Vector-valued robust stochastic control

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Abstract: We study a dynamic stochastic control problem subject to Knightian uncertainty with multi-objective (vector-valued) criteria. Assuming the preferences across expected multi-loss vectors are represented by a given, yet general, preorder, we address the model uncertainty by adopting a robust or minimax perspective, minimizing expected loss across the worst-case model. For loss functions taking scalar values, there is no ambiguity in interpreting supremum and infimum. In contrast, major challenges for multi-loss control problems include properly defining and interpreting the notions of supremum and infimum, as well as addressing their non-uniqueness. To deal with these, we employ the notion of an ideal point vector-valued supremum for the robust part of the problem, while we view the control part as a multi-objective (or vector) optimization problem. Using a set-valued framework, we derive both a weak and a strong version of the dynamic programming principle (DPP) or Bellman equations for two appropriately chosen value functions: the collection of all worst expected losses across all feasible actions, and for its upper image. The weak version of Bellman's principle is proved under minimal assumptions. To establish a stronger version of DPP, we introduce the rectangularity property with respect to a general preorder. We also show that the weak minimizers obey the time consistency property. Finally, we study the important particular case of component-wise partial order of vectors, and conclude with some illustrative examples motivated by financial problems.

Keywords: set-valued control; model uncertainty; stochastic robust control; multi-objective criteria; Bellman's principle; dynamic programming; rectangularity property; weak minimizers; time consistency; Knightian uncertainty.
MSC2020: Primary 90C39; Secondary 90C29, 93E03

1 Introduction

Model uncertainty refers to the challenge of accurately modeling the dynamics of the underlying stochastic system. This uncertainty may arise from various sources, such as incomplete information, ambiguity or presence of unobservable factors. It was first discussed by Knight [Kni21] and is also referred to as Knightian uncertainty.

A particularly important field extensively involving Knightian uncertainty is the control of stochastic systems subject to model uncertainty. Here, the controller does not know the true law of the underlying stochastic process a priori but knows it belongs to a known family of probability laws. As such, the controller faces not only the randomness of the controlled system, but also the Knightian uncertainty. A significant body of literature has addressed the challenge of model

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uncertainty, particularly driven by issues in finance and economics where flawed models can lead to erroneous investment decisions, ineffective risk management strategies, and inaccurate pricing of financial instruments.

One approach to tackle this challenge is through robust optimization, which seeks controls that perform well across various possible models. Other approaches include adaptive control, Bayesian control, adaptive-robust control, and strong robust control. For a brief overview of these approaches, we refer the reader to [BCC+19]. Additionally, recent advancements in approximation methods using machine learning have led to the development of new efficient methods for solving these control problems [KTW24]. Substantial advancements have been achieved, encompassing highly abstract scenarios such as general multidimensional state spaces, discrete and continuous time frameworks, finite and infinite horizon settings, and finite or infinite model uncertainty spaces. These developments have also established connections to partial differential equations and backward stochastic differential equations. However, it is important to note that all these achievements assume that the loss or reward functions take real, one dimensional, values; we refer to this as the scalar case.

On the other hand, there is a growing body of literature on multi-valued or set-valued stochastic control problems, with prominent applications in mathematical finance and economics. In [RU20], the authors study certainty equivalent and utility indifference pricing for incomplete preferences using vector optimization, stemming from [Nau06], which is dedicated to the representation of incomplete preferences. Multi-portfolios and markets with transaction costs in a dynamic setup are studied in [FR13, LR14, FR14, FR21, AF20] using multi-valued or set-valued risk measures. A novel approach to dealing with (scalar) time-inconsistent stochastic control problems by viewing them as vector-valued problems was introduced in [KR21], which in particular was applied to portfolio optimization problems; see also [KRC22] for applications to the maximization of dynamic acceptability indices. In [IZ23] the authors derive Hamilton-Jacobi-Bellman Equations for set valued stochastic control problems.

The literature on multi-objective optimization problems under model uncertainty is predominantly focused on the static case. Specifically, a robust approach to static multi-objective problems under model uncertainty has been explored through two main streams of research. The first stream, exemplified by works such as [KL12, FW14], addresses robust multi-objective problems by employing a robustified objective. The second stream, represented by studies like [EIS14, IKK+14], utilizes a set-optimization approach. For an overview of existing results in static robust multi-objective (or vector) optimization, we refer the reader to the surveys [IS15, WD16].

To the best of our knowledge, this study is the first attempt to investigate (dynamic) stochastic control problems subject to model or Knightian uncertainty, involving multi-objective (vector-valued) criteria. These results will serve as a foundation for addressing time-inconsistent problems subject to model uncertainty through the lens of a set-valued framework, as well as for studying multi-portfolio allocation problems under model uncertainty.

Our approach assumes that the decision maker's preferences across expected (multi-)loss vectors are represented by a given, yet general, preorder. We address model uncertainty by adopting a robust or minimax perspective, minimizing expected loss across the worst-case model. For loss functions taking real (or scalar) values, there is no ambiguity in interpreting supremum and infimum. In contrast, one major challenge for multi-loss control problems is to properly define and interpret the notion of supremum and infimum. Another key difficulty is that usually these suprema and infima are not unique.

To deal with these obstacles, **first**, we employ the notion of *ideal point vector-valued supremum* of an arbitrary collection of vectors in $\mathbb{R}^d$ with respect to a preorder, see [Löh11, Example 1.8]. We also introduce its dynamic, or conditional, version, and provide sufficient conditions for its existence. Using this notion of the supremum for the robust part, we treat the control part of the problem as a multi-objective (or vector) optimization problem with respect to the preorder.

The **second** novel contribution is derivation of a *version of dynamic programming principle* (DPP) or Bellman equations. For multi-objective or set-optimization control problems there is no universal or unified definition of a value function. Recent works on this topic have illustrated the advantage of utilizing set-valued mappings as value functions, see e.g. [KR21, FRZ22, HV20, FR17, IZ24, IZ23]. In broad terms, the DPP corresponds to order relations between sets. Similar to [KR21], we consider two appropriately chosen value functions: the first, the collection of all worst expected losses across all feasible actions, and, the second, its upper image. Without any additional assumptions, we derive the corresponding backward recursive inclusions for the robust problem; see Theorem 3.2, the weak Bellman's principle.

For the scalar, one dimensional, stochastic robust control problems, it is well understood that the DPP hinges on the so called rectangularity property of the set of probability measures describing the model uncertainty, and we refer to [Sha16] and references therein. The **third** key contribution of our study is the introduction of *rectangularity property* with respect to a general preorder. Leveraging this we prove a stronger version of Bellman's principle of optimality; see Theorem 4.3 and Theorem 4.4.

Using a series of results, we show that while the vector-valued supremum may not be unique, the Bellman's principle is invariant with respect to chosen supremum. **Fourth**, we introduce the notion of the weak minimizers for stochastic vector-valued problems, and show that a weak minimizer of the robust problems is time consistent. Finally, we discuss an important particular case of component-wise partial order of vectors, and conclude with some illustrative examples.

The paper is organized as follows. In Section 2 we set the stage by introducing the underlying stochastic model and briefly reviewing the fundamental concepts associated with vector preorders (Section 2.1) and vector optimization problems (Section 2.3). Section 2.2 is dedicated to the notion of supremum of collection of vectors with respect to a preorder. We discuss several important properties of this supremum, such as existence, uniqueness and monotonicity. Section 3 is dedicated to weak Bellman's principle of optimality for the main stochastic control problem. Starting with precise formulation of the problem, we define the set-valued candidates for the value function, and derive a weak version of DPP; Theorem 3.2. In Section 4, we introduce the notion of rectangularity with respect to a preorder, and prove strong versions of DPP, Theorem 4.3 and Theorem 4.4. In Section 4.1, we relate the strong DPP and the weak minimizers with time consistency property of the solution. In Section 5 we discuss a particular but important and natural preorder – component-wise partial order. Last section is devoted to examples.

We are confident that the concepts outlined in this manuscript will form the basis for exploring general stochastic control problems under model uncertainty for multi-valued or set-valued criteria. Among some important avenues for imminent exploration, we highlight: other approaches to model uncertainty such as adaptive control, adaptive-robust control, Bayesian control; more general objectives such as risk-reward or dynamic risk; development of numerical solutions and computationally feasible algorithms to solve these problems. Furthermore, in light of recent advancements in reinforcement learning methodology, it would be advantageous to investigate the aforementioned problems within the context of Markov Decision Processes (MDP). Unlike the scalar case, however, there exists no straightforward mapping between MDP and our framework.

2 Preliminaries

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, $T \in \mathbb{N}$ be a fixed time horizon, and let $\mathcal{T} = \{0, 1, 2, \dots, T\}$, $\mathcal{T}' = \{0, 1, 2, \dots, T-1\}$. Consider a non-empty compact set $\Theta \subseteq \mathbb{R}^k$, which will play the role of the parameter space throughout.

On the space $(\Omega, \mathscr{F})$ we consider a random factor process $Z = \{Z_t,\ t \in \mathcal{T}\}$ taking values in some measurable space $(\mathbb{D}, \mathscr{D})$. We postulate that this process is observed, and we denote by $\mathbb{F} = (\mathscr{F}_t, t \in \mathcal{T})$ its natural filtration. In the sequel, we assume that all processes are $\mathbb{F}$-adapted. The (true) law of Z is unknown to the controller, and assumed to be generated by a probability measure[1] belonging to a parameterized family $\mathbf{Q}(\Theta) := \{\mathbb{Q}_\theta : \theta \in \Theta\}$ of probability measures on $(\Omega, \mathscr{F})$. We assume that the elements of $\mathbf{Q}(\Theta)$ are absolutely continuous with respect $\mathbb{P}$. We denote by $\mathbb{E}^\theta$, respectively $\mathbb{E}^\theta_t$, the expectation, respectively the conditional expectation given $\mathscr{F}_t$, with respect to the probability $\mathbb{Q}_\theta$. We will denote by $\mathbb{Q}_{\theta^*}$ the probability that generates the true, but unknown, law of Z, that is, $\theta^* \in \Theta$ is the unknown true parameter. In what follows all equalities, inequalities and inclusions will be understood in $\mathbb{P}$-a.s. sense.

Let $(\mathbb{A}, \mathcal{A})$ be a Polish space endowed with its Borel σ-algebra, $\mathbb{A}$ denoting the set of control values. Let $(\mathbb{S}, \mathscr{S})$ be a measurable (state) space, with $\mathbb{S}$ standing for the state space of the state process $S = (S_t)_{t \in \mathcal{T}}$. Namely, S is an $\mathbb{F}$-adapted $(\mathbb{S}, \mathscr{S})$-valued process. The set of feasible controls is determined through the set-valued mappings $U_t : \mathbb{S} \rightrightarrows \mathbb{A}$ with non-empty closed values, and with measurable graph

$$\mathrm{Gr}(U_t) := \{(s, a) \in \mathbb{S} \times \mathbb{A} : a \in U_t(s)\} \in \mathscr{S} \otimes \mathscr{A}.$$

for $t \in \mathcal{T}'$. An admissible control φ is an $\mathbb{F}$-adapted process such that $\varphi_t \in U_t(S_t)$, for all $t \in \mathcal{T}'$. We denote by $\mathfrak{A}$ the set of all admissible control processes. We further assume that the state process follows controlled dynamics

$$S_{t+1} = F(t, S_t, \varphi_t, Z_{t+1}),\ t \in \mathcal{T}',\ S_0 = s, \tag{2.1}$$

where $\varphi \in \mathfrak{A}$, and $F : \mathcal{T} \times \mathbb{S} \times \mathbb{A} \times \mathbb{D} \to \mathbb{S}$ is a jointly measurable function. We will denote by S^φ the controlled state process corresponding to the control φ. With slight abuse of notations we will also use the notation $S^{\varphi_t}_{t+1} := F(t, S_t, \varphi_t, Z_{t+1})$.

Given an $\mathscr{F}_t$-measurable state $S_t \in \mathscr{L}_t(\mathbb{S}) := L^0(\Omega, \mathscr{F}_t, \mathbb{P}; \mathbb{S})$, the set of one-step time t admissible controls is defined by $\mathfrak{A}_t(S_t) := \{\varphi_t \in \mathscr{L}_t(\mathbb{A}) \mid \varphi_t \in U_t(S_t)\ \mathbb{P}\text{-a.s.}\}$, and respectively, the set of admissible (tail) controls $\varphi = (\varphi_t, \dots, \varphi_{T-1})$ starting at time t is given by

$$\begin{aligned}
\mathfrak{A}^t(S_t) : &= \{(\varphi_t, \dots, \varphi_{T-1}) \mid \varphi_j \in \mathfrak{A}_j(S_j),\ S_{j+1} = F(j, S_j, \varphi_j, Z_{j+1}),\ j = t, \dots, T-1\} \\
&= \left\{(\varphi_t, \dots, \varphi_{T-1}) \mid \varphi_t \in \mathfrak{A}_t(S_t),\ (\varphi_{t+1}, \dots, \varphi_{T-1}) \in \mathfrak{A}^{t+1}(F(t, S_t, \varphi_t, Z_{t+1}))\right\}.
\end{aligned}$$

We remark that by Kuratowski-Ryll-Nardzewski Theorem [AB06, Chapter 18] or by a version of Aumann Theorem [AB06, Theorem 18.26] on measurable selectors, the sets $\mathfrak{A}_t(\cdot), \mathfrak{A}^t(\cdot), \mathfrak{A}$ are not empty.

We now consider a bounded, measurable, multi-loss/multi-cost function $\ell : \mathbb{S} \to \mathbb{R}^d$. Hence, for each admissible strategy $\varphi \in \mathfrak{A}$ and unknown parameter $\theta \in \Theta$, the expected loss $\mathbb{E}^\theta\left[\ell(S^\varphi_T)\right]$ is multivariate, and to optimize this expected loss, one has to clarify how we compare (random) vectors, see Section 2.1. Note that, in light of the above assumptions, the multi-loss $\ell(S^\varphi_T) \in \mathscr{L}_T(\mathbb{R}^d) := L^\infty(\Omega, \mathscr{F}_T, \mathbb{P}; \mathbb{R}^d)$ is a bounded random vector.

[1] By the law of Z we mean the push-forward measure $\mathbb{Q}_\theta \circ Z^{-1}$.

Our aim is to apply the robust approach to model uncertainty for a problem with multiple objectives. That is, we want to understand and solve a problem of the form

$$“\inf_{\varphi\in\mathfrak{A}}\sup_{\theta\in\Theta}\mathbb{E}^{\theta}\left[\ell(S_T^{\varphi})\right]”. \tag{2.2}$$

To interpret this problem meaningfully, we first need to clarify how we understand suprema over vectors that we address in Section 2.2.

2.1 Vector preorder and partial order

An order relation on $\mathbb{R}^d$ is a *vector preorder* if it is reflexive ($x\preceq x$), transitive ($x\preceq y$ and $y\preceq z$ imply $x\preceq z$) and compatible with the vector space structure of $\mathbb{R}^d$ ($x\preceq y$ implies $x+z\preceq y+z$ and $\alpha x\preceq\alpha y$ for $\alpha\geq 0$ and $z\in\mathbb{R}^d$). A *vector partial order* is a vector preorder that is additionally antisymmetric ($x\preceq y$ and $y\preceq x$ imply $x=y$).

An order relation $\preceq$ generates an *ordering cone* $C_{\preceq}=\{x\in\mathbb{R}^d:0\preceq x\}$. Similarly, a set $C\subseteq\mathbb{R}^d$ can be used to define an order relation $\preceq_C$ as $x\preceq_C y$ whenever $y-x\in C$. Recall that a cone C is called *pointed* if $C\cap(-C)=\{0\}$ and it is called *solid* if it has a non-empty interior with respect to norm $\|x\|=\max_i|x_i|$ in $\mathbb{R}^d$. Next we summarize some well-known facts about order relations and their ordering cones:

- For any set $C\subseteq\mathbb{R}^d$ it holds $C_{\preceq_C}=C$. Moreover, for an arbitrary order relation that is compatible with the vector space structure, the order relation $\preceq_{C_{\preceq}}$ coincides with $\preceq$.
- If $\preceq$ is a vector preorder, then $C_{\preceq}$ is a convex cone with $0\in C_{\preceq}$. If $\preceq$ is a vector partial order, then $C_{\preceq}$ is a pointed convex cone.
- If C is a convex cone with $0\in C$, then $\preceq_C$ is a vector preorder. If C is a pointed convex cone, then $\preceq_C$ is a vector partial order.

Throughout the paper, we make the following standing assumption:

Assumption 1. *The fixed cone $C\subseteq\mathbb{R}^d$ is*[2] *convex, solid and closed with respect to $\|\cdot\|$. The corresponding vector preorder is denoted simply by $\preceq$. Compatibility with the vector space is assumed throughout and $\preceq$ is referred to as preoder for the sake of brevity.*

Preorder (or partial order) $\preceq$ on $\mathbb{R}^d$ naturally extends to a preorder (or partial order) on the space $\mathscr{L}_t(\mathbb{R}^d):=L^\infty(\Omega,\mathscr{F}_t,\mathbb{P};\mathbb{R}^d)$ of $\mathscr{F}_t$-measurable and essentially bounded (classes of equivalence of) random variables with values in $\mathbb{R}^d$, endowed with[3] the norm $\|X\|:=\max_i\|X_i\|_\infty$, where $\|X_i\|_\infty$ is the usual ess sup norm in $L^\infty(\Omega,\mathscr{F}_t,\mathbb{P})$. In the following, $\mathscr{L}_t(C)\subseteq\mathscr{L}_t(\mathbb{R}^d)$ denotes the set of $\mathscr{F}_t$-measurable random vectors with values in C. We use $\mathscr{L}_t(C)$ to define the order relation $\preceq^t$, for $X,Y\in\mathscr{L}_t(\mathbb{R}^d)$ we say that $X\preceq^t Y$ if and only if $Y-X\in\mathscr{L}_t(C)$, which implies $X(\omega)\preceq Y(\omega)$, $\mathbb{P}$-a.s. It is easily verified that properties of $\preceq$ are inherited by the order relation $\preceq^t$ (in the $\mathbb{P}$-a.s. and conditional sense) and $\preceq^t$ is a preorder (or partial order) on $\mathscr{L}_t(\mathbb{R}^d)$.

In order to compare sets of vectors, a given preorder on the space of vectors can be used to define various order relations on the space of sets, see e.g. [HHL+15]. Within this work we focus

[2]These assumptions are standard in vector optimization. Some of the technical auxiliary results presented in this paper, such as Lemmas 2.1, 2.5, and 2.6, remain valid under weaker assumptions. Lemma 2.1 is true, for example, when the cone is not solid.

[3]With slight abuse of notations, we use the same notations for the norm in $\mathbb{R}^d$ and $\mathscr{L}_t(\mathbb{R}^d)$.

on two canonical set order relations generated by the vector preorder $\preceq$ and its ordering cone C. For sets $A, B \subseteq \mathbb{R}^d$ we define

$$A \preccurlyeq B \quad \Leftrightarrow \quad B \subseteq A + C,$$
and
$$A \curlyeqprec B \quad \Leftrightarrow \quad A \subseteq B - C.$$

Similarly, we define the order relations $\preccurlyeq^t$ and $\curlyeqprec^t$ for subsets of $\mathscr{L}_t(\mathbb{R}^d)$. We remark that minimization under the preorder $\preceq$ is closely related to the set order relation $\preccurlyeq$; see Subsection 2.3.

Lemma 2.1. *The preorder is compatible with the conditional expectation operator, that is, if $X, Y \in \mathscr{L}_{t+1}(\mathbb{R}^d)$ and $X \preceq^{t+1} Y$, then $\mathbb{E}_t^\theta[X] \preceq^t \mathbb{E}_t^\theta[Y]$ for any $\theta \in \Theta$.*

Proof. Given $X \preceq^{t+1} Y$, the random variable $Z := Y - X \in C$, $\mathbb{P}$-a.s. Thus, for any fixed $\theta \in \Theta$ it holds $Z \in C$, $\mathbb{Q}_\theta$-a.s., that is the set $\Omega^Z := \{Z \notin C\}$ has $\mathbb{Q}_\theta$ measure zero.

Let $C^+ := \{z \in \mathbb{R}^d : \langle z, c\rangle \geq 0, \forall c \in C\}$ be the dual cone of C. Then, for any $b \in C^+$ and all $\omega \notin \Omega^Z$ we have $\langle b, Z(\omega)\rangle \geq 0$. Hence, for all $\omega \notin \Omega^Z$ and any $b \in C^+$ we have $\langle b, \mathbb{E}_t^\theta[Z](\omega)\rangle = \mathbb{E}_t^\theta[\langle b, Z\rangle](\omega) \geq 0$ and, thus, $\mathbb{E}_t^\theta[Z] \in C^{++}$ $\mathbb{P}-$a.s. Since C is convex and closed, $C^{++} = C$, and hence $\mathbb{E}_t^\theta[Y] - \mathbb{E}_t^\theta[X] = \mathbb{E}_t^\theta[Z] \in \mathscr{L}_t(C)$. The proof is complete. □

One of the simplest and most widely used partial orders on $\mathbb{R}^d$ is generated by the ordering cone $C = \mathbb{R}^d_+ = \{x \in \mathbb{R}^d : x_i \geq 0, i = 1, \dots, d\}$. We denote this partial order simply by $\leq$ since it corresponds to coordinate-wise comparison of vectors. Section 5 is dedicated to this partial order.

2.2 Supremum of vectors

In simplified terms, the robust control approach is based on the following idea: for each choice of a control determine the worst-case model (the supremum in (2.2)), then optimize across all feasible controls (the infimum in (2.2)). For loss functions ℓ taking real values, there is no ambiguity in interpreting supremum and infimum. In contrast, for multi-loss problems, with ℓ taking values in $\mathbb{R}^d$, one first needs to clarify how to interpret a supremum of a collection of (random) vectors. In this regards, vector (and set) optimization literature, including (deterministic) robust multi-objective optimization, provides two canonical ways to think about the supremum:

- *as a vector* with the property of being the lowest upper bound of the collection of vectors. We will refer to this vector-valued notion of the supremum as *ideal point* supremum, properly defined below. As shown later, even for finite collection of vectors, often no member of the collection is an upper bound for all other members. In terms of model uncertainty, a vector-valued supremum (assuming it exists) will often not correspond to any market model, but rather be an idealized scenario, hence the name ideal point. This approach corresponds to the robustified objective proposed in [KL12, FW14] for static robust multi-objective problems.

- *as sets* using set optimization framework. An upper bound would be a so-called lower set; if one considers finite collections, then a set of maximizers can also serve the purpose. Recall that order relations extended to set relations allow us to compare sets; minimizing a set-valued supremum would yield a set optimization problem. This is the basis of the second approach to robust multi-objective optimization explored in [EIS14, IKK+14].

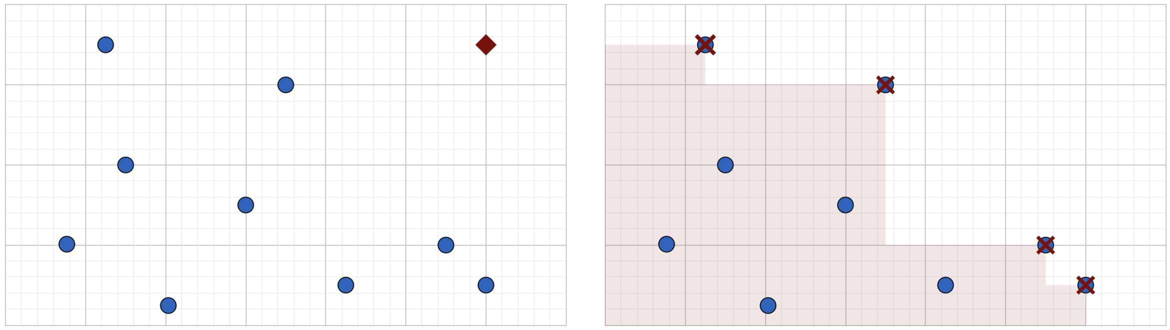

Figure 1: The blue circles represent a set of vectors $A \subseteq \mathbb{R}^2$ over which we take supremum with respect to coordinate-wise order. Left: Red square depicts the *vector* smallest upper bound for the collection A. Right: Red crossed circles represent the maximal (non-dominated) elements, while the shaded area is the lower set they generate that is a *set* smallest upper bound for A.

These two approaches to define supremum are illustrated in Figure 1. Among the two, the ideal point approach is generally more conservative. For further comparisons of the two approaches for *static* robust vector optimization we refer the reader to surveys [IS15, WD16].

In this study, our focus is on the first approach – a vector-valued, ideal point, supremum. We chose to start with this approach due to its tractability, leading to a vector or multi-objective optimization problem, rather than set-valued optimization problems. The second alternative, where the robust problem results in a set optimization problem, is a challenge that we defer to future work.

Definition 2.2. Given a collection of (random) vectors $\{X^\theta : \theta \in \Theta\} \subseteq \mathscr{L}_t(\mathbb{R}^d)$, *a supremum* with respect to the preorder $\preceq^t$ is a (random) vector $V \in \mathscr{L}_t(\mathbb{R}^d)$ satisfying:

(i) $X^\theta \preceq^t V$, for all $\theta \in \Theta$;

(ii) If for some $A \in \mathscr{L}_t(\mathbb{R}^d)$ we have $X^\theta \preceq^t A$ for all $\theta \in \Theta$, then $V \preceq^t A$.

The set of all such suprema will be denoted by $\underset{\theta\in\Theta}{\text{V-sup}}^t X^\theta$.

We note that in contrast to a supremum of a subset of $\mathbb{R}$, a supremum in the sense of Definition 2.2 may not exist, and when it exists it may be not unique. See Example 2.4 below that illustrates this. Later in this section, we provide sufficient conditions for the existence of a suprema as well as for its uniqueness.

In what follows, we will denote by $\underset{\theta\in\Theta}{\text{v-sup}}^t X^\theta$ one arbitrary chosen element from the set $\underset{\theta\in\Theta}{\text{V-sup}}^t X^\theta$. Unless otherwise stated, the obtained results do not depend on the choice of the suprema, see Lemma 2.3. If the time instance t is clear, then we may simply write $\text{v-sup}_{\theta\in\Theta}$ and $\text{V-sup}_{\theta\in\Theta}$.

The two properties in the Definition 2.2 correspond to the usual requirement that the *supremum* is (i) an *upper bound* and (ii) the *smallest* among all upper bounds in $\mathscr{L}_t(\mathbb{R}^d)$. Conditions (i) and (ii) can be equivalently stated in terms of the ordering cone $\mathscr{L}_t(C)$, respectively, as follows:

(i) $V \in \bigcap_{\theta\in\Theta} (X^\theta + \mathscr{L}_t(C))$,

(ii) $\bigcap_{\theta\in\Theta} (X^\theta + \mathscr{L}_t(C)) \subseteq V + \mathscr{L}_t(C)$.

Lemma 2.3. *For any* $V, W \in \underset{\theta\in\Theta}{\text{V-sup}}^t X^\theta$*, we have that*

(a) $V \preceq^t W$ *and* $W \preceq^t V$*;*

(b) $V \pm \mathscr{L}_t(C) = W \pm \mathscr{L}_t(C)$*, and thus* $V \pm \mathscr{L}_t(C) = \underset{\theta\in\Theta}{\text{V-sup}}^t X^\theta \pm \mathscr{L}_t(C)$*;*

(c) If $\preceq^t$ *is a vector partial order, then* $\text{v-sup}_\theta X^\theta$ *with respect to* $\preceq^t$ *is unique* $\mathbb{P}$*-a.s, if it exists.*

(d) Assume that $\underset{\theta\in\Theta}{\text{V-sup}}^t X^\theta \neq \emptyset$*. Then for all* $b \in \mathscr{L}_t(\mathbb{R}^d)$ *it holds* $\underset{\theta\in\Theta}{\text{V-sup}}^t (X^\theta + b) = \left(\underset{\theta\in\Theta}{\text{V-sup}}^t X^\theta\right) + b$*.*

Proof. (a) follows from the definition of v-sup. By (a) and definition of the preorder we get $V + \mathscr{L}_t(C) = W + \mathscr{L}_t(C)$. Consequently $V - \mathscr{L}_t(C) = W - \mathscr{L}_t(C)$. If $\preceq^t$ is a vector partial order, then $\preceq^t$ is antisymmetric which implies (c). Finally, (d) follows from compatibility of the preorder $\preceq^t$ with the vector structure. $\square$

Similarly, one could consider the notion of *ideal point* infimum defined as $\text{v-inf}_{\theta\in\Theta} X^\theta := -\text{v-sup}_{\theta\in\Theta}(-X^\theta)$. However, in this work we use the *ideal point* supremum only for the robust part (inner sup in (2.2)), while we treat the outer inf in (2.2) as minimization of a vector-valued objective function in the standard sense of vector optimization, see Section 2.3.

Example 2.4. i) To illustrate that an ideal point supremum *may not exist*, consider the partial order generated by the convex, pointed, polyhedral ordering cone

$$C = \text{cone}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix}, \begin{pmatrix}0.75\\0.75\\1\end{pmatrix}, \begin{pmatrix}-0.75\\0.75\\1\end{pmatrix}, \begin{pmatrix}-0.75\\-0.75\\1\end{pmatrix}, \begin{pmatrix}0.75\\-0.75\\1\end{pmatrix}\right\}$$

alongside two points, $X = (0,0,0)^T$ and $Y = (0.25, -0.25, 0)^T$. It can be verified directly that the intersection $(X+C)\cap(Y+C)$ is a polyhedron with three vertices rather than a shifted cone; see Figure 2 for a graphical illustration. Therefore, a supremum of vectors X and Y with respect to $\preceq_C$ does not exist.

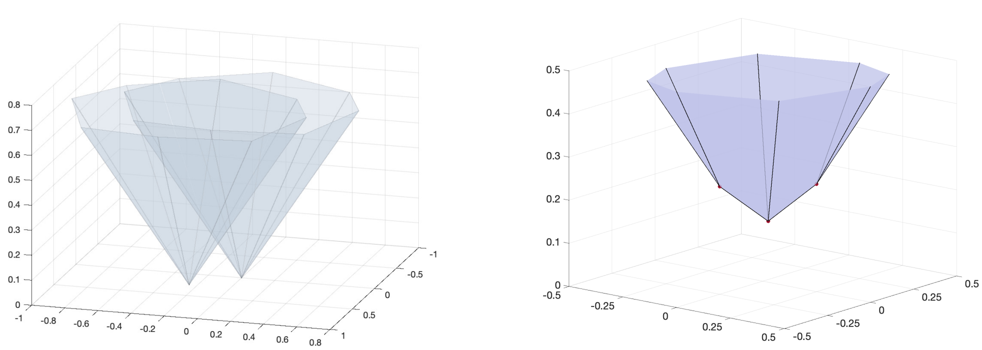

Figure 2: Shifted cones $X+C$ and $Y+C$ (left) and their intersection $(X+C)\cap(Y+C)$ (right).

ii) To illustrate that the ideal point supremum *may not be unique*, consider a preorder generated by a (convex and polyhedral) ordering cone that is a half-space,

$$C = \Big\{ x \in \mathbb{R}^d \ : \ \langle w, x \rangle \geq 0 \Big\}$$

for some $w \in \mathbb{R}^d \setminus \{0\}$. Then for any bounded collection $\{X^\theta \in \mathbb{R}^d \ : \ \theta \in \Theta\}$ all points on the hyperplane

$$\Big\{ x \in \mathbb{R}^d \ : \ \langle w, x \rangle = \sup\{\langle w, X^\theta \rangle : \theta \in \Theta\} \Big\}$$

are suprema of $\{X^\theta \ : \ \theta \in \Theta\}$ with respect to the preorder generated by C. □

Next we study the existence of an ideal point supremum.

Lemma 2.5. *Assume that $\{X^\theta : \theta \in \Theta\} \subseteq \mathscr{L}_t(\mathbb{R}^d)$ is bounded. Then $\bigcap_{\theta \in \Theta} (X^\theta + \mathscr{L}_t(C)) \neq \emptyset$.*

Proof. For a constant $r > 0$, let $B_r := \{x \in \mathbb{R}^d \ : \ \|x\| \leq r\}$ and $\mathcal{B}_r = \{X \in \mathscr{L}_t(\mathbb{R}^d) \ : \ \|X\| \leq r \ \ \mathbb{P}\text{-a.s.}\}$. Since C is solid, there exists $k \in \operatorname{Int} C$ and without loss of generality we assume that $k - B_1 \subseteq C$. Subsequently, for the constant random vector $K(\omega) := k$, $\omega \in \Omega$, it holds $K - \mathcal{B}_1 \subseteq \mathscr{L}_t(C)$ as well as $rK - \mathcal{B}_r \subseteq \mathscr{L}_t(C)$, since C is a cone.

Since $\{X^\theta : \theta \in \Theta\}$ is bounded in $\mathscr{L}_t(\mathbb{R}^d)$, there exists a constant $\bar{r} > 0$ such that for all $\theta \in \Theta$, $X^\theta \in \mathcal{B}_{\bar{r}}$, which combined with $\bar{r}K - \mathcal{B}_{\bar{r}} \subseteq \mathscr{L}_t(C)$, imply that for every $\theta \in \Theta$, we get $\bar{r}K \in X^\theta + \mathscr{L}_t(C)$. This concludes the proof. □

We recall that a cone C is non-trivial, if $\{0\} \subsetneq C \subsetneq \mathbb{R}^d$, and it is polyhedral if it is equal to the intersection of finite number of halfspaces, where a half space is a set of the form $H^+_\alpha(s) = \{z \in \mathbb{R}^d : \langle a, z \rangle \geq \alpha\}$ for some $a \in \mathbb{R}^d \setminus \{0\}$ and $\alpha \in \mathbb{R}$.

Proposition 2.6. *Suppose that the ordering cone C is non-trivial and it is a polyhedral such that the dual cone $C^+ := \{b \in \mathbb{R}^d : \langle b, c \rangle \geq 0, \ \forall c \in C\}$ is generated by some linearly independent vectors $b_1, \ldots, b_k \in \mathbb{R}^d \setminus \{0\}$. Additionally, assume that $X^\theta(\omega)$ is jointly measurable on $\Theta \times \Omega$, the function $\theta \mapsto X^\theta(\omega)$ is lower semi-continuous (l.s.c.) for a.s. $\omega \in \Omega$, and $\{X^\theta : \theta \in \Theta\}$ is bounded in $\mathscr{L}_t(\mathbb{R}^d)$. Then, there exists* $\text{v-sup}_{\theta \in \Theta} X^\theta$.

Proof. For a deterministic $a \in \mathbb{R}^d \setminus \{0\}$ and random scalar $\alpha \in \mathscr{L}_t(\mathbb{R})$ we denote by $\mathcal{H}^+_\alpha(a) = \{Z \in \mathscr{L}_t(\mathbb{R}^d) : \langle a, Z \rangle \geq \alpha, \ \mathbb{P}\text{-a.s.}\}$ and $\mathcal{H}^=_\alpha(a) = \{Z \in \mathscr{L}_t(\mathbb{R}^d) : \langle a, Z \rangle = \alpha, \ \mathbb{P}\text{-a.s.}\}$ the half-space and hyperplane in the space of random vectors $\mathscr{L}_t(\mathbb{R}^d)$, respectively. In view of the assumptions, the polyhedral convex cone C and the induced cone $\mathscr{L}_t(C)$ are of the form

$$C = \bigcap_{i=1}^{k} H^+_0(b_i), \qquad \mathscr{L}_t(C) = \bigcap_{i=1}^{k} \mathcal{H}^+_0(b_i) = \{Z \in \mathscr{L}_t(\mathbb{R}^d) \ : \ \langle b_i, Z \rangle \geq 0, \ \mathbb{P}\text{-a.s.}, \ i = 1, \ldots, k\}.$$

We recall that all (in)equalities and inclusions hold in the $\mathbb{P}$-a.s. sense. First, for a fixed $\theta \in \Theta$ we show that $X^\theta + \mathscr{L}_t(C) = \bigcap_{i=1,\ldots,k} \mathcal{H}^+_{\langle b_i, X^\theta \rangle}(b_i)$. Take arbitrary $\bar{c} \in \mathscr{L}_t(C)$, for $i = 1, \ldots, k$ it holds $\langle b_i, X^\theta + \bar{c} \rangle \geq \langle b_i, X^\theta \rangle$, which shows that $X^\theta + \mathscr{L}_t(C) \subseteq \bigcap_{i=1,\ldots,k} \mathcal{H}^+_{\langle b_i, X^\theta \rangle}(b_i)$. Now take $Z \in \bigcap_{i=1,\ldots,k} \mathcal{H}^+_{\langle b_i, X^\theta \rangle}(b_i)$, then for $i = 1, \ldots, k$ it holds $\langle b_i, Z - X^\theta \rangle \geq 0$, so $Z - X^\theta \in \mathscr{L}_t(C)$ and $X^\theta + \mathscr{L}_t(C) \supseteq \bigcap_{i=1,\ldots,k} \mathcal{H}^+_{\langle b_i, X^\theta \rangle}(b_i)$.

Secondly, in view of Lemma 2.5, $\bigcap_{\theta\in\Theta}(X^\theta+\mathscr{L}_t(C))$ is non-empty. Let $Z\in\bigcap_{\theta\in\Theta}(X^\theta+\mathscr{L}_t(C))$. Then, for every fixed $\theta\in\Theta$,

$$\langle b_i, Z\rangle \geq \langle b_i, X^\theta\rangle \quad \mathbb{P}-\text{a.s.}$$

Note that there exists an at most countable subset $\Theta_0\subseteq\Theta$, which is also dense in Θ. Since Θ_0 is countable,

$$\langle b_i, Z\rangle \geq \langle b_i, X^\theta\rangle,\ \theta\in\Theta_0, \quad \mathbb{P}-\text{a.s.} \tag{2.3}$$

For any $\theta\in\Theta$, there exists $\{\theta_n\}_{n\in\mathbb{N}}\subseteq\Theta_0$, such that $\theta_n\to\theta$. Since X^θ is l.s.c. in θ, so is $\langle b_i, X^\theta\rangle$, and thus by (2.3), with probability one we have

$$\langle b_i, Z\rangle \geq \langle b_i, X^\theta\rangle, \quad \theta\in\Theta.$$

Since Θ is compact, $\langle b_i, X^\theta\rangle$ is l.s.c in θ and jointly measurable, then, in view of Theorem [BS78, Theorem 7.32] there exist $\alpha_i\in\mathscr{L}_t(\mathbb{R})$ such that $\alpha_i(\omega):=\sup\left\{\langle b_i, X^\theta(\omega)\rangle : \theta\in\Theta\right\}$. Note that the finiteness of α_i is guaranteed by boundedness of the set $\{X^\theta:\theta\in\Theta\}$. Consequently,

$$\langle b_i, Z\rangle \geq \alpha_i,$$

with probability one. This implies that

$$\bigcap_{\theta\in\Theta}(X^\theta+\mathscr{L}_t(C))\subseteq\bigcap_{i=1}^k\mathcal{H}^+_{\alpha_i}(b_i). \tag{2.4}$$

Converse inclusion is clear.

Third, we consider the intersection of corresponding hyperplanes,

$$\bigcap_{i=1}^k\mathcal{H}^=_{\alpha_i}(b_i)=\left\{Z\in\mathscr{L}_t(\mathbb{R}^d)\quad|\quad\langle b_i, Z\rangle=\alpha_i,\ \mathbb{P}\text{-a.s.},\quad i=1,\dots,k\right\}.$$

Since $b_1,\dots,b_k$ are linearly independent, the system of linear equations $\langle b_i, Z(\omega)\rangle=\alpha_i(\omega),\quad i=1,\dots,k$ has an $\mathscr{F}_t$-measurable solution for arbitrary $\mathscr{F}_t$-measurable right-hand side $(\alpha_1,\dots,\alpha_k)$. Therefore, $\bigcap\limits_{i=1,\dots,k}\mathcal{H}^=_{\alpha_i}(b_i)\neq\emptyset$, $\mathbb{P}$-a.s..

Fourth, we show that for arbitrary $V\in\bigcap\limits_{i=1,\dots,k}\mathcal{H}^=_{\alpha_i}(b_i)$ we have $\bigcap\limits_{i=1,\dots,k}\mathcal{H}^+_{\alpha_i}(b_i)=V+\mathscr{L}_t(C)$. Take any $Z\in\bigcap\limits_{i=1,\dots,k}\mathcal{H}^+_{\alpha_i}(b_i)$, then for all $i=1,\dots,k$ it holds $\langle b_i, Z-V\rangle\geq 0$, so $Z-V\in\mathscr{L}_t(C)$ and $\bigcap\limits_{i=1,\dots,k}\mathcal{H}^+_{\alpha_i}(b_i)\subseteq V+\mathscr{L}_t(C)$. On the other hand, for any $\bar{c}\in\mathscr{L}_t(C)$ we have for all $i=1,\dots,k$ that $\langle b_i, V+\bar{c}\rangle\geq\langle b_i, V\rangle=\alpha_i$, so $\bigcap\limits_{i=1,\dots,k}\mathcal{H}^+_{\alpha_i}(b_i)\supseteq V+\mathscr{L}_t(C)$.

Therefore, there exists a random vector $V\in\mathscr{L}_t(\mathbb{R}^d)$ satisfying

$$V\in\bigcap_{\theta\in\Theta}(X^\theta+\mathscr{L}_t(C))\quad\text{ and }\quad\bigcap_{\theta\in\Theta}(X^\theta+\mathscr{L}_t(C))\subseteq V+\mathscr{L}_t(C),$$

which means that V is a supremum of $\{X^\theta\}_{\theta\in\Theta}$ with respect to $\preceq^t$ in the sense of Definition 2.2. $\square$

Remark 2.7.

(i) In the bi-objective case $d = 2$, each solid, closed, convex and non-trivial cone $C \subseteq \mathbb{R}^2$ is either a half-space or an intersection of two half-spaces. Therefore, under the additional assumptions of Proposition 2.6 on family X^θ, the ideal point supremum exists in the bi-objective case. Moreover, for any pair $x, y \in \mathbb{R}^2$ we can explicitly compute the vector $z \in \mathbb{R}^2$ such that $(x+C) \cap (y+C) = z + C$ through a solution of two linear equations. By analogous arguments, v-sup exists and can be explicitly computed for any finite collection of two-dimensional random vectors.

(ii) If the ordering cone is $C = \mathbb{R}^d_+$, which corresponds to the component-wise partial order $\leq$, then the supremum exists, is unique and can be computed explicitly in a component-wise fashion, for any bounded family of vectors; see Section 5 for more details.

(iii) For a *partial* order satisfying the assumptions of Proposition 2.6 the order relation can be reduced to a component-wise partial order. Indeed, store the linearly independent vectors b_i generating the dual cone in a matrix B. In case of a partial order, the ordering cone C is pointed, and therefore the matrix B is square and invertible. It remains to note that

$$x \preceq_C y \quad \Leftrightarrow \quad Bx \leq By.$$

(iv) Proposition 2.6 is a sufficient, but not a necessary condition for existence of ideal point supremum. For example, consider the non-solid cone $C = \{x \in \mathbb{R}^3 : x_1, x_2 \geq 0, x_3 = 0\}$. For a compact collection of vectors $\{x^\theta : \theta \in \Theta\} \subseteq \{x \in \mathbb{R}^3 : x_3 = \alpha\}$ from the hyperplane $H^=_\alpha(e_3)$ a vector-valued supremum exists.

(v) The geometry of the ordering cone C determines how conservative is the ideal point supremum. When C is a halfspace, there are elements in $\{X^\theta : \theta \in \Theta\}$ infinitesimally close to the set of suprema, see Example 2.4(ii). In contrast, if C is 'narrow', all points in $\{X^\theta : \theta \in \Theta\}$ can be far from the set of the suprema, in which case the ideal point approach is quite conservative.

The remainder of this section provides additional properties of v-sup needed in the sequel, such as monotonicity and equivalent characterizations.

Lemma 2.8. *Assume that $\{X^\theta\}_{\theta\in\Theta}, \{Y^\theta\}_{\theta\in\Theta} \subseteq \mathscr{L}_t(\mathbb{R}^d)$, and $X^\theta \preceq^t Y^\theta$ for all $\theta \in \Theta$, then $\operatorname*{v-sup}_{\theta\in\Theta} X^\theta \preceq^t \operatorname*{v-sup}_{\theta\in\Theta} Y^\theta$, assuming that both suprema exist.*

Proof. For any $\theta \in \Theta$, by transitivity $X^\theta \preceq^t Y^\theta$ and $Y^\theta \preceq^t \operatorname*{v-sup}_{\theta\in\Theta} Y^\theta$ imply $X^\theta \preceq^t \operatorname*{v-sup}_{\theta\in\Theta} Y^\theta$. Thus, $\operatorname*{v-sup}_{\theta\in\Theta} Y^\theta$ is an upper bound of $\{X^\theta\}_{\theta\in\Theta} \subseteq \mathscr{L}_t(\mathbb{R}^d)$. The result follows from Definition 2.2(ii). □

Lemma 2.9. *Let $\{X^\theta : \theta \in \Theta\} \subseteq \mathscr{L}_{t+1}(\mathbb{R}^d)$, and assume that $V, W \in \operatorname*{V-sup}^{t+1}_{\theta\in\Theta} X^\theta$. Then*

$$\operatorname*{V-sup}_{\theta\in\Theta}{}^t \mathbb{E}^\theta_t[V] + \mathscr{L}_t(C) = \operatorname*{V-sup}_{\theta\in\Theta}{}^t \mathbb{E}^\theta_t[W] + \mathscr{L}_t(C),$$

assuming both sets are not empty.

Proof. By Lemma 2.3(a), Lemma 2.1 and Lemma 2.8,

$$\operatorname*{v-sup}_{\theta\in\Theta}{}^t \mathbb{E}^\theta_t[V] \preceq^t \operatorname*{v-sup}_{\theta\in\Theta}{}^t \mathbb{E}^\theta_t[W], \quad \text{and} \quad \operatorname*{v-sup}_{\theta\in\Theta}{}^t \mathbb{E}^\theta_t[W] \preceq^t \operatorname*{v-sup}_{\theta\in\Theta}{}^t \mathbb{E}^\theta_t[V],$$

and note that all elements exist. Consequently, by the definition of $\preceq^t$, we get

$$\underset{\theta\in\Theta}{\text{v-sup}}^t\mathbb{E}_t^\theta[V] + \mathscr{L}_t(C) = \underset{\theta\in\Theta}{\text{v-sup}}^t\mathbb{E}_t^\theta[W] + \mathscr{L}_t(C),$$

and in view of Lemma 2.3(b) the proof is complete. □

2.3 Vector optimization problem

In this section, we discuss how to interpret the optimization with respect to feasible controls $\varphi \in \mathfrak{A}$ (the infimum in (2.2)) in the context of our multi-objective robust stochastic control problem. Since our choice for supremum across models was a vector, the infimum across controls will be viewed as a vector optimization problem (VOP).

A VOP, for a fixed time $t \in \mathcal{T}'$, is a problem of the form

$$\text{minimize} \;\; F(\varphi) \quad \text{with respect to} \quad \preceq^t \quad \text{subject to} \;\; \varphi \in \mathfrak{A}^t, \tag{VOP}$$

where $\mathfrak{A}^t$ is a non-empty feasible set, $F : \mathfrak{A}^t \to \mathscr{L}_t(\mathbb{R}^d)$ is a vector-valued mapping, and $\preceq^t$ is a fixed preorder on $\mathscr{L}_t(\mathbb{R}^d)$ with the corresponding ordering cone $\mathscr{L}_t(C)$. The *image of the feasible set* $\mathfrak{A}^t$ is the set $F[\mathfrak{A}^t] = \{F(\varphi) \, : \, \varphi \in \mathfrak{A}^t\}$ and the *upper image* of (VOP) is $\mathcal{P} = \text{cl}\,(F[\mathfrak{A}^t] + \mathscr{L}_t(C))$. These two sets, the image of the feasible set and the upper image, play important roles within vector optimization theory and will serve as two candidates for value function in subsequent sections.

A control $\bar{\varphi} \in \mathfrak{A}^t$ is called a *minimizer* of (VOP) if $(F(\bar{\varphi}) - \mathscr{L}_t(C) \setminus \{0\}) \cap F[\mathfrak{A}^t] = \emptyset$, and it is called a *weak minimizer* of (VOP) if $(F(\bar{\varphi}) - \text{Int}(\mathscr{L}_t(C))) \cap F[\mathfrak{A}^t] = \emptyset$. In various streams of literature, a (weak) minimizer is also referred to as a *(weakly) Pareto optimal point* or an *(weakly) efficient point*. The set of all images of (weak) minimizers of (VOP) is often known as the *(weak) Pareto frontier* or the *(weakly) efficient frontier*.

In numerous vector optimization problems the Pareto frontier is non-trivial containing mutually incomparable points with respect to considered preorder. This highlights the challenges in defining what a *solution* of (VOP) should be. In certain contexts, solving a VOP can be understood as finding one Pareto optimal point, while in other contexts, one aims to identify (or approximate) the entire Pareto frontier or the upper image. Consequently, it does not come as a surprise that the current literature contains several concepts of solutions for VOPs based on case by case relevant properties (e.g. bounded versus unbounded, convex versus non-convex) of the VOPs themselves. We refer the reader to [Löh11, Jah04], and references therein, for more details.

In this work, our main objective is to establish a dynamic programming principle for vector-valued robust stochastic problems. As such, we will be interested in an appropriately defined 'value function', for which, as it turns out, the relevant objects are the image of the feasible set and the upper image, while the particular notion of the solutions of VOP is less important.

Remark 2.10. We note that for the space of bounded random vectors $\mathscr{L}_t(\mathbb{R}^d) := L^\infty(\Omega, \mathscr{F}_t, \mathbb{P}; \mathbb{R}^d)$, the interior of the cone $\mathscr{L}_t(C)$ is nonempty. However, it is known that this is not true for the space of p-integrable random vectors when $p \in [1, \infty)$. In these cases, the notion of weak minimality has to be adjusted, for example by using quasi-interior approach [BL92]. For the sake of tractability, we omit these technical details in this manuscript.

3 Weak set-valued Bellman's principle

In what follows, we make the following standing assumption on the preorder $\preceq$:

Assumption 2. *Given any $t \in \mathcal{T}$ and any $\mathscr{F}_T$-measurable state random variable S, the collections of random vectors $\{\mathbb{E}_t^\theta[\ell(S)] : \theta \in \Theta\} \subseteq \mathscr{L}_t(\mathbb{R}^d)$ admits a supremum* v-sup *compatible with $\preceq^t$.*

The fulfillment of this assumption can be guaranteed, for example, by assuming that the stochastic kernels corresponding to the conditional expectation are appropriately measurable and l.s.c. in θ, and the cone C satisfies the assumptions of Proposition 2.6.

Now, we can formulate time t robust vector optimization problem

$$\begin{aligned} &\text{minimize} \quad && \operatorname*{v\text{-}sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[\ell(S_T^\varphi)] \quad \text{with respect to } \preceq^t \\ &\text{subject to} \quad && \varphi \in \mathfrak{A}^t(S_t), \end{aligned} \tag{3.1}$$

where, we recall that $\mathfrak{A}^t(S_t)$ is the set of admissible controls $\varphi = (\varphi_t, \dots, \varphi_{T-1})$ starting at time t, and given an $\mathscr{F}_t$-measurable state $S_t \in \mathscr{L}_t(\mathbb{S})$. Given the *ideal point* (vector-valued) understanding of a supremum, problem (3.1) is a well-defined (stochastic) vector optimization problem.

We are interested in dynamic programming principle and time consistency property of the family of these robust problems. Dynamic programming principle in the context of non-standard problems without model uncertainty has been explored in the literature for multi-objective stochastic control problems in [KR21], Nash equilibria of non-zero sum games in [FRZ22] and the computation of multivariate risk measures in [FR17]. The developed Bellman's principle obtained for these non-standard problems is based on a set-valued notion of a value function. In line with this, we explore and establish the dynamic programming principle for problem (3.1) in terms of two closely related set-valued candidates for a value function: the image set and the upper image.

The first set-valued candidate for the value function that we take is the image of the feasible set of (3.1), which should be viewed as a time-indexed family of set-valued function $\mathscr{L}_t(\mathbb{S}) \ni S_t \mapsto \mathcal{V}_t(S_t) \subseteq \mathscr{L}_t(\mathbb{R}^d)$ for $t \in \mathcal{T}$. For the terminal time T and state $S_T \in \mathscr{L}_T(\mathbb{S})$ we define $\mathcal{V}_T(S_T) := \ell(S_T)$, and for all other times $t = T-1, \dots, 0$ and any states $S_t \in \mathscr{L}_t(\mathbb{S})$ we set

$$\begin{aligned} \mathcal{V}_t(S_t) &:= \left\{ \operatorname*{v\text{-}sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[\ell(S_T)] \ : \ \varphi \in \mathfrak{A}^t(S_t),\ S_{j+1} = F(j, S_j, \varphi_j, Z_{j+1}),\ j = t, \dots, T-1 \right\} \\ &= \bigcup_{\varphi\in\mathfrak{A}^t(S_t)} \operatorname*{v\text{-}sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[\ell(S_T^\varphi)]. \end{aligned}$$

Note that the value function consists of suprema over all time t feasible strategies. Since $\mathfrak{A} \neq \emptyset$, the set $\mathcal{V}_t(S_t)$ is also not empty. We recall that if v-sup is not unique, we take (arbitrary) one of them. The main results of this paper, the weak and the strong set-valued Bellman's principle, correspond to set relations of value function(s). In view of Lemma 2.3 these results do not depend on the choice of the v-sup.

Remark 3.1. We note that the value function attains as value a *set of random vectors* $\mathcal{V}_t(S_t) \subseteq \mathscr{L}_t(\mathbb{R}^r)$. Therefore, an inclusion $X \in \mathcal{V}_t(S_t)$ implies existence of (at least one) corresponding admissible control $\varphi^X \in \mathfrak{A}^t(S_t)$. Such admissible control is an $\mathbb{F}$-adapted process, so in particular ω-measurable.

The second considered set-valued candidate for the value function is the upper image defined for all $t \in \mathcal{T}$ and all states $S_t \in \mathscr{L}_t(\mathbb{S})$ as

$$\mathcal{P}_t(S_t) := \operatorname{cl}\left(\mathcal{V}_t(S_t) + \mathscr{L}_t(C)\right).$$

Observe that in view of Lemma 2.3 the upper image $\mathcal{P}$ does not depend on the particular v-sup included in $\mathcal{V}$. The set relation $\preccurlyeq$ is the natural order relation for an upper image, an element of a complete lattice of upper sets. The upper image $\mathcal{P}_t$ can be interpreted as a (set-optimization) infimum of the vector optimization problem, see [KR21, Section 5.1]. The advantage of working with $\mathcal{V}$ lies in its interpretability and the possibility to establish results under both set relations $\preccurlyeq$ and $\curlyeqprec$.

By analogy to the scalar case, we aim to derive Bellman-type backward recursions, starting with the value function $\mathcal{V}$. Let us consider one-step optimization problems where the value (the image set) of the problem at time $t+1$ determines the constraints of the problem at time t, for which we define the auxiliary set-valued function $\mathcal{W}$ defined recursively as

$$\mathcal{W}_T(S_T) := \ell(S_T),$$
$$\mathcal{W}_t(S_t) := \left\{ \operatorname*{v-sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[X] \ : \ \varphi_t \in \mathfrak{A}_t(S_t),\ X \in \mathcal{W}_{t+1}(F(t,S_t,\varphi_t,Z_{t+1})) \right\}, \quad t = T-1,\dots,0.$$

Observe that $\mathcal{W}_t(S_t)$ is the image of the feasible set of the one-step optimization problem

$$\begin{aligned} \text{minimize} \quad & \operatorname*{v-sup}_{\theta\in\Theta}{}^t \mathbb{E}_t^\theta[X] \quad \text{with respect to } \preceq^t \\ \text{subject to} \quad & \varphi_t \in \mathfrak{A}_t(S_t),\ X \in \mathcal{W}_{t+1}(F(t,S_t,\varphi_t,Z_{t+1})). \end{aligned} \tag{3.2}$$

Additionally, we define an upper image of the above one-step problem as

$$\mathcal{R}_t(S_t) := \operatorname{cl}\left(\mathcal{W}_t(S_t) + \mathscr{L}_t(C)\right)$$

for all $t \in \mathcal{T}'$ and all states S_t.

We next derive order relation(s) between $\mathcal{V}_t$ and $\mathcal{W}_t$, as well as between $\mathcal{P}_t$ and $\mathcal{R}_t$, which represent Bellman-type relations for the robust problem (3.1) and its one-step version (3.2). In presence of model uncertainty, the value function $\mathcal{V}$ and the set-valued function $\mathcal{W}$ do not need to coincide as illustrated in Example 6.1, Section 6.

Theorem 3.2. *For every time $t \in \mathcal{T}'$ and every state S_t we have*

$$\mathcal{V}_t(S_t) \preccurlyeq^t \mathcal{W}_t(S_t), \qquad \textit{i.e.} \qquad \mathcal{W}_t(S_t) \subseteq \mathcal{V}_t(S_t) + \mathscr{L}_t(C), \tag{3.3}$$

as well as

$$\mathcal{V}_t(S_t) \curlyeqprec^t \mathcal{W}_t(S_t), \qquad \textit{i.e.} \qquad \mathcal{V}_t(S_t) \subseteq \mathcal{W}_t(S_t) - \mathscr{L}_t(C). \tag{3.4}$$

Moreover, for every $t \in \mathcal{T}'$ and every state S_t it holds

$$\mathcal{P}_t(S_t) \preccurlyeq^t \mathcal{R}_t(S_t) \qquad \textit{i.e.} \qquad \mathcal{R}_t(S_t) \subseteq \mathcal{P}_t(S_t). \tag{3.5}$$

The proof of Theorem 3.2 follows after an intermediate Lemma 3.3 below.

We recall that the set order relation $\preccurlyeq$ is naturally connected to vector minimization problems, and, especially, to the upper image of a vector optimization problems. Therefore, the relations (3.3) and (3.5) represent *the weak set-valued Bellman's principle* for the problem (3.1). In contrast, the order relation $\curlyeqprec$ would be appropriate for vector maximization problems. Nevertheless, for the sake of completeness, we also provide relation (3.4) as an additional bounds between $\mathcal{V}_t(\cdot)$ and $\mathcal{W}_t(\cdot)$.

In order to prove Theorem 3.2, we introduce an auxiliary one-step problem at time $t \in \mathcal{T}'$, by assuming that that the value function $\mathcal{V}_{t+1}$ for time $t+1$ is known,

$$\begin{aligned} \text{minimize} \quad & \operatorname*{v-sup}_{\theta\in\Theta}{}^t \mathbb{E}_t^\theta[X], \quad \text{with respect to } \preceq^t \\ \text{subject to:} \quad & \varphi_t \in \mathfrak{A}_t(S_t),\ X \in \mathcal{V}_{t+1}(F(t,S_t,\varphi_t,Z_{t+1})). \end{aligned} \tag{3.6}$$

We establish a connection between the image set of the feasible set of the auxiliary problem (3.6) and the value function $\mathcal{V}_t$.

Lemma 3.3. *For every time $t = 0, 1, \dots, T-1$ and every state S_t it holds*

$$\left\{ \underset{\theta\in\Theta}{\text{v-sup}}^t \mathbb{E}_t^\theta[X] \ : \ \varphi_t \in \mathfrak{A}_t(S_t),\ X \in \mathcal{V}_{t+1}(F(t, S_t, \varphi_t, Z_{t+1})) \right\} \ \subseteq \ \mathcal{V}_t(S_t) + \mathscr{L}_t(C), \tag{3.7}$$

as well as

$$\mathcal{V}_t(S_t) \ \subseteq \ \left\{ \underset{\theta\in\Theta}{\text{v-sup}}^t \mathbb{E}_t^\theta[X] \ : \ \varphi_t \in \mathfrak{A}_t(S_t),\ X \in \mathcal{V}_{t+1}(F(t, S_t, \varphi_t, Z_{t+1})) \right\} - \mathscr{L}_t(C) \tag{3.8}$$

Proof. We start by noting that in view of Lemma 2.3(b)

$$\mathcal{V}_t(S_t) + \mathscr{L}_t(C) = \bigcup_{\varphi\in\mathfrak{A}^t(S_t)} \left(\underset{\theta\in\Theta}{\text{V-sup}}\, \mathbb{E}_t^\theta[\ell(S_T^\varphi)] + \mathscr{L}_t(C) \right), \tag{3.9}$$

and the right-hand side does not depend on the choice of v-sup.

To prove (3.7), take arbitrary $\varphi_t \in \mathfrak{A}_t(S_t)$ and $X \in \mathcal{V}_{t+1}(S_{t+1}^{\varphi_t})$, where $S_{t+1}^{\varphi_t} = F(t, S_t, \varphi_t, Z_{t+1})$. For $X \in \mathcal{V}_{t+1}(S_{t+1}^{\varphi_t})$ there must exist a corresponding feasible control $\varphi^X \in \mathfrak{A}^{t+1}(S_{t+1}^{\varphi_t})$. Thus, a control $\bar{\varphi} := (\varphi_t, \varphi_{t+1}^X, \dots, \varphi_{T-1}^X) \in \mathfrak{A}^t(S_t)$ generates the terminal state $S_T^{\bar{\varphi}} = S_T^{\varphi^X}$.

By the definition of the supremum, since $X \in \underset{\theta\in\Theta}{\text{V-sup}}^{t+1} \mathbb{E}_{t+1}^\theta[\ell(S_T^{\bar{\varphi}})]$, we have

$$\mathbb{E}_{t+1}^\theta \left[\ell(S_T^{\bar{\varphi}}) \right] \preceq^{t+1} X, \quad \forall \theta \in \Theta.$$

Using Lemma 2.1 and Lemma 2.8, it follows

$$\underset{\theta\in\Theta}{\text{v-sup}}\, \mathbb{E}_t^\theta \left[\ell(S_T^{\bar{\varphi}}) \right] \preceq^t \underset{\theta\in\Theta}{\text{v-sup}}\, \mathbb{E}_t^\theta \left[X \right].$$

Consequently, since $\bar{\varphi} \in \mathfrak{A}^t(S_t)$ and by (3.9), we deduce

$$\underset{\theta\in\Theta}{\text{v-sup}}\, \mathbb{E}_t^\theta \left[X \right] \in \underset{\theta\in\Theta}{\text{v-sup}}\, \mathbb{E}_t^\theta \left[\ell(S_T^{\bar{\varphi}}) \right] + \mathscr{L}_t(C) \subseteq \mathcal{V}_t(S_t) + \mathscr{L}_t(C).$$

Hence, (3.7) is proved.

To prove (3.8), let $V \in \mathcal{V}_t(S_t)$. Then $V = \underset{\theta\in\Theta}{\text{v-sup}}^t \mathbb{E}_t^\theta[\ell(S_T^\varphi)]$ for some feasible control $\varphi = (\varphi_t, \tilde{\varphi}) \in \mathfrak{A}^t(S_t)$, where $\tilde{\varphi} := (\varphi_{t+1}, \dots, \varphi_{T-1}) \in \mathfrak{A}^{t+1}(S_{t+1}^{\varphi_t})$, with assumed notation $S_{t+1}^{\varphi_t} = F(t, S_t, \varphi_t, Z_{t+1})$. Note that $S_T^\varphi = S_T^{\tilde{\varphi}}$.

Let us denote by $X := \underset{\theta\in\Theta}{\text{v-sup}}^{t+1} \mathbb{E}_{t+1}^\theta[\ell(S_T^{\tilde{\varphi}})] \in \mathcal{V}_{t+1}(S_{t+1}^{\varphi_t})$. By the definition of the supremum $\mathbb{E}_{t+1}^\theta[\ell(S_T^\varphi)] \preceq^{t+1} X$ for all $\theta \in \Theta$. Hence, by Lemma 2.1 and Lemma 2.8

$$\underset{\theta\in\Theta}{\text{v-sup}}^t \mathbb{E}_t^\theta \left[\ell(S_T^\varphi) \right] \preceq^t \underset{\theta\in\Theta}{\text{v-sup}}^t \mathbb{E}_t^\theta \left[X \right],$$

which implies that

$$\begin{aligned} \underset{\theta\in\Theta}{\text{v-sup}}^t \mathbb{E}_t^\theta \left[\ell(S_T^\varphi) \right] &\in \left\{ \underset{\theta\in\Theta}{\text{v-sup}}^t \mathbb{E}_t^\theta \left[X \right] \right\} - \mathscr{L}_t(C) \\ &\subseteq \ \left\{ \underset{\theta\in\Theta}{\text{v-sup}}^t \mathbb{E}_t^\theta[X] \ : \ \varphi_t \in \mathfrak{A}_t(S_t),\ X \in \mathcal{V}_{t+1}(F(t, S_t, \varphi_t, Z_{t+1})) \right\} - \mathscr{L}_t(C), \end{aligned}$$

and (3.8) is proved. The proof is complete. $\square$

Proof of Theorem 3.2. We will prove set relations (3.3) and (3.4) by backward induction. By definition $\mathcal{V}_T(S_T) = \ell(S_T) = \mathcal{W}_T(S_T)$.

First, we prove (3.3). As an induction hypothesis assume that the inclusion $\mathcal{W}_{t+1}(S_{t+1}) \subseteq \mathcal{V}_{t+1}(S_{t+1}) + \mathscr{L}_{t+1}(C)$ holds for any state S_{t+1}. Let $W \in \mathcal{W}_t(S_t)$. Then $W \in \operatorname*{V\text{-}sup}^t_{\theta\in\Theta} \mathbb{E}_t^\theta[X]$ for some $\varphi_t \in \mathfrak{A}_t(S_t)$ and $X \in \mathcal{W}_{t+1}(S_{t+1}^{\varphi_t})$, where $S_{t+1}^{\varphi_t} = F(t, S_t, \varphi_t, Z_{t+1})$.

According to the induction hypothesis it holds $X \in \mathcal{W}_{t+1}(S_{t+1}^{\varphi_t}) \subseteq \mathcal{V}_{t+1}(S_{t+1}^{\varphi_t}) + \mathscr{L}_t(C)$, therefore, there exists $V \in \mathcal{V}_{t+1}(S_{t+1}^{\varphi_t})$ such that $V \preceq^t X$. Then, by Assumption 2 and Lemmas 2.1, 2.8 and 2.3(a) it follows

$$\operatorname*{v\text{-}sup}_{\theta\in\Theta}{}^t\mathbb{E}_t^\theta[V] \preceq^t \operatorname*{v\text{-}sup}_{\theta\in\Theta}{}^t\mathbb{E}_t^\theta[X] \preceq^t W.$$

This and the inclusion (3.7) from Lemma 3.3 yield

$$\begin{aligned}
W &\in \operatorname*{v\text{-}sup}_{\theta\in\Theta}{}^t\mathbb{E}_t^\theta[V] + \mathscr{L}_t(C) \\
&\subseteq \left\{ \operatorname*{v\text{-}sup}_{\theta\in\Theta}{}^t\mathbb{E}_t^\theta[V] \ : \ \varphi_t \in \mathfrak{A}_t(S_t),\ V \in \mathcal{V}_{t+1}(F(t, S_t, \varphi_t, Z_{t+1})) \right\} + \mathscr{L}_t(C) \\
&\subseteq \ \mathcal{V}_t(S_t) + \mathscr{L}_t(C),
\end{aligned}$$

which proves (3.3).

Moreover, (3.5) follows from (3.3) and the definition of the upper images.

Let us next prove (3.4). As an induction hypothesis assume $\mathcal{V}_{t+1}(S_{t+1}) \subseteq \mathcal{W}_{t+1}(S_{t+1}) - \mathscr{L}_{t+1}(C)$ holds for all states S_{t+1}. Let $V \in \mathcal{V}_t(S_t)$ and recall the set relation (3.8) proven in Lemma 3.3. According to (3.8), for given $V \in \mathcal{V}_t(S_t)$ there exists $\varphi_t \in \mathfrak{A}_t(S_t)$ and $X \in \mathcal{V}_{t+1}(S_{t+1}^{\varphi_t})$, with $S_{t+1}^{\varphi_t} = F(t, S_t, \varphi_t, Z_{t+1})$, such that

$$V \preceq^t \operatorname*{v\text{-}sup}_{\theta\in\Theta}{}^t\mathbb{E}_t^\theta[X].$$

On the other hand, according to the induction hypothesis, for given $X \in \mathcal{V}_{t+1}(S_{t+1}^{\varphi_t}) \subseteq \mathcal{W}_{t+1}(S_{t+1}^{\varphi_t}) - \mathscr{L}_{t+1}(C)$ there exists $W \in \mathcal{W}_{t+1}(S_{t+1}^{\varphi_t})$ such that $X \preceq^{t+1} W$. From Assumption 2 and Lemmas 2.1 and 2.8 it follows

$$V \preceq^t \operatorname*{v\text{-}sup}_{\theta\in\Theta}{}^t\mathbb{E}_t^\theta[X] \preceq^t \operatorname*{v\text{-}sup}_{\theta\in\Theta}{}^t\mathbb{E}_t^\theta[W].$$

Therefore,

$$V \in \left\{ \operatorname*{v\text{-}sup}_{\theta\in\Theta}{}^t\mathbb{E}_t^\theta[W] \right\} - \mathscr{L}_t(C) \subseteq \mathcal{W}_t(S_t) - \mathscr{L}_t(C),$$

which proves (3.4). ◻

We conclude this section with results on recurrence of the sets $\mathcal{P}$ and $\mathcal{R}$.

Proposition 3.4. *For all times $t \in \mathcal{T}'$ and all states S_t it holds*

$$\mathcal{R}_t(S_t) = \operatorname{cl}\left\{ \operatorname*{v\text{-}sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[X] \ : \ \varphi_t \in \mathfrak{A}_t(S_t),\ X \in \mathcal{R}_{t+1}(F(t, S_t, \varphi_t, Z_{t+1})) \right\}, \tag{3.10}$$

as well as

$$\left\{ \operatorname*{v\text{-}sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[X] \ : \ \varphi_t \in \mathfrak{A}_t(S_t),\ X \in \mathcal{P}_{t+1}(F(t, S_t, \varphi_t, Z_{t+1})) \right\} \subseteq \mathcal{P}_t(S_t). \tag{3.11}$$

The proof of Proposition 3.4 is given after Lemma 3.5.

The one-step inclusion (3.11) of the upper image $\mathcal{P}$ is a version of the auxiliary result (3.7) from Lemma 3.3 for upper image. Let us shortly discuss (3.10). The auxiliary upper image $\mathcal{R}$ arises from the one-step problem (3.2), which is constructed from the image-valued function $\mathcal{W}$. When considering upper image as a value function, we should use time $t+1$ upper image as an 'input' to construct the upper image at time t. Relation (3.10) shows that this is indeed the case for $\mathcal{R}$.

Lemma 3.5. *Consider $t \in \mathcal{T}'$, $\mathcal{V} \subseteq \mathscr{L}_t(\mathbb{R}^d)$, $k \in \operatorname{Int}(C)$ and set $K(\omega) = k$ for all $\omega \in \Omega$. Then it holds*

$$\operatorname{cl}(\mathcal{V} + \mathscr{L}_t(C)) + K \subseteq \mathcal{V} + \mathscr{L}_t(C).$$

Proof. We denote $B_\epsilon := \left\{x \in \mathbb{R}^d \,:\, \|x\| \leq \epsilon\right\}$ and $\mathcal{B}_\epsilon := \left\{X \in \mathscr{L}_t(\mathbb{R}^d) \,:\, \|X\| \leq \epsilon \quad \mathbb{P}\text{-a.s.}\right\}$. For $k \in \operatorname{Int}(C)$ there exists $\epsilon > 0$ such that $k + B_\epsilon \subseteq C$ and, therefore, $K + \mathcal{B}_\epsilon \subseteq \mathscr{L}_t(C)$.

Consider $X \in \operatorname{cl}(\mathcal{V} + \mathscr{L}_t(C))$. Then, there exists $\{X^n\}_{n\in\mathbb{N}} \subseteq \mathcal{V} + \mathscr{L}_t(C)$ such that for any $\epsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$ it holds

$$\|X - X^n\|_\infty \leq \epsilon \quad \mathbb{P}\text{-a.s.}, \qquad \text{i.e.} \qquad X - X^n \in \mathcal{B}_\epsilon.$$

Therefore, for any $n \geq N_0$ we obtain

$$X + K = X^n + (K + (X - X^n)) \subseteq (\mathcal{V} + \mathscr{L}_t(C)) + \mathscr{L}_t(C) \subseteq \mathcal{V} + \mathscr{L}_t(C).$$

□

Proof of Proposition 3.4. Since C is a solid cone, there exists $k \in \operatorname{Int}(C)$ and we define a constant random vector $K(\omega) := k$ for all $\omega \in \Omega$.

First, we address (3.10). We claim that

$$\mathcal{W}_t(S_t) + \mathscr{L}_t(C) = \left\{ \operatorname*{V-sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[X] \,:\, \varphi_t \in \mathfrak{A}_t(S_t),\, X \in \mathcal{W}_{t+1}(F(t, S_t, \varphi_t, Z_{t+1})) + \mathscr{L}_{t+1}(C) \right\}. \tag{3.12}$$

Indeed, the inclusion $\subseteq$ follows from the definition of $\mathcal{W}$, the inclusion $\mathscr{L}_t(C) \subseteq \mathscr{L}_{t+1}(C)$, and Lemma 2.3(d). For the opposite inclusion, take arbitrary $\varphi_t \in \mathfrak{A}_t(S_t), X \in \mathcal{W}_{t+1}(F(t, S_t, \varphi_t, Z_{t+1}))$ and $c \in \mathscr{L}_{t+1}(C)$. Then, $X \preceq^{t+1} X + c$, and in view of Lemmas 2.1 and 2.8, we have $\operatorname*{v-sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[X] \preceq^t \operatorname*{v-sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[X + c]$, which gives the desired inclusion, and (3.12) is proved.

From the definition of $\mathcal{R}$ and (3.12) we obtain the inclusion

$$\begin{aligned} \mathcal{R}_t(S_t) &= \operatorname{cl}\left\{ \operatorname*{v-sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[X] \;:\; \varphi_t \in \mathfrak{A}_t(S_t),\; X \in \mathcal{W}_{t+1}(F(t, S_t, \varphi_t, Z_{t+1})) + \mathscr{L}_{t+1}(C) \right\} \\ &\subseteq \operatorname{cl}\left\{ \operatorname*{v-sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[X] \;:\; \varphi_t \in \mathfrak{A}_t(S_t),\; X \in \mathcal{R}_{t+1}(F(t, S_t, \varphi_t, Z_{t+1})) \right\}. \end{aligned}$$

To prove the opposite inclusion, consider an arbitrary element of the right-hand side of (3.10) without the closure, which corresponds to some $\varphi_t \in \mathfrak{A}_t(S_t)$ and $X \in \mathcal{R}_{t+1}(S_{t+1}^{\varphi_t})$, where $S_{t+1}^{\varphi_t} = F(t, S_t, \varphi_t, Z_{t+1})$. For all $n \in \mathbb{N}$ we define

$$X^n := X + \frac{1}{n}K \in \mathcal{R}_{t+1}(S_{t+1}^{\varphi_t}) + \frac{1}{n}K \subseteq \mathcal{W}_{t+1}(S_{t+1}^{\varphi_t}) + \mathscr{L}_{t+1}(C),$$

where the last inclusion follows by Lemma 3.5. Then, in view of Lemma 2.3(d) and (3.12), for all $n \in \mathbb{N}$ it holds

$$\operatorname*{v\text{-}sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[X] + \frac{1}{n}K = \operatorname*{v\text{-}sup}_{\theta\in\Theta} \mathbb{E}_t^\theta\left[X + \frac{1}{n}K\right] \in \mathcal{W}_t(S_t) + \mathscr{L}_t(C).$$

Therefore,

$$\operatorname*{v\text{-}sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[X] = \lim_{n\to\infty} \operatorname*{v\text{-}sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[X] + \frac{1}{n}K \in \operatorname{cl}(\mathcal{W}(S_t) + \mathscr{L}_t(C)) = \mathcal{R}_t(S_t),$$

which proves the inclusion $\supseteq$ in (3.10).

Now we prove (3.11). Observe that by (3.7), combined with Lemmas 2.1 and 2.8, it follows

$$\left\{\operatorname*{v\text{-}sup^t}_{\theta\in\Theta} \mathbb{E}_t^\theta[X] \ : \ \varphi_t \in \mathfrak{A}_t(S_t),\ X \in \mathcal{V}_{t+1}(F(t, S_t, \varphi_t, Z_{t+1})) + \mathscr{L}_{t+1}(C)\right\} \subseteq \mathcal{V}_t(S_t) + \mathscr{L}_t(C). \tag{3.13}$$

Consider an arbitrary element from the left-hand side of (3.11), which corresponds to some $\varphi_t \in \mathfrak{A}_t(S_t)$ and $X \in \mathcal{P}_{t+1}(S_{t+1}^{\varphi_t})$, where $S_{t+1}^{\varphi_t} = F(t, S_t, \varphi_t, Z_{t+1})$. For all $n \in \mathbb{N}$ we define

$$X^n := X + \frac{1}{n}K \in \mathcal{P}_{t+1}(S_{t+1}^{\varphi_t}) + \frac{1}{n}K \subseteq \mathcal{V}_{t+1}(S_{t+1}^{\varphi_t}) + \mathscr{L}_{t+1}(C),$$

where the last inclusion follows by Lemma 3.5. Then, in view of Lemma 2.3(d) and (3.13), for all $n \in \mathbb{N}$ it holds

$$\operatorname*{v\text{-}sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[X] + \frac{1}{n}K = \operatorname*{v\text{-}sup}_{\theta\in\Theta} \mathbb{E}_t^\theta\left[X + \frac{1}{n}K\right] \in \mathcal{V}_t(S_t) + \mathscr{L}_t(C).$$

Therefore,

$$\operatorname*{v\text{-}sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[X] = \lim_{n\to\infty} \operatorname*{v\text{-}sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[X] + \frac{1}{n}K \in \operatorname{cl}(\mathcal{V}(S_t) + \mathscr{L}_t(C)) = \mathcal{P}_t(S_t)$$

and (3.11) is proved. □

4 Strong set-valued Bellman's principle

It is known from [KR21] that for problem (3.1) without model uncertainty, a stronger form of the set-valued Bellman's principle than Theorems 3.2 holds. We establish such result for the robust problem (3.1) by assuming additionally that the family of models satisfies the rectangularity property with respect to the order relation introduced next.

Definition 4.1. We say that the family of models Θ is $\preceq$*-rectangular* if for all times $t = 0, 1, \dots, T-1$ and all random vectors $X \in \mathscr{L}_T(\mathbb{R}^d)$ it holds

$$\operatorname*{v\text{-}sup^t}_{\theta\in\Theta} \mathbb{E}_t^\theta\left[\operatorname*{v\text{-}sup^{t+1}}_{\theta\in\Theta} \mathbb{E}_{t+1}^\theta[X]\right] \preceq^t \operatorname*{v\text{-}sup^t}_{\theta\in\Theta} \mathbb{E}_t^\theta[X]. \tag{4.1}$$

Remark 4.2.

(i) Definition 4.1 is well-posed, in the sense that (4.1) does not depend on the choices of v-sup's. Indeed, in view of Lemma 2.9, (4.1) is invariant with respect to the choice of v-sup^{t+1}. On the other hand, by Lemma 2.3(a) and transitivity of the $\preceq^t$, we have that (4.1) does not depend on the choices of v-sup^t.

(ii) By transitivity and Lemma 2.8, $\preceq$-rectangularity given by (4.1) also implies the following nested form of this property

$$\underset{\theta\in\Theta}{\text{v-sup}}^t\mathbb{E}_t^\theta\left[\underset{\theta\in\Theta}{\text{v-sup}}^{t+1}\mathbb{E}_{t+1}^\theta\left[\dots\underset{\theta\in\Theta}{\text{v-sup}}^{T-1}\mathbb{E}_{T-1}^\theta[X]\dots\right]\right]\preceq^t\underset{\theta\in\Theta}{\text{v-sup}}^t\mathbb{E}_t^\theta[X].$$

A corresponding nested formulation was used in [Sha16] to define rectangularity property in the context of one-dimensional random variables and standard suprema, see next section.

(iii) For a general preorder $\preceq$, constructing a $\preceq$-rectangular family of models as well as verifying $\preceq$-rectangularity property remain challenging problems. In [Sha16], the author studies these questions for one-dimensional random variables, and provides a recursive construction of rectangular sets of probability measures. In Section 5 we show that a similar constructions implies $\leq$-rectangularity in $\mathbb{R}^d$, which allows to explore the component-wise setting further.

(iv) Using properties of suprema, we always have

$$\underset{\theta\in\Theta}{\text{v-sup}}^t\mathbb{E}_t^\theta[X]\preceq^t\underset{\theta\in\Theta}{\text{v-sup}}^t\mathbb{E}_t^\theta\left[\underset{\theta\in\Theta}{\text{v-sup}}^{t+1}\mathbb{E}_{t+1}^\theta[X]\right].$$

Hence, if $\preceq$ is a partial order (i.e. also antisymmetric), then the induced preorders $\preceq^t$ are also antisymmetric and thus partial orders, and, therefore, the $\preceq$-rectangularity property takes the form

$$\underset{\theta\in\Theta}{\text{v-sup}}^t\mathbb{E}_t^\theta[X]=\underset{\theta\in\Theta}{\text{v-sup}}^t\mathbb{E}_t^\theta\left[\underset{\theta\in\Theta}{\text{v-sup}}^{t+1}\mathbb{E}_{t+1}^\theta[X]\right].\tag{4.2}$$

The following Theorem gives order relation between the value function $\mathcal{V}$ and the auxiliary function $\mathcal{W}$ assuming the rectangularity property. This represents the strong set-valued Bellman's principle for the robust control problem (3.1) when interpreting value function in terms of the image of the feasible set. Theorem 4.4 provides the corresponding result for upper images.

Theorem 4.3. *Assume that the family of models Θ is $\preceq$-rectangular. Then for every time $t\in\mathcal{T}$ and every state S_t we have*

$$\mathcal{W}_t(S_t)\preccurlyeq^t\mathcal{V}_t(S_t),\qquad i.e.\qquad \mathcal{V}_t(S_t)\subseteq\mathcal{W}_t(S_t)+\mathscr{L}_t(C),$$

as well as

$$\mathcal{W}_t(S_t)\preccurlyeq^t\mathcal{V}_t(S_t),\qquad i.e.\qquad \mathcal{W}_t(S_t)\subseteq\mathcal{V}_t(S_t)-\mathscr{L}_t(C).$$

Proof. We prove both set relations by a backward induction. Recall that at time T it holds $\mathcal{W}_T(S_T)=\mathcal{V}_T(S_T)$ by definition.

Let us prove that $\mathcal{V}_t(S_t)\subseteq\mathcal{W}_t(S_t)+\mathscr{L}_t(C)$. As an induction hypothesis assume $\mathcal{V}_{t+1}(S_{t+1})\subseteq\mathcal{W}_{t+1}(S_{t+1})+\mathscr{L}_{t+1}(C)$ holds across all states S_{t+1}. Take an arbitrary $V\in\mathcal{V}_t(S_t)$. Then, there exists $(\varphi_t,\tilde\varphi)\in\mathfrak{A}^t(S_t)$ such that $V\in\underset{\theta\in\Theta}{\text{V-sup}}^t\mathbb{E}_t^\theta[\ell(S_T^\varphi)]$, where $\tilde\varphi=(\varphi_{t+1},\dots,\varphi_{T-1})\in\mathfrak{A}^{t+1}(S_{t+1}^{\varphi_t})$ with assumed notation $S_{t+1}^{\varphi_t}=F(t,S_t,\varphi_t,Z_{t+1})$. Note that $S_T^\varphi=S_T^{\tilde\varphi}$. Hence, $\underset{\theta\in\Theta}{\text{v-sup}}^{t+1}\mathbb{E}_{t+1}^\theta[\ell(S_T^{\tilde\varphi})]\in\mathcal{V}_{t+1}(S_{t+1}^{\varphi_t})+\mathscr{L}_t(C)$ and, by induction hypothesis and convexity of the cone $\mathscr{L}_{t+1}(C)$, we obtain

$$\underset{\theta\in\Theta}{\text{v-sup}}^{t+1}\mathbb{E}_{t+1}^\theta[\ell(S_T^{\tilde\varphi})]\in\mathcal{W}_{t+1}(S_{t+1}^{\varphi_t})+\mathscr{L}_t(C).$$

Consequently, there exists $W \in \mathcal{W}_{t+1}(S_{t+1}^{\varphi_t})$ such that

$$W \preceq^{t+1} \underset{\theta\in\Theta}{\text{v-sup}}^{t+1}\mathbb{E}_{t+1}^{\theta}[\ell(S_T^{\varphi})],$$

where we use the fact that $S_T^{\varphi} = S_T^{\tilde{\varphi}}$. By Assumption 2, Lemmas 2.1, 2.3(a), and 2.8, and the $\preceq$-rectangularity, we have

$$\underset{\theta\in\Theta}{\text{v-sup}}^{t}\mathbb{E}_t^{\theta}\left[W\right] \preceq^t \underset{\theta\in\Theta}{\text{v-sup}}^{t}\mathbb{E}_t^{\theta}\left[\underset{\theta\in\Theta}{\text{v-sup}}^{t+1}\mathbb{E}_{t+1}^{\theta}[\ell(S_T^{\varphi})]\right] \preceq^t \underset{\theta\in\Theta}{\text{v-sup}}^{t}\mathbb{E}_t^{\theta}\left[\ell(S_T^{\varphi})\right] \preceq^t V.$$

By Lemma 2.3(b) it holds $\underset{\theta\in\Theta}{\text{v-sup}}^{t}\mathbb{E}_t^{\theta}\left[W\right] \in \mathcal{W}_t(S_t) + \mathscr{L}_t(C)$. From above, by transitivity of $\preceq^t$ and convexity of the cone $\mathscr{L}_t(C)$, it follows $V \in \mathcal{W}_t(S_t) + \mathscr{L}_t(C)$, which proves the claim.

Now, let us prove $\mathcal{W}_t(S_t) \subseteq \mathcal{V}_t(S_t) - \mathscr{L}_t(C)$. As an induction hypothesis assume $\mathcal{W}_{t+1}(S_{t+1}) \subseteq \mathcal{V}_{t+1}(S_{t+1}) - \mathscr{L}_{t+1}(C)$ holds across all state S_{t+1}. Take an arbitrary element $W \in \mathcal{W}_t(S_t)$. Then $W \in \underset{\theta\in\Theta}{\text{V-sup}}^{t}\mathbb{E}_t^{\theta}[X]$ for some corresponding $\varphi_t \in \mathfrak{A}_t(S_t)$ and $X \in \mathcal{W}_{t+1}(S_{t+1}^{\varphi_t})$ with assumed notation $S_{t+1}^{\varphi_t} = F(t, S_t, \varphi_t, Z_{t+1})$. By the induction hypothesis $X \in \mathcal{V}_{t+1}(S_{t+1}^{\varphi_t}) - \mathscr{L}_{t+1}(C)$. Therefore, there exists a strategy $\tilde{\varphi} \in \mathfrak{A}^{t+1}(S_{t+1}^{\varphi_t})$ such that

$$X \preceq^{t+1} \underset{\theta\in\Theta}{\text{v-sup}}^{t+1}\mathbb{E}_{t+1}^{\theta}[\ell(S_T^{\tilde{\varphi}})].$$

Next, we consider the combined control $\varphi = (\varphi_t, \tilde{\varphi}) \in \mathfrak{A}^t(S_t)$ and note that $S_T^{\varphi} = S_T^{\tilde{\varphi}}$. By Lemmas 2.1, 2.3(a), and 2.8 and $\preceq$-rectangularity we deduce

$$W \preceq^t \underset{\theta\in\Theta}{\text{v-sup}}^{t}\mathbb{E}_t^{\theta}[X] \preceq^t \underset{\theta\in\Theta}{\text{v-sup}}^{t}\mathbb{E}_t^{\theta}\left[\underset{\theta\in\Theta}{\text{v-sup}}^{t+1}\mathbb{E}_{t+1}^{\theta}[\ell(S_T^{\varphi})]\right] \preceq^t \underset{\theta\in\Theta}{\text{v-sup}}^{t}\mathbb{E}_t^{\theta}\left[\ell(S_T^{\varphi})\right].$$

By Lemma 2.3(b) it holds $\underset{\theta\in\Theta}{\text{v-sup}}^{t}\mathbb{E}_t^{\theta}\left[\ell(S_T^{\varphi})\right] \in \mathcal{V}_t(S_t) - \mathscr{L}_t(C)$. From above, by transitivity of $\preceq^t$ and convexity of the cone $\mathscr{L}_t(C)$, it follows $W \in \mathcal{V}_t(S_t) - \mathscr{L}_t(C)$, which proves the claim. □

Theorem 4.4. *Assume that the family of models Θ is $\preceq$-rectangular. Then, for every time $t \in \mathcal{T}'$ and every state S_t we have*

$$\mathcal{P}_t(S_t) = \mathcal{R}_t(S_t),$$

which gives

$$\mathcal{P}_t(S_t) = \text{cl}\left\{\underset{\theta\in\Theta}{\text{v-sup}}\,\mathbb{E}_t^{\theta}[X] \;:\; \varphi_t \in \mathfrak{A}_t(S_t),\ X \in \mathcal{P}_{t+1}(F(t, S_t, \varphi_t, Z_{t+1}))\right\}. \tag{4.3}$$

Proof. From the weak set-valued Bellman's principle, Theorem 3.2, we know $\mathcal{P}_t(S_t) \supseteq \mathcal{R}_t(S_t)$. On the other hand, by Theorem 4.3, $\mathcal{V}_t(S_t) \subseteq \mathcal{W}_t(S_t) + \mathscr{L}_t(C)$, so $\mathcal{P}_t(S_t) \subseteq \mathcal{R}_t(S_t)$ follows. Therefore, we obtain $\mathcal{P}_t(S_t) = \mathcal{R}_t(S_t)$. Combining (3.10) from Proposition 3.4 with $\mathcal{P}_t(S_t) = \mathcal{R}_t(S_t)$ yields the remaining result. □

Observe that, assuming rectangularity property, we have $\mathcal{W}_t(S_t) \preccurlyeq^t \mathcal{V}_t(S_t)$ as well as $\mathcal{V}_t(S_t) \preccurlyeq^t \mathcal{W}_t(S_t)$ (see Theorem 3.2). Note that, in general, this does not mean that $\mathcal{V}_t(S_t) = \mathcal{W}_t(S_t)$. However, as we next show, if additionally $\preceq$ is a partial order, then the value functions $\mathcal{V}$ and $\mathcal{W}$ coincide. Similar to (4.3), we also obtain explicit recursive formulas for the image-valued value function $\mathcal{V}$.

Theorem 4.5. *Assume that the preorder $\preceq$ is antisymmetic and the family of models Θ has the $\preceq$-rectangularity property. Then, for every time $t = 0, 1, \ldots, T-1$ and every state S_t it holds*

$$\mathcal{V}_t(S_t) = \mathcal{W}_t(S_t),$$

which yields

$$\mathcal{V}_t(S_t) = \left\{ \operatorname*{v-sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[X] \ : \ \varphi_t \in \mathfrak{A}_t(S_t),\ X \in \mathcal{V}_{t+1}(F(t, S_t, \varphi_t, Z_{t+1})) \right\}.$$

Proof. Recall that in the case of a partial order, the supremum operator v-sup is uniquely defined and the $\preceq$-rectangularity property translates to (4.2). This result can also be proven by backward induction. By definition $\mathcal{V}_T(S_T) = \mathcal{W}_T(S_T)$. As an induction hypothesis assume $\mathcal{V}_{t+1}(S_{t+1}) = \mathcal{W}_{t+1}(S_{t+1})$ holds for all states S_{t+1}. Observe that $\varphi = (\varphi_t, \widetilde{\varphi}) \in \mathfrak{A}^t(S_t)$ is a feasible control if and only if $\varphi_t \in \mathfrak{A}_t(S_t)$ and $\widetilde{\varphi} \in \mathfrak{A}^{t+1}(F(t, S_t, \varphi_t, Z_{t+1}))$.

Combining the recursiveness of the set of feasible controls, uniqueness of the supremum, $\preceq$-rectangularity (4.2) and the induction hypothesis yields

$$\begin{aligned}
\mathcal{V}_t(S_t) &= \left\{ \operatorname*{v-sup}_{\theta\in\Theta}{}^t \mathbb{E}_t^\theta[\ell(S_T^\varphi)] \ : \ (\varphi_t, \widetilde{\varphi}) \in \mathfrak{A}^t(S_t) \right\} \\
&= \left\{ \operatorname*{v-sup}_{\theta\in\Theta}{}^t \mathbb{E}_t^\theta[\ell(S_T^\varphi)] \ : \ \varphi_t \in \mathfrak{A}_t(S_t),\ \widetilde{\varphi} \in \mathfrak{A}^{t+1}(F(t, S_t, \varphi_t, Z_{t+1})) \right\} \\
&= \left\{ \operatorname*{v-sup}_{\theta\in\Theta}{}^t \mathbb{E}_t^\theta\left[\operatorname*{v-sup}_{\theta\in\Theta}{}^{t+1} \mathbb{E}_{t+1}^\theta[\ell(S_T^\varphi)] \right] \ : \ \varphi_t \in \mathfrak{A}_t(S_t),\ \widetilde{\varphi} \in \mathfrak{A}^{t+1}(F(t, S_t, \varphi_t, Z_{t+1})) \right\} \\
&= \left\{ \operatorname*{v-sup}_{\theta\in\Theta}{}^t \mathbb{E}_t^\theta\left[X\right] \ : \ \varphi_t \in \mathfrak{A}_t(S_t),\ X \in \mathcal{V}_{t+1}(F(t, S_t, \varphi_t, Z_{t+1})) \right\} \\
&= \left\{ \operatorname*{v-sup}_{\theta\in\Theta}{}^t \mathbb{E}_t^\theta\left[X\right] \ : \ \varphi_t \in \mathfrak{A}_t(S_t),\ X \in \mathcal{W}_{t+1}(F(t, S_t, \varphi_t, Z_{t+1})) \right\} = \mathcal{W}_t(S_t).
\end{aligned}$$

Finally, substituting $\mathcal{V}_{t+1}(\cdot) = \mathcal{W}_{t+1}(\cdot)$ into the last line gives the remaining result. □

The dynamic programming results can be extended to problems with running costs as well as to backward recursions between time t and $t+s$. For the sake of clarity and simplicity of notations, we focus on problems with terminal cost and on one-step backward recursions. All results and their proofs can be generalized to problems with running costs and multi-step recursions. We outline here the resulting strong set-valued Bellman's principle – in terms of the upper images – assuming the rectangularity property.

For the robust problem with a running multi-cost

$$\text{minimize} \qquad \operatorname*{v-sup}_{\theta\in\Theta} \mathbb{E}_t^\theta\left[\sum_{s=t}^{T} \ell_s(S_s^\varphi) \right] \qquad \text{w.r.t.} \ \preceq^t \ \text{s.t.} \qquad \varphi \in \mathfrak{A}^t(S_t),$$

the strong set-valued Bellman's principle takes the form

$$\mathcal{P}_t(S_t) = \operatorname{cl}\left\{ \ell_t(S_t) + \operatorname*{v-sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[X] \ : \ \varphi_t \in \mathfrak{A}_t(S_t),\ X \in \mathcal{P}_{t+1}(F(t, S_t, \varphi_t, Z_{t+1})) \right\}.$$

Similarly, the s-step strong dynamic programming principle has the form

$$\mathcal{P}_t(S_t) = \operatorname{cl}\left\{ \operatorname*{v-sup}_{\theta\in\Theta} \mathbb{E}_t^\theta[X] \ : \ (\varphi_t, \ldots, \varphi_{t+s-1}) \in \mathfrak{A}_{[t,t+s)}(S_t),\ X \in \mathcal{P}_{t+s}(S_{t+s}^\varphi) \right\},$$

where $\mathfrak{A}_{[t,t+s)}(S_t)$ is the set of admissible controls on time interval $[t, t+s)$ and state S_{t+s}^φ is derived from S_t via (2.1).

4.1 On time consistency and weak minimizers

In the previous sections we derived dynamic programming principle (DPP) for the robust vector-valued problem – the weak and the strong set-valued Bellman's principles – for which we used two appropriate notions of value functions.

Time consistency of a stochastic control problem is a property known to be closely related to the validity of DPP in standard settings. In broad terms, time consistency property means that 'an optimal solution remains optimal as time passes.' The time consistency property for vector-valued problem without model uncertainty was explored in [KR21]. It was shown that the appropriate notion of an 'optimal solution' connected to the time consistency of a vector-valued problem is the weak minimizer.

Next we explore the time consistency, in the sense of weak minimizers, for the robust vector-valued problem (3.1). We show that time consistency property holds granted that the rectangularity property holds.

Theorem 4.6. *Assume that the family of models Θ is $\preceq$-rectangular. If $\varphi = (\varphi_t, \varphi_{t+1}, \dots, \varphi_{T-1}) \in \mathfrak{A}^t(S_t)$ is a weak minimizer of the robust problem at time t with initial state S_t, then $(\varphi_{t+1}, \dots, \varphi_{T-1}) \in \mathfrak{A}^{t+1}(F(t, S_t, \varphi_t, Z_{t+1}))$ is a weak minimizer of the robust problem at time $t+1$ with initial state $F(t, S_t, \varphi_t, Z_{t+1})$.*

Proof of Theorem 4.6 follows after Lemma 4.7.

Lemma 4.7. *Let $z \in \operatorname{Int}(C)$ be a fixed (non-zero) vector, and let $Y \in \operatorname{Int}(\mathscr{L}_t(C))$. Then, there exists $\varepsilon > 0$ such that $\varepsilon z \preceq^t Y$.*

Proof. Since $Y \in \operatorname{Int}(\mathscr{L}_t(C))$, then there exits $\delta > 0$ such that $\mathcal{B}_\delta(Y) \subseteq \mathscr{L}_t(C)$. Then, for $\varepsilon = \frac{1}{\|z\|}\delta > 0$, we have $-\varepsilon z \in B_\delta(0)$ and, consequently, $Y \in \varepsilon z + C$ holds $\mathbb{P}$-a.s.. □

Proof of Theorem 4.6. We denote by $S^\varphi_{t+1} := F(t, S_t, \varphi_t, Z_{t+1})$ and the corresponding terminal state generated via (2.1) as S^φ_T.

We will prove the assertion by contradiction. Assume that $\varphi \in \mathfrak{A}^t(S_t)$ is a weak minimizer at time t, and $(\varphi_{t+1}, \dots, \varphi_{T-1}) \in \mathfrak{A}^{t+1}(S^\varphi_{t+1})$ is not a weak minimizer at time $t+1$. Then there exists a feasible control $(\psi_{t+1}, \dots, \psi_{T-1}) \in \mathfrak{A}^{t+1}(S^\varphi_{t+1})$ such that

$$Y := \underset{\theta\in\Theta}{\text{v-sup}}^{t+1} \mathbb{E}^\theta_{t+1}[\ell(S^\varphi_T)] - \underset{\theta\in\Theta}{\text{v-sup}}^{t+1} \mathbb{E}^\theta_{t+1}[\ell(S^\psi_T)] \in \operatorname{Int}(\mathscr{L}_{t+1}(C)).$$

In view of Lemma 4.7, there exist $z \in \operatorname{Int}(C)$, and $\epsilon > 0$ such that $\epsilon Z \preceq^{t+1} Y$, where $Z(\omega) := z$. Consequently,

$$\underset{\theta\in\Theta}{\text{v-sup}}^{t+1} \mathbb{E}^\theta_{t+1}[\ell(S^\psi_T)] + \epsilon Z \preceq^{t+1} \underset{\theta\in\Theta}{\text{v-sup}}^{t+1} \mathbb{E}^\theta_{t+1}[\ell(S^\varphi_T)].$$

Lemmas 2.1, 2.3 and 2.8 then imply

$$\underset{\theta\in\Theta}{\text{v-sup}}^{t} \mathbb{E}^\theta_{t}\left[\underset{\theta\in\Theta}{\text{v-sup}}^{t+1} \mathbb{E}^\theta_{t+1}[\ell(S^\psi_T)]\right] + \epsilon Z \preceq^{t} \underset{\theta\in\Theta}{\text{v-sup}}^{t} \mathbb{E}^\theta_{t}\left[\underset{\theta\in\Theta}{\text{v-sup}}^{t+1} \mathbb{E}^\theta_{t+1}[\ell(S^\varphi_T)]\right]$$

and rectangularity alongside properties of supremum then gives

$$\underset{\theta\in\Theta}{\text{v-sup}}^{t} \mathbb{E}^\theta_{t}\left[\ell(S^\psi_T)\right] + \epsilon Z \preceq^{t} \underset{\theta\in\Theta}{\text{v-sup}}^{t} \mathbb{E}^\theta_{t}\left[\ell(S^\varphi_T)\right].$$

Observe that a control $\psi = (\varphi_t, \psi_{t+1}, \ldots, \psi_{T-1}) \in \mathfrak{A}^t(S_t)$ is feasible at time t with initial state S_t. Therefore, the above provides a contradiction to φ being a weak minimizer as for $\psi \in \mathfrak{A}^t(S_t)$ we have

$$\underset{\theta\in\Theta}{\text{v-sup}}^t \mathbb{E}_t^\theta\left[\ell(S_T^\varphi)\right] \in \underset{\theta\in\Theta}{\text{v-sup}}^t \mathbb{E}_t^\theta\left[\ell(S_T^\psi)\right] + \epsilon Z + \mathscr{L}_t(C).$$

Note that since $z \in \mathrm{Int}(\mathrm{C})$, then $\epsilon Z \in \mathrm{Int}(\mathscr{L}_t(C))$, and hence $\epsilon Z + \mathscr{L}_t(C) \subseteq \mathrm{Int}(\mathscr{L}_t(C))$. From the above, we obtain

$$\underset{\theta\in\Theta}{\text{v-sup}}^t \mathbb{E}_t^\theta\left[\ell(S_T^\varphi)\right] \in \underset{\theta\in\Theta}{\text{v-sup}}^t \mathbb{E}_t^\theta\left[\ell(S_T^\psi)\right] + \mathrm{Int}(\mathscr{L}_t(C)),$$

which contradicts that φ is a weak minimizer. □

With strong Bellman's principle Theorem 4.3 at hand, and the time consistency property Theorem 4.6, the proof of the existence of a weak minimizer boils down to establishing the existence of an one-step $\mathscr{F}_t$-weak minimizer. This conditional vector optimization problem is beyond the goals of the current study.

5 Dynamic programming under the component-wise partial order

This section is dedicated to a particular, but important, case of the partial order $\leq$ corresponding to the ordering cone $C = \mathbb{R}^d_+$. This partial order, as well as its extension $\leq^t$ to the space of random vectors, corresponds to the natural component-wise comparison of (random) vectors, $X \leq^t Y$ iff for all $i = 1, \ldots, d$ it holds $X_i \leq Y_i \quad \mathbb{P}$-a.s.

We recall from Definition 2.2 that the vector-valued supremum is (i) an *upper bound* and (ii) the *smallest* among upper bounds. In case of the component-wise order we not only have existence and uniqueness, but also an explicit formula.

Lemma 5.1. *Assume that $\{X^\theta : \theta \in \Theta\} \subset \mathscr{L}_t(\mathbb{R}^d)$ is bounded. Then, there exists $\mathbb{P}$-a.s. unique supremum with respect to $\leq^t$ given by*

$$V_i := \operatorname*{ess\,sup}_{\theta\in\Theta}{}^t X_i^\theta, \qquad i = 1, \ldots, d, \tag{5.1}$$

where $\operatorname{ess\,sup}^t$ denotes the conditional essential supremum.

Proof. Observe that, since random vectors X^θ are $\mathcal{F}_t$-measurable, conditional essential suprema in (5.1) coincides with essential suprema.

Since $\{X^\theta : \theta \in \Theta\}$ is a bounded set, there exists $r > 0$ such that for all $\theta \in \Theta$ and all $i = 1, \ldots, d$ it holds $X_i^\theta \geq -r$. Therefore, by [KS98, Theorem A.3] for each $i = 1, \ldots, d$ there exists essential supremum of a family of nonnegative random variables $\{X_i^\theta + r : \theta \in \Theta\}$ and, subsequently, $V_i = \operatorname{ess\,sup}_{\theta\in\Theta}{}^t X_i^\theta = \left(\operatorname{ess\,sup}_{\theta\in\Theta}{}^t (X_i^\theta + r)\right) - r$ is well-defined.

By definition of essential supremum, see [KS98, Definition A.1], for each $i = 1, \ldots, d$ it holds

(i) $\forall \theta \in \Theta \ : \ X_i^\theta \leq V_i \ \ \mathbb{P}$-a.s.

(ii) $\left(\forall \theta \in \Theta \ : \ X_i^\theta \leq Y \ \ \mathbb{P}\text{-a.s.}\right) \ \Rightarrow \ V_i \leq Y \ \ \mathbb{P}$-a.s.

This shows that the random vector $V \in \mathscr{L}_t(\mathbb{R}^d)$ satisfies Definition 2.2 for the partial order $\leq^t$. Uniqueness of the vector supremum follows from Lemma 2.3 since $\leq^t$ is a partial order. □

Remark 5.2.

(i) Observe that, given that the multi-loss function ℓ is bounded, the expected multi-loss $\{\mathbb{E}_t^\theta[\ell(S_T^\varphi)] : \theta \in \Theta\}$ is a bounded set for any control φ.

(ii) The approach of this paper is motivated by the stream of literature on (static) robust multi-objective optimization using a vector-valued (or ideal point) notion of supremum of a collection of vectors. In particular, the supremum operator (5.1) under the component-wise partial order corresponds to the *robustified objective* proposed in [FW14]. Results of this section illustrate that the notion naturally extends to the dynamic setting and measurability is preserved.

(iii) It is worth mentioning that similar statements and constructions of supremum hold true for any (solid, convex) ordering cone C in dimension $d = 2$, see Remark 2.7. Hence, the important case of problems with dual objectives can be addressed using a similar approach as outlined in this section.

Ass standing modeling assumptions remain unchanged and we study the robust multi-objective stochastic control problem,

$$\begin{aligned} &\text{minimize} && \operatorname*{v\text{-}sup}_{\theta\in\Theta}{}^{t}\, \mathbb{E}_t^\theta[\ell(S_T^\varphi)] \quad \text{with respect to } \leq^t \\ &\text{subject to} && \varphi \in \mathfrak{A}^t(S_t). \end{aligned} \tag{5.2}$$

From Section 3 we know that the family of robust multi-objective problems satisfies the weak set-valued Bellman's principle and under a rectangularity assumption on the family Θ it also satisfies strong set-valued Bellman's principle. Verifying $\preceq$-rectangularity of Θ under a general preorder $\preceq$ as well as constructing a $\preceq$-rectangular family of market models is an open challenge. However, the situation simplifies for the component-wise order, where known results on (scalar) rectangularity can be directly used.

Definition 5.3 ([Sha16])**.** We say that the family of models Θ is *m-rectangular* if for all times $t = 0, 1, \dots, T-1$ and all random variables $X \in \mathscr{L}_T(\mathbb{R})$ it holds

$$\operatorname*{ess\,sup}_{\theta\in\Theta}{}^{t}\mathbb{E}_t^\theta\left[\operatorname*{ess\,sup}_{\theta\in\Theta}{}^{t+1}\mathbb{E}_{t+1}^\theta[X]\right] = \operatorname*{ess\,sup}_{\theta\in\Theta}{}^{t}\mathbb{E}_t^\theta[X]. \tag{5.3}$$

Lemma 5.4. *The family of models Θ is m-rectangular if and only if it is $\leq$-rectangular.*

Proof. Both implications follow from the component-wise construction (5.1) of supremum with respect to $\leq^t$. ☐

Remark 5.5. [Sha16] provides an approach for recursive constructions of m-rectangular family of probability measures from marginals. The same approach can, therefore, be used to construct $\leq$-rectangular family Θ.

Let us now summarize the results derived in Section 3 within a context of coordinate-wise order. Recall that central role is played by the value function

$$\mathcal{V}_t(S_t) = \bigcup_{\varphi\in\mathfrak{A}} \operatorname*{v\text{-}sup}_{\theta\in\Theta} \mathbb{E}_t^\theta\left[\ell(S_T^\varphi)\right]$$

consisting of the suprema over all feasible strategies, alongside the recursive function $\mathcal{W}_t(S_t)$, respectively the upper images $\mathcal{P}_t(S_t) = \operatorname{cl}\left(\mathcal{V}_t(S_t) + \mathscr{L}_t(\mathbb{R}^d_+)\right)$ and $\mathcal{R}_t(S_t) = \operatorname{cl}\left(\mathcal{W}_t(S_t) + \mathscr{L}_t(\mathbb{R}^d_+)\right)$.

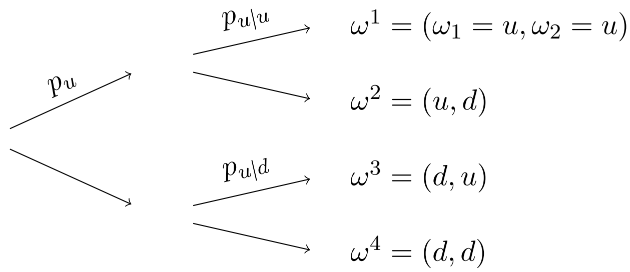

Figure 3: Tree diagram for filtered probability space in Example 6.1.

Corollary 5.6. For every time $t \in \mathcal{T}'$ and every state S_t we have

$$\mathcal{W}_t(S_t) \subseteq \mathcal{V}_t(S_t) + \mathscr{L}_t(\mathbb{R}^d_+), \qquad \mathcal{V}_t(S_t) \subseteq \mathcal{W}_t(S_t) - \mathscr{L}_t(\mathbb{R}^d_+) \qquad \text{and} \qquad \mathcal{R}_t(S_t) \subseteq \mathcal{P}_t(S_t).$$

If, additionally, the set of models Θ is m-rectangular, then for every time $t \in \mathcal{T}'$ and every state S_t

$$\mathcal{V}_t(S_t) = \left\{ \operatorname*{v-sup}_{\theta\in\Theta} \mathbb{E}^\theta_t[X] \ : \ \varphi_t \in \mathfrak{A}_t(S_t),\ X \in \mathcal{V}_{t+1}(F(t, S_t, \varphi_t, Z_{t+1})) \right\},$$
$$\mathcal{P}_t(S_t) = \operatorname{cl} \left\{ \operatorname*{v-sup}_{\theta\in\Theta} \mathbb{E}^\theta_t[X] \ : \ \varphi_t \in \mathfrak{A}_t(S_t),\ X \in \mathcal{P}_{t+1}(F(t, S_t, \varphi_t, Z_{t+1})) \right\}.$$

6 Example

In this section we present some illustrative examples that depict some key points of the study.

Example 6.1. In this example, we illustrate the importance of rectangularity property. Consider a two period model $t \in \{0, 1, 2\}$, and a binomial tree setup. That is, $\Omega = \{\omega^1, \omega^2, \omega^3, \omega^4\} = \{(\omega_1, \omega_2) : \omega_1, \omega_2 \in \{u, d\}\}$ that is endowed with filtration $\{\mathscr{F}_t\}$ generated by the stochastic factor Z that goes up, or down at each time step, perhaps with different transition probabilities at different steps and/or nodes, see the diagram in Figure 3. A probability measure $\mathbb{P}$ on this filtered probability space is identified by the quantities

$$p_u = \mathbb{P}(\omega_1 = u), \quad p_{u|u} = \mathbb{P}(\omega_2 = u | \omega_1 = u), \quad p_{u|d} = \mathbb{P}(\omega_2 = u | \omega_1 = d),$$

which in turn would determine uniquely the (marginal) distributions of the stochastic factor Z.

Let us consider two models, given by

$$\begin{aligned} \mathbb{P}_{\theta_1} : &\quad p^1_u = 1/4, \quad p^1_{u|u} = p^1_{u|d} = 1/2 \\ \mathbb{P}_{\theta_2} : &\quad p^2_u = 1/2, \quad p^2_{u|u} = p^2_{u|d} = 3/4. \end{aligned}$$

We use $\mathbb{P}_{\theta_1}$ and $\mathbb{P}_{\theta_2}$ to construct an m-rectangular family of models $\Theta = \{\theta_1, \ldots, \theta_8\}$, given in Table 1, for example by following [Sha16] and build probabilities by exhausting all possible combinations of conditional probabilities. Additionally, we also take (smaller) family of models $\Theta^0 = \{\theta_1, \theta_2, \theta_5, \theta_8\}$. Family Θ^0 collects models under which the the events $\{\omega_1 = u\}$ and $\{\omega_2 = u\}$ are independent. We note that Θ^0 is not m-rectangular.

Assume that there are only two admissible strategies $\varphi, \psi \in \mathfrak{A}$ that generate two possible outcomes S^φ_T, S^ψ_T of the two-dimensional terminal state $S_T \in \mathscr{L}_T(\mathbb{R}^2)$, with specific values given in Table 2. For the sake of brevity we omit here presenting the controlled dynamics or the dynamics

	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8
p_u^θ	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$p_{u\|u}^\theta$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{3}{4}$
$p_{u\|d}^\theta$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4}$

Table 1: Set of probability measures satisfying m-rectangularity property.

	ω^1	ω^2	ω^3	ω^4
$S_T^\varphi(\omega^i)$	$\begin{pmatrix}8\\0\end{pmatrix}$	$\begin{pmatrix}0\\8\end{pmatrix}$	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}8\\8\end{pmatrix}$
$S_T^\psi(\omega^i)$	$\begin{pmatrix}0\\8\end{pmatrix}$	$\begin{pmatrix}0\\0\end{pmatrix}$	$\begin{pmatrix}6\\0\end{pmatrix}$	$\begin{pmatrix}6\\8\end{pmatrix}$

Table 2: Terminal values corresponding to two strategies.

of the stochastic factors. Within this example we use the identity function $\ell : \mathbb{R}^2 \to \mathbb{R}^2$, $\ell(S) = S$ as the multi-loss function, and we assume the component-wise order $\leq$.

By direct computation we find $\mathbb{E}_t^\theta[\ell(S_T^\varphi)]$ as well as $\mathbb{E}_t^\theta[\ell(S_T^\psi)]$ for all $\theta \in \Theta$ and $t = 0, 1$; see Tables 3 and 4. Recalling the component-wise structure of the supremum operator (5.1) and applying further direct computation, we can verify that for $\bar\Theta \in \{\Theta, \Theta^0\}$ it holds

$$\operatorname*{v-sup}_{\theta\in\bar\Theta} \mathbb{E}_1^\theta[\ell(S_T^\varphi)] = \begin{cases} \begin{pmatrix}6\\4\end{pmatrix} & \omega_1 = u \\ \begin{pmatrix}4\\4\end{pmatrix} & \omega_1 = d \end{cases} \qquad \text{and} \qquad \operatorname*{v-sup}_{\theta\in\bar\Theta} \mathbb{E}_1^\theta[\ell(S_T^\psi)] = \begin{cases} \begin{pmatrix}0\\6\end{pmatrix} & \omega_1 = u \\ \begin{pmatrix}6\\4\end{pmatrix} & \omega_1 = d \end{cases}$$

as well as

$$\operatorname*{v-sup}_{\theta\in\bar\Theta} \mathbb{E}_0^\theta\left[\operatorname*{v-sup}_{\theta\in\bar\Theta} \mathbb{E}_1^\theta[\ell(S_T^\varphi)]\right] = \begin{pmatrix}5\\4\end{pmatrix} \qquad \text{and} \qquad \operatorname*{v-sup}_{\theta\in\bar\Theta} \mathbb{E}_0^\theta\left[\operatorname*{v-sup}_{\theta\in\bar\Theta} \mathbb{E}_1^\theta[\ell(S_T^\psi)]\right] = \begin{pmatrix}4.5\\5\end{pmatrix}.$$

This shows that for both families of models $\bar\Theta \in \{\Theta, \Theta^0\}$, the recursive value function is

$$\mathcal{W}_0^{\bar\Theta}(S_0) = \left\{\operatorname*{v-sup}_{\theta\in\bar\Theta} \mathbb{E}_0^\theta\left[\operatorname*{v-sup}_{\theta\in\bar\Theta} \mathbb{E}_1^\theta[\ell(S_T^\varphi)]\right], \operatorname*{v-sup}_{\theta\in\bar\Theta} \mathbb{E}_0^\theta\left[\operatorname*{v-sup}_{\theta\in\bar\Theta} \mathbb{E}_1^\theta[\ell(S_T^\psi)]\right]\right\} = \left\{\begin{pmatrix}5\\4\end{pmatrix}, \begin{pmatrix}4.5\\5\end{pmatrix}\right\}.$$

However, we do not obtain the same value functions $\mathcal{V}$. For the full set of models Θ we get

$$\mathcal{V}_0^{\Theta}(S_0) = \left\{\operatorname*{v-sup}_{\theta\in\Theta} \mathbb{E}_0^\theta[\ell(S_T^\varphi)],\ \operatorname*{v-sup}_{\theta\in\Theta} \mathbb{E}_0^\theta[\ell(S_T^\psi)]\right\} = \left\{\begin{pmatrix}5\\4\end{pmatrix}, \begin{pmatrix}4.5\\5\end{pmatrix}\right\},$$

while for family Θ^0 we obtain

$$\mathcal{V}_0^{\Theta^0}(S_0) = \left\{\operatorname*{v-sup}_{\theta\in\Theta^0} \mathbb{E}_0^\theta[\ell(S_T^\varphi)],\ \operatorname*{v-sup}_{\theta\in\Theta^0} \mathbb{E}_0^\theta[\ell(S_T^\psi)]\right\} = \left\{\begin{pmatrix}4\\4\end{pmatrix}, \begin{pmatrix}4.5\\4\end{pmatrix}\right\}.$$

This illustrates the difference between (non-rectangular) family of models Θ^0 and $\leq$-rectangular family of models Θ. Only the weak set-valued Bellman's principle is satisfied for Θ^0, it holds $\mathcal{W}_0^{\Theta^0} \subsetneq \mathcal{V}_0^{\Theta^0} + \mathbb{R}^2_+$ and $\mathcal{V}_0^{\Theta^0} \subsetneq \mathcal{W}_0^{\Theta^0} - \mathbb{R}^2_+$. For the $\leq$-rectangular family Θ, the strong set-valued Bellman's principle holds as $\mathcal{V}_0^{\Theta} = \mathcal{W}_0^{\Theta}$.

θ	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	
$\mathbb{E}_1^\theta[\ell(S_2^\varphi)]$	$\begin{pmatrix}4\\4\end{pmatrix}$	$\begin{pmatrix}6\\2\end{pmatrix}$	$\begin{pmatrix}6\\2\end{pmatrix}$	$\begin{pmatrix}4\\4\end{pmatrix}$	θ_1	θ_4	θ_3	θ_2	$\omega_1 = u$
	$\begin{pmatrix}4\\4\end{pmatrix}$	$\begin{pmatrix}2\\2\end{pmatrix}$	$\begin{pmatrix}4\\4\end{pmatrix}$	$\begin{pmatrix}2\\2\end{pmatrix}$					$\omega_1 = d$
$\mathbb{E}_0^\theta[\ell(S_2^\varphi)]$	$\begin{pmatrix}4\\4\end{pmatrix}$	$\begin{pmatrix}4\\2\end{pmatrix}$	$\begin{pmatrix}5\\3\end{pmatrix}$	$\begin{pmatrix}3\\3\end{pmatrix}$	$\begin{pmatrix}4\\4\end{pmatrix}$	$\begin{pmatrix}2.5\\2.5\end{pmatrix}$	$\begin{pmatrix}4.5\\3.5\end{pmatrix}$	$\begin{pmatrix}3\\2\end{pmatrix}$	

Table 3: Conditional expectations for different models corresponding to strategy φ.

θ	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6	θ_7	θ_8	
$\mathbb{E}_1^\theta[\ell(S_T^\psi)]$	$\begin{pmatrix}0\\4\end{pmatrix}$	$\begin{pmatrix}0\\6\end{pmatrix}$	$\begin{pmatrix}0\\6\end{pmatrix}$	$\begin{pmatrix}0\\4\end{pmatrix}$	θ_1	θ_4	θ_3	θ_2	$\omega_1 = u$
	$\begin{pmatrix}6\\4\end{pmatrix}$	$\begin{pmatrix}6\\2\end{pmatrix}$	$\begin{pmatrix}6\\4\end{pmatrix}$	$\begin{pmatrix}6\\2\end{pmatrix}$					$\omega_1 = d$
$\mathbb{E}_0^\theta[\ell(S_T^\psi)]$	$\begin{pmatrix}4.5\\4\end{pmatrix}$	$\begin{pmatrix}3\\4\end{pmatrix}$	$\begin{pmatrix}3\\5\end{pmatrix}$	$\begin{pmatrix}3\\3\end{pmatrix}$	$\begin{pmatrix}3\\4\end{pmatrix}$	$\begin{pmatrix}4.5\\2.5\end{pmatrix}$	$\begin{pmatrix}4.5\\4.5\end{pmatrix}$	$\begin{pmatrix}4.5\\3\end{pmatrix}$	

Table 4: Conditional expectations for different models corresponding to strategy ψ.

Example 6.2. We present an application to portfolio optimization problem. Consider an investor that holds a collection of portfolios, with wealth process represented by a vector $S_t^\varphi = (S_t^{1,\varphi}, \dots, S_t^{d,\varphi}) \in \mathbb{R}^d$, where $S_t^{j,\varphi}$ is the wealth of the j-th portfolio corresponding to a self-financing trading strategy φ that additionally may satisfy various constraints, such as short-selling constraints, turn-over constraints, etc. For example, each component could be a portfolio in a different currency, or in a different market with fundamentally different risk characteristics. See, for instance, [FR13, Section 5] for more details and motivations for studying a similar setup, instead of the traditional scalar formulations of the problem. The investor may have different preferences for each market, and hence use different utility functions ℓ_j for each component S^j, and overall optimizing the multi-objective $\left[\mathbb{E}^\theta[\ell_1(S_T^\varphi)], \dots, \mathbb{E}^\theta[\ell_d(S_T^\varphi)]\right]^\intercal$. Our motivation to study this formulation of portfolio optimization stems from the emerging fintech theme of digital financial advice for investment management and trading, known as robo-advising; see the survey [DR21]. Traditionally, the robo-advising problem is formulated as one period mean-variance Markowitz portfolio optimization or its variations such as Black-Litterman model [KBL23]. The investor's risk profile (risk-aversion or risk tolerance coefficient) is elicited through questionnaires and assumed to be fixed and known between interaction times with the robo-advisor. Arguably, for unsophisticated and small investors, who constitute the vast majority of robo-advising platform users, finding the risk tolerance coefficient is a notoriously difficult problem, especially when combining several markets or classes of assets [ACRLS20]. A multi-valued formulation would allow: (a) eliciting the risk tolerance for major portfolio components (equity market, emerging markets, real estate market) and viewing the optimal portfolio as a point on the efficient frontier, potentially chosen by the investor through another questionnaire, and (b) dealing with inherently time-inconsistent problems. Model uncertainty would permit the use of tractable models while assuming uncertainty about the parametric characteristics of the driving stochastic factors. Detailed numerical implementation of these ideas using market data is outside the scope of this manuscript.

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