathrm{ext}\, C_s} \langle t, \xi\rangle = \langle s, \xi^+\rangle$ for every $\xi \in \mathbb{R}^n$. Hence, for every lower set S we have $\varphi_S(\xi) = \sup_{s\in S} \langle s, \xi^+\rangle = \varphi_S(\xi^+)$ for $\xi \in \mathbb{R}^n$. This equivalence follows from Proposition 5.6 with $\eta = \xi^+$.

(iii)⇒(iv): Let $w \in \mathbb{C}^n$, $z \in \mathbb{C}^{*n}$ and assume that $z - w \in \mathbb{C}^{*n}$. Since $\varphi_S(\xi) - \varphi_S(\eta) \leq \varphi_S(\xi - \eta)$ and $\varphi(\xi) \leq \sigma_S \|\xi\|_\infty$ for every $\xi, \eta \in \mathbb{R}^n$ and $|\log^+ x - \log^+ y| \leq |x - y|$ for every $x, y \in \mathbb{R}_+$, **(iii)** implies that

$$\begin{aligned} H_S(z - w) - H_S(z) &= \varphi_S(\mathrm{Log}^+(z - w)) - \varphi_S(\mathrm{Log}^+(z)) \\ &\leq \sigma_S \max_j |\log^+ |z_j - w_j| - \log^+ |z_j|| \\ &\leq \sigma_S \max_j ||z_j - w_j| - |z_j|| \leq \sigma_S \|w\|_\infty. \end{aligned}$$

By continuity of the function H_S the inequality follows.

(iv)⇒(v): If $u \leq c_u + H_S$, then $u(\cdot - w) \leq c_u + \sigma_S \|w\|_\infty + H_S$ for every $w \in \mathbb{C}^n$.

(v)⇒(vi): It is sufficient to show that $H_S * \psi \in \mathcal{L}^S(\mathbb{C}^n)$. We observe that for every $\gamma \in]0, 1[$ the Riemann sum $A_\gamma = \sum_{\alpha\in\mathbb{Z}^{2n}} \psi(\gamma\alpha)\gamma^{-2n}$ tends to $1 = \int_{\mathbb{C}^n} \psi \, d\lambda$ as $\gamma \to 0$. This implies that the function $u_\gamma \colon \mathbb{C}^n \to \mathbb{R}$ defined by the Riemann sum

$$u_\gamma(z) = \sum_{\alpha\in\mathbb{Z}^{2n}} H_S(z - \gamma\alpha)\psi(\gamma\alpha)\gamma^{-2n}/A_\gamma$$

tends to $H_S * \psi$ as $\gamma \to 0$. By **(v)** the function u_γ is in $\mathcal{L}^S(\mathbb{C}^n)$ for it is a convex combination of functions in $\mathcal{L}^S(\mathbb{C}^n)$ and we have $u_\gamma - c_\gamma \leq V^S_{\overline{\mathbb{D}}^n} = H_S$, where $c_\gamma = \sup_{\overline{\mathbb{D}}^n} u_\gamma$. Furthermore, since $H_S \in \mathcal{C}(\mathbb{C}^n)$ the convergence is locally uniform, so by Proposition 4.3 we have $H_S * \psi(z) \leq \sup_{\overline{\mathbb{D}}^n} H_S * \psi + H_S(z)$ for $z \in \mathbb{C}^n$ and conclude that $H_S * \psi \in \mathcal{L}^S(\mathbb{C}^n)$.

(vi)⇒(iii): We begin by taking $\psi(\zeta) = \chi(\zeta_1)\cdots\chi(\zeta_n)$, where $0 \leq \chi \in \mathcal{C}_0^\infty(\mathbb{C})$ is rotationally invariant, $\operatorname{supp}\chi \subset \overline{\mathbb{D}}$, and $\int_{\mathbb{C}} \chi \, d\lambda = 1$ and observe that with this choice of ψ we have $(f * \psi)(z) = \sum_{j=1}^n (f_j * \chi)(z_j)$ for any locally integrable function of the form $f(z) = \sum_{j=1}^n f_j(z_j)$.

Let $\eta \in \mathbb{R}^{*n}$ have at least one strictly negative coordinate, enumerate the coordinates so that $\eta_j < 0$ for $j = 1, \dots, \ell$ and $\eta_j > 0$ for $j = \ell+1, \dots, n$, and take $s \in S$ such that $\varphi_S(\eta) = \langle s, \eta \rangle$. Then for every $t > 0$ we have

$$\begin{aligned} H_S * \psi(e^{t\eta}) - H_S(e^{t\eta}) &= \int_{\mathbb{C}^n} \varphi_S(\operatorname{Log}(e^{t\eta} - \zeta))\, \psi(\zeta)\, d\lambda(\zeta) - \langle s, t\eta \rangle \\ &\geq \int_{\mathbb{C}^n} \langle s, \operatorname{Log}(e^{t\eta} - \zeta) - t\eta \rangle\, \psi(\zeta)\, d\lambda(\zeta) \\ &= \sum_{j=1}^n s_j \int_{\mathbb{D}} \big(\log|e^{t\eta_j} - \zeta_j| - t\eta_j\big)\, \chi(\zeta_j)\, d\lambda(\zeta_j) \\ &= -t\langle s', \eta' \rangle + \sum_{j=1}^{\ell} s_j \int_{\mathbb{D}} \log|e^{t\eta_j} - \zeta_j| \chi(\zeta_j)\, d\lambda(\zeta_j) \\ &\quad + \sum_{j=\ell+1}^{n} s_j \int_{\mathbb{D}} \log|1 - e^{-t\eta_j}\zeta_j| \chi(\zeta_j)\, d\lambda(\zeta_j). \end{aligned}$$

We let $v = \log|\cdot| \in \mathcal{SH}(\mathbb{C}) \cap \mathcal{H}(\mathbb{C}^*)$. Then for $j = 1, \dots, \ell$

$$\int_{\mathbb{D}} \log|e^{t\eta_j} - \zeta_j|\, \chi(\zeta_j)\, d\lambda(\zeta_j) = (v * \chi)(e^{t\eta_j}) \to v * \chi(0), \quad t \to +\infty.$$

Since $v \in \mathcal{H}(\mathbb{C}^*)$, χ is rotationally invariant, and $\operatorname{supp}\chi \subset \overline{\mathbb{D}}$, the mean value property gives for $j = \ell+1, \dots, n$

$$\int_{\mathbb{D}} \log|1 - e^{-t\eta_j}\zeta_j|\, \chi(\zeta_j)\, d\lambda(\zeta_j) = \log 1 = 0.$$

Hence,

$$H_S * \psi(e^{t\eta}) - H_S(e^{t\eta}) \geq -t\langle s', \eta' \rangle + \sum_{j=1}^{\ell} s_j (v * \chi)(e^{t\eta_j}). \tag{5.5}$$

Assume that **(iii)** does not hold and take $\xi_0 \in \mathbb{R}^n$ such that $\varphi_S(\xi_0) < \varphi_S(\xi_0^+)$. Then at least one coordinate of ξ_0 is strictly negative. By continuity, we can choose $\xi_0 \in \mathbb{R}^{*n}$, and we can renumber the coordinates so that $\xi_{0,j} < 0$ for $j = 1, \dots, \ell$ and $\xi_{0,j} > 0$ for $j = \ell+1, \dots, n$. By continuity there exists an open neighborhood U of ξ_0^+ such that $\varphi_S(\xi_0) < \varphi_S(\eta)$ for every $\eta \in U$. We fix $\eta = (\eta', \xi_0'') \in U$ with $\eta_j < 0$ for $j = 1, \dots, \ell$. There exists a point $s = (s', s'') \in \partial S$, such that $\langle s, \eta \rangle = \langle s', \eta' \rangle + \langle s'', \eta'' \rangle = \varphi_S(\eta)$. We have

$\langle s', \eta' \rangle \leq 0$. Equality is excluded for it would imply $s' = 0'$ and $\varphi_S(\eta) = \langle s'', \xi_0'' \rangle = \langle (0', s''), \xi_0 \rangle = \langle s, \xi_0 \rangle \leq \varphi_S(\xi_0)$, contradicting the choice of η. Hence, $\langle s', \eta' \rangle < 0$, and the estimate (5.5) implies that $H_S * \psi - H_S$ is unbounded, contradicting **(vi)**. □

6 Monge-Ampère masses

The equilibrium measure for a bounded non-pluripolar set $E \subset \mathbb{C}^n$ is the Monge-Ampère operator of V_E, defined as $\mu_E = (dd^c V_E^*)^n$ where $d^c = i(\overline{\partial} - \partial)$ and $(dd^c V_E^*)^n = dd^c V_E^* \wedge \cdots \wedge dd^c V_E^*$ is defined in terms of currents. Similarly denote the Monge-Ampère measure of $V_{E,q}^{S*}$ by

$$\mu_{E,q}^S = \left(dd^c V_{E,q}^{S*}\right)^n.$$

Theorem 6.1. *Let $S \subset \mathbb{R}_+^n$ be compact and convex with $0 \in S$, $E \subset \mathbb{C}^n$ be closed, and q be an admissible weight on E. Then*

(i) $\operatorname{supp} \mu_{E,q}^S \subseteq \{z \in E\,;\, V_{E,q}^{S*}(z) \geq q(z)\}$, *and*

(ii) $\{z \in E\,;\, V_{E,q}^{S*}(z) > q(z)\}$ *is pluripolar.*

Proof. **(i)** We need to prove that $V_{E,q}^{S*}$ is maximal in $U = (\mathbb{C}^n \setminus E) \cup \{z \in E\,;\, V_{E,q}^{S*}(z) < q(z)\}$, which is open. Take $a \in U$. If $a \in \mathbb{C}^n \setminus E$ then we take $r > 0$ such that $B(a,r) \in \mathbb{C}^n \setminus E$. If on the other hand $a \in E$ and $V_{E,q}^{S*}(a) < q(a)$ then by upper semicontinuity of $V_{E,q}^{S*}$ and lower semicontinuity of q there exists $r > 0$ such that

$$\sup_{\zeta \in B(a,r)} V_{E,q}^{S*}(\zeta) < \inf_{\zeta \in E \cap B(a,r)} q(\zeta).$$

We need to prove that the restriction of $\mu_{E,q}^S$ to some $B(a,r)$ is the zero measure. We let $V \in \mathcal{PSH}(\mathbb{C}^n)$ be the function given by

$$V(z) = \begin{cases} V_{E,q}^{S*}(z), & z \in \mathbb{C}^n \setminus B(a,r), \\ u(z), & z \in B(a,r), \end{cases}$$

where u is the Perron-Bremermann function on $B(a,r)$ with boundary values $V_{E,q}^{S*}$, i.e.,

$$u(z) = \sup\{v(z)\,;\, v \in \mathcal{PSH}(B(a,r)),\, v^* \leq V_{E,q}^{S*} \text{ on } \partial B(a,r)\}.$$

Then $V_{E,q}^{S*} \leq V$ and since V only deviates from $V_{E,q}^{S*}$ on a compact set we have $V \in \mathcal{L}^S(\mathbb{C}^n)$ by Proposition 4.6. Furthermore, in the case $V_{E,q}^{S*}(a) < q(a)$ the maximum principle implies

$$V(z) \leq \sup_{\zeta \in B(a,r)} V_{E,q}^{S*}(\zeta) < \inf_{\zeta \in B(a,r)} q(\zeta) \leq q(z), \quad z \in B(a,r),$$

and it implies that $V \leq q$ on E. Hence, $V = V_{E,q}^{S*}$ and, by Klimek [19, Theorem 4.4.1], $\mu_{E,q}^S = \big(dd^c V_{E,q}^{S*}\big)^n = 0$ on $B(a,r)$.

(ii) Since $V_{E,q}^S \leq q$ on E we have $\{z \in E\,;\, q(z) < +\infty\} \subseteq \{z \in \mathbb{C}^n\,;\, V_{E,q}^S < +\infty\}$ Since q is admissible, the left-hand side is non-pluripolar, and then so is the right-hand side. Since $\mathcal{L}^S(\mathbb{C}^n)/\varphi_S(\mathbf{1}) \subset \mathcal{L}(\mathbb{C}^n)$ it follows from Klimek [19, Proposition 5.2.1 and Theorem 5.2.4 and 4.7.6] the set $\{z \in E\,;\, q < V_{E,q}^{S*}(z)\} \subseteq \{z \in \mathbb{C}^n\,;\, V_{E,q}^S(z) < V_{E,q}^{S*}(z)\}$ is pluripolar. □

The complex Monge-Ampère mass of H_S can be described in terms of the real Monge-Ampère mass of φ_S. Let $U_\ell = H_S * \psi_{\delta_\ell} \searrow H_S$, where $0 < \delta_\ell \searrow 0$ and $\psi_{\delta_\ell}(\zeta) = \delta_\ell^{-2n}\psi(\zeta/\delta_\ell)$, $\psi(\zeta) = \chi(\zeta_1)\cdots\chi(\zeta_n)$, where $0 \leq \chi \in \mathcal{C}_0^\infty(\mathbb{C})$ is rotationally invariant, $\operatorname{supp}\chi \subset \overline{\mathbb{D}}$, and $\int_{\mathbb{C}} \chi\, d\lambda = 1$. Then $U_\ell \in \mathcal{C}^\infty(\mathbb{C}^n) \cap \mathcal{PSH}(\mathbb{C}^n)$ and $U_\ell(z) = U_\ell(|z_1|, \dots, |z_n|)$ holds. We set $v_\ell(\xi) = U_\ell(e^{\xi_1}, \dots, e^{\xi_n})$. Since

$$\frac{\partial^2 U_\ell(z)}{\partial z_j \partial \overline{z}_k} = \frac{1}{4 z_j \overline{z}_k} \cdot \frac{\partial^2 v_\ell(\xi)}{\partial \xi_j \partial \xi_k}, \quad z \in \mathbb{C}^{*n},$$

it follows that

$$\det\left(\frac{\partial^2 U_\ell(z)}{\partial z_j \partial \overline{z}_k}\right) = \frac{1}{4^n |z_1 \cdots z_n|^2} \det\left(\frac{\partial^2 v_\ell(\xi)}{\partial \xi_j \partial \xi_k}\right)\Bigg|_{\xi = \operatorname{Log} z}.$$

With $z_j = e^{\xi_j + i\theta_j}$, $\theta_j \in [0, 2\pi]$ we have

$$\left(\tfrac{i}{2}\right)^n dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n = |z_1 \cdots z_n|^2 d\xi d\theta$$

and on $\mathbb{C}^{*n}$ we have

$$(dd^c U_\ell)^n = n! \det\left(\frac{\partial^2 v_\ell(\xi)}{\partial \xi_j \partial \xi_k}\right) d\xi d\theta.$$

The real Monge-Ampère measure of v, denoted $\mathcal{MA}_{\mathbb{R}}(v)$, is defined for $\mathcal{C}^2$ function by

$$\mathcal{MA}_{\mathbb{R}}(v) = \det\left(\frac{\partial^2 v(\xi)}{\partial \xi_j \partial \xi_k}\right) d\xi,$$

and extended to convex functions by locally uniform limits. This is done in Figalli [14, Proposition 2.6].

Letting $\ell \to \infty$, we have, for every Borel set $E \subset \mathbb{R}^n$,

$$\begin{aligned} \int_{\operatorname{Log}^{-1}(E)} (dd^c H_S)^n &= n! \big(\mathcal{MA}_{\mathbb{R}}(\varphi_S) \otimes d\theta\big)(E \times [0, 2\pi]^n) \\ &= (2\pi)^n n!\, \mathcal{MA}_{\mathbb{R}}(\varphi_S)(E). \end{aligned}$$

In particular

$$\int_{\mathbb{T}^n} (dd^c H_S)^n = (2\pi)^n n!\, \mathcal{MA}_{\mathbb{R}}(\varphi_S)(\{0\}). \tag{6.1}$$

This will be useful for our next result. Its proof is borrowed from Rashkovskii [27, Theorem 3.4].

Theorem 6.2. *Let $S \subset \mathbb{R}^n_+$ be compact and convex with $0 \in S$, $E \subset \mathbb{C}^n$ be closed, and q be an admissible weight on E. Then*

$$\mu^S_{E,q}(\mathbb{C}^n) = (2\pi)^n n! \operatorname{vol}(S),$$

where vol *denotes the euclidean volume.*

Proof. By Proposition 4.5, $V^S_{K,q} - H_S$ is bounded. The comparison principle, Klimek [19, Theorem 3.7.1]. then implies that

$$\mu^S_{K,q}(\mathbb{C}^n) = \int_{\mathbb{C}^n} (dd^c H_S)^n = \mu^S_{\mathbb{T}^n,0}(\mathbb{C}^n).$$

By Theorem 6.1 **(i)**, $\mu^S_{\mathbb{T}^n,0}(\mathbb{C}^n) = \int_{\mathbb{T}^n} (dd^c H_S)^n$. We already established in (6.1) that

$$\int_{\mathbb{T}^n} (dd^c H_S)^n = (2\pi)^n n! \mathcal{MA}_{\mathbb{R}}(\varphi_S)(\{0\})$$

By Blocki [9], see also Figalli [14], we have

$$\mathcal{MA}_{\mathbb{R}}(\varphi_S)(\{0\}) = \operatorname{vol}(\{s \in \mathbb{R}^n \,;\, \langle s, \xi\rangle \le \varphi_S(\xi),\ \forall \xi \in \mathbb{R}^n\}) = \operatorname{vol}(S).$$

□

7 Characterization of polynomials by L^2-estimates

In this section we study characterization of the polynomial spaces $\mathcal{P}^S_m(\mathbb{C}^n)$ with weighted L^2-norms of entire functions. Recall that the Liouville type Theorem 3.6 states that if $f \in \mathcal{O}(\mathbb{C}^n)$ and $|f|e^{-mH_S - a\log^+\|\cdot\|_\infty} \in L^\infty(\mathbb{C}^n)$ for some $a \in [0, d_m[$, where d_m is the distance between mS and $\mathbb{N}^n \setminus mS$ in the L^1-norm, then $f \in \mathcal{P}^S_m(\mathbb{C}^n)$. We take a measurable function $\psi \colon \mathbb{C}^n \to \overline{\mathbb{R}}$ and let $L^2(\mathbb{C}^n, \psi)$ denote the space of all measurable $f \colon \mathbb{C}^n \to \mathbb{C}$ such that

$$\|f\|^2_\psi = \int_{\mathbb{C}^n} |f|^2 e^{-\psi}\, d\lambda < +\infty. \tag{7.1}$$

Proposition 7.1. *Let $f \in L^2(\mathbb{C}^n, \psi) \cap \mathcal{O}(\mathbb{C}^n)$ for some measurable $\psi \colon \mathbb{C}^n \to \overline{\mathbb{R}}$ with power series expansion $f(z) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha$ at the origin. Then for every polyannulus $A_{\sigma,\tau} = \{\zeta \in \mathbb{C}^n \,;\, e^{\sigma_j} \le |\zeta_j| < e^{\tau_j}\}$ in $\mathbb{C}^n$, where $\sigma, \tau \in \mathbb{R}^n$, $\sigma_j < \tau_j$ for $j = 1, \dots, n$, with volume $v(A_{\sigma,\tau}) = \pi^n \prod_{j=1}^n (e^{2\tau_j} - e^{2\sigma_j})$, we have*

$$|a_\alpha| \le \frac{\|f\|_\psi}{v(A_{\sigma,\tau})} \left(\int_{A_{\sigma,\tau}} \frac{e^{\psi(\zeta)}}{|\zeta^\alpha|^2}\, d\lambda(\zeta) \right)^{1/2}. \tag{7.2}$$

Furthermore, if ψ is rotationally invariant in each variable ζ_j, then for $\chi(\xi) = \frac{1}{2}\psi(e^{\xi_1}, \dots, e^{\xi_n})$ and $K_{\sigma,\tau} = \prod_{j=1}^n [\sigma_j, \tau_j] \subset \mathbb{R}^n$ we have

$$|a_\alpha| \le \frac{\|f\|_\psi}{\prod_{j=1}^n (1 - e^{-2(\tau_j - \sigma_j)})} e^{-\langle \mathbf{1}, \tau\rangle} \left(\int_{K_{\sigma,\tau}} e^{2(\chi(\xi) - \langle \alpha, \xi\rangle)}\, d\lambda(\xi) \right)^{1/2}. \tag{7.3}$$

Proof. By the Cauchy formula for derivatives

$$\begin{aligned} a_\alpha &= \frac{1}{(2\pi i)^n} \int_{C_r} \frac{f(\zeta)}{\zeta^\alpha} \cdot \frac{d\zeta_1 \cdots d\zeta_n}{\zeta_1 \cdots \zeta_n} \\ &= \frac{1}{(2\pi)^n} \int_{[-\pi,\pi]^n} \frac{f(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})}{r^\alpha e^{i\langle \alpha, \theta \rangle}}\, d\theta_1 \cdots d\theta_n, \end{aligned}$$

where $C_r = \{z \in \mathbb{C}^n \,;\, |z_j| = r_j\}$, is any polycircle with center 0 and polyradius $r \in \mathbb{R}_+^{*n}$. We parametrize C_r by $[-\pi, \pi]^n \ni \theta \mapsto (r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})$, multiply the integral by $r_1 \cdots r_n \, dr_1 \cdots dr_n$, integrate with respect to r_j over $[e^{\sigma_j}, e^{\tau_j}]$, note that $\int_{[e^{\sigma_j}, e^{\tau_j}]} r_j \, dr_j = \frac{1}{2}(e^{2\tau_j} - e^{2\sigma_j})$, set $L_{\sigma,\tau} = \prod_{j=1}^n \big([e^{\sigma_j}, e^{\tau_j}] \times [-\pi, \pi]\big)$, and get

$$\begin{aligned} a_\alpha &= \frac{1}{v(A_{\sigma,\tau})} \int_{L_{\sigma,\tau}} \frac{f(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})}{r^\alpha e^{i\langle \alpha, \theta \rangle}} (r_1 \, dr_1 d\theta_1) \cdots (r_n \, dr_n d\theta_n) \\ &= \frac{1}{v(A_{\sigma,\tau})} \int_{A_{\sigma,\tau}} \frac{f(\zeta)}{\zeta^\alpha} \, d\lambda(\zeta) = \frac{1}{v(A_{\sigma,\tau})} \int_{A_{\sigma,\tau}} f(\zeta) e^{-\psi(\zeta)/2} \cdot \frac{e^{\psi(\zeta)/2}}{\zeta^\alpha} \, d\lambda(\zeta). \end{aligned}$$

Now (7.2) follows from the Cauchy-Schwarz inequality and (7.3) by integrating over the angular variables and also using the fact that $v(A_{\sigma,\tau}) = \pi^n \prod_{j=1}^n (e^{2\tau_j} - e^{2\sigma_j})$. □

Theorem 7.2. *Let S be a compact convex subset of $\mathbb{R}_+^n$ with $0 \in S$, $m \in \mathbb{N}^*$, and $d_m = d(mS, \mathbb{N}^n \setminus mS)$ denote the euclidean distance between mS and $\mathbb{N}^n \setminus mS$. If $f \in \mathcal{O}(\mathbb{C}^n)$ and for some $a \in [0, d_m[$*

$$\int_{\mathbb{C}^n} |f|^2 (1 + |\zeta|^2)^{-a} e^{-2mH_S} \, d\lambda < +\infty, \tag{7.4}$$

then $f \in \mathcal{P}_m^{\widehat{S}_\Gamma}(\mathbb{C}^n)$, where $\widehat{S}_\Gamma$ is the Γ-hull of S and $\Gamma = \Gamma_a$ consists of all ξ such that the angle between the vectors $\mathbf{1} = (1, \dots, 1)$ and ξ is $\leq \arccos(-(d_m - a)/\sqrt{n})$.

Observe that the largest possible d_m is 1 and smallest possible a is 0, which implies that the largest possible opening angle of the cone Γ is $\arccos(-1/\sqrt{n})$ in the case $d_m = 1$ and $\Gamma = \mathbb{R}^n \setminus \mathbb{R}_-^{*n}$. If a_0 is the infimum of a such that (7.4) holds, then $\Gamma_{a_0} = \cup_{a > a_0} \Gamma_a$. Therefore f is a polynomial with exponents in $m\widehat{S}_{\Gamma(a_0)} = \cap_{a > a_0} m\widehat{S}_{\Gamma(a)}$.

We are interested in conditions on cones $\Lambda \subseteq \Gamma$ which guarantee that $\widehat{S}_\Lambda = S$:

Corollary 7.3. *The function f in Theorem 7.2 is in $\mathcal{P}_m^S(\mathbb{C}^n)$ in the cases:*

(i) *$S = \widehat{S}_\Lambda$ for some cone Λ contained in $\{\xi \in \mathbb{R}^n \,;\, \langle \mathbf{1}, \xi \rangle \geq 0\}$.*

(ii) *S is a lower set, that is $S = \widehat{S}_{\mathbb{R}_+^n}$.*

(iii) *$(mS) \cap \mathbb{N}^n = (m\widehat{S}_\Gamma) \cap \mathbb{N}^n$.*

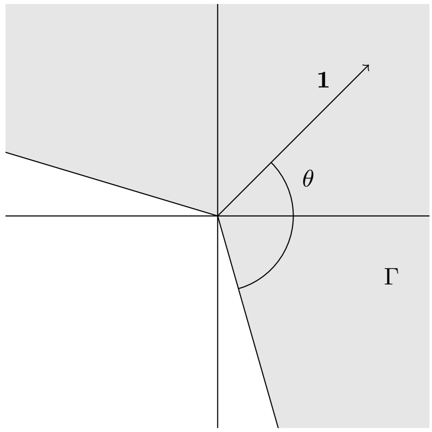

Figure 1: The cone Γ has opening angle $\theta = \arccos(-(d_m - a)/\sqrt{n})$

Proof of Theorem 7.2. Let $f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$ be the Taylor expansion of f at the origin. We need to show that $a_\alpha = 0$ for every $\alpha \in \mathbb{N}^n \setminus m\widehat{S}_\Gamma$. Since $\alpha \notin m\widehat{S}_\Gamma$, there exists $\tau \in \Gamma$ such that $|\tau| = 1$ and $\langle \alpha, \tau \rangle > m\varphi_S(\tau)$. By rotating τ we may assume that τ is an interior point of Γ. Since the angle between τ and $\mathbf{1}$ is $\leq \arccos(-(d_m - a)/\sqrt{n})$ we have $-\langle \mathbf{1}, \tau \rangle < d_m - a$. We choose $\varepsilon > 0$ such that $d_m - a - \varepsilon > 0$, and $-\langle \mathbf{1}, \tau \rangle < d_m - a - \varepsilon$. Recall that $\langle \alpha, \tau \rangle - m\varphi_S(\tau)$ is the euclidean distance from α to the supporting hyperplane $\{x\,;\, \langle x, \tau \rangle = m\varphi_S(\tau)\}$, so by assumption $m\varphi_S(\tau) - \langle \alpha, \tau \rangle \leq -d_m$. Hence

$$-\langle \mathbf{1}, \tau \rangle + m\varphi_S(\tau) - \langle \alpha, \tau \rangle < -a - \varepsilon.$$

We take $\sigma \in \mathbb{R}^n \setminus \{0\}$ with $\sigma_j < \tau_j$ for $j = 1, \dots, n$ such that

$$-\langle \mathbf{1}, \tau \rangle + m\varphi_S(\xi) - \langle \alpha, \xi \rangle < -(a + \varepsilon)|\xi|, \quad \xi \in K_{\sigma,\tau} = \prod_{j=1}^{n} [\sigma_j, \tau_j].$$

By homogeneity we get

$$-t\langle \mathbf{1}, \tau \rangle + m\varphi_S(\xi) - \langle \alpha, \xi \rangle < -(a + \varepsilon)|\xi|, \quad t > 0,\ \xi \in tK_{\sigma,\tau}.$$

Let $\xi_j = \log|\zeta_j|$ and observe that $(1 + |\zeta|^2)^a \leq (n+1)^a \max\{1, \|\zeta\|_\infty^{2a}\}$. From this inequality and (7.4) it follows that $f \in L^2(\mathbb{C}^n, \psi)$, where

$$\tfrac{1}{2}\psi(\zeta) = a \log \|\zeta\|_\infty + mH_S(\zeta) = a\|\xi\|_\infty + m\varphi_S(\xi).$$

We set $\chi(\xi) = \frac{1}{2}\psi(e^{\xi_1}, \dots, e^{\xi_n})$. Then

$$-t\langle \mathbf{1}, \tau \rangle + \chi(\xi) - \langle \alpha, \xi \rangle < -\varepsilon|\xi|, \quad t > 0,\ \xi \in tK_{\sigma,\tau}.$$

The estimate (7.3) with $tK_{\sigma,\tau}$ in the role of $K_{\sigma,\tau}$ gives

$$\begin{aligned}|a_\alpha| &\le \frac{\|f\|_\psi}{\prod_{j=1}^n(1-e^{-2(\tau_j-\sigma_j)t})} e^{-t\langle \mathbf{1},\tau\rangle}\bigg(\int_{tK_{\sigma,\tau}} e^{2(\chi(\xi)-\langle\alpha,\xi\rangle)}\,d\lambda(\xi)\bigg)^{1/2} \\ &\le \frac{\|f\|_\psi}{\prod_{j=1}^n(1-e^{-2(\tau_j-\sigma_j)t})} e^{-\varepsilon|\sigma|t} t^{n/2} v(K_{\sigma,\tau})^{1/2} \to 0, \quad t\to+\infty,\end{aligned}$$

and we conclude that $a_\alpha = 0$. □

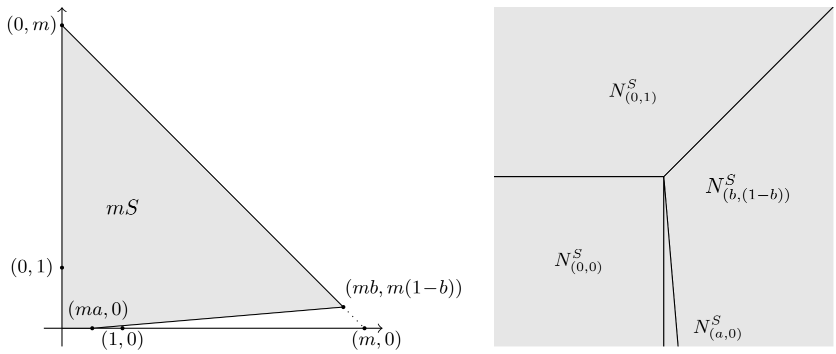

Figure 2: (a) The set S. (b) The normal cones N^S_s of the extreme points of S.

The Γ-hull $\widehat{S}_\Gamma$ can not be replaced by S in Theorem 7.2:

Example 7.4. Let $m \ge 4$ and $S \subseteq \mathbb{R}^2_+$ be the quadrilateral

$$S = \mathrm{ch}\{(0,0),(a,0),(b,1-b),(0,1)\}.$$

where $0 < a < 1/m$, $0 < a < b < 1$, $m(1-b) < 1$, and $(b-a)/(1-b) > m-2-am$. Then $(1,0),(2,0),\dots,(m-3,0) \notin mS$, but the calculations below show that $\|p_k\|_{2mH_S} < +\infty$ for $p_k(z) = z^{(k,0)} = z_1^k$ with $k = 1,\dots,m-3$.

Since the map $(\mathbb{R}\times]-\pi,\pi[)^2 \to \mathbb{C}^2$, $(\xi_1,\theta_1,\xi_2,\theta_2) \mapsto (e^{\xi_1+i\theta_1}, e^{\xi_2+i\theta_2})$ has the Jacobi determinant $e^{2\xi_1+2\xi_2}$, we have

$$\|p_k\|^2_{2mH_S} = \int_{\mathbb{C}^2} |z_1|^{2k} e^{-2mH_S(z)}\,d\lambda(z) = 4\pi^2 \int_{\mathbb{R}^2} e^{2(k+1)\xi_1+2\xi_2-2m\varphi_S(\xi)}\,d\xi_1 d\xi_2,$$

$$\varphi_S(\xi) = \max_{x\in\mathrm{ext}(S)}\langle x,\xi\rangle = \begin{cases} 0, & \xi \in N^S_{(0,0)}, \\ a\xi_1, & \xi\in N^S_{(a,0)}, \\ b\xi_1+(1-b)\xi_2, & \xi\in N^S_{(b,(1-b))}, \\ \xi_2, & \xi\in N^S_{(0,1)}. \end{cases}$$

We split the integral over $\mathbb{R}^2$ into the sum of the integrals over the normal cones at the extreme points of S, which we calculate as

$$\int_{N^S_{(0,0)}} e^{2(k+1)\xi_1+2\xi_2}\, d\xi = \int_{-\infty}^{0} e^{2(k+1)\xi_1}\, d\xi_1 \int_{-\infty}^{0} e^{2\xi_2}\, d\xi_2 = \frac{1}{4(k+1)},$$

$$\int_{N^S_{(a,0)}} e^{2(k+1)\xi_1+2\xi_2-2ma\xi_1}\, d\xi = \int_0^{\infty} e^{2(k+1-ma)\xi_1} \int_{-\infty}^{-\xi_1(b-a)/(1-b)} e^{2\xi_2}\, d\xi_2\, d\xi_1$$
$$= \frac{1}{4((b-a)/(1-b)+ma-1-k)},$$

$$\int_{N^S_{(b,1-b)}} e^{2(k+1)\xi_1+2\xi_2-2m(b\xi_1+(1-b)\xi_2)}\, d\xi$$
$$= \int_0^{\infty} e^{2(k+1-mb)\xi_1} \int_{-\xi_1(b-a)/(1-b)}^{\xi_1} e^{2(1-m(1-b))\xi_2}\, d\xi_2\, d\xi_1$$
$$= \frac{1}{4(1-m(1-b))}\left(\frac{1}{m-2-k} + \frac{1}{(b-a)/(1-b)+ma-1-k}\right),$$

$$\int_{N^S_{(0,1)}} e^{2(k+1)\xi_1+2\xi_2-2m\xi_2}\, d\xi = \int_0^{\infty} e^{2(1-m)\xi_2} \int_{-\infty}^{\xi_2} e^{2(k+1)\xi_1}\, d\xi_1\, d\xi_2$$
$$= \frac{1}{4(k+1)(m-2-k)}.$$

This shows that (7.4) is satisfied with p_k in the role of f, but $p_k \notin \mathcal{P}^S_m(\mathbb{C}^n)$.

8 Pullbacks of polynomial classes by polynomial maps

Let $S \subset \mathbb{R}^n_+$ be compact and convex with $0 \in S$ and $f = (f_1, \dots, f_n)\colon \mathbb{C}^\ell \to \mathbb{C}^n$ be a polynomial map. If $f_j(z) = \sum_{\alpha\in\mathbb{N}^\ell} a_{j,\alpha} z^\alpha$, $I_j = \{\alpha \in \mathbb{N}^\ell\,;\, a_{j,\alpha} \neq 0\}$, and $S_j \subset \mathbb{R}^\ell_+$ be the convex hull of I_j. Then $f_j \in \mathcal{P}^{S_j}_1(\mathbb{C}^\ell)$ and for every $p(w) = \sum_{\beta\in(mS)\cap\mathbb{N}^n} b_\beta w^\beta$ in $\mathcal{P}^S_m(\mathbb{C}^n)$ we have

$$(f^*p)(z) = \sum_{\beta\in(mS)\cap\mathbb{N}^n} b_\beta f_1(z)^{\beta_1} \cdots f_n(z)^{\beta_n}, \quad z \in \mathbb{C}^\ell.$$

By Theorem 3.6 we have $|f_j(z)| \leq e^{c_j + H_{S_j}(z)}$, which implies that

$$|(f^*p)(z)| \leq \sum_{\beta\in(mS)\cap\mathbb{N}^n} |b_\beta| e^{\langle c,\beta\rangle + \beta_1 H_{S_1}(z)+\cdots+\beta_n H_{S_n}(z)}$$
$$\leq \Big(\sum_{\beta\in(mS)\cap\mathbb{N}^n} |b_\beta| e^{\langle c,\beta\rangle}\Big) e^{m\varphi_S(\varphi_{S_1}(\operatorname{Log} z),\dots,\varphi_{S_n}(\operatorname{Log} z))}, \quad z \in \mathbb{C}^{\ell *}.$$

We have for every $\xi \in \mathbb{R}^\ell$ that

$$\begin{aligned}\varphi_S(\varphi_{S_1}(\xi), \dots, \varphi_{S_n}(\xi)) &= \sup\{\langle x_1 s_1 + \cdots + x_n s_n, \xi\rangle; \, x \in S, \, s_j \in S_j\} \\ &= \sup_{x\in S} \varphi_{x_1 S_1 + \cdots + x_n S_n}(\xi) = \varphi_{S'}(\xi),\end{aligned}$$

where $S' = \cup_{x\in S} x_1 S_1 + \cdots + x_n S_n$. The set $S' \subset \mathbb{R}^\ell_+$ is compact and convex.

Proposition 8.1. *Assume that $\overline{S \cap \mathbb{Q}^n} = S$. Then with the notation above S' is the smallest compact convex subset T of $\mathbb{R}^\ell_+$ with $0 \in T$ for which $f^*\big(\mathcal{P}^S_m(\mathbb{C}^n)\big) \subseteq \mathcal{P}^T_m(\mathbb{C}^\ell)$ for all $m \in \mathbb{N}$.*

Proof. Assume that $T \subsetneq S'$ such that $f^*\big(\mathcal{P}^S_m(\mathbb{C}^n)\big) \subseteq \mathcal{P}^T_m(\mathbb{C}^\ell)$ for every $m \in \mathbb{N}$. Then there exists $\xi \in \mathbb{R}^\ell$ such that $\varphi_T(\xi) < \varphi_{S'}(\xi)$ and, since $S \cap \mathbb{Q}^n$ is dense in S, we can choose $r \in S \cap \mathbb{Q}^n$ such that

$$\varphi_T(\xi) < r_1\langle \alpha_1, \xi\rangle + \cdots + r_n\langle \alpha_n, \xi\rangle \le \varphi_{S'}(\xi). \tag{8.1}$$

Now we fix $m \in \mathbb{N}$ such that $\gamma = mr \in \mathbb{N}^n$, define $p \in \mathcal{P}^S_\mu(\mathbb{C}^n)$ by $p(w) = w^\gamma$. By Theorem 3.6, there exists a constant c_p such that

$$|f^*p(z)| \le e^{c_p + mH_T(z)}, \quad z \in \mathbb{C}^\ell. \tag{8.2}$$

Since S_j is a convex polytope in $\mathbb{R}^\ell_+$, the set $\cap_{j=1}^n \cup_{\alpha\in \mathrm{ext} S_j} \mathring{N}^{S_j}_\alpha$ is dense in $\mathbb{R}^\ell$, where $\mathring{N}^{S_j}_\alpha$ is the interior of $N^{S_j}_\alpha$. Hence ξ may be chosen from $\mathring{N}^{S_j}_{\alpha_j}$ where $\alpha_j \in \mathrm{ext} S_j$ and $\varphi_{S_j}(\xi) = \langle \alpha_j, \xi\rangle > \langle \alpha, \xi\rangle$ for every $\alpha \in I_j \setminus \{\alpha_j\}$. All the α_j are from $\mathbb{N}^\ell$ by the definition of S_j. We define the sequence $(\zeta_k)_{k\in\mathbb{N}}$ in $\mathbb{C}^\ell$ by $\zeta_k = (e^{k\xi_1}, \dots, e^{k\xi_\ell})$ and we will show that (8.1) implies that the sequence $\big(f^*p(\zeta_k)e^{-mH_T(\zeta_k)}\big)_{k\in\mathbb{N}}$ is unbounded, contradicting the estimate (8.2). First we observe that $\zeta_k^{\alpha_j} = e^{k\langle \alpha_j, \xi\rangle}$ and then that (8.1) implies

$$(\zeta_k^{\alpha_1}, \dots, \zeta_k^{\alpha_n})^\gamma e^{-mH_T(\zeta_k)} = e^{km(r_1\langle \alpha_1,\xi\rangle + \cdots + r_n\langle \alpha_n, \xi\rangle - \varphi_T(\xi))} \to +\infty, \tag{8.3}$$

as $k \to +\infty$. Next we observe that

$$f_j(\zeta_k)/\zeta_k^{\alpha_j} = a_{j,\alpha_j} + \sum_{\alpha\in I_j\setminus\{\alpha_j\}} a_{j,\alpha} e^{-k(\langle \alpha_j,\xi\rangle - \langle \alpha, \xi\rangle)} \to a_{j,\alpha_j} \neq 0, \tag{8.4}$$

$$f^*p(\zeta_k)/(\zeta_k^{\alpha_1}, \dots, \zeta_k^{\alpha_n})^\gamma = (f_1(\zeta_k)/\zeta_k^{\alpha_1})^{\gamma_1} \cdots (f_n(\zeta_k)/\zeta_k^{\alpha_n})^{\gamma_n}. \tag{8.5}$$

By combining (8.3), (8.4), and (8.5), we see that $\big(f^*p(\zeta_k)e^{-mH_T(\zeta_k)}\big)_{k\in\mathbb{N}}$ is unbounded. □

Assume now that f is a proper map and that q is a given admissible weight function on a compact set $K \subseteq \mathbb{C}^n$. Then f^*q is lower semicontinuous and

$$\{z \in f^{-1}(K)\,;\, f^*q(z) < +\infty\} = f^{-1}\big(\{w \in K\,;\, q(w) < +\infty\}\big).$$

Since inverse images of non-pluripolar sets by proper maps are non-pluripolar it follows that f^*q is an admissible weight on $f^{-1}(K)$. Furthermore, we have

$$\|f^*pe^{-mf^*q}\|_{f^{-1}(K)} = \|pe^{-mq}\|_K.$$

From Proposition 8.1 we conclude that $f^*\big(\Phi^S_{K,q,m}\big) \le \Phi^{S'}_{f^{-1}(K),f^*q,m}$, for every $m \in \mathbb{N}^*$, consequently $f^*\big(\Phi^S_{K,q}\big) \le \Phi^{S'}_{f^{-1}(K),f^*q}$ and equality holds if $f^*\colon \mathcal{P}^S_m(\mathbb{C}^n) \to \mathcal{P}^{S'}_m(\mathbb{C}^\ell)$ is surjective.

Next we look at the pullback of Lelong classes. Let $u \in \mathcal{L}^S(\mathbb{C}^n)$, say $u \le c_u + H_S$. Then for every $z \in \mathbb{C}^\ell$ with $\mathrm{Log} f(z) \in \mathbb{C}^{*n}$ we have

$$\begin{aligned}(f^*u)(z) &\le c_u + H_S(f(z)) = c_u + \varphi_S(\log|f_1(z)|, \dots, \log|f_1(z)|)\\ &\le c_u + \varphi_S(c_1 + \varphi_{S_1}(\mathrm{Log} z), \dots, c_n + \varphi_{S_n}(\mathrm{Log} z))\\ &\le c_u + \varphi_S(c) + \varphi_S(\varphi_{S_1}(\mathrm{Log} z), \dots, \varphi_{S_n}(\mathrm{Log} z))\\ &= c_u + \varphi_S(c) + H_{S'}(z)\end{aligned}$$

and conclude that $f^*u \in \mathcal{L}^{S'}(\mathbb{C}^\ell)$. If $u \le q$ on K, then $f^*u \le f^*q$ on $f^{-1}(K)$, we have $f^*\big(V^S_{K,q}\big) \le V^{S'}_{f^{-1}(K),f^*q}$ and that equality holds if $f^*\colon \mathcal{L}^S(\mathbb{C}^n) \to \mathcal{L}^{S'}(\mathbb{C}^\ell)$ is surjective.

When $\ell = n$ we have the following weighted transformation rule, which is a generalization of Klimek [19, Theorem 5.3.1] and Perera [25, Theorem 1]:

Proposition 8.2. *If $f\colon \mathbb{C}^n \to \mathbb{C}^n$ is proper, the following are equivalent:*

- **(i)** *The difference $f^*H_S - H_{S'}$ is bounded, that is $f^*H_S \in \mathcal{L}^{S'}_+(\mathbb{C}^n)$.*
- **(ii)** *For every compact set $K \subset \mathbb{C}^n$ and every admissible weight q on K we have $f^*V^S_{K,q} = V^{S'}_{f^{-1}(K),\, f^*q}$.*

Proof. **(i)⇒(ii):** Let $u \in \mathcal{L}^{S'}(\mathbb{C}^n)$ with $u|_{f^{-1}(K)} \le f^*q$. By Klimek [19, Theorem 2.9.26], the function $v(z) = \max\{u(w)\colon w \in f^{-1}(z)\}$ is plurisubharmonic on $\mathbb{C}^n$. Let c be a constant such that $f^*H_S - H_{S'} \ge -c$. Then $v|_K \le q$ and

$$\begin{aligned}v(z) &\le \max_{w\in f^{-1}(z)} H_{S'}(w) + c_u\\ &\le \max_{w\in f^{-1}(z)} f^*H_S(w) + c + c_u\\ &= H_S(z) + c + c_u.\end{aligned}$$

It follows that $u \le f^*v \le f^*V^S_{K,q}$.

(ii)⇒(i): Let $c = \max_{w\in f^{-1}(\overline{\mathbb{D}}^n)} H_{S'}(w)$ and $c' = \max_{w\in f(\overline{\mathbb{D}}^n)} H_S(w)$. Then

$$H_{S'} - c \le V^{S'}_{f^{-1}(\overline{\mathbb{D}}^n)} = f^*V^S_{\overline{\mathbb{D}}^n} = f^*H_S$$

and

$$H_{S'} = V^{S'}_{\mathbb{D}^n} \ge V^{S'}_{f^{-1}(f(\overline{\mathbb{D}}^n))} = f^*V^S_{f(\overline{\mathbb{D}}^n)} \ge f^*H_S - c'.$$

□

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Paper II

Polynomials with exponents in compact convex sets and associated weighted extremal functions - Approximations and regularity

Bergur Snorrason

Abstract

We study various regularization operators on plurisubharmonic functions that preserve Lelong classes with growth given by certain compact convex sets. The purpose is to show that the weighted Siciak-Zakharyuta functions associated with these Lelong classes are lower semicontinuous.

1 Introduction

A common tool in pluripotential theory is regularization by convolution. If $u \in \mathcal{PSH}(\Omega)$, for some open $\Omega \subset \mathbb{C}^n$, then $u * \chi_\delta \in \mathcal{PSH}(\Omega_\delta) \cap \mathcal{C}^\infty(\Omega_\delta)$ for $\delta > 0$, where

$$u * \chi_\delta(z) = \int_{\mathbb{C}^n} u(z-w)\chi_\delta(w)\, d\lambda(w), \quad z \in \Omega_\delta.$$

Here $\Omega_\delta = \{z \in \Omega\,; B(z,\delta) \subset \Omega\}$, where $B(z,\delta)$ is the open euclidean ball with center z and radius δ, χ_δ is a standard smoothing kernel in $\mathbb{C}^n$ with support in the closed ball $\overline{B}(0,\delta)$, and $\mathcal{C}^\infty(\Omega_\delta)$ is the family of smooth function on Ω_δ. We also have that $u * \chi_\delta \searrow u$ as $\delta \searrow 0$. In the specific setting of $\Omega = \mathbb{C}^n$ we have that $\Omega_\delta = \mathbb{C}^n$, so every function in $\mathcal{PSH}(\mathbb{C}^n)$ can approximated by a decreasing sequence in $\mathcal{PSH}(\mathbb{C}^n) \cap \mathcal{C}^\infty(\mathbb{C}^n)$. For details on smoothing by convolution see Theorem 2.9.2 in Klimek [9].

In some cases it is necessary to choose a method for regularizing plurisubharmonic functions that preserves a certain subclass of $\mathcal{PSH}(\mathbb{C}^n)$. One such case is the Lelong class $\mathcal{L}(\mathbb{C}^n)$. We say that $u \in \mathcal{PSH}(\mathbb{C}^n)$ belongs to the Lelong class, denoted by $\mathcal{L}(\mathbb{C}^n)$, if $u(z) \leq \log^+ \|z\|_\infty + c_u$ for some constant c_u, where $\|\cdot\|_\infty$ is the supremum norm in $\mathbb{C}^n$ and $\log^+ x = \max\{0, \log x\}$. A powerful tool in the study of pluripotential theory is the Siciak-Zakharyuta function of $E \subset \mathbb{C}^n$, defined by

$$V_E(z) = \sup\{u(z)\,; u \in \mathcal{L}(\mathbb{C}^n), u|_E \leq 0\}, \quad z \in \mathbb{C}^n.$$

One can show that smoothing with a standard smoothing kernel preserves the Lelong class, namely if $u \in \mathcal{L}(\mathbb{C}^n)$ then $u * \chi_\delta \in \mathcal{L}(\mathbb{C}^n)$. This is then used to show that V_K is lower semicontinuous for compact $K \subset \mathbb{C}^n$. See Section 5.1 in [9].

In the recent study of pluripotential theory related to convex sets the Lelong classes are generalized by describing the growth more freely. We fix a compact convex set $0 \in S \subset \mathbb{R}^n_+$ and recall that the *supporting function of* S, given by $\varphi_S(\xi) = \sup_{x \in S}\langle x, \xi\rangle$, is positively homogeneous and convex, which is equivalent to $\varphi_S(t\xi) = t\varphi_S(\xi)$ and $\varphi_S(\xi+\eta) \leq \varphi_S(\xi) + \varphi_S(\eta)$, for $t \in \mathbb{R}_+$ and $\xi, \eta \in \mathbb{R}^n$. We define the *logarithmic supporting function of* S by

$$H_S(z) = \begin{cases} \varphi_S(\log|z_1|, \dots, \log|z_n|), & z \in \mathbb{C}^{*n}, \\ \varlimsup\limits_{\mathbb{C}^{*n} \ni w \to z} H_S(w), & z \in \mathbb{C}^n \setminus \mathbb{C}^{*n}, \end{cases}$$

where $\mathbb{C}^{*n} = (\mathbb{C}^*)^n$ and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. This function is continuous and plurisubharmonic on $\mathbb{C}^n$ and its behavior on the coordinate hyperplanes $\mathbb{C}^n \setminus \mathbb{C}^{*n}$ can be described using logarithmic supporting functions of sets of lower dimensions. See Propositions 3.3 and 3.4 in [11]. We define the *Lelong class given by* S as the set of all $u \in \mathcal{PSH}(\mathbb{C}^n)$ satisfying $u \leq H_S + c_u$, for some constant c_u, and we denote it by $\mathcal{L}^S(\mathbb{C}^n)$. Similarly, we define $\mathcal{L}^S_+(\mathbb{C}^n)$ as the set of all functions $u \in \mathcal{L}^S(\mathbb{C}^n)$ satisfying $H_S + c_u \leq u$, for some constant c_u. This leads to our definition of the *Siciak-Zakharyuta function of* $E \subset \mathbb{C}^n$ *and* S *with weight* $q\colon E \to \mathbb{R} \cup \{+\infty\}$,

$$V^S_{E,q}(z) = \sup\{u(z)\,;\, u \in \mathcal{L}^S(\mathbb{C}^n), u|_E \leq q\}, \quad z \in \mathbb{C}^n.$$

We also write $V^S_K = V^S_{K,0}$, $V_{K,q} = V^{\Sigma}_{K,q}$, and $V_K = V^{\Sigma}_{K,0}$, where Σ is the standard simplex in $\mathbb{R}^n$ given by $\Sigma = \operatorname{ch}\{0, e_1, \dots, e_n\}$, where $\operatorname{ch} A$ denotes the closed convex hull of the set A.

We say that the weight q is *admissible* if q is lower semicontinuous, the set $\{z \in E\,;\, q(z) < +\infty\}$ is non-pluripolar, and $\lim\limits_{E \ni z, |z| \to \infty} (H_S(z) - q(z)) = -\infty$, if E is unbounded.

Fundamental results of the Siciak-Zakharyuta function are developed in [11]. There it is proven that $V^S_{K,q}$ is lower semicontinuous on $\mathbb{C}^{*n}$, but it is not proven that it is lower semicontinuous on all of $\mathbb{C}^n$. The obstruction to proving semi continuity on $\mathbb{C}^n$ is a suitable method of approximating functions in $\mathcal{L}^S(\mathbb{C}^n)$ by functions in $\mathcal{L}^S(\mathbb{C}^n) \cap \mathcal{C}(\mathbb{C}^n)$, where $\mathcal{C}(\mathbb{C}^n)$ denotes all continuous function on $\mathbb{C}^n$, since smoothing by convolution with a standard smoothing kernel does not always preserve $\mathcal{L}^S(\mathbb{C}^n)$. In fact, $\mathcal{L}^S(\mathbb{C}^n)$ is preserved by the standard convolution operator if and only if S is a *lower set*, that is when S is such that for all $x \in S$ we have that $[0, x_1] \times \cdots \times [0, x_n] \subset S$. See Theorem 5.8 in [11]. To remedy this we consider other regularization operators.

In Section 2 we consider a generalization of an operator used by Siciak [15, Proposition 1.3].

Theorem 1.1. *Let* $S \subset \mathbb{R}^n_+$ *be a lower set,* $u \in \mathcal{L}^S(\mathbb{C}^n)$ *be bounded below,* μ *be a distance function on* $\mathbb{C}^n$*,* $u \leq H_S + c_u$ *for some constant* c_u*, and* $\delta \in]0, \sigma_S^{-1} e^{c_u} r_\mu[$*, where* $\sigma_S = \varphi_S(1, \dots, 1)$ *and* $r_\mu = \inf_{|z|=1} \mu(z) > 0$*. Then*

$$R^a_{\mu,\delta} u(z) = -\log \inf_{w \in \mathbb{C}^n} \{e^{-u(w)} + \delta^{-1}\mu(z - w)\}, \quad z \in \mathbb{C}^n, \tag{1.1}$$

is in $\mathcal{L}^S(\mathbb{C}^n) \cap \mathcal{C}(\mathbb{C}^n)$ *and* $R^a_{\mu,\delta} u \searrow u$ *as* $\delta \searrow 0$.

With $\mu = |\cdot|$, Siciak proved that $R^a_{\mu,\delta} u \in \mathcal{L}(\mathbb{C}^n) \cap \mathcal{C}(\mathbb{C}^n)$ and that $R^a_{\mu,\delta} u \searrow u$ as $\delta \searrow 0$. In [1] this process is referred to as *Ferrier approximation*, since Siciak developed it using a result from Ferrier [2]. Here it is attributed to Siciak, as Ferrier was not concerned with approximations. It is claimed in [1, Proposition 3.1] that if $\mu = |\cdot|$, $u \in \mathcal{L}^S(\mathbb{C}^n)$, and S is such that it contains a neighborhood of 0 in $\mathbb{R}^n_+$, then $R^a_{\mu,\delta} u \in \mathcal{L}^S(\mathbb{C}^n) \cap \mathcal{C}(\mathbb{C}^n)$. The following example shows that this claim is false.

Example 1.2. Let $S = \operatorname{ch}\{(0,0),(a,0),(0,a),(b,a)\} = \operatorname{ch}(a\Sigma \cup \{(b,a)\})$, $a > 0$, $b > a(a+1)$, and $\mu = |\cdot|$. Then

$$H_S(z_1,z_2) = \max\{b \log|z_1| + a\log|z_2|, a\log^+\|z\|_\infty\}, \quad z_1, z_2 \in \mathbb{C}^*.$$

So, with $r = b/(a+1) > a$, we have that

$$H_S(\zeta, |\zeta|^{-r}) \geq \log|\zeta|^{b-ra}, \quad \zeta \in \mathbb{C}^*.$$

Setting $w = (\zeta, |\zeta|^{-r})$ in (1.1) yields

$$\begin{aligned} R^a_{\mu,\delta} H_S(\zeta,0) &\geq -\log(e^{-H_S(\zeta,|\zeta|^{-r})} + \delta^{-1}|\zeta|^{-r}) \\ &\geq -\log(|\zeta|^{ar-b} + \delta^{-1}|\zeta|^{-r}) \\ &= -\log(|\zeta|^{r(a+1)-b} + \delta^{-1}) + r\log|\zeta| \\ &= -\log(1+\delta^{-1}) + r\log|\zeta|. \end{aligned}$$

So $R^a_{\mu,\delta} H_S(\zeta,0) - H_S(\zeta,0) \geq (r-a)\log|\zeta| - \log(1+\delta^{-1})$ is not bounded above, since $r - a > 0$, and hence $R^a_{\mu,\delta} H_S$ is not in $\mathcal{L}^S(\mathbb{C}^n)$.

The infimum in the operator $R^a_{\mu,\delta}$ is an *infimal convolution* of the functions e^{-u} and $\delta^{-1}|\cdot|$. Details on infimal convolutions and its application in convexity theory can be found in Rockafellar [14]. Applications in complex analysis can be found in Kiselman [8] and Halvarsson [3, 4, 5].

In Section 3 we consider the following *supremal convolution*.

Theorem 1.3. *Let* $0 \in S \subset \mathbb{R}^n_+$ *be compact convex,* $1_n = (1,\dots,1) \in \mathbb{N}^n$, $\sigma_S = \varphi_S(1_n)$, $\delta \in]0, \sigma_S^{-1}[$, $u \in \mathcal{L}^S(\mathbb{C}^n)$ *be locally bounded, and set*

$$R^b_\delta u(z) = \sup_{w \in \mathbb{C}^n} \{u(Zw) - \delta^{-1}\log(\|w - 1_n\|_\infty + 1)\}, \quad z \in \mathbb{C}^n,$$

where $Zw = (z_1w_1, \dots, z_nw_n)$. *Then* $R^b_\delta u \in \mathcal{L}^S(\mathbb{C}^n)$, $R^b_\delta u$ *is continuous on* $\mathbb{C}^{*n}$, *and* $R^b_\delta u \searrow u$ *as* $\delta \searrow 0$.

The meaning of Zw is justified in Section 3. We note that σ_S may be 0. This only happens when $S = \{0\}$, which is a case of no interest, since then $\mathcal{L}^S(\mathbb{C}^n)$ would only contain the constant functions.

In Section 4 we consider two related operators. The first one was developed in Section 5 in [11].

Theorem 1.4. *Let $0 \in S \subset \mathbb{R}^n_+$ be compact convex, $\delta \in]0,1[$, $u \in \mathcal{L}^S(\mathbb{C}^n)$, and set*

$$R^c_\delta u(z) = \int_{\mathbb{C}^n} u(Zw)\psi_\delta(w)\, d\lambda(w), \quad z \in \mathbb{C}^n, \tag{1.2}$$

where $\psi_\delta(w) = \delta^{-2n}\psi((w - 1_n)/\delta)$ and ψ is a smoothing kernel that is rotationally invariant in each variable. Then $R^c_\delta u \in \mathcal{L}^S(\mathbb{C}^n)$, $R^c_\delta u$ is smooth on $\mathbb{C}^{*n}$, and $R^c_\delta u \searrow u$ as $\delta \searrow 0$. If u is locally bounded below and $\lim_{z\to a} u_z = u_a$, in $\mathcal{PSH}(\mathbb{C}^n)$, for all $a \in \mathbb{C}^n$, where u_z is given by $u_z(w) = u(Zw)$, for $w \in \mathbb{C}^n$, then $R^c_\delta u \in \mathcal{C}(\mathbb{C}^n)$.

The assumption that u is locally bounded below ensure that $R^c_\delta u > -\infty$. Taking $u(z) = \log|z_1|$ we have $R^c_\delta u(z) = -\infty$, for $z \in \{0\} \times \mathbb{C}^{n-1}$. So $R^c_\delta u$ fails to be continuous.

A possible approach to showing continuity on all of $\mathbb{C}^n$ is to regularize e^u instead of u. The second operator in Section 4 is given by $\log R^c_\delta e^u$. It must be shown that this function is plurisubharmonic.

Theorem 1.5. *Let $0 \in S \subset \mathbb{R}^n_+$ be compact convex, $\delta \in]0,1[$, $u \in \mathcal{L}^S(\mathbb{C}^n)$, and set*

$$R^d_\delta u(z) = \log \int_{\mathbb{C}^n} e^{u(Zw)}\psi_\delta(w)\, d\lambda(w), \quad z \in \mathbb{C}^n. \tag{1.3}$$

Then $R^d_\delta u \in \mathcal{L}^S(\mathbb{C}^n)$, $R^d_\delta u$ is smooth on $\mathbb{C}^{*n}$, and $R^d_\delta u \searrow u$ as $\delta \searrow 0$.

In a recent paper of Perera [12] a classical results on the regularity of V_K is considered, and attempts are made to generalize it to the setting of $V^S_{K,q}$. Perera [12] claims to give sufficient condition for $V^S_{K,q}$ to be Hölder continuous on $\mathbb{C}^n$, with the only assumption on S that it is a convex body. In Section 5 we show that this can not hold without additional restrictions to S. The main result, Corollary 5.2, is that if S is not a lower set then $\mathcal{L}^S_+(\mathbb{C}^n)$ contains no uniformly continuous functions. So $V^S_{K,q}$ can not be Hölder continuous if S is not a lower set, since all Hölder continuous functions are uniformly continuous and $V^{S*}_{K,q} \in \mathcal{L}^S_+(\mathbb{C}^n)$.

The main motivation for the study of these regularizations is proving that $V^S_{K,q}$ is lower semicontinuous on $\mathbb{C}^n$, not just $\mathbb{C}^{*n}$, for compact $K \subset \mathbb{C}^n$ and admissible q. We fix an open set $\Omega \in \mathbb{C}^n$ and note that, if for all $u \in \mathcal{L}^S(\mathbb{C}^n)$ there exists a decreasing sequence $(u_j)_{j\in\mathbb{N}}$ in $\mathcal{L}^S(\mathbb{C}^n)$, with $u_j|_\Omega$ continuous, such that $u_j \searrow u$, then $V^S_{K,q}$ is lower semicontinuous on Ω. If we take $\varepsilon > 0$ and assume that $u \le q$ then we can find $j_z \in \mathbb{N}$ for all $z \in K$ such that $u_j(z) < q(z) + \varepsilon$ for all $j > j_z$. Since $u_j - q$ is upper semicontinuous there exist open sets U_z for all $z \in K$ such that $u_j < q + \varepsilon$ in U_z for $j > j_z$. The open sets U_z are a covering of K so we can find $z_1, \dots, z_\ell$ such that $K \subset U_{z_1} \cup \cdots \cup U_{z_\ell}$. Setting $k = \max\{j_{z_1}, \dots, j_{z_\ell}\}$ we get that $u_j - \varepsilon < q$ on K, for $j > k$. So $V^S_{K,q}$ is given as the supremum over a family of functions continuous on Ω, and is therefore lower semicontinuous on Ω.

In this paper we consider regularizations that work for $\Omega = \mathbb{C}^{*n}$. Whether these, or any other, regularizations work on all of $\mathbb{C}^n$ remains an open question.

2 Infimal convolutions

Recall that Siciak's infimal convolution was given by

$$R^a_{\mu,\delta}u(z) = -\log \inf_{w\in\mathbb{C}^n}\{e^{-u(w)} + \delta^{-1}\mu(z-w)\}, \quad z \in \mathbb{C}^n,$$

with $\mu = |\cdot|$. In Ferrier [2] it is proven that $R^a_{\mu,\delta}u$ is plurisubharmonic and continuous when u is plurisubharmonic, $\delta = 1$, and $\mu = |\cdot|$. Generalizing for $\delta > 0$ is not significant. In Siciak [15] it is shown that if

$$\log(|z|+1) - c_u \leq u(z) \leq \log(|z|+1) + c_u, \quad z \in \mathbb{C}^n,$$

for some constant c_u, then $R^a_{\mu,\delta}u(z) \leq \log(|z|+1) + c_u$ for $\delta < e^{c_u}$, and consequently that $R^a_{\mu,\delta}u \in \mathcal{L}(\mathbb{C}^n)$. He also shows that $R^a_{\mu,\delta}u \searrow u$ when $\delta \searrow 0$. The fact that $R^a_{\mu,\delta}u$ is plurisubharmonic is not obvious, but follows from geometric arguments. We will recall some useful results on pseudoconvex domains before going through the proof.

Let $u \in \mathcal{PSH}(\mathbb{C}^n)$. Recall that $\Omega = \{z \in \mathbb{C}^n\,; u(z) < c\}$ is pseudoconvex. To see this take compact $K \subset \Omega$ and note that $u \leq c - \delta$ on K, for some $\delta > 0$, so $u \leq c - \delta$ on $\widehat{K}_{\mathcal{PSH}(\mathbb{C}^n)}$, which is compact. So we have that $\widehat{K}_{\mathcal{PSH}(\Omega)} \subset \widehat{K}_{\mathcal{PSH}(\mathbb{C}^n)} \subset \Omega$, which implies that $\widehat{K}_{\mathcal{PSH}(\Omega)}$ is relatively compact in Ω, so Ω is pseudoconvex.

We call a continuous function $\mu\colon \mathbb{C}^n \to \mathbb{R}_+$ a *distance function* if **(i)** $\mu(z) = 0$ if and only if $z = 0$ and **(ii)** $\mu(tz) = |t|\mu(z)$ for $z \in \mathbb{C}^n$ and $t \in \mathbb{C}$. Note that by **(ii)** we have constants $r_\mu = \inf_{|z|=1}\mu(z)$ and $s_\mu = \sup_{|z|=1}\mu(z)$ such that $r_\mu|z| \leq \mu(z) \leq s_\mu|z|$, for $z \in \mathbb{C}^n$, and by **(i)** we have that $r_\mu > 0$. For an open $\Omega \subset \mathbb{C}^n$ we define the μ-distance to the boundary by

$$\mu_\Omega(z) = \inf_{w\notin\Omega}\mu(z-w), \quad z \in \Omega,$$

and note that it is a continuous function. Pseudoconvex domains are classically characterized by distances, namely an open $\Omega \subset \mathbb{C}^n$ is pseudoconvex if and only if $-\log\mu_\Omega$ is plurisubharmonic for every (or some) distance function μ. See Theorem 2.10.4 in Klimek [9].

Proof of Theorem 1.1. To start off we show that $R^a_{\mu,\delta}u$ is continuous and plurisubharmonic. We let

$$\Omega = \{(z,a) \in \mathbb{C}^n \times \mathbb{C}\,; |a| < e^{-u(z)}\}$$

and note that Ω is a sublevel set of $(z,a) \mapsto u(z) + \log|a|$, so it is pseudoconvex. We define a distance function $\widehat{\mu}$ on $\mathbb{C}^{n+1}$ by $\widehat{\mu}(z,a) = |a| + \delta^{-1}\mu(z)$, and see that, for $(z,a) \in \mathbb{C}^n \times \mathbb{C}$,

$$\widehat{\mu}_\Omega(z,a) = \inf\{|a-b| + \delta^{-1}\mu(z-w)\,; (w,b) \in \mathbb{C}^n \times \mathbb{C}, |b| \geq e^{-u(w)}\}.$$

By Theorem 2.10.4 in [9] we have that $-\log\widehat{\mu}_\Omega$ is continuous and plurisubharmonic on $\Omega \supset \mathbb{C}^n \times \{0\}$, so $\mathbb{C}^n \ni z \mapsto -\log\widehat{\mu}_\Omega(z,0)$ is continuous and plurisubharmonic and

$$\begin{aligned}\widehat{\mu}_\Omega(z,0) &= \inf\{|b| + \delta^{-1}\mu(z-w)\,;(w,b)\in\mathbb{C}^n\times\mathbb{C}, |b|\geq e^{-u(w)}\}\\ &= \inf_{w\in\mathbb{C}^n}\{e^{-u(w)} + \delta^{-1}\mu(z-w)\} = R^a_{\mu,\delta}u(z), \qquad z\in\mathbb{C}^n.\end{aligned}$$

So $R^a_{\mu,\delta}u$ is plurisubharmonic and continuous.

Since S is a lower set we have that $H_S(z-w)\leq H_S(z)+\sigma_S|w|$. See Theorem 5.8 in [11]. So, for $z\in\mathbb{C}^n$,

$$\begin{aligned}R^a_{\mu,\delta}u(z) &\leq -\log\inf_{w\in\mathbb{C}^n}\{e^{-H_S(z-w)} + \delta^{-1}e^{c_u}r_\mu|w|\} + c_u\\ &\leq -\log\inf_{w\in\mathbb{C}^n}\{e^{-H_S(z)-\sigma_S|w|} + \delta^{-1}e^{c_u}r_\mu|w|\} + c_u\\ &\leq H_S(z) - \log\inf_{w\in\mathbb{C}^n}\{e^{-\sigma_S|w|} + \delta^{-1}e^{c_u}r_\mu|w|\} + c_u.\end{aligned}$$

Note that if $f\colon \mathbb{R}_+\to\mathbb{R}$, $f(x) = e^{-ax}+bx$ for $a,b>0$, then $f'(x) = b - ae^{-ax} > b-a$. So f is increasing if $a<b$. We have that $\delta < \sigma_S^{-1}e^{c_u}r_\mu$, implying that $\sigma_S < \delta^{-1}e^{c_u}r_\mu$ so the last infimum is obtained at $w=0$. So $R^a_{\mu,\delta}u(z)\leq H_S(z)+c_u$ and $R^a_{\mu,\delta}u\in\mathcal{L}^S(\mathbb{C}^n)$.

We let $\delta_j\searrow 0$, and set $u_{\delta,w}(z) = -\log(e^{-u(z-w)}+\delta^{-1}\mu(w))$ and

$$K_j = \{w\in\mathbb{C}^n\,; u_{\delta_j,w}(z)\geq u(z)\},$$

Since u is bounded below we have $\underline{\lim}_{|z|\to\infty} u_{\delta_j,w}(z) = -\infty$, and therefore K_j is bounded. We also have that u is upper semicontinuous, so K_j is closed, and consequently compact. We also have that $u_{\delta_j,w}(z)$ is upper semicontinuous as a function of w, so there exists $w_j\in K_j$ such that $R^a_{\mu,\delta_j}u(z) = u_{\delta_j,w_j}(z)$. Note the $u_{\delta,w}(z)\searrow-\infty$ as $\delta\searrow 0$ for all $z\in\mathbb{C}^n$ and $w\in\mathbb{C}^{n*}$. So $\cap_j K_j = \{0\}$ and $w_j\to 0$ as $j\to\infty$, and consequently, for $z\in\mathbb{C}^n$,

$$u(z)\leq\lim_{j\to\infty}R^a_{\mu,\delta_j}u(z) = \lim_{j\to\infty}u_{\delta_j,w_j}(z)\leq\overline{\lim_{j\to\infty}}\,u(z-w_j)\leq u(z),$$

so $R^a_{\mu,\delta}u\searrow u$, as $\delta\searrow 0$. □

3 Supremal convolutions

Let us recall some topological properties of $\mathcal{PSH}(\Omega)$, with the goal of determining when $\sup\mathcal{F}$ is upper semicontinuous, for some family $\mathcal{F}\subset\mathcal{PSH}(\Omega)$ that is locally upper bounded. For more details see Hörmander [6, Theorems 4.1.8-9] and [7, Theorem 3.2.11-13]. Recall that $\mathcal{PSH}(\Omega)$ is the family of plurisubharmonic functions on Ω that are not identically $-\infty$ on any connected component of Ω, so

$$\mathcal{PSH}(\Omega)\subset L^1_{\mathrm{loc}}(\Omega)\subset\mathcal{D}'(\Omega).$$

The topology $\mathcal{PSH}(\Omega)$ inherits from the weak topology on $\mathcal{D}'(\Omega)$ coincides with topology it inherits from $L^1_{\mathrm{loc}}(\Omega)$ as a Fréchet space topology with semi-norms

$f \mapsto \int_K |f|\, d\lambda$, for compact K. With this topology $\mathcal{PSH}(\Omega)$ is a closed subspace of $L^1_{\text{loc}}(\Omega)$, so it is a complete metrizable space.

Furthermore, $\mathcal{PSH}(\Omega)$ has a Montel property which says that every sequence $(u_j)_{j\in\mathbb{N}}$ in $\mathcal{PSH}(\Omega)$, that is locally bounded above and does not converge to $-\infty$ uniformly on every compact subset of Ω has a convergent subsequence with limit in $\mathcal{PSH}(\Omega)$.

Sometimes $\sup \mathcal{F} = \sup \mathcal{F}_0$ for some $\mathcal{F}_0 \subset \mathcal{F}$ containing a minimal element. The Montel property then states that $\mathcal{F}_0$ is relatively compact. If, in addition, $\mathcal{F}_0$ is closed then $\sup \mathcal{F}$ is upper semicontinuous. This follows from Proposition 2.1 in Sigurdsson [17], which we include for the convenience of the reader.

Lemma 3.1. *Let $\Omega \subset \mathbb{C}^n$ be open and $\mathcal{F}$ be a compact family in $\mathcal{PSH}(\Omega)$. Then $\sup \mathcal{F}$ is upper semicontinuous, and consequently plurisubharmonic.*

Proof. To start off the proof we recall some essential properties of integral averages. If $f \in L^1_{\text{loc}}(\Omega)$ we let $M_r f(z)$ denote the integral average of f over a euclidean ball with center z and radius $r > 0$, where we assume that the euclidean distance from $z \in \Omega$ to the boundary of Ω is strictly less than r. We have that $L^1_{\text{loc}}(\Omega) \times \mathbb{C}^n \times \mathbb{R}^*_+, (f, z, r) \mapsto M_r f(z)$ is continuous.

Let $u = \sup \mathcal{F}$, $z_0 \in \Omega$, and $a \in \mathbb{R}$ such that $u(z_0) < a$. If we can show that there exists a neighborhood V of z_0 such that $u|_V < a$ then u is upper semicontinuous. Let $\varepsilon > 0$ such that $u(z_0) < a - \varepsilon$ and $v_0 \in \mathcal{F}$, and take $r_0 > 0$ such that

$$v_0(z_0) \le M_{r_0} v_0(z_0) \le a - \varepsilon.$$

By the continuity of $(f, z) \mapsto M_{r_0} f(z)$ there exists an open neighborhood U_0 of v_0 in $\mathcal{PSH}(\Omega)$ and an open neighborhood V_0 of z_0 such that

$$M_{r_0} v(z) < a - \varepsilon, \quad v \in U_0,\ z \in V_0.$$

The mean value property implies that $v(z) < a - \varepsilon$ for $v \in U_0$ and $z \in V_0$. Since v_0 was arbitrary and $\mathcal{F}$ is compact there exists a finite covering $U_1, \dots, U_\ell$ of $\mathcal{F}$ and open neighborhoods $V_1, \dots, V_\ell$ of z_0 such that $v(z) < a - \varepsilon$, for $v \in U_j$ and $z \in V_j$. If we set $V = \cap_j V_j$, then $v(z) < a - \varepsilon$ for all $v \in \mathcal{F}$ and $z \in V$. So $u|_V < a$ and u is therefore upper semicontinuous. □

As usual we let $0 \in S \subset \mathbb{R}^n_+$ be compact and convex, $1_n = (1, \dots, 1) \in \mathbb{N}^n$, and $\sigma_S = \varphi_S(1_n)$. We will allow us a slight abuse of notation by identifying a vector in $\mathbb{C}^n$ denoted by a lower case letter with a diagonal matrix denoted by the corresponding upper case letter. Thus we identify $a \in \mathbb{C}^n$ with the diagonal matrix A with diagonal a. In this notation the subadditivity of the supporting function of S implies that

$$\begin{aligned} H_S(Zw) = H_S(Wz) &= H_S(z_1 w_1, \dots, z_n w_n) \\ &\le H_S(z) + H_S(w), \qquad z, w \in \mathbb{C}^n, \end{aligned} \tag{3.1}$$

Proof of Theorem 1.3. Recall that we are studying the operator defined by

$$R_\delta^b u(z) = \sup_{w\in\mathbb{C}^n}\{u(Zw) - \delta^{-1}\log(\|w-1_n\|_\infty + 1)\}$$
$$= \log \sup_{w\in\mathbb{C}^n} \frac{e^{u(Zw)}}{(\|w-1_n\|_\infty+1)^{1/\delta}}, \qquad z\in\mathbb{C}^n.$$

With $u_{\delta,w}(z) = u(Zw) - \delta^{-1}\log(\|w-1_n\|_\infty + 1)$, which is plurisubharmonic, we have that $R_\delta^b u = \sup_{w\in\mathbb{C}^n} u_{\delta,w}$. The assumption that u is locally bounded ensures that $u_{\delta,w}\in\mathcal{PSH}(\mathbb{C}^n)$. For every $\delta>0$ we have that $u_{\delta,1_n} = u$ so $R_\delta^b u \geq u$. Additionally,

$$H_S(w) \leq \sigma_S \log^+\|w\|_\infty \leq \sigma_S\log(\|w-1_n\|_\infty+1), \quad w\in\mathbb{C}^n.$$

This, along with (3.1), implies that if $u\leq H_S + c_u$ then

$$R_\delta^b u \leq H_S + \sup_{w\in\mathbb{C}^n}\{(\sigma_S - \delta^{-1})\log(\|w-1_n\|_\infty+1)\} + c_u.$$

Since $\delta < \sigma_S^{-1}$, we have $R_\delta^b u\leq H_S + c_u$. We also have, for all $\gamma>0$ and $z\in\mathbb{C}^n$,

$$R_\delta^b u(z) \leq \max\left\{\sup_{w\in 1_n+\gamma\overline{\mathbb{D}}^n} u(Zw), H_S(z) + c_u + (\sigma_S-\delta^{-1})\log(\gamma+1)\right\}.$$

With $z\in\mathbb{C}^n$ fixed and $\varepsilon>0$, we may take γ small enough that $u(Zw)\leq u(z)+\varepsilon$ for all $w\in 1_n+\gamma\overline{\mathbb{D}}^n$, since u is upper semicontinuous. Consequently,

$$R_\delta^b u \leq \max\{u+\varepsilon, H_S + c_u + (\sigma_S-\delta^{-1})\log(\gamma+1)\}.$$

The second term converges to $-\infty$ as $\delta\searrow 0$, so, since $R_\delta^b u\geq u$, we have that $R_\delta^b u \searrow u$ as $\delta\searrow 0$.

We now observe that $R_\delta^b u = \sup \mathcal{F}_{u,\delta}$ for $\mathcal{F}_{u,\delta} = \{\max\{u, u_{\delta,w}\}\,;\, w\in\mathbb{C}^n\}\subset \mathcal{PSH}(\mathbb{C}^n)$. All functions in $\mathcal{F}_{u,\delta}$ are bounded above by $H_S + c_u$ and u is a minimal element of $\mathcal{F}_{u,\delta}$, so $\mathcal{F}_{u,\delta}$ is relatively compact by the Montel property. To show that $\mathcal{F}_{u,\delta}$ is closed in $\mathcal{PSH}(\mathbb{C}^n)$, and thus that $R_\delta^b u$ is plurisubharmonic, we note that

$$u_{\delta,w} \leq H_S + c_u + (\sigma_S-\delta^{-1})\log^+\|w\|_\infty, \quad w\in\mathbb{C}^n,$$

and recall as well that u is assumed to be locally bounded below. So fixing $a\in\mathbb{C}^n$ and $U = B(z,1)$ as the euclidean ball with center z and radius 1, and setting $M_2 < M_1$ as constants such that $M_2\leq u(z) - c_u$ and $H_S(z)\leq M_1$, for $z\in U$, where $c_u>0$ is such that $u\leq H_S + c_u$, we have that

$$u_{\delta,w}(z) - u(z) \leq M_1 - M_2 + (\sigma_S - \delta^{-1})\log^+\|w\|_\infty, \quad z\in U,\ w\in\mathbb{C}^n.$$

So if $w\in\mathbb{C}^n$ is such that $\log^+\|w\|_\infty \geq (M_1-M_2)/(\delta^{-1}-\sigma_S) > 0$, then $u_{\delta,w}(z)\leq u(z)$, for $z\in U$. So, since $u\leq R_\delta^b u$, we have that

$$R_\delta^b u(z) = \sup_{w\in\bar{B}(z,r)} u_{\delta,w}(z), \quad z\in U,$$

where $r = \max\{1, e^{(M_1-M_2)/(\delta^{-1}-\sigma_S)}\}$. If we take a sequence $(w_j)_{j\in\mathbb{N}}$ in $\mathbb{C}^n$ such that $v_{\delta,w_j} \to v$ in $\mathcal{PSH}(\mathbb{C}^n)$, where $v_{\delta,w} = \max\{u, u_{\delta,w}\}$, we need to show that $v \in \mathcal{F}_{u,\delta}$. If $(w_j)_{j\in\mathbb{N}}$ is unbounded then we can pick a subsequence $(w_{j_k})_{k\in\mathbb{N}} \subset \mathbb{C}^n \setminus \overline{B}(z,r)$. Since $v_{\delta,w_{j_k}} = u$, we have that $v = u \in \mathcal{F}_{u,\delta}$. If $(w_j)_{j\in\mathbb{N}}$ is bounded we can pick a convergent subsequence $(w_{j_k})_{k\in\mathbb{N}}$ with limit w. By the upper semicontinuity of $(z,w) \mapsto v_{\delta,w}$ we have that $\overline{\lim}_{k\to\infty} v_{\delta,w_{j_k}} = v_{\delta,w}$, and by Theorem 4.1.9 in [6] we have that $\overline{\lim}_{k\to\infty} v_{\delta,w_{j_k}} = v$ almost everywhere. So $v = v_{\delta,w}$ almost everywhere and, since they are plurisubharmonic, we have that $v = v_{\delta,w} \in \mathcal{F}_{u,\delta}$. Consequently, $\mathcal{F}_{u,\delta}$ is compact and, by Lemma 3.1, $R^b_\delta u$ is plurisubharmonic.

All that remains to be proven is that $R^b_\delta u$ is lower semicontinuous. By a change of variables, we have, for $z \in \mathbb{C}^{*n}$,

$$\begin{aligned} R^b_\delta u(z) &= \sup_{w\in\mathbb{C}^n} \{u(Zw) - \delta^{-1}\log(\|w - 1_n\|_\infty + 1)\} \\ &= \sup_{w\in\mathbb{C}^n} \{u(w) - \delta^{-1}\log(\|Z^{-1}w - 1_n\|_\infty + 1)\}. \end{aligned}$$

So we set $v_{\delta,w}(z) = u(w) - \delta^{-1}\log(\|Z^{-1}w - 1_n\|_\infty + 1)$, and note that $v_{\delta,w}$ is continuous on $\mathbb{C}^{*n}$ and $R^b_\delta u(z) = \sup_{w\in\mathbb{C}^n} v_{\delta,w}(z)$, for $z \in \mathbb{C}^{*n}$. So $R^b_\delta u$ is given as the supremum of continuous functions on the open set $\mathbb{C}^{*n}$, and is consequently lower semicontinuous on $\mathbb{C}^{*n}$. See Lemma 2.3.2 in Klimek [9]. □

4 Integral convolutions over diagonal matrices

Let us recall that the regularization from Theorem 1.4 was define by

$$\begin{aligned} R^c_\delta u(z) &= \int_{\mathbb{C}^n} u(Az)\psi_\delta(A)\, d\lambda(A) = \int_{\mathbb{C}^n} u((I+\delta B)z)\psi(B)\, d\lambda(B) \qquad (4.1) \\ &= \int_{\mathbb{C}^n} u((1+\delta w_1)z_1, \dots, (1+\delta w_n)z_n)\psi(w)\, d\lambda(w), \quad z \in \mathbb{C}^n, \end{aligned}$$

where $\psi \in \mathcal{C}^\infty_0(\mathbb{C}^n)$ is rotationally symmetric in each variable and $\int_{\mathbb{C}^n} \psi\, d\lambda = 1$, and $\psi_\delta(z) = \delta^{-2n}\psi((z - 1_n)/\delta)$.

Proof of Theorem 1.4. By the Fubini-Tonelli theorem $R^c_\delta \colon L^1_{\text{loc}}(\mathbb{C}^n) \to L^1_{\text{loc}}(\mathbb{C}^n)$ and $R^c_\delta u \to u$, in the L^1_{loc} topology, as $\delta \searrow 0$. We also have that $R^c_\delta \colon \mathcal{PSH}(\mathbb{C}^n) \to \mathcal{PSH}(\mathbb{C}^n)$. Proposition 3.2 in [11] states that $H_S(1_n+\delta w) \leq \delta\sigma_S$, for all $w \in \overline{\mathbb{D}}^n$, so $R^c_\delta \colon \mathcal{L}^S(\mathbb{C}^n) \to \mathcal{L}^S(\mathbb{C}^n)$, for all compact convex $S \subset \mathbb{R}^n_+$ containing S. In fact, if $u \in \mathcal{L}^S(\mathbb{C}^n)$ and c_u is a constant such that $u \leq H_S + c_u$ then, by (3.1),

$$\begin{aligned} R^c_\delta(z) &\leq \int_{\mathbb{C}^n} (H_S(z) + H_S(1_n + \delta w) + c_u)\psi_\delta(w)\, d\lambda(w) \\ &= H_S(z) + C\sigma_S\delta + c_u, \qquad z \in \mathbb{C}^n, \end{aligned}$$

for some constant C. The linear map $\mathbb{C}^n \to \mathbb{C}^n, A \mapsto Az$ has the Jacobi determinant $|z_1 \cdots z_n|^2$, so

$$R^c_\delta u(z) = \int_{\mathbb{C}^n} u(w)\psi_\delta(Z^{-1}w)|z_1 \cdots z_n|^{-2}\, d\lambda(w), \quad z \in \mathbb{C}^{*n}.$$

By applying Lebesgue's theorem on dominated convergence we may differentiate by z_j under the integral sign arbitrarily often. So if $u \in L^1_{\mathrm{loc}}(\mathbb{C}^n)$ then $R^c_\delta u \in \mathcal{C}^\infty(\mathbb{C}^{*n})$.

Note that $R^c_\delta u(z) = U_z(\psi_\delta)$, where U_z is the distribution associated to u_z, which is locally integrable since u is locally bounded. Let us fix $a \in \mathbb{C}^n$. Since $u_z \to u_a$, in $\mathcal{PSH}(\mathbb{C}^n)$, as $z \to a$, we have that

$$\lim_{z\to a} R^c_\delta u(z) = \lim_{z\to a} U_z(\psi_\delta) = U_a(\psi_\delta) = R^c_\delta u(a),$$

showing that $R^c_\delta u$ is continuous at a. □

Recall that the operator in Theorem 1.5 is given by

$$R^d_\delta u(z) = \log R^c_\delta e^u(z) = \log \int_{\mathbb{C}^n} e^{u(Za)} \psi_\delta(a)\, d\lambda(a), \quad z \in \mathbb{C}^n.$$

We have already shown that $e^{R^d_\delta u} \searrow e^u$ as $\delta \searrow 0$. We also have that

$$R^d_\delta u \le H_S + c_u + \log \int_{\mathbb{C}^n} e^{H_S(a)} \psi_\delta(A)\, d\lambda(A),$$

if c_u is a constant such that $u \le H_S + c_u$. To show that $R^d_\delta u$ is plurisubharmonic we recall that following result.

Lemma 4.1. *Let $\Omega \subset \mathbb{C}$ and $u\colon \Omega \to [0, \infty[$. Then $\log u \in \mathcal{SH}(\Omega)$ if and only if $z \mapsto u(z)e^{2\operatorname{Re}(\tau z)} \in \mathcal{SH}(\Omega)$ for all $\tau \in \mathbb{C}$.*

Proof. To simplify notation we denote by $\operatorname{Re}(\tau\cdot)$ the function $z \mapsto \operatorname{Re}(\tau z)$. If $\log u$ is subharmonic then so is $e^{\log u + 2\operatorname{Re}(\tau\cdot)}$, since $2\operatorname{Re}(\tau\cdot)$ is subharmonic, for all $\tau \in \mathbb{C}$.

So we assume $ue^{2\operatorname{Re}(\tau\cdot)}$ is subharmonic for all $\tau \in \mathbb{C}$. We further more assume u is smooth and $u > 0$. We have that

$$\Delta \log u = 4\frac{\partial}{\partial z}\left(\frac{1}{u}\frac{\partial u}{\partial \bar z}\right) = \frac{\Delta u}{u} - \frac{4}{u^2}\frac{\partial u}{\partial z}\frac{\partial u}{\partial \bar z} = \frac{u\Delta u - |\nabla u|^2}{u^2}.$$

For all $\tau \in \mathbb{C}$ we also have that

$$\begin{aligned} 0 \le \Delta(ue^{2\operatorname{Re}(\tau\cdot)}) &= 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z}(ue^{2\operatorname{Re}(\tau\cdot)}) = 4e^{2\operatorname{Re}(\tau\cdot)}\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial \bar z} + \bar\tau u\right) \\ &= 4ue^{2\operatorname{Re}(\tau\cdot)}\left(\frac{\Delta u}{4u} + \frac{\tau}{u}\frac{\partial u}{\partial \bar z} + \frac{\bar\tau}{u}\frac{\partial u}{\partial z} + |\tau|^2\right) \\ &= 4ue^{2\operatorname{Re}(\tau\cdot)}\left(\frac{\Delta u}{4u} + \left|\tau + \frac{1}{u}\frac{\partial u}{\partial z}\right|^2 - \frac{1}{u^2}\frac{\partial u}{\partial z}\frac{\partial u}{\partial \bar z}\right). \end{aligned}$$

As this holds for all $\tau \in \mathbb{C}$ and $u > 0$ we have that

$$0 \le \frac{\Delta u}{4u} - \frac{1}{u^2}\frac{\partial u}{\partial z}\frac{\partial u}{\partial \bar z} = \Delta \log u,$$

so $\log u$ is subharmonic. For a general $u \geq 0$ we set $v_\delta = (u+\delta) * \chi_\delta > 0$, where χ_δ is a standard smoothing kernel. Then, for $z \in \mathbb{C}$,

$$\int_{B(z,r)} v_\delta(w) e^{2\operatorname{Re}(\tau w)}\, d\lambda(w) \geq \int_{B(z,r)} (u * \chi_\delta)(w) e^{2\operatorname{Re}(\tau w)}\, d\lambda(w) + \delta e^{2\operatorname{Re}(\tau z)}$$

and, by the Fubini-Tonelli theorem,

$$\int_{B(z,r)} (u*\chi_\delta)(w) e^{2\operatorname{Re}(\tau w)}\, d\lambda(w) = \int_{\mathbb{C}} \chi_\delta(a) \int_{B(z,r)} e^{2\operatorname{Re}(\tau w)} u(w-a)\, d\lambda(w) d\lambda(a)$$
$$\geq e^{2\operatorname{Re}(\tau z)} \int_{\mathbb{C}} u(z-a)\chi_\delta(a)\, d\lambda(a) = u * \chi_\delta(z) e^{2\operatorname{Re}(\tau z)},$$

so $v_\delta e^{2\operatorname{Re}(\tau \cdot)}$ satisfies the submean inequality, and is therefore subharmonic. We have already shown that this implies that $\log v_\delta$ is subharmonic, and since $\log v_\delta \searrow \log u$ as $\delta \searrow 0$, $\log u$ must also be subharmonic. See Theorem 2.4.6 in Ransford [13]. □

Proof of Theorem 1.5. What remains is to show that $R^d_\delta u$ is plurisubharmonic. We fix $a, b \in \mathbb{C}^n$, $\tau \in \mathbb{C}$, and set, for $\zeta \in \mathbb{C}$,

$$v(\zeta) = R^c_\delta e^u(a+\zeta b) e^{2\operatorname{Re}(\tau\zeta)} = \int_{\mathbb{C}^n} e^{u(W(a+\zeta b))+2\operatorname{Re}(\tau\zeta)} \psi_\delta(W)\, d\lambda(W).$$

By [13, Theorem 2.4.8] we have that $v \in \mathcal{SH}(\mathbb{C})$. So, by Lemma 4.1, $\zeta \mapsto R^d_\delta u(a+\zeta b)$ is subharmonic and, consequently, $R^d_\delta u$ is plurisubharmonic. □

5 Uniform continuity in Lelong classes

Classically, sufficient conditions on K such that V_K is Hölder continuous on $\mathbb{C}^n$ have been studied. See, for example, Siciak [16]. In Perera [12] this study is continued for $V^S_{K,q}$. Her main result, stated in the abstract, claims that if S is a convex body, and K and q are sufficiently regular then $V^S_{K,q}$ is α-Hölder continuous on $\mathbb{C}^n$. The main aim of this section is showing that this result can not be true if S is not a lower set, no matter what regularity we impose on K and q. We will do this by showing that if S is not a lower set then $V^S_{K,q}$ will never be uniformly continuous. Let us begin by recalling the definition of some classes of regularity.

Let $U \subset \mathbb{C}^n$ be an open set and f be a function on U. We say that f is *uniformly continuous* if for $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x+y) - f(x)| < \varepsilon$ for all $x, y \in U$ such that $x + y \in U$ and $|y| < \delta$. We say that f is *α-Hölder continuous*, for $0 < \alpha \leq 1$ if

$$|f(x) - f(y)| \leq C|x-y|^\alpha, \quad x, y \in U,$$

for some constant C. We also say the f is *Hölder continuous* if it is α-Hölder continuous for some $0 < \alpha \leq 1$. We say that f is *Lipschitz continuous* with *Lipschitz constant C* if

$$|f(x) - f(y)| \leq C|x-y|, \quad x, y \in U.$$

The Lipschitz continuous functions coincide with 1-Hölder continuous functions, and the Hölder continuous functions are uniformly continuous. We can describe H_S in these terms, depending on whether S is a lower set or not.

Theorem 5.1. *Let $0 \in S \subset \mathbb{R}^n_+$ be compact and convex. If S is a lower set then H_S is Lipschitz continuous with Lipschitz constant $\sigma_S = \varphi_S(1_n)$. If S is not a lower set then H_S is not uniformly continuous.*

Proof. First we assume S is a lower set. Then, by Theorem 5.8 in [11],

$$H_S(z) = \varphi_S(\log^+|z_1|, \dots, \log^+|z_n|), \quad z \in \mathbb{C}^n,$$

where $\log^+ x = \max\{0, \log x\}$, so $\log^+ 0 = 0$. Note that $\log^+ x$ is Lipschitz continuous with Lipschitz constant 1 and φ_S is Lipschitz continuous with Lipschitz constant σ_S. To see this note that

$$\varphi_S(\xi) = \varphi_S(\xi - \eta + \eta) \leq \varphi_S(\xi - \eta) + \varphi_S(\eta), \quad \xi, \eta \in \mathbb{R}^n_+,$$

so $\varphi_S(\xi) - \varphi_S(\eta) \leq \varphi_S(\xi - \eta)$ and consequently

$$\varphi_S(\xi) - \varphi_S(\eta) \leq \varphi_S(\xi - \eta) \leq \sigma_S\|\xi - \eta\|_\infty \leq \sigma_S|\xi - \eta|, \quad \xi, \eta \in \mathbb{R}.$$

With $\operatorname{Log}^+ z = (\log^+|z_1|, \dots, \log^+|z_n|)$, we have, for $z, w \in \mathbb{C}^n$,

$$\begin{aligned}|H_S(z) - H_S(w)| &= |\varphi_S(\operatorname{Log}^+ z) - \varphi_S(\operatorname{Log}^+ w)| \\ &\leq \sigma_S|\operatorname{Log}^+ z - \operatorname{Log}^+ w| \leq \sigma_S|z - w|.\end{aligned}$$

Before continuing recall that $\varphi_S \leq \varphi_T$ if and only if $S \subset T$. So $H_S \leq H_T$ if and only if $S \subset T$. It will prove useful to us that if S is not a subset of T, then $H_S - H_T$ is not bounded above.

Now assume S is not a lower set. To show that H_S is not uniformly continuous it is sufficient to show that, for all $\delta > 0$ we have $w \in \delta\overline{\mathbb{D}}^n$ such that $z \mapsto H_S(z + w) - H_S(z)$ is unbounded. After possibly rearranging the variables, there exists $s = (s', s'') \in \mathbb{R}^\ell_+ \times \mathbb{R}^{n-\ell}_+$ such that $s \in S$ and $(s', 0) \notin S$. By Proposition 3.3 in [11] we have that $H_S(z', 0) = H_T(z')$, $z' \in \mathbb{C}^\ell$, where $0 \in T \subset \mathbb{R}^\ell_+$ is compact convex and $s' \notin T$. So, with $L = \operatorname{ch}\{0, s'\} \subset \mathbb{R}^\ell_+$, we have that $H_L - H_T$ is not bounded above. Note that, for $z' \in \mathbb{C}^{*\ell} \setminus \mathbb{D}^\ell$,

$$\begin{aligned}H_S(z', \delta, \dots, \delta) &= \sup_{x \in S}\big(x_1 \log|z'_1| + \dots + x_\ell \log|z'_\ell| + (x_{\ell+1} + \dots + x_n)\log\delta\big) \\ &\geq s_1 \log|z'_1| + \dots + s_\ell \log|z'_\ell| + (s_{\ell+1} + \dots + s_n)\log\delta \\ &= H_L(z') + (s_{\ell+1} + \dots + s_n)\log\delta.\end{aligned}$$

So

$$H_S(z', \delta, \dots, \delta) - H_S(z', 0, \dots, 0) \geq H_L(z') - H_T(z') + C_\delta, \quad z' \in \mathbb{C}^\ell \setminus \mathbb{D}^\ell,$$

where $C_\delta = (s_{\ell+1} + \dots + s_n)\log\delta$, and consequently $z \mapsto H_S(z + w) - H_S(z)$ is not bounded for $w = (0, \dots, 0, \delta, \dots, \delta)$. To be clear, $w_1 = \dots = w_\ell = 0$ and $w_{\ell+1} = \dots = w_n = \delta$. This shows that H_S is not uniformly continuous. □

We know that $V^S_{\mathbb{T}^n} = V^S_{\overline{\mathbb{D}}^n} = H_S$, where $\mathbb{T} = \{z \in \mathbb{C}\,; |z| = 1\}$. See Proposition 4.3 in [11]. However, the previous proposition does not provide a counterexample to the main result in [12], since $\mathbb{D}^n$ does not satisfy the boundary condition required by the Proposition and $\mathbb{T}^n$ is not the closure of an open set. A consequence of the previous result will however suffice.

Corollary 5.2. *Let $0 \in S \subset \mathbb{R}^n_+$ be compact and convex. If S is not a lower set then $\mathcal{L}^S_+(\mathbb{C}^n)$ contains no uniformly continuous functions.*

Proof. Let $u \in \mathcal{L}^S_+(\mathbb{C}^n)$ and c_u be a constant such that

$$H_S(z) - c_u \leq u(z) \leq H_S(z) + c_u, \quad z \in \mathbb{C}^n.$$

We then have

$$u(z+w) - u(z) \geq H_S(z+w) - H_S(z) - 2c_u, \quad z, w \in \mathbb{C}^n.$$

As shown in the previous Proposition, there exists $w \in \delta\overline{\mathbb{D}}^n$, for every $\delta > 0$, such that $z \mapsto u(z+w) - u(z)$ is unbounded, and thus u is not uniformly continuous. □

By Proposition 4.5 in [11] we have that $V^{S*}_{K,q} \in \mathcal{L}^S_+(\mathbb{C}^n)$, for all compact convex $S \subset \mathbb{R}^n_+$ containing 0, compact $K \subset \mathbb{C}^n$ and admissible q on K. So if S is not a lower set then $V^S_{K,q}$ can not be uniformly continuous, and thus it can not be Hölder continuous.

An error in [12] is in the proof of Lemma 2.4. A major step in the proof involves finding a constant C_w, depending on w, such that $V^S_{\mathbb{C}^n,q}(z+w) \leq H_S(z) + C_w$, for $z \in \mathbb{C}^n$, where q has been extended to $\mathbb{C}^n$ by standard methods of extending Hölder continuous functions. This is in contradiction to Theorem 5.8 in [11], where it is shown that $\mathcal{L}^S(\mathbb{C}^n)$ is translation invariant if and only if S is a lower set.

We will now assume that $n = 2$, since this is done in the proof of Lemma 2.4 in [12]. Where the proof fails is assuming that $H_S(z_1, z_2) = \sup_{x \in S} x_2 \log|z_2|$, if $|z_1| < 1$. We can see this is not true by an explicit counterexample. Let $S = \mathrm{ch}\{(0,0), (1,1), (1,0)\}$. Then $\sup_{x\in S} x_2 \log|z_2| = \log|z_2|$. However, if $|z_2| > 1$ and $z_1 = 1/z_2$ then $H_S(z_1, z_2) = \sup_{x \in S}(x_2 - x_1)\log|z_2| = 0$, since $x_2 - x_1 \leq 0$ for all $x \in S$.

It is worth noting that other results in [12] need correcting, as they depend on wrong results from other papers. Corollary 1.3 depends on [1, Proposition 3.1], which is shown to be incorrect in Example 1.2 herein. Proposition 2.3 also depends on [1, Proposition 3.1], as well as depending on Levenberg and Perera [10, Proposition 2.2]. Proposition 2.2 in [10] relies on constructing a strictly plurisubharmonic function in $\mathcal{L}^S_+(\mathbb{C}^n)$. This construction is the content of Section 3 in the paper. First it is done under the assumption that S is a convex polytope, that is $S = \mathrm{ch}\{v_1, \dots, v_\ell\} \subset \mathbb{R}^n_+$, where $v_1 = 0$. The specific function chosen is

$$h_S(z) = \log\Big(\sum_{j=1}^{\ell} |z|^{v_j}\Big), \quad z \in \mathbb{C}^n.$$

In the case where S is not a convex polytope, but still contains a neighborhood of 0, Levenberg and Perera [10] take a decreasing sequence of convex polytopes S_j such that $\cap_j S_j = S$. They then claim that h_{S_j} is a decreasing sequence and use its limit as a candidate function. But the sequence is not decreasing. This can be seen by considering the point 1_n. In fact, $h_{S_j}(1_n) = \log(\#\operatorname{ext} S_j)$, where $\#A$ denotes the number of elements in the set A. So if S is not a convex polytope then $h_{S_j}(1_n) \to +\infty$, as $j \to +\infty$. An alternative to [10, Proposition 2.2] can be found in in [18, Lemma 2.1 and Proposition 2.2], which apply when S contains a neighborhood of 0.

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Paper III

Polynomials with exponents in compact convex sets and associated weighted extremal functions - Generalized product property

Bergur Snorrason

Abstract

A famous result of Siciak is how the Siciak-Zakharyuta functions, sometimes called global extremal functions or pluricomplex Green functions with a pole at infinity, of two sets relate to the Siciak-Zakharyuta function of their cartesian product. In this paper Siciak's result is generalized to the setting of Siciak-Zakharyuta functions with growth given by a compact convex set, along with discussing why this generalization does not work in the weighted setting.

1 Introduction

Let $\mathcal{L}(\mathbb{C}^n)$ denote the Lelong class in $\mathbb{C}^n$, consisting of all $u \in \mathcal{PSH}(\mathbb{C}^n)$ such that $u(z) \leq \log^+ |z| + c_u$, for $z \in \mathbb{C}^n$ and some constant c_u. For every compact $K \subset \mathbb{C}^n$ we define the Siciak-Zakharyuta function of K by

$$V_K(z) = \sup\{u(z)\,;\, u \in \mathcal{L}(\mathbb{C}^n), u|_K \leq 0\}, \quad z \in \mathbb{C}^n.$$

Siciak proved in [9] a product formula for these functions. Namely, if $K_j \subset \mathbb{C}^{n_j}$, for $j = 1, \dots, \ell$, $K = K_1 \times \cdots \times K_\ell$, and $n = n_1 + \cdots + n_\ell$, then

$$V_K(z) = \max\{V_{K_1}(z_1), \dots, V_{K_\ell}(z_\ell)\}, \quad z = (z_1, \dots, z_\ell) \in \mathbb{C}^n, z_j \in \mathbb{C}^{n_j}.$$

See Klimek [3], Theorem 5.1.8.

Bos and Levenberg in [1] continued the study of pluripotential theory related to convex sets by, among other results, generalizing Siciak's product formula. To state the result we define, for $S \subset \mathbb{R}^n_+$ compact convex and containing 0, the *logarithmic supporting function* by

$$H_S(z) = \varphi_S(\mathrm{Log}(z)), \quad z \in \mathbb{C}^{*n},$$

where $\varphi_S(\xi) = \sup_{x \in S}\langle x, \xi\rangle$ is the supporting function of S and $\mathrm{Log}(z) = (\log|z_1|, \dots, \log|z_n|)$. We then extend the definition of H_S to $\mathbb{C}^n$ by

$$H_S(z) = \varlimsup_{\mathbb{C}^{*n} \ni w \to z} H_S(w), \quad z \in \mathbb{C}^n.$$

This allows us to define the Lelong class $\mathcal{L}^S(\mathbb{C}^n)$ with respect to S consisting of $u \in \mathcal{PSH}(\mathbb{C}^n)$ such that $u(z) \leq H_S(z) + c_u$ for some constant c_u. We also define $\mathcal{L}^S_+(\mathbb{C}^n)$ as the family of those functions $u \in \mathcal{L}^S(\mathbb{C}^n)$ such that $H_S - c_u \leq u \leq H_S + c_u$. For every compact $K \subset \mathbb{C}^n$ and function q on K we can then define the weighted Siciak-Zakharyuta function of K, S, and q by

$$V^S_{K,q}(z) = \sup\{u(z)\,; u \in \mathcal{L}^S(\mathbb{C}^n), u|_K \leq q\}, \quad z \in \mathbb{C}^n.$$

In the case where $q = 0$ we omit it in the subscript. The superscript is omitted when S is the standard simplex in $\mathbb{R}^n$, that is $S = \Sigma = \mathrm{ch}\{0, e_1, \dots, e_n\}$, where $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$ and ch is denotes the closed convex hull. This is justified since $V^\Sigma_{K,0} = V_K$. The main result of this paper is the following.

Theorem 1.1. *Let $K_j \subset \mathbb{C}^{n_j}$ be compact and non-pluripolar for $j = 1, \dots, \ell$, and $K = K_1 \times \cdots \times K_\ell$, $n = n_1 + \cdots + n_\ell$, $T \subset \mathbb{R}^\ell_+$ be compact convex, $S_j \subset \mathbb{R}^{n_j}_+$ be compact convex containing* 0*, for $j = 1, \dots, \ell$, and $S \subset \mathbb{R}^n_+$ be given by*

$$S = \bigcup_{x \in T} (x_1 S_1) \times \cdots \times (x_\ell S_\ell).$$

Then

$$V^S_K(z) = \varphi_T(V^{S_1}_{K_1}(z_1), \dots, V^{S_\ell}_{K_\ell}(z_\ell)), \quad z = (z_1, \dots, z_n),\ z_j \in \mathbb{C}^{n_j}.$$

Taking $T = \mathrm{ch}\{e_1, \dots, e_\ell\}$ and $S_j = \Sigma_j$, where Σ_j is the standard simplex in $\mathbb{R}^{n_j}$, we get $S = \Sigma_n$ so this generalizes Siciak's original result, since in this case $\varphi_T(\xi) = \max\{\xi_1, \dots, \xi_\ell\}$. This result also generalizes Theorem 1.1 in Nguyen and Long [8]. They prove the following.

Corollary 1.2. *Let $K_1 \subset \mathbb{C}^{n_1}$ and $K_2 \subset \mathbb{C}^{n_2}$ be compact and non-pluripolar, $K = K_1 \times K_2$, $S_1 \subset \mathbb{R}^{n_1}_+$ and $S_2 \subset \mathbb{R}^{n_2}_+$ be compact convex and containing a neighborhood of* 0 *in $\mathbb{R}^{n_1}_+$ and $\mathbb{R}^{n_2}_+$, respectively, and $S = \mathrm{ch}((S_1 \times \{0\}) \cup (\{0\} \times S_2))$. Then*

$$V^S_K(z) = \max\{V^{S_1}_{K_1}(z_1), V^{S_2}_{K_2}(z_2)\}, \quad z \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}.$$

In [8] it is not assumed that S_1 and S_2 contain a neighborhood of their respective origins, but their proof requires it, see Proposition 2.2 herein.

Another corollary of Theorem 1.1 is Proposition 2.4 in Bos and Levenberg [1]. To state this result we recall the definition of a lower set, which can be seen as the real counterpart to a complete Reinhardt domain. A compact convex $S \subset \mathbb{R}^n_+$, with $0 \in S$, is said to be a *lower set* if for all $s \in S$ the box $[0, s_1] \times \cdots \times [0, s_n]$ is contained in S. Theorem 5.8 in [6] gives several equivalent characterizations for this property. One of these is that S is a lower set if and only if $\varphi_S(\xi) = \varphi_S(\xi^+)$, for all $\xi \in \mathbb{R}^n$, where $\xi^+_j = \max\{0, \xi_j\}$. We refer to the smallest lower set containing S as the *lower hull of S* denoted by $\widehat{S}$, and note that S is a lower set if and only if $S = \widehat{S}$. The supporting function of $\widehat{S}$ is therefore given by

$\varphi_{\widehat{S}}(\xi) = \varphi_S(\xi^+)$. If we assume $\ell = n$ and $S_j = [0,1]$ in Theorem 1.1 we have that

$$\varphi_S(\xi) = \varphi_T(\varphi_{[0,1]}(\xi_1), \dots, \varphi_{[0,1]}(\xi_n)) = \varphi_T(\xi_1^+, \dots, \xi_n^+) = \varphi_{\widehat{T}}(\xi)$$

since $\varphi_{[a,b]}(\xi) = \max\{a\xi, b\xi\}$. So $S = \widehat{T}$, since supporting functions uniquely determine compact convex sets. This is clarified further in Section 2. So setting $\ell = n$ and $S_j = [0,1]$ leads to the following, which is a generalization of Proposition 2.4 in [1].

Corollary 1.3. *Let $K_1, \dots, K_n \subset \mathbb{C}$ be compact and non-polar, $K = K_1 \times \cdots \times K_n$, and $S \subset \mathbb{R}^n_+$ convex compact and containing 0. Then*

$$V_K^{\widehat{S}}(z) = \varphi_S(V_{K_1}(z_1), \dots, V_{K_n}(z_n)), \quad z \in \mathbb{C}^n.$$

In [1] this is proven in the setting where S is a lower set. Then the formula becomes

$$V_K^S(z) = \varphi_S(V_{K_1}(z_1), \dots, V_{K_n}(z_n)), \quad z \in \mathbb{C}^n. \tag{1.1}$$

Levenberg and Perera [4] claim to prove that the formula also holds if we only assume that $a\Sigma \subset S$ for some $a > 0$, where Σ is the standard simplex in $\mathbb{R}^n$. Subsequently Nguyen and Long [8] claimed it holds under the relaxed condition that S is a convex body, that is when the interior of S is not empty. These results can not hold. Both make the erroneous assumption that the right hand side of (1.1) is in $\mathcal{L}^S(\mathbb{C}^n)$, but Theorem 1.1 tells us that it is in $\mathcal{L}_+^{\widehat{S}}(\mathbb{C}^n)$. We can also show that these result are wrong by an explicit counterexample.

Let $K_1 = K_2 = \overline{\mathbb{D}}$ and $K = K_1 \times K_2 = \overline{\mathbb{D}}^2$. By Proposition 4.3 in [6], we have that $V_K^S = H_S$, for every $0 \in S \subset \mathbb{R}^n_+$ compact and convex. If we set $S = \mathrm{ch}\{(0,0),(1,0),(1,1),(0,a)\}$ then $\varphi_S(\xi) = \max\{\xi_1^+, (\xi_1+\xi_2)^+, a\xi_2^+\}$, for all $\xi \in \mathbb{R}^2$, and thus

$$H_S(z) = \max\{\log^+|z_1|, (\log|z_1| + \log|z_2|)^+, a\log^+|z_2|\},$$

for $z \in \mathbb{C}^2$. But

$$\varphi_S(V_{\overline{\mathbb{D}}}(z_1), V_{\overline{\mathbb{D}}}(z_2)) = \max\{\log^+|z_1|, \log^+|z_1| + \log^+|z_2|, a\log^+|z_2|\},$$

for $z \in \mathbb{C}^2$, and, for $\zeta \in \mathbb{C}$ with $|\zeta| > 1$,

$$V_K^S(\zeta^{-1}, \zeta) = H_S(\zeta^{-1}, \zeta) = a\log|\zeta| < \log|\zeta| = \varphi_S(V_{\overline{\mathbb{D}}}(\zeta), V_{\overline{\mathbb{D}}}(\zeta^{-1})),$$

when $a < 1$.

We get more corollaries when we have an explicit formula for φ_T. One immediate example is taking $T = \{(1,1)\}$, since then $\varphi_T(\xi) = \xi_1 + \xi_2$ and $S = S_1 \times S_2$.

Corollary 1.4. *Let $K_1 \subset \mathbb{C}^{n_1}$ and $K_2 \subset \mathbb{C}^{n_2}$ be compact and non-pluripolar, and $S_1 \subset \mathbb{R}^{n_1}_+$ and $S_2 \subset \mathbb{R}^{n_2}_+$ be compact convex and containing 0. Then*

$$V_{K_1 \times K_2}^{S_1 \times S_2}(z) = V_{K_1}^{S_1}(z_1) + V_{K_2}^{S_2}(z_2), \quad z \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}.$$

It is well known that $\varphi_T(\xi) = \|\xi^+\|_q$ if $T = B \cap \mathbb{R}^n_+$, where B is the unit ball with respect to the norm $\|\cdot\|_p$ and $p, q \geq 1$ satisfy $1/p + 1/q = 1$, see the discussion of equations (2.2.12) in [2].

Corollary 1.5. *Let $n_1, \dots, n_\ell$ be natural numbers, $n = n_1 + \cdots + n_\ell$, $K_j \subset \mathbb{C}^{n_j}$ be compact and non-pluripolar for $j = 1, \dots, \ell$, and $K = K_1 \times \cdots \times K_\ell$, $S_j \subset \mathbb{R}^{n_j}_+$ be compact convex and containing 0, for $j = 1, \dots, \ell$, and $S \subset \mathbb{R}^n_+$ be given by*

$$S = \bigcup_{\substack{x \in \mathbb{R}^\ell_+ \\ \|x\|_p \leq 1}} (x_1 S_1) \times \cdots \times (x_\ell S_\ell).$$

Then

$$V^S_K(z)^q = V^{S_1}_{K_1}(z_1)^q + \cdots + V^{S_\ell}_{K_\ell}(z_\ell)^q, \quad z = (z_1, \dots, z_\ell),\ z_j \in \mathbb{C}^{n_j}.$$

Note that $B \cap \mathbb{R}^n_+$ is a lower set so the previous result becomes particularly explicit when $\ell = n$.

Corollary 1.6. *Let $p, q > 1$ with $1/p + 1/q = 1$, $K_j \subset \mathbb{C}$ be compact and non-pluripolar for $j = 1, \dots, n$, and $K = K_1 \times \cdots \times K_n$, $S \subset \mathbb{R}^n_+$ is given by $S = \{x \in \mathbb{R}^n_+\,;\, \|x\|_p \leq 1\}$. Then*

$$V^S_K(z)^q = V_{K_1}(z_1)^q + \cdots + V_{K_n}(z_n)^q, \quad z \in \mathbb{C}^n.$$

A natural question is if it is possible to generalize Theorem 1.1 to the weighted case. The answer turns out to be negative, as is shown in Propositions 4.1 and 4.2. As a follow up, we will look into when the sublevel sets of V^S_K are not convex, even if K is convex.

For general results on weighted Siciak-Zakharyuta functions and their properties see [6] and the references therein. See also [5] and [7].

2 Background

This section is an overview of required fundamental results from [6]. We will use the notation $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\mathbb{C}^{*n} = (\mathbb{C}^*)^n$. We fix a compact convex $S \subset \mathbb{R}^n_+$ such that $0 \in S$. Recall that we define the logarithmic supporting function of S by

$$H_S(z) = \begin{cases} \varphi_S(\operatorname{Log}(z)), & z \in \mathbb{C}^{*n}, \\ \varlimsup\limits_{\mathbb{C}^{*n} \ni w \to z} H_S(w), & z \notin \mathbb{C}^{*n}, \end{cases}$$

where $\varphi_S(\xi) = \sup_{x \in S} \langle x, \xi \rangle$ and $\operatorname{Log}(z) = (\log|z_1|, \dots, \log|z_n|)$. We then define the Lelong class $\mathcal{L}^S(\mathbb{C}^n)$ with respect to S as those $u \in \mathcal{PSH}(\mathbb{C}^n)$ such that $u \leq H_S + c_u$, for some constant c_u. For a function q on a compact $K \subset \mathbb{C}^n$ we define the Siciak-Zakharyuta function of K, q, and S by

$$V^S_{K,q}(z) = \sup\{u(z)\,;\, u \in \mathcal{L}^S(\mathbb{C}^n), u|_K \leq q\}, \quad z \in \mathbb{C}^n.$$

The function q is referred to as a *weight* and is said to be *admissible* if it is lower semicontinuous and the set $\{z \in K\,; q(z) < +\infty\}$ is non-pluripolar. Let now $K \subset \mathbb{C}^n$ be compact and q an admissible weight on K. By Proposition 5.4 in [6] we have that $V^S_{K,q}$ is lower semicontinuous on $\mathbb{C}^{*n}$ and if furthermore $V^{S*}_{K,q} \le q$ on K then $V^{S*}_{K,q} = V^S_{K,q} \in \mathcal{L}^S(\mathbb{C}^n)$, and consequently, $V^S_{K,q}$ is continuous on $\mathbb{C}^{*n}$. Here $V^{S*}_{K,q}$ denotes the upper semicontinuous regularization of $V^S_{K,q}$. The assumption that $V^{S*}_{K,q} \le q$ is not restrictive, since $V^{S*}_{K+\varepsilon\overline{\mathbb{D}}^n,q'} \le q'$, where q' is a continuous (and thus admissible) weight on $K + \varepsilon\overline{\mathbb{D}}^n$, and $V^S_{K_j,q_j} \nearrow V^S_{K,q}$ if $K_j \searrow K$ and $q_j \nearrow q$. See Lemma 5.2, and Propositions 5.3 and 4.8 in [6]. The Siciak-Zakharyuta functions are also continuous under decreasing sequences in S under some conditions. Namely, if $T_j \searrow S$, such that $V^{T_j*}_{K,q} \le q$ for some $j \in \mathbb{N}$, then $V^{T_j}_{K,q} \searrow V^S_{K,q}$. See Proposition 4.8 in [6].

Fundamental to this study is that we can explicitly determine some Siciak-Zakharyuta functions. By Proposition 4.3 in [6] we know that $V^S_K = H_S$ if

$$\mathbb{T}^n \subset K \subset \{z \in \mathbb{C}^n\,; H_S(z) = 0\},$$

where $\mathbb{T}$ is the unit circle in $\mathbb{C}$. An example of such a K is $\overline{\mathbb{D}}^n$.

Lemma 2.2 in Nguyen and Long [8] contains a comparison result that can be used, in certain cases, to characterize the Siciak-Zakharyuta functions in terms of maximality. For the convenience of the reader we include it here, along with their proof.

Lemma 2.1. *Let $u, v \in \mathcal{PSH}(\mathbb{C}^n)$ such that:*

- **(i)** $u \le v$ *on* K,
- **(ii)** $\underline{\lim}_{|z|\to\infty} v(z) = +\infty$,
- **(iii)** $\sup_{z\in\mathbb{C}^n} u(z) - v(z) < +\infty$,
- **(iv)** v *is maximal on* $\mathbb{C}^n \setminus K$,
- **(v)** v *is bounded from below on* K,

then $u \le v$ *on* $\mathbb{C}^n$.

Proof. By **(v)** we may assume that $v \ge 0$ on K. We now fix $\lambda > 1$ and note that, by **(iii)**, there exists a constant C such that $u \le v + C$. By **(ii)** and the upper semicontinuity of v we can take $R > 0$ such that $v(z) > C(\lambda - 1)^{-1}$, for $z \in U_R$, where $U_R = \{z \in \mathbb{C}^n\,; |z| \ge R\}$. We then have

$$u(z) \le v(z) + C = \lambda v(z) + (1-\lambda)v(z) + C \le \lambda v(z), \quad z \in U_R.$$

By the positivity of v on K and **(i)** we have that $u \le \lambda v$ on $K \cup U_R$. Note that, by **(iv)**, v is maximal on $\mathbb{C}^n \setminus (K \cup U_R)$ so $u \le \lambda v$ on $\mathbb{C}^n$. This holds for all $\lambda > 1$ so we conclude that $u \le v$. □

Proposition 2.2. *Let $S \subset \mathbb{R}^n_+$ be compact and convex with $0 \in S$ such that S contains a neighborhood of 0 in $\mathbb{R}^n_+$, $K \subset \mathbb{C}^n$ compact, q an admissible weight on K, and $v \in \mathcal{L}^S_+(\mathbb{C}^n)$. If $V^{S*}_{K,q} \le v \le q$ on K and v is maximal on $\mathbb{C}^n \setminus K$ then $v = V^{S*}_{K,q}$ on $\mathbb{C}^n$.*

Proof. Since $v \in \mathcal{L}^S_+(\mathbb{C}^n)$ and $v \le q$ on K it is clear that $v \le V^{S*}_{K,q}$. By assumption S contains a neighborhood of 0 in $\mathbb{R}^n_+$, that is there exists $a > 0$ such that $a\Sigma \subset S$, so

$$v \ge H_S + C \ge a \log^+ \|\cdot\|_\infty + C.$$

This implies that $\underline{\lim}_{|z|\to\infty} v(z) = +\infty$ and v is bounded below on K. Hence, v and $u = V^{S*}_{K,q}$ satisfy the conditions of Lemma 2.1 and thus $V^{S*}_{K,q} \le v$. □

In [8] Nguyen and Long do not include the assumption that S contains a neighborhood of 0 in $\mathbb{R}^n_+$. So the proof of their Theorem 1.1 is incomplete.

Central to the proof of our main result is the Siciak-Zakharyuta theorem. The version we require can be found in [5], as Theorem 1.1, and is restated here for the convenience of the reader. The Siciak-Zakharyuta theorem relates $V^S_{K,q}$ to an extremal function given by polynomials. This function, the Siciak extremal function, is defined by

$$\Phi^S_{K,q}(z) = \overline{\lim_{m\to\infty}} \sup\{|p(z)|^{1/m} ; p \in \mathcal{P}^S_m(\mathbb{C}^n), pe^{-mq}|_K \le 1\}, \quad z \in \mathbb{C}^n,$$

where $\mathcal{P}^S_m(\mathbb{C}^n)$ consists of all polynomials p of the form $p(z) = \sum_{\alpha\in(mS)\cap\mathbb{N}^n} c_\alpha z^\alpha$.

Theorem 2.3 ([5], Theorem 1.1)**.** *Let $S \subset \mathbb{R}^n_+$ be compact convex and containing 0, $K \subset \mathbb{C}^n$ be compact, and q an admissible weight on K. Then*

$$V^S_{K,q}(z) = \log \Phi^S_{K,q}(z), \quad z \in \mathbb{C}^{*n},$$

if and only if $S = \overline{S \cap \mathbb{Q}^n}$.

Let us now turn our attention to the generalization of the product property. Recall that if $A \subset \mathbb{R}^n$ is compact and convex then its supporting function $\varphi_A(\xi) = \sup_{x\in A}\langle x, \xi\rangle$ is positively homogeneous and convex, that is $\varphi_A(t\xi) = t\varphi_A(\xi)$ and $\varphi_A(\xi+\eta) \le \varphi_A(\xi)+\varphi_A(\eta)$, for $t \in \mathbb{R}_+$ and $\eta, \xi \in \mathbb{R}^n$. Furthermore, if $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous and convex then it is the supporting function of precisely one compact convex set. See Theorem 2.2.8 in [2]. Let $0 \in S_j \subset \mathbb{R}^{n_j}$ be compact convex for $j = 1, \dots, \ell$ and $T \subset \mathbb{R}^\ell_+$ compact convex. Note that we do not assume that $0 \in T$. By this assumption φ_T is increasing in each variable, so if $u_j \in \mathcal{L}^{S_j}(\mathbb{C}^n)$ for $j = 1, \dots, \ell$, then

$$\begin{aligned} u(z) &= \varphi_T(u_1(z_1), \dots, u_\ell(z_\ell)) \\ &\le \varphi_T(H_{S_1}(z_1), \dots, H_{S_\ell}(z_\ell)) + \varphi_T(c), \quad z \in \mathbb{C}^n, \end{aligned}$$

for $c \in \mathbb{R}^\ell$. The convexity of φ_T implies that $u \in \mathcal{PSH}(\mathbb{C}^n)$. This leads us to define $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ by

$$\varphi(\xi) = \varphi_T(\varphi_{S_1}(\xi_1), \dots, \varphi_{S_\ell}(\xi_\ell)), \quad \xi = (\xi_1, \dots, \xi_\ell),\ \xi_j \in \mathbb{R}^{n_j},$$

for $j = 1, \dots, \ell$. Note that φ is positively homogeneous and convex so it is the supporting function of some compact convex $S \subset \mathbb{R}^n$. We have thus shown that $u \in \mathcal{L}^S(\mathbb{C}^n)$ and, since $\varphi_T(0) = 0$,

$$\varphi_T(V_{K_1}^{S_1}(z_1), \dots, V_{K_\ell}^{S_\ell}(z_\ell)) \le V_K^S(z), \quad z = (z_1, \dots, z_\ell),\ z_j \in \mathbb{C}^{n_j}, \tag{2.1}$$

where $K = K_1 \times \dots \times K_\ell$.

To determine an explicit formula for S we set

$$\widetilde{S}_j = \{0_1\} \times \dots \times \{0_{j-1}\} \times S_j \times \{0_{j+1}\} \times \dots \times \{0_\ell\},$$

where 0_j is the origin of $\mathbb{R}^{n_j}$. For $\xi = (\xi_1, \dots, \xi_\ell) \in \mathbb{R}^n$ we have $\varphi_{\widetilde{S}_j}(\xi) = \varphi_{S_j}(\xi_j)$, for $j = 1, \dots, \ell$, so $\varphi_S = \varphi_T(\varphi_{\widetilde{S}_1}, \dots, \varphi_{\widetilde{S}_\ell})$. For compact convex sets $A, B \subset \mathbb{R}^n$ and $a > 0$, we have that $a\varphi_A + \varphi_B = \varphi_{aA+B}$, and if $(A_\alpha)_{\alpha \in I}$, $I \neq \emptyset$, is a family of compact sets, $A = \operatorname{ch} \cup_{\alpha \in I} A_\alpha$, and

$$\varphi(\xi) = \sup_{\alpha \in I} \varphi_{A_\alpha}(\xi), \quad \xi \in \mathbb{R}^n,$$

is bounded for every $\xi \in \mathbb{R}^n$, then φ is the supporting function of A. We therefore have

$$\begin{aligned} \varphi_S(\xi) &= \sup_{x \in T}(x_1\varphi_{\widetilde{S}_1}(\xi) + \dots + x_\ell\varphi_{\widetilde{S}_\ell}(\xi)) \\ &= \sup_{x \in T}(\varphi_{x_1\widetilde{S}_1 + \dots + x_\ell\widetilde{S}_\ell}(\xi)), \qquad \xi \in \mathbb{R}^n, \end{aligned}$$

so $S = \operatorname{ch} \cup_{x \in T}(x_1\widetilde{S}_1 + \dots + x_\ell\widetilde{S}_\ell)$. Actually, the union is convex:

Lemma 2.4. *Let $A \subset \mathbb{R}_+^\ell$ and $B_1, \dots, B_\ell \subset \mathbb{R}^n$ be convex sets. Then*

$$C = \bigcup_{a \in A} a_1B_1 + \dots + a_\ell B_\ell$$

is a convex subset of $\mathbb{R}^n$. If $A, B_1, \dots, B_\ell$ are compact, then C is compact.

Proof. Let $c_1 = x_1w_1 + \dots + x_\ell w_\ell, c_2 = y_1z_1 + \dots + y_\ell z_\ell \in C$ and $t \in [0, 1]$, where $x = (x_1, \dots, x_\ell)$, $y = (y_1, \dots, y_\ell) \in A$ and $w_j, z_j \in B_j$ for $j = 1, \dots, \ell$. Then

$$(1-t)c_1 + tc_2 = (1-t)x_1w_1 + ty_1z_1 + \dots + (1-t)x_\ell w_\ell + ty_\ell z_\ell.$$

Since B_j is convex we have $(1-t)x_jw_j + ty_jz_j \in ((1-t)x_j + ty_j)B_j$, so for $a = (1-t)x + ty$ we have $(1-t)c_1 + tc_2 \in a_1B_1 + \dots + a_\ell B_\ell \subset C$. The last statement follows by a simple sequence argument. □

Finally note that $\varphi_S = \varphi_T(\varphi_{S_1}, \dots, \varphi_{S_\ell}) \ge 0$. It is well known that the supporting function of a compact convex set is positive if and only if the set contains the origin.

So, in conclusion, $S \subset \mathbb{R}^n_+$ is compact and convex, contains 0, is given by

$$S = \bigcup_{x \in T} x_1 \widetilde{S}_1 + \cdots + x_\ell \widetilde{S}_\ell = \bigcup_{x \in T} (x_1 S_1) \times \cdots \times (x_\ell S_\ell), \tag{2.2}$$

has the supporting function

$$\varphi_S(\xi) = \varphi_T(\varphi_{S_1}(\xi_1), \ldots, \varphi_{S_\ell}(\xi_\ell)), \quad \xi = (\xi_1, \ldots, \xi_\ell),\ \xi_j \in \mathbb{R}^{n_j}, \tag{2.3}$$

and the logarithmic supporting function

$$H_S(z) = \varphi_T(H_{S_1}(z_1), \ldots, H_{S_\ell}(z_\ell)), \quad z = (z_1, \ldots, z_\ell),\ z_j \in \mathbb{C}^{n_j}.$$

3 Proof of the main result

We will prove Theorem 1.1 by applying the Siciak-Zakharyuta theorem along with a product variant of the Bernstein-Walsh inequality. Recall that the Bernstein-Walsh inequality states that

$$|f(z)| \le \|f\|_{L^\infty(K)} e^{m V_K^S(z)}, \quad z \in \mathbb{C}^n,$$

for $0 \in S \subset \mathbb{R}^n_+$ compact and convex, $K \subset \mathbb{C}^n$ non-pluripolar compact, and $f \in \mathcal{P}_m^S(\mathbb{C}^n)$. This is a consequence of Theorem 3.6 in [6], namely that f, holomorphic on $\mathbb{C}^n$, is in $\mathcal{P}_m^S(\mathbb{C}^n)$ if and only if $\log |f|^{1/m} \in \mathcal{L}^S(\mathbb{C}^n)$.

The proof of the product variant of the Bernstein-Walsh inequality follows Klimek's proof of Theorem 5.1.8 in [3] and Siciak's proof of Proposition 5.9 in [9], in a similar way as the proof of Bos and Levenberg [1].

Proposition 3.1. *Let $K_j \subset \mathbb{C}^{n_j}$ be compact and non-pluripolar for $j = 1, \ldots, \ell$, and $K = K_1 \times \cdots \times K_\ell$, $n = n_1 + \cdots + n_\ell$, $T \subset \mathbb{R}^\ell_+$ be compact convex, $S_j \subset \mathbb{R}^{n_j}_+$ be compact convex and containing* 0*, for $j = 1, \ldots, \ell$, $S \subset \mathbb{R}^n_+$ be given by* (2.2)*, and $f \in \mathcal{P}_m^S(\mathbb{C}^n)$. Then*

$$|f(z)| \le \|f\|_{L^\infty(K)} e^{m \varphi_T(V_{K_1}^{S_1}(z_1), \ldots, V_{K_\ell}^{S_\ell}(z_\ell))}, \quad z = (z_1, \ldots, z_\ell),\ z_j \in \mathbb{C}^{n_j}.$$

Proof. We will assume that $\overline{S_j \cap \mathbb{Q}^{n_j}} \neq \{0\}$ for $j = 1, \ldots, \ell$. If this were not the case then $f(z)$, $z \in \mathbb{C}^n$, would not depend on $z_j \in \mathbb{C}^{n_j}$. We fix an $r > 0$ and let $G_j = K_j + r\mathbb{D}^{n_j}$ for $j = 1, \ldots, \ell$ and $\Omega_r = G_1 \times \cdots \times G_\ell$.

We now need to define an orthonormal basis for $\mathcal{P}^{S_j}(\mathbb{C}^{n_j})$ for $j = 1, \ldots, \ell$, as subspaces of $L^2(G_j)$. To do this we define $\rho_j \colon \mathbb{R}_+ S_j \cap \mathbb{N}^{n_j} \to \mathbb{R}_+$ by

$$\rho_j(\alpha_j) = \inf\{t \in \mathbb{R}\,;\, \alpha_j \in t S_j\}$$

and let $\kappa_j \colon \mathbb{N} \to \mathbb{R}_+ S_j \cap \mathbb{N}^{n_j}$ be a bijection such that $\rho_j(\kappa_j(k)) \le \rho_j(\kappa_j(k+1))$ for all $k \in \mathbb{N}$. We then set $e_k = z^{\kappa_j(k)}$, apply the Gram-Schmidt process to $(e_k)_{k \in \mathbb{N}}$ to get $(\hat{e}_k)_{k \in \mathbb{N}}$, and define $p_{j,\alpha_j} = \hat{e}_{\kappa_j^{-1}(\alpha_j)}$, for all $\alpha_j \in \mathbb{R}_+ S_j \cap \mathbb{N}^{n_j}$. This construction implies that if $z_j^{\alpha_j} = \sum_k c_{j,k} p_{j,\beta_k}(z_j)$ then $p_{j,\beta_k} \in \mathcal{P}_1^{\rho_j(\alpha_j) S_j}(\mathbb{C}^{n_j})$ for all $k \in \mathbb{N}$ such that $c_{j,k} \neq 0$ and $z_j \in \mathbb{C}^{n_j}$.

Now we define $p_\alpha = p_{1,\alpha_1} \dots p_{\ell,\alpha_\ell}$ for $\alpha = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{R}_+ S \cap \mathbb{N}^n$, where $\alpha_j \in \mathbb{N}^{n_j}$ for $j = 1, \dots, \ell$. We now need to show that $\{p_\alpha\,; \alpha \in \mathbb{R}_+ S \cap \mathbb{N}^n\}$ is a basis for $\mathcal{P}^S(\mathbb{C}^n)$. First we show that $p_\alpha \in \mathcal{P}^S(\mathbb{C}^n)$ for all $\alpha \in \mathbb{R}_+ S \cap \mathbb{N}^n$, and then we show that the span of $\{p_\alpha\,; \alpha \in \mathbb{R}_+ S \cap \mathbb{N}^n\}$ is $\mathcal{P}^S(\mathbb{C}^n)$.

Let $m \in \mathbb{N}$ and $\alpha = (\alpha_1, \dots, \alpha_\ell) \in mS \cap \mathbb{N}^n$, where $\alpha_j \in \mathbb{R}_+^{n_j}$ for $j = 1, \dots, \ell$. By the definition of S there exists $x \in T$ such that $\alpha_j \in m x_j S_j$ for $j = 1, \dots, \ell$. By construction $p_{j,\alpha_j} \in \mathcal{P}_m^{x_j S_j}(\mathbb{C}^{n_j})$ for $j = 1, \dots, \ell$, so for some $C > 0$, and all $z \in \mathbb{C}^n$,

$$\begin{aligned} |p_\alpha(z)| &= |p_{1,\alpha_1}(z_1)| \dots |p_{\ell,\alpha_\ell}(z_\ell)| \\ &\le C e^{mH_{x_1 S_1}(z_1)} \dots e^{mH_{x_\ell S_\ell}(z_\ell)} \\ &= C e^{m(x_1 H_{S_1}(z_1) + \dots + x_\ell H_{S_\ell}(z_\ell))} \\ &\le C e^{m\varphi_T(H_{S_1}(z_1), \dots, H_{S_\ell}(z_\ell))} \\ &= C e^{mH_S(z)}, \qquad z \in \mathbb{C}^n, \end{aligned}$$

so we infer $p_\alpha \in \mathcal{P}_m^S(\mathbb{C}^n)$.

To show that $\{p_\alpha\,; \alpha \in \mathbb{R}_+ S \cap \mathbb{N}^n\}$ spans $\mathcal{P}^S(\mathbb{C}^n)$ it is sufficient to show that z^α belongs to the span for all $\alpha \in \mathbb{R}_+ S \cap \mathbb{N}^n$. We let $m \in \mathbb{N}$ and $\alpha = (\alpha_1, \dots, \alpha_\ell) \in mS \cap \mathbb{N}^n$, where $\alpha_j \in \mathbb{R}_+^{n_j}$. Then there exists $x \in T$ such that $\alpha_j \in m x_j S_j$ for $j = 1, \dots, \ell$. For $j = 1, \dots, \ell$ we let $\beta_{j,k} \in \mathbb{R}_+ S_j \cap \mathbb{N}^{n_j}$ such that

$$z_j^{\alpha_j} = \sum_{k=1}^{d_j} c_{j,k} p_{j,\beta_{j,k}}(z_j), \quad z_j \in \mathbb{C}^{n_j},$$

where $c_{j,k} \in \mathbb{C}^*$, for $k = 1, \dots, d_j$. By construction of the bases we have, for $j = 1, \dots, \ell$, that $\beta_{j,k} \in m x_j S_j$ for $k = 1, \dots, d_j$. This then implies that $(\beta_{1,k_1}, \dots, \beta_{j,k_j}) \in mS$ for $k_j = 1, \dots, d_j$ and $j = 1, \dots, \ell$, and thus that $p_{1,\beta_{1,k_1}} \dots p_{\ell,\beta_{\ell,k_\ell}} \in \{p_\alpha\,; \alpha \in \mathbb{R}_+ S \cap \mathbb{N}^n\}$. We also have that

$$z^\alpha = \sum_{k_1=1}^{d_1} \cdots \sum_{k_j=1}^{d_j} c'_{k_1,\dots,k_\ell} p_{1,\beta_{1,k_1}}(z_1) \dots p_{\ell,\beta_{\ell,k_\ell}}(z_\ell), \quad z \in \mathbb{C}^n,$$

where $c'_{k_1,\dots,k_\ell} = c_{1,k_1} \dots c_{\ell,k_\ell}$. So $\{p_\alpha\,; \alpha \in \mathbb{R}_+ S \cap \mathbb{N}^n\}$ spans $\mathcal{P}^S(\mathbb{C}^n)$. We note as well that $\{p_\alpha\,; \alpha \in \mathbb{R}_+ S \cap \mathbb{N}^n\}$ is an orthonormal basis for $\mathcal{P}^S(\mathbb{C}^n)$ as a subspace of $L^2(\Omega_r)$.

We can now write

$$f(z) = \sum_{\alpha \in mS \cap \mathbb{N}^n} c_\alpha p_\alpha(z), \quad z \in \mathbb{C}^n, \tag{3.1}$$

where $c_\alpha = \langle f, p_\alpha \rangle$. By the Cauchy-Schwarz inequality $|c_\alpha| \le C_r \|f\|_{L^\infty(\Omega_r)}$, where $C_r = \mathrm{Vol}(\Omega_r)^{1/2}$. Let us now fix $z = (z_1, \dots, z_\ell) \in \mathbb{C}^n$, where $z_j \in \mathbb{C}^{n_j}$, for $j = 1, \dots, \ell$. If $\alpha = (\alpha_1, \dots, \alpha_\ell)$, where $\alpha_j \in m x_j S_j$ for $j = 1, \dots, \ell$ and $x \in T$, we get, by the submean value property (in each variable), that

$$|p_{j,\alpha_j}(a_j)|^2 \le (\pi r^2)^{-n_j} \int_{r\mathbb{D}^{n_j} + a_j} |p_{j,\alpha_j}(\zeta)|^2 d\lambda(\zeta) \le (\pi r^2)^{-n_j},$$

for $a_j \in K_j$ and $j = 1, \dots, \ell$. By the Bernstein-Walsh inequality

$$\begin{aligned}|p_\alpha(z)| &\leq (\pi r^2)^{-n/2} e^{mV_{K_1}^{x_1 S_1}(z_1)} \cdots e^{mV_{K_\ell}^{x_\ell S_\ell}(z_\ell)} \\ &= \pi^{-n/2} r^{-n} e^{m(x_1 V_{K_1}^{S_1}(z_1) + \cdots + x_\ell V_{K_\ell}^{S_\ell}(z_\ell))} \\ &\leq \pi^{-n/2} r^{-n} e^{m\varphi_T(V_{K_1}^{S_1}(z_1), \dots, V_{K_\ell}^{S_\ell}(z_\ell))}.\end{aligned}$$

Since $mS \subset m\sigma_S \Sigma \subset [0, m\sigma_S]^n$, the number of terms in the sum in equation (3.1) is no greater than $(m\sigma_S + 1)^n$, where $\sigma_S = \varphi_S(1, \dots, 1)$. So

$$|f(z)| \leq C_r \|f\|_{L^\infty(\Omega_r)} \pi^{-n/2} r^{-n} e^{m\varphi_T(V_{K_1}^{S_1}(z_1), \dots, V_{K_\ell}^{S_\ell}(z_\ell))} (m\sigma_S + 1)^n. \tag{3.2}$$

Applying the above inequality on $f^t \in \mathcal{P}_{mt}^S(\mathbb{C}^n)$, for $t \in \mathbb{N}$, we get

$$|f(z)|^t \leq C_r \|f\|_{L^\infty(\Omega_r)}^t \pi^{-n/2} r^{-n} e^{mt\varphi_T(V_{K_1}^{S_1}(z_1), \dots, V_{K_\ell}^{S_\ell}(z_\ell))} (mt\sigma_S + 1)^n.$$

Taking the t-th root improves the estimate in (3.2) to

$$|f(z)| \leq \|f\|_{L^\infty(\Omega_r)} e^{m\varphi_T(V_{K_1}^{S_1}(z_1), \dots, V_{K_\ell}^{S_\ell}(z_\ell))} \left(C_r \pi^{-n/2} r^{-n} (mt\sigma_S + 1)^n \right)^{1/t}.$$

We can now take the limit as t goes to infinity and then as r goes to zero to get

$$|f(z)| \leq \|f\|_{L^\infty(K)} e^{m\varphi_T(V_{K_1}^{S_1}(z_1), \dots, V_{K_\ell}^{S_\ell}(z_\ell))},$$

concluding the proof. □

Theorem 1.1 now follows from the Siciak-Zakharyuta Theorem 2.3, along with some regularization arguments discussed in Section 2, which we will now restate. Take $K \subset \mathbb{C}^n$ non-pluripolar and $0 \in S \subset \mathbb{R}_+^n$ compact and convex. Setting $K_\varepsilon = K + \varepsilon\overline{\mathbb{D}}^n$ we have that $V_{K_\varepsilon}^{S*} \leq 0$ on K and $V_{K_\varepsilon}^S \nearrow V_K^S$ when $\varepsilon \searrow 0$, see Propositions 4.8 and 5.3 in [6]. If $T_j \subset \mathbb{R}_+^n$ is a decreasing sequence of convex compact sets containing 0 then $V_K^{T_j} \searrow V_K^S$, when $j \to \infty$, if $V_K^{T_k*} \leq 0$ on K, for some $k \in \mathbb{N}$, see Proposition 4.8 in [6].

Proof of Theorem 1.1. Recall that, by (2.1), we have, for $z \in \mathbb{C}^n$,

$$\varphi_T(V_{K_1}^{S_1}(z_1), \dots, V_{K_\ell}^{S_\ell}(z_\ell)) \leq V_K^S(z),$$

so the goal is to prove the inverse inequality. To do this we use Theorem 2.3 and Proposition 3.1. For Theorem 2.3 to apply we assume some regularity on S and $V_K^S, V_{K_1}^{S_1}, \dots, V_{K_\ell}^{S_\ell}$. These assumptions are then relaxed using regularization arguments.

To start off we assume that $S = \overline{S \cap \mathbb{Q}^n}$ and $V_K^S, V_{K_1}^{S_1}, \dots, V_{K_\ell}^{S_\ell}$ are all plurisubharmonic. The second assumption implies that $\varphi_T(V_{K_1}^{S_1}, \dots, V_{K_\ell}^{S_\ell})$ is plurisubharmonic. Theorem 2.3 implies that for $z \in \mathbb{C}^{*n}$

$$V_K^S(z) = \varlimsup_{m \to \infty} \sup\{\log |f(z)|^{1/m} ; f \in \mathcal{P}_m^S(\mathbb{C}^n), f|_K \leq 1\},$$

which, with Proposition 3.1, yields, for $z \in \mathbb{C}^{*n}$,

$$V_K^S(z) \le \varphi_T(V_{K_1}^{S_1}(z_1), \dots, V_{K_\ell}^{S_\ell}(z_\ell)).$$

By assumption V_K^S and $\varphi_T(V_{K_1}^{S_1}(z_1), \dots, V_{K_\ell}^{S_\ell}(z_\ell))$ are plurisubharmonic, so

$$\begin{aligned} V_K^S(z) &= \varlimsup_{\mathbb{C}^{*n}\ni w\to z} V_K^S(w) \le \varlimsup_{\mathbb{C}^{*n}\ni w\to z} \varphi_T(V_{K_1}^{S_1}(w_1), \dots, V_{K_\ell}^{S_\ell}(w_\ell)) \\ &= \varphi_T(V_{K_1}^{S_1}(z_1), \dots, V_{K_\ell}^{S_\ell}(z_\ell)), \qquad z \in \mathbb{C}^n, \end{aligned}$$

since $\mathbb{C}^n \setminus \mathbb{C}^{*n}$ is pluripolar. Equality then follows from (2.1).

We now drop the assumptions on S and instead assume that $T \cap \mathbb{R}^{*\ell} \neq \emptyset$. Let $K_{j,\varepsilon} = K_j + \varepsilon\overline{\mathbb{D}}^{n_j}$, $K_\varepsilon = K + \varepsilon\overline{\mathbb{D}}^n = K_{1,\varepsilon} \times \cdots \times K_{\ell,\varepsilon}$, $S_{j,k} = \mathrm{ch}\{(1/k)\Sigma_j \cup S_j\}$, for $k = 1, 2, 3, \dots$, and

$$S_k' = \bigcup_{x\in T} (x_1 S_{1,k}) \times \cdots \times (x_\ell S_{\ell,k}) \subset \mathbb{R}_+^n.$$

If $x \in T \cap \mathbb{R}^{*\ell}$ then $(x_1 S_{1,k}) \times \cdots \times (x_\ell S_{\ell,k}) \subset S_k'$. Since $x_1 S_{1,k}, \dots, x_\ell S_{\ell,k}$ are all convex bodies, S_k' is also a convex body. Consequently, $\overline{S_k' \cap \mathbb{Q}^n} = S_k'$ and

$$V_{K_\varepsilon}^{S_k'}(z) = \varphi_T(V_{K_{1,\varepsilon}}^{S_{1,k}}(z_1), \dots, V_{K_{\ell,\varepsilon}}^{S_{\ell,k}}(z_\ell)), \quad z \in \mathbb{C}^n.$$

So, by the continuity of φ_T, we have

$$\begin{aligned} V_{K_\varepsilon}^S(z) &= \lim_{k\to\infty} V_{K_\varepsilon}^{S_k'}(z) = \lim_{k\to\infty} \varphi_T(V_{K_{1,\varepsilon}}^{S_{1,k}}(z_1), \dots, V_{K_{\ell,\varepsilon}}^{S_{\ell,k}}(z_\ell)) \\ &= \varphi_T(V_{K_{1,\varepsilon}}^{S_1}(z_1), \dots, V_{K_{\ell,\varepsilon}}^{S_\ell}(z_\ell)), \qquad z \in \mathbb{C}^n. \end{aligned}$$

and

$$\begin{aligned} V_K^S(z) &= \lim_{\varepsilon\to 0} V_{K_\varepsilon}^S(z) = \lim_{\varepsilon\to 0} \varphi_T(V_{K_{1,\varepsilon}}^{S_1}(z_1), \dots, V_{K_{\ell,\varepsilon}}^{S_\ell}(z_\ell)) \\ &= \varphi_T(V_{K_1}^{S_1}(z_1), \dots, V_{K_\ell}^{S_\ell}(z_\ell)), \qquad z \in \mathbb{C}^n. \end{aligned}$$

Lastly we assume $T \cap \mathbb{R}^{*\ell} = \emptyset$ and $T \neq \{0\}$. If $T = \{0\}$ then $V_K^S = 0$ and $\varphi_T = 0$ so there is nothing to prove. By rearranging the coordinates, we can assume that $T = A \times \{0\}$, where $A \subset \mathbb{R}_+^k$ satisfies that $A \cap \mathbb{R}^{*k} \neq \emptyset$ and $k < \ell$. Note that $\varphi_T(\xi) = \varphi_A(\xi_1, \dots, \xi_k)$, so $\varphi_S(\xi) = \varphi_{A'}(\xi_1, \dots, \xi_\nu)$, where $\nu = n_1 + \cdots + n_k$ and

$$A' = \bigcup_{x\in A} (x_1 S_1) \times \cdots \times (x_k S_k).$$

We then get, by the Liouville theorem for subharmonic functions, that functions in $\mathcal{L}^S(\mathbb{C}^n)$ only depend on their first ν variables. So

$$\begin{aligned} V_K^S(z) &= V_{\tilde{K}}^{A'}(z) = \varphi_A(V_{K_1}^{S_1}(z_1), \dots, V_{K_k}^{S_k}(z_k)) \\ &= \varphi_T(V_{K_1}^{S_1}(z_1), \dots, V_{K_\ell}^{S_\ell}(z_\ell)), \qquad z \in \mathbb{C}^n, \end{aligned}$$

where $\tilde{K} = K_1 \times \cdots \times K_k$. □

4 The weighted case

We would like to prove a version of Theorem 1.1 which includes weights. One approach is to try to find the correct representation of q such that

$$V^S_{K,q}(z) = \varphi_T(V^{S_1}_{K_1,q_1}(z_1), \dots, V^{S_\ell}_{K_\ell,q_\ell}(z_\ell)), \quad z \in \mathbb{C}^n,$$

holds. A natural first guess is to take $q = \varphi_T(q_1, \dots, q_\ell)$. The following results will show that this choice of q is not correct. We will then show that no choice of q will work in the generality of Theorem 1.1, since the function $\varphi_T(V^{S_1}_{K_1,q_1}(z_1), \dots, V^{S_\ell}_{K_\ell,q_\ell}(z_\ell))$ may fail to be maximal outside of K, which is a necessary condition for $V^S_{K,q}$, for any choice of an admissible weight q. See [6], Theorem 6.1.

Proposition 4.1. *Let $n_1, \dots, n_\ell$ be natural numbers, $K_j \subset \mathbb{C}^{n_j}$ be compact and non-pluripolar for $j = 1, \dots, \ell$, and $T \subset \mathbb{R}^\ell_+$ be compact convex and containing more than one point, and $\{0\} \neq S_j \subset \mathbb{R}^{n_j}_+$ be compact convex and containing 0, for $j = 1, \dots, \ell$, such that $V^{S_j *}_{K_j} = 0$, on K_j, for $j = 1, \dots, \ell$. Then there exist admissible weights $q_1, \dots, q_\ell$ on $K_1, \dots, K_\ell$ respectively, such that, for some $z \in \mathbb{C}^n$,*

$$V^S_{K,q}(z) > \varphi_T(V^{S_1}_{K_1,q_1}(z_1), \dots, V^{S_\ell}_{K_\ell,q_\ell}(z_\ell)),$$

where $K = K_1 \times \cdots \times K_\ell$, $q = \varphi_T(q_1, \dots, q_\ell)$, and S is given by equation (2.2). Furthermore, if T is a convex body then the previous statement holds for all constant weights $q_j < 0$, $j = 1, \dots, \ell$.

Proof. Recall that if $\eta \in \mathbb{R}^\ell$ with $|\eta| = 1$ then $\{x \in \mathbb{R}^\ell\,; \langle x, \eta\rangle = \varphi_T(\eta)\}$ and $\{x \in \mathbb{R}^\ell\,; \langle x, \eta\rangle = -\varphi_T(-\eta)\}$ are supporting hyperplanes of S with outward normals η and $-\eta$, respectively. The distance between these parallel hyperplanes is $\varphi_T(\eta) + \varphi_T(-\eta)$, since $0 = \varphi_T(0) \leq \varphi_T(\eta) + \varphi_T(-\eta)$. So $\operatorname{diam} T \geq \varphi_T(\eta) + \varphi_T(-\eta)$, where $\operatorname{diam} T$ denotes the diameter of T. If $L \subset T$ is a line segment parallel to η_L, with $|\eta_L| = 1$, then $\varphi_T(\eta_L) + \varphi_T(-\eta_L) \geq \varphi_L(\eta_L) + \varphi_L(-\eta_L) = \operatorname{diam} L$. We can take L such that $\operatorname{diam} L = \operatorname{diam} T$, so

$$\operatorname{diam} T = \sup_{|\eta|=1} (\varphi_T(\eta) + \varphi_T(-\eta)).$$

Consequently, since T contains more than one point, we can find $\eta \in \mathbb{R}^\ell_+$ such that $\varphi_T(\eta) + \varphi_T(-\eta) > 0$.

Since $V^{S_j *}_{K_j} = 0$ on K_j, Proposition 5.4 in [6] implies that $V^{S_j}_{K_j}$ is continuous on $\mathbb{C}^{*n_j}$ and consequently $V^{S_j}_{K_j}(\mathbb{C}^{n_j}) = \mathbb{R}_+$.

Now let $\eta \in \mathbb{R}^\ell_+$ such that $\varphi_T(\eta) + \varphi_T(-\eta) > 0$, $z_j \in \mathbb{C}^{n_j}$ be such that $V^{S_j}_{K_j}(z_j) = \eta_j$, and $q_j(w) = -\eta_j$, for $w \in K_j$ and $j = 1, \dots, \ell$. Then $V^{S_j}_{K_j,q_j}(z_j) = 0$ and

$$\varphi_T(V^{S_1}_{K_1,q_1}(z_1), \dots, V^{S_\ell}_{K_\ell,q_\ell}(z_\ell)) = 0.$$

But, we have by Theorem 1.1 that

$$V^S_{K,q}(z) = V^S_K(z) + \varphi_T(q_1, \dots, q_\ell) = \varphi_T(V^{S_1}_{K_1}(z_1), \dots, V^{S_\ell}_{K_\ell}(z_\ell)) + \varphi_T(-\eta)$$

$$= \varphi_T(\eta) + \varphi_T(-\eta) > 0.$$

So $V^S_{K,q}(z) > \varphi_T(V^{S_1}_{K_1,q_1}(z_1), \dots, V^{S_\ell}_{K_\ell,q_\ell}(z_\ell))$.

Now assume that T is a convex body and $q_j < 0$ are given constant weights. Then T contains a Euclidean ball with radius r and thus

$$\varphi_T(\eta) + \varphi_T(-\eta) \geq 2r|\eta|,$$

for all $\eta \in \mathbb{R}^\ell$. So we set $\eta_j = -q_j$ and proceed as previously in the proof. □

Note that all the weights in the previous theorem are negative. Counterexamples with positive weights can, however, be inferred. Take $T = \Sigma$. Then $\varphi_T(\xi) = \max\{\xi_1, \dots, \xi_\ell\}$. For $a \in \mathbb{R}$ we have that $V^S_{K,q} + a = V^S_{K,q+a}$ and

$$\varphi_T(V^{S_1}_{K_1,q_1}, \dots, V^{S_\ell}_{K_\ell,q_\ell}) + a = \varphi_T(V^{S_1}_{K_1,q_1+a}, \dots, V^{S_\ell}_{K_\ell,q_\ell+a}).$$

Now we turn to the question of whether there is any way to choose q to attain an equality. Recall that, for compact $K \subset \mathbb{C}^n$, compact convex $S \subset \mathbb{R}^n_+$ such that $0 \in S$ and $a\Sigma \subset S$ for some $a > 0$, and admissible weight q on K, Proposition 2.2 states that the only function $v \in \mathcal{L}^S_+(\mathbb{C}^n)$ that is maximal on $\mathbb{C}^n \setminus K$ and agrees with $V^{S*}_{K,q}$ on K, is $V^{S*}_{K,q}$ itself. This enables us to show that $\varphi_S(V^{S_1}_{K_1,q_1}(z_1), \dots, V^{S_\ell}_{K_\ell,q_\ell}(z_\ell))$ is not generally maximal outside of $K_1 \times \cdots \times K_\ell$.

Proposition 4.2. *Let $n_1, \dots, n_\ell$ be natural numbers, $K_j \subset \mathbb{C}^{n_j}$ be compact and non-pluripolar for $j = 1, \dots, \ell$, and $T \subset \mathbb{R}^\ell_+$ be compact convex body, $S_j \subset \mathbb{R}^{n_j}_+$ be compact convex and containing 0 with $a\Sigma_j \subset S_j$ for some a, for $j = 1, \dots, \ell$, such that $V^{S_j*}_{K_j} = 0$ on K_j, for $j = 1, \dots, \ell$, and $q_j < 0$ is a constant weight on K_j, for $j = 1, \dots, \ell$. Then*

$$\begin{aligned} V(z) &= \varphi_T(V^{S_1}_{K_1,q_1}(z_1), \dots, V^{S_\ell}_{K_\ell,q_\ell}(z_\ell)) \\ &= \varphi_T(V^{S_1}_{K_1}(z_1) + q_1, \dots, V^{S_\ell}_{K_\ell}(z_\ell) + q_\ell), \quad z \in \mathbb{C}^n, \end{aligned}$$

is not maximal on $\mathbb{C}^n \setminus K$.

Proof. Let $K = K_1 \times \cdots \times K_\ell$ and S be given by (2.2). We have that $V^{S_j}_{K_j,q_j} \in \mathcal{L}^{S_j}_+(\mathbb{C}^{n_j})$, see Proposition 4.5 in [6], so there exists a constant c such that $H_{S_j} - c \leq V^{S_j}_{K_j,q_j}$. Since $\varphi_T(\xi) \leq \varphi_T(\xi - \eta) + \varphi_T(\eta)$ holds for all $\xi, \eta \in \mathbb{R}^n$, we have

$$H_S - \varphi_T(c, \dots, c) \leq \varphi_T(H_{S_1} - c, \dots, H_{S_\ell} - c) \leq V.$$

So $V|_K = \varphi_T(q_1, \dots, q_\ell)$ and $V \in \mathcal{L}^S_+(\mathbb{C}^n)$. If V were maximal outside of K then, by Proposition 2.2, we would have $V = V^S_{K,q}$ where $q = \varphi_T(q_1, \dots, q_\ell)$, contradicting Proposition 4.1. So V is not maximal outside of K. □

5 Convexity of sublevel sets

Theorem 1.2 in [8] states that for $t > 0$, convex body $0 \in S \subset \mathbb{R}^n_+$, and compact convex $K \subset \mathbb{C}^n$, the sublevel set $\{z \in \mathbb{C}^n\,; V^S_K(z) < t\}$ is convex. In this section we will show that this does not hold in the state generality. The error in the proof in [8] is in the first equality on page 516, where $\max\{a,b\}+\max\{c,d\} = \max\{a+c,b+d\}$ is used. This identity does not hold generally. To describe for which sets S the result can not hold we define, for $x \in \mathbb{R}_+^{*n}$, the *simplex given by* x as

$$\Sigma_x = \mathrm{ch}\{0, x_1 e_1, \dots, x_n e_n\} = \{\xi \in \mathbb{R}_+^n\,;\xi_1/x_1 + \cdots + \xi_n/x_n \le 1\},$$

where $e_1, \dots, e_n$ is the standard basis for $\mathbb{R}^n$.

Proposition 5.1. *Let $S \neq \{0\}$ be a compact convex subset of $\mathbb{R}_+^n$ containing 0 that is not a simplex. Then there exists $t_0 > 0$ such that the sublevel set*

$$\{z \in \mathbb{C}^n\,;H_S(z) \le t\}$$

is not convex for all $t > t_0$.

Proof. Assume first that S contains a neighborhood of 0 in $\mathbb{R}_+^n$ and define $x \in \mathbb{R}_+^n$ by $x_j = \max\{t \in \mathbb{R}\,;te_j \in S\} > 0$. By assumption there is an $s \in S$ such that $s \notin \Sigma_x$, and consequently $s_1/x_1 + \cdots + s_n/x_n > 1$. For $a > 1$ we have by Proposition 3.3 in [6] that $H_S(a^{1/x_j}e_j) = \log a$, for $j = 1, \dots, n$, and

$$H_S(a^{1/x_1}/n, \dots, a^{1/x_n}/n) \ge s_1 \log(a^{1/x_1}/n) + \cdots + s_n \log(a^{1/x_n}/n)$$

$$= (s_1/x_1 + \cdots + s_n/x_n)\log a - (s_1 + \cdots + s_n)\log n.$$

Setting

$$t_0 = \frac{(s_1 + \cdots + s_n)\log n}{s_1/x_1 + \cdots + s_n/x_n - 1} > 0,$$

$t > t_0$, and $a = e^t$, we have that $\log a > t_0$ and consequently

$$H_S(a^{1/x_1}/n, \dots, a^{1/x_n}/n) > \log a = H_S(a^{1/x_j}e_j)$$

for $j = 1, \dots, n$. Since $t = \log a$ we have that $a^{1/x_1}e_j, \dots, a^{1/x_n}e_n$ are all in the sublevel set $\{z \in \mathbb{C}^n\,;H_S(z) \le t\}$, but their average is not. So the sublevel set is not convex.

Now assume S does not contain a neighborhood of 0. Then, by possibly rearranging the variables, we may assume that $H_S(\zeta, 0, \dots, 0) = 0$ for $\zeta \in \mathbb{C}$. Since $S \neq \{0\}$ we may assume that there is an $s \in S$ such that $s_1 > 0$, since otherwise we could write $S = \{0\} \times T$ reducing the problem to a lower dimension. Let us now fix $t > 0$ and note that

$$H_S(\zeta/2, 1/2, \dots, 1/2) \ge s_1 \log|\zeta| - (s_1 + s_2 + \cdots + s_n)\log 2,$$

for $\zeta \in \mathbb{C}$, so we can choose $\tau \in \mathbb{C}$ such that $H_S(\tau/2, 1/2, \dots, 1/2) > t$. We have that $H_S(\tau, 0, \dots, 0) = H_S(0, 1, \dots, 1) = 0$, so both $(\tau, 0, \dots, 0)$ and $H_S(0, 1, \dots, 1)$ are in the sublevel set $\{z \in \mathbb{C}^n\,;H_S(z) \le t\}$ but their average is not. So the sublevel set is not convex. □

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Paper IV

Polynomials with exponents in compact convex sets and associated weighted extremal functions - The Bernstein-Walsh-Siciak theorem

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Abstract

We generalize the Bernstein-Walsh-Siciak theorem on polynomial approximation in $\mathbb{C}^n$ to the case where the polynomial ring $\mathcal{P}(\mathbb{C}^n)$ is replaced by a subring $\mathcal{P}^S(\mathbb{C}^n)$ consisting of all polynomials with exponents restricted to sets mS, where S is a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$ and $m = 0, 1, 2, 3, \dots$, and uniform estimates of error in the approximation are replaced by weighted uniform estimates with respect to an admissible weight function.

1 Introduction

The Runge-Oka-Weil theorem states that if K is a compact polynomially convex subset of $\mathbb{C}^n$ and f is a holomorphic function in some neighborhood of K, then f can be approximated uniformly on K by polynomials. We let $\mathcal{P}_m(\mathbb{C}^n)$ denote the space of polynomials of degree $\leq m$ in n complex variables and let

$$d_{K,m}(f) = \inf\{\|f - p\|_K\,;\, p \in \mathcal{P}_m(\mathbb{C}^n)\}$$

denote the smallest error in an approximation of f by polynomials of degree $\leq m$, i.e., the distance from f to $\mathcal{P}_m(\mathbb{C}^n)$ in the supremum norm $\|\cdot\|_K$ on $\mathcal{C}(K)$. Then the Runge-Oka-Weil theorem is equivalent to stating that

$$\lim_{m\to\infty} d_{K,m}(f) = 0.$$

The Bernstein-Walsh-Siciak theorem states that

$$\varlimsup_{m\to\infty} d_{K,m}(f)^{1/m} \leq \frac{1}{R}$$

if and only if f has a holomorphic extension to $X_R = \{z \in \mathbb{C}^n\,;\, \Phi_K(z) < R\}$, where $\Phi_K = \varlimsup_{m\to\infty} \Phi_{K,m}$ and $\Phi_{K,m} = \sup\{|p|^{1/m}\,;\, p \in \mathcal{P}_m(\mathbb{C}^n), \|p\|_K \leq 1\}$ are the Siciak functions of the set K and it is assumed that Φ_K is continuous and $R \geq 1$. For more on this result see Siciak [7, §10].

In this paper, we generalize the Bernstein-Walsh-Siciak theorem where $\mathcal{P}_m(\mathbb{C}^n)$ is replaced by the space $\mathcal{P}^S_m(\mathbb{C}^n)$ of all polynomials p of the form

$$p(z) = \sum_{\alpha \in (mS)\cap \mathbb{N}^n} a_\alpha z^\alpha, \quad z \in \mathbb{C}^n,$$

for a given compact convex subset S of $\mathbb{R}^n_+$ with $0 \in S$, and $\mathcal{P}(\mathbb{C}^n)$ is replaced by $\mathcal{P}^S(\mathbb{C}^n) = \cup_{m\in\mathbb{N}} \mathcal{P}^S_m(\mathbb{C}^n)$. For every function $q\colon E \to \mathbb{R} \cup \{+\infty\}$ defined on a subset E of $\mathbb{C}^n$ and $m = 1, 2, 3, \dots$ we define the *Siciak functions with respect to S, E, q, and m* by

$$\Phi^S_{E,q,m}(z) = \sup\{|p(z)|^{1/m}\,;\, p \in \mathcal{P}^S_m(\mathbb{C}^n), \|pe^{-mq}\|_E \le 1\}, \quad z \in \mathbb{C}^n,$$

and the Siciak function with respect to S, E, and q by $\Phi^S_{E,q} = \overline{\lim}_{m\to\infty} \Phi^S_{E,q,m}$. By Proposition 2.2 in [6] we have that

$$\Phi^S_{E,q}(z) = \sup_{m\in\mathbb{N}} \Phi^S_{E,q,m}(z) = \lim_{m\to\infty} \Phi^S_{E,q,m}(z), \quad z \in \mathbb{C}^n, \tag{1.1}$$

and if q is bounded below and $\Phi^S_{E,q}$ is continuous on a compact $X \subset \mathbb{C}^n$, then the convergence is uniform on X.

The Lelong class with respect to S, denoted by $\mathcal{L}^S(\mathbb{C}^n)$, is defined in terms of the supporting function φ_S of S, given by $\varphi_S(\xi) = \sup_{s\in S}\langle s, \xi\rangle$ for $\xi \in \mathbb{R}^n$, and the map $\operatorname{Log}\colon \mathbb{C}^{*n} \to \mathbb{R}^n$, given by $\operatorname{Log} z = (\log|z_1|, \dots, \log|z_n|)$. We define the *logarithmic supporting function* $H_S \in \mathcal{PSH}(\mathbb{C}^n)$ of S by

$$H_S = \varphi_S \circ \operatorname{Log}$$

on $\mathbb{C}^{*n}$ and extend the definition to the coordinate hyperplanes by the formula

$$H_S(z) = \overline{\lim_{\mathbb{C}^{*n}\ni w\to z}} H_S(w), \quad z \in \mathbb{C}^n \setminus \mathbb{C}^{*n}.$$

The Lelong class $\mathcal{L}^S(\mathbb{C}^n)$ is defined as the set of all $u \in \mathcal{PSH}(\mathbb{C}^n)$ satisfying a growth estimate of the form $u \le c_u + H_S$ for some constant c_u. The *Siciak-Zakharyuta function with respect to S, $E \subset \mathbb{C}^n$, and $q\colon E \to \mathbb{R} \cup \{+\infty\}$* is defined by

$$V^S_{E,q}(z) = \sup\{u(z)\,;\, u \in \mathcal{L}^S(\mathbb{C}^n), u|_E \le q\}, \quad z \in \mathbb{C}^n.$$

Finally, the distance from a bounded function f on E to $\mathcal{P}^S_m(\mathbb{C}^n)$ with respect to the supremum norm on E with weight e^{-mq} is defined by

$$d^S_{E,q,m}(f) = \inf\{\|(f-p)e^{-mq}\|_E\,;\, p \in \mathcal{P}^S_m(\mathbb{C}^n)\}.$$

We call the sequence $(d^S_{E,q,m}(f))_{m\in\mathbb{N}}$ the *approximation numbers of f on E with respect to S and q.*

Observe that the standard simplex $\Sigma = \operatorname{ch}\{0, e_1, \dots, e_n\}$ has the supporting function $\varphi_\Sigma(\xi) = \max\{\xi_1^+ \dots, \xi_n^+\}$ where $\xi_j^+ = \max\{\xi_j, 0\}$ and that the logarithmic supporting function is $H_\Sigma(z) = \log^+ \|z\|_\infty$, which implies that the Lelong class $\mathcal{L}^\Sigma(\mathbb{C}^n)$ is the standard Lelong class $\mathcal{L}(\mathbb{C}^n)$, see for example Section 5 in Klimek [4], and the polynomial space $\mathcal{P}_m^\Sigma(\mathbb{C}^n)$ is $\mathcal{P}_m(\mathbb{C}^n)$. We drop the superscript S in the case $S = \Sigma$ and the subscript q in the case $q = 0$.

The function $q\colon E \to \mathbb{R} \cup \{+\infty\}$ is said to be an *admissible weight with respect to* S *on* E if q is lower semi-continuous, the set $\{z \in E\,;\, q(z) < +\infty\}$ is non-pluripolar, and for E unbounded $\lim_{E\ni z,|z|\to\infty}(H_S(z) - q(z)) = -\infty$.

By Proposition 3.6 in [6], an entire function p is in $\mathcal{P}_m^S(\mathbb{C}^n)$ if and only if $\log|p|^{1/m}$ is in $\mathcal{L}^S(\mathbb{C}^n)$. This implies that $\log \Phi_{E,q}^S \le V_{E,q}^S$. In Corollary 4.7 in [6], we have examples where equality does not hold and in Theorem 1.1 in [5] it is proved that for an admissible weight q on a closed set $E \subset \mathbb{C}^n$ the equality $V_{E,q}^S = \log \Phi_{E,q}^S$ holds on $\mathbb{C}^{*n}$ if and only if $S \cap \mathbb{Q}^n$ is dense in S.

Bos and Levenberg [1, Theorem 3.1] generalized the Bernstein-Walsh-Siciak theorem to the case where the polynomial ring is $\mathcal{P}^S(\mathbb{C}^n)$ and the weight is $q = 0$, with the assumption that V_K^S is continuous and S is a lower set. We say that S is a *lower set* if for all $x \in S$ the box $[0, x_1] \times \cdots \times [0, x_n] \subset S$. Lower sets can be described in terms of their supporting functions, logarithmic supporting functions, and respective Lelong classes. See Theorem 5.8 in [6] and Theorem 5.1 in [10].

We have not been able to find a generalization of the Bernstein-Walsh-Siciak theorem, with weighted uniform estimates, in the form of an equivalence statement. We have to separate our results into two parts, where in the first part we assume that the approximation numbers $d_{K,q,m}^S(f)$ decrease exponentially with m. One reason for this separation is the different influences S and q have. The added detail to Theorem 1.1 is due to the weight q, while most of the added assumptions in Theorem 1.2 are because of S.

Theorem 1.1. *Let $S \subset \mathbb{R}_+^n$ be a compact convex set with $0 \in S$, q be an admissible weight on a compact subset K of $\mathbb{C}^n$, such that $V_{K,q}^{S*} \le q$ on K, and for every $r > 0$ define $X_r = \{z \in \mathbb{C}^n\,;\, V_{K,q}^S(z) < \log r\}$. Let $f\colon K \to \mathbb{C}$ be bounded, assume that*

$$L = \{z \in K\,;\, \lim_{m\to\infty} d_{K,q,m}^S(f) e^{mq(z)} = 0\} \neq \varnothing$$

and that for some $R > 0$, $K \subset X_R$, and

$$\varlimsup_{m\to\infty} d_{K,q,m}^S(f)^{1/m} \le \frac{1}{R}. \tag{1.2}$$

Then the following hold:

(i) *For every $\gamma > 0$, such that $K \subset X_{R-\gamma}$, the function $f|_L$ extends to a holomorphic function $F_\gamma \in \mathcal{O}(X_{R-\gamma})$.*

(ii) *If X is an open component of X_R and $L_X = L \cap X$ is non-pluripolar, then $f|_{L_X}$ extends to a unique holomorphic function on X.*

(iii) *If $q < \log R$ then $L = K$ and, consequently, there exists $F \in \mathcal{O}(X_R)$ such that $F|_K = f$.*

If p_m are such that $\|(f - p_m)e^{-mq}\|_K$ is close to $d^S_{K,q,m}(f)$, as in Section 2, then L denotes the set where $p_m \to f$, pointwise. If q is bounded above then $p_m \to f$ uniformly on L.

The assumption that $V^{S*}_{K,q} \le q$ on K implies that $V^S_{K,q}$ is upper semicontinuous and that the sublevel sets X_r are all open. See Proposition 4.5 in [6]. Here $V^{S*}_{K,q}(z) = \overline{\lim}_{w\to z} V^S_{K,q}(w)$ denotes the *upper semicontinuous regularization* of $V^S_{K,q}$.

For the converse statement, we need the concept of a hull of a compact convex subset S of $\mathbb{R}^n_+$ with respect to a cone Γ,

$$\widehat{S}_\Gamma = \{x \in \mathbb{R}^n_+ \,;\, \langle x, \xi\rangle \le \varphi_S(\xi), \forall \xi \in \Gamma\}. \tag{1.3}$$

We have $S = \widehat{S}_{\mathbb{R}^n}$ for every compact convex subset S of $\mathbb{R}^n_+$ and if $\Gamma_1 \subseteq \Gamma_2$ then $\widehat{S}_{\Gamma_2} \subseteq \widehat{S}_{\Gamma_1}$. Since $S \subset \mathbb{R}^n_+$ and $0 \in S$, we have that $\varphi_S = 0$ on $\mathbb{R}^n_-$ and $S = \widehat{S}_{(\mathbb{R}^n\setminus\mathbb{R}^n_-)\cup\{0\}}$.

Theorem 1.2. *Let $S \subset \mathbb{R}^n_+$ be a compact convex set with $0 \in S$ and $S = \overline{S \cap \mathbb{Q}^n}$, q be an admissible weight on a compact subset K of $\mathbb{C}^n$, and $R > 0$. Assume that $V^S_{K,q}$ is continuous, X_R is bounded, and $K \subset X_R$. Let $d_m = d(mS, \mathbb{N}^n \setminus mS)$ be the euclidean distance between mS and $\mathbb{N}^n \setminus mS$. If $f \in \mathcal{O}(X_R)$ then*

$$\overline{\lim_{m\to\infty}}\, d^{\widehat{S}_{\Gamma_m}}_{K,q,m}(f)^{1/m} \le \frac{1}{R}, \tag{1.4}$$

where $\Gamma_m = \{\xi \in \mathbb{R}^n \,;\, \langle \mathbf{1}, \xi\rangle \ge -\frac{1}{2}d_m|\xi|\}$, and the hull $\widehat{S}_{\Gamma_m}$ with respect to Γ_m is defined by (1.3).

In the special case when S is a lower set we have $\widehat{S}_{\mathbb{R}^n_+} = S$ by Theorem 5.8 in [6]. So $S \subset \widehat{S}_{\Gamma_m} \subset \widehat{S}_{\mathbb{R}^n_+} = S$, since $\mathbb{R}^n_+ \subset \Gamma_m$. It is possible that S is a lower set, but not a convex body, that is S may be a lower set with an empty interior. These lower sets, however, are not interesting in their own right, since they can be written, after possibly rearranging the variables, as $S = T \times \{0\}^{n-\ell}$, where $T \subset \mathbb{R}^\ell_+$ is lower body. In this case $\varphi_S(\xi) = \varphi_T(\xi_1, \dots, \xi_\ell)$, so the function in $\mathcal{L}^S(\mathbb{C}^n)$ only depend on the first ℓ variables. When S is a lower body we also have that the sublevel sets X_r are bounded, for $r > 0$. This holds since there exists $a > 0$ such that $a\Sigma \subset S$, so $H_S \ge a \log^+ \|\cdot\|_\infty$. By Proposition 4.5 in [6], there exists a constant c such that $V^S_{K,q} \ge H_S + c \ge a \log \|\cdot\|_\infty + c$. So when S is a lower body the sublevel sets of $V^S_{K,q}$ are bounded.

Corollary 1.3. *Let S be a lower set, q be an admissible weight on a compact subset K of $\mathbb{C}^n$ and assume that $V^S_{K,q}$ is continuous and $K \subset X_R$ for some $R > 0$. If $f \in \mathcal{O}(X_R)$, then* (1.2) *holds.*

A Bernstein-Walsh-Siciak theorem for a lower set S and weight $q = 0$ is proved in Bos and Levenberg [1, Theorem 3.1]. Their result follows from Corollary 1.3, since if $q = 0$ then $K \subset X_R$ if and only if $R > 1$.

Corollary 1.4. *Let $R > 1$, S be a lower set, and $K \subset \mathbb{C}^n$ be a compact set such that V_K^S is continuous. Then $f \in \mathcal{O}(X_R)$ if and only if $\varlimsup_{m\to\infty} d_{K,m}^S(f)^{1/m} \leq 1/R$.*

Setting $S = \Sigma$ in Corollary 1.3 we get a weighted Bernstein-Walsh-Siciak theorem. We include it to justify our definition of the approximation numbers $d_{K,q,m}^S(f)$. In [2, Theorem 2.4] a weighted Bernstein-Walsh theorem is proved, for $n = 1$, using approximation by weighted polynomials. They use $d'_{K,q,m}(f) = \inf\{\|f - e^{-mq}p\|\,;\, p \in \mathcal{P}_m(\mathbb{C}^n)\}$, with the assumption that e^{-q} extends as an entire function which is non-vanishing on K.

Approximations of this form have been studied for decades, but it does not match some of our intuitions for $V_{K,q}^S$. To see this we note that $d'_{K,q,m}(f) = d_{K,m}(f)$ when $q = a$ is any constant weight, whereas $d_{K,q,m}^S(f) = e^{-ma}d_{K,m}^S(f)$ under the same assumption. In this case, $d_{K,q,m}^S(f)$ are a closer match for the classical Bernstein-Walsh-Siciak theorem, since $V_{K,q}^S = V_K^S + a$.

Corollary 1.5. *Let $R > 0$ and q be an admissible weight on a compact subset K of $\mathbb{C}^n$, and assume that $V_{K,q}^* \leq q$ and $K \subset X_R$. If $f \in \mathcal{O}(X_R)$, then $\varlimsup_{m\to\infty} d_{K,q,m}(f)^{1/m} \leq 1/R$.*

Section 4 discusses lower bounds for the distances from mS to $\mathbb{N}^n \setminus mS$. This is done to justify some technical decisions in the proof of Theorem 1.2. They are also interesting in their own right since these distances appear in results that give sufficient conditions for entire functions to belong to certain polynomial classes. See Theorems 3.6 and 7.2 in [6].

This paper is part of a series of papers studying the Lelong class $\mathcal{L}^S(\mathbb{C}^n)$, with the aim of relating it to polynomial approximations using polynomials from $\mathcal{P}^S(\mathbb{C}^n)$. This self-contained exposition began in [6], with a focus on fundamental results that could be useful in the following papers.

A commonly used tool in pluripotential theory is approximation by integral convolution with a smoothing kernel. The Lelong class $\mathcal{L}(\mathbb{C}^n)$ is closed under such smoothing, whereas $\mathcal{L}^S(\mathbb{C}^n)$ may not be. In [6, Theorem 5.8] it is shown that $\mathcal{L}^S(\mathbb{C}^n)$ is closed under such smoothing if and only if S is a lower set. Methods of approximation under which $\mathcal{L}^S(\mathbb{C}^n)$ is always closed are considered in [11].

Siciak [8, Proposition 5.9] proved a product formula for the Siciak-Zakharyuta functions, relating the behavior of the Siciak-Zakharyuta function of a cartesian product with the Siciak-Zakharyuta function of each term. This formula is greatly generalized in [10], along with showing that a weighted version of the formula is not possible.

The Siciak-Zakharyuta theorem relates the Siciak-Zakharyuta functions with the Siciak functions. It states that $V_K = \log \Phi_K$. In [5] it is showed that $V_{K,q}^S = \log \Phi_{K,q}^S$ on $\mathbb{C}^{*n}$ if and only if $S \cap \mathbb{Q}^n$ is dense in S. This result will play an important role in the proof of Theorem 1.2.

In [9] generalizations of the Runge-Oka-Weil theorem are considered. Namely, the paper considers when holomorphic functions can be approximated by polynomials in $\mathcal{P}^S(\mathbb{C}^n)$. The hull of compact subsets of $\mathbb{C}^n$ with respect to $\mathcal{P}^S(\mathbb{C}^n)$

is relevant, as in the classical Runge-Oka-Weil theorem. This hull is the same as the classical polynomial hull when S contains a neighborhood of 0 in $\mathbb{R}^n_+$, but is otherwise more complicated, as it may not be bounded. Techniques from [9] appear in the proof of Theorem 1.2.

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2 Proof of Theorem 1.1

The proof is a modification of a classical argument which is the easy part of the equivalence in the original Bernstein-Walsh theorem.

Proof of Theorem 1.1. For a sequence $\varepsilon_m \searrow 0$ we can, by (1.2), find a sequence of polynomials $(p_m)_{m\in\mathbb{N}}$, with $p_0 = 0$ and $p_m \in \mathcal{P}^S_m(\mathbb{C}^n)$, such that

$$|f(z) - p_m(z)| \leq (1+\varepsilon_m) d^S_{K,q,m}(f) e^{mq(z)}, \quad z \in K. \tag{2.1}$$

Then, for every $\gamma \in]0, R[$ such that $K \subset X_{R-\gamma}$, there exists a constant $A_\gamma > 0$, such that

$$d^S_{K,q,m}(f) \leq \|(f-p_m)e^{-mq}\|_K \leq \frac{A_\gamma(1+\varepsilon_m)^m}{(R-\gamma)^m}, \quad m \in \mathbb{N}.$$

For every $j \in \mathbb{N}^*$ and every $z \in K$ we have

$$\begin{aligned} |p_j(z) - p_{j-1}(z)| &\leq |f(z) - p_j(z)| + |f(z) - p_{j-1}(z)| \\ &\leq \frac{A_\gamma(1+\varepsilon_{j-1})^j}{(R-\gamma)^j}\Big(1 + \frac{R-\gamma}{e^{q(z)}}\Big) \cdot e^{jq(z)}. \end{aligned}$$

Since $q \in \mathcal{LSC}(K)$ takes its minimum a at some point in K, we have

$$\frac{1}{j}\log\big((R-\gamma)^j|p_j(z) - p_{j-1}(z)|/(1+\varepsilon_{j-1})^j B_\gamma\big) \leq q(z), \quad z \in K,$$

where $B_\gamma = A_\gamma(1 + (R-\gamma)/e^a)$. By the definition of $V^S_{K,q}$ this implies that

$$\frac{1}{j}\log\big((R-\gamma)^j|p_j(z) - p_{j-1}(z)|/(1+\varepsilon_{j-1})^j B_\gamma\big) \leq V^S_{K,q}(z), \quad z \in \mathbb{C}^n,$$

and consequently

$$\big|p_j(z) - p_{j-1}(z)\big| \leq \frac{B_\gamma(1+\varepsilon_{j-1})^j e^{jV^S_{K,q}(z)}}{(R-\gamma)^j}, \quad z \in \mathbb{C}^n.$$

If $0 < \varrho < 1$ such that $K \subset X_{\varrho(R-\gamma)}$, then $V^S_{K,q}(z) \leq \log(\varrho(R-\gamma))$ for $z \in X_{\varrho(R-\gamma))}$, which implies that

$$|p_j(z) - p_{j-1}(z)| \leq B_\gamma\big((1+\varepsilon_{j-1})\varrho\big)^j, \quad z \in X_{\varrho(R-\gamma)},$$

and this estimate implies that $p_m = \sum_{j=1}^m (p_j - p_{j-1})$ converges locally uniformly on $X_{R-\gamma}$ to a holomorphic function F_γ. By (2.1), we have $F_\gamma = f$ on L, proving **(i)**. Point **(ii)** then follows from the identity principle for holomorphic functions. For **(iii)** we assume $\sup_K q < \log R$ and let $z \in K$ and $R_0 > 0$ such that $q < \log R_0 < \log R$. Then $e^{q(z)} < R_0$, so

$$\varlimsup_{m\to\infty} \Big(d^S_{K,q,m}(f)e^{mq(z)}\Big)^{1/m} < R_0 \varlimsup_{m\to\infty} d^S_{K,q,m}(f)^{1/m} \leq \frac{R_0}{R} < 1.$$

Therefore, $\lim_{m\to\infty} d^S_{K,q,m}(f)e^{mq(z)} = 0$, since $\sum_{m=0}^\infty d^S_{K,q,m}(f)e^{mq(z)}$ converges by the root test. Since this holds for all $z \in K$ we have that $L = K$. □

3 Proof of Theorem 1.2

Proving (1.4) involves constructing polynomials that approximate f sufficiently well. This is done by solving $\bar\partial$-equations using Hörmander's L^2-estimates.

Proof of Theorem 1.2. It is sufficient to construct a sequence of polynomials $(p_m)_{m\in\mathbb{N}}$ such that $p_m \in \mathcal{P}_m^{\widehat{S}_{\Gamma_m}}(\mathbb{C}^n)$ and

$$\varlimsup_{m\to\infty} \|(f-p_m)e^{-mq}\|_K^{1/m} \leq 1/R. \tag{3.1}$$

The polynomial p_m will be of the form $p_m = \chi f - u_m$ where χ is a cut-off function with support in X_R, equal to 1 on $\overline{X}_{R-\gamma}$, for some $\gamma \in]0,R[$, and u_m is a solution to $\bar\partial u_m = f\bar\partial\chi$ satisfying the weighted L^2 estimate

$$\int_{\mathbb{C}^n} |u_m|^2(1+|w|^2)^{-a_m}e^{-2mV_m}\,d\lambda \leq \frac{1}{a_m}\int_{\mathbb{C}^n} |f\bar\partial\chi|^2(1+|w|^2)^{-a_m+2}e^{-2mV_m}\,d\lambda, \tag{3.2}$$

where $(V_m)_{m\in\mathbb{N}}$ is a sequence in $\mathcal{L}^S(\mathbb{C}^n)$ and $(a_m)_{m\in\mathbb{N}}$ is a sequence of strictly positive numbers such that

$$\varliminf_{m\to\infty} a_m^{1/m} = 1. \tag{3.3}$$

For the choice of these sequences we follow Section 5 in Sigurðardóttir [9].

Since $V^S_{K,q}$ is continuous, the Siciak-Zakharyuta theorem, Theorem 1.1 in [5], implies that $\log\Phi^S_{K,q} = V^S_{K,q}$ and by (1.1) we have that $\log\Phi^S_{K,q,m} \to V^S_{K,q}$ locally uniformly on $\mathbb{C}^n$.

The subset $S_m = \mathrm{ch}(S \cap \frac{1}{m}\mathbb{N}^n)$ of S is a polytope with rational vertices and $\mathcal{P}_m^{S_m}(\mathbb{C}^n) = \mathcal{P}_m^S(\mathbb{C}^n)$. Hence $\Phi^{S_m}_{K,q,m} = \Phi^S_{K,q,m}$, and

$$\log\Phi^S_{K,q,m} = \log\Phi^{S_m}_{K,q,m} \leq V^{S_m}_{K,q} \leq V^S_{K,q}.$$

Consequently, $V_{K,q}^{S_m} \to V_{K,q}^{S}$ uniformly on compact subsets of $\mathbb{C}^n$. For simplicity, we set $a_m = \frac{1}{2}d(mS_m, \mathbb{N}^n \setminus mS_m)$, $V_m = V_{K,q}^{S_m}$, $\psi_m = 2mV_m + a_m \log(1+|\cdot|^2)$, and $\eta_m = \psi_m - 2\log(1+|\cdot|^2)$. We note that $V_{K,q}^{S_m *} \le V_{K,q}^{S} \le q$ on K, so V_m is upper semicontinuous, and therefore plurisubharmonic. By Theorem 4.2.6 in Hörmander [3] there exists a solution u_m to the equation $\bar{\partial} u_m = f\bar{\partial}\chi$ such that (3.2) holds. By Corollary 5.4 in Sigurðardóttir [9] we have that (3.3) holds. By Proposition 4.5 in [6] we have $V_m \le V_{K,q}^{S} \le H_S + c$, for some constant c, so by Theorem 7.2 in [6] it follows that $p_m \in \mathcal{P}_m^{\widehat{S}_{\Gamma_m}}(\mathbb{C}^n)$.

We turn our attention to finding an upper bound for $\|(f-p_m)e^{-mq}\|_K$. To this end we take $\varepsilon > 0$, $\gamma \in]0, R[$, and m_0 such that $V_m > V_{K,q}^{S} - \varepsilon$, on the compact set $\overline{X}_R + \overline{B}(0,1)$, for $m \ge m_0$. By the continuity of $V_{K,q}^{S}$ and the compactness of K, we can take $\delta \in]0,1]$ small enough that $B(z,\delta) \subset X_{R-\gamma}$, for all $z \in K$, and $V_{K,q}^{S}(w) < q(z) + \varepsilon/2$ for all $w \in B(z,\delta)$ and $z \in K$. Since $\chi|_K = 1$, we have that $p_m = f - u_m$, on K. So, by the Cauchy-Schwarz inequality,

$$
\begin{aligned}
|f(z) - p_m(z)| = |u_m(z)| &\le \Omega_{2n}^{-1}\delta^{-2n} \int_{B(z,\delta)} |u_m|\, d\lambda \\
&\le \Omega_{2n}^{-1}\delta^{-2n} \Big(\int_{B(z,\delta)} |u_m|^2 e^{-\psi_m}\, d\lambda \int_{B(z,\delta)} e^{\psi_m}\, d\lambda \Big)^{1/2}, \quad z \in K,
\end{aligned}
$$

where Ω_{2n} denotes the volume of the unit ball in $\mathbb{R}^{2n}$. Since $0 < a_m < 1$, we have

$$
\begin{aligned}
\int_{B(z,\delta)} e^{\psi_m}\, d\lambda &\le \int_{B(z,\delta)} e^{2mV_{K,q}^{S}} (1+|\cdot|^2)\, d\lambda \\
&\le e^{2mq(z)+m\varepsilon} \int_{X_R} (1+|\cdot|^2)\, d\lambda, \quad z \in K,
\end{aligned}
$$

and by (3.2), we have

$$
\begin{aligned}
\int_{B(z,\delta)} |u_m|^2 e^{-\psi_m}\, d\lambda &\le \frac{1}{a_m} \int_{M_\gamma} |f\bar{\partial}\chi|^2 e^{-\eta_m}\, d\lambda \\
&\le \frac{e^{m\varepsilon}}{a_m (R-\gamma)^{2m}} \int_{M_\gamma} |f\bar{\partial}\chi|^2 (1+|\cdot|^2)^2\, d\lambda, \quad z \in K,
\end{aligned}
$$

where $M_\gamma = \operatorname{supp}\bar{\partial}\chi$. The last step follows from the fact that $\chi|_{X_{R-\gamma}} = 1$, so $V_{K,q}^{S} > \log(R-\gamma)$ on M_γ, and thus $V_m > \log(R-\gamma) - \varepsilon$. Combining these inequalities we have that

$$
|f(z) - p_m(z)| e^{-mq(z)} \le \frac{C_{\varepsilon,\gamma} e^{m\varepsilon}}{a_m^{1/2} (R-\gamma)^m}, \quad m \ge m_0,
$$

where $C_{\varepsilon,\gamma}$ is a constant that does not depend on m, and finally, by (3.3),

$$
\varlimsup_{m\to\infty} d_{K,m}^{S}(f)^{1/m} \le \varlimsup_{m\to\infty} \frac{C_{\varepsilon,\gamma}^{1/m} e^{\varepsilon}}{a_m^{1/(2m)} (R-\gamma)} = \frac{e^{\varepsilon}}{R-\gamma}.
$$

Since $\varepsilon > 0$ and $\gamma \in]0, R[$ are arbitrary the estimate (3.1) follows. □

4 Distances from mS to the integer lattice

The presence of S_m in the proof of Theorem 1.2 is to obtain control of the constants a_m in (3.2). If S was used instead of S_m we would need to study

$$a = \varliminf_{m\to\infty} d(mS, \mathbb{N}^n \setminus mS)^{1/m}. \tag{4.1}$$

The distances $d(mS, \mathbb{N}^n \setminus mS)$ come from Theorems 3.6 and 7.2 in [6], which give sufficient conditions for an entire function to belong to certain polynomial classes, depending on S. Lemma 5.3 in [9] gives the lower bound

$$d(S, \mathbb{N}^n \setminus S) \geq \frac{1}{\sqrt{n}(n-1)!M^{n-1}}$$

when S is a polytope with vertices in $\mathbb{N}^n \cap [0, M]^n$. In this section we will find a lower bound for $d(mS, \mathbb{N}^n \setminus mS)$, that depends on S but not m, when S is polytope with vertices in $\mathbb{Q}^n_+$.

Recall that for every convex set $A \subset \mathbb{R}^n$ and every $m \in \mathbb{N}^*$ we have that $mA = \sum_{j=1}^n A = A + \cdots + A$ with m terms in the right hand side. Denoting that extremal set of A by $\operatorname{ext} A$ and with $x \in (n+1)A$, the Minkowski theorem [3, Theorem 2.1.9] tells us that we can find $a_0, \ldots, a_n$ in $\operatorname{ext} A$ such that

$$x = \lambda_0 a_0 + \cdots + \lambda_n a_n, \quad \lambda_j \geq 0,\ j = 0, \ldots, n, \quad \sum_{k=0}^n \lambda_k = n + 1.$$

By renumbering $a_0, \ldots, a_n$ we may assume that $\lambda_0 \geq 1$, which implies that we can write $x = t + a_0$, where $t = (\lambda_0 - 1)a_0 + \lambda_1 a_1 + \cdots + \lambda_n a_n$ and $a_0 \in \operatorname{ext} A$. By induction

$$mA = nA + \sum_{j=1}^{m-n} \operatorname{ext} A, \quad m > n,$$

and if we set $T = (1/n)A$, then

$$mA = mnT = nT + \sum_{j=1}^{mn-n} \operatorname{ext} T = A + \frac{1}{n} \sum_{j=1}^{mn-n} \operatorname{ext} A, \quad m > 1. \tag{4.2}$$

Proposition 4.1. *Let $S = \operatorname{ch}\{v_1, \ldots, v_N\}$ be a convex polytope in $\mathbb{R}^n_+$ with rational vertices $v_j \in \mathbb{Q}^n_+$. Then*

$$d(mS, \mathbb{N}^n \setminus mS) \geq \frac{1}{nq} d(nqS, \mathbb{N}^n \setminus nqS), \quad m \in \mathbb{N}^*, \tag{4.3}$$

where q is the common denominator for all the coordinates of $v_1, \ldots, v_n$.

Proof. Let $d_m = d(mS, \mathbb{N}^n \setminus mS)$. Then $v_j \in (1/q)\mathbb{N}^n$ for $j = 1, \ldots, N$. Let $s \in mS$ and $u \in \mathbb{N}^n \setminus mS$ such that $d_m = d(s, u)$. By (4.2) we have $s = t + h$ where $t \in S$ and

$$h \in \frac{1}{n} \sum_{j=1}^{mn-n} \operatorname{ext} S \subset \frac{1}{nq} \mathbb{N}^n.$$

Since $u \in \mathbb{N}^n \setminus mS \subseteq (1/(nq))\mathbb{N}^n$ we have $u-h \in (1/(nq))\mathbb{N}^n$. We have $u-h \notin S$ for otherwise (4.2) would imply that $u = (u-h) + h \in S$ which does not hold. Hence

$$d_m = d(s,u) = d(s-h, u-h) = d(t, u-h) \geq d(S, (1/(nq))\mathbb{N}^n \setminus S),$$

concluding the proof. □

The inclusion of S_m in the proof of Theorem 1.2 would be unnecessary if we could show that a in (1.2) was always 1. This is not the case. In fact, we can explicitly construct a lower set S such that $a = 0$. Let $f\colon [0,1] \to [0,1]$, given by

$$f(t) = 1 - e^{-ct^{-b}+c}, \quad t \in [0,1],$$

where $b > 1$ and $c > 1 + 1/b$, and

$$S = \{x = (x_1, x_2) \in \mathbb{R}^2\,;\, 0 \leq x_2 \leq f(x_1), 0 \leq x_1 \leq 1\}.$$

For $t \in]0,1[$, we have

$$f'(t) = bct^{-b-1}(f(t) - 1) \quad \text{and} \quad f''(t) = bct^{-2b-2}(bc - (b+1)t^b)(f(t) - 1).$$

Since $f(t) < 1$ for $t \in]0,1[$, we have that f is decreasing and concave on $[0,1]$, so S is convex. Furthermore, $d(S, (\delta, 1)) \leq d((\delta, f(\delta), (\delta, 1)) = 1 - f(\delta)$, so

$$d(mS, \mathbb{N}^n \setminus mS) = md(S, (1/m)\mathbb{N}^2 \setminus S) \leq m(1 - f(1/m)).$$

Consequently, we have that

$$d(mS, \mathbb{N}^n \setminus mS)^{1/m} \leq m^{1/m}(1 - f(1/m))^{1/m} \leq m^{1/m} e^{-cm^{b-1}+c/m} \to 0,$$

as $m \to +\infty$. So a in (4.1) is 0.

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