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Scaling limits of weighted random tree-like planar maps

Daniel Amankwah

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Scaling limits of weighted random tree-like planar maps

Daniel Amankwah

Dissertation submitted in partial fulfillment of a
Philosophiae Doctor degree in Mathematics

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Abstract

The topic for this thesis lies in the scope of random planar maps. We investigate scaling limits of some models of discrete planar maps which are by construction “tree-like”. Specifically, we consider some models of “Halin-like” maps and series-parallel maps. The main aim is to study how some fundamental limiting objects appear as scaling limits of these models, hence affirming their universality properties. The Brownian Continuum Random tree (CRT), originally introduced by David Aldous is one of such limiting objects and has been known to be the limit of various different discrete models of uniformly sampled and weighted planar maps. The stable looptrees, due to Curien and Kortchemski is also of significant interest in this thesis. They are known to arise as scaling limit of models of tree-like maps for the case when each face in the maps is assigned a heavy tailed weight so that a typical face is in the domain of attraction of a stable law. One motivation for this thesis is to understand how these convergences generically happen for maps characterized via exclusion of minors. To our knowledge, series-parallel maps known to be characterized by not containing K_4 as a minor is the largest known collection of maps, defined by the exclusion of minors, which admits the CRT as a scaling limit.

Ágrip(in Icelandic)

Í ritgerðinni er fjallað um slembin sléttukort. Við fáumst við flokk sléttukorta sem líkjast trjám og rannsökum skölunarmarkgildi þeirra. Nánar tiltekið skoðum við svokölluð Halin-kort og rað-hliðtengd kort. Meginmarkmiðið er að sýna að ákveðin slembin firðrúm eru skölunarmarkildi þessara líkana og undirstrika þar með allsherjareiginleika þeirra. Brownska tré Aldous (CRT) er eitt slíkra slembifirðrúma en sýnt hefur verið að það er skölunarmarkgildi ýmissa ólíkra strjálla líkana af jafndreifðum slembikortum. Stöðug lykkjutré, sem kynnt voru til sögunnar af Curien og Kortchemski eru annað dæmi. Þau reynast vera skölunarmarkgildi ýmissi slembikorta sem líkjast trjám, þegar þunghaladreifing er sett á möskvastærðir þeirra þannig að dæmigerður möskvi hefur slembna stærð sem dregst að stöðugri dreifingu. Annað markmið er að skilja hvernig mismunandi skölunarmarkgildi koma fram í flokkum jafndreifðra neta sem skilgreindir eru þannig að netin innihalda ekki hlutnet grannmóta tilteknum netum. Rað-hliðtengd net, sem má skilgreina þannig að þau eru net sem innihalda ekki hlutnet grannnmóta K_4, eru að okkar vitneskju stærsti slíki flokkur sem hefur CRT sem skölunarmarkgildi.

For Priscilla, Julia and Wilna.

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Paper I: **D. Amankwah** and S. Stefánsson, 2023, On scaling limits of random Halin-like maps. *MATHEMATICA SCANDINAVICA*, Vol. 129, Issue 3, pp. 507–542.

Paper II: **D. Amankwah**, J. Björnberg, S. Stefánsson, B. Stufler and J. Turunen, 2025, The scaling limit of random 2-connected series-parallel maps. *arXiv:2503.19705*.

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1 Introduction

An Introduction to the Introduction

This thesis presents the research findings of the author and collaborators within the scope of the so-called random discrete structures, specifically focusing on the study of random planar maps. This field has received significant attention in the probability theory literature within the past two decades, see e.g. [**Le Gall 2014**; **Miermont 2014**].

We deemed it appropriate to provide a comprehensive introduction to the research field as a preparation for the original results presented in papers I and II, inserted in this thesis. This includes introducing objects of interest, relevant questions that arise, and overviews of the technical tools that are typically employed. In Subsection 1.1 and 1.2, we introduce the fundamental objects and their well-known results, aiming to motivate and provide essential background for readers new to the topic.

A summary of the results by the author and collaborators is presented in Section 2. Subsection 2.1 provides an overview of the author's joint work with Sigurdur Örn Stefánsson on the scaling limits of random Halin-like maps. The full version of this work, titled Scaling Limits of Random Halin-Like Maps, is published in *Mathematica Scandinavica* (see [**Amankwah and Stefansson 2023**]). Subsection 2.2 summarizes a collaboration between the author, Jakob Björnberg, Sigurdur Örn Stefánsson, Benedikt Stufler, and Joonas Turunen, which introduces the study of the scaling limits of random 2-connected series-parallel maps. A draft of this work is available on arXiv.

1.1 The limits of Planar Maps

1.1.1 On "large" planar maps

This section may be carefully skipped by a reader who has basic a idea of what planar maps are. Here, we provide some standard definitions in order to motivate the study of these objects.

The pair (V,E) is a *multigraph* if V is any set and E is a multiset whose elements are doubleton subsets of V, i.e. $E = \{\{x,y\} : x,y \in V\}$. The set V may be finite or infinite and is called the *vertex set*. The set E is called the *edge set*. One has to note that an element of E may appear with multiplicity (called *multiple edge*), also an element in E may contain a single element of V (called *loop*). Henceforth, we will simply refer to a multigraph as a *graph*.

Consider a graph $G = (V,E)$. A 2-*cell embedding* of G on a closed, connected surface Σ is defined as the representation of G as a union of simple arcs so that the following

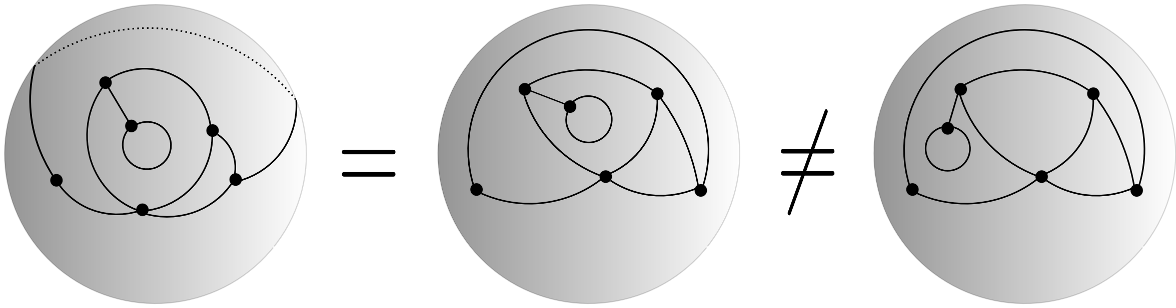


Figure 1.1. The first two pictures are representations of same planar map. The third picture has the same underlying graph structure as the first two but is a different planar map.

are satisfied;

- There is an injective map, $f : V \to \Sigma$ such that, each vertex $x \in V$ corresponds to a point $f(x) \in \Sigma$.
- Each edge $\{x, y\} \in E$ is identified by an arc between $f(x)$ and $f(y)$ in Σ. One can think of the arcs as homeomorphic images of $[0, 1]$.
- When arcs intersect, they only do so at their endpoints.
- Each connected component of the complement of all arcs is homeomorphic to an open disc. Such a component is called a *face*.

Definition 1.1 *A planar map is a finite connected planar graph equipped with a 2-cell embedding in the 2-dimensional sphere, considered up to continuous deformations.*

It can be noticed that planarity and connectedness are both implied by the 2-cell embedding in the 2-sphere.

A planar map will naturally have a vertex set and an edge set so that concepts of graphs can be translated accordingly. For example, vertices u and v are said to be *neighbours* if $\{u, v\}$ is contained in the edge set. However, as an embedding, planar maps inherit additional properties. An immediate notion is that of *faces*. It makes sense to mention faces in the context of planar maps because a face is invariant under orientation-preserving homeomorphisms of the 2-sphere. The vertices and edges that lie on the boundary of a given face are said to be *adjacent* to that face. A vertex can be adjacent to any number of faces(this number is finite if the planar map is locally finite). An edge on the other hand can be adjacent to either one or two faces.

An edge can be assigned two possible orientations. This makes sense since the $2-$sphere is an oriented surface. If an edge is oriented, one can observe the face that lies on its left and that lies on its right. These two faces may coincide. By convention, we will say an edge is incident to the face that lies to its right. The number of edges that are incident to a given face f is called the *degree* of the face (denoted by $\mathsf{deg}(f)$), counted with multiplicity.

We refer to a planar map as *rooted* if it has a distinguished and oriented edge called the root. Rooting a planar map makes it easier to handle combinatorially since it eliminates symmetry issues. Also, the root gives a starting point for recursive decompositions.

1.1.2 Historical Background

The study of planar maps can be linked back to when attempts were made to solve the well-known 4-colour problem. Interests have ever since increased in various aspects of mathematics and physics.

Planar maps have been studied broadly in combinatorics, mostly due to Tutte's [**Tutte 1963**] initial formal set up of an enumeration theory. This theory produced generating functions for various classes of rooted planar maps. Several studies have since been done even in areas that are not directly linked to graph theory. These areas for example may span from *dessins d'enfant* to knots [**Bonahon 2009**], also to Matrix integrals [**Zvonkin 1997**].

In recent physics literature, graphs embedded in surfaces have been studied in the context of two-dimensional quantum gravity where they have been used to model random discretised geometries, see e.g. [**Ambjorn and Durhuus and Jonsson 1997**].

It is a known fact that a (long) simple random walk on $\mathbb{Z}$ can be made to appear as a one-dimensional Brownian motion by rescaling over a long time. Analogously, a natural question that arise is how a large planar map can be represented as an approximation of some continuous random surface, which is now known as the Brownian map [**Le Gall 2013**; **Miermont 2013**].

Many of the advances made in the combinatorial, probabilistic and physics applications of planar maps have been possible via some bijective approaches. Cori and Vauquelin [**Cori and Vauquelin 1981**] introduced such a bijection for quadrangulations (planar maps whose faces are all of degree 4). This bijection was made much more known by Schaeffer[**Schaeffer 1998**]. Bouttier, Di Francesco and Guitter[**Bouttier and Francesco and Guitter 2004**] extended the Cori - Vauquelin - Schaeffer bijection to much more general planar maps.

1.1.3 Scaling limits

Consider a well-defined class of combinatorial objects quantified according to some *size* parameter. One would like to study some asymptotic properties of members of this class having large sizes. Specifically, one would like to answer the following question: Assume a member of the class of size n is drawn uniformly or according to a probability measure that is proportional to its "weight". If such samplings are compared in some sense, is it possible to define a limit as $n \to \infty$?

The way to approach these kind of questions would usually depend on the properties of the class of objects under consideration and on the accessibility of the tools for comparing "distances" between members. In a broad sense, samplings from the class of combinatorial objects are normalised (mostly by some extremal statistics). This is to ensure that they do not blow up with increasing size, hence making them comparable. This operation also renders them to be viewed as a detailed approximation of a continuous limit.

When the class of combinatorial objects of interest consists of graphs or graphs possessing additional structures (planar maps, labelled planar maps etc.), it is useful to explore with the usual graph metric of graphs (i.e the graph distance on its vertex set). The objects are viewed as metric spaces and are compared via the Gromov-Hausdorff distance.

Let (E_1,d_1),(E_2,d_2) be two metric spaces. The Gromov-Hausdorff distance between E_1 and E_2 is

$$\mathrm{d_{GH}}(E_1,E_2) = \inf_{\substack{(E,d)\,\text{metric space}\\ \varphi_1:E_1\to E\\ \varphi_2:E_2\to E}} \mathrm{d}^E_{\mathrm{Hau}}(\varphi_1(E_1),\varphi_2(E_2))$$

where φ_1 and φ_2 are isometric embeddings (not necessarily surjective). Recall that the Hausdorff distance between two nonempty subsets K and K' of the metric space E is

$$\mathrm{d}^E_{\mathrm{Hau}}(K,K') = \inf\{\varepsilon > 0, K \subseteq K'_\varepsilon \text{ and } K' \subseteq K_\varepsilon\}$$

where K_ε denotes the ε-neighborhood of K. An equivalent definition of the Gromov-Hausdorff distance involves correspondences. A correspondence between two metric spaces (E_1,d_1) and (E_2,d_2) is a subset $\mathscr{R}$ of $E_1 \times E_2$ such that for every $x_1 \in E_1$, there exists at least one point $x_2 \in E_2$ such that $(x_1,x_2) \in \mathscr{R}$ and conversely for every $y_2 \in E_2$, there exists at least one point $y_1 \in E_1$ such that $(y_1,y_2) \in \mathscr{R}$. The distortion of the correspondence $\mathscr{R}$ is defined by

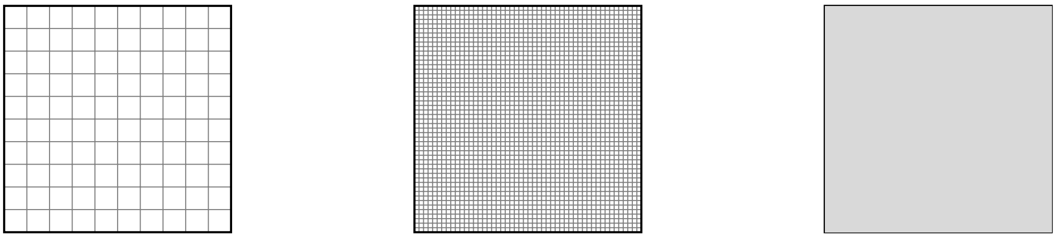
$$\mathrm{dis}(\mathscr{R}) = \sup\{|d_1(x_1,y_1) - d_2(x_2,y_2)| \colon (x_1,x_2),(y_1,y_2) \in \mathscr{R}\}.$$

The Gromov-Hausdorff distance can be written in terms of correspondences by the formula

$$\mathrm{d_{GH}}(E_1,E_2) = \frac{1}{2}\inf\{\mathrm{dis}(\mathscr{R})\}$$

where the infimum is over all correspondences $\mathscr{R}$ between E_1 and E_2. We refer the reader to [**Burago and Burago and Ivanov 2001**] for details about this metric.

Example 1.2 *An obvious way to illustrate a scaling limit is by taking the $n \times n$ grid graph, $L(n,n)$. It can be viewed as a metric space $(V(L(n,n)),d_{gr})$, where d_{gr} is the usual graph distance. One can deduce that the sequence of metric spaces $\left(V(L(n,n)),\frac{1}{n}\cdot d_{gr}\right)_{n>0}$ converges towards the set $[0,1]^2$ in the Gromov-Hausdorff sense. In order to see this, we embed the metric space $\left(V(L(n,n)),\frac{1}{n}\cdot d_{gr}\right)$ into $\{\frac{i}{n}|\ 1 \le i \le n\}^2$. We then construct* *a correspondence between* $\left(V(L(n,n)), \frac{1}{n} \cdot d_{gr}\right)$ *and* $\{\frac{i}{n} |\ 1 \leq i \leq n\}^2$ *as follows; a point* $x_1 \in V(L(n,n))$ *is in correspondence with a point* $a_1 \in \{\frac{i}{n} |\ 1 \leq i \leq n\}^2$ *if there exists an edge of* $\left(V(L(n,n)), \frac{1}{n} \cdot d_{gr}\right)$ *that contains both* x_1 *and* a_1. *By definition, this is a correspondence. The distortion of this correspondence is bounded above by* $\frac{2}{n}$ *so we have that* $\mathrm{d_{GH}}(\left(V(L(n,n)), \frac{1}{n} \cdot d_{gr}\right), \{\frac{i}{n} |\ 1 \leq i \leq n\}^2) \leq \frac{2}{n}$.



(a) $n = 10$ *(b)* $n = 50$ *(c) Scaling Limit*

Figure 1.2. Visualization of the scaling limit of the $n \times n$ *grid as* $n \to \infty$.

It can be shown that $(\mathbb{K}, \mathrm{d_{GH}})$ is a metric space, where $\mathbb{K}$ is a space of equivalence classes of non-empty compact metric spaces. The proof that $\mathrm{d_{GH}}$ is indeed a metric on $\mathbb{K}$ can be found in Thm 7.3.30 [**Burago and Burago and Ivanov 2001**]. Also, it can be showed that $(\mathbb{K}, \mathrm{d_{GH}})$ is a polish space (i.e. complete and separable). We can also consider the pointed version of the metric space $(\mathbb{K}, \mathrm{d_{GH}})$, denoted by $(\mathbb{K}^\bullet, \mathrm{d_{GH}})$. In this case $\mathbb{K}^\bullet$ is viewed as a set of pointed compact metric spaces up to isometries.

A good number of works have been done under the umbrella of the so called scaling limit for various classes of random planar maps. For example the following have been shown to converge to scalar multiple of the Brownian map;

- Uniform random quadrangulations with n faces [**Le Gall 2013**; **Miermont 2013**].
- Uniform p-angulations (in the case for $p = 3$ or p even) [**Le Gall 2013**].
- Uniform random planar maps with n edges [**Bettinelli et al. 2013**].
- Random critical Boltzmann planar maps in which the degree of a typical face belongs to the domain of attraction of a stable law with index $\alpha \in (1, 2]$ (in the case when $\alpha = 2$) [**Marzouk 2018**].

1.1.4 Local limits

A different approach to exploring the asymptotics of large elements of the class of combinatorial objects is via the *local limit*. In the context of rooted planar maps or (alternatively) *pointed* planar maps, say, the interest in studying what a large planar map in a certain class looks like locally around its root(either the root vertex or the root edge). In a similar fashion as with the scaling limit, we require the right tool for comparing two objects in the class. We utilise the so-called *local distance* in this case.

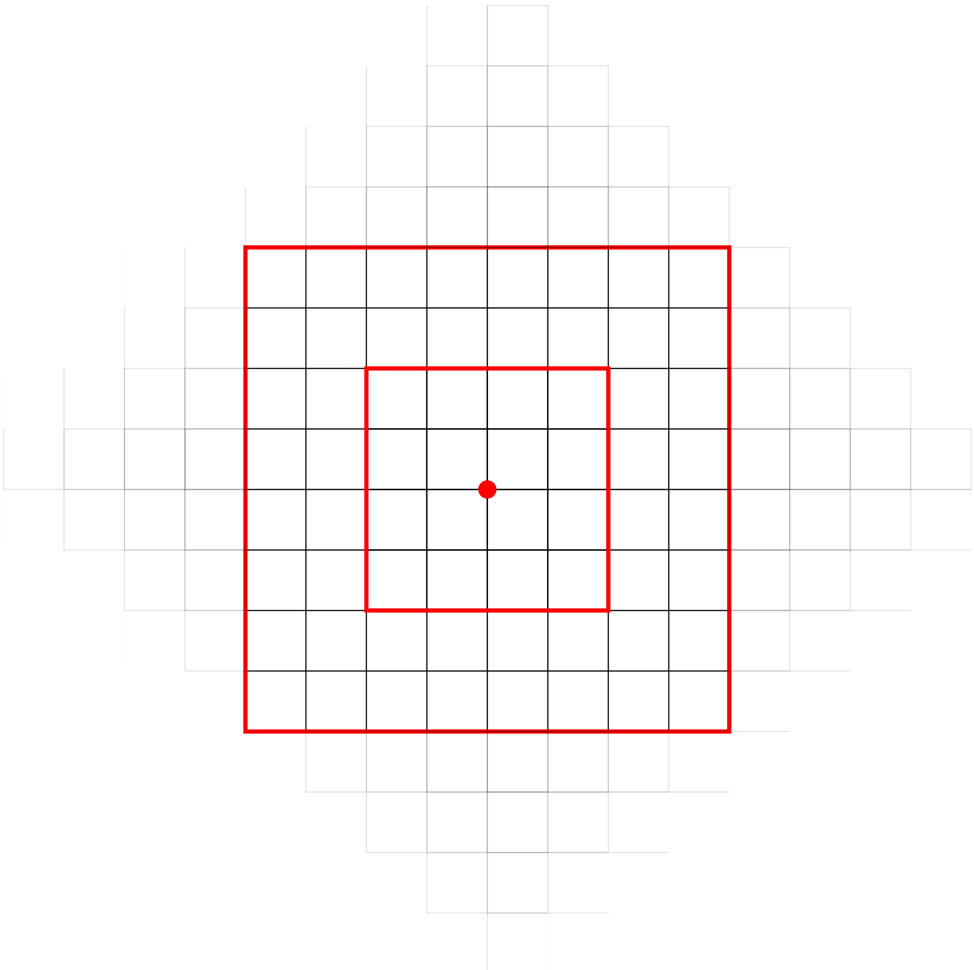
Consider a planar map M pointed at a vertex v. We will denote by $[M]_r$ *ball centered at* v *with radius* r, where $r \in \mathbb{N}$. i.e. the map obtained by deleting all vertices u of M so that $d_{\text{gr}}(u,v) > r$, including all the edges incident to such vertices.

Given two planar maps M and M', pointed in v_M and $v_{M'}$ respectively, we set

$$\mathrm{d_{loc}}(M,M') = (1 + \sup\{r \in \mathbb{N} : [M]_r = [M']_r\})^{-1}.$$

In words, we say that two pointed planar maps are "near" in the sense of the local distance $\mathrm{d_{loc}}$ if their respective centered balls are the same up to large distances. Assume that there exists a limit in distribution, in the sense of the local distance $\mathrm{d_{loc}}$ of a given sequence of random pointed planar maps $(M_n)_{n\in\mathbb{N}}$, and denote such a limit by M_∞. One may view the random variable, $[M_\infty]_r$ for each non-negative integer r as the limit of the laws of the centered balls, $[M_n]_r$ as n approaches infinity. Analogously, one can say that M_∞ is the local limit of the sequence $(M_n)_{n\in\mathbb{N}}$. In the literature, the size measure of a planar map n are usually taken to be the number of edges, vertices, faces or leaves.

Example 1.3 *For a simple example illustrating the intuition of local limits, consider the $2n+1 \times 2n+1$ grid, pointed at the vertex $(n+1,n+1)$. The local limit of this sequence of planar maps when $n \to \infty$ is clearly given by the infinite two-dimensional lattice $\mathbb{Z}^2$ pointed in one of its vertices. This is because balls of radius r in $\mathbb{Z}^2$ are the same as those in the finite grids for large n.*



1.2 Trees

1.2.1 Rooted plane trees

Definition 1.4 *A rooted plane tree is a single face planar map. We will henceforth simply refer to a rooted plane tree as a plane tree.*

In the following, we will give an equivalent combinatorial definition of a plane tree. Let $\mathbb{N} = \{1,2,...\}$ and define the set of finite sequences by

$$\mathscr{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n,$$

where by convention $\mathbb{N}^0 = \{\emptyset\}$. An element of $\mathscr{U}$ is thus a sequence $u = (u^1,...,u^n)$ of elements of $\mathbb{N}$, and we set $|u| = n$, so that $|u|$ represents the generation of u. If $u = (u^1,...,u^m)$ and $v = (v^1,...,v^n)$ belong to $\mathscr{U}$, we write the concatenation of u and v as $uv = (u^1,...,u^m,v^1,...,v^n)$.

A plane tree T is a finite or infinite subset of $\mathscr{U}$ such that,

1. $\emptyset \in T$.
2. if $v \in T$ and $v = uj$ for some $j \in \mathbb{N}$ then $u \in T$.
3. for every $u \in T$, there exists an integer $k_u(T) \geq 0$ (the number of children of u) such that for every $j \in \mathbb{N}$, $uj \in T$ if and only if $1 \leq j \leq k_u(T)$.

The vertex $\emptyset$ is called the root vertex and the edge $\{\emptyset,(1)\}$ is called the root edge. Let us denote the set of all plane trees by $\mathbf{T}$ and the set of plane trees with n vertices($n \geq 1$) by $\mathbf{T}_n$. For $T \in \mathbf{T}$, we will view each of its vertex as an individual of a population for which T is the geneological tree. If $T \in \mathbf{T}$ and $u \in T$, we define the shift of T at u by $\text{shift}_u T = \{v \in \mathscr{U} : uv \in T\}$ which is itself a tree. The total progeny of T (total number of vertices of T) will be denoted by $\zeta(T)$. The maximal generation of the tree T is called its height and we denote it by $\mathsf{Height}(T)$.

Given a plane tree T , define the contour sequence of T

$$c = (c_0, c_1, \ldots, c_{2(\zeta(T)-1)})$$

as the sequence of vertices encountered when moving along the contour of T in a clockwise direction, starting and ending at $c_0 = c_{2(\zeta(T)-1)} = \emptyset$. The contour sequence is in a one-to-one correspondence with the corners around the vertices in T, and we use the convention that the corner clockwise from the root vertex is c_0. The tree T can be encoded using its contour sequence via the so-called *contour function*. The function is denoted by C_T and it maps c_i, $i \in \{0,1,\ldots,2(\zeta(T)-1)\}$, to the generation of the vertex adjacent to the c_i-th corner encountered in the sequence c. We have that $C_T(c_0) = C_T(2(\zeta(T)-1)) = 0$ and $|C_T(i) - C_T(i-1)| = 1$ for all $i \in \{0,1,\ldots,2(\zeta(T)-1)\}$. C_T defines a *Dyck path* of length $2(\zeta(T)-1)$.

It is known that the mapping $T \mapsto (C_T(c_0), C_T(c_1), \ldots, C_T(2(n-1)))$ is a bijection between the set $\mathbf{T}_n$ and the set of all Dyck paths of lenght $2(n-1)$. The set of plane

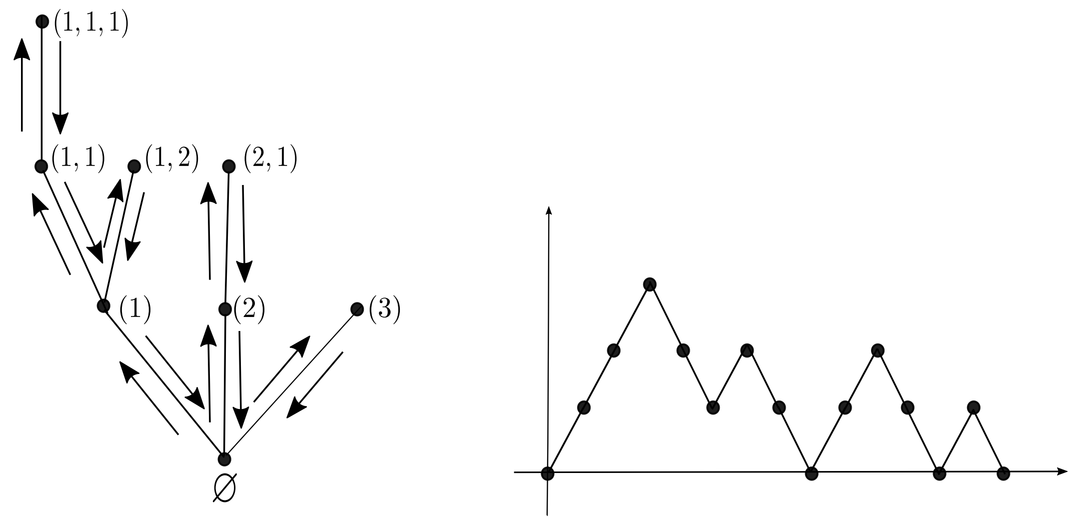


Figure 1.3. A tree T with its associated contour function C_T.

trees with n vertices are therefore enumerated by the $(n-1)$-th Catalan number. i.e

$$\#\mathbf{T}_n = \mathrm{Cat}(n-1) = \frac{1}{n}\binom{2(n-1)}{n-1} \tag{1}$$

1.2.2 Real trees

Consider a finite plane tree T whose associated contour function is given by $C_T : \{c_0, c_1, \ldots, c_{2(\zeta(T)-1)}\} \to \mathbb{N}$. One may extend the mapping C_T so that both domain and co-domain are non-negative real numbers. We do this by setting $C_T(x)$ to 0 for $x = 0,\ 2(\zeta(T)-1)$ and $C_T(x) \geq 0$ for $x \in (0, 2(\zeta(T)-1))$. We linearly interpolate C_T to obtain a continuous function. An example is illustrated in Fig. 1.3.

For positive $x, y \in \mathbb{R}$, define

$$d_{C_T}(x,y) := C_T(x) + C_T(y) - 2 \inf_{z \in [x \wedge y, x \vee y]} C_T(z) \tag{2}$$

The function d_{C_T} is clearly symmetric. Also the triangle inequality is verifiable for d_{C_T}(one would need to consider cases based on the position of the triple (x,y,z) say). The map $(x,y) \mapsto d_{C_T}(x,y)$ is therefore a pseudometric on $\mathbb{R}$. It is natural to consider the quotient space $\mathscr{T}_{C_T} = \mathbb{R}_+/\sim$ where $\sim$ is an equivalence relation defined as follows:

$$x \sim y \quad \text{iff} \quad d_{C_T}(x,y) = 0. \tag{3}$$

d_{C_T} induces a metric on $\mathscr{T}_{C_T}$ so that $(\mathscr{T}_{C_T}, d_{C_T})$ is a compact metric space. It is possible to embed $(\mathscr{T}_{C_T}, d_{C_T})$ in the plane to obtain a tree, however there arise some limitations in fully recovering T. A point q in $\mathscr{T}_{C_T}$ may represent a vertex in T whose degree is the number of connected components in $\mathscr{T}_{C_T} \setminus \{q\}$. However, vertices with exactly one child may be hard to handle. Also, two vertices share a common edge if one can find a path in $(\mathscr{T}_{C_T}, d_{C_T})$ between the two that does not contain any other vertex. The ordering of the children of a vertex is lost in this description.

The pair $(\mathscr{T}_{C_T}, d_{C_T})$ satisfies the definition of the so-called *real* tree given below.

Definition 1.5 *A compact metric space $(\mathscr{T}, d)$ is called a real tree if the following two conditions are satisfied for each pair of points $a, a' \in \mathscr{T}$, .*

- *there is a unique isometric map, $\pi_{a,a'} : [0, d(a,a')] \to \mathscr{T}$ such that $\pi_{a,a'}(0) = a$ and $\pi_{a,a'}(d(a,a')) = a'$.*
- *assuming $q : [0,1] \to \mathscr{T}$ is a continuous injective map such that $q(0) = a$ and $q(1) = a'$, the image $q([0,1])$ coincides with the image $\pi_{a,a'}([0, d(a,a')])$.*

The two conditions in Definition 1.5 ensure that given two points in a real tree, there exists a unique geodesic path between them. The first part describes path-connectedness whiles the second eliminates occurence of cycles.
Various properties of a plane tree can be translated into that of a real tree, $(\mathscr{T}, d)$. The degree of a point $a \in \mathscr{T}$ is the cardinality of the set of connected components of $\mathscr{T} \setminus \{a\}$. If $\mathscr{T} \setminus \{a\}$ has at least 3 connected components, a is called a *branching point*, and a is a leaf if $\mathscr{T} \setminus \{a\}$ is connected. Points in a real tree may have infinite degree.
The geneological structure of a real tree can also be described. Let v be the root of $(\mathscr{T}, d)$. For $a, a' \in \mathscr{T}$, consider the images $\pi_{v,a}([0,|a|])$ and $\pi_{v,a'}([0,|a'|])$ of $\pi_{v,a}$ and $\pi_{v,a'}$ respectively, where $|\cdot|$ is the *height* function mimicking the graph distance from the root. There is a unique a'' so that the intersection of the two images is given by $\pi_{v,a''}([0,|a''|])$. a'' will be written as $a \wedge a'$ and it is called the *least common ancestor* of a and a'. If $a'' = a$, we say a is an *ancestor* of a'.

The construction of real tree with the function C_T can be generalised to a construction via any continuous function $h : [0,1] \to [0,\infty[$ satisfying $h(0) = h(1) = 0$.

Proposition 1.6 *The set of metric spaces $(\mathscr{T}_h, d_h)$ forms the set of compact real trees, where $h : [0,1] \to [0,\infty[$ is any continuous function satisfying $h(0) = h(1) = 0$ and $\mathscr{T}_h = [0,1]/\sim$ for $x \sim y \iff d_h(x,y) = 0$, where*

$$d_h(x,y) = h(x) + h(y) - 2 \inf_{z \in [x \wedge y, x \vee y]} h(z).$$

One can also say that non-negative valued continuous functions encode compact real trees. A useful result known for comparing real trees encoded by two different functions, h_1 and h_2 is given as follows:

Lemma 1.7 *Let h_1 and h_2 be two continuous functions from $[0,1]$ to $[0,\infty[$ and satisfying,*

$h_1(0) = h_2(0) = h_1(1) = h_2(1) = 0$. *Then*

$$\mathrm{d_{GH}}(\mathscr{T}_{h_1}, \mathscr{T}_{h_2}) \le \sup_{t\in[0,1]} |h_1(t) - h_2(t)|$$

From Lemma 1.7, one can deduce that the convergence of continuous functions for the supremum norm on $C([0,1],\mathbb{R})$ implies the convergence of their corresponding real trees. This can indeed provide a tool for convergence results of the sequence of some class of random trees from the convergence (in law) of some stochastic processes.

1.2.3 Bienaymé-Galton-Watson trees

We proceed under this part by introducing randomness to the already established setting of plane trees. A well-known notion of random plane trees have been described via Galton-Watson trees since the 19th century. It is however worth knowing that Galton-Watson trees were first introduced in a study to obtain the probability of extinction for some family names. The model has found itself in diverse applications in pure and applied mathematics.

Let μ be a probability measure on $\mathbb{Z}_+$ such that $\mu(1) < 1$. We call μ the offspring distribution. The *Galton-Watson measure* with offspring distribution μ is the probability measure GW_μ on $\mathbf{T}$ such that:

1. $\mathrm{GW}_\mu(k_\emptyset = j) = \mu(j)$ for $j \ge 0$,
2. If τ is distributed by GW_μ, then for every $j \ge 1$ with $\mu(j) > 0$, conditionally on $\{k_\emptyset = j\}$, the subtrees $\mathrm{shift}_1\tau, \ldots, \mathrm{shift}_j\tau$ are i.i.d. with distribution GW_μ.

One can think of this as a tree that starts with a single individual(root) then grows such that each individual already present in the tree has a number of offspring according to the measure μ, independently of all others.

If in addition, we have that

$$\sum_{n>0} n\mu(n) \le 1 \tag{4}$$

(μ is critical or subcritical), then τ is almost surely finite and

$$\mathrm{GW}_\mu(\tau = T) = \prod_{u\in T} \mu(k_u(T)) \tag{5}$$

[**Le Gall 2005**, Prop 1.4].

Example 1.8 *Let μ be the geometric law, i.e. $\mu(n) = 2^{-n-1}$ for $n = 0, 1, \ldots$ Clearly, it is a critical offspring distribution and*

$$\mathrm{GW}_\mu(\tau = T) = 2^{-\sum_{u\in T} k_u(T)+1} = 2^{-2\zeta(T)+1}. \tag{6}$$

It can thus be seen that the set of geometric Galton-Watson trees conditioned to have n vertices are uniformly distributed over $\mathbf{T}_n$.

In order to study, for example, the asymptotics of this random tree, it is useful to observe the properties of its corresponding contour function. For a critical geometric Galton-Watson tree T, it is known that its corresponding contour function, $(C_T(c_i))_{0\leq i\leq 2(\zeta(T)-1)}$ is an excursion of a symmetric random walk on $\mathbb{Z}$. In other words, the law of $(C_T(c_i))_{0\leq i\leq 2(\zeta(T)-1)}$ is exactly that of the random sequence of integers $(S(i))_{0\leq i\leq k}$, where $S(0)=0$ and $S(i+1)-S(i)\sim \text{Bernoulli}(\frac{1}{2})$ with $k=\inf\{i : S(i)=-1\}-1$.

In this thesis, we will consider a 2-type Galton-Watson tree (specifically when we consider random non-crossing trees, see [**Kortchemski and Marzouk 2016**]). We will make the following modification to what we have above in (5). Let $\mu_\varnothing=(\mu_\varnothing(k))_{k\geq 1}$ be another offspring distribution and define

$$\mathrm{GW}_{\mu,\mu_\varnothing}(\tau=T)=\mu_\varnothing(k_\varnothing(T))\prod_{u\in T\setminus\{\varnothing\}}\mu(k_u(T)). \tag{7}$$

In other words, the root receives offspring according to $\mu_\varnothing$ and other vertices receive offspring according to μ.

1.2.4 Simply generated trees

Consider a finite plane tree, T and let $v\in T$. We will denote the outdegree of v by $\mathsf{out}_T(v)$(this is basically the number of children of v in T). Let $\mathbf{w}=(w(k))_{k\geq 0}$ be a fixed weight sequence and define the weight of T by

$$W(T):=\prod_{v\in T}w(\mathsf{out}_T(v)). \tag{8}$$

Plane trees associated with such weights are called *simply generated trees* and were introduced by Meir and Moon [**Meir and Moon 1978**].

Now we let τ_n be a random element of $\mathbf{T}_n$ such that

$$\mathbb{P}(\tau_n=T)=\frac{W(T)}{\sum_{T'\in\mathbf{T}_n}W(T')}. \tag{9}$$

We can build a connection between simply generated trees and Galton-Watson trees as follows: we let the weight sequence $\mathbf{w}$ be a probability distribution on $\{0,1,2,\ldots\}$ i.e $\sum_{k=0}^{\infty}w(k)=1$. Let ξ be a random variable with $\mathbb{P}(\xi=k)=w(k)$ so that a finite tree τ is a random Galton-Watson tree with offspring distribution ξ. One can deduce that the simply generated tree τ_n is the same as the Galton-Watson tree τ conditioned on $\zeta(\tau)=n$.

The weight sequence, $(w(k))$ can be tilted to obtain another weight sequence $\tilde{w}(k)$ so that they both define the same simply generated tree, τ_n. This is done by taking $a,b>0$ so that

$$\tilde{w}(k):=ab^k w(k). \tag{10}$$

$w(k)$ and $\tilde{w}(k)$ are in other words said to be *equivalent*. The significance of this tilting is that, in several cases, it can be done in a way such that the equivalent probability distribution $\tilde{w}(k)_{k\geq 0}$ has mean 1. In that case, τ_n becomes a critical Galton-Watson tree conditional on $\zeta(\tau) = n$. There are however cases where the weight sequence is not equivalent to any critical weight sequence. In such cases it may be equivalent to a subcritical one.

These possible cases can be characterized algebraically as follows: Let the generating function of the weight sequence be

$$\phi(z) := \sum_{k=0}^{\infty} w_k z^k \tag{11}$$

and let its radius of convergence be $\rho_\phi \in [0, \infty]$.

For $t \in \mathbb{R}$ such that $\phi(t) < \infty$, let

$$\psi(t) := \frac{t\phi'(t)}{\phi(t)} \tag{12}$$

and let

$$\nu := \psi(\rho_\phi) = \lim_{t \nearrow \rho_\phi} \psi(t). \tag{13}$$

When $\rho_\phi > 0$, the limit in (13) exists and the above cases are characterized as follows. If $\nu \geq 1$, then there is a unique solution, τ to the equation $\psi(t) = 1$. In this case the weight sequence, **w** describes a simply generated tree distributed like a critical Galton-Watson tree conditioned on having n vertices. If $0 < \nu < 1$, we set $\tau := \rho_\phi$ and in this case **w** describes a simply generated tree that corresponds to a subcritical Galton-Watson tree with a heavy-tailed offspring distribution.
When $\rho_\phi = 0$, we set $\nu = 0$ and $\tau = 0$. **w** then describes a simply generated tree with superexponential branching weights.

1.2.5 The Brownian tree

The *Brownian tree* can be seen as a random real tree (i.e. a real tree constructed via a Brownian excursion). Technically, one will define a Brownian tree as a random variable, taking values in the set $\mathbb{K}$ (or $\mathbb{K}^\bullet$ when considering a rooted real tree), that appears as the scaling limit of several classes of random trees and random planar maps. It was first introduced by Aldous in [**Aldous1 1991**] where he showed that a critical Galton-Watson tree whose offspring distribution has a finite variance converges in distribution torwards the Brownian tree, when conditioned to be large. In the literature, the Brownian tree is also called the continuum random tree (CRT).

The CRT will appear in the first part of this work, which is devoted to studying the scaling limit of various regimes of "Halin-like maps". The regime where the CRT appears, will involve showing that these maps up to some constant factor admit the same scaling limit as uniform random plane trees. We do not discuss the construction of the CRT in detail here, however there are several surveys and lecture notes that do so. For example in [**Le Gall 2013**], the proofs of the most important results of the CRT, Theorem 1.9 and Corollary 1.10, including some extensions are given.

One may follow a sequel in the previous sections to obtain a scaling of a uniform random trees: given a uniform plane tree τ_n (with size n), we can obtain a random contour function, C_{τ_n} known to be a random Dyck path of length $2(n-1)$. If such a contour functions is viewed as random variables taking value in the metric space, $(C([0,1],\mathbb{R}),\|\cdot\|_\infty)$, we could build a random real tree as in section 1.2.2. Following Lemma 1.7 and the argument given in the paragraph below, one could conclude that the random real tree is the scaling limit of the uniform random plane trees.

One would like to determine the limit of a simple symmetric random walk conditioned on having a length $2p$ if time is shrunk by p and space is shrunk by $\sqrt{p}$.
In order to state the explicit convergence result, we define the normalized Brownian excursion. We recall that an excursion of a one-dimensional Brownian motion, $(\mathscr{B}_t)_{t\geq 0}$ say is a maximal open interval (α,β) such that $|\mathscr{B}_t|>0$ for $t\in(\alpha,\beta)$. Almost surely, a well-defined first excursion of $\mathscr{B}_t$ having a length of at least 1 occurs. Let (a,b) be that excursion, where a and b are random non-negative real numbers with $\mathscr{B}_a=\mathscr{B}_b=0$. We denote the *normalized Brownian excursion* by the process $\mathbf{e}_t$, defined as

$$\mathbf{e}_t=\frac{1}{(a-b)^{\frac{1}{2}}}|\mathscr{B}_{a+t(b-a)}|,$$

for $t\in[0,1]$.

It is important to mention that the conditioned Itô measure $\mathbf{n}_{(1)}$ is the law of $\mathbf{e}_t$. One can find the formal definition of the Itô measure and its conditioned version in Chapter VI of the book [**Revuz and Yor 1999**]. Their connection with the contour funtions of large Galton-Watson trees can be found in [**Le Gall 2005**].

The following result gives the convergence of a sequence of contour functions torwards the normalized Brownian excursion for general critical Galton-Watson trees.

Theorem 1.9 (Aldous 1991) *Let τ be a Galton-Watson tree with offspring distribution μ such that μ is a critical probability measure with finite variance σ^2, and let τ_p be the random plane tree τ conditioned on having size p; let C_{τ_p} be the contour function of τ_p (viewed as a random variable taking values in $C^0(\mathbb{R},\mathbb{R})$. Then we have that*

$$\left(\frac{\sigma}{2p^{1/2}}\cdot C_{\tau_p}(2pt)\right)_{t\in[0,1]}\xrightarrow[p\to\infty]{(d)}\mathbf{e}_t,$$

for the uniform norm on $C([0,1],\mathbb{R})$.

A convergence result torwards real trees is given as follows:

Corollary 1.10 *Consider a Galton-Watson tree, τ satisfying the hypothesis of Theorem 1.9. The convergence in distribution of following random pointed metric space, viewed as a random element of $(\mathbb{K}^\bullet,\mathrm{d}_{\mathrm{GH}})$, holds as follows:*

$$\left(V(\tau_p), \frac{\sigma}{2p^{1/2}} \cdot \mathrm{d_{gr}}\right)_{t\in[0,1]} \xrightarrow[p\to\infty]{(d)} (\mathscr{T}_{\mathbf{e}}, \mathrm{d}_{\mathbf{e}}),$$

where $(\mathscr{T}_{\mathbf{e}}, d_{\mathbf{e}})$ *is also viewed as a random element of* $(\mathbb{K}^\bullet, \mathrm{d_{GH}})$*, its distinguished vertex being the image of 0 via the natural projection of* $[0,1]$.

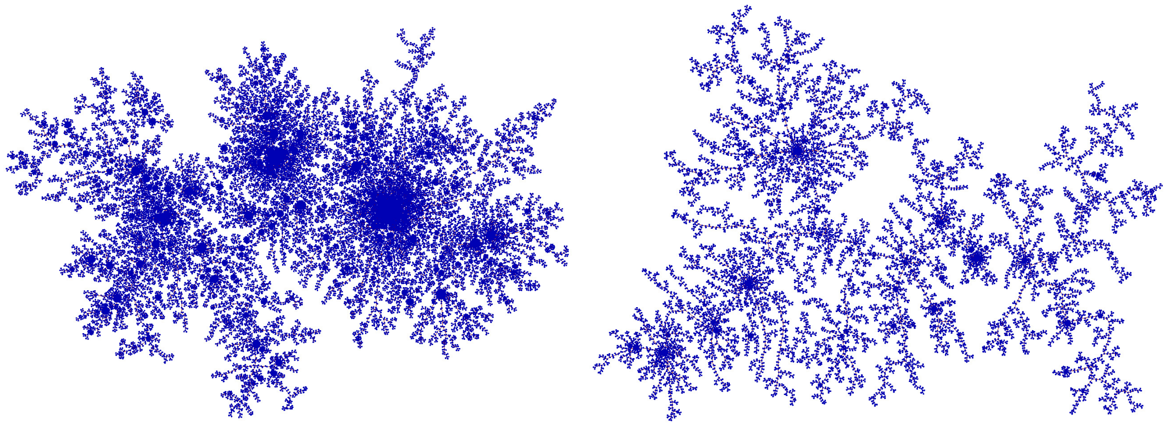
The CRT poses some important properties which include the fact that it has Hausdorff dimension 2. Also, it has a universality property which is reminiscent of that of Brownian motion. Numerous classes of uniform random trees are known in the literature to converge to the CRT. A few of these include the Cayley trees [**Aldous 1993**], unordered binary trees [**Marckert and Miermont 2011**] and unordered unrooted binary trees [**Stufler 2019**]. Some classes of random planar maps have been shown to converge to the CRT. Bettinelli demonstrated that uniform quadrangulations with n faces and a fixed boundary of size $\gg \sqrt{n}$ converge to the CRT [**Bettinelli 2015**]. Janson and Stefánsson showed that Boltzmann-type bipartite maps with n edges and a face of macroscopic degree converge in distribution to the CRT in the Gromov–Hausdorff sense [**Janson and Stefansson 2015**]. Additionally, random dissections [**Curien and Haas and Kortchemski 2015**] and random outerplanar maps, under various modifications, are also known to converge to the CRT [**Stufler 2014**; **Caraceni 2016**; **Stefansson and Stufler 2019**]. As part of this thesis, we show that certain random Halin-like maps [**Amankwah and Stefansson 2023**] and uniform random series-parallel maps also belong to the universality class of the CRT.

1.2.6 Stable looptrees

The generalization of convergence of Galton-Watson trees having their offspring distribution in the domain of attraction of a stable law with index $\alpha \in (1,2]$ were given by Duquesne and Le Gall (see [**Duquesne 2003**; **Duquesne and Le Gall 2002**]). They showed that these trees, with a properly rescaled graph distance admit a weak limit which is referred to as the α-stable tree, $\mathscr{T}_\alpha$. One recovers the Brownian tree for $\alpha = 2$, precisely $\mathscr{T}_2 = \sqrt{2} \cdot \mathscr{T}_{\mathbf{e}}$. When $\alpha < 2$, the stable trees are qualitatively different from the Brownian CRT in the sense that they possess a countably infinite number of vertices of infinite degree. These vertices are called *branching points*.

Curien and Kortchemski introduced the study of a family of random compact metric spaces $(\mathscr{L}_\alpha)_{1<\alpha<2}$, called *stable looptrees* (see [**Curien and Kortchemski 2014**]). On the discrete level, looptrees are graphs constructed on the vertex set of a plane tree, by connecting each vertex to its leftmost child by an edge, connecting adjacent siblings by edges, and finally connecting the rightmost child to its parent by an edge. The space $\mathscr{L}_\alpha$ may be informally described in the same spirit by replacing each branching point in the α-stable tree $\mathscr{T}_\alpha$ by a circle of length proportional to this degree. However, no two circles of $\mathscr{L}_\alpha$ share a common point. This construction therefore does not form a compact metric space. One has to go through some technicalities of taking closure (see **Curien and Kortchemski 2014**]). This approach makes use of the so called excursion of an α-stable spectrally positive Lévy process $X^{\mathrm{exc},\alpha}$ which is known for encoding $\mathscr{T}_\alpha$. For a formal definition of this process see for example [**Duquesne 2003**].

Stable looptrees later appeared in the study of site percolation on the Uniform Infinite



(a) *(b)*

Figure 1.4. Approximate simulation of stable trees by Igor Kortchemski. (a) is an $\alpha = 1.1$ stable tree whilst (b) is an $\alpha = 1.5$ stable tree.

Planar Triangulation (UIPT). In a companion paper, Curien and Kortchemski paper demonstrated that, for a site percolation with parameter $p = \frac{1}{2}$, the boundary of the convex hull of the connected component of the origin, when properly rescaled, converges in distribution to the stable looptree $\mathscr{L}_{\frac{3}{2}}$ [**Curien and Kortchemski 2015**]. In a certain phase regime, Richier proved that the boundary of a face-weighted Boltzmann planar map, whose face degree is in the domain of attraction of a stable law with index $\alpha \in (1,2)$, admits the stable looptree as a scaling limit, [**Richier 2018**]. Some classes of Boltzmann outerplanar maps with n vertices, when scaled by $n^{\frac{1}{\alpha}}$, are known to converge towards the stable looptree [**Stefansson and Stufler 2019**]. In a recent work, Sénizergues, Stefánsson and stufler intoduced the so-called decorated stable trees which are constucted from α-stable trees by blowing branching points into random metric spaces, rather than just deterministic circles in the case of stable looptrees [**Senizergues and Stefansson and Stufler 2023**].

2 Summary of papers

2.1 Paper I

In this collaborative work [**Amankwah and Stefansson 2023**] we consider certain classes of Boltzmann-measured random planar maps which are obtained from finite trees with marked corners (which we call skeleton trees) by adding an edge between two consecutive marks to form a boundary. When the trees have at least four vertices, none of which has degree two and only the leaves are marked, one obtains a *Halin map*. Two cases of this model are considered, namely:

- ($C^{\star}$) Each vertex in the skeleton tree H° of the map H has **exactly** one marked corner. The first corner counterclockwise from the root edge is marked.
- ($C^{\#}$) Each vertex in the skeleton tree H° of the map H has **at least** one marked corner. The first corner counterclockwise from the root edge is marked.

The map obtained for case $C^{\star}$ is the so-called noncrossing dissection of a polygon, whereas that obtained for case $C^{\#}$ is an outerplanar map whose blocks are noncrossing dissections. In both models, the maps are sampled according to the Boltzmann measure by assigning weights to their faces. In one case, we chose the weights so that they are in the domain of attraction of a stable law with index $\alpha \in (1,2)$. The maps arising from both models are then shown (according to Theorems 1.2 and 1.4 of [**Amankwah and Stefansson 2023**]) to converge after proper rescaling to the so-called α-stable looptree in the Gromov-Hausdorff sense. In another case, we chose the weights so that they have finite exponential moments. We show (according to Theorems 1.1 and 1.3 of [**Amankwah and Stefansson 2023**]) in this case that the maps converge after proper rescaling to the continuum random tree(CRT) in the Gromov-Hausdorff sense.

2.2 Paper II

In this work [**Amankwah et al. 2025**], we study the scaling limits of random series-parallel maps. A finite connected graph is called a *series-parallel graph* if it can be obtained from a single edge by repeatedly subdividing and doubling edges. We define a *series-parallel map* as an embedding of a series-parallel graph in the plane, up to continuous deformations. Uniformly sampled series-parallel maps with n edges belong to the class of "tree-like" maps. Specifically, they are shown in [**Stufler 2020**] to converge weakly, in the Gromov-Hausdorff sense, to a multiple of Aldous' continuum random tree (CRT) after rescaling distances by $n^{-1/2}$.

In this work, however, we focus on the two-connected blocks of series-parallel maps.

It turns out that certain weighted randomly sampled two-connected series-parallel maps with n edges converge, as $n \to \infty$, with distances rescaled by $n^{-1/2}$, to a multiple of the CRT.

There is a natural bijection between the set of plane trees with n leaves and the set of series-parallel maps with n edges, allowing us to describe series-parallel maps in terms of Galton-Watson trees and simply generated trees. Additionally, the maps admit a decomposition into *blobs*, which are maximal substructures separated by bottlenecks. Furthermore, geodesics between vertices within the same blob remain entirely within that blob. These structural properties, combined with a Markov chain argument introduced by Curien, Haas, and Kortchemski in [**Curien and Haas and Kortchemski 2015**], enable us to prove the main convergence result.

Paper I

ON SCALING LIMITS OF RANDOM HALIN-LIKE MAPS

by

Daniel Amankwah & Sigurdur Örn Stefánsson

Abstract. — We consider maps which are constructed from plane trees by assigning marks to the corners of each vertex and then connecting each pair of consecutive marks on their contour by a single edge. A measure is defined on the set of such maps by assigning Boltzmann weights to the faces. When every vertex has exactly one marked corner, these maps are dissections of a polygon which are bijectively related to non-crossing trees. When every vertex has at least one marked corner, the maps are outerplanar and each of its two-connected component is bijectively related to a non-crossing tree. We study the scaling limits of the maps under these conditions and establish that for certain choices of the weights the scaling limits are either the Brownian CRT or the α-stable looptrees of Curien and Kortchemski.

1. Introduction

A planar map is an embedding of a planar graph onto a 2-dimensional sphere up to continuous deformations. We refer to a planar map as rooted if it has a distinguished and oriented edge. Rooting a planar map makes it easier to handle combinatorially since it eliminates symmetry issues. Also, the root gives a starting point for recursive decomposition. The tail of the root edge will be called the root vertex. The connected components of the complement of a planar map are called faces. If f is a face of a planar map M, we will denote its degree (the number of edge sides that surround it, counted with multiplicity) by $\mathsf{deg}_M(f)$ or just $\mathsf{deg}(f)$ when there is no ambiguity. A rooted map with a single face is called a plane tree.

Research into asymptotic behavior of random planar maps has been active in the last two decades. This has been motivated in part by the connection with two-dimensional Liouville quantum gravity(see [**24**, **26**] together with references for detailed account). Much of the advances in this field have been made possible by the use of bijections between specific classes of planar maps and certain plane trees. See for example [**10**, **27**, **7**]. This is due to the fact that plane trees are easier to analyse since they are coded by functions on which one can apply the standard tools from analysis and probability theory. In [2, 3, 4], Aldous gives an exposition on the scaling limit of various classes of discrete plane trees conditioned to be large. For a Galton-Watson tree conditioned to be large, whose offspring distribution has mean 1 and a finite variance, he showed that the scaling limit, in the Gromov-Hausdorff sense (see Section 4.3) is a continuous random tree called the Brownian continuum random tree (CRT). The CRT is often denoted by $\mathcal{T}_{\mathbf{e}}$ where $\mathbf{e}$ is a standard Brownian excursion of duration 1 from which the tree may be constructed. It admits a universality property in the sense that it is the limit of several classes of trees and maps including dissections of polygons and outerplanar maps, which are of interest in this work. A dissection of a polygon is a two-connected planar map, all of whose vertices lie on the boundary of the unbounded face. Outerplanar maps are maps whose two-connected components are dissections. Curien, Haas and Kortchemski [11] established such limit theorems in the case of random dissections and this was first shown by Caraceni [8] in the case of uniform random outerplanar maps and later generalized by Stufler [31].

Generalizations to Galton-Watson trees whose offspring disribution is in the domain of attraction of a stable law with index $\alpha \in (1,2]$ were given by Duquesne and Le Gall (see [13, 14]). They showed that these trees, with a properly rescaled graph distance admit a weak limit which is referred to as the α-stable tree, $\mathcal{T}_\alpha$. In the case $\alpha = 2$ we recover the Brownian CRT, more precisely $\mathcal{T}_2 = \sqrt{2}\mathcal{T}_{\mathbf{e}}$. The α-stable trees are qualitatively different from the Brownian CRT when $\alpha < 2$, since then they possess a countably infinite number of vertices of infinite degree.

Curien and Kortchemski introduced the study of a family of random compact metric spaces $(\mathcal{L}_\alpha)_{1<\alpha<2}$ called α-stable looptrees (see [12]). On the discrete level, looptrees are graphs constructed on the vertex set of a plane tree by connecting each vertex to its leftmost child by an edge, connecting adjacent siblings by edges, and finally connecting the rightmost child to its parent by an edge (see Fig. 1). The space $\mathcal{L}_\alpha$ may be informally described in the same spirit by replacing each vertex of large degree in the α-stable tree $\mathcal{T}_\alpha$, by a circle of length proportional to this degree. Curien and Kortchemski gave sufficient

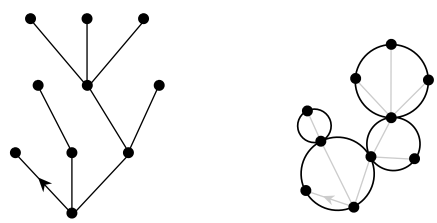


Figure 1. A looptree associated to a plane tree.

conditions for which a random sequence of discrete looptrees converges in distribution to the stable looptree. As an application they showed that the random dissection of a polygon sampled according to a Boltzmann distribution on the face degrees obeying the relevant conditions, converges, once properly rescaled, to the stable looptree for the Gromov-Hausdorff topology. Furthermore, Stefánsson and Stufler used this invariance principle to show that under certain conditions, properly rescaled Boltzmann face-weighted outerplanar maps also converge to stable looptrees [**29**].

These examples have the common feature that the objects involved are tree-like and when their faces are forced to have large degrees, in the limit, they appear as circles arranged along a tree structure. We are interested in understanding how generic this behavior is and provide in this paper additional examples of families of maps which share this feature of being tree-like and converging to the Brownian CRT or α-stable looptrees.

1.1. Halin-like maps. — We will assume (using the stereographic projection) that our planar maps are drawn in the infinite plane. The maps then have exactly one unbounded face which will be denoted by f_∞. The maps we consider in this work are defined as follows: Start with a rooted plane tree and assign marks to some of the corners around its vertices. Denote this marked tree by H°. Then go around the contour of the tree and connect each marked corner with an edge to the next marked corner on the contour. Denote the resulting map by H. We call H° the *skeleton* of H. We use the convention that the root of H lies on the boundary of the unbounded face and is the directed edge from the first to the second marked corner encountered in a clockwise exploration around H°, starting from the root vertex. The face containing the root edge is called the root face and is denoted by f_0.

In the special case when the skeleton H° has at least four edges and no vertex of degree two, and all its leaves but no other vertices receive a mark, the corresponding graph H has been referred to as a Halin graph in the literature, see e.g. [**17**] and Fig. 2. Due to this connection we refer to the maps considered in the current work as *Halin-like maps*. There are other models which bear similarity with Halin-like maps such as graphs with a given surplus of edges which appear as components in the critical Erdös-Rényi random graph [**1**] and unicellular maps, which are generalizations of trees on surfaces of arbitrary genus, see e.g. [**15**].

In the current work we will consider separately two special cases in which we put the following assumption on the Halin-like maps which are best described as conditions on the marks of the skeleton tree:

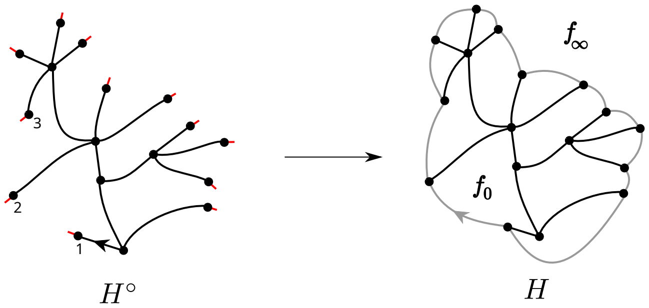


Figure 2. The skeleton tree H° has no vertices of degree two and each leaf has a mark but not other vertices. The corresponding map H is then a Halin graph. The marks on corners in H° are indicated by red line segments and the root edges are indciated by arrows.

($C^\star$) Each vertex in the skeleton tree H° of the map H has **exactly** one marked corner. The first corner counterclockwise from the root edge is marked.

($C^\#$) Each vertex in the skeleton tree H° of the map H has **at least** one marked corner. The first corner counterclockwise from the root edge is marked.

Under condition $C^\star$, the skeleton tree H° is a so-called *non-crossing tree* (NCT), see e.g. [**22**] in the context of scaling limits. The corresponding map H is then a dissection of a polygon which we will refer to as a *non-crossing dissection* (NCD), see Fig. 3. Under condition $C^\#$, the map H is an outerplanar map whose blocks are NCDs. We will refer to such an outerplanar map as a non-crossing outerplanar map (NCO), see Fig. 4.

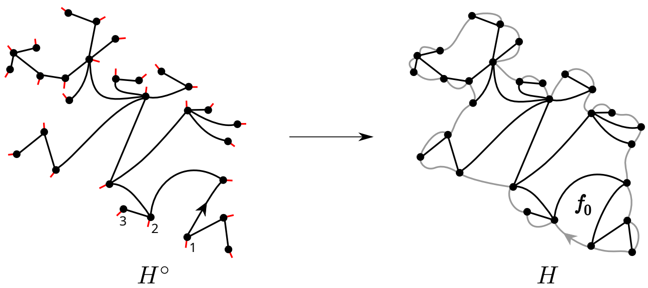


Figure 3. The skeleton tree H° satisfying condition $C^\star$ and its corresponding map H. The tree is a non-crossing tree and the map is a non-crossing dissection.

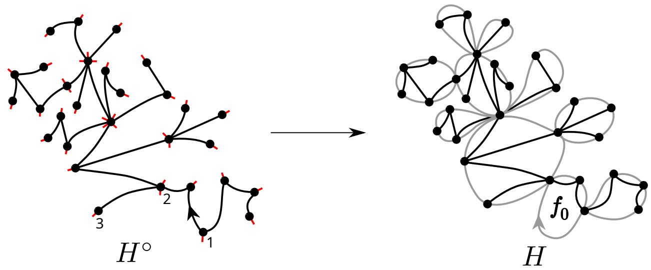


Figure 4. The skeleton tree H° satisfying condition $C^\#$ and its corresponding map H which is a non-crossing outerplanar map.

1.2. The main result. — For every integer $n \geq 1$, denote by $\mathbb{H}_n^*$ the set of Halin-like maps satisfying (C^*), $* \in \{\star, \#\}$ and having n bounded faces. When it is clear under which assumption we are working, we will leave out the $*$ from the notation. Let $H \in \mathbb{H}_n^*$ and fix a sequence $w = (w(k) : k \geq 2)$ of non-negative *face weights*. We define the weight of the map H by

$$W(H) = \prod_{\substack{f \text{ face of } H \\ f \neq f_\infty}} w(\mathsf{deg}(f))$$

and the normalization

$$Z_n^* = \sum_{H' \in \mathbb{H}_n^*} W(H').$$

If $Z_n^* > 0$, we define a random element $\mathcal{H}_n^{w,*}$ in $\mathbb{H}_n^*$ by

$$\mathbb{P}(\mathcal{H}_n^{w,*} = H) = \frac{1}{Z_n^*} W(H) \qquad \text{for all } H \in \mathbb{H}_n^*. \tag{1.1}$$

We view $\mathcal{H}_n^{w,*}$ as a random metric space equipped with the graph distance as its metric (where every edge of $\mathcal{H}_n^{w,*}$ has unit length).

It will prove convenient to define the two measures μ and $\mu_\varnothing$ on $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ by

$$\mu(k) = a(k+1)w(k+2)b^k, \ k \geq 0 \text{ and } \mu_\varnothing(k) = cw(k+1)b^k, \ k \geq 1 \tag{1.2}$$

where $a, b, c > 0$. We will assume throughout this paper that the generating function of the sequence w has a non-zero radius of convergence

$$\rho_w = \left(\limsup_{k \to \infty} w(k)^{1/k} \right)^{-1}.$$

This will allow us to define for each $b \in (0, \rho_w)$

$$a = \left(\sum_{k=0}^{\infty} (k+1) w(k+2) b^k \right)^{-1} \qquad \text{and} \qquad c = \left(\sum_{k=1}^{\infty} w(k+1) b^k \right)^{-1} \tag{1.3}$$

so that μ and $\mu_\varnothing$ are probability measures. The motivation for defining the measures μ and $\mu_\varnothing$ is that they are equivalent (in the sense of Equation (3.4)) to offspring distributions of non-root vertices and the root resp. in the weak dual tree of the Halin-like map under condition $(C^\star)$. This will allow us to use the theory of branching processes to understand the distribution of the maps. The shift by 2 in μ and 1 in $\mu_\varnothing$ is due to the deletion of 1 edge when taking the weak dual and substraction of 1 for non-root vertices since we consider only the number of offspring. The factor $k+1$ in μ comes from the fact that we can choose a location for the edge which was deleted when taking the weak dual in $k+1$ different ways.

When μ and $\mu_\varnothing$ are probability measures, denote their mean and variance by m_μ and σ_μ^2 respectively. In order to guarantee that $Z_n^* > 0$ for sufficiently large n we assume that $\mu(k)$ is aperiodic. The general case may also be considered but the results are essentially the same and we aim to keep the notation simple.

The main results of the current work are the following four scaling limit theorems for $\mathcal{H}_n^{w,*}$. Theorems 1.1 and 1.3 show that if the faces are not forced to be too large, $n^{-1/2} \cdot \mathcal{H}_n^{w,*}$ converges to a constant multiple (depending on $*$) of the Brownian CRT $\mathcal{T}_{\mathbf{e}}$. The notation $s \cdot \mathcal{H}_n^w$ represents the metric space obtained from $\mathcal{H}_n^w$ by multiplying all distances by $s > 0$. Theorems 1.2 and 1.4 however show that it is possible to choose the weights w such that faces in the maps become large when n is large and the maps, properly rescaled, converge in distribution towards stable looptrees. We use techniques from four different references in the proof of the theorems. Note that the convergence is understood as the weak convergence of measures in the Gromov-Hausdorff topology on the space of compact metric spaces (see Section 4.3 for the definition).

In the next two Theorems we assume that condition $(C^\star)$ holds and that μ and $\mu_\varnothing$ are probability measures on $\mathbb{Z}_+$ defined by (1.2) and (1.3).

Theorem 1.1. — *Assume condition $(C^\star)$. If there is a $b \in (0, \rho_w]$ and a $\lambda > 0$ such that $m_\mu = 1$ and $\sum_{k=0}^\infty e^{\lambda k}\mu(k) < \infty$ then there is a constant $c^\star(\mu) > 0$ such that*

$$n^{-1/2} \cdot \mathcal{H}_n^{w,\star} \xrightarrow[n\to\infty]{(d)} c^\star(\mu) \cdot \mathcal{T}_{\mathbf{e}}.$$

The proof, which is given in Section 4.4, follows from the Markov-chain argument developed in [11], which shows that the ratio of lengths of geodesics in large NCTs and the corresponding paths in their weak dual tree is given by

$$c_{\text{geo}}(\mu) = \frac{1}{4}\left(\sigma_\mu^2 + 1 + \frac{\gamma(2\mathbb{Z}_+ + 1)(1 - 2\gamma(\mathbb{Z}))}{1 - \gamma(2\mathbb{Z}_+) + \gamma(2\mathbb{Z}_+ + 1)}\right)$$

where $\gamma(j) = \frac{j\mu(j)}{j+1}$ and for any $A \subseteq \mathbb{Z}_+ = \{0, 1, \ldots\}$, $\gamma(A) = \sum_{j\in A}\gamma(j)$. By the results of Aldous, the weak dual tree of $\mathcal{H}_n^{w,\star}$, rescaled by $n^{-1/2}$, converges towards a multiple $c_{\text{tree}}(\mu) := 2\sigma_\mu^{-1}$ of the Brownian CRT. Combining these results yields $c^\star(\mu) = c_{\text{tree}}(\mu) \cdot c_{\text{geo}}(\mu)$.

It should be possible to relax the exponential moment condition on μ and only assume finite variance, see e.g. [23], but we do not pursue this further.

Theorem 1.2. — *Assume condition* $(C^\star)$*. If there is a* $b \in (0, \rho_w]$ *such that* $m_\mu = 1$ *and* μ *belongs to the domain of attraction of an* α*-stable law then there is a sequence of positive numbers* $b_n \to \infty$ *such that*

$$b_n^{-1} \cdot \mathcal{H}_n^{w,\star} \xrightarrow[n\to\infty]{(d)} \mathcal{L}_\alpha.$$

A probability measure μ is said to be in the domain of attraction of an α-stable law, with $\alpha \in (1, 2)$, if

$$\mu([j, \infty)) = j^{-\alpha} L(j),$$

where $L : \mathbb{R}_+ \to \mathbb{R}_+$ is a slowly varying function, i.e. $L(x) > 0$ and $\lim_{x\to\infty} \frac{L(tx)}{L(x)} = 1$ for all $t > 0$ (see [6] for details). In this case, the variance of μ is infinite. Common examples of slowly varying functions are constant functions, powers of logarithms, and iterated logarithms. By [20, Theorem 1.10], it holds that

$$b_n = n^{1/\alpha} \Lambda(n) \tag{1.4}$$

where Λ is a slowly varying function. An explicit example of a measure μ which satisfies the conditions of the theorem is

$$\mu(k) = ck^{-1-\alpha}, \qquad c > 0$$

in which case

$$b_n = \frac{n^{\frac{1}{\alpha}}}{(c|\Gamma(-\alpha)|)^{\frac{1}{\alpha}}}.$$

We refer the reader to e.g. [12, 20] for more information on random variables in the domain of attraction of stable laws, in a relevant context. The proof of Theorem 1.2, which is given in Section 4.5, follows directly from the invariance principle presented in [12, Theorem 4.1], along with an argument showing that the Gromov-Hausdorff distance between the NCD and the looptree of its weak dual tree is sufficiently small.

In the next two Theorems we assume that μ and $\mu_\varnothing$ are defined by (1.2), but they are not necessarily probability measures. Before the statement of the Theorems we require some notation. Define the generating functions

$$g^\mu(z) = \sum_{n=0}^{\infty} \mu(n) z^n \qquad \text{and} \qquad g^\mu_\varnothing(z) = \sum_{n=0}^{\infty} \mu_\varnothing(n) z^n$$

and denote their radius of convergence by ρ_μ. Define

$$\nu_\mu = \lim_{x \nearrow \rho_\mu} \frac{x(g^\mu)'(x)}{g^\mu(x)}$$

as the largest possible mean that μ can have, when b is varied. Let

$$A_\mu = \rho_\mu \frac{g^\mu_\varnothing(\rho_\mu)}{g^\mu(\rho_\mu)} \qquad \text{and} \qquad B_\mu = \rho_\mu \frac{(g^\mu_\varnothing)'(\rho_\mu)}{g^\mu(\rho_\mu)}.$$

Theorem 1.3. — *Assume condition* $(C^\#)$. *If one of the three cases*

$$\begin{cases} \nu_\mu \geq 1, \\ 0 < \nu_\mu < 1, \rho_\mu \geq 1, \\ 0 < \nu_\mu < 1, \frac{A_\mu(1-\nu_\mu)+\rho_\mu B_\mu}{(1-A_\mu)(1-\nu_\mu)} > 1 \end{cases}$$

holds then there is a constant $c^\# > 0$ *such that*

$$n^{-1/2} \cdot \mathcal{H}_n^{w,\#} \xrightarrow[n\to\infty]{(d)} c^\# \cdot \mathcal{T}_{\mathbf{e}}.$$

The proof, which is given in Section 4.6, follows from Theorem 6.60 in [**31**] which holds for random decorated trees (or enriched trees), satisfying certain conditions. The NCO is viewed as a size conditioned Galton-Watson tree $\mathcal{T}$ decorated by face weighted finite sequences of NCDs. We do not provide an explicit formula for the constant $c^\#$, but it may be calculated in terms of expected distances in face weighted finite sequences of NCDs, see Section 7.11 in [**31**]. The formula for its value will depend on which of the above three cases holds.

It will be apparent in the proof of Theorem 1.3 that the above conditions guarantee that the offspring distribution of $\mathcal{T}$ has finite exponential moments. We remark again that this exponential moment condition could possibly be relaxed to only assuming that the offspring distribution has finite variance.

Theorem 1.4. — *Assume condition* $(C^\#)$. *Fix* $\alpha \in (1,2)$ *and assume that*

$$\mu(k) = L(k)k^{-\alpha-1}r^{-k} \tag{1.5}$$

with $r = \rho_\mu < 1$ *and* L *slowly varying. If* $\nu_\mu < 1$ *and*

$$\frac{A_\mu(1-\nu_\mu)+\rho_\mu B_\mu}{(1-A_\mu)(1-\nu_\mu)} = 1$$

Then

$$c_n^{-1} \cdot \mathcal{H}_n^{w,\#} \xrightarrow[n\to\infty]{(d)} \mathcal{L}_\alpha$$

where

$$c_n = \left(\frac{|\Gamma(-\alpha)| g^\mu(r) B_\mu}{(1-\nu_\mu)(1-A_\mu)} \right)^{1/\alpha} (L_n n)^{1/\alpha}.$$

The proof, which is presented in Section 4.7 again involves representing $\mathcal{H}_n^{w,\#}$ as a size conditioned Galton-Watson tree, decorated by face weighted finite sequences of NCDs. The result then follows from similar arguments given explicitly for outerplanar maps, in [29]. However, we will instead apply the recent more general invariance principle for decorated trees presented in [28, Theorem 5.1].

Remark 1.5. — *We have not been able to relax conditions* ($C^\star$) *and* ($C^\#$) *in the current work. If a vertex has for example no marked corners it will not be on the boundary of* f_∞ *in the Halin-like map and we lose the connection to dissections and outer-planar maps. There are certain exceptions to this, e.g. when* H *is a Halin-map which satisfies that each internal vertex is connected to a leaf. Then only the leaves are marked and by contracting the edges containing the leaves one arrives again at a Halin-like map which is outerplanar. One may show that this contraction does not affect the Gromov-Hausdorff distance too much and thus one gets similar invariance principles as above.*

2. Halin-like maps and equivalent representations by trees

In this section we give formal definitions of plane trees and Halin-like maps and show that when they satisfy condition ($C^\star$) or ($C^\#$), they are equivalent to plane trees or enriched trees respectively. These connections will allow us to relate them to size conditioned Galton-Watson trees in a suitable sense which gives access to known invariance principles.

2.1. Rooted plane trees and Halin maps. — We briefly recall the formalism of rooted plane trees (or rooted ordered trees). Let $\mathbb{N} = \{1, 2, ...\}$ and define the set of finite sequences by

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n,$$

where by convention $\mathbb{N}^0 = \{\varnothing\}$. An element of $\mathcal{U}$ is thus a sequence $u = (u^1, ..., u^n)$ of elements of $\mathbb{N}$. If $u = (u^1, ..., u^m)$ and $v = (v^1, ..., v^n)$ belong to $\mathcal{U}$, we write the concatenation of u and v as $uv = (u^1, ..., u^m, v^1, ..., v^n)$.

A plane tree T is a finite or infinite subset of $\mathcal{U}$ such that,

1. $\varnothing \in T$.
2. if $v \in T$ and $v = uj$ for some $j \in \mathbb{N}$ then $u \in T$.
3. for every $u \in T$, there exists an integer $\mathsf{out}_T(u) \geq 0$ (the outdegree or number of children of u) such that for every $j \in \mathbb{N}$, $uj \in T$ if and only if $1 \leq j \leq \mathsf{out}_T(u)$.

The vertex $\varnothing$ is called the root vertex and the edge $\{\varnothing, (1)\}$ is called the root edge. The letter T will sometimes be suppressed in $\mathsf{out}_T(u)$ when it is clear from the context which tree is being referred to. Let us denote the set of all plane trees by $\mathbf{T}$ and the set of plane trees with n vertices by $\mathbf{T}_n$. For $T \in \mathbf{T}$, we will view each of its vertices as an individual of a population whose T is the geneological tree. If $T \in \mathbf{T}$ and $u \in T$, we define the shift of T at u by $\mathsf{shift}_u T = \{v \in \mathcal{U} : uv \in T\}$ which is itself a tree. The total progeny of T (total number of vertices of T) will be denoted by $|T|$. The maximal generation of the tree T is called its height and we denote it by $\mathsf{Height}(T)$.

Given a plane tree T with k edges, define the contour sequence of T

$$c = (c_0, c_1, \ldots, c_{2k})$$

as the sequence of vertices encountered when moving along the contour of T in a clockwise direction, starting and ending at $c_0 = c_{2k} = \varnothing$. The contour sequence is in a one-to-one correspondence with the corners around the vertices in T, and we use the convention that the corner clockwise from the root vertex is c_0.

We define marks on the corners of the tree T by specifying a sequence $m = (m_0, \ldots, m_{2k})$ of 0's and 1's. The corner c_i receives a mark if $m_i = 1$ but otherwise it does not. Let $|m| = \sum_i m_i$ denote the number of marks.

Definition 2.1. — *Let (T, m) be a rooted plane tree T with corners marked by a sequence m. Let $v_1, \cdots, v_{|m|}$ be the sequence of marked corners in the same order as they appear in the contour sequence. A Halin-like map H, is a map constructed by adding to T the edges $\{v_i, v_{i+1}\}$, $1 \leq i \leq |m| - 1$, and the edge $\{v_{|m|}, v_1\}$. We refer to (T, m) as the* skeleton tree *of H and use the notation $H^\circ := (T, m)$. The root edge of H is the directed edge (v_1, v_2) and the root face of H is the bounded face containing the root edge. The tree T is referred to as the shape of the marked tree (T, m) and will be denoted by $\mathcal{S}(T, m) = T$.*

The Halin maps considered in this work are assumed to satisfy condition $(C^\star)$ or $(C^\#)$ in the introduction. The set of Halin maps with n bounded faces, satisfying condition C^*, is denoted by $\mathbb{H}_n^*$, $* \in \{\star, \#\}$.

2.2. Condition $(C^\star)$ and non-crossing trees. — Under condition $(C^\star)$ the skeleton H° is a non-crossing tree and the corresponding map is a non-crossing dissection. We denote the set of non-crossing trees with n vertices by $\mathbf{NCT}_n$ and the set of non-crossing dissections with n vertices by $\mathbf{NCD}_n$. There is another bijection between NCDs and NCTs which is defined by taking the weak dual of the NCDs: First, let $\hat{H}$ be the usual geometric dual and if f is a face in H let $\hat{f}$ be its corresponding vertex in the dual. The weak dual, denoted by

$\bar{H}$, is obtained from $\hat{H}$ by removing from it the vertex $\hat{f}_\infty$ and all the edges $\{\cdot, \hat{f}_\infty\}$ containing it. Since H is a dissection of a polygon, $\bar{H}$ is clearly a tree. We define its root edge as the leftmost edge emanating from the vertex $\hat{f}_0$ dual to the root face. Moreover, each vertex in $\bar{H}$, has a unique corner from which an edge $\{\cdot, \hat{f}_\infty\}$ was removed and therefore we may view the weak dual as a non-crossing tree, see Fig. 5. We have thus arrived at the following

Lemma 2.2. — *The function $\phi_n : \mathbb{H}_n^\star \to \boldsymbol{NCT}_n$, $H \mapsto \bar{H}$, is a bijection. If f is a face in $H \in \mathbb{H}_n^\star$ and u the corresponding vertex in $\bar{H}$ then*

$$\mathsf{out}_{\bar{H}}(u) = \begin{cases} \mathsf{deg}_H(f) - 1 & \text{if } u = \varnothing, \\ \mathsf{deg}_H(f) - 2 & \text{otherwise.} \end{cases}$$

The reason for the shift in degrees is that $\mathsf{out}(u)$ is an outdegree (accounting for -1 for each non-root vertex) and the deletion of edges when taking the weak dual (accounting for -1 for all vertices). The bijection ϕ_n furthermore defines a bijection between the set of dissections and trees of arbitrary finite sizes which we will denote by ϕ.

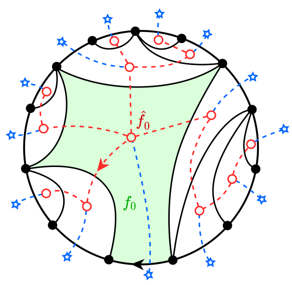


Figure 5. A NCD in black and its weak dual tree in dashed red. The root face is depicted in green. The augmented tree is obtained by adding the dashed blue edges.

Later we will require a slightly modified version of the weak dual which is obtained by adding back to $\bar{H}$ the edges dual to the boundary of H but in such a way that each of them gets a separate endpoint in f_∞. This is the same notion of a dual as is used in [11]. Equivalently, given a $T \in \mathbf{NCT}_n$, add edges to each of its marked corners which have one of their endpoints at the corresponding corner and the other one at a new leaf, see Fig. 5. We will call these trees *augmented* NCTs and denote them by $T^\bullet$. These play the role of the *planted trees* used in [11].

2.3. Condition $(C^{\#})$ and enriched trees. — Under condition $(C^{\#})$ the Halin-maps still have the property that each vertex lies on the boundary of the face f_∞, however they are not necessarily two-connected. Therefore, they will in general be outerplanar maps. An outerplanar map may be decomposed into an underlying plane tree whose vertices are decorated by sequences of dissections. This representation is explained in detail in [29, Lemma 4.3] in the formalism of *enriched trees* (or decorated trees) where it is used to obtain scaling limit results for face weighted outerplanar maps. We will follow the same approach, but in the current case the dissections involved are non-crossing dissections (NCDs) and therefore we will need to understand some of their properties in more detail.

We give a short description of this representation below. Let $\mathbf{NCD}'_n$ be the set of NCDs with n vertices where we do not count the root vertex. Pictorially we will represent the root vertex with a $*$. Now define the set of finite sequences of size n by

$$\mathbf{SD}_n = \bigcup_{k\geq 0} \bigcup_{\substack{n_1+\cdots+n_k=n \\ n_i\geq 1,\ 1\leq i\leq k}} \{(D_{n_1}, D_{n_2}, \ldots, D_{n_k}) \ : \ D_{n_i} \in \mathbf{NCD}'_{n_i}, 1 \leq i \leq k\}.$$

An element from $\mathbf{SD}_n$ is viewed as a connected map by identifying the root vertices of each element in the sequence. Its unbounded face is denoted by f_∞. The vertices of $S \in \mathbf{SD}_n$ are ordered by their occurence on the boundary of the unbounded face in a clockwise direction from the root-vertex. We will denote them by $s_0(S), s_1(S), \ldots, s_n(S)$ where $s_0(S)$ is the root-vertex. The directed edge $(s_0(S), s_1(S))$ is taken to be the root vertex.

Now, let $T \in \mathbf{T}_n$ be a plane tree with n vertices. For each vertex $u \in T$ let S_u be an element from $\mathbf{SD}_{\mathsf{out}(u)}$. Note that if u is a leaf, i.e. $\mathsf{out}(u) = 0$, then S_u is the empty sequence. We construct a map as follows: For each non-leaf vertex u and each of its non-leaf children ui, identify $s_i(S_u)$ and $s_0(S_{ui})$. The resulting object is a non-crossing outerplanar map (NCO), see Fig. 6 for an example. We denote this map by O. The construction is reversible, i.e. O may be decomposed into an underlying plane tree T where each vertex u is decorated by an object from $\mathbf{SD}_{\mathsf{out}(u)}$. Therefore, we have a bijection between the set of NCOs and trees decorated by finite sequence of NCDs. We will therefore sometimes write $O = (T, (S_u)_{u\in T})$.

3. Simply generated trees and Galton-Watson trees

In the previous section we showed that certain Halin-like maps may be described in terms of trees and enriched trees. In this section we show how this allows us to understand the corresponding random Halin-like maps in terms of simply generated trees and Galton-Watson trees.

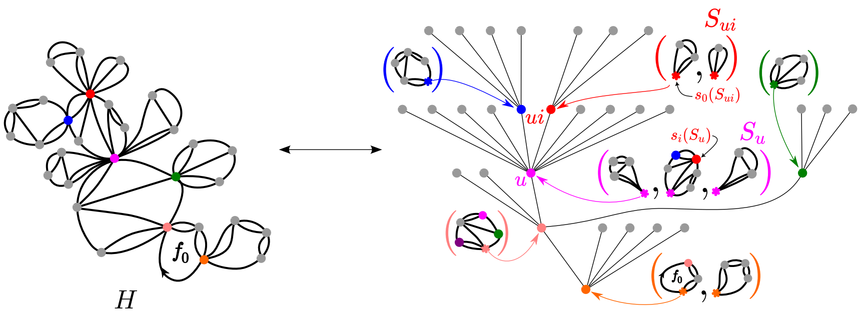


Figure 6. **Left**: A NCO, the same as in Fig. 4. **Right:** A corresponding decorated tree where the decorations are sequences of NCDs. Grey vertices are decorated by empty sequences.

Let μ be a probability measure on $\mathbb{Z}_+$ such that $\mu(1) < 1$. We call μ, the offspring distribution. The *Galton-Watson measure* with offspring distribution μ is the probability measure, GW^μ on $\mathbf{T}$ such that:

1. $\mathrm{GW}^\mu(\mathsf{out}(\varnothing) = j) = \mu(j)$ for $j \geq 0$,
2. If τ is distributed by GW^μ, then for every $j \geq 1$ with $\mu > 0$, conditionally on $\{\mathsf{out}(\varnothing) = j\}$, the subtrees $\mathsf{shift}_1\tau, \dots, \mathsf{shift}_j\tau$ are i.i.d. with distribution GW^μ.

Informally, the tree starts as a single root and then grows such that each individual already present in the tree has a number of offspring according to the measure μ, independent of all others.

If μ has mean less than or equal to 1 then the tree τ is almost surely finite and

$$\mathrm{GW}^\mu(\tau = T) = \prod_{u \in T} \mu(\mathsf{out}_\tau(u)). \tag{3.1}$$

In order to relate the Galton-Watson measure to random non-crossing trees we require the following modification. Let $\mu_\varnothing$ be another offspring distribution and define

$$\mathrm{GW}^{\mu,\mu_\varnothing}(\tau = T) = \mu_\varnothing(\mathsf{out}_\tau(\varnothing)) \prod_{u \in T \setminus \{\varnothing\}} \mu(\mathsf{out}_\tau(u)). \tag{3.2}$$

We interpret this such that the root has offspring according to the measure $\mu_\varnothing$ and all other individuals have offspring according to μ, independent of others.

Another related model of random trees is the following: Let $\eta = (\eta(i))_{i \geq 0}$ be a sequence of non-negative numbers, not necessarily probabilities. Let $T \in \mathbf{T}_n$ and define the weight of T by

$$\eta(T) = \prod_{u \in T} \eta(\mathsf{out}_T(u)).$$

A *simply generated tree* on n vertices is a random plane tree in $\mathbf{T}_n$ selected with a probability proportional to its weight $\eta(T)$. We list some important properties of simply generated trees below and refer to Janson's comprehensive survey [18] for details. Define the *partition function*

$$Z_n^\eta = \sum_{T \in \mathbf{T}_n} \eta(T)$$

and the generating series

$$\mathcal{Z}^\eta(z) = \sum_{n=1}^{\infty} Z_n^\eta z^n, \qquad g^\eta(z) = \sum_{n=0}^{\infty} \eta(n) z^n \tag{3.3}$$

and denote their radii of convergence by R_η and ρ_η respectively. In the current work we will assume that $\rho_\eta > 0$ in which case it is possible, by *tilting* the weights η, to assume that they are probabilities. Tilting refers to defining new weights

$$(\hat{\eta}(k))_{k\geq 0} = (ab^k \eta(k))_{k \geq 0}, \qquad a, b > 0 \tag{3.4}$$

and noting that these define the same simply generated trees as the original weights since

$$\hat{\eta}(T) = \prod_{u \in T} ab^{\mathsf{out}_T(u)} \eta(\mathsf{out}_T(u)) = a^{|T|} b^{|T|-1} \eta(T). \tag{3.5}$$

Then, the simply generated tree has the same distribution as a GW^η tree conditioned on having n vertices which we will denote by GW_n^η.

A standard result is the following relation between the generating series

$$\mathcal{Z}^\eta(z) = z g^\eta(\mathcal{Z}^\eta(z)). \tag{3.6}$$

Define

$$\nu_\eta = \lim_{x \nearrow \rho_\eta} \mathbf{m}_\eta(x) \qquad \text{with } \mathbf{m}_\eta(x) = \frac{x (g^\eta)'(x)}{g^\eta(x)}$$

and

$$\tau_\eta = \begin{cases} \text{unique solution to } \mathbf{m}_\eta(x) = 1 & \text{if } \nu_\eta \geq 1 \\ \rho_\eta & \text{if } \nu_\eta < 1. \end{cases} \tag{3.7}$$

Then it holds that

$$\tau_\eta = \mathcal{Z}^\eta(R_\eta) \qquad \text{and} \qquad R_\eta = \frac{\tau_\eta}{g^\eta(\tau^\eta)}, \tag{3.8}$$

see e.g. [18, Remarks 7.4 and 7.5]. The parameter τ_η determines tilted weights so that the offspring distribution of the associated Galton-Watson tree has mean $\mathbf{m}_\eta(\tau_\eta) = 1$, if possible. Otherwise the mean $\mathbf{m}_\eta(\tau_\eta) = \nu_\eta < 1$ is the largest attainable. For these weights, we define their variance by

$$\sigma_\eta^2 = \tau_\eta \mathbf{m}'(\tau_\eta).$$

When $\rho_\eta > 0$ we have the three cases:

$$\text{I) } \nu_\eta > 1 \quad \text{II) } \nu_\eta = 1 \quad \text{III) } \nu_\eta < 1 \tag{3.9}$$

in which the large simply generated trees have different properties. In case I), the offspring distribution has finite exponential moments and the random trees, rescaled by $n^{-1/2}$, converge towards a multiple of Aldous' Brownian CRT. In case II) the trees also converge towards the CRT when $\sigma_\eta < \infty$ but when $\sigma_\eta = \infty$ one needs to assume the regularity condition that the weights are in the domain of attraction of an α-stable law, with $\alpha \in (1, 2]$. In that case, the trees, rescaled by $n^{-\frac{\alpha-1}{\alpha}}$ (up to slowly varying functions), converge towards the α-stable tree [**13, 14**]. Finally, in case III) there is no interesting scaling limit [**21**] but there appears a unique vertex of degree $(1-\nu_\eta)n$ in the trees when they become large. This phenomenon has been referred to as *condensation* [**18, 19**]. All the three cases above will be relevant when we consider the scaling limits of the Halin-like maps.

It is also possible to define modified simply generated trees which correspond to Galton-Watson trees distributed by $\mathrm{GW}^{\eta,\eta_\varnothing}$ conditioned on having n vertices, denoted by $\mathrm{GW}_n^{\eta,\eta_\varnothing}$. First assume that η and $\eta_\varnothing$ are sequences of non-negative weights, not necessarily probabilities. Define the weight of a tree T by

$$\eta_\varnothing(T) = \eta_\varnothing(\mathsf{out}_T(\varnothing)) \prod_{u \in T\setminus\{\varnothing\}} \eta(\mathsf{out}_T(u)), \tag{3.10}$$

the partition function

$$Z^\eta_{\varnothing,n} = \sum_{T\in\mathbf{T}_n} \eta_\varnothing(T)$$

the generating series

$$\mathcal{Z}^\eta_\varnothing(z) = \sum_{n=1}^{\infty} Z^\eta_{\varnothing,n} z^n, \qquad g^\eta_\varnothing(z) = \sum_{n=0}^{\infty} \eta_\varnothing(n) z^n$$

and denote their radii of convergence by $R_{\varnothing,\eta}$ and $\rho_{\varnothing,\eta}$ respectively. A modified simply generated tree with weights η and $\eta_\varnothing$ is a random tree chosen with probability proportional to its weight $\eta_\varnothing$. The tilting in (3.4) may be applied simultaneously to η and $\eta_\varnothing$ without changing the distribution. We will, unless otherwise stated, assume that μ is given by (1.2) from which it follows that $\rho_{\varnothing,\eta} = \rho_\eta$.

We now have the pair of equations (3.6) and

$$\mathcal{Z}^\eta_\varnothing(z) = z g^\eta_\varnothing(\mathcal{Z}^\eta(z)) \tag{3.11}$$

which immediately yields that $R_{\varnothing,\eta} = R_\eta$, since $\rho_{\varnothing,\eta} = \rho_\eta$. We will still consider the three cases given by the criteria in (3.9) when we study the modified Galton-Watson trees. Note that these criteria do not depend on the probabilities $\eta_\varnothing$.

3.1. Random non-crossing trees as modified Galton-Watson trees. — We will now explain how random non-crossing trees are related to the modified Galton-Watson trees distributed by $\mathrm{GW}^{\mu,\mu_\varnothing}$ and then we will present some results which show that these trees are in many aspects very close to GW^μ trees. We start by stating the following result which is due to Kortchemski and Marzouk, in the context of non-crossing trees.

Theorem 3.1 (Kortchemski & Marzouk, Theorem 18 [**22**])
Assume that condition $(C^\star)$ holds. Then

$$\mathcal{S}(\phi_n(\mathcal{H}_n^{w,\star})) \stackrel{d}{=} \mathrm{GW}_n^{\mu,\mu_\varnothing}$$

where μ and $\mu_\varnothing$ are probability measures defined by (1.2) and (1.3), ϕ_n denotes the bijection in Lemma (2.2) and $\mathcal{S}$ maps a non-crossing tree to its shape.

The proof follows from the fact that $\mathcal{H}_n^{w,\star}$ is a face weighted dissection and the outdegrees in $\phi_n(\mathcal{H}_n^{w,\star})$ are related to the face degrees in $\mathcal{H}_n^{w,\star}$ by Lemma 2.2.

The next lemma is analogous to [**22**, Proposition 23] which applies in the case when μ is critical and is in the domain of attraction of a stable law with index $\alpha \in (1, 2]$, which belongs to cases I and II in (3.9). The current version shows that a similar result holds when $\nu_\mu < 1$ which belongs to case III in (3.9). These results both show that when the two type trees $\mathrm{GW}_n^{\mu,\mu_\varnothing}$ are large, the degree of the root converges towards a finite random variable and that exactly one of the subtrees attached to the root has mass $M_n = n - o(1)$ with high probability. In particular, this large subtree, conditionally on M_n, behaves as a single type $\mathrm{GW}_{M_n}^\mu$ tree. This allows us to conclude that many of the asymptotic properties of the two type trees are the same as those of the single type trees. In particular, when $\nu_\mu < 1$, condensation will occur in the largest subtree of the root (but not at the root itself).

Let N_n denote the degree of the root and M_n denote the size of the largest subtree of the root in a random tree distributed by $\mathrm{GW}_n^{\mu,\mu_\varnothing}$. Let $\mathcal{N}$ be a random variable with distribution

$$\mathbb{P}(\mathcal{N} = k) = \frac{k\mu_\varnothing(k)}{\sum_{j\geq 1} j\mu_\varnothing(j)}$$

for $k \geq 1$. Let $(\tau_i)_{i\geq 0}$ be a sequence of independent trees distributed as GW^μ. When μ is critical or subcritical, $|\tau_i|$ is a.s. finite.

LEMMA 3.2. — *Fix $\alpha > 1$ and let*

$$\mu(k) = L(k)k^{-\alpha-1} \tag{3.12}$$

be probabilities with mean $m = \nu_\mu < 1$ where L is slowly varying. Let $\mu_\varnothing$ be the probabilities related to μ via (1.2)and (1.3). Then

$$N_n \xrightarrow[n\to\infty]{(d)} \mathcal{N}$$

and

$$n - 1 - M_n \xrightarrow[n\to\infty]{(d)} \sum_{i=1}^{\mathcal{N}-1} |\tau_i|.$$

Proof. — This can be shown with direct but somewhat lengthy calculations but instead we use a recent general result by Stufler on Gibbs-partitions with a critical composition scheme [**32**, Theorem 3.11, (i)]. We consider Equation (3.11) (with η replaced by μ) as such a composition scheme where $\mathcal{Z}^\mu_\varnothing$, $g^\mu_\varnothing$ and $\mathcal{Z}^\mu$ correspond respectively to U, V and W in [**32**]. In order to satisfy the requirements of this theorem, it suffices to have

$$g'_\varnothing(\mathcal{Z}^\mu(R_\mu)) < \infty, \tag{3.13}$$

$$\rho_\mu = \mathcal{Z}^\mu(R_\mu), \tag{3.14}$$

$$\mu_\varnothing(k) = \frac{c}{a} L_1(k)k^{-(\alpha+2)}\rho_\mu^{-k} \quad \text{and} \tag{3.15}$$

$$Z^\mu_n = \rho_\mu L_2(n)R_\mu^{-n}((1-\nu_\mu)n)^{-(\alpha+1)} \tag{3.16}$$

for slowly varying functions L_1 and L_2. First note that $\rho_{\varnothing,\mu} = \rho_\mu = 1$ and by (1.2) and (1.3) it holds that

$$\mu_\varnothing(k) \sim \frac{c}{a} L(k)k^{-\alpha-2}.$$

Therefore (3.15) holds with $L_1(k) \sim L(k)$ as $k \to \infty$. Since $\nu_\mu < 1$ we are in case III in (3.9) and therefore (3.14) holds which further gives $\mathcal{Z}^\mu(R_\mu) = 1$. Equation (3.13) then reads $g'_\varnothing(1) < \infty$ which holds due to (3.15). Finally, Equation (3.16) follows from [**21**, Equation 14] with $L_2(n) \sim L(n)$ as $n \to \infty$. □

3.2. Random non-crossing outerplanar maps as random enriched trees. — Recall the definition $\mathbf{SD}_n$ of the finite sequences of NCDs in Section 2.3. Let $w = (w(k) : k \geq 2)$ be a weight sequence and assume μ is related to w through Equation (1.2). Let $S \in \mathbf{SD}_n$ and define its weight as

$$W(S) = \prod_{\substack{f \text{ face of } S \\ f \neq f_\infty}} w(\mathsf{deg}(f))$$

and the normalization by

$$Z_n^{\mu,\mathrm{seq}} = \sum_{S' \in \mathbf{SD}_n} W(S')$$

with the convention that $Z_0^{\mu,\mathrm{seq}} = 1$. Let $\mathcal{SD}_n^w$ be the random element in $\mathbf{SD}_n$ distributed by $W(\cdot)/Z_n^{\mu,\mathrm{seq}}$. Define the generating function

$$\mathcal{Z}^{\mu,\mathrm{seq}}(z) = \sum_{n=0}^{\infty} Z_n^{\mu,\mathrm{seq}} z^n.$$

The next result shows how we may construct a random non-crossing outerplanar map by viewing it as an enriched tree. The statement is essentially the same as in Lemma 4.2 in [**29**], taking into account that the dissections are non-crossing in the current case. The proof follows directly from the representation of the NCO as a tree decorated by finite sequences of NCDs, explained in Section 2.3.

Lemma 3.3. — *Assume that condition* $(C^{\#})$ *holds. Let* $\mathcal{T}_n^\eta$ *be an* η*-simply generated tree, where* $\eta(j) = Z_j^{\mu,seq}$ *for all* $j \geq 0$*. Conditionally on* $\mathcal{T}_n^\eta$*, for each vertex* $u \in \mathcal{T}_n^\eta$*, sample* β_u^w *independently according to the distribution of* $\mathcal{SD}^w_{\mathsf{out}(u)}$*. Then*

$$\mathcal{H}_n^{w,\#} \overset{d}{=} (\mathcal{T}_n^\eta, (\beta_u^w)_{u \in \mathcal{T}_n^\eta}).$$

In order to apply Lemma 3.2 to the study of scaling limits we need to understand how the weights w affect the behaviour of β_n^w and $\mathcal{T}_n^\eta$ for n large. Each element in the finite sequence β_n^w is a NCD which has a NCT as its weak dual. Thus, the possible cases which we consider for the weights w will still be of the form given in (3.9) (in our case, μ has the role of η in (3.9)). We then find the following cases for the weights η according to the choice of μ (or equivalently w). We note that this result is analogous to the one for outerplanar maps obtained in [**31**, Lemma 2.62], but in the current case the dissections are non-crossing. The proof will be very similar but we include it for completeness.

Lemma 3.4. — *Let* μ *and* $\mu_\varnothing$ *be defined as in* (1.2) *and* $\eta(j) = Z_j^{\mu,seq}$*,* $j \geq 0$*. It holds that*

$$\nu_\eta = \begin{cases} \infty & \text{if } \ \nu_\mu \geq 1, \\ \infty & \text{if } \ 0 < \nu_\mu < 1, \ \rho_\mu \geq 1, \\ \frac{A_\mu(1-\nu_\mu)+\rho_\mu B_\mu}{(1-A_\mu)(1-\nu_\mu)} & \text{if } \ 0 < \nu_\mu < 1, \ \rho_\mu < 1 \end{cases}$$

where

$$A_\mu = \rho_\mu \frac{g_\varnothing^\mu(\rho_\mu)}{g^\mu(\rho_\mu)} \qquad \text{and} \qquad B_\mu = \rho_\mu \frac{(g_\varnothing^\mu)'(\rho_\mu)}{g^\mu(\rho_\mu)}.$$

Proof. — By Lemma 3.3 we may write

$$\mathcal{Z}^{\eta}(z) = zg^{\eta}(\mathcal{Z}^{\eta}(z)) \tag{3.17}$$

with

$$g^{\eta}(z) = \mathcal{Z}^{\mu,\mathrm{seq}}(z) = \frac{1}{1-\mathcal{Z}^{\mu}_{\varnothing}(z)}. \tag{3.18}$$

The second equality follows from a standard argument (see e.g. [**31**]) which relates the generating function of the partition function for finite sequences of NCTs to the generating function of the partion function for NCTs. It follows that

$$\rho_{\eta} = \min\{R_{\mu}, y\} \tag{3.19}$$

where y is the unique solution to $\mathcal{Z}^{\mu}_{\varnothing}(y) = 1$. Recall that

$$\nu_{\eta} = \lim_{x \nearrow \rho_{\eta}} \mathbf{m}_{\eta}(x)$$

where

$$\mathbf{m}_{\eta}(x) = \frac{x(g^{\eta})'(x)}{g^{\eta}(x)} = \frac{x(\mathcal{Z}^{\mu}_{\varnothing})'(x)}{1-\mathcal{Z}^{\mu}_{\varnothing}(x)} = x(\mathcal{Z}^{\mu}_{\varnothing})'(x)\sum_{n=0}^{\infty}(\mathcal{Z}^{\mu}_{\varnothing}(x))^{n}. \tag{3.20}$$

Differentiating (3.6) and (3.11) we get

$$\begin{aligned}(\mathcal{Z}^{\mu})'(x) &= g^{\mu}(\mathcal{Z}^{\mu}(x)) + x(g^{\mu})'(\mathcal{Z}^{\mu}(x))(\mathcal{Z}^{\mu})'(x)\\ (\mathcal{Z}^{\mu}_{\varnothing})'(x) &= g^{\mu}_{\varnothing}(\mathcal{Z}^{\mu}(x)) + x(g^{\mu}_{\varnothing})'(\mathcal{Z}^{\mu}(x))(\mathcal{Z}^{\mu})'(x)\end{aligned}$$

which solved together for $(\mathcal{Z}^{\mu}_{\varnothing})'(x)$ yields

$$(\mathcal{Z}^{\mu}_{\varnothing})'(x) = g^{\mu}_{\varnothing}(\mathcal{Z}^{\mu}(x)) + \frac{x(g^{\mu}_{\varnothing})'(\mathcal{Z}^{\mu}(x))g^{\mu}(\mathcal{Z}^{\mu}(x))}{1-x(g^{\mu})'(\mathcal{Z}^{\mu}(x))}. \tag{3.21}$$

We first assume that $\nu_{\mu} \geq 1$ and show that $\nu_{\eta} = \infty$. By (3.20), we may view $\mathbf{m}_{\eta}(x)$ as a power series in x with positive coefficent whose radius of convergence equals

$$r = \min\{R_{\mu}, y\} = \rho_{\eta}$$

where y is the unique value such that $\mathcal{Z}^{\mu}_{\varnothing}(y) = 1$. It is therefore sufficient to show that $\mathbf{m}_{\eta}(\rho_{\eta}) = \infty$. Since $\nu_{\mu} \geq 1$ then by (3.7) there is a $\tau_{\mu} \leq \rho_{\mu} = \rho_{\varnothing,\mu}$ such that $\mathbf{m}_{\mu}(\tau_{\mu}) = 1$ and we also recall the relations (3.8) which state that $\tau_{\mu} = \mathcal{Z}^{\mu}(R_{\mu})$ and $R_{\mu} = \tau_{\mu}/g^{\mu}(\tau_{\mu})$. Thus,

$$\begin{aligned}(\mathcal{Z}^{\mu}_{\varnothing})'(R_{\mu}) &= g^{\mu}_{\varnothing}(\tau_{\mu}) + \frac{R_{\mu}(g^{\mu}_{\varnothing})'(\tau_{\mu})g^{\mu}(\tau_{\mu})}{1-R_{\mu}(g^{\mu})'(\tau_{\mu})}\\ &= g^{\mu}_{\varnothing}(\tau_{\mu}) + \frac{\tau_{\mu}(g^{\mu}_{\varnothing})'(\tau_{\mu})}{1-\mathbf{m}_{\mu}(\tau_{\mu})} = \infty.\end{aligned}$$

From this, (3.19) and (3.20) it follows that $\mathbf{m}_{\eta}(\rho_{\eta}) = \infty$.

Next, assume that $0 < \nu_\mu < 1$. Then by (3.7) $\tau_\mu = \rho_\mu$. Firstly, if $\rho_\mu \geq 1$ then $\mathcal{Z}^\mu(R_\mu) = \tau_\mu \geq 1$ and therefore there is a $y \leq R_\mu$ such that $\mathcal{Z}^\mu(y) = 1$. Thus $\rho_\eta = y$ and we see immediately from (3.20) that $\mathbf{m}_\eta(\rho_\eta) = \infty$ and thus $\nu_\eta = \infty$. Finally, if $\rho_\mu < 1$ then $\rho_\eta = R_\mu$ and we get from (3.11), (3.20) and (3.21) that

$$\nu_\eta = \mathbf{m}_\eta(\rho_\eta) = \mathbf{m}_\eta(R_\mu) = \frac{R_\mu(\mathcal{Z}^\mu_\varnothing)'(R_\mu)}{1 - \mathcal{Z}^\mu_\varnothing(R_\mu)}$$
$$= \frac{\tau_\mu}{g^\mu(\tau_\mu)}\left[\frac{g^\mu_\varnothing(\tau_\mu) + \frac{\tau_\mu (g^\mu_\varnothing)'(\tau_\mu)}{1-\nu_\mu}}{1 - \tau_\mu \frac{g^\mu_\varnothing(\tau_\mu)}{g^\mu(\tau_\mu)}}\right]$$

The result follows by recalling that $\tau_\mu = \rho_\mu$ in the current case. □

REMARK 3.5. — *If $\mu = \mu_\varnothing$ then $A_\mu = \rho_\mu$ and $B_\mu = \nu_\mu$ and we recover the formula in* [**31**].

4. Proofs of the limit theorems

In this section we provide more details for the proofs of Theorems 1.1 - 1.4. In each case we show how existing techniques may be adapted to the current situation. In the first three subsections we introduce some important definitions. Firstly, we introduce the *Łukasiewicz path* of a tree which is a central object in the proofs, secondly we define the notion of a discrete looptree and finally we recall the definition of the Gromov-Hausdorff metric.

4.1. Łukasiewicz path. — Let T be a plane tree whose vertices are listed in depth first-search-order(or lexicographical order), $\varnothing = v(0) < v(1) < \ldots < v(|T|-1)$. The Łukasiewicz path $W(T) = (W_j(T), 0 \leq j \leq |T|)$ is defined by $W_0(T) := 0$ and

$$W_j(T) := \sum_{i=0}^{j-1} (\mathsf{out}(v(i)) - 1)$$

for $1 \leq j \leq |T|$). Figure 7 gives an example of a tree encoded by a Łukasiewicz path. W is viewed as a càdlàg function (right continuous with left limits) by assuming it is constant in between successive integers. It may be seen that $W_j(T) \geq 0$ for $0 \leq j \leq |T|-1$, but $W_{|T|}(T) = -1$.

Consider a sequence (τ_n) distributed by GW^μ_n. If μ is in the domain of attraction of a stable law with index $\alpha \in (1,2)$, then

$$\left(\frac{1}{b_n} W_{\lfloor nt \rfloor}(\tau_n); 0 \leq t \leq 1\right) \xrightarrow[n\to\infty]{(d)} X^{exc,\alpha}, \tag{4.1}$$

where $X^{exc,\alpha}$ is an excursion of an α-stable spectrally positive Lévy process (see [**13, 14**]). Background information on Lévy process can be found in [**5**] and for

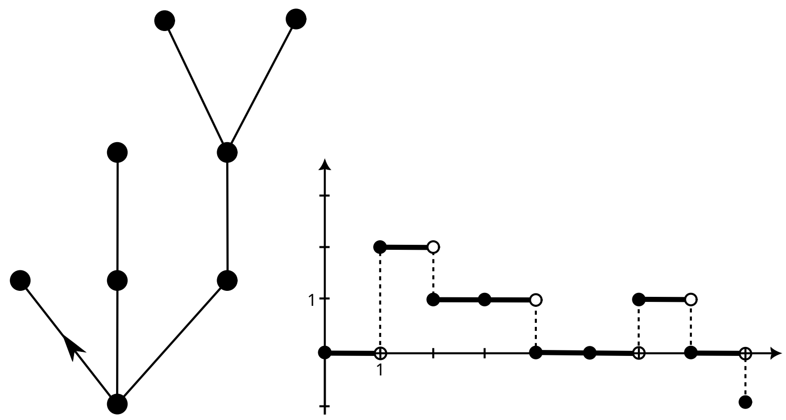


Figure 7. A tree encoded by a Łukasiewicz path.

more details, see [12]. This process lives on the Skorokhod space $D([0,1],\mathbb{R})$ of real valued càdlàg functions.

4.2. Discrete looptrees. — A discrete looptree is constructed on the vertex set of a plane tree, T as follows: For each vertex v of outdegree $k \geq 1$, add the edges $\{v, v1\}$ and $\{v, vk\}$ (with repetition if $k = 1$), and add the edges $\{vi, v(i+1)\}$, for $i = 1, \ldots, k-1$. See Fig. 1 for an illustration. We denote the associated looptree by $\mathsf{Loop}(T)$. $\mathsf{Loop}(T)$ will be viewed as a metric space equipped with the graph distance as its metric (each edge with length one).

4.3. The Gromov-Hausdorff topology. — Let (E_1, d_1),(E_2, d_2) be two compact metric spaces. The Gromov-Hausdorff distance between E_1 and E_2 is

$$\mathrm{d_{GH}}(E_1, E_2) = \inf_{\substack{(E,d)\text{ metric space}\\ \varphi_1: E_1 \to E\\ \varphi_2: E_2 \to E}} \mathrm{d}^E_{\mathrm{Hau}}(\varphi_1(E_1), \varphi_2(E_2))$$

where φ_1 and φ_2 are isometries. Recall that the Hausdorff distance between two compact subsets, K and K' of the metric space E is

$$\mathrm{d}^E_{\mathrm{Hau}}(K, K') = \inf\{\varepsilon > 0 \ : \ K \subseteq K'_\varepsilon \text{ and } K' \subseteq K_\varepsilon\}$$

where K_ε denotes the ε-neighborhood of K. An equivalent definition of the Gromov-Hausdorff distance involves correspondences. A correspondence between two metric spaces, (E_1, d_1) and (E_2, d_2) is a subset $\mathcal{R}$ of $E_1 \times E_2$ such that for every $x_1 \in E_1$, there exists at least one point $x_2 \in E_2$ such that $(x_1, x_2) \in \mathcal{R}$ and conversely for every $y_2 \in E_2$, there exists at least one point $y_1 \in E_1$ such that $(y_1, y_2) \in \mathcal{R}$. The distortion of the correspondence $\mathcal{R}$ is defined by

$$\mathrm{dis}(\mathcal{R}) = \sup\{|d_1(x_1, y_1) - d_2(x_2, y_2)| : (x_1, x_2), (y_1, y_2) \in \mathcal{R}\}.$$

The Gromov-Hausdorff distance can be written in term of correspondences by the formula,

$$\mathrm{d_{GH}}(E_1, E_2) = \frac{1}{2}\inf\{\mathrm{dis}(\mathcal{R})\}$$

where the infimum is over all correspondences $\mathcal{R}$ between E_1 and E_2. We refer the reader to [9] for details about this metric.

Let (X, d) be a compact metric and denote the diameter of X by $\mathrm{diam}_d(X)$. We will often suppress d from the notation if it is the intrinsic metric or graph metric on X. If G is a graph we denote the vertex set of G by V_G (or sometimes by G) and the graph metric on V_G by d_G. The following result will be used repeatedly and its proof is included for completeness.

Lemma 4.1. — *Let G be a graph and H and $\bar{H}$ be subgraphs of G such that $G = H \cup \bar{H}$ and $H \cap \bar{H} = e$ is a single edge $e = \{v_1, v_2\}$ (with the possibility that $v_1 = v_2$). Then*

$$d_{\mathrm{GH}}((V_G, d_G), (V_H, d_H)) \leq \mathrm{diam}_{d_{\bar{H}}}(V_{\bar{H}}).$$

Proof. — The identity map on V_G maps (V_H, d_H) isometrically to (V_G, d_G) since $H \cap \bar{H}$ is a single edge. Therefore

$$\begin{aligned} d_{\mathrm{GH}}((V_G, d_G), (V_H, d_H)) &\leq d^G_{\mathrm{Hau}}(V_G, V_H) = \inf\{\varepsilon \geq 0 \ : \ V_G \subseteq (V_H)_\varepsilon\} \\ &= \max_{v \in \{v_1, v_2\}} \max_{w \in V_{\bar{H}}} d_{\bar{H}}(v, w) \leq \mathrm{diam}_{\bar{H}}(V_{\bar{H}}). \end{aligned}$$

□

4.4. Proof of Theorem 1.1. — We rely on the Markov-chain argument established in [11]. This argument involves introducing many new concepts and instead of repeating all their definitions here, we aim to point out how our case is different and in which way that modifies the result.

Let $\mathcal{T}_n^w = \phi(\mathcal{H}_n^{w,\star})$ be the weak dual of $\mathcal{H}_n^{w,\star}$ and choose μ and $\mu_\varnothing$ as in (1.2) such that they are probability measures satisfying the assumptions of Theorem 1.1. By Theorem 3.1, the shape $\mathcal{S}(\mathcal{T}_n^w)$ is distributed as $\mathrm{GW}_n^{\mu,\mu_\varnothing}$. Recall that $\mathcal{T}_n^{w,\bullet}$ denotes the augmented tree obtained by attaching leaves by an edge to each corner of $\mathcal{T}_n^w$. In [11], the setup is similar, they consider a random face weighted dissection $\mathcal{D}_n^\mu$, (not necessarily non-crossing), whose augmented weak dual $\hat{\mathcal{T}}_n^{\mu,\bullet} := \phi(\mathcal{D}_n^\mu)^\bullet$ is distributed as a planted GW^μ tree conditioned on having n leaves. The difference to our case is thus only that we consider an augmented, modified $(\mu, \mu_\varnothing)$-Galton-Watson tree instead of a planted normal

μ-Galton-Watson tree and we condition on the number of vertices instead of the number of leaves.

Kortchemski and Marzouk [22, Theorem 21 and Remark 26] showed that

$$n^{-1/2} \cdot \mathcal{T}_n^w \xrightarrow[n\to\infty]{(d)} c_{\text{tree}}(\mu) \cdot \mathcal{T}_{\mathbf{e}}$$

with $c_{\text{tree}}(\mu) := 2\sigma_\mu^{-1}$. This is the same as Aldous' well-known result but for the modified Galton-Watson tree $\mathrm{GW}_n^{\mu,\mu_\varnothing}$. Since

$$d_{\mathrm{GH}}(\mathcal{T}_n^w, \mathcal{T}_n^{w,\bullet}) \leq 1$$

one obtains similarly that

$$n^{-1/2} \cdot \mathcal{T}_n^{w,\bullet} \xrightarrow[n\to\infty]{(d)} c_{\text{tree}}(\mu) \cdot \mathcal{T}_{\mathbf{e}}. \tag{4.2}$$

We note that the constant $c_{\text{tree}}(\mu)$ is different from the one in [11] which is only due to the fact that they condition on the number of leaves.

The key argument in [11] involves showing how geodesics in a dissection $\mathcal{D}$ may be found from geodesics in the corresponding augmented weak dual tree $\hat{\mathcal{T}}^\bullet = \phi(\mathcal{D})^\bullet$ by an iterative local construction. This comparison of geodesics works deterministically but is then applied to the random objects. The first comparison is made on the local limits of the two objects so we start by describing the local limit in the current case.

First, define the size-biased distributions

$$\mu^*(k) = k\mu(k), \qquad k \geq 1$$

and

$$\mu_\varnothing^*(k) = \frac{k\mu_\varnothing(k)}{\sum_{j\geq 1} j\mu_\varnothing(j)}, \qquad k \geq 1.$$

Then define an infinite tree $\mathcal{T}_\infty^w$ as follows. Let $(W_1, W_2, \ldots)$ be an infinite path, often referred to as a *spine*. The vertex W_1 has offspring according to $\mu_\varnothing^*$, one of which is chosen uniformly at random to be W_2. Each W_j, $j \geq 2$ has independently offspring according to μ^*, one of which is chosen uniformly at random to be W_{j+1}. From each of the offspring, excluding $W_1, W_2, \ldots$, we grow independent GW^μ trees, see Fig. 8.

The augmented infinite tree $\mathcal{T}_\infty^{w,\bullet}$ is obtained by first sampling $\mathcal{T}_\infty^w$ and then independently for each vertex different from W_1, a corner is sampled uniformly and a new leaf attached to it by an edge. A leaf, denoted by W_0, is also attached to W_1 at the corner which is counter-clockwise from the root-edge. This last step is what is referred to as *planting* in [11].

Let $k \geq 0$ be an integer and denote by $[T]_k$ the subtree of T consisting of its first k generations. By a minor adjustment of standard arguments concerning

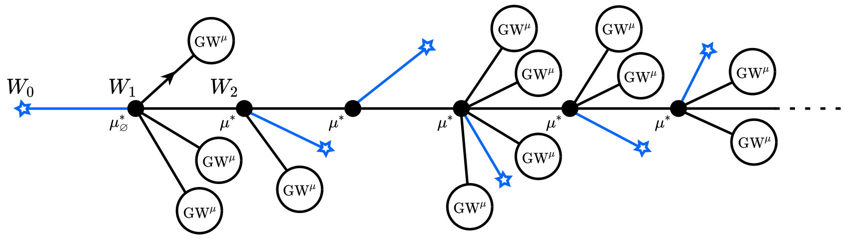


Figure 8. The tree $\mathcal{T}^w_\infty$. Its augmentation $\mathcal{T}^{w,\bullet}_\infty$ is depicted by blue edges and star-shaped vertices.

local limits (see e.g. [18, Theorem 7.1]), it holds that

$$[\mathcal{T}^{w,\bullet}_n]_k \xrightarrow[n\to\infty]{(d)} [\mathcal{T}^{w,\bullet}_\infty]_k.$$

There is a corresponding infinite dual dissection $\mathcal{H}^{w,\star}_\infty$, and we denote the edges in that dissection which are dual to (W_i, W_{i+1}) by $e_i = (e^L_i, e^R_i)$, i ≥ 0 where e^L_i (resp. e^R_i) denotes the vertex to the left (resp. right) of the spine.

Consider the random sequence $(S_n)_{n\geq 0}$ defined by

$$S_n = \min\{d^\star(e^L_0, e^L_n), d^\star(e^L_0, e^R_n)\}$$

where $d^\star$ denotes the graph distance in $\mathcal{H}^{w,\star}_\infty$. The process $(S_n)_{n\geq 0}$ provides information on distances between points in the dissection, as a function of the number of steps n taken along the spine in the tree. In order to find the value S_n, a geodesic whose length is S_n is constructed step by step. This geodesic has to go through either e^L_i or e^R_i for each $i \geq 0$. Assuming we have constructed the first i steps, in order to construct the next one we need to know whether we went through e^L_i or e^R_i or whether it is still undecided. The information on this position is stored in a variable P_i (position) which takes one of the three values: L (left), R (right) or U (undecided). We refer to [11, Section 2.2] for the precise description.

Let $X_0 = 0$, $X_n = S_n - S_{n-1}$, $n \geq 1$ denote the increments of S_n. It is shown in [11] that $(X_n, P_n)_{n\geq 0}$ is a Markov-chain whose transition probabilities only depend on the value of P_n. The sequence $(S_n, P_n)_{n\geq 0}$ is called a Markov additive process with driving chain $(P_n)_{n\geq}$. Step i of the chain can be found by comparing the number of children of W_i to the left and to the right of the spine, including the additional leaf added in the augmentation. Denote the number of children of W_{n+1} in $\mathcal{T}^{w,\bullet}_\infty$ which lie to the left (resp. right) of the spine by by G_n (resp. D_n), and the total number of its children by $C_{n+1} = G_n + D_n + 1$.

By comparing with the transition probabilities in [11, pages 11-12] we find that

$$\begin{aligned}
\mathbb{P}(X_{n+1} = i, P_{n+1} = R \mid P_n = R) &= \mathbb{P}(G_n \geq i, D_n = i) \\
\mathbb{P}(X_{n+1} = i, P_{n+1} = L \mid P_n = R) &= \mathbb{P}(G_n \geq i+1, D_n = i-1) \\
\mathbb{P}(X_{n+1} = i, P_{n+1} = U \mid P_n = R) &= \mathbb{P}(G_n = i, D_n = i-1) \\
\mathbb{P}(X_{n+1} = i, P_{n+1} = R \mid P_n = U) &= \mathbb{P}(G_n \geq i+1, D_n = i) \\
\mathbb{P}(X_{n+1} = i, P_{n+1} = U \mid P_n = U) &= \mathbb{P}(G_n = i, D_n = i).
\end{aligned}$$

By symmetry between left and right, the first four lines also give the transition probabilities when L and R switch roles.

Recall that $C_{n+1} = G_n + D_n + 1$, and that for $n \geq 1$

$$\mathbb{P}(C_{n+1} - 1 = j) = \mu^*(j) = j\mu(j)$$

and by symmetry between left and right

$$\mathbb{P}(G_n = i \mid C_{n+1} = j) = \mathbb{P}(D_n = i \mid C_{n+1} = j) = \frac{1}{j}\mathbb{1}\{0 \leq i \leq j-1\}.$$

From this we can find formulas for the transition probabilities in terms of μ. For easier notation, define $\gamma(j) = j\mu(j)/(j+1)$ and for any $A \subseteq \mathbb{Z}_+ = \{0, 1, \ldots\}$, and any $i \in \mathbb{Z}_+$ let

$$\gamma(A) = \sum_{j \in A} \gamma(j) \qquad \text{and} \qquad \bar{\gamma}(i) = \sum_{j \geq i} \gamma(j).$$

Then we find for example that

$$\begin{aligned}
\mathbb{P}(G_n \geq i, D_n = i) &= \sum_{j \geq 1} \mathbb{P}(G_n \geq i, D_n = i \mid C_{n+1} - 1 = j)\mathbb{P}(C_{n+1} - 1 = j) \\
&= \sum_{j \geq 2i} \mathbb{P}(D_n = i \mid C_{n+1} = j+1)\mu^*(j) \\
&= \sum_{j \geq 2i} \frac{j}{j+1}\mu(j) = \bar{\gamma}(2i)
\end{aligned}$$

and with similar calculations we get

$$\begin{aligned}
\mathbb{P}(G_n \geq i+1, D_n = i-1) &= \bar{\gamma}(2i)\mathbb{1}\{i \geq 1\}, \\
\mathbb{P}(G_n = i, D_n = i-1) &= \gamma(2i-1)\mathbb{1}\{i \geq 1\}, \\
\mathbb{P}(G_n \geq i+1, D_n = i) &= \bar{\gamma}(2i+1), \\
\mathbb{P}(G_n = i, D_n = i) &= \gamma(2i).
\end{aligned}$$

Note that these formulas only hold for $n \geq 1$ since the degree distribution of W_1 is different from the others. However, the first step X_1 does not affect the asymptotic behaviour of S_n and therefore we leave out the details.

Because of symmetry between left and right we proceed as in [11] and say that a state P_n is determined, denoted by $P_n = D$ if and only if $P_n \in \{L, R\}$. Then

$$\begin{aligned}
\mathbb{P}(X_{n+1} = i, P_{n+1} = D \mid P_n = D) &= \bar{\gamma}(2i) + \bar{\gamma}(2i)\mathbb{1}\{i \geq 1\}\\
\mathbb{P}(X_{n+1} = i, P_{n+1} = U \mid P_n = D) &= \gamma(2i-1)\mathbb{1}\{i \geq 1\}\\
\mathbb{P}(X_{n+1} = i, P_{n+1} = D \mid P_n = U) &= 2\bar{\gamma}(2i+1)\\
\mathbb{P}(X_{n+1} = i, P_{n+1} = U \mid P_n = U) &= \gamma(2i).
\end{aligned}$$

By summing over i in the above formulas we find the transition probabilites of the driving chain $(P_n)_{n\geq 0}$ to be

$$\begin{aligned}
\mathbb{P}(P_{n+1} = U \mid P_n = D) &= \gamma(2\mathbb{Z}_+ + 1) = 1 - \mathbb{P}(P_{n+1} = D \mid P_n = D)\\
\mathbb{P}(P_{n+1} = U \mid P_n = U) &= \gamma(2\mathbb{Z}_+) = 1 - \mathbb{P}(P_{n+1} = D \mid P_n = U).
\end{aligned}$$

This allows us to find the stationary state of the driving chain

$$\pi(D) = \frac{1-\gamma(2\mathbb{Z}_+)}{1-\gamma(2\mathbb{Z}_+) + \gamma(2\mathbb{Z}_+ + 1)}, \qquad \pi(U) = \frac{\gamma(2\mathbb{Z}_+ + 1)}{1-\gamma(2\mathbb{Z}_+) + \gamma(2\mathbb{Z}_+ + 1)}$$

as long as $\gamma(2\mathbb{Z}_+ + 1) > 0$. The constant $c_{\text{geo}}(\mu)$ appearing in Theorem 1.1 may now be calculated by finding the expected value of the step distribution X_n, for $n \geq 2$, when the driving chain is in its stationary state. Using this, along with (4.2), allows us to complete the proof. However, we require a few additional technical details which are the same, mutatis mutandis, as in [11, pages 12-18] so we leave out the details. We therefore conclude by calculating

$$\begin{aligned}
c_{\text{geo}}(\mu) = \mathbb{E}_\pi(X_2) &= \sum_{i\geq 0} i\mathbb{P}(X_2 = i \mid P_1 = D)\pi(D) + i\mathbb{P}(X_2 = i \mid P_1 = U)\pi(U)\\
&= \frac{1}{1-\gamma(2\mathbb{Z}_+) + \gamma(2\mathbb{Z}_+ + 1)}\Bigg(\Big(\sum_{i\geq 1} i\bar{\gamma}(2i) + \sum_{i\geq 1} i\bar{\gamma}(2i-1)\Big)(1-\gamma(2\mathbb{Z}_+))\\
&+ \Big(\sum_{i\geq 0} i\bar{\gamma}(2i+1) + \sum_{i\geq 0} i\bar{\gamma}(2i)\Big)\gamma(2\mathbb{Z}_+ + 1)\Bigg).
\end{aligned}$$

By swapping the sum over i and the sum appearing in $\bar{\gamma}$ in each term we find that

$$\begin{aligned}\sum_{i\geq 1} i\bar{\gamma}(2i) + \sum_{i\geq 1} i\bar{\gamma}(2i-1) &= \frac{1}{2}\sum_{j\geq 1}\gamma(j)\left(\left\lfloor\frac{j}{2}\right\rfloor\left(\left\lfloor\frac{j}{2}\right\rfloor+1\right)+\left\lfloor\frac{j+1}{2}\right\rfloor\left(\left\lfloor\frac{j+1}{2}\right\rfloor+1\right)\right)\\ &= \frac{1}{2}\sum_{j\geq 1}\gamma(j)\left(\frac{1}{2}j(j+1)+\frac{1}{2}\mathbb{1}\{j \text{ odd}\}\right)\\ &= \frac{1}{4}\sum_{j\geq 1} j^2\mu(j) + \frac{1}{4}\gamma(2\mathbb{Z}_+ + 1)\\ &= \frac{\sigma_\mu^2 + 1 + \gamma(2\mathbb{Z}_+ + 1)}{4}\end{aligned}$$

and similarly

$$\left(\sum_{i\geq 0} i\bar{\gamma}(2i+1) + \sum_{i\geq 0} i\bar{\gamma}(2i)\right) = \frac{\sigma_\mu^2 + 1 - \gamma(\mathbb{Z}_+) - \gamma(2\mathbb{Z}_+ + 1)}{4}$$

which finally yields

$$c_{\text{geo}}(\mu) = \frac{1}{4}\left(\sigma_\mu^2 + 1 + \frac{\gamma(2\mathbb{Z}_+ + 1)(1 - 2\gamma(\mathbb{Z}))}{1 - \gamma(2\mathbb{Z}_+) + \gamma(2\mathbb{Z}_+ + 1)}\right).$$

□

4.5. Proof of Theorem 1.2. — The following invariance theorem gives a sufficient condition on a sequence of random plane trees $(\tau_n)_{n\geq 1}$ that ensures that the associated looptrees, $(\mathsf{Loop}(\tau_n))_{n\geq 1}$ appropriately rescaled, converge towards the α-stable looptree $\mathcal{L}_\alpha$. Informally, one can view the α-stable looptree $\mathcal{L}_\alpha$ as "$\mathsf{Loop}(\mathcal{T}_\alpha)$", where $\mathcal{T}_\alpha$ is the α-stable tree mentioned in the introduction. The precise definition requires more work and we refer to [12, Sec. 2] for the details of the construction.

THEOREM 4.2 (CURIEN & KORTCHEMSKI, [12]). — *Let $(\tau_n)_{n\geq 1}$ be a sequence of random trees such that there exists a sequence $(b_n)_{n\geq 0}$ of positive real numbers satisfying*

(i) $\left(\frac{1}{b_n} W_{\lfloor|\tau_n|t\rfloor}(\tau_n); 0 \leq t \leq 1\right) \xrightarrow[n\to\infty]{(d)} X^{exc,\alpha}$,

(ii) $\frac{1}{b_n}\mathsf{Height}(\tau_n) \xrightarrow[n\to\infty]{(\mathbb{P})} 0$

where the first convergence holds in distribution for the Skorokhod topology on $D([0,1],\mathbb{R})$ and the second convergence holds in probability. Then the convergence

$$\frac{1}{b_n}\cdot \mathsf{Loop}(\tau_n) \xrightarrow[n\to\infty]{(d)} \mathcal{L}_\alpha$$

holds in distribution for the Gromov-Hausdorff topology.

Let μ and $\mu_\varnothing$ be given by (1.2) and chosen such that they are probability measures satisfying the assumptions of Theorem 1.2. First of all, we apply this invariance principle to $\mathcal{T}_n^w = \mathcal{S}(\phi_n(\mathcal{H}_n^{w,\star}))$, the shape of the weak dual of $\mathcal{H}_n^{w,\star}$ which by Proposition 3.1 is distributed as $\mathrm{GW}_n^{\mu,\mu_\varnothing}$. Condition (i) is satisfied for $\mathcal{T}_n^w$ by [**22**, Theorem 21] whose proof involves comparing $\mathrm{GW}_n^{\mu,\mu_\varnothing}$ with GW_n^{μ} and using that (i) holds for the latter by standard results [**13**]. Property (ii) holds since the height of $\mathcal{T}_n^w$ is of order n/b_n and that b_n is regularly varying with index $\alpha < 2$. The former statement follows again from comparing $\mathrm{GW}_n^{\mu,\mu_\varnothing}$ with GW_n^{μ}, see the discussion in [**22**, Remark 26]. We therefore have

$$\frac{1}{b_n} \cdot \mathsf{Loop}(\mathcal{T}_n^w) \xrightarrow[n\to\infty]{(d)} \mathcal{L}_\alpha.$$

Finally, we can bound the Gromov–Hausdorff distance between $\mathsf{Loop}(\mathcal{T}_n^w)$ and $\mathcal{H}_n^{w,\star}$. By [**12**, Equation (4.10)] it holds deterministically (by a short argument) that the Gromov–Hausdorff distance between a dissection and the looptree of its weak dual tree is roughly bounded by its height. More precisely we have

$$\frac{1}{b_n} d_{\mathrm{GH}}(\mathcal{H}_n^{w,\star}, \mathsf{Loop}(\mathcal{T}_n^w)) \leq \frac{1}{b_n} \left(\mathsf{Height}(\mathcal{T}_n^w) + 2\right).$$

The expression on the right hand side converges to 0 in probability since $\mathcal{T}_n^w$ satisfies (ii) in Theorem 4.2, which concludes the proof.

□

4.6. Proof of Theorem 1.3. — Recall that the random NCO can in this case be sampled as described in Lemma 3.3. An underlying tree is first sampled according to branching weights η given by $\eta(j) = Z_j^{\mu,seq}$. Each vertex of the underlying tree is then replaced by an i.i.d. copy of a finite sequence of w-face weighted dissections of the appropriate size. If the conditions in Theorem 1.3 are satisfied then, by Lemma 3.4, $\nu_\eta > 1$ which means that we are in case I) in (3.9). This means that the branching weights η of the underlying tree have a finite exponential moment, a condition which is referred to as Ia) in [**31**]. In addition, some very mild conditions on the decorations [**31**, Conditions 1,2 and 3, page 83] are required which is easy to verify in the current case. We may then apply [**31**, Theorem 6.60] which completes the proof.

□

4.7. Proof of Theorem 1.4. — As in the previous subsection we rely on the description of the NCO as a decorated tree. In order to prove the theorem we apply the recent general invariance principle for decorated stable trees given in [**28**, Theorem 5.1]. In order to apply this result we need to show that Conditions D1, D2, T1, T2 and B1 in [**28**, Section 3.1] are satisfied (the other conditions stated there are only needed when measures are also considered). Since their statements are lengthy and require the introduction of more notation we will let it suffice to refer to them here. In the current case it trivially holds that

$$\operatorname{diam}(\mathcal{SD}_n^w) \leq n/2. \tag{4.3}$$

Denote the circle of unit circumference by C^1. Then by definition

$$\operatorname{diam}(C^1) = 1/2. \tag{4.4}$$

It will be sufficient to prove the two lemmas

Lemma 4.3. — *The weights η of the underlying tree are equivalent, in the sense of* (3.4)*, to a probability measure $\hat{\eta}$ which is critical and belongs to the domain of attraction of an α-stable law, $\alpha \in (1,2)$. In particular,*

$$\hat{\eta}(j) \sim \frac{g^\mu(r)B_\mu}{1-A_\mu} L(j)((1-\nu_\mu)j)^{-\alpha-1}$$

where L is the same slowly varying function as in (3.12)*.*

Lemma 4.4. — *The following convergence holds in distribution in the Gromov-Hausdorff sense*

$$n^{-1} \cdot \mathcal{SD}_n^w \to (1-\nu_\mu) \cdot C^1.$$

Lemma 4.3 implies that T1 and T2 hold and along with (4.3) it further implies that condition B1 holds (see [**28**, Proposition 3.2 with $\gamma = 1$ and Section 3.3]). Lemma 4.4 implies that D1 holds and along with (4.4) it further implies that D2 holds.

These Lemmas give a rough idea for the validity of Theorem 1.4. By Lemma 4.3, it holds that

$$\hat{\eta}(j,\infty) \sim \frac{g^\mu(r)B_\mu}{\alpha(1-A_\mu)} L(j)((1-\nu_\mu)j)^{-\alpha-1}$$

and therefore $\hat{\eta}$ belongs to the domain of attraction of a stable law with index α. Thus, the finite underlying tree will converge towards an α-stable tree once rescaled by

$$\begin{aligned} C_n &= (\alpha|\Gamma(-\alpha)|)^{1/\alpha} \inf\{x \geq 0 \ : \ \hat{\eta}(x,\infty) \leq 1/n\} \\ &\sim \left(\frac{|\Gamma(-\alpha)| g^\mu(r) B_\mu}{(1-\nu_\mu)(1-A_\mu)}\right)^{1/\alpha} (1-\nu_\mu)^{-1}(L(n)n)^{1/\alpha}, \end{aligned}$$

see e.g. [**29**, Proof of Lemma 5.1] for similar calculations. By Lemma 4.4, a typical large decoration converges towards a circle of circumference $1-\nu_\mu$. Hence, up to some technical details which are taken care of in the other conditions, the decorated tree converges towards a multiple $(1-\nu_\mu)$ of an α-stable looptree. Combining these results gives the correct scaling of distances in the decorated trees in the theorem

$$c_n = (1 - \nu_\mu)C_n.$$

Proof of Lemma 4.3. — Define the probability measure

$$\hat{\eta}(j) = \frac{Z_j^{\mu,seq} R_\mu^j}{\mathcal{Z}^{\mu,seq}(R_\mu)}$$

which is a tilt of $\eta(j) = Z_j^{\mu,seq}$. The conditions of the Theorem state that μ satisfies $\nu_\mu < 1$ and

$$\mu(k) = L(k)k^{-\alpha-1}r^{-k}$$

with $\alpha \in (1,2)$, $r = \rho_\mu$ and L slowly varying, and that

$$\frac{A_\mu(1-\nu_\mu) + \rho_\mu B_\mu}{(1-A_\mu)(1-\nu_\mu)} = 1.$$

In this case it holds that

$$\mathcal{Z}^\mu(R_\mu) = r = \rho_\mu < 1 \qquad \text{and} \qquad \mathcal{Z}_\varnothing^\mu(R_\mu) = r\frac{g_\varnothing^\mu(r)}{g^\mu(r)} = A_\mu < 1. \tag{4.5}$$

In the following, we will use the notation $[z^j]\left\{\sum_{j=0}^\infty a_n z^n\right\} = a_j$. Recall the expression (3.16) for $Z_n^\mu = [z^n]\{\mathcal{Z}^\mu(z)\}$. By (3.11) we have

$$\begin{aligned}[z^j]\left\{\mathcal{Z}_\varnothing^\mu(z)\right\} &= [z^{j-1}]\left\{g_\varnothing^\mu(\mathcal{Z}^\mu(z))\right\} \\ &\sim (g_\varnothing^\mu)'(\rho_\mu)[z^{j-1}]\left\{\mathcal{Z}^\mu(z)\right\} = R_\mu^{-1}B_\mu[z^{j-1}]\left\{\mathcal{Z}^\mu(z)\right\}\end{aligned}$$

where we used [16, Theorem 4.30] in the second last step. In the same way, we find that

$$\begin{aligned}\eta(j) = Z_j^{\mu,seq} = [z^j]\{\mathcal{Z}^{\mu,seq}(z)\} &= [z^j]\left\{\frac{1}{1-\mathcal{Z}_\varnothing^\mu(z)}\right\} \\ &\sim \frac{1}{(1-A_\mu)^2}[z^j]\{\mathcal{Z}_\varnothing^\mu(z)\} \\ &\sim \frac{B_\mu}{R_\mu(1-A_\mu)^2}[z^{j-1}]\left\{\mathcal{Z}^\mu(z)\right\} \\ &\sim \frac{rB_\mu}{(1-A_\mu)^2}L(j)R_\mu^{-j-1}((1-\nu_\mu)j)^{-\alpha-1} \end{aligned} \tag{4.6}$$

where in the last step we used $[z^{j-1}]\mathcal{Z}^\mu(z) = Z_{j-1}^\mu$ and (3.16). Then

$$\hat{\eta}(j) \sim \frac{rB_\mu}{R_\mu(1-A_\mu)}L(j)((1-\nu_\mu)j)^{-\alpha-1} = \frac{g^\mu(r)B_\mu}{(1-A_\mu)}L(j)((1-\nu_\mu)j)^{-\alpha-1}$$

which shows that $\hat{\eta}$ is in the domain of attraction of a stable law with index α. Finally, we calculate using (3.21),

$$\sum_{j=0}^{\infty} j\hat{\eta}(j) = \frac{R_\mu(\mathcal{Z}^{\mu,seq})'(R_\mu)}{\mathcal{Z}^{\mu,seq}(R_\mu)} = \frac{R_\mu}{1-A_\mu}\left(g^\mu_\varnothing(r) + \frac{r(g^\mu_\varnothing)'(r)}{1-\nu_\mu}\right)$$
$$= \frac{A_\mu(1-\nu_\mu) + rB_\mu}{(1-A_\mu)(1-\nu_\mu)} = 1$$

which shows that $\hat{\eta}$ is critical. □

Proof of Lemma 4.4. — Since $\nu_\mu < 1$, the decorations, consisting of a finite sequence of NCDs, are in a so-called condensation phase, see e.g. [**18, 19, 21**]. This means that exactly one face, call it $f_{\max}$, will have a size close to $(1-\nu_\mu)n$ where n is the total size of the NCDs in the sequence, and all other faces are uniformly small. The geometry of a large NCD may then be shown to be close to that of a circle, see Fig. 9. We now make this more precise.

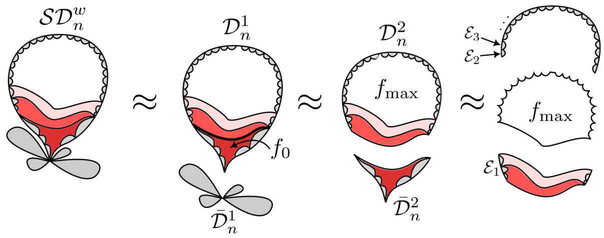


Figure 9. The figure demonstrates how one may show in three steps that $\mathcal{SD}^w_n$ is close to the circle graph $f_{\max}$ of size approximately $(1-\nu_\mu)n$, in the Gromov-Hausdorff sense. The shaded regions represent small dissections.

Denote the largest element of the finite sequence $\mathcal{SD}^w_n$ by $\mathcal{D}^1_n$ (in case of a tie, let it be the first largest one) and let $\bar{\mathcal{D}}^1_n$ denote the remaining part. Denote the root face of $\mathcal{D}^1_n$ by f_0. One may decompose $\mathcal{D}^1_n$ into the face f_0 and a number of sub-dissections which have an edge in common with f_0. Denote the largest such sub-dissection by $\mathcal{D}^2_n$ (in case of a tie, choose e.g. the first one in a clockwise order) and denote the remaining part by $\bar{\mathcal{D}}^2_n$. Denote the face in $\mathcal{D}^2_n$ of largest degree by $f_{\max}$. By abuse of notation we also let $f_{\max}$ denote the circle graph consisting of the edges on the boundary of $f_{\max}$. We will show that $\mathcal{SD}^w_n$ is close to $f_{\max}$ by going through the two intermediate dissections $\mathcal{D}^1_n$ and $\mathcal{D}^2_n$, see Fig. 9. Since $\mathsf{deg}(f_{\max})$ is with high probability of size close to

$(1-\nu_\mu)n$ this will conclude the proof. More precisely we will show that

$$n^{-1}\cdot d_{\mathrm{GH}}(\mathcal{SD}^w_n,\mathcal{D}^1_n)\xrightarrow[n\to\infty]{(\mathbb{P})}0, \tag{4.7}$$

$$n^{-1}\cdot d_{\mathrm{GH}}(\mathcal{D}^1_n,\mathcal{D}^2_n)\xrightarrow[n\to\infty]{(\mathbb{P})}0, \tag{4.8}$$

$$n^{-1}\cdot d_{\mathrm{GH}}(\mathcal{D}^2_n,f_{\max})\xrightarrow[n\to\infty]{(\mathbb{P})}0 \qquad \text{and} \tag{4.9}$$

$$d_{\mathrm{GH}}(n^{-1}\cdot f_{\max},(1-\nu_\mu)\cdot C^1)\xrightarrow[n\to\infty]{(\mathbb{P})}0 \tag{4.10}$$

and the result then follows from the triangle inequality.

We start by proving (4.7). By Lemma 4.1

$$d_{\mathrm{GH}}(\mathcal{SD}^w_n,\mathcal{D}^1_n)\le \mathrm{diam}(\bar{\mathcal{D}}^1_n)\le |\bar{\mathcal{D}}^1_n|, \tag{4.11}$$

where $|G|$ denotes the number of vertices in the graph G. Since $\mathcal{Z}^\mu_\varnothing(R_\mu)<1$, it follows from (3.18) and a general result on Gibbs-partitions with a subcritical composition scheme [**30**, Theorem 3.1] that $|\bar{\mathcal{D}}^1_n|$ converges in total variation towards a finite random variable which by (4.11) concludes the proof of (4.7). It also follows that $\frac{|\mathcal{D}^1_n|}{n}\to 1$ in probability.

By Lemma 4.1

$$d_{\mathrm{GH}}(\mathcal{D}^1_n,\mathcal{D}^2_n)\le \mathrm{diam}(\bar{\mathcal{D}}^2_n)\le |\bar{\mathcal{D}}^2_n|. \tag{4.12}$$

Let $\hat{\mu}$ denote the probability distribution with mean ν_μ obtained by tilting the weights μ and choose $\hat{\mu}_\varnothing$ by simultaneously tilting $\mu_\varnothing$ according to (1.2). Again, using [**30**, Theorem 3.1] it holds that conditionally on $|\mathcal{D}^1_n|$, $\mathcal{D}^1_n$ is distributed as a w-weighted NCD and thus the shape of its weak dual is distributed as $\mathrm{GW}^{\hat{\mu},\hat{\mu}_\varnothing}_{|\mathcal{D}^1_n|}$. Therefore, since $\frac{|\mathcal{D}^1_n|}{n}\to 1$ in probability, it holds by Lemma 3.2 that $|\bar{\mathcal{D}}^2_n|$ converges towards a finite random variable when $n\to\infty$ which along with (4.12) proves (4.8). It also follows that $\frac{|\mathcal{D}^2_n|}{n}\to 1$ in probability.

Applying Lemma 3.2 again allows us to conclude that conditionally on $|\mathcal{D}^2_n|$, $\mathcal{D}^2_n$ is distributed like a NCD whose weak dual has a shape which is a one type Galton-Watson tree $\mathrm{GW}^{\hat{\mu}}_{|\mathcal{D}^2_n|}$. Let us now decompose $\mathcal{D}^2_n$ into the face $f_{\max}$ and the dissections which share an edge with this face. Denote these dissections by $(\mathcal{E}_i)_{i=1}^{\mathsf{deg}(f_{\max})}$ in clockwise order from the edge which belonged to f_0, see Fig. 9. By a straigtforward generalization of Lemma 4.1

$$\begin{aligned} d_{\mathrm{GH}}(\mathcal{D}^2_n,f_{\max}) &\le \max\{\mathrm{diam}(\mathcal{E}_i)\ :\ 1\le i\le \mathsf{deg}(f_{\max})\}\\ &\le \max\{|\mathcal{E}_i|\ :\ 1\le i\le \mathsf{deg}(f_{\max})\}. \end{aligned} \tag{4.13}$$

It follows from the bijection with trees and the fact that $\nu_\mu<1$ that there is a slowly varying sequence $(\tilde{L}_n)_{n\ge1}$ such that

$$\max\{|\mathcal{E}_i|\ :\ 1\le i\le \mathsf{deg}(f_{\max})\}=O_p(\tilde{L}_n n^{1/\alpha}), \tag{4.14}$$

and

$$\mathsf{deg}(f_{\max}) = (1-\nu_\mu)n + O_p(\tilde{L}_n n^{1/\alpha}) \tag{4.15}$$

see [21, Theorem 1 and Corollary 1], where we have used $|\mathcal{D}_n^2|/n \to 1$ in probability to replace $|\mathcal{D}_n^2|$ by n. The estimates (4.13) and (4.15) prove (4.9).

Finally we prove (4.10). We construct a correspondence between $f_{\max}$ and C^1 by dividing the latter into $\mathsf{deg}(f_{\max})$ number of arcs of equal length and letting subsequent points of $f_{\max}$ correspond to subsequent arcs of C^1. By bounding its distortion we find that

$$d_{\mathrm{GH}}(n^{-1}\cdot f_{\max}, (1-\nu_\mu)\cdot C^1) \le \left|\frac{\mathsf{deg}(f_{\max})}{n} - (1-\nu_\mu)\right| + O(\mathsf{deg}(f_{\max})^{-1}) \to 0$$

in probability as $n \to \infty$ by (4.15) which proves (4.10).

□

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Paper II

THE SCALING LIMIT OF RANDOM 2-CONNECTED SERIES-PARALLEL MAPS

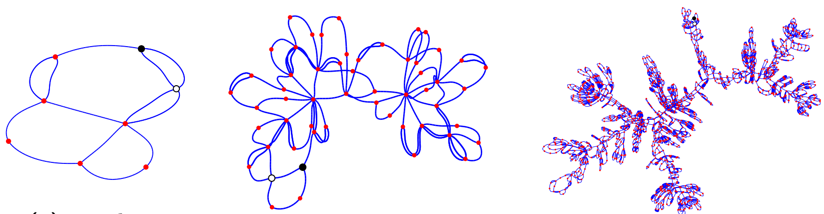
by

Daniel Amankwah, Jakob Björnberg, Sigurdur Örn Stefánsson, Benedikt Stufler & Joonas Turunen

Abstract. — A finite graph embedded in the plane is called a series-parallel map if it can be obtained from a finite tree by repeatedly subdividing and doubling edges. We study the scaling limit of weighted random two-connected series-parallel maps with n edges and show that under some integrability conditions on these weights, the maps with distances rescaled by a factor $n^{-1/2}$ converge to a constant multiple of Aldous' continuum random tree (CRT) in the Gromov–Hausdorff sense. The proof relies on a bijection between a set of trees with n leaves and a set of series-parallel maps with n edges, which enables one to compare geodesics in the maps and in the corresponding trees via a Markov chain argument introduced by Curien, Haas and Kortchemski (2015).

1. Introduction

A finite connected graph is called a *series-parallel graph* if it can be obtained from a finite tree by repeatedly subdividing or doubling edges. It may equivalently be characterized as a graph that does not contain the complete graph



(a) 13 edges. **(b)** 102 edges. **(c)** 2352 edges.

Figure 1. A simulation of uniform 2-connected series-parallel maps with different numbers of edges. The endpoints of the root edge are represented by $\bullet$ and $\circ$.

K_4 as a minor. In this work we consider series-parallel *maps*, which are series-parallel graphs embedded in the plane, viewed up to continuous deformations.

Specifically, we consider series-parallel maps which are *two-connected*, and we are interested in their *scaling limits*. Without the condition of two-connectivity, uniformly sampled series-parallel maps have Aldous' continuum random tree (CRT) as their scaling limit, as easily follows from a general theorem of Stufler [**18**, Thm. 6.62]. More preciesely, uniform samples of such maps with n edges, and distances rescaled by $n^{-1/2}$, converge weakly in the Gromov–Hausdorff sense to a multiple of the CRT. The essence of the proof lies in decomposing the map into two-connected components, or *blocks*, and using a coupling with simply generated trees where each block of the map corresponds to a vertex in the tree. As the tree scales at the order $n^{1/2}$, and since blocks are typically small, after rescaling distances by $n^{-1/2}$ on average each block contributes a constant length to the geodesic. Thus, the global shape of the rescaled map is approximated by the tree, stretched by a constant factor.

In this work, however, we would like to understand the structure of the blocks themselves. Thus, we focus our attention on *two-connected* series-parallel maps. These do not enjoy the same type of coupling to simply generated trees and prove more difficult to handle. Our main result is that, nevertheless, uniformly sampled two-connected series-parallel maps with n edges converge, as $n \to \infty$ with distances rescaled by $n^{-1/2}$, to a multiple of the CRT. Figure 1 shows simulations of these uniformly sampled maps and suggests that they become tree-like with an increasing number of edges.

Analogous results have been established before for several tree-like random maps, which can often be characterized through excluded minors. Most fundamental is the original result of Aldous, showing that uniform trees with n edges converge to the CRT when rescaled by $n^{-1/2}$ [**1, 2, 3**]. Trees may be characterized by not containing K_3 as a minor. Other notable cases are various classes of outerplanar graphs. A graph is called *outerplanar* if it may be embedded in the plane so that every vertex lies on the boundary of the external face; equivalently, they are graphs not containing K_4 or the complete bipartite graph $K_{3,2}$ as minors. In [**8**], Caraceni established that uniformly sampled outerplanar maps (properly rescaled) converge to the CRT. Stufler [**18**] generalized this result to a family of face-weighted outerplanar maps, using the aforementioned coupling with simply generated trees. Note that outerplanar maps are series-parallel maps, since series-parallel maps have K_4 as an excluded minor, and outerplanar maps further exclude $K_{3,2}$ as a minor.

If one additionally imposes the condition of two-connectivity on outerplanar maps, one obtains *dissections of a polygon*. Random, face weighted dissections of polygons were studied by Curien, Haas and Kortchemski [9] who showed, under a suitable condition on the weights, that the scaling limit is a constant multiple of the CRT. Their proof involves relating the dissections bijectively to trees and using a clever Markov chain argument to compare lengths of geodesics in the dissections and their corresponding trees.

One of the main aims of the above studies is to understand how universally the CRT appears as a scaling limit for different models of random graphs or maps. Note that planar maps themselves, given by excluding K_5 and $K_{3,3}$ as minors, behave vastly differently. Uniform planar maps with n edges, when rescaled by $n^{-1/4}$, admit the so-called *Brownian sphere* as a scaling limit [4] which is different from the CRT. See Table 1 for a summary of these results.

Map	Excluded minors	Diameter	Scaling limit
Plane trees	K_3	$n^{1/2}$	CRT
Outerplanar maps	K_4, $K_{2,3}$	$n^{1/2}$	CRT
Series-parallel maps	K_4	$n^{1/2}$	CRT
Planar maps	$K_5, K_{3,3}$	$n^{1/4}$	Brownian sphere

Table 1. Scaling limits of uniform maps with n edges characterized by exclusion of minors.

The proof of our main scaling limit result follows the strategy developed by Curien, Haas and Kortchemski for random dissections [9]. We use a bijection with certain plane trees (see Section 2.3 for details), which enables us to describe the distribution of the random maps with n edges in terms of a Bienaymé–Galton–Watson (BGW) tree conditioned on having n leaves. We then adapt the Markov chain argument from [9] for comparing geodesics in the map and in the corresponding tree.

Moreover, the relation with BGW-trees allows us to consider a weighted version of the random maps in a natural way by assigning a general offspring distribution μ to the BGW-trees. We will assume that this offspring distribution is critical and that it has finite exponential moments (is light-tailed), which includes the uniform distribution as a special case. The precise definition of the weighted series-parallel maps is given in Definition 2.2 in Section 2.6.

The version of CRT used in this work is the one constructed from a normalised Brownian excursion $\mathbf{e}$ (see [13]) denoted by $\mathcal{T}_{\mathbf{e}}$. Let SP_n^μ denote the random, weighted, rooted, 2-connected series-parallel map with n non-root edges corresponding to a critical offspring distribution μ of a BGW-process possessing finite exponential moments (see Definition 2.2). The following is our main result:

Theorem 1.1. — *There exists a constant* $c_\mu > 0$ *such that*

$$(\mathsf{SP}_n^\mu, c_\mu n^{-1/2} d_{\mathsf{SP}_n}) \xrightarrow{d} (\mathcal{T}_{\mathrm{e}}, d_{\mathcal{T}_{\mathrm{e}}})$$

in the Gromov–Hausdorff sense.

We also obtain optimal tail-bounds for the diameter $\mathrm{D}(\mathsf{SP}_n^\mu)$:

Theorem 1.2. — *There exist constants* $C, c > 0$ *such that for all* n *and all* $x > 0$

$$\mathbb{P}(\mathrm{D}(\mathsf{SP}_n^\mu) > x) \leq C \exp(-cx^2/n).$$

Remark 1.3. — *We will consider other closely related classes of series-parallel maps referred to as series-parallel networks, their definitions are given in Subsection 2.1. The two previous theorems also apply to random weighted series-parallel networks with the same scaling constant* c_μ.

2. Decomposing and sampling series-parallel maps

2.1. Rooted, two-connected series-parallel maps. — A planar map is a drawing of a finite connected graph in the plane so that edges do not cross except possibly at their endpoints, viewed up to orientation preserving homeomorphisms of the plane. A series-parallel (SP) map is a planar map which does not contain the complete graph K_4 as a minor. The objects of interest in this paper are two-connected SP-maps. Another way to describe them is to start from a double edge and repeatedly subdivide or double edges.

We will assign a direction to one of the edges of the map and refer to it as the *root* of the map. The root will be denoted by $e = (*_0, *_1)$. For concreteness, we draw the maps so that the root edge lies on the boundary of the unbounded face and is directed in a clockwise manner along that face. See Fig. 2 (left) for an example. Denote the set of rooted two-connected SP-maps with $n \geq 1$ non-

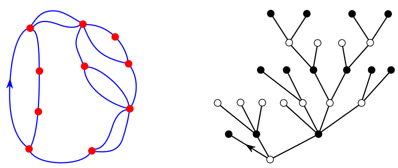


Figure 2. Left: An example of a rooted, two-connected SP-map. The directed root edge is indicated by an arrow. Right: Its corresponding labeled plane tree where $\circ$ denotes the label P and $\bullet$ denotes the label S.

root edges by $\mathcal{SP}_n$ and the set of all such finite maps by $\mathcal{SP} = \cup_{n\geq 1}\mathcal{SP}_n$. We deliberately leave out the map with the single root edge later on for notational convenience .

In order to be able to describe a decomposition of the maps into series and parallel components, we require two additional definitions. Let $M \in \mathcal{SP}_n$ and let M' be the map obtained by removing the root edge from M, keeping the endpoints $*_0, *_1$ as marked vertices of M'. Define for all $n \geq 2$

$$\mathcal{S}_n = \{M' \mid M \in \mathcal{SP}_n, M' \text{ is not two-connected}\},$$
$$\mathcal{P}_n = \{M' \mid M \in \mathcal{SP}_n, M' \text{ is two-connected}\},$$

Also set

$$\mathcal{S} = \cup_{n\geq 2}\mathcal{S}_n, \qquad \mathcal{P} = \cup_{n\geq 2}\mathcal{P}_n, \qquad \text{and} \qquad \mathcal{N} = \mathcal{S}\cup\mathcal{P}\cup\{e\}. \tag{2.1}$$

The operation of removing or adding back the root edge defines a bijection

$$\partial_{\text{Map}} : \mathcal{SP} \to \mathcal{N}, \tag{2.2}$$

see Fig. 3. We refer to the maps in $\mathcal{N}$ as *series-parallel (SP) networks*.

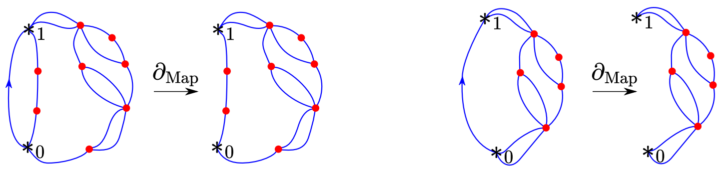


Figure 3. The root edge is removed from a rooted two-connected SP-map. The resulting map is either two-connected (left) or it is not (right).

2.2. Plane trees and labels. — Recall the standard definition of the Ulam–Harris tree: set $\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$, where $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{N}^0 = \{\varnothing\}$ by convention. If $u = u^1, \dots, u^m$ and $v = v^1, \dots, v^n$ are elements of $\mathcal{U}$, we write the concatenation of u and v as $uv = u^1, \dots, u^m, v^1, \dots, v^n$.

Definition 2.1. — *A* plane tree T *is a finite or infinite subset of* $\mathcal{U}$ *such that*

1. $\varnothing \in T$*,*
2. *if* $v \in T$ *and* $v = uj$ *for some* $j \in \mathbb{N}$ *then* $u \in T$*.*
3. *for every* $u \in T$*, there exists an integer* $\text{out}_T(u) \geq 0$ *(the outdegree or number of children of* u*) such that for every* $j \in \mathbb{N}$*,* $uj \in T$ *if and only if* $1 \leq j \leq \text{out}_T(u)$*.*

The element $\varnothing \in \mathcal{U}$ is called the *root vertex* and the pair $\{\varnothing, 1\}$ is called the *root edge*. One may view plane trees as planar maps with a directed root edge which do not contain a cycle. A vertex of outdegree 0 is called a *leaf* and an edge incident to a leaf is called a *leaf edge*. The trivial tree consisting only of the root vertex $\varnothing$, which is then a leaf, will be denoted by $\bullet$.

We denote by $\mathcal{T}$ the set of *finite* plane trees which have no vertices of outdegree 1, and by $\mathcal{T}_n \subseteq \mathcal{T}$ the set of such trees with $n \geq 1$ leaves. We will apply labels S and P to the trees in $\mathcal{T}$ (related to the series and parallel decompositions of maps). We use the notation $\bar{S} = P$ and $\bar{P} = S$ when we need to swap labels. If $(T, \ell) \in \mathcal{T} \times \{S, P\}$ and $T \neq \bullet$ we say that T is labeled by ℓ and we assign the label ℓ to vertices in every even generation and $\bar{\ell}$ to vertices in every odd generation. In the case when $T = \bullet$ we take $(\bullet, P)$ and $(\bullet, S)$ to be equivalent and simply denote them by $\bullet$ and say that $\bullet$ is neither labeled by P nor S. Define $\mathcal{TP} = (\mathcal{T} \setminus \{\bullet\}) \times \{P\}$ and $\mathcal{TS} = (\mathcal{T} \setminus \{\bullet\}) \times \{S\}$ as the sets of nontrivial trees labeled by P and S, respectively, and let

$$\mathcal{TN} := \mathcal{TP} \cup \mathcal{TS} \cup \{\bullet\}. \tag{2.3}$$

The corresponding sets of trees with n leaves are denoted using the lower index n.

Next define $\mathcal{T}^* \subseteq \mathcal{T}$ as the set of trees in $\mathcal{T}$ such that the leftmost child of the root is a leaf, and let $\mathcal{T}_n^*$ denote the set of such trees with $n+1$ leaves (i.e. n leaves not counting the leftmost child of the root). A tree $T \in \mathcal{T}^*$ is commonly called a *planted tree*. We take trees in $\mathcal{T}^*$ to be labelled so that vertices in even generations have label P and vertices in odd generations have label S. There is a useful bijection

$$\partial_{\mathrm{Tree}} : \mathcal{T}^* \to \mathcal{TN} \tag{2.4}$$

which is described as follows. Let $T \in \mathcal{T}_n^*$. If the outdegree of the root in T is strictly greater that 2, remove the leftmost child of the root (along with the edge adjacent to it). Note that the resulting tree is still labeled by P, so this yields an element in $\mathcal{TP}_n$. If the outdegree of the root in T equals 2, then remove the root and its leftmost child along with their adjacent edges. The resulting tree is labeled by S and is thus an element in $\mathcal{TS}_n$, see Fig. 4. In the case when T has exactly 1 leaf, let $\partial_{\mathrm{Tree}} T = \bullet$.

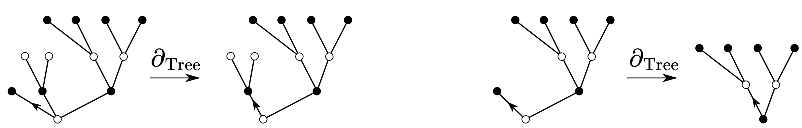


Figure 4. The removal of the first child of the root. Here, the vertices $\circ$ correspond to label P and the vertices $\bullet$ to label S.

2.3. Bijection between SP-maps and labeled trees. — Both SP-networks $\mathcal{N}$ (2.1) and labelled trees $\mathcal{TN}$ (2.3) may be described in the language of combinatorial species [**19**]. This provides a natural framework for defining bijections $\varphi : \mathcal{TN} \to \mathcal{N}$ and $\varphi^* : \mathcal{T}^* \to \mathcal{SP}$, as we now describe.

First, an element of $\mathcal{N}$ is either the single edge $e = (*_0, *_1)$, or a parallel network or a series network:

$$\mathcal{N} = e + \mathcal{P} + \mathcal{S}. \tag{2.5}$$

The elements of $\mathcal{P}$ are called parallel networks and are constructed from a parallel composition of at least two non-parallel networks:

$$\mathcal{P} = \mathrm{SEQ}_{\geq 2}(\mathcal{N} - \mathcal{P}) \tag{2.6}$$

where the combinatorial species $\mathrm{SEQ}_{\geq 2}$ are the linear orders of length greater than or equal to two. The elements of $\mathcal{S}$ are called series networks and are constructed from a series composition of at least two non-series networks:

$$\mathcal{S} = \mathrm{SEQ}_{\geq 2}(\mathcal{N} - \mathcal{S}). \tag{2.7}$$

The single edge network is treated seperately in the above definitions since it is in some sense neither series nor parallel.

Similarly, a tree from $\mathcal{TN}$ is either the unlabeled single leaf $\bullet$ or a tree labeled by P or a tree labeled by S:

$$\mathcal{TN} = \bullet + \mathcal{TP} + \mathcal{TS}. \tag{2.8}$$

A tree labeled by P is constructed by a composition of at least 2 trees from $\mathcal{T}_{\mathcal{N}}$ which are not labeled by P:

$$\mathcal{TP} = \mathrm{SEQ}_{\geq 2}(\mathcal{TN} - \mathcal{TP}). \tag{2.9}$$

A tree labeled by S is constructed by a composition of at least 2 trees from $\mathcal{T}_{\mathcal{N}}$ which are not labeled by S:

$$\mathcal{TS} = \mathrm{SEQ}_{\geq 2}(\mathcal{TN} - \mathcal{TS}). \tag{2.10}$$

The combinatorial identities (2.5), (2.6), (2.7) and (2.8), (2.9), (2.10) naturally define a bijection

$$\varphi : \mathcal{TN} \to \mathcal{N}.$$

Moreover, networks with n edges are in bijection with the set of labeled trees with n leaves and networks in $\mathcal{P}$ and $\mathcal{S}$ are in bijection with labeled trees in $\mathcal{TP}$ and $\mathcal{TS}$ respectively. Combining φ with the bijections (2.2) and (2.4) allows us to define a bijection $\varphi^* : \mathcal{T}^* \to \mathcal{SP}$ as in Fig. 5.

It will be useful to have the following recursive description of φ^*. Given a labelled tree $(T, \ell) \in \mathcal{T} \times \{S, P\}$, first assign a single edge to each of the leaves. Then, start forming networks by combining the edges corresponding to sibling leaves in series or in parallel, as determined by the label of their

$$\begin{tikzcd} \mathcal{T}^* \arrow[r, "\varphi^*"] \arrow[d, "\partial_{\text{Tree}}"'] & \mathcal{SP} \arrow[d, "\partial_{\text{Map}}"] \\ \mathcal{TN} \arrow[r, "\varphi"] & \mathcal{N} \end{tikzcd}$$

Figure 5. Definition of φ^* from φ.

common parent. Proceeding recursively towards the root of the tree, each vertex corresponds to a series or a parallel network which is constructed from the subtree of its descendants. Networks corresponding to vertices with a common parent are connected in series if that parent has label S, respectively in parallel if it has label P. To be definite, parallel connections are made in the same order left-to-right as the order of the corresponding siblings, while series connections are made so that the left-to-right order of the siblings gives the bottom-to-top order of connections. Restricting this construction to $\mathcal{T}^*$ gives the bijection φ^*.

2.4. Blobs. — We now define what we call *blobs*. Informally, blobs are parts of M which are separated by bottlenecks, in the sense that a geodesic between two vertices within the same blob stays entirely within that blob. See Fig. 6 for an illustration of the description that follows.

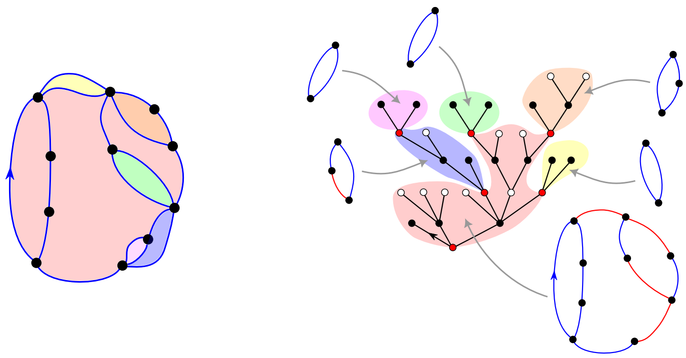


Figure 6. Left: An example of a rooted, two-connected SP-map divided into blobs. Right: Its corresponding planted plane tree, labeled by P, divided into segments. Here, $\circ$ and $\color{red}{\bullet}$ denote the label P and $\bullet$ denotes the label S. Each segment corresponds to a network and the red edges depicted in these networks are the places where networks corresponding to fringe subtrees are inserted.

Suppose that we are given a rooted 2-connected SP-map $M \in \mathcal{SP}$ and the corresponding planted tree $T^* = (\varphi^*)^{-1}(M) \in \mathcal{T}^*$ (which we recall is labeled by P). We colour a vertex of T^* red if it is marked P and has at least one child that is a leaf. In particular, the root of T^* is coloured red. This way, the tree T^* is decomposed into (maximal) subtrees whose roots are red, and where any red non-root vertex is a leaf. We call these subtrees *segments*.

We define the *blobs* of M to be the networks obtained by applying the recursive construction defining φ^* (described at the end of Section 2.3) to the segments. We moreover declare the blobs to have an oriented red placeholder edge for each red non-root vertex in the segment, which marks the location where we would insert the network constructed from the corresponding fringe subtree in T^*.

The orientation of each red edge is naturally inherited from the orientation of the edge in the tree from the root to the red vertex, together with the orientation of the root edge of the SP-map, where the tree is explored in its usual lexicographic order. We assign placeholder south and north poles to each of the blobs as the start- and endpoints of the oriented edge that corresponds to the red edge assigned to the red leaf of the corresponding parent segment. This way, the map M gets decomposed into blobs, where the placeholder poles of each blob are connected by an edge in M.

We perform a similar decomposition of networks N into blobs, where all definitions are analogous except that we demand the root vertex of the corresponding tree T to be coloured red.

2.5. Bienaymé–Galton–Watson trees and simply generated trees. — The main focus of this work is uniformly distributed two-connected SP-maps. However, the bijective correspondence with trees which is explained in the previous section allows us easily to consider a more general measure on the maps by assigning weights to the trees in a natural way and pushing these forward to the maps using the bijection.

Let $\mu = (\mu(k))_{k\geq 0}$ be a sequence of probabilities with $\mu(1) = 0$. Let BGW^μ denote the distribution of a Bienaymé–Galton–Watson process with offspring distribution μ. If μ has mean less than or equal to 1 then the tree τ is almost surely finite and, for any finite tree T,

$$\mathrm{BGW}^\mu(\tau = T) = \prod_{u\in T} \mu(\mathrm{out}_\tau(u)). \tag{2.11}$$

Let BGW^μ_n be the corresponding measure conditioned on the number of leaves being n. The random trees distributed by BGW^μ and BGW^μ_n will be denoted by T^μ and T^μ_n respectively.

In order to relate the BGW-trees to the random maps, it will be useful to consider another (almost) equivalent model of random trees called *simply generated trees with leaves as atoms*. Let $\eta = (\eta(i))_{i\geq 0}$ be a sequence of non-negative numbers called *weights* and assume that $\eta(1) = 0$. Let $T \in \mathcal{T}_n$ and define the weight of T by

$$\eta(T) = \prod_{u \in T} \eta(\mathrm{out}_T(u)).$$

A *simply generated tree* on n leaves is a random plane tree τ with values in $\mathcal{T}_n$ with a probability distribution

$$\mathbb{P}(\tau = T) = \frac{\eta(T)}{\sum_{T' \in \mathcal{T}_n} \eta(T')}, \qquad T \in \mathcal{T}_n. \tag{2.12}$$

The literature contains many results on simply generated trees with a fixed number of vertices, and for these we refer to [11] for details. The current case, where instead the number of leaves is fixed, is considered in e.g. [18]. Below we outline the main results which we require.

For $a, b > 0$, let $\eta^{a,b}$ be another weight sequence defined by

$$\eta^{a,b}(k) = \begin{cases} a\eta(0) & k = 0 \\ b^{k-1}\eta(k) & k \geq 2. \end{cases} \tag{2.13}$$

These weights are equivalent to η in the sense that they define the same measure as in (2.12). Define the generating series

$$g^\eta(z) = \sum_{n=0}^{\infty} \eta(n) z^n \tag{2.14}$$

and denote its radius of convergence by ρ_η. We will assume that the weights η are chosen such that $\rho_\eta > 0$. Then for any $b > 0$ such that $\frac{g^\eta(b)-\eta(0)}{\eta(0)b} < 1$ one may choose

$$a = \frac{1}{\eta(0)} - \frac{g^\eta(b) - \eta(0)}{\eta(0)b}$$

in which case $\eta^{a,b}$ is a probability measure which we will denote by $\eta^{(b)}$. Denote its mean by

$$\mathbf{m}_b = \sum_{k=2}^{\infty} k\eta^{(b)}(k) = \sum_{k=2}^{\infty} k\eta(k) b^{k-1} = (g^\eta)'(b).$$

The mean is a strictly increasing function of b and $\mathbf{m}_0 = 0$. In the current work we will assume that there is a $b < \rho_\eta$ such that $\mathbf{m}_b = 1$ which is then necessarily unique. The corresponding probability measure will be denoted by μ and the condition $\mathbf{m}_b = 1$ is referred to as *criticality*. The condition $b < \rho_\eta$ furthermore guarantees that μ has finite exponential moments. It is straightforward to check that the simply generated tree with weights η has the same distribution as T_n^μ.

2.6. Weighted trees and maps. — Let η be a weight sequence such that there exists a corresponding critical offspring distribution μ. Generate a random planted tree, denoted by $\mathsf{T}_n^{\mu,*}$, as follows:

1. Let T_n^{μ} have distribution BGW_n^{μ}.
2. Flip a fair coin and label T_n^{μ} by P or S according to the outcome of the coin flip.
3. Plant the resulting labeled tree, i.e. set $\mathsf{T}_n^{\mu,*} := \partial_{\mathrm{Tree}}^{-1}\mathsf{T}_n^{\mu}$.

Definition 2.2. — *The random, weighted, rooted, 2-connected SP-map with n non-root edges is defined as*

$$\mathsf{SP}_n^{\mu} := \varphi^*(\mathsf{T}_n^{\mu,*}).$$

The corresponding random network is denoted by

$$\mathsf{N}_n^{\mu} = \partial_{\mathrm{Map}}\mathsf{SP}_n^{\mu} = \varphi(\mathsf{T}_n^{\mu}).$$

The networks obtained by conditioning N_n^{μ} on being a series or a parallel network (i.e. on the outcome of the coin flip above) are denoted by S_n^{μ} and P_n^{μ} respectively.

In order to lighten the notation, we will often suppress the μ and write e.g. $\mathsf{SP}_n = \mathsf{SP}_n^{\mu}$, $\mathsf{N}_n = \mathsf{N}_n^{\mu}$, $\mathsf{T} = \mathsf{T}^{\mu}$, $\mathsf{T}_n^* = \mathsf{T}_n^{\mu,*}$ and $\mathsf{T}_n = \mathsf{T}_n^{\mu}$.

Example 2.3. — *The uniformly distributed map SP_n is obtained by choosing $\eta(k) = 1$ for all $k \geq 0$, $k \neq 1$. In this case one finds that*

$$\mu(0) = 2 - \sqrt{2}, \quad \mu_1 = 0, \quad \mu(k) = \left(1 - \frac{1}{\sqrt{2}}\right)^{k-1}, \qquad k \geq 2. \tag{2.15}$$

Example 2.4. — *Let $N \geq 2$ be an integer. The map SP_n, which is distributed uniformly among those where each series and each parallel composition in the network N_n consists of exactly N components, is obtained by choosing the weights $\eta(0) = \eta(N)$ and all other weights equal to 0. In the corresponding labeled tree T_n, every vertex has outdegree 0 or N and the root has outdegree 1 or N. In this case one finds that $\mu(0) = 1 - \frac{1}{N}$, $\mu(N) = \frac{1}{N}$.*

3. Proofs of the main results

We now turn to the proofs of Theorems 1.1 and 1.2. The main method is to compare lengths of geodesics between vertices in a series-parallel map and between corresponding vertices in its associated tree. For concreteness, we will work with the random parallel networks P_n. The associated tree is then a BGW-tree $\mathsf{T}_n = \mathsf{T}_n^{\mu}$ labelled by P. The proof can easily be adapted to the maps S_n, N_n and SP_n with exactly the same outcome.

Let $\mathsf{T} = \mathsf{T}^{\mu}$ be an unconditioned BGW-tree and let ξ be a random variable with distribution μ. We will use the *local limit* $\hat{\mathsf{T}}$ of T_n (as $n \to \infty$), which is an infinite random tree with a unique half-infinite path starting from the root, called its *spine*. In $\hat{\mathsf{T}}$ non-spine vertices have offspring according to independent copies of ξ, whereas spine vertices have offspring according to independent copies of the size-biased random variable $\hat{\xi}$ with distribution $\mathbb{P}(\hat{\xi} = k) = k\mathbb{P}(\xi = k)$, and the successor spine vertex is chosen uniformly at random among these children. For further details, we refer to Janson's extensive review [11].

For each $\ell \geq 1$, let $\mathsf{T}^{(\ell)}$ denote the tree obtained by deleting all descendants of the $(\ell+1)$st spine vertex of $\hat{\mathsf{T}}$ and identifying the $(\ell+1)$st spine vertex with the root of an independent copy of T. We view all the random trees T_n, T, $\hat{\mathsf{T}}$, and $\hat{\mathsf{T}}^{(\ell)}$ for $\ell \geq 1$ as being coloured (i.e. some vertices are red) and labeled by P or S according to the fair coin flip as defined in Section 2.

We may define infinite random series-parallel maps/networks SP, N, P and S in a natural way by extending the functions φ and φ^* to infinite trees. The maps/networks may then be viewed as the weak local limits of the corresponding sequences of finite maps/networks. These infinite objects are not strictly needed in the following arguments. However, it is convenient to refer to them occasionally, so we provide some details of their construction in the appendix.

We may generate T by starting with a root segment and a list of i.i.d. non-root segments. Likewise, $\hat{\mathsf{T}}$ can be generated from a root segment, a list of i.i.d. normal non-root segments, and a list of i.i.d. spine non-root segments. The tree $\hat{\mathsf{T}}^{(\ell)}$ additionally has a mixed segment containing the tip of the spine.

Lemma 3.1. — *The number of vertices in each type of segment (root, spine and normal) has finite exponential moments.*

Proof. — The number of vertices in each random segment may be stochastically bounded by two times the total population of a subcritical branching process (with possibly a modified root degree) whose offspring distribution is light-tailed. The total population of such a process is light-tailed itself by [11, Thm. 18.1].

We detail this argument for the root segment of T^{μ}. The proofs for all other types of segments are analogous. We construct a BGW-tree T^{ν} with offspring probabilites $(\nu_n)_n$ from T^{μ} as follows. Let T' be the root segment of T^{μ}. The tree T^{ν} is defined as the tree with vertex set consisting of the vertices in even generations in T' and vertices in T^{ν} are connected by an edge if and only if one is the grandchild of the other in T'. Note that since T' has no vertices of outdegree 1, the number of vertices in T' is less than two times the number of its leaves and T^ν has the same number of leaves as T'. Therefore, the number of vertices in T' is bounded by two times the number of leaves in T^ν.

Another way to describe T^ν is the following. Let v be a vertex in T^μ at an even generation whose children we have not yet generated and assume that v is also present in T^ν. If v either has no children in T^μ or it has more than one child and some of its children have no children, then we say that v has no children in T^ν. On the other hand, if v has children in T^μ and all of its children also have children, then we add these grandchildren as the children of v in T^ν. From this description, we find that

$$\begin{aligned}
\nu(0) &= 1 - \sum_{n=2}^{\infty} \mu(n)(1-\mu(0))^n,\\
\nu(1) &= 0,\\
\nu(k) &= \sum_{n=2}^{\infty} \mu(n) \sum_{i_1+\cdots+i_n=k} \prod_{j=1}^{n} \mu(i_j)\mathbb{1}\{i_j > 0\}, \quad k \geq 2.
\end{aligned}$$

The generating function g of the probabilities $(\nu_n)_n$ satisfies

$$g(z) = \nu(0) + h(h(z)) \tag{3.1}$$

where $h(z) = f(z) - \mu_0$ and f is the generating function of the sequence $(\mu_n)_n$. We then have

$$g'(1) = h'(h(1))h'(1) = h'(1-\mu(0)) = f'(1-\mu(0)) < 1$$

since $f'(1) = 1$. Thus the ν-process is subcritical.

Since μ has finite exponential moments, we find that ν also has finite exponential moments. Thus, the total population of T^ν (and hence also the size of the root segment of T) has finite exponential moments by [**11**, Thm. 18.1]. □

Corollary 3.2. — *There are constants $C, c > 0$ such that the probability for the maximal size of a segment in T_n to be larger than x is bounded by $Cn^{5/2}\exp(-cx)$ uniformly for all $n \geq 2$ and $x > 0$.*

Proof. — The probability for T to have n leaves satisfies

$$\mathbb{P}\left(|\mathsf{T}| = n\right) \sim c_{\mathrm{cond}} n^{-3/2} \tag{3.2}$$

for some constant $c_{\mathrm{cond}} > 0$, see for example [**12**]. Let $(X_i)_i$ denote the sizes of the segments in T. There are at most n segments in T_n, hence the probability for the maximal size of a segment in T_n to be larger than $x > 0$ satisfies

$$\begin{aligned}
\mathbb{P}\left(\max_i X_i > x \;\middle|\; |\mathsf{T}| = n\right) &= \frac{\mathbb{P}\left(\bigcup_i \{X_i > x\}, |\mathsf{T}| = n\right)}{\mathbb{P}\left(|\mathsf{T}| = n\right)}\\
&\leq c_{\mathrm{cond}}^{-1} n^{3/2} \sum_{i=1}^{n} \mathbb{P}(X_i > x),
\end{aligned}$$

where $|T|$ denote the number of leaves in T. By Lemma 3.1, each segment of the unconditioned tree has a finite exponential moment and the segments are identically distributed, except possibly the root segment. Thus each term in the sum is bounded by a constant times e^{-cx}, for some $c > 0$, and the result follows. □

In the infinite map P corresponding to the tree $\hat{\mathsf{T}}$, the blobs $B_0, B_1, \ldots$ corresponding to the spine segments are called *spine blobs*. The spine blobs are independent and the non-root spine blobs $(B_i)_{i \geq 1}$ are identically distributed. For each $\ell \geq 1$, consider the graph distance X_ℓ from the south pole of B_0 to the south pole of B_ℓ.

Lemma 3.3. — *There exists a constant $\eta > 0$ such that for each $\epsilon > 0$ there are constants $C, c > 0$ with*

$$\mathbb{P}(|X_\ell - \eta\ell| > \epsilon\ell) \leq C \exp(-c\ell)$$

uniformly for all $\ell \geq 1$.

Proof. — Consider the graph distance Y_ℓ from the south pole v of B_1 to the south pole of B_ℓ. This way, $|X_\ell - Y_\ell|$ is bounded by the size of B_0, which has finite exponential moments by Lemma 3.1. It hence suffices to show that there exists a constant $\eta > 0$ such that for each $\epsilon > 0$, there are constants $C, c > 0$ such that for all $\ell \geq 1$

$$\mathbb{P}(|Y_\ell - \eta\ell| > \epsilon\ell) \leq C \exp(-c\ell). \tag{3.3}$$

Since the poles of B_ℓ are joined by an edge, the graph distances from v to either of them (north or south) differ by at most 1. Hence, the increment $Y_{\ell+1} - Y_\ell$ only depends on this difference $S_\ell \in \{-1, 0, 1\}$ and B_ℓ. Thus $(Y_\ell, S_\ell)_{\ell \geq 1}$ is a Markov additive process with a driving chain $(S_\ell)_{\ell \geq 1}$ whose state space has three elements. Inequality (3.3) now immediately follows from a general large deviation result for Markov additive processes [**10**, Thm. 5.1] whose conditions are satisfied since the state space of the driving chain under consideration is finite [**10**, Rem. 3.5, Sec 7 (ii)]. □

Remark 3.4. — *We leave out details concerning the exact transition probabilities of the Markov additive process in the above proof but refer to the paper by Curien, Haas and Kortchemski* [**9**] *for a similar situation. In our case, the increments of the additive component $(Y_\ell)_{\ell \geq 1}$ of the chain are not as explicit as in* [**9**] *(they are given in terms of distances between certain vertices in a spine-blob) and therefore it is less useful to write them down.*

LEMMA 3.5. — *Let Z_ℓ be the number of segments encountered on the spine from the root of $\hat{\mathsf{T}}$ to the $(\ell+1)$st spine vertex. There exists a constant $\kappa > 0$ such that for each $\epsilon > 0$ there are constants $C, c > 0$ such that for all $\ell \geq 1$*

$$\mathbb{P}(|Z_\ell - \kappa\ell| > \epsilon\ell) \leq C\exp(-c\ell).$$

Proof. — Let E_m denote the number of edges on the spine we need to cross in order to get from the root vertex to the start of the $(m+1)$st segment. Set $E_0 = 0$. Hence the differences $E_m - E_{m-1}$ are independent for $\ell \geq 1$ and identically distributed for all $m \geq 2$. They have a geometric distribution since they may be found by flipping independent coins to check whether the next even vertex on the spine is a red vertex in which case the segment stops.

We make use of the following inequality given for example in [15, Example 1.4]: Let S_n denote the sum of n i.i.d. real-valued centred random variables with finite exponential moments. Then there are constants $\lambda_0, c > 0$ such that for all $n \in \mathbb{N}$, $x > 0$ and $0 \leq \lambda \leq \lambda_0$ it holds that

$$\mathbb{P}(|S_n| \geq x) \leq 2\exp(cn\lambda^2 - \lambda x). \tag{3.4}$$

It follows that there exists $\kappa^{-1} > 0$ such that for each $\delta > 0$ there exist constants $C, c > 0$ with

$$\mathbb{P}(|E_m - \kappa^{-1}m| > \delta m) \leq C\exp(-cm), \qquad \text{for all } m \geq 1.$$

As $\delta > 0$ is arbitrary, it follows by choosing $\delta = \pm\left(1 - \frac{1}{1\pm\epsilon}\right)\kappa^{-1}$ that for any $\epsilon > 0$ there exist constants $C', c' > 0$ with

$$\mathbb{P}(\ell \notin [E_{\lfloor\kappa\ell(1-\epsilon)\rfloor}, E_{\lfloor\kappa\ell(1+\epsilon)\rfloor}]) \leq C'\exp(-c'\ell).$$

Since $\ell \notin [E_{\lfloor\kappa\ell(1-\epsilon)\rfloor}, E_{\lfloor\kappa\ell(1+\epsilon)\rfloor}$ is equivalent to $|Z_\ell - \kappa\ell| > \kappa\epsilon\ell$, the proof is complete. □

Before proceeding with the proof, we recall some facts about the Gromov–Hausdorff metric d_{GH} on the space of isometry classes of compact metric spaces. Informally, two compact metric spaces are close in the Gromov–Hausdorff metric d_{GH} if they may be isometrically embedded in the third space Z so that their Hausdorff distance in Z is small. Here, we will use a representation of d_{GH} in terms of distortions of correspondences, see e.g. [7]. A correspondence $\mathcal{R}$ between two metric spaces (A, d_A) and (B, d_B) is a subset $\mathcal{R} \subseteq A \times B$ such that for any $a \in A$ there is a $b \in B$ such that $(a, b) \in \mathcal{R}$, and for any $b \in B$ there is an $a \in A$ such that $(a, b) \in \mathcal{R}$. A distortion of a correspondence $\mathcal{R}$ is defined by

$$\mathrm{dis}(\mathcal{R}) = \sup\{|d_A(a_1, a_2) - d_B(b_1, b_2)| \ : \ (a_1, b_1), (a_2, b_2) \in \mathcal{R}\}.$$

The Gromov–Hausdorff metric may then be written as

$$d_{\mathrm{GH}}(A,B) = \frac{1}{2}\inf_{\mathcal{R}} \mathrm{dis}(\mathcal{R}).$$

By the main theorem of [12, 16], there exists a constant $c_{\mathrm{tree}} > 0$ such that

$$(\mathsf{T}_n, c_{\mathrm{tree}} n^{-1/2} d_{\mathsf{T}_n}) \xrightarrow{d} (\mathcal{T}_{\mathrm{e}}, d_{\mathcal{T}_{\mathrm{e}}}) \tag{3.5}$$

in the Gromov–Hausdorff topology, with $(\mathcal{T}_{\mathrm{e}}, d_{\mathcal{T}_{\mathrm{e}}})$ denoting Aldous' Brownian tree. The rest of the proof involves defining a suitable correspondence between the spaces P_n and T_n, and showing that the distortion of that correspondence, with properly rescaled metrics, converges to 0 in probability.

We define a correspondence R_n between P_n and T_n as follows. Note that there is a bijective relationship between blobs in P_n and segments in T_n. We let $(u,v) \in \mathsf{R}_n$ if and only if there is a blob in P_n which contains u and a corresponding segment in T_n which contains v. In that case we write $u \sim v$.

Let $\mathrm{h}_{\mathsf{P}_n}(u)$ denote the graph distance in the map P_n from the south pole $*_0$ to u, and $\mathrm{h}_{\mathsf{T}_n}(v)$ the height of the vertex v in T_n. The following lemma is the first step in relating distances in P_n to distances in T_n.

Lemma 3.6. — *Let η be the constant in Lemma 3.3, κ the constant in Lemma 3.5, and let $\delta > 0$ be arbitrary. With probability tending to one as $n \to \infty$, there exist no vertices $u \in \mathsf{P}_n$ and $v \in \mathsf{T}_n$ such that $u \sim v$ and*

$$|\mathrm{h}_{\mathsf{P}_n}(u) - \eta\kappa \mathrm{h}_{\mathsf{T}_n}(v)| \geq \delta\sqrt{n}. \tag{3.6}$$

Proof. — Let $\epsilon > 0$ be given. Let $H_n = H\sqrt{n}$ for some $H > 0$. By (3.5) we may choose H sufficiently large such that the height $\mathrm{H}(\mathsf{T}_n)$ satisfies

$$\mathbb{P}(\mathrm{H}(\mathsf{T}_n) > H_n) \leq \epsilon$$

for all n. Let us call a vertex v in a tree τ *bad* if it corresponds to some vertex u in the associated map P such that inequality (3.6) holds (with obvious replacements of P_n and T_n by P and τ). Let $\mathcal{E}(\tau)$ denote the property that a tree τ contains a bad vertex. Let us denote the tip of the spine in $\hat{\mathsf{T}}^{(\ell)}$ by U_ℓ. Given any finite plane tree τ with a vertex u having height ℓ, it holds that

$$\mathbb{P}\left((\hat{\mathsf{T}}^{(\ell)}, U_\ell) = (\tau, u)\right) = \mathbb{P}(\mathsf{T} = \tau)$$

(see e.g. the short derivation in [17, Appendix A3]). Hence, using (3.2) we obtain

$$\begin{aligned}\mathbb{P}(\mathcal{E}) &\leq \epsilon + \mathbb{P}(\mathcal{E}(T_n), \mathrm{H}(\mathsf{T}_n) \leq H_n) \\ &\leq \epsilon + \frac{2n}{\mathbb{P}(|\mathsf{T}| = n)} \sum_{\ell=0}^{\lfloor H_n \rfloor} \mathbb{P}(U_\ell \text{ is bad}) \\ &= \epsilon + O(n^{5/2}) \sum_{\ell=0}^{\lfloor H_n \rfloor} \mathbb{P}(U_\ell \text{ is bad}).\end{aligned}$$

The second inequality is explained as follows. We may write

$$\begin{aligned}&\mathbb{P}(\mathcal{E}(T_n), \mathrm{H}(\mathsf{T}_n) \leq H\sqrt{n})\mathbb{P}(|\mathsf{T}| = n) = \sum_{\tau \in \mathcal{T}} \mathbb{1}\{\mathcal{E}(\tau), \mathrm{H}(\tau) \leq H_n, |\tau| = n\}\mathbb{P}(\mathsf{T} = \tau) \\ &\leq \sum_{\tau \in \mathcal{T}, u \in \tau} \mathbb{1}\{u \text{ is bad}, \mathrm{H}(\tau) \leq H_n, |\tau| = n\}\mathbb{P}(\mathsf{T} = \tau) \\ &= \sum_{\tau \in \mathcal{T}, u \in \tau} \sum_{\ell=0}^{\mathrm{H}(\tau)} \mathbb{1}\{u \text{ is bad}, \mathrm{H}(\tau) \leq H_n, |\tau| = n, \mathrm{h}_\tau(u) = \ell\}\mathbb{P}\left((\hat{\mathsf{T}}^{(\ell)}, U_\ell) = (\tau, u)\right) \\ &\leq 2n \sum_{\ell=0}^{\lfloor H_n \rfloor} \sum_{\tau \in \mathcal{T}} \mathbb{P}\left(\hat{\mathsf{T}}^{(\ell)} = \tau, U_\ell \text{ is bad}\right) = 2n \sum_{\ell=0}^{\lfloor H_n \rfloor} \mathbb{P}\left(U_\ell \text{ is bad}\right).\end{aligned}$$

For the last inequality we used the fact that if a plane tree has n leaves and no inner vertex with exactly one child then it has at most $2n$ vertices in total.

If u is a vertex in the map corresponding to $\hat{\mathsf{T}}^{(\ell)}$ such that u corresponds to the tip of the spine of $\hat{\mathsf{T}}^{(\ell)}$, then the height of u may be bounded by the total number of leaves in the root segments of the subtrees attached to the spine vertices of $\hat{\mathsf{T}}^{(\ell)}$. Denote by W_i the total size of all the root segments in the subtrees attached to the spine vertex at distance i from the root, where $0 \leq i \leq \ell$. Note that they are independent and i.i.d for $i < \ell$. At each of these spine vertices the total number of subtrees is distributed as $\hat{\xi} - 1$ or $\hat{\xi}$ (at the tip of the spine) and is thus light-tailed. Furthermore, the total size of the root segment of each of the subtrees is light-tailed by Lemma 3.1. We conclude that W_i is light-tailed for all $0 \leq i \leq \ell$.

If $\ell < \delta/(2\eta\kappa)\sqrt{n}$, then the event that U_ℓ is bad implies that the height of u is larger than $\delta\sqrt{n}$, and therefore that

$$\sum_{i=0}^{\ell} W_i \geq \delta\sqrt{n}.$$

Since the W_i are i.i.d and light-tailed, it follows by the medium deviation inequality (3.4) that if we take $h > 0$ small enough then there exists constants

$C_1, c_1 > 0$ such that uniformly for all n and $1 \le \ell \le h\sqrt{n}$

$$\mathbb{P}(U_\ell \text{ is bad}) \le \mathbb{P}\left(\sum_{i=0}^{h\sqrt{n}} W_i \ge \delta\sqrt{n}\right) \le C_1 \exp(-c_1\sqrt{n}). \tag{3.7}$$

This ensures

$$O(n^{5/2}) \sum_{\ell=0}^{\lfloor h\sqrt{n} \rfloor} \mathbb{P}(U_\ell \text{ is bad}) = O(n^3)\exp(-c_1\sqrt{n}) = \exp(-\Theta(\sqrt{n})).$$

Furthermore, uniformly for $h\sqrt{n} \le \ell \le H\sqrt{n}$ we know by Lemma 3.5 that for any $c > 0$ we have $|Z_\ell - \kappa\ell| \le c\ell$ with probability at least $1 - \exp(\Theta(\sqrt{n}))$, which allows us to apply Lemma 3.3 to obtain for any $c > 0$ that $|X_{Z_\ell} - \eta\kappa\ell| \le c\ell$ again with probability at least $1 - \exp(\Theta(\sqrt{n}))$. Recall that we constructed $\hat{\mathsf{T}}^{(\ell)}$ from $\hat{\mathsf{T}}$ by replacing the descendants of the tip of the spine, therefore creating a "mixed" segment that contains the tip of the spine. Hence, in the map corresponding to $\hat{\mathsf{T}}^{(\ell)}$ any vertex corresponding to the tip of the spine has a height that differs from X_{Z_ℓ} at most by a summand that by Lemma 3.1 has finite exponential moments. Thus, we obtain

$$O(n^{5/2}) \sum_{h\sqrt{n} \le \ell \le H\sqrt{n}} \mathbb{P}(U_\ell \text{ is bad}) = O(n^3)\exp(-\Theta(\sqrt{n})) = \exp(-\Theta(\sqrt{n})).$$

In summary,

$$\mathbb{P}(\mathcal{E}) \le \epsilon + \exp(-\Theta(\sqrt{n})).$$

Since $\epsilon > 0$ was arbitrary this completes the proof. □

We are now ready to prove our main theorems.

Proof of Theorem 1.1. — By Corollary 3.2 and Lemma 3.6 we know that, for some constant $C > 0$ and some deterministic sequence $t_n = o(\sqrt{n})$, we have with high probability that the largest blob in P_n has diameter at most $C \log n$, and whenever a vertex u in P_n corresponds to a vertex v in T_n, then

$$|\mathrm{h}_{\mathsf{P}_n}(u) - \eta\kappa \mathrm{h}_{\mathsf{T}_n}(v)| \le t_n. \tag{3.8}$$

Now, let u_1, u_2 be vertices in P_n and let v_1 and v_2 be any vertices in T_n such that v_i corresponds to u_i for $i = 1, 2$. If v_3 denotes the lowest common ancestor of v_1 and v_2 in T_n, then

$$d_{\mathsf{T}_n}(v_1, v_2) = \mathrm{h}_{\mathsf{T}_n}(v_1) + \mathrm{h}_{\mathsf{T}_n}(v_2) - 2\mathrm{h}_{\mathsf{T}_n}(v_3).$$

Any geodesic in P_n between u_1 and u_2 must pass through a blob corresponding to v_3. Letting u_3 denote some vertex in this blob, it follows that

$$d_{\mathsf{P}_n}(u_1, u_2) = \mathrm{h}_{\mathsf{P}_n}(u_1) + \mathrm{h}_{\mathsf{P}_n}(u_2) - 2\mathrm{h}_{\mathsf{P}_n}(u_3) + O(\log n).$$

By Inequality (3.8) it follows that

$$d_{\mathsf{P}_n}(u_1, u_2) = \eta\kappa d_{\mathsf{T}_n}(v_1, v_2) + o(\sqrt{n}).$$

By Equation (3.5) it follows that

$$(\mathsf{P}_n, \frac{c_{\mathrm{tree}}}{\eta\kappa} n^{-1/2} d_{\mathsf{P}_n}) \xrightarrow{d} (\mathcal{T}_{\mathrm{e}}, d_{\mathcal{T}_{\mathrm{e}}}) \tag{3.9}$$

in the Gromov–Hausdorff topology. □

Proof of Theorem 1.2. — Since the diameter is bounded by twice the height with respect to the origin of the root edge, it suffices to show that there exist $C, c > 0$ with

$$\mathbb{P}(\mathrm{H}(\mathsf{P}_n) > x\sqrt{n}) \leq C \exp(-cx^2).$$

By possibly adjusting the constants C, c, it suffices to show this inequality for all $x \geq 1$. Furthermore, the left-hand side equals to zero if $x \geq \sqrt{n}$. Hence, we only need to treat the case $1 \leq x \leq \sqrt{n}$, which we are now going to do.

We now use the same notation and argue analogously as in the proof of Lemma 3.6, with the exception that we call a vertex v in T_n *bad* if it corresponds to some vertex u in P_n such its height in P_n is larger than $x\sqrt{n}$. Analogously, we define for any $\ell \geq 1$ whether the tip of the spine of $\hat{\mathsf{T}}^{(\ell)}$ is *bad*, and we let $\mathcal{E}$ denote the event that T_n contains a bad vertex.

By [**18**, Lem. 6.61] there exist $C_1, c_1 > 0$ with

$$\mathbb{P}(\mathrm{H}(\mathsf{T}_n) > y\sqrt{n}) \leq C_1 \exp(-c_1 y^2)$$

for any $y > 0$. Hence, it follows as in the proof of Lemma 3.6 that for any fixed $h > 0$

$$\begin{aligned}\mathbb{P}(\mathcal{E}) &\leq \mathbb{P}(\mathrm{H}(\mathsf{T}_n) > (x/h)\sqrt{n}) + \mathbb{P}(\mathcal{E}, \mathrm{H}(\mathsf{T}_n) \leq (x/h)\sqrt{n}) \\ &\leq C_1 \exp(-c_1 x^2/h^2) + O(n^{5/2}) \sum_{\ell=0}^{\lfloor (x/h)\sqrt{n} \rfloor} \mathbb{P}(U_\ell \text{ is bad}).\end{aligned}$$

Furthermore, applying Inequality (3.4) as before yields that there exist constants $\lambda_0, \mu_{\mathrm{blob}}, c_2 > 0$ such that for all $n \in \mathbb{N}$ and $0 \leq \lambda \leq \lambda_0$ it holds that

$$\mathbb{P}(U_\ell \text{ is bad}) \leq 2\exp(c\ell\lambda^2 - \lambda(x\sqrt{n} - \ell\mu_{\mathrm{blob}})).$$

As $h > 0$ was fixed but arbitrary, we may take h large enough such that $\mu_{\mathrm{blob}}/h < 1/2$. As $x \geq 1$, it follows that uniformly for $\ell \leq (x/h)\sqrt{n}$

$$c\ell\lambda^2 - \lambda(x\sqrt{n} - \ell\mu_{\mathrm{blob}}) \leq c\ell\lambda^2 - \lambda x\sqrt{n}/2 = -x\Theta(\sqrt{n}).$$

Hence,

$$\mathbb{P}(\mathrm{H}(\mathsf{P}_n) > x\sqrt{n}) = \mathbb{P}(\mathcal{E}) \leq C_1 \exp(-c_1 x^2/h^2) + \exp(-x\Theta(\sqrt{n})).$$

Since we assumed $1 \leq x \leq \sqrt{n}$, it follows that

$$\mathbb{P}(\mathrm{H}(\mathsf{P}_n) > x\sqrt{n}) \leq C_3 \exp(-c_3 x^2)$$

for some constants $C_3, c_3 > 0$ that do not depend on n. □

4. Conclusions and related classes

Our results imply in particular a scaling limit of uniformly sampled 2-connected rooted series-parallel maps. From a different viewpoint, it has been shown in [5] that rooted series-parallel maps are in bijection with permutations avoiding the patterns 2413 and 3142. As a side result, it was also shown that a bipolar planar map admits a unique bipolar orientation precisely when it is series-parallel. These observations link series-parallel maps to the large framework of separable permutations and their connections to the Liouville quantum gravity, which has enjoyed remarkable developments during the last few years. For example, it is conjectured in [6] that conformally embedded rooted series-parallel maps converge towards the 2-LQG quantum sphere. Therefore, we expect interesting research directions following from our work.

Apart from planar maps, we may also consider planar graphs subject to constraints. The terminology for graph classes is a bit at odds with the one used here for planar maps. The class of series-parallel graphs refers to (simple) graphs that do not admit the K_4 as a minor, which may very well be separable (that is, not 2-connected). This class is an important example of a subcritical graph class and hence admits the Brownian tree as scaling limit by [14].

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Appendix A. Local metrics and continuity

Recall the sets $\mathcal{T}$ (finite plane trees with no vertices of outdegree 1) and $\mathcal{T}^*$ (trees in $\mathcal{T}$ where the leftmost child of the root is a leaf) from Section 2.2. Denote by $\overline{\mathcal{T}}$ the set of (possibly infinite) plane trees which have no vertices of outdegree 1, and by $\mathcal{T}_\infty = \overline{\mathcal{T}} \backslash \mathcal{T}$ the infinite trees. Denote by $\overline{\mathcal{T}}^*$ the set of trees in $\overline{\mathcal{T}}$ such that the leftmost child of the root is a leaf and let $\mathcal{T}^*_\infty = \overline{\mathcal{T}}^* \setminus \mathcal{T}^*$. As for $\mathcal{T}^*$, we take trees in $\overline{\mathcal{T}}^*$ to be labelled so that vertices in even generations have label P and vertices in odd generations have label S.

We define *local metrics* and *local topologies* on $\overline{\mathcal{T}}$, $\overline{\mathcal{T}}^*$ and $\mathcal{SP}$ as follows. First, for any graph G with a distinguished vertex v we let $B_r(G; v)$ denote the subgraph of G spanned by vertices at distance $\leq r$ from the vertex v. For two graphs G and G' belonging to some set of graphs $\mathcal{G}$, we let

$$d_{\mathcal{G}}(G, G') = \frac{1}{1 + \max\{r \in \mathbb{N} : B_r(G; v) = B_r(G'; v)\}}.$$

The topology generated by this metric is called the *local topology* with respect to v. In the case when $\mathcal{G}$ is one of the sets $\overline{\mathcal{T}}$, $\overline{\mathcal{T}}^*$ we take $v = \varnothing$ and use the notation $d_{\mathcal{T}}$ for the metric in both cases. When $\mathcal{G}$ is the set $\mathcal{SP}$ we use $v = *_0$.

Using the local metric we can define *infinite* series-parallel maps. Namely, we let $\overline{\mathcal{SP}}$ denote the completion of $\mathcal{SP}$ with respect to the metric $d_{\mathcal{SP}}$. Then $\mathcal{SP}_\infty := \overline{\mathcal{SP}} \setminus \mathcal{SP}$ is the set of infinite series-parallel maps, which are formally equivalence classes of Cauchy sequences of finite series-parallel maps. We now show how to extend the bijection $\varphi^* : \mathcal{T}^* \to \mathcal{SP}$ of Section 2.3 to infinite trees and maps.

A *spine* of an infinite tree is an infinite path $(V(0), V(1), V(2), ...)$ such that $V(0) = \varnothing$ and $V(i+1)$ is a child of $V(i)$ for all $i \geq 0$. At every vertex $V(i)$ on the spine, there is one subtree rooted on the left of the spine and another one on the right of the spine, which we refer to as the left and right *outgrowths* of the spine. Let $\hat{\mathcal{T}}^*_\infty$ be the subset of $\mathcal{T}^*_\infty$ consisting of trees with a *unique* spine, and such that this unique spine has infinitely many outgrowths in odd generations on both the left and the right.

Now let $T \in \hat{\mathcal{T}}^*_\infty$ with unique spine $(V_T(0), V_T(1), V_T(2), ...)$. We inductively define a sequence $(i_k, j_k)_{k \geq 1}$ of pairs of *odd* integers $i_k \leq j_k$ as follows, see Figure 7 for an illustration. First, i_1 is the smallest odd integer i so that $V_T(i)$ has a non-empty outgrowth on the right of the spine, and j_1 is the smallest odd integer $j \geq i_1$ so that $V_T(j)$ has a non-empty outgrowth on the left of the spine. Inductively, i_{k+1} is the smallest odd integer $i > j_k$ so that $V_T(i)$ has a non-empty outgrowth on the right of the spine, and j_{k+1} is the smallest odd integer $j \geq i_{k+1}$ so that $V_T(j)$ has a non-empty outgrown on the left of the spine. We call $(V_T(i_k), V_T(j_k))_{k \geq 1}$ the *RL-pairs* of T. Since i_k, j_k are odd, $V_T(i_k), V_T(j_k)$ are labelled by S. Note that $\hat{\mathcal{T}}^*_\infty$ is defined so that T has infinitely many RL-pairs.

Proposition A.1. — *Let $[T]_m := B_m(T; \varnothing)$ denote the first m generations of T.*

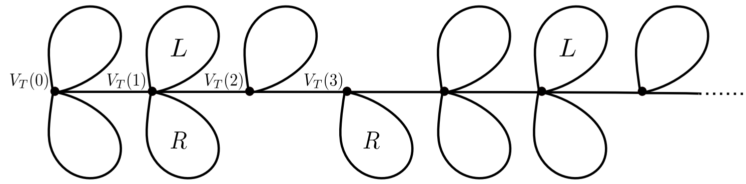

Figure 7. An infinite tree $T \in \hat{\mathcal{T}}^*_\infty$ with its first two RL-pairs.

(a) If $T \in \hat{\mathcal{T}}^*_\infty$ then $\varphi^*([T]_m)$ is a Cauchy sequence with respect to $d_{\mathcal{SP}}$. Thus $\varphi^*(T) := \lim_{m\to\infty} \varphi^*([T]_m)$ gives a well-defined function $\mathcal{T}^* \cup \hat{\mathcal{T}}^*_\infty \to \overline{\mathcal{SP}}$.

(b) If a sequence $(T_n)_{n\geq 1}$ in $\mathcal{T}^* \cup \hat{\mathcal{T}}^*_\infty$ converges to $T \in \mathcal{T}^* \cup \hat{\mathcal{T}}^*_\infty$ with respect to the metric $d_{\mathcal{T}}$ then $\varphi^*(T_n) \to \varphi^*(T)$ with respect to $d_{\mathcal{SP}}$.

Proof. — The key observation is that if $T \in \hat{\mathcal{T}}^*_\infty$ and $v \in T$ is a leaf that is beyond the first r RL-pairs, then for any m large enough such that $v \in [T]_m$, the edge $\mathbf{e}_v$ corresponding to v in $\varphi^*([T]_m)$ is at least at distance r from both poles $*_0$, $*_1$. This follows from the recursive description of φ^*, where the non-empty outgrowths of each RL-pair become non-empty networks placed in series on either side between $\mathbf{e}_v$ and the poles $*_0$, $*_1$. Each of these networks contributes to the distance from the poles to $\mathbf{e}_v$ at least by one, and to the total distance at least by r. Due to this observation, the proof can be completed as follows.

(a) If $[T]_m$ contains the first $r+1$ RL-pairs for all $m > m_0(r)$ then $\forall m, m' > m_0(r)$, $B_r(\varphi^*([T]_m) = B_r(\varphi^*([T]_{m'}))$ since all the edges beyond the first $r+1$ RL-pairs do not belong to $B_r(\varphi^*([T]_m))$.

(b) Let $(T_n)_{n\geq 1}$ be a sequence in $\overline{\mathcal{T}}^*$ converging to $T \in \mathcal{T}^* \cup \hat{\mathcal{T}}^*_\infty$. If $T \in \mathcal{T}^*$ then $T_n = T$ for every n large enough so that $\varphi^*(T_n) \to \varphi^*(T)$ is obvious. Now we assume that $T \in \hat{\mathcal{T}}^*_\infty$. Let R be such that $[T]_R$ contains the first $r+1$ RL-pairs. There exists a number $n_0(R)$ so that for some $n > n_0(R)$, $[T_n]_R = [T]_R$. For such n it holds that $B_r(\varphi^*([T_n]_R)) = B_r(\varphi^*([T]_R))$. □

Since the local limit $\hat{\mathsf{T}}$ of the trees T_n almost surely belongs to $\hat{\mathcal{T}}^*_\infty$, it follows that the local limit $\mathsf{SP}_\infty = \varphi^*(\hat{\mathsf{T}})$ of SP_n is a well-defined element of $\mathcal{SP}_\infty$.

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