

Dissertation for the degree of Doctor of Philosophy

# Polynomial approximation on Stein manifolds and the Monge-Ampère operator

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# Abstract

The first main theme of this thesis is the approximation of holomorphic functions on Stein manifolds by polynomials. In our setting a polynomial is an entire function whose absolute value is bounded from above by  $\exp(a\psi + b)$  on the complement of a compact set, where  $a, b$  are positive constants and  $\psi$  is a given plurisubharmonic exhaustion function. In particular we generalize the Bernstein-Walsh-Siciak theorem to a certain class of Stein manifolds, which describes the equivalence between the possible holomorphic continuation of a function  $f$  defined on a compact set  $K$  in  $\mathbb{C}^n$  to the rapidity of the best uniform approximation of  $f$  on  $K$  by polynomials. We also generalize Winiarski's theorem, which relates the growth rate of an entire function on  $\mathbb{C}^n$  to its best uniform approximation by polynomials on a compact set.

If a plurisubharmonic exhaustion function  $\psi$  on a Stein manifold  $X$  satisfies the homogeneous Monge-Ampère equation outside a compact subset of  $X$ , i.e. if  $\psi$  is a parabolic potential on  $X$ , then the polynomial spaces defined by  $\psi$  are finite dimensional. Therefore the problem of constructing parabolic potentials arises naturally in the theory of polynomial approximation on Stein manifolds.

The second main theme of this thesis is the complex Monge-Ampère operator. In particular we derive formulas for the Monge-Ampère measures of functions of the form  $\log |\Phi|_c$ , where  $\Phi$  is a holomorphic map on a complex manifold of dimension  $n$  with values in  $\mathbb{C}^{n+1} \setminus \{0\}$  and  $|\cdot|_c$  is the Lie-norm in  $\mathbb{C}^{n+1}$ .



# Ágrip (in Icelandic)

Fyrsta meginþema þessarar ritgerðar er margliðunálganir á fágudum föllum á Stein víðáttum. Hér lítum við á margliðu sem heilt fágað fall hvers algildi er takmarkað að ofan af falli af gerðinni  $\exp(a\psi + b)$  á fyllimengi þjappaðs mengis, þar sem  $a, b$  eru jákvæðir fastar og  $\psi$  er einhver gefin fjölundirþýð tæming. Sér í lagi þá alhæfum við Bernstein-Walsh-Siciak setninguna þannig að hún nái yfir ákveðinn nýjan flokk af Stein víðáttum. Rifjum upp að Bernstein-Walsh-Siciak setningin lýsir venslunum á milli mögulegrar fágaðrar framlengingar á falli  $f$  sem er skilgreint á þjöppuðu mengi  $K$  í  $\mathbb{C}^n$  og hraðanum á bestu nálgun á  $f$  í jöfnum mæli á  $K$  með margliðum. Við alhæfum líka setningu Winiarski sem lýsir venslunum á milli vaxtahraða heils fágaðs falls á  $\mathbb{C}^n$  og bestu nálgun  $f$  í jöfnum mæli á þjöppuðu mengi með margliðum.

Ef fjölundirþýð tæming  $\psi$  á Stein víðáttu  $X$  uppfyllir óhliðruðu Monge-Ampère jöfnuna á fyllimengi þjappaðs mengis í  $X$ , þ.e. ef  $\psi$  er fleygert mætti á  $X$ , þá eru margliðurúmin sem  $\psi$  skilgreinir af endanlegri vídd. Þess vegna framlengjast rannsóknir á margliðunálgunum á Stein víðáttum náttúrulega yfir í rannsóknir á Monge-Ampère virkjanum.

Seinna meginþema þessarar ritgerðar er Monge-Ampère virkinn. Sér í lagi leiðum við út jöfnur fyrir Monge-Ampère mál falla af gerðinni  $\log |\Phi|_c$  þar sem að  $\Phi$  er fágud vörpun á fágadri víðáttu af vídd  $n$  sem tekur gildi sín í  $\mathbb{C}^n \setminus \{0\}$  og  $|\cdot|_c$  er Lie-staðallinn á  $\mathbb{C}^{n+1}$ .



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# 1

## Introduction

The theory of polynomial approximation has long been a classical subject, dating back at least to the early 18th century when Gregory, Taylor and Maclaurin formulated and researched the concept of a Taylor series. It wasn't until the late 19th century, in the year 1885, when Weierstraß proved that every continuous function on a closed interval in  $\mathbb{R}$  can be uniformly approximated by polynomials [81]. A prosperous year indeed, since in 1885 Runge also published his celebrated result on uniform polynomial approximation in the complex plane [64]. The generalization of Runge's theorem to several variables was proven by Oka and Weil in 1935-1936 [59, 82]. Because of its relevance to this thesis we state it here:

**Theorem 1.0.1** *If  $K \subset \mathbb{C}^n$  is compact and polynomially convex then every function  $f$  holomorphic in a neighborhood of  $K$  can be uniformly approximated by polynomials on  $K$ .*

In 1930 Bernstein considered the line segment  $K = [-1, 1]$  in  $\mathbb{C}$ . He proved a precise quantitative version of the Runge-Oka-Weil theorem for this special case. Later, Walsh [80] generalized this result with the line segment replaced by more general compact subsets of  $\mathbb{C}$ , see also [61, Theorem 6.3.1]. Finally, in 1962, Siciak [70] generalizes this result to several variables. For a more detailed survey on the history of polynomial approximation we refer to [7, 52].

Bernstein, Walsh and Siciak all considered the decay of the sequence  $d_K(f, \mathcal{P}_m) := \inf\{\|f - p\|_K; p \in \mathcal{P}_m\}$  of the best uniform approximation of  $f$  on  $K$  by polynomials of degree less than or equal to  $m$ , denoted by  $\mathcal{P}_m$ . In the most general case when  $K$

is any compact subset of  $\mathbb{C}^n$ , Siciak proved that

$$\limsup_{m \rightarrow \infty} (d_K(f, \mathcal{P}_m))^{1/m} \leq \frac{1}{L} \quad (1.1)$$

where  $L > 1$  is a constant depending on the size of the neighborhood around  $K$  on which  $f$  is holomorphic. To make the description of the parameter  $L$  more precise we must introduce the concept of an extremal function.

The *Siciak-Zakharyuta extremal function* for a compact set  $K \subset \mathbb{C}^n$  was originally introduced by Zakharyuta in [84] and is defined by the equation

$$V_K(z) = \sup\{u(z); u \in \mathcal{L}, u \leq 0 \text{ on } K\}, \quad z \in \mathbb{C}^n.$$

Here  $\mathcal{L}$  is the *Lelong class* of plurisubharmonic (psh) functions on  $\mathbb{C}^n$  which satisfy the growth condition  $u(z) \leq \log |z| + O(1)$  as  $|z| \rightarrow \infty$ . In two papers [71, 84] Siciak and Zakharyuta showed that the extremal function has another representation, namely

$$V_K(z) = \log \left( \sup\{|p(z)|^{\frac{1}{\deg p}}; p \text{ a non-constant polynomial such that } \|p\|_K \leq 1\} \right).$$

From this representation of the extremal function it is relatively simple to show that if inequality (1.1) is true, then  $f$  extends as a holomorphic function to the set  $K_L := \{z \in \mathbb{C}^n; V_K(z) < \log(L)\}$ . It turns out that the converse is true as well, under the extra assumption that  $V_K$  is continuous, although this result is considerably more difficult to prove. This is the result of Siciak [70].

One of the main topics of this thesis is to generalize Siciak's theorem to more general Stein manifolds. We do this in Chapter 3, which is based on paper [75]. It is probably not immediately clear to many readers how such a generalization might look like. Indeed, since polynomials on  $\mathbb{C}^n$  are usually defined in terms of the canonical global coordinate system, it is not completely straight forward to generalize this concept to a more general manifold. To do this we must introduce more background.

Recall that, by the theorem of Liouville, the polynomials on  $\mathbb{C}^n$  can be characterized in terms of their growth rate. More precisely, an entire function  $f$  on  $\mathbb{C}^n$  is a polynomial of degree  $\leq m$  if and only if there exists a constant  $C > 0$  such that  $\log |f| \leq m \log(1 + |z|) + C$  on  $\mathbb{C}^n$ . We generalize the concept of a polynomial with this characterization in mind. For a general manifold  $X$  with a psh exhaustion function  $\psi$  we define the  *$\psi$ -polynomials* on  $X$  to be entire functions  $f$  on  $X$  satisfying the

inequality

$$\log |f(z)| \leq t\psi^+(z) + C, \quad z \in X \quad (1.2)$$

for some  $t, C > 0$ , where  $\psi^+ = \max\{0, \psi\}$  is the positive part of  $\psi$ . The set of  $\psi$ -polynomials is denoted by  $\mathcal{P}^\psi$  and for  $t > 0$  fixed we denote by  $\mathcal{P}_t^\psi$  the set of  $\psi$ -polynomials satisfying inequality (1.2) for some  $C > 0$ .

Following Stoll [77] we say that a manifold  $X$  is *S-parabolic* if it possesses a plurisubharmonic exhaustion function  $\tau$  which is maximal outside a compact subset of  $X$ . Such an exhaustion function is called a *parabolic potential* or a *special exhaustion function*. In [85] Zeriahi proves that if  $\tau$  is a parabolic potential then the polynomial spaces  $\mathcal{P}_t^\tau$  have finite dimension. Therefore *S-parabolic* manifolds are of particular interest in the study of polynomial approximation on Stein manifolds. In their work [4–6], Aytuna and Sadullaev consider the Fréchet-space  $\mathcal{O}(X)$  of holomorphic functions on  $X$ . They construct an example of an *S-parabolic* manifold where the set of  $\tau$ -polynomials consists only of the constant, and another example where the polynomial space is non-trivial but still not dense in  $\mathcal{O}(X)$ . Zeriahi [85, 86] introduces analogues of classical pluripotential theory to *S-parabolic* Stein manifolds. He generalizes the theorem of Siciak and a theorem of Winiarski [83] to algebraic varieties.

Now our generalization of Siciak’s theorem in Chapter 3 should be more clear, we simply do the approximation with  $\psi$ -polynomials instead of the classical ones. Of course this is not always possible since, as previously stated, the polynomials are not always dense in  $\mathcal{O}(X)$ . Therefore we must make some assumptions on the exhaustion function  $\psi$ . In this context we consider the metric defined by the  $(1, 1)$ -form  $dd^c e^\psi$ . We have to make particular assumptions on the growth rate of its Ricci curvature and we also have to demand holomorphic functions to satisfy a sort of sub mean-value inequality with respect to this metric. We make our assumptions more precise in Section 3.2. In Section 3.3 we then look at several examples of exhaustion functions on Stein manifolds satisfying all our assumptions.

Observe that if inequality (1.1) holds for every  $L > 0$  then, by the previously mentioned theorem of Siciak,  $f$  is the restriction to  $K$  of an entire function, also denoted by  $f$ . If  $f$  is of finite order  $\varrho$  and of finite type  $\sigma$  with respect to  $\varrho$ , then Winiarski [83] proved a more precise estimate for the best uniform approximation of  $f$  on  $K$  (see Theorem 2.3.6). In Chapter 3 we also generalize Winiarski’s theorem to more general manifolds.

It is well known that a locally bounded psh function is maximal if and only if it satisfies the homogeneous Monge-Ampère equation. Therefore the problem of constructing examples of parabolic potentials on manifolds naturally brings us to the study of the Monge-Ampère operator. In Chapter 4 we turn our attention in this direction. This chapter is based on paper [76].

In Chapter 4 we consider functions of the form  $\log |\Phi|_c$  and calculate their Monge-Ampère measure. Here  $\Phi : X \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$  is a non-vanishing holomorphic map on a complex manifold of dimension  $n$  and  $|\cdot|_c$  is a specific norm on  $\mathbb{C}^{n+1}$  called the Lie norm, see Theorem 4.0.1. It turns out that the support of the Monge-Ampère measure  $(dd^c \log |\Phi|_c)^n$  is a subset of  $\Phi^{-1}(\mathbb{C}\mathbb{R}^{n+1})$ . This means that if  $X$  is a Stein manifold of dimension  $n$  and  $\Phi : X \rightarrow \mathbb{C}^{n+1}$  is a proper holomorphic map such that  $\Phi^{-1}(\mathbb{C}\mathbb{R}^{n+1})$  is compact, then  $X$  is  $S$ -parabolic with parabolic potential  $\log |\Phi|_c$ . Let us elaborate on this idea.

The *Lie norm* on  $\mathbb{C}^N$  is the largest norm on  $\mathbb{C}^N$  which extends the Euclidean norm  $|\cdot|$  on  $\mathbb{R}^N$  to a complex norm on  $\mathbb{C}^N$ . It thus dominates the standard hermitian norm on  $\mathbb{C}^N$ . It is given by an explicit formula in Proposition 4.1.2, which shows that it is a  $C^\infty$  function on  $\mathbb{C}^N \setminus \mathbb{C}\mathbb{R}^N$ .

In order to understand when we can expect to find a parabolic potential of the form  $\log |\Phi|_c$ , we look at a slightly more general problem. Let  $\Phi : X \rightarrow Y$  be a holomorphic map on a manifold  $X$  of dimension  $n$  into a manifold  $Y$  of dimension  $N$  and let  $h$  be a psh function on  $Y$ . If  $h$  is an exhaustion and  $\Phi$  is a proper map, then  $h \circ \Phi$  is an exhaustion on  $X$ . If  $h$  is a  $C^2$  function on an open subset  $V$  of  $Y$  then  $h$  is maximal on  $V$  if and only if the Levi form of  $h$  is degenerate at every point  $\zeta \in V$ . Recall that the *Levi form* of  $h$  is the hermitian form which in local coordinates  $\zeta = (\zeta_1, \dots, \zeta_N)$  is given by

$$w \mapsto \mathcal{L}_h(\zeta, w) = \sum_{j,k=1}^N \frac{\partial^2 h}{\partial \zeta_j \partial \bar{\zeta}_k}(\zeta) w_j \bar{w}_k, \quad w = (w_1, \dots, w_N) \in T_\zeta Y.$$

By a simple application of the chain rule we have

$$\mathcal{L}_{h \circ \Phi}(z, w) = \mathcal{L}_h(\Phi(z), d\Phi(z)(w)), \quad z \in \Phi^{-1}(V), w \in T_z X.$$

We see that if for every  $z \in \Phi^{-1}(V)$  there exists a tangent vector  $w \neq 0$  of  $X$  at  $z$  which is mapped by  $d\Phi(z)$  to the kernel of the Levi form of  $h$  at  $\Phi(z)$ , then  $h \circ \Phi$  is maximal on  $\Phi^{-1}(V)$ . In particular, if at every  $\zeta \in V$  the kernel of the Levi form of  $h$  at  $\zeta$  has dimension at least  $k \geq 1$  and  $N = n + k - 1$ , then it follows by the

rank-nullity theorem from linear algebra that  $h \circ \Phi$  is maximal on  $\Phi^{-1}(V)$ .

By Bos, Levenberg, Ma'u, and Piazzon [21] the kernel of the Levi form of  $\log|\cdot|_c$  on  $\mathbb{C}^N$  at every  $\zeta \neq 0$  is spanned by  $\zeta$  and  $\bar{\zeta}$ . These vectors are linearly independent if and only if  $\zeta$  is in  $\mathbb{C}^N \setminus \mathbb{C}\mathbb{R}^N$ , so the kernel is two dimensional at every  $\zeta \in \mathbb{C}^N \setminus \mathbb{C}\mathbb{R}^N$ . Hence  $\log|\Phi|_c$  is maximal on  $\Phi^{-1}(\mathbb{C}^{n+1} \setminus \mathbb{C}\mathbb{R}^{n+1})$  for every holomorphic map  $\Phi: X \rightarrow \mathbb{C}^{n+1}$  on a manifold of dimension  $n$ . These are the advantages of the Lie norm rather than, say, the usual Euclidean norm for constructing maximal psh functions. For a general  $\Phi: X \rightarrow \mathbb{C}^{n+1}$  the function  $\log|\Phi|$  is not maximal on any open subset of  $X$ .

## Outline

The outline of this thesis is as follows.

**Chapter 2:** We take a quick look at the background necessary for the rest of the thesis. In particular we discuss maximal psh functions and the Monge-Ampère operator. We introduce the Siciak-Zakharyuta extremal function and review some of its most important properties. Finally we discuss Stein manifolds and recall the main properties of  $S$ -parabolic manifolds.

**Chapter 3:** In Section 3.1 we introduce some preliminaries and key concepts necessary for the sections that follow. We discuss  $\psi$ -polynomials and their properties. We review the concept of an abstract Lelong class, which was introduced by Zeriahi in [87], and for a general psh exhaustion function  $\psi$  on  $X$  we also consider the space  $\mathcal{L}_\psi$  of psh functions  $u$  on  $X$  satisfying the growth estimate  $u \leq \psi^+ + C$ . We characterize the set of  $\psi$  such that  $\mathcal{L}_\psi$  is an abstract Lelong class in the sense of Zeriahi. With a clear understanding of Lelong classes we then proceed to define extremal functions on manifolds and discuss their properties.

In Section 3.2 we present two main results of this thesis. Those are the generalizations of the theorems of Sicak and Winiarski previously discussed to more general manifolds and concern the uniform approximation of holomorphic functions by  $\psi$ -polynomials on compact sets. Such an approximation is not always possible so we must make particular assumptions on the curvature of the metric defined by the  $(1,1)$ -form  $dd^c e^\psi$ . More precisely, we assume that  $\psi$  is *Ricci compensable* and *induces an integral estimate for holomorphic functions*; terms that we introduce in detail in this section.

In Section 3.3 we look at examples of plurisubharmonic exhaustion functions which satisfy the assumptions of our main results from the preceding section. That is, we look at several examples of Stein manifolds and plurisubharmonic exhaustion func-

tions  $\psi$  which are Ricci compensable and induce an integral estimate for holomorphic functions.

In Section 3.4 we prove all results stated in Section 3.2.

**Chapter 4:** The main result from Chapter 4, Theorem 4.0.1, is the description of the Monge-Ampère measure of functions of the form  $\log |\Phi|_c$  where  $\Phi : X \rightarrow \mathbb{C}^{n+1}$  is holomorphic on an  $n$  dimensional manifold  $X$  and  $|\cdot|_c$  is the Lie-norm.

In Section 4.1 we review the general theory of the cross norm  $\|\cdot\|_c$  on the complexification  $V_{\mathbb{C}}$  of a real normed space  $(V, \|\cdot\|)$ . For an inner product space  $(V, \langle \cdot, \cdot \rangle)$  with norm  $|\cdot|$  we derive a simple formula for the cross norm,  $|\zeta|_c = |a| + |b|$ , where the vector  $\zeta \in V_{\mathbb{C}}$  is represented as  $\zeta = e^{i\theta}(a + ib)$ , with  $\theta \in \mathbb{R}$ ,  $a, b \in V$ ,  $\langle a, b \rangle = 0$ , and  $|b| \leq |a|$ . From this formula we give a new proof of the explicit formulas for  $|\zeta|_c$  originally proved by Drużkowski in [36]. If  $|\cdot|$  is the Euclidean norm on  $\mathbb{R}^N$ , then  $|\cdot|_c$  is called the Lie norm on  $\mathbb{C}^N$ .

In Section 4.2 we first take a look at the set  $\mathbb{C}\mathbb{R}^N$  from a few different viewpoints. Then we look at the representation of  $\zeta \in \mathbb{C}^N$  as  $\zeta = e^{i\theta}(a + ib)$  with  $\theta \in \mathbb{R}$ ,  $a, b \in \mathbb{R}^N$ ,  $\langle a, b \rangle = 0$ , and  $|b| \leq |a|$ , which is very useful in our calculations. We derive a formula for the pullback of the volume form  $dV$  on  $\mathbb{C}^N$  to the  $2N$ -dimensional real manifold  $\{(\theta, a, b); \langle a, b \rangle = 0, |b| < |a|\}$ . This formula is applied in Section 4.5 for calculating weak limits of Monge-Ampère measures.

In Sections 4.3 and 4.4 we calculate the Siciak-Zakharyuta extremal function and the corresponding equilibrium measure for a few compact subsets  $K$  of  $\mathbb{R}^n \subset \mathbb{C}^n$ . The extremal function has been extensively studied for several decades and, in particular, authors have shown interest in the case when  $K$  is a compact subset of  $\mathbb{R}^N$  [10, 14, 18, 22–24, 54]. Among our examples is a result that motivated this study [21].

As a preparation for the proof of Theorem 4.0.1 we analyse in Section 4.5 the Levi form of the maximal function  $h_\varepsilon = \log v_\varepsilon$ , where the family of functions,  $v_\varepsilon(\zeta) = (|a| + \varepsilon|b|)^{\frac{1}{2}} + (|b| + \varepsilon|a|)^{\frac{1}{2}}$ ,  $\varepsilon \geq 0$ , regularizes the Lie norm  $v(\zeta) = |\zeta|_c = |a| + |b|$ .

Finally, in Section 4.6 we complete the proof of Theorem 4.0.1 by calculating the Monge-Ampère measure  $(dd^c(\log |\Phi|_c))^n$  as the weak limit of  $(dd^c(\log v_\varepsilon \circ \Phi))^n$ .

# 2

## Background

### 2.1 Basic definitions in pluripotential theory

To introduce notation, we start this section by defining the most common differential operators in complex analysis. Let  $f$  be a  $C^1$  function defined on a domain  $\Omega$  in  $\mathbb{C}^n$ . The differential of  $f$  splits canonically into a holomorphic part and an antiholomorphic part  $df = \partial f + \bar{\partial} f$  where

$$\partial f := \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j \quad \text{and} \quad \bar{\partial} f := \sum_{j=1}^n \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j.$$

We define the operator  $d^c := i(\bar{\partial} - \partial)$  and note that we have  $dd^c = 2i\partial\bar{\partial}$ . The standard Euclidean volume form on  $\mathbb{C}^n$  is defined as

$$dV := \bigwedge_{j=1}^n (dx_j \wedge dy_j) = \bigwedge_{j=1}^n \frac{i}{2} dz_j \wedge d\bar{z}_j.$$

**Definition 2.1.1** Let  $\Omega$  be a domain in  $\mathbb{C}$ . We say that a function  $h : \Omega \rightarrow \mathbb{R}$  is *harmonic* if it is  $C^2$  and  $dd^c h \equiv 0$  on  $\Omega$ . We say that an upper semicontinuous function  $u : \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$  is *subharmonic* if for every relatively compact open subset  $U$  of  $\Omega$  and every continuous function  $h : \bar{U} \rightarrow \mathbb{R}$ , which is harmonic on  $U$  and satisfies  $u \leq h$  on  $\partial U$ , we have  $u \leq h$  on  $U$ . We denote by  $\mathcal{SH}(\Omega)$  the set of subharmonic functions on  $\Omega$  not identically equal to  $-\infty$  on any connected component of  $\Omega$ .

In other words, the subharmonic functions are precisely the functions which are always dominated by the harmonic ones. Conversely, using the solution to the Dirichlet problem, it is relatively easy to show that the harmonic functions are the only subharmonic functions which dominate every other subharmonic function in the following sense:

**Proposition 2.1.2** *Let  $u$  be a subharmonic function on  $\Omega$  and assume that for every relatively compact open subset  $U$  of  $\Omega$  and every upper semicontinuous function  $v : \bar{U} \rightarrow \mathbb{R} \cup \{-\infty\}$ , which is subharmonic on  $U$  and satisfies  $v \leq u$  on  $\partial U$ , we have  $v \leq u$  on  $U$ . Then  $u$  is harmonic on  $\Omega$ .*

It is a well known fact that if  $u$  is a  $C^2$  function on  $\Omega \subset \mathbb{C}$  then it is subharmonic if and only if the  $(1,1)$ -form

$$dd^c u = \Delta u dV = \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dV = 4 \frac{\partial^2 u}{\partial z \partial \bar{z}} dV$$

is positive. For our purposes this is a far more convenient characterization of subharmonic functions than its defining properties. Fortunately this statement generalizes to more irregular functions as well.

**Proposition 2.1.3** *Let  $\Omega \subset \mathbb{C}$  be a domain and  $u \in L^1_{loc}(\Omega)$ . Then  $u$  is almost everywhere equal to a subharmonic function  $\tilde{u}$  if and only if  $dd^c u$  is positive in the sense of currents. Moreover, if  $dd^c u \geq 0$  then the function  $\tilde{u}$  is uniquely determined. If  $dd^c u \equiv 0$  on  $\Omega$  then  $\tilde{u}$  is harmonic (and in particular  $C^2$ ).*

Every harmonic function on a simply connected domain in  $\mathbb{C}$  is the real part of a holomorphic function. Conversely, the real part of a holomorphic function  $f$  is harmonic. Because of this (and for many other reasons) the study of harmonic and subharmonic functions has been of high importance when studying holomorphic functions on  $\mathbb{C}$ . By Hartogs theorem on separate analyticity a function  $f$  is holomorphic on a domain  $\Omega \subset \mathbb{C}^n$  iff it is holomorphic on the intersection of  $\Omega$  with every complex line (see [50, Section 2.4] for proof). A complex line is here a set of the form  $\{\zeta a + b; \zeta \in \mathbb{C}\} \subset \mathbb{C}^n$  for some  $a \in \mathbb{C}^n \setminus \{0\}$ ,  $b \in \mathbb{C}^n$ . With this fact in mind it is natural to generalize the concepts of subharmonic and harmonic functions to  $\mathbb{C}^n$  analogously.

**Definition 2.1.4** We say that an upper semicontinuous function  $u$  on a domain  $\Omega \subset \mathbb{C}^n$  is plurisubharmonic (pluriharmonic) and write psh (ph) for short, if it is subharmonic (harmonic) on the intersection of  $\Omega$  with every complex line. We denote

by  $\mathcal{PSH}(\Omega)$  ( $\mathcal{PH}(\Omega)$ ) the set of all psh (ph) functions on  $\Omega$  which are not identically equal to  $-\infty$  on any connected component of  $\Omega$ .

Let  $\Omega \subset \mathbb{C}^n$  be a domain. In the light of Proposition 2.1.3 we see that a function  $u \in L^1_{\text{loc}}(\Omega)$  is almost everywhere equal to a plurisubharmonic function  $\tilde{u}$  if and only if the  $(1, 1)$ -form  $dd^c u$  is positive in the sense of currents. In the case when  $u$  is  $C^2$  this means that the Levi form of  $u$ , defined by the equation

$$w \rightarrow \mathcal{L}_u(\zeta, w) := \sum_{j,k=1}^n \frac{\partial^2 u}{\partial \zeta_j \partial \bar{\zeta}_k}(\zeta) w_j \bar{w}_k, \quad \zeta \in \Omega, w \in \mathbb{C}^n,$$

is positive semidefinite if and only if  $u$  is psh. We end this section with an important regularization theorem for psh functions.

**Proposition 2.1.5** *Let  $u$  be psh on  $\Omega \subset \mathbb{C}^n$ . Then there exists a sequence of domains  $\Omega_1 \subset \Omega_2 \subset \dots$  with  $\Omega = \cup_m \Omega_m$  and  $C^\infty$  psh functions  $u_m$  on  $\Omega_m$  such that  $u_m(z) \searrow u(z)$  for every  $z \in \Omega$ .*

## 2.2 Maximal psh functions and the complex Monge-Ampère operator

### 2.2.1 Maximal psh functions

We have already seen that in one complex dimension the (pluri)harmonic functions are precisely the functions which always dominate the (pluri)subharmonic functions in the sense of Definition 2.1.1 and Proposition 2.1.2. The same can not be said about pluriharmonic and plurisubharmonic functions in several complex dimensions. Indeed, the class of pluriharmonic functions is far too small to satisfy the equivalent property. Therefore we introduce a new larger subclass of the plurisubharmonic functions.

Let  $\Omega$  be a domain in  $\mathbb{C}^n$  and  $u : \Omega \rightarrow \mathbb{R}$  be plurisubharmonic. We say that  $u$  is *maximal* if for every relatively compact subset  $G$  of  $\Omega$ , written  $G \Subset \Omega$ , and every plurisubharmonic function  $v$  on  $G$  satisfying

$$\liminf_{z \rightarrow \xi} (u(z) - v(z)) \geq 0, \quad \text{for all } \xi \in \partial G \tag{2.1}$$

we have  $u \geq v$  on  $G$ . We denote by  $\mathcal{MPSH}(\Omega)$  the set of all maximal psh functions which are not identically equal to  $-\infty$  on any connected component of  $\Omega$ .

It is clear that every ph function is a maximal psh function. Moreover, by Proposition 2.1.2, we see that in one complex dimension the maximal psh functions coincide with the harmonic ones. Although this is the case, many properties of harmonic functions are not inherited by maximal psh functions. For instance while harmonic functions are smooth, maximal psh functions are in general not even continuous. The lack of regularity of maximal plurisubharmonic functions can make them considerably more difficult to deal with.

**Example 2.2.1** Let  $w : \mathbb{C} \rightarrow \mathbb{R}$  be any discontinuous subharmonic function and define the function  $u : \mathbb{C}^2 \rightarrow \mathbb{R}$  as  $u(z_1, z_2) = w(z_1)$ . Let  $G$  be relatively compact in  $\mathbb{C}^2$  and  $v$  be any psh function satisfying inequality (2.1). For every fixed  $a \in \mathbb{C}$  the function  $z \rightarrow u(a, z)$  is constant and therefore harmonic on  $G_a := \{(a, z) \in G; z \in \mathbb{C}\}$ . Moreover we have  $v \leq u$  on  $\partial G_a$  so  $v \leq u$  on  $G_a$ . Since  $G = \cup_a G_a$  it follows that  $v \leq u$  on  $G$  and therefore  $u$  is a maximal psh function.

Working along the same lines, it is easy to provide examples of many more maximal psh functions. Indeed, let  $u$  be a psh function on a domain  $\Omega$  and suppose there exists a family of single dimensional open and proper submanifolds  $M_j \subset \Omega$ ,  $j \in J$  such that  $\cup_{j \in J} M_j = \Omega$  and such that  $u$  is harmonic on  $M_j$  for each  $j \in J$ . Then by an argument similar to the one in the previous example we can see that  $u$  is maximal on  $\Omega$ .

**Example 2.2.2** Let  $\|\cdot\|$  be any norm in  $\mathbb{C}^n$  and define  $u : \mathbb{C}^n \setminus \{0\} \rightarrow \mathbb{R}$  as  $u(z) = \log \|z\|$ . For any point  $a \in \mathbb{C}^n \setminus \{0\}$  the single variable function  $\zeta \rightarrow u(\zeta a)$  is harmonic on  $\mathbb{C} \setminus \{0\}$ . This means that  $u$  is maximal.

For convenience we include these equivalent descriptions of maximality.

**Proposition 2.2.3** [65, Proposition 17.2] *Let  $u$  be a psh function on  $\Omega \subset \mathbb{C}^n$ . The following statements are equivalent:*

- (i)  $u$  is maximal on  $\Omega$ ;
- (ii) for any psh function  $v$  on  $\Omega$  such that for each  $\varepsilon > 0$  there exists a compact set  $K \subset \Omega$  such that  $u - v \geq -\varepsilon$  on  $\Omega \setminus K$ , we have  $u \geq v$  on  $\Omega$ ;
- (iii) for any  $G \Subset \Omega$  and any psh function  $v$  on  $\Omega$  such that  $u \geq v$  on  $\partial G$ , we have  $u \geq v$  on  $G$ ;
- (iv) for any  $G \Subset \Omega$  and any psh function  $v$  on  $\Omega$  such that

$$\liminf_{z \rightarrow \xi, z \in G} (u(z) - v(z)) \geq 0, \quad \text{for all } \xi \in \partial G,$$

we have  $u \geq v$  on  $G$ ;

(v) for any  $G \Subset \Omega$  and any function  $v$  which is upper semicontinuous on  $\overline{G}$ , psh on  $G$  and satisfies  $u \geq v$  on  $\partial G$ , we have  $u \geq v$  on  $G$ .

## 2.2.2 The Monge-Ampère operator

The Monge-Ampère measure of  $u \in C^2 \cap \mathcal{PSH}(\Omega)$  is defined by

$$(dd^c u)^n = \underbrace{(dd^c u) \wedge \dots \wedge (dd^c u)}_{n \text{ times}} = 4^n n! \det \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right)_{j,k} dV.$$

By the fundamental work of Bedford and Taylor [12, 13] one can define the Monge-Ampère measure of a function  $u \in L_{\text{loc}}^\infty \cap \mathcal{PSH}(\Omega)$  as

$$(dd^c u)^n := \lim_{m \rightarrow \infty} (dd^c u_m)^n$$

where the limit is taken in the weak\* topology of measures and  $(u_m)$  is any sequence of smooth psh functions on  $\Omega$  satisfying  $u_m \searrow u$  pointwise. Demailly [34] extended this definition to psh functions which are locally bounded away from a compact set. It was shown by Shiffman and Taylor (see Appendix 1 of [73]) that the Monge-Ampère operator can not be extended to  $\mathcal{PSH}(\Omega)$  in a way such that  $(dd^c u)^n$  is a regular Borel measure for all psh functions  $u$ , see also [48, Example 3.1] and [27, Example 4]. In [15, 16, 28] Błocki and Cegrell give a complete characterization of those plurisubharmonic functions  $u$  for which one can well define  $(dd^c u)^n$  as a regular Borel measure, see also [29].

In the single dimensional case we have  $(dd^c u)^1 = \Delta u dV$  and therefore the Monge-Ampère operator is a generalization to several variables of the single dimensional Laplace operator. Moreover, when  $n = 1$ , we have  $dd^c u \geq 0$  if and only if  $u$  is subharmonic and  $dd^c u = 0$  if and only if  $u$  is harmonic.

Although the Monge-Ampère operator is continuous under monotone convergence, it is not continuous under general pointwise convergence [25], nor is it continuous under the  $L_{\text{loc}}^1$  convergence [26]. To emphasize the second statement we look at an example.

**Example 2.2.4** Consider the function

$$u(z) = \log \left( \max\{1, |z_1|, \dots, |z_n|\} \right), \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

For a fixed  $z$  with  $|z_j| < 1$  for some  $j$  the single variable function

$$\lambda \rightarrow u(z_1, \dots, z_{j-1}, z_j + \lambda, z_{j+1}, \dots, z_n)$$

is constant, and hence harmonic, in a small neighborhood of  $0 \in \mathbb{C}$ . Therefore  $(dd^c u)^n = 0$  on  $\{|z_j| < 1 \text{ for some } j\}$ . For a fixed  $z$  with  $\|z\|_\infty := \max\{|z_1|, \dots, |z_n|\} > 1$  the single variable function

$$\lambda \rightarrow u((1 + \lambda)z), \quad \lambda \in \mathbb{D}\left(0, \frac{1 - \|z\|_\infty}{\|z\|_\infty}\right),$$

is harmonic. We can conclude that the support of  $(dd^c u)^n$  lies on the torus  $\mathbb{T}^n = \{1 = |z_1| = \dots = |z_n|\}$ . By symmetry, and by the fact that the function  $u$  is radial in each variable, we can see that  $(dd^c u)^n = c\lambda$ , where  $\lambda$  is the normalized Lebesgue measure on  $\mathbb{T}^n$  and  $c \geq 0$  is some constant. By Theorem 2.3.8, we get  $c = (2\pi)^n$ . Now consider the functions

$$u_m(z) = \frac{1}{2m} \log(1 + |z_1^m + \dots + z_n^m|^2), \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

The sequence  $(u_m)$  is locally bounded and it converges pointwise almost everywhere to  $u$ . Therefore it converges to  $u$  in  $L^1_{\text{loc}}$ . Moreover we have

$$dd^c u_m = \frac{m}{(1 + |z_1^m + \dots + z_n^m|^2)^2} \sum_{j,k} z_j^{m-1} \bar{z}_k^{m-1} dz_j \wedge d\bar{z}_k.$$

The matrix  $(z_j^{m-1} \bar{z}_k^{m-1})_{j,k}$  is easily seen to have rank 1, so if  $n > 1$  we have  $(dd^c u_m)^n = 0$ . In particular  $\lim_{m \rightarrow \infty} (dd^c u_m)^n \neq (dd^c u)^n$ .

### 2.2.3 Comparison theorems

A proof of a more general version of the following theorem can be found in [15, Proposition 2.2].

**Theorem 2.2.5** *Let  $\Omega \subset \mathbb{C}^n$  be open and  $u \in \mathcal{PSH} \cap L^\infty_{\text{loc}}(\Omega)$ . Then  $u$  is maximal if and only if  $(dd^c u)^n = 0$ .*

A characterization like this of possibly unbounded maximal psh functions is not known. Indeed, it is not even known whether or not maximality is a local notion for arbitrary psh functions. Theorem 2.2.5 provides a convenient method of producing examples of maximal psh functions. If  $u$  is a psh function on  $\Omega$  and  $\Omega$  can locally be foliated by analytic disks such that the restriction of  $u$  to each disk is harmonic,

then Theorem 2.2.5 implies that  $u$  is maximal. Taylor and Kalka [11] showed that the converse is partly true as well. More precisely they showed that if  $u$  is maximal,  $C^3$  and  $(dd^c u)^{n-1} \neq 0$  on  $\Omega$  then such a foliation exists.

The following theorem is usually referred to as the comparison principle and it was originally proven by Bedford and Taylor [13]. It is widely considered to be one of the most important tools in pluripotential theory.

**Theorem 2.2.6** [49, Theorem 3.7.1] *Let  $\Omega \subset \mathbb{C}^n$  be open and bounded. Let  $u, v \in \mathcal{P}\mathcal{SH} \cap L^\infty(\Omega)$  be such that*

$$\liminf_{z \rightarrow \xi} (u(z) - v(z)) \geq 0, \quad \xi \in \partial\Omega.$$

*Then*

$$\int_{\{u < v\}} (dd^c v)^n \leq \int_{\{u < v\}} (dd^c u)^n.$$

To emphasize the effectiveness of the comparison principle we show how it can be used to prove one direction of Theorem 2.2.5, i.e. that  $(dd^c u)^n = 0$  on  $\Omega$  implies that  $u$  is maximal in  $\Omega$ . In fact we prove a more general version of this statement.

**Corollary 2.2.7** *Assume  $u, v \in \mathcal{P}\mathcal{SH} \cap L^\infty_{\text{loc}}(\Omega)$  are such that*

$$\liminf_{z \rightarrow \xi} (u(z) - v(z)) \geq 0, \quad \xi \in \partial\Omega.$$

*If  $(dd^c u)^n \leq (dd^c v)^n$  on  $\Omega$  then  $v \leq u$  in  $\Omega$ .*

*Proof.* Let  $R > 0$  be large enough such that  $\rho(z) := |z|^2 - R^2$  is negative on  $\Omega$  and define  $v_\varepsilon := v + \varepsilon\rho$ . It is a simple fact that  $(dd^c(f + g))^n \geq (dd^c f)^n + (dd^c g)^n$  for all locally bounded and psh  $f, g$ . Indeed, this inequality is clear if  $f, g$  are smooth and the general case follows by smooth decreasing sequences (see [49, Corollary 3.4.9] for more details). Therefore we have

$$(dd^c v_\varepsilon)^n \geq (dd^c v)^n + \varepsilon^n (dd^c \rho)^n \geq (dd^c u)^n + \varepsilon^n 4^n n! dV \quad (2.2)$$

where  $dV$  is the Lebesgue measure on  $\mathbb{C}^n$ . By the comparison principle and by (2.2) we have

$$\int_{\{u < v_\varepsilon\}} (dd^c u)^n \geq \int_{\{u < v_\varepsilon\}} (dd^c v_\varepsilon)^n \geq \int_{\{u < v_\varepsilon\}} (dd^c u)^n + \varepsilon^n 4^n n! \int_{\{u < v_\varepsilon\}} dV. \quad (2.3)$$

By subtracting  $\int_{\{u < v_\varepsilon\}} (dd^c u)^n$  from both sides of inequality (2.3) we can infer that the set  $\{u < v_\varepsilon\}$  has Lebesgue measure 0. Since  $\{u < v\} = \cup_{j \in \mathbb{N}} \{u < v_{1/j}\}$  we see that  $\{u < v\}$  has Lebesgue measure 0. By the sub-mean value inequality we can then see that  $v \leq u$  on  $\Omega$ .  $\square$

## 2.3 The Siciak-Zakharyuta extremal function

### 2.3.1 Definition and basic properties

The Lelong class of plurisubharmonic functions on  $\mathbb{C}^n$  is defined as follows,

$$\mathcal{L} := \{u \in \mathcal{PSH}(\mathbb{C}^n); \exists C > 0 \text{ such that } u(z) \leq \log^+ |z| + C, z \in \mathbb{C}^n\},$$

where  $\log^+ := \max\{0, \log\}$  is the positive part of the logarithm. We also define

$$\mathcal{L}^+ := \{u \in \mathcal{L}; \exists C \text{ such that } \log^+ |z| \leq u^+(z) + C, z \in \mathbb{C}^n\}.$$

We include an important result about the Lelong class on  $\mathbb{C}^n$ .

**Proposition 2.3.1** [49, Theorem 5.2.4] *If  $E \subset \mathbb{C}^n$  is pluripolar, i.e. if there exists a function  $v \in \mathcal{PSH}(\mathbb{C}^n)$  such that  $v|_E = -\infty$ , then there exists  $u \in \mathcal{L}$  such that  $u|_E = -\infty$ .*

For any compact set  $K \subset \mathbb{C}^n$  we define the extremal function of  $K$  as

$$V_K(z) := \sup\{u(z) : u \in \mathcal{L}, u \leq 0 \text{ on } K\}, \quad z \in \mathbb{C}^n,$$

and we denote by  $V_K^*$  the upper semi-continuous regularization of  $V_K$ . In dimension  $n = 1$  the extremal function  $V_K^*$  coincides with the well known generalized Green function on  $\mathbb{C} \setminus K$  for any compact and simply connected set  $K$ . The extremal function was introduced by Zakharyuta in [84].

**Remark.** The extremal function can be defined analogously for any set  $E \subset \mathbb{C}^n$ . In this thesis we only consider the extremal function for compact sets.

**Example 2.3.2** Let  $\|\cdot\|$  be any norm in  $\mathbb{C}^n$  and denote by  $K$  the closed unit ball with respect to this norm. Then we have

$$V_K(z) = \log^+ \|z\|, \quad z \in \mathbb{C}^n.$$

Indeed, it is clear that the function  $\log^+ \|z\|$  equals 0 on  $K$  and is a member of  $\mathcal{L}$  so  $\log^+ \|\cdot\| \leq V_K(z)$ . For the opposite inequality, let  $u \in \mathcal{L}$  be such that  $u \leq 0$  on  $K$  and let  $z_0 \notin K$  be fixed. The single variable function

$$v(\lambda) := u(\lambda z_0) - \log^+ \|\lambda z_0\|, \quad \lambda \in \mathbb{C} \setminus \mathbb{D}(0, \frac{1}{\|z_0\|})$$

is subharmonic on  $\mathbb{C} \setminus \overline{\mathbb{D}}(0, \|z_0\|^{-1})$  and non-positive on  $\partial\mathbb{D}(0, \frac{1}{\|z_0\|})$ . Moreover, the function  $v$  is bounded from above so it extends as a psh function to the Riemann-sphere  $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}(0, \|z_0\|^{-1})$ . By the maximum principle we have  $v(\lambda) \leq 0$  for  $\lambda \in \widehat{\mathbb{C}} \setminus \mathbb{D}(0, \|z_0\|^{-1})$ . In particular, taking  $\lambda = 1$ , we have  $u(z_0) \leq \log^+ \|z_0\|$ . Since  $u$  and  $z_0$  were arbitrary, we have  $V_K \leq \log^+ \|\cdot\|$ .

We now list a few properties of the extremal function, all of which can be found in [49, Chapter 5].

**Proposition 2.3.3** *Let  $K$  be compact. If  $K$  is pluripolar, then  $V_K = \infty$  on  $\mathbb{C}^n \setminus L$  for some pluripolar set  $L$ . Otherwise*

- (i)  $V_K^*$  is a member of  $\mathcal{L}^+$ ;
- (ii) for any sequence of compact sets  $K_1 \supset K_2 \supset \dots$  such that  $K = \bigcap K_j$  we have  $\lim_{j \rightarrow \infty} V_{K_j} \nearrow V_K$  at each point of  $\mathbb{C}^n$ ;
- (iii) for any neighborhood  $U$  of  $K$  there exists a compact  $\tilde{K}$  such that  $K \subseteq \tilde{K} \subset U$  and  $V_{\tilde{K}}$  is continuous.

### 2.3.2 Polynomial approximation

For  $m \in \mathbb{N}$  we denote by  $\mathcal{P}_m$  the set of complex polynomials  $p : \mathbb{C}^n \rightarrow \mathbb{C}$  of degree less than or equal to  $m$  and we write  $\mathcal{P} := \bigcup_{m \in \mathbb{N}} \mathcal{P}_m$ . We define the Siciak extremal function for a compact set  $K \subset \mathbb{C}^n$  as

$$\Phi_K(z) := \sup\{|p(z)|^{1/\deg p}; p \in \mathcal{P}, \deg p \geq 1, |p| \leq 1 \text{ on } K\}, \quad z \in \mathbb{C}^n.$$

In two papers [71, 84] Siciak and Zakharyuta showed that the functions  $V_K$  and  $\Phi_K$  are practically equivalent.

**Theorem 2.3.4** *Let  $K \subset \mathbb{C}^n$  be compact. Then  $V_K = \log \Phi_K$ . Moreover, the function  $V_K$  vanishes precisely on the polynomially convex hull of  $K$ .*

The Bernstein-Walsh inequality follows almost immediately from Theorem 2.3.4. Namely if  $K \subset \mathbb{C}^n$  is non-pluripolar and compact then

$$|p(z)| \leq \|p\|_K e^{mV_K(z)}, \quad z \in \mathbb{C}^n, \quad p \in \mathcal{P}_m.$$

The Siciak extremal function was originally introduced in [70] in order to extend classical results of approximation and interpolation to holomorphic functions of several complex variables. Indeed, in the paper he proves the following theorem.

**Theorem 2.3.5** *Let  $K \subset \mathbb{C}^n$  be compact,  $f$  be a function on  $K$  and assume  $V_K$  is continuous. Then  $f$  is the restriction to  $K$  of a holomorphic function on  $K_L := \{z \in \mathbb{C}^n; V_K(z) < \log(L)\}$  if and only if*

$$\limsup_{m \rightarrow \infty} (d_K(f, \mathcal{P}_m))^{\frac{1}{m}} \leq \frac{1}{L}, \quad (2.4)$$

where  $d_K(f, \mathcal{P}_m) := \inf_{p \in \mathcal{P}_m} \|f - p\|_K$  is the best uniform approximation of  $f$  on  $K$  by polynomials in  $\mathcal{P}_m$ .

The requirement of the continuity of  $V_K$  in Theorem 2.3.5 is arguably not very restrictive. To see why, suppose that  $f$  is a function holomorphic on a neighborhood  $U$  of a compact set  $K$  and we want to uniformly approximate  $f$  on  $K$  by polynomials. If  $V_K$  is not continuous, then by Proposition 2.3.3 we can simply find a slightly bigger set  $\tilde{K}$  such that  $V_{\tilde{K}}$  is continuous and then do the approximation. Goncar [40–42] and Cirka [30, 31] have proven theorems in a similar spirit, regarding approximation by rational functions on  $\mathbb{C}^n$ .

Observe that if inequality (2.4) holds for every  $L > 0$ , then  $f$  is the restriction to  $K$  of an entire function, also denoted by  $f$ . If  $f$  is of finite order  $\varrho$  and of finite type  $\sigma$  with respect to  $\varrho$ , then Winiarski [83] proved a more precise estimate of  $d_K(f, \mathcal{P}_m)^{1/m}$ . Denote by  $B_r \subset \mathbb{C}^n$  the ball centered at 0 with radius  $r$ .

**Theorem 2.3.6** *Let  $f$  be a function on the compact set  $K \subset \mathbb{C}^n$ . Then  $f$  is the restriction to  $K$  of an entire function, also denoted by  $f$ , satisfying the growth estimates*

$$\limsup_{r \rightarrow \infty} \frac{\log^+ \log \|f\|_{B_r}}{\log(r)} \leq \varrho \quad \text{and} \quad \limsup_{r \rightarrow \infty} \frac{\log \|f\|_{B_r}}{r^\varrho} \leq \sigma,$$

if and only if

$$\limsup_{m \rightarrow \infty} m^{1/\varrho} (d_K(f, \mathcal{P}_m))^{1/m} \leq (e\sigma\varrho)^{1/\varrho}.$$

### 2.3.3 The equilibrium measure

If the compact set  $K \subset \mathbb{C}^n$  is not pluripolar we define the (complex) equilibrium measure of  $K$  as  $\mu_K := (dd^c V_K^*)^n$ . The equilibrium measure of  $K$  is actually supported on  $K$ , or equivalently:

**Theorem 2.3.7** *The extremal function  $V_K^*$  is maximal on  $\mathbb{C}^n \setminus K$ . Conversely, if  $u \in \mathcal{L}^+$ ,  $(dd^c u)^n = 0$  on  $\mathbb{C}^n \setminus K$  and if there exists a pluripolar set  $G$  such that  $u = 0$  on  $K \setminus G$ , then  $u = V_K^*$ .*

*Proof.* For the first statement, let  $s$  be a psh function on  $\Omega \Subset \mathbb{C}^n \setminus K$  satisfying

$$\limsup_{z \rightarrow \xi, z \in \Omega} s(z) \leq V_K^*(\xi), \quad \xi \in \partial\Omega.$$

The function

$$v := \begin{cases} \max\{s, V_K^*\} & \text{in } \Omega, \\ V_K^* & \text{in } \mathbb{C}^n \setminus \Omega, \end{cases}$$

is a member of  $\mathcal{L}$  and vanishes on  $K \setminus F$ , where  $F = \{z \in K, V_K^*(z) > 0\}$ . The set  $F$  is pluripolar, and therefore, by Proposition 2.3.1, there exists  $w \in \mathcal{L}$  such that  $w|_F = -\infty$  and  $w \leq 0$  on  $K \cup \bar{\Omega}$ . For any  $\varepsilon > 0$  the function  $(1 - \varepsilon)v + \varepsilon w$  is a member of  $\mathcal{L}$  and  $(1 - \varepsilon)v + \varepsilon w \leq 0$  on  $K$  so  $(1 - \varepsilon)v + \varepsilon w \leq V_K^*$  on  $\mathbb{C}^n$ . By letting  $\varepsilon \rightarrow 0$  we see that  $s \leq V_K^*$  almost everywhere in  $\Omega$ , and hence everywhere in  $\Omega$ .

For the second statement, let  $u$  be as described in the theorem,  $w$  be as in the previous paragraph and  $\varepsilon > 0$ . The function  $(1 - 2\varepsilon)V_K^* + \varepsilon w$  is a member of  $(1 - \varepsilon)\mathcal{L}$  and since  $u \in \mathcal{L}^+$  we can find  $R$  large enough such that  $u \geq (1 - 2\varepsilon)V_K^* + \varepsilon w$  on  $\partial B(0, R)$ . We also have  $u \geq (1 - 2\varepsilon)V_K^* + \varepsilon w$  on  $K$  so by maximality we have  $u \geq (1 - 2\varepsilon)V_K^* + \varepsilon w$  on  $B(0, R)$ . By letting  $R \rightarrow \infty$  we see that the same inequality holds on  $\mathbb{C}^n$  and by then letting  $\varepsilon \rightarrow 0$  we see that  $u \geq V_K^*$  almost everywhere on  $\mathbb{C}^n$ , and hence everywhere on  $\mathbb{C}^n$ . For the opposite inequality, let  $r \in \mathcal{L}$  be such that  $r|_G = -\infty$  and  $r|_K \leq 0$ . Then we clearly have  $(1 - \varepsilon)u + \varepsilon r \leq V_K^*$  on  $\mathbb{C}^n$  and by letting  $\varepsilon \rightarrow 0$  we see that  $u \leq V_K^*$  on  $\mathbb{C}^n$ . □

Using the comparison principle it can be seen that the total mass of the equilibrium measure is  $\int_K \mu_K = (2\pi)^n$  for any compact set  $K \subset \mathbb{C}^n$ . Indeed, a much stronger statement for the entire class  $\mathcal{L}^+$  is true.

**Theorem 2.3.8** *Let  $u, v \in L_{\text{loc}}^\infty(\mathbb{C}^n)$  and assume there exists  $C > 0$  such that  $v \leq$*

$u^+ + C$  on  $\mathbb{C}^n$ , then  $\int_{\mathbb{C}^n} (dd^c v)^n \leq \int_{\mathbb{C}^n} (dd^c u)^n$ . Moreover, if  $w \in \mathcal{L}^+ \cap L_{\text{loc}}^\infty(\mathbb{C}^n)$  then  $\int_{\mathbb{C}^n} (dd^c w)^n = (2\pi)^n$ .

*Proof.* Let  $\varepsilon > 0$  and  $M > 0$  be fixed. Let  $R > 0$  be large enough such that  $v + M \leq (1 + \varepsilon)u$  on  $\mathbb{C}^n \setminus B(0, R)$ . By the comparison principle we have

$$\begin{aligned} \int_{B(0, R) \cap \{(1+\varepsilon)u < v+M\}} (dd^c v)^n &\leq (1 + \varepsilon)^n \int_{B(0, R) \cap \{(1+\varepsilon)u < v+M\}} (dd^c u)^n \\ &\leq (1 + \varepsilon)^n \int_{\mathbb{C}^n} (dd^c u)^n. \end{aligned}$$

The first statement follows by letting  $R \rightarrow \infty$ , then letting  $\varepsilon \rightarrow 0$  and finally letting  $M \rightarrow \infty$ .

From the first statement it is clear that  $\int_{\mathbb{C}^n} (dd^c w)^n$  equals a constant independent of  $w \in \mathcal{L}^+$ . Therefore it is sufficient to calculate the total Monge-Ampère mass of just one member of  $\mathcal{L}^+$ . We refer to [35, Example (3.8)] to see that

$$\int_{\mathbb{C}^n} \left( dd^c \frac{\log(1+|z|^2)}{2} \right)^n = (2\pi)^n$$

where  $|\cdot|$  is the Euclidean norm on  $\mathbb{C}^n$ . □

### 2.3.4 Compact subsets of $\mathbb{R}^n \subset \mathbb{C}^n$

The Joukovski transformation  $J : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$  is defined by the formula  $J(\zeta) = \frac{1}{2}(\zeta + \zeta^{-1})$ . For  $\zeta = re^{i\theta}$  we have

$$J(re^{i\theta}) = \frac{1}{2}(r + r^{-1}) \cos(\theta) + \frac{i}{2}(r - r^{-1}) \sin(\theta)$$

and therefore we see that  $J$  maps the boundary of the disk  $\partial\mathbb{D}(0, r)$  to an ellipse with center at the origin and semiaxes  $\frac{1}{2}(r \pm r^{-1})$ . Moreover,  $J$  maps  $\mathbb{C} \setminus \overline{\mathbb{D}}(0, 1)$  bijectively to  $\mathbb{C} \setminus [-1, 1]$  and its inverse is given by the formula

$$J^{-1}(z) = z + (z^2 - 1)^{1/2}, \quad z \in \mathbb{C} \setminus [-1, 1]$$

where the square root is chosen such that  $J^{-1}(x) > 1$  for  $x > 1$ . The function  $z \rightarrow \log |J^{-1}(z)|$  is clearly harmonic and goes to zero as  $z$  goes to  $[-1, 1]$ . Moreover this function is a member of  $\mathcal{L}^+(\mathbb{C})$  and therefore, by Theorem 2.3.7, we have the following result.

**Proposition 2.3.9** *The extremal function of the line segment  $K = [-1, 1]$  in  $\mathbb{C}$  is  $V_K(z) = \log |J^{-1}(z)|$ .*

Now let  $K$  be a convex symmetric body in  $\mathbb{R}^n \subset \mathbb{C}^n$  containing the origin, i.e. assume that  $K = -K$  and that the interior of  $K$  is not empty. The dual body of  $K$  is by definition the set

$$K^* := \{y \in \mathbb{R}^n; \langle x, y \rangle \leq 1 \text{ for all } x \in K\}.$$

Lundin [54] (see also Baran [8–10]) showed that the extremal function of every convex symmetric body  $K$  in  $\mathbb{R}^n \subset \mathbb{C}^n$  can be expressed in terms of the Joukovski transformation and the dual body of  $K$ . The special case of the unit ball was independently discovered by Sadullaev [67].

**Theorem 2.3.10** *Let  $K \subset \mathbb{R}^n \subset \mathbb{C}^n$  be a convex symmetric body containing the origin. Then*

$$V_K(z) = \sup\{\log |J^{-1}\langle z, w \rangle|; w \in \text{extr } K^*\}$$

where  $\text{extr } K^*$  denotes the set of points in  $K^*$  which are not a mid-point of any non-trivial line segment in  $K^*$ .

Applying Theorem 2.3.10 it is possible to work out the extremal function for the unit ball in  $\mathbb{R}^n$  (see for example [49, Theorem 5.4.6]). In [53] Lundin also calculated its equilibrium measure.

**Proposition 2.3.11** *The extremal function for the unit ball  $K$  in  $\mathbb{R}^n$  is*

$$V_K(z) = \frac{1}{2} \log J^{-1}(|z|^2 + |z_1|^2 + \dots + |z_n|^2 - 1)$$

and its equilibrium measure is

$$(dd^c V_K)^n|_K = \frac{2^n n! \Omega_n}{\sqrt{1 - x_1^2 - \dots - x_n^2}} dx_1 \wedge \dots \wedge dx_n$$

where  $\Omega_n$  is the volume of the unit ball in  $\mathbb{R}^n$ .

Many authors have improved on the results of Lundin and Baran, see for example [14, 18, 22–24].

## 2.4 Parabolic manifolds

### 2.4.1 Stein manifolds

A complex analytic manifold  $X$  of dimension  $n$  which is countable at infinity is said to be *Stein* if

- (i)  $X$  is holomorphically convex, i.e. the holomorphically convex hull of any compact set  $K \subset X$ ,

$$\hat{K} := \{z \in X; |f(z)| \leq \|f\|_K, \text{ for all } f \in \mathcal{O}(X)\}$$

is compact in  $X$ ;

- (ii)  $\mathcal{O}(X)$  separates the points of  $X$ , i.e. for all  $z_1, z_2 \in X$  with  $z_1 \neq z_2$  there exists  $f \in \mathcal{O}(X)$  such that  $f(z_1) \neq f(z_2)$ ;
- (iii) in a neighborhood of any  $z \in X$  there exists a coordinate chart consisting of functions from  $\mathcal{O}(X)$ .

The underlying motivation for the definition of Stein manifolds is arguably the structure of domains of holomorphy in  $\mathbb{C}^n$ . Therefore many theorems concerning domains of holomorphy can be generalized to Stein manifolds. In particular the following Oka-Weil theorem generalizes the classical Runge theorem.

**Theorem 2.4.1** [46, Corollary 5.2.9.] *If  $K \subset X$  is compact,  $K = \hat{K}$  and  $f$  is a function holomorphic in a neighborhood of  $K$ , then  $f$  can be uniformly approximated on  $K$  by functions in  $\mathcal{O}(X)$ .*

Like domains of holomorphy, Stein manifolds possess a plurisubharmonic exhaustion function:

**Theorem 2.4.2** [46, Theorem 5.2.10.] *A complex manifold  $X$  is Stein if and only if there exists a strictly psh function  $\varphi \in C^\infty(X)$  such that for all  $c \in \mathbb{R}$  the set  $\{z \in X; \varphi(z) < c\}$  is relatively compact in  $X$ .*

An important characterization of Stein manifolds of dimension  $n$  is that they are properly holomorphically embeddable in  $\mathbb{C}^N$  for  $N$  large enough. This was originally proven by Remmert [62]. A precise estimate for the lowest possible dimension  $N$  in terms of  $n$  was obtained by Eliashberg and Gromov in [37] and improved by one dimension for odd values of  $n$  by Schürmann in [68] (see also [39]).

**Theorem 2.4.3** *An  $n$ -dimensional Stein manifold can be properly holomorphically embedded into  $\mathbb{C}^N$  with  $N = n + [n/2] + 1$ , where  $[·]$  denotes the integer part of a real number.*

In [38] Forster provides an example, showing that these values of  $N$  are minimal for every  $n > 1$ .

### 2.4.2 Notions of parabolicity

The uniformization theorem states that a simply connected complex manifold of dimension  $n = 1$  is conformally equivalent to the Riemann sphere, the unit disk or the complex plane. Moreover, it states that every one dimensional complex manifold admits a universal covering by one of these manifolds. A non-compact manifold is said to be

- (i) *hyperbolic* if its universal cover is the unit disk;
- (ii) *parabolic* if its universal cover is the complex plane.

In several complex dimension there is no theorem analogous to the uniformization theorem and therefore the classification of Stein manifolds into parabolic and hyperbolic ones is not as straight forward. Many authors classify them in terms of the triviality (parabolic type) and non-triviality (hyperbolic type) of the Kobayashi or the Caratheodory metrics. We classify the parabolic ones in terms of the existence of non-constant psh functions.

**Definition 2.4.4** A Stein manifold  $X$  is said to be parabolic if the constants are the only bounded psh functions on  $X$ .

Parabolic manifolds play an important role in the structure theory of Fréchet spaces of analytic functions on Stein manifolds, especially in finding continuous extension operators for analytic functions from complex submanifolds [1–3, 5, 79]. Following the notation of Stoll [77], we also define  $S$ -parabolic manifolds:

**Definition 2.4.5** Let  $X$  be a Stein manifold. We say that a psh exhaustion function  $\tau$  on  $X$  is a parabolic potential if it is maximal outside a compact subset of  $X$ . We say that the manifold  $X$  is  $S$ -parabolic if there exists a parabolic potential on  $X$ . If there exists a continuous parabolic potential  $\tau$  on  $X$  then we say that  $X$  is  $S^*$ -parabolic.

All the notions of parabolicity mentioned in this section are known to be equivalent in dimension  $n = 1$ . The  $S^*$ -parabolic manifolds are trivially  $S$ -parabolic and it is

relatively simple to show that  $S$ -parabolic manifolds of any dimension are parabolic, see for example [5]. The converse statements are important open questions, i.e. it is not known whether or not parabolicity implies  $S$ -parabolicity or if  $S$ -parabolicity implies  $S^*$ -parabolicity for dimensions  $n > 1$ . In [85,86] Zeriahi introduces analogues of classical pluripotential theory to  $S$ -parabolic Stein spaces. He also generalizes Theorem 2.3.5 and Theorem 2.3.6 to algebraic varieties.

**Example 2.4.6** We provide three examples of  $S^*$ -parabolic manifolds.

- (i) The complex Euclidean space  $\mathbb{C}^n$  is  $S^*$ -parabolic with a parabolic potential  $\tau = \log^+ |z|$ .
- (ii) Let  $X \subset \mathbb{C}^N$  be an algebraic manifold of dimension  $n$ . By Rudin [63], after a linear change of variables, we can assume that  $X$  is a subset of

$$\{(z_1, \dots, z_n, z_{n+1}, \dots, z_N) = (z', z'') \in \mathbb{C}^N; |z''| \leq A(1 + |z'|)^B\}$$

for some positive constants  $A, B$ . This implies that the function  $\tau := \log |z'|$  is an exhaustion function. It is easy to see that  $\tau$  is maximal on  $X \setminus \tau^{-1}(-\infty)$  so  $\tau$  is a parabolic potential.

- (iii) Let  $F : \mathbb{C}^n \rightarrow \mathbb{C}$  be a Weierstrass polynomial, i.e. let  $F$  be of the form

$$F(z) = z_n^k + z_n^{k-1} f_{k-1}(z') + \dots + z_n f_1(z') + f_0(z') \quad z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$$

where  $f_j : \mathbb{C}^{n-1} \rightarrow \mathbb{C}$  are entire functions. Write  $X = \mathbb{C}^n \setminus \{F = 0\}$ . Then  $X$  is an  $S^*$ -parabolic manifold with a parabolic potential

$$\tau(z) = \log (|z'|^2 + |F(z) - 1|^2) - \log |F(z)|, \quad z \in X.$$

Indeed, it is easy to see that  $\tau$  is maximal on  $X \setminus \tau^{-1}(-\infty)$ . To see that  $\tau$  is actually an exhaustion function we refer to [5, Theorem 4.1].

### 2.4.3 On curvature

We end this section with a quick review of the Ricci curvature and a result of Demailly from [33]. Let  $\omega$  be a *Kähler form* on a complex manifold  $X$ , i.e. assume that  $\omega$  is a closed and smooth strictly positive  $(1,1)$ -form on  $X$ . As is customary we abuse notation by representing the metric whose fundamental form is  $\omega$  with the same symbol. The Ricci curvature tensor for a general metric  $\omega$  is usually defined in terms

of the Riemann curvature tensor, but since we only consider Kähler metrics we can represent it in a way which is much more convenient for our purposes. Indeed, if

$$\omega = \frac{i}{2} \sum_{1 \leq j, k \leq n} \omega_{j, \bar{k}} dz_j \wedge d\bar{z}_k$$

with respect to some coordinate chart  $z = (z_1, \dots, z_n)$  defined on  $\mathbb{D}_z^n$  then the Ricci curvature of  $\omega$  is the  $(1, 1)$ -form

$$\text{Ricci}(\omega) := -\frac{1}{2} dd^c \log (\text{Det } \omega_{j, \bar{k}}).$$

To show that this representation is independent of the choice of coordinates, suppose that  $\zeta = (\zeta_1, \dots, \zeta_n)$  is some other coordinate chart defined on  $\mathbb{D}_\zeta^n$  and  $\tau : \mathbb{D}_\zeta^n \rightarrow \mathbb{D}_z^n$  is the transition map between them. Then  $\omega = \sum \tilde{\omega}_{j, \bar{k}} d\zeta_j \wedge d\bar{\zeta}_k$  where the matrices  $(\tilde{\omega}_{j, \bar{k}})$  and  $(\omega_{j, \bar{k}})$  are related by the equality  $(\tilde{\omega}_{j, \bar{k}}) = (J_\tau)^*(\omega_{j, \bar{k}})(J_\tau)$  and  $J_\tau$  is the Jacobian of  $\tau$ . The matrix  $J_\tau$  is non-singular so the function  $\log |\text{Det } J_\tau|^2$  is pluriharmonic and therefore

$$dd^c \log (\text{Det } \tilde{\omega}_{j, \bar{k}}) = dd^c \left( \log (\text{Det } \omega_{j, \bar{k}}) + \log |\text{Det } J_\tau|^2 \right) = dd^c \log (\text{Det } \omega_{j, \bar{k}}).$$

In [33] Demailly gives a characterization of the algebraic varieties among Stein spaces in terms of the existence of a particular plurisubharmonic exhaustion function  $\psi$  and the Ricci curvature of  $dd^c e^\psi$ .

**Theorem 2.4.7** [33, Theoreme 9.1] *Let  $X$  be a complex analytic space of dimension  $n$  containing a finite number of singular points. Then  $X$  is an affine algebraic variety if and only if there exists a  $C^\infty$  strictly psh exhaustion function  $\psi$  on  $X$  satisfying:*

- (i)  $\int_X (dd^c \psi)^n < \infty$ ;
- (ii) there exists a function  $\theta \in L^1_{\text{loc}}(X) \cap C^0(X_{\text{reg}})$  with values in  $\mathbb{R}$  which is bounded from above by  $A\psi + B$  for some constants  $A, B > 0$  and

$$\frac{1}{2} dd^c \theta + \text{Ricci}(dd^c e^\psi) \geq 0;$$

- (iii)  $\psi$  has only a finite number of critical points in  $X_{\text{reg}}$ .



# 3

## Rapid polynomial approximation on Stein manifolds

The main theme of this chapter is the uniform approximation of holomorphic functions on compact sets and the Siciak-Zakharyuta extremal function. We can look at this theme from several perspectives; some authors have recently considered approximation by different subclasses of the polynomials than the usual ones [20, 78], others have considered different expressions of the extremal function and the equilibrium measure, for instance by disc formulas [51, 56, 57] or by random polynomials [17, 19, 45, 69].

We consider approximation on Stein manifolds with polynomials which are defined in terms of their growth rate. The main results of this chapter, Theorem 3.2.2 and Theorem 3.2.5, are generalizations of the theorems of Siciak and Winiarski (Theorems 2.3.5 and 2.3.6).

### 3.1 Preliminaries

Since polynomials on the Euclidean space  $\mathbb{C}^n$  are usually defined in terms of the canonical global coordinate system, it is not completely straight forward to generalize this concept to a more general manifold  $X$ . If  $X$  is an analytic submanifold of  $\mathbb{C}^N$

then one might consider to define the polynomials on  $X$  to be the restrictions of polynomials on  $\mathbb{C}^N$  to  $X$ . For our purposes, this definition is not very useful; for instance if  $K \subset X$  is not pluripolar, the corresponding extremal function

$$\Phi_K(z) := \{|p(z)|^{1/\deg(p)}; p \in \mathcal{P}(\mathbb{C}^M), \deg p \geq 1, \|p\|_K \leq 1\}, \quad z \in X$$

is locally bounded if and only if  $X$  is algebraic. This was proven by Sadullaev in [66]. Instead we define polynomials on  $X$  in terms of a predetermined psh exhaustion function on  $X$ .

**Definition 3.1.1** Let  $\psi$  be a psh exhaustion function on  $X$ . We say that a function  $f \in \mathcal{O}(X)$  is a  $\psi$ -polynomial if there exist constants  $t$  and  $C$  such that

$$\log |f(z)| \leq t\psi^+(z) + C, \quad z \in X. \quad (3.1)$$

We denote by  $\mathcal{P}^\psi$  the space of  $\psi$ -polynomials on  $X$  and for a fixed  $t > 0$  we denote by  $\mathcal{P}_t^\psi$  the set of  $\psi$ -polynomials on  $X$  satisfying inequality (3.1) for some constant  $C$ . If  $f$  is a  $\psi$ -polynomial on  $X$  then the  $\psi$ -degree of  $f$  is

$$\deg_\psi(f) := \inf\{t > 0; f \in \mathcal{P}_t^\psi\}.$$

Note that if  $X = \mathbb{C}^n$  and  $\tau(z) = \log |z|$  then, by Liouville's theorem, the notion of a  $\tau$ -polynomial coincides with the classical notion of a polynomial. The polynomial spaces  $\mathcal{P}_t^\tau$  are of particular interest when the function  $\tau$  is a parabolic potential because then they are of finite dimension. More specifically we have:

**Proposition 3.1.2** [85, Théorème 4.8] *If  $\tau$  is a parabolic potential on the  $n$ -dimensional manifold  $X$ , then there exists a constant  $M$  such that*

$$\dim \mathcal{P}_m^\tau \leq \binom{n + mM}{n}, \quad m \in \mathbb{N}.$$

Before we prove Proposition 3.1.2 we must introduce other concepts; the proof can be found at the end of this section.

In [85, 86] Zeriahi considers the case when  $X$  is an affine algebraic variety and proves theorems similar to the theorems of Oka-Weil and Siciak. In [4–6] Aytuna and Sadullaev consider the polynomial space  $\mathcal{P}^\tau$  when  $\tau$  is a parabolic potential. They construct an example where the polynomial space  $\mathcal{P}^\tau$  only consists of the constant functions, and another one where  $\mathcal{P}^\tau$  is not trivial, but still not dense in the Fréchet-

space of holomorphic functions  $\mathcal{O}(X)$ . In [87] Zeriahi introduces the notion of an abstract Lelong classes on complex spaces.

**Definition 3.1.3** Let  $\mathcal{L} \subset \mathcal{PSH}(X)$  be a class of psh functions on a complex space  $X$ . We say that  $\mathcal{L}$  is an abstract Lelong class on  $X$  if

- (i)  $\mathcal{L}$  contains all the (real) constants and is translation invariant, i.e. if  $u \in \mathcal{L}$  and  $\alpha \in \mathbb{R}$  then  $u + \alpha \in \mathcal{L}$ ;
- (ii)  $\mathcal{L}$  is a closed subset of  $\mathcal{PSH}(X)$  in the  $L^1_{\text{loc}}$  topology;
- (iii) for every subfamily  $\mathcal{M} \subset \mathcal{L}$  either the set

$$B_{\mathcal{M}} := \{x \in X; \sup_{v \in \mathcal{M}} v(x) < \infty\}$$

is pluripolar in  $X$ , or the family  $\mathcal{M}$  is locally bounded from above in  $X$ .

In this thesis we are most interested in one particular class of psh functions; namely for any psh exhaustion function  $\psi$  on  $X$  we define the  $\psi$ -Lelong class on  $X$  to be the set

$$\mathcal{L}_{\psi} := \{u \in \text{PSH}(X); \exists C \geq 0 \text{ such that } u \leq \psi^+ + C \text{ on } X\}$$

and we define

$$\mathcal{L}_{\psi}^+ := \{u \in \mathcal{L}_{\psi}; \exists C \geq 0 \text{ such that } \psi \leq u^+ + C \text{ on } X\}.$$

We can characterize the set of psh exhaustion functions  $\psi$  such that  $\mathcal{L}_{\psi}$  is an abstract Lelong class in the following way:

**Proposition 3.1.4** *Let  $\psi$  be a psh exhaustion function on  $X$ . If  $\mathcal{L}_{\psi}$  contains a parabolic potential, then  $\mathcal{L}_{\psi}$  is an abstract Lelong class. Conversely if  $\mathcal{L}_{\psi}$  is an abstract Lelong class, and there exists a non-pluripolar compact set  $K$  such that the function*

$$V_{K,\psi}(z) := \sup\{v(z); v \in \mathcal{L}_{\psi}, v|_K \leq 0\}, \quad z \in X \quad (3.2)$$

*satisfies the equation  $V_{K,\psi} = V_{K,\psi}^*$  on  $K$ , then  $V_{K,\psi}$  is maximal on  $X \setminus K$  and therefore is a parabolic potential on  $X$ . Moreover  $V_{K,\psi}$  is a member of  $\mathcal{L}_{\psi}$  and  $V_{K,\psi} = V_{K,\psi}^*$  on  $X$ .*

*Proof.* The class  $\mathcal{L}_\psi$  clearly contains all the constants and is translation invariant. Suppose there exists a parabolic potential  $\tau$  in  $\mathcal{L}_\psi^+$  and denote by  $K$  the closure of the set  $\{z \in X; \tau(z) \leq 0\}$ ; by adding a constant to  $\tau$  we can assume that  $K$  is not empty and we can assume that it is maximal on  $X \setminus K$ . Using methods similar to the ones used in the proof of Theorem 2.3.7, we can show that

$$w(z) \leq \sup_K w + \tau(z), \quad w \in \mathcal{L}_\psi, z \in X. \quad (3.3)$$

Now let  $\mathcal{M} \subset \mathcal{L}_\psi$  be such that  $B_{\mathcal{M}}$  is not pluripolar and write  $u(z) := \sup\{w(z); w \in \mathcal{M}\}$ . We want to show that  $u$  is locally bounded from above on  $X$ . By inequality (3.3) it is enough to show that  $\sup_{z \in K} u(z) < \infty$ . Assume the opposite holds and let  $(w_j)_{j \in \mathbb{N}}$  be a sequence of functions in  $\mathcal{M}$  such that  $m_j := \sup_K w_j \geq 2^j$ . From inequality (3.3) we can see that the sequence  $(w_j - m_j)$  is locally bounded from above on  $X$ . Observe that at some point  $z_0 \in X$  we have  $\limsup_{j \rightarrow \infty} (w_j(z_0) - m_j) > -1$ , because otherwise we would have  $\limsup_{j \rightarrow \infty} (w_j(z) - m_j) \leq -1$  at each point  $z \in X$  and, by Hartogs lemma, this would imply that  $\sup_{z \in K} (w_j(z) - m_j) \leq -1/2$  for all sufficiently large  $j$  contradicting the definition of the constants  $m_j$ . Taking a subsequence if necessary, we can assume that  $w_j(z_0) - m_j \geq -1$  for all  $j$ . Now the function

$$w(z) := \sum_{j=1}^{\infty} 2^{-j} (w_j(z) - m_j), \quad z \in X$$

is psh on  $X$ . By definition of the set  $B_{\mathcal{M}}$  we have  $u(z) < \infty$  for all  $z \in B_{\mathcal{M}}$  and since  $m_j \geq 2^j$  we have

$$w(z) \leq \sum_{j=1}^{\infty} 2^{-j} (u(z) - m_j) = u(z) - \sum_{j=1}^{\infty} 2^{-j} m_j = -\infty, \quad z \in B_{\mathcal{M}}.$$

Moreover we have  $w(z_0) \geq \sum_{j=1}^{\infty} (-2^{-j}) = -1$  which means that  $w$  is not identically equal to  $-\infty$  and therefore  $B_{\mathcal{M}}$  is pluripolar, contradicting our original assumption.

Conversely suppose  $\mathcal{L}_\psi^+$  is an abstract Lelong class and let  $K$  as described in the proposition. By property (iii) of Definition 3.1.3 the function  $V_{K,\psi}^*$  is locally bounded on  $X$  and by property (ii) we see that it is a member of  $\mathcal{L}_\psi^+$ . Since  $V_{K,\psi}^* = V_{K,\psi} = 0$  on  $K$ ,  $V_{K,\psi}^*$  is a member of the set on the right hand side of (3.2) so  $V_{K,\psi}^* \leq V_{K,\psi}$  on  $X$ . Clearly the opposite equality holds as well so we have  $V_{K,\psi} = V_{K,\psi}^*$  on  $X$ . To show that  $V_{K,\psi}$  is maximal on  $X \setminus K$  suppose that  $s$  is a psh function on  $\Omega \Subset X \setminus K$

satisfying

$$\limsup_{z \rightarrow \xi, z \in \Omega} s(z) \leq V_K^*(\xi), \quad \xi \in \partial\Omega.$$

The function

$$v := \begin{cases} \max\{s, V_K^*\} & \text{in } \Omega, \\ V_K^* & \text{in } \mathbb{C}^n \setminus \Omega, \end{cases}$$

is a member of  $\mathcal{L}_\psi$  and vanishes on  $K$  so  $v \leq V_{K,\psi}$  on  $X$ . Therefore we see that  $s \leq V_{K,\psi}$  on  $\Omega$ . This means that  $V_{K,\psi}$  is maximal on  $X \setminus K$ . □

The function  $V_{K,\psi}$  from Proposition 3.1.4 is called the  $\psi$ -*extremal function* (or just the extremal function if the context is clear) for  $K$ . A more general version of the following proposition is proved in [87, Theorem 2.1.2].

**Proposition 3.1.5** *The following statements are equivalent:*

- (i) *The class  $\mathcal{L}_\psi$  is an abstract Lelong class;*
- (ii) *there exists a non-pluripolar compact set  $K \subset X$  such that  $V_{K,\psi}$  is locally bounded on  $X$ ;*
- (iii) *the function  $V_{K,\psi}$  is locally bounded on  $X$  for every non-pluripolar compact set  $K \subset X$ .*

For future reference we include the following result regarding the regularity of the extremal function.

**Proposition 3.1.6** [85, Théorème 4.2] *If  $\tau$  is a continuous parabolic potential on  $X$ ,  $K \subset X$  is compact and pluriregular, i.e.  $V_{K,\tau}^* = 0$  on  $K$ , then  $V_{K,\tau}$  is continuous on  $X$ .*

Finally we define the function

$$\Phi_{K,\psi}(z) = \sup\{|f(z)|^{1/t}; f \in \mathcal{P}_t^\psi, \|f\|_K \leq 1, t > 0\}. \quad (3.4)$$

In the case when  $X = \mathbb{C}^n$  and  $\psi(z) = \log|z|$  the function  $\Phi_K := \Phi_{K,\psi}$  was originally introduced by Siciak [70] in order to extend classical results of approximation and interpolation to holomorphic functions of several complex variables. Later, Zakharyuta [84] defined the extremal function  $V_K := V_{K,\psi}$  with  $X = \mathbb{C}^n$  and  $\psi = \log|z|$ .

It is well known that  $\log \Phi_K = V_K$  for every compact  $K \subset \mathbb{C}^n$  (see for example [49, Theorem 5.1.7]), but on a more general manifold such an equality might not be true, even if we assume  $\psi$  to be a parabolic potential. Indeed, as mentioned before, there exists an example of a parabolic potential  $\psi$  on a manifold  $X$  such that  $\mathcal{P}^\psi$  consists only of the constants [6], in which case we have  $\log \Phi_{K,\psi} \equiv 0$ . In general we have  $\log \Phi_{K,\psi} \leq V_{K,\psi}$ .

Now consider a  $\psi$ -polynomial  $p$  on  $X$  of degree  $t$  and a non-pluripolar set  $K$ . The following Bernstein-Walsh inequality follows almost directly from the definition of  $\Phi_{K,\psi}$ :

$$|p(z)| \leq \|p\|_K (\Phi_{K,\psi}(z))^t \leq \|p\|_K e^{tV_{K,\psi}(z)}, \quad z \in X. \quad (3.5)$$

We end this section by proving an earlier statement regarding the dimension of the polynomial spaces  $\mathcal{P}_m^\tau$  when  $\tau$  is a parabolic potential.

*Proof of Proposition 3.1.2.* Let  $U' = B(0, 3)$  be the ball with radius 3, centered at the origin in  $\mathbb{C}^n$ , write  $U'_0 = B(0, 2)$  and  $K' = \overline{B}(0, 1)$ . Then  $V_{K'}(z) = \log^+ |z|$  for  $z \in \mathbb{C}^n$  and  $U'_0 = \{z \in \mathbb{C}^n; V_{K'}(z) < \log(2)\}$ .

Let  $z_0$  be a fixed point in  $X$  and  $H : U \rightarrow U'$  be a biholomorphism defined in a neighborhood of  $z_0$ . Write  $U_0 = H^{-1}(U'_0)$  and  $K = H^{-1}(K')$ . The set  $K$  is not pluripolar so by Proposition 3.1.4 the function  $V_{K,\tau}$  is locally bounded and

$$r := \sup_{z \in \overline{U}_0} e^{V_{K,\tau}(z)} < \infty.$$

For each polynomial  $p \in \mathcal{P}_m^\tau$  write  $\tilde{p} := p \circ H^{-1} \in \mathcal{O}(U')$ . Since  $\tilde{p}$  is holomorphic in a neighborhood of  $U'_0 = \{z \in \mathbb{C}^n; V_{K'}(z) < \log(2)\}$ , then by a result of Plèsniak [60], applied with  $L = 2$  we can find a constant  $C$  such that

$$d_{K'}(\tilde{p}, \mathcal{P}_k(\mathbb{C}^n)) \leq C2^{-k} \|\tilde{p}\|_{U'_0}, \quad p \in \mathcal{P}^\tau, k \in \mathbb{N}.$$

By the Bernstein-Walsh inequality (3.5) above we then have

$$d_{K'}(\tilde{p}, \mathcal{P}_k(\mathbb{C}^n)) \leq C2^{-k} \|\tilde{p}\|_{U'_0} = C2^{-k} \|p\|_{U_0} \leq C2^{-k} r^m \|p\|_K$$

for all  $m \in \mathbb{N}$ ,  $p \in \mathcal{P}_m^\tau$  and  $k \in \mathbb{N}$ . In particular if we choose  $M \in \mathbb{N}$  large enough such that  $C2^{-M}r < 1$ , then

$$d_{K'}(\tilde{p}, \mathcal{P}_{mM}(\mathbb{C}^n)) < \|p\|_K, \quad \text{for all } m \in \mathbb{N} \text{ and } p \in \mathcal{P}_m^\tau. \quad (3.6)$$

We claim that

$$\dim \mathcal{P}_m^\tau \leq \dim \mathcal{P}_{mM}(\mathbb{C}^n) = \binom{n+mM}{n}.$$

To derive a contradiction, suppose that  $\dim \mathcal{P}_m^\tau > \dim \mathcal{P}_{mM}(\mathbb{C}^n)$  and write  $\tilde{\mathcal{P}}_m^\tau = \{\tilde{p}; p \in \mathcal{P}_m^\tau\}$ . By considering  $\mathcal{P}_{mM}(\mathbb{C}^n)$  and  $\tilde{\mathcal{P}}_m^\tau$  as linear subspaces of  $C(K')$  with the sup-norm  $\|\cdot\|_{K'}$ , then a result of Krein, Krasnoselsky and Milman (see [72, Lemma 6.1]) implies that since  $\dim \tilde{\mathcal{P}}_m^\tau > \dim \mathcal{P}_{mM}(\mathbb{C}^n)$ , there exists a polynomial  $p_0 \in \mathcal{P}_m^\tau$  with the property that

$$\|\tilde{p}_0\|_{K'} \leq \|\tilde{p}_0 - q\|_{K'} \quad \text{for all } q \in \mathcal{P}_{mM}(\mathbb{C}^n).$$

In particular we have  $\|p_0\|_K = \|\tilde{p}_0\|_{K'} \leq d_{K'}(\tilde{p}, \mathcal{P}_{mM}(\mathbb{C}^n))$  which contradicts inequality (3.6).

## 3.2 Results

In this section we present the main results of this chapter. All results are proven in Section 3.4. Recall that if  $\omega$  is a Kähler-form on  $X$  with coefficients  $\omega_{j,\bar{k}}$  with respect to a given coordinate system then the Ricci curvature of  $\omega$  is given as follows

$$\text{Ricci}(\omega) = -\frac{1}{2} dd^c \log(\text{Det}(\omega_{j,\bar{k}})).$$

For any  $z \in X$ ,  $r > 0$  we denote by  $B(z, r, \omega)$  the geodesic ball with center  $z$  and radius  $r$  with respect to the metric  $\omega$ .

**Definition 3.2.1** Let  $\psi$  be a psh exhaustion function on the  $n$ -dimensional manifold  $X$  and assume that the  $(1,1)$ -form  $\frac{1}{4} dd^c e^\psi$  is smooth and strictly positive outside a compact subset  $S$  of  $X$ .

- (i) Let  $\theta \in \text{PSH}(X)$ . We say that  $\theta$  is a Ricci compensator for  $\psi$  if it is continuous, strictly psh in a neighborhood of  $S$ ,

$$|\theta| \leq A\psi^+ + B \quad \text{on } X \tag{3.7}$$

for some constants  $A, B > 0$  and

$$\frac{1}{4} dd^c \theta + \text{Ricci} \left( \frac{1}{4} dd^c e^\psi \right) \geq 0, \quad \text{on } X \setminus S.$$

If there exists a Ricci compensator for  $\psi$ , then we say that  $\psi$  is Ricci compensable.

- (ii) We say that  $\psi$  induces an integral estimate for holomorphic functions if for every  $\delta > 0$  there are constants  $A, B$  such that for every  $z \in X \setminus S$  and every function  $F \in \mathcal{O}(\overline{B}(z, \delta, \frac{1}{4}dd^c e^\psi))$

$$|F(z)|^2 \leq e^{A\psi^+(z)+B} \int_{B(z, \delta, \frac{1}{4}dd^c e^\psi) \setminus S} |F|^2 \left(\frac{1}{4}dd^c e^\psi\right)^n. \quad (3.8)$$

To help the readers familiarize themselves with these new concepts, consider the simple example  $X = \mathbb{C}^n$  and  $\psi = \log |z|^2$ . In this case

$$\frac{1}{4}dd^c e^\psi = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$$

is the standard Euclidean metric on  $\mathbb{C}^n$  which is Ricci-flat. The function  $\theta = 0$  is therefore a Ricci compensator so  $\psi$  is Ricci compensable. Moreover, if we take  $A = B = 0$ , inequality (3.8) is simply the sub mean value inequality for holomorphic functions on  $\mathbb{C}^n$  so  $\psi$  clearly induces an integral estimate for holomorphic functions.

For a given psh exhaustion function  $\psi$ , if we can explicitly calculate the Ricci curvature of  $\frac{1}{4}dd^c e^\psi$ , then in some cases it immediately becomes apparent whether or not there exists a Ricci compensator for  $\psi$ ; we see this in Section 3.3. In this sense, verifying whether or not  $\psi$  is Ricci compensable can be a straight forward task. Though it might not be immediately clear how to verify whether or not  $\psi$  induces an integral estimate for holomorphic functions. The method we use to do that in Section 3.3 involves comparing the metric  $\frac{1}{4}dd^c e^\psi$  with the Euclidean one in a local coordinate chart and using the sub mean value theorem for holomorphic functions.

We present the first main result of this chapter.

**Theorem 3.2.2** *Let  $\psi$  be a psh exhaustion function on  $X$  which is Ricci compensable and induces an integral estimate for holomorphic functions. Let  $K \subset X$  be compact and  $\varphi \in \mathcal{L}_\psi^+$  be a continuous function satisfying  $\varphi|_K \leq 0$ . Then for every  $L \in ]1, \infty[$  and every function  $f$  holomorphic on  $K_L := \{z \in X; \varphi(z) < \log(L)\}$  we have*

$$\limsup_{t \rightarrow \infty} \left( d_K(f, \mathcal{P}_t^\psi) \right)^{1/t} \leq L^{-1}. \quad (3.9)$$

Notice that if  $\mathcal{L}_\psi^+$  contains a parabolic potential  $\tau$  then by Proposition 3.1.2 we

see that the polynomial spaces  $\mathcal{P}_t^\psi = \mathcal{P}_t^\tau$  have finite dimension so Theorem 3.2.2 is, in some sense, the strongest when this is the case. We should note though, that for a given parabolic potential  $\tau$  it is not always possible to find  $\psi \in \mathcal{L}_\tau$  satisfying the properties of Definition 3.2.1 (for instance if  $\mathcal{P}^\tau$  consists only of constants). We then need larger polynomial spaces if we want to apply Theorem 3.2.2.

We should at this point consider the similarities between the assumptions made in Theorem 3.2.2 and the ones made in Demailly's result, Theorem 2.4.7. Indeed the concept of a Ricci compensator highly resembles assumption (ii) of Theorem 2.4.7. This means that if we want to find examples of non-algebraic manifolds and corresponding psh exhaustion functions which are Ricci compensable and induce an integral estimate for holomorphic functions, we should only consider functions which fail to satisfy at least one of the other assumptions of Demailly's theorem. Notice that while Demailly requires  $\psi$  to be  $C^\infty$  and strictly psh, we require  $e^\psi$  to be  $C^\infty$  and strictly psh outside a compact set, which is a weaker assumption. Beside the regularity condition we can also consider functions who fail to satisfy either (i) or (iii) from Theorem 2.4.7.

If  $X$  is an algebraic manifold then there exists an abundance of psh exhaustion functions on  $X$  which are Ricci compensable and induce an integral estimate for holomorphic functions, see Section 3.3.

If  $\mathcal{L}_\psi$  is an abstract Lelong class then the function  $V_{K,\psi}$  is locally bounded for every non-pluripolar set  $K$ . If  $V_{K,\psi}$  happens to be continuous as well, then we can take  $\varphi$  to be equal to  $V_{K,\psi}$  in Theorem 3.2.2. If we make the stronger assumption, that  $\psi \in \mathcal{L}_\tau^+$  for a continuous parabolic potential  $\tau$  then Proposition 3.1.6 gives a criterion to check the continuity of  $V_{K,\psi} = V_{K,\tau}$ . Of course the subject of the continuity of the extremal function or the existence of a continuous parabolic potential is a delicate one. As mentioned in Section 2.4.2, we do not even know if  $S$ -parabolic manifolds are  $S^*$ -parabolic. In either case, if we take  $\varphi = V_{K,\psi}$  in Theorem 3.2.2 then the converse is true as well. Indeed the following proposition makes no assumption on the continuity of  $V_{K,\psi}$ .

**Proposition 3.2.3** *If  $\mathcal{L}_\psi$  is an abstract Lelong class and  $f : K \rightarrow \mathbb{C}$  is any function such that*

$$\limsup_{t \rightarrow \infty} \left( d_K(f, \mathcal{P}_t^\psi) \right)^{1/t} \leq L^{-1} \quad (3.10)$$

*for some  $L > 1$ , then  $f$  is the restriction to  $K$  of a function holomorphic on the set  $K_L = \{z \in X; V_{K,\psi}(z) < \log(L)\}$ .*

Coming back to the comparison of the functions  $V_{K,\psi}$  and  $\log(\Phi_{K,\psi})$ , we have the following result.

**Proposition 3.2.4** *Assume that  $\mathcal{L}_\psi$  is an abstract Lelong class. Further assume that*

- (i) *for any  $L > 1$  and any function  $f$  holomorphic on the set  $K_L$  inequality (3.9) is true;*
- (ii) *any connected component of any sublevel set of the function  $\Phi_{K,\psi}$  contains a connected component of  $K$ .*

Then  $V_{K,\psi}^* = \log(\Phi_{K,\psi}^*)$ .

Observe that if inequality (3.10) holds for every  $L > 0$ , then  $f$  is the restriction to  $K$  of an entire function, also denoted by  $f$ . If  $f$  is of finite order  $\varrho$  and of finite type  $\sigma$  with respect to  $\varrho$ , then we have a more precise estimate of  $d_K(f, \mathcal{P}_t^\psi)^{1/t}$ . More precisely, we have a generalization of a theorem of Winiarski [83] for the special case  $X = \mathbb{C}^n$  and  $\psi = \log|z|$ .

**Theorem 3.2.5** *Assume  $\psi$  is Ricci compensable and induces an integral estimate for holomorphic function. Further assume that for every  $r > 0$  there exist constants  $A, B$  such that*

$$\int_{\{\psi(z) < \log(L)\} \setminus S} \left(\frac{1}{4} dd^c e^\psi\right)^n \leq e^{AL^r + B}, \quad L > 1, \quad (3.11)$$

where  $S$  is the same as in Definition 3.2.1. Let  $K \subset X$  be compact and  $\varphi \in \mathcal{L}_\psi^+$  be a continuous function on  $X$  satisfying  $\varphi|_K \leq 0$ . Then for any entire function  $f$  on  $X$  satisfying the growth estimates

$$\limsup_{r \rightarrow \infty} \frac{\log^+ \log \|f\|_{\{\varphi \leq \log(r)\}}}{\log(r)} \leq \varrho \quad \text{and} \quad \limsup_{r \rightarrow \infty} \frac{\log \|f\|_{\{\varphi \leq \log(r)\}}}{r^\varrho} \leq \sigma, \quad (3.12)$$

for some  $\varrho > 0$ ,  $\sigma \geq 0$ , we have

$$\limsup_{t \rightarrow \infty} t^{1/\varrho} (d_K(f, \mathcal{P}_t^\psi))^{1/t} \leq (e\sigma\varrho)^{1/\varrho}. \quad (3.13)$$

If  $\mathcal{L}_\psi$  is an abstract Lelong class and  $\varphi := V_{K,\psi}$  then the converse holds as well, i.e. if  $f$  is a function on  $K$  and inequality (3.13) holds, then  $f$  extends to an entire function on  $X$  and inequalities (3.12) are true with  $\varphi$  replaced by  $V_{K,\psi}$ .

Theorems 3.2.2 and 3.2.5 are based on a third main result, in which we give an estimate for  $d_K(f, \mathcal{P}_t^\psi)$  for fixed  $t$ .

**Theorem 3.2.6** *Let  $\psi$  be a psh exhaustion on  $X$  such that  $\frac{1}{4}dd^c e^\psi$  is smooth and strictly positive outside a compact set  $S$  and assume  $\psi$  induces an integral estimate for holomorphic functions. Let  $K \subset X$  be compact and  $\varphi$  be a continuous psh function on  $X$  satisfying:*

- (i) *There is a constant  $t_0 > 0$  such that  $t_0 dd^c \varphi + \text{Ricci}(\frac{1}{4}dd^c e^\psi) \geq 0$  on  $X \setminus S$  and  $dd^c \varphi > 0$  on  $S$ ,*
- (ii)  $\varphi \in \mathcal{L}_\psi$ ,
- (iii)  $\varphi|_K \leq 0$ .

Let  $L > 1$ ,  $f$  be a function holomorphic on the set  $\{z \in X; \varphi(z) < \log(L)\}$  and  $\varepsilon \in ]0, (L-1)/2[$ . Then there are constants  $l, T_0$  neither depending on  $f$  nor  $L$  and a constant  $M$  not depending on  $f$ , such that for any  $t \geq T_0$  we have

$$d_K(f, \mathcal{P}_t^\psi) \leq M \|f\|_{\{\varphi \leq \log(L-\varepsilon/2)\}} \left( \frac{1+\varepsilon}{L-\varepsilon} \right)^{t-l}. \quad (3.14)$$

**Remark.** If  $L > 1$  is large enough, we can replace the constant  $M$  from inequality (3.14) with a more precise quantity. See Lemma 3.4.4 for more details.

### 3.3 Examples

In this section we denote by

$$\omega_0 = \frac{i}{2} \sum_{j=1}^n dz_j \wedge d\bar{z}_j$$

the Euclidean metric on  $\mathbb{C}^n$  and we denote by  $B(z, r) = B(z, r, \omega_0) \subset \mathbb{C}^n$  the Euclidean ball with center  $z$  and radius  $r$ .

We start this section by proving a few propositions. We then apply them to construct a few examples of psh exhaustion functions  $\psi$  on Stein manifolds which are Ricci compensable and induce an integral estimate for holomorphic functions.

**Proposition 3.3.1** *Let  $\psi$  be a psh exhaustion on  $X$  such that  $\frac{1}{4}dd^c e^\psi$  is smooth and strictly positive outside a compact set  $S$ . If there exist functions  $\varepsilon, M : X \rightarrow \mathbb{R}_+$  and constants  $A, B > 0$  satisfying*

$$\varepsilon(z) \geq e^{-(A\psi^+(z)+B)} \quad \text{and} \quad M(z) \leq e^{A\psi^+(z)+B}, \quad z \in X,$$

such that for every  $z \in X \setminus S$ , there is a coordinate patch  $\xi : B(0, \varepsilon(z)) \rightarrow X$  with  $\xi(0) = z$ ,  $\xi^*(\frac{1}{4}dd^c e^\psi) \leq M(z)\omega_0$  and  $\omega_0^n \leq M(z)(\xi^*\frac{1}{4}dd^c e^\psi)^n$  on  $B(0, \varepsilon(z))$ , then  $\psi$  induces an integral estimate for holomorphic functions.

*Proof.* Let  $\delta > 0$  and write  $r(z) = \delta_1 \min\{\varepsilon(z), (M(z))^{-1}\}$  where  $\delta_1 = \min\{1, \delta\}$ . Since  $\xi^*(\frac{1}{4}dd^c e^\psi) \leq M(z)\omega_0$  on  $B(0, \varepsilon(z))$  we have

$$\begin{aligned} B(0, r(z)) &= B(0, \delta_1, \max\{\varepsilon^{-1}(z), M(z)\}\omega_0) \\ &\subset B(0, \delta_1, \xi^*\frac{1}{4}dd^c e^\psi) \subset \xi^{-1}B(z, \delta, \frac{1}{4}dd^c e^\psi). \end{aligned}$$

Let  $F$  be a function holomorphic in a neighborhood of the ball  $\overline{B}(z, \delta, \frac{1}{4}dd^c e^\psi)$  and denote by  $v_{2n}$  the volume of the unit ball of dimension  $2n$ . Then by the sub-mean-value inequality on  $\mathbb{C}^n$  we have

$$\begin{aligned} |F(z)|^2 &= |F \circ \xi(0)|^2 \leq \frac{1}{v_{2n}(r(z))^{2n}} \int_{B(0, r(z))} |F \circ \xi|^2 \omega_0^n \\ &\leq \frac{\max\{\varepsilon^{-1}(z), M(z)\}^{2n}}{v_{2n}\delta_1^{2n}} \int_{B(0, r(z))} |F \circ \xi|^2 M(z) (\xi^*\frac{1}{4}dd^c e^\psi)^n \\ &\leq \frac{\max\{\varepsilon^{-1}(z), M(z)\}^{2n} M(z)}{v_{2n}\delta_1^{2n}} \int_{B(z, \delta, \frac{1}{4}dd^c e^\psi)} F(\frac{1}{4}dd^c e^\psi)^n. \end{aligned}$$

By assumption of the growth rate of  $\varepsilon$  and  $M$  we get the result.  $\square$

**Proposition 3.3.2** Assume  $\psi$  is an exhaustion function of the form

$$\psi(z) = \log(|g_1(z)|^2 + \cdots + |g_m(z)|^2), \quad z \in X,$$

for some holomorphic functions  $g_1, \dots, g_m \in \mathcal{O}(X)$ .

(i) The Ricci curvature of  $\frac{1}{4}dd^c e^\psi$  is

$$\text{Ricci}(\frac{1}{4}dd^c e^\psi) = -\frac{1}{2}dd^c \log \left( \sum_{1 \leq j_1 < \dots < j_n \leq m} |\text{Det}(\text{Jac}(g_{j_1}, \dots, g_{j_n}))|^2 \right).$$

where the sum is taken over every subcollection  $\{j_1 \leq \dots \leq j_n\} \subset \{1, \dots, m\}$  of size  $n$  and  $\text{Jac}(g_{j_1}, \dots, g_{j_n})$  is the Jacobian of  $g_{j_1}, \dots, g_{j_n}$  with respect to any local coordinate system.

(ii) If there exist constants  $A, B > 0$  such that for every  $z^0 \in X$  there exists a subcollection  $\{g_{j_1}, \dots, g_{j_n}\} \subset \{g_1, \dots, g_m\}$  and a neighborhood  $V$  of  $z^0$  such

that  $(g_{j_1}, \dots, g_{j_n})$  maps  $V$  bijectively to an open ball in  $\mathbb{C}^n$  of radius  $\varepsilon(z^0) \geq e^{-(A\psi^+(z^0)+B)}$  and such that for every  $g_k \in \{g_1, \dots, g_m\} \setminus \{g_{j_1}, \dots, g_{j_n}\}$  we have

$$dd^c |g_k|^2 \leq e^{A\psi^+ + B} dd^c \sum_{s=1}^n |g_{j_s}|^2 \quad \text{on } V, \quad (3.15)$$

then  $\psi$  induces an integral estimate for holomorphic functions.

*Proof.* (i) Let  $(z_1, \dots, z_n)$  be some local coordinate chart. We have

$$\frac{1}{4} dd^c e^\psi = \frac{i}{2} \sum_{j,k,r} \frac{\partial g_r}{\partial z_j} \overline{\frac{\partial g_r}{\partial z_k}} dz_j \wedge d\bar{z}_k$$

and therefore, by the Cauchy-Binet formula, we have

$$\begin{aligned} \left(\frac{1}{4} dd^c e^\psi\right)^n &= \text{Det} \left( \sum_r \frac{\partial g_r}{\partial z_j} \overline{\frac{\partial g_r}{\partial z_k}} \right)_{j,k} dV \\ &= \left( \sum_{1 \leq j_1 < \dots < j_n \leq m} |\text{Det}(\text{Jac}(g_{j_1}, \dots, g_{j_n}))|^2 \right) dV. \end{aligned}$$

The result now follows from definition of the Ricci curvature.

(ii) This is a special case of Proposition 3.3.1. Indeed, for each  $z^0 \in X$  define  $\xi$  as the inverse of the map  $z \rightarrow (g_{j_1}(z), \dots, g_{j_n}(z))$ . Then we have

$$\xi^* \left( \frac{1}{4} dd^c e^\psi \right) = \omega_0 + \xi^* \sum_{\substack{k \neq j_s \\ \text{for any } s}} \frac{1}{4} dd^c |g_k|^2 \leq (1 + e^{(m-n)(A\psi^+ + B)}) \omega_0$$

and  $(\xi^* \frac{1}{4} dd^c e^\psi)^n \geq \omega_0^n$  on  $B(0, \varepsilon(z^0))$  so the assumptions of Proposition 3.3.1 are satisfied. □

**Proposition 3.3.3** *Let  $\eta : Y \rightarrow X$  be a holomorphic covering of  $X$  with finite fibers and  $\psi$  be a psh exhaustion function on  $X$  such that  $\frac{1}{4} dd^c e^\psi$  is smooth and strictly positive outside a compact subset of  $X$ . Then  $\frac{1}{4} dd^c e^{\eta^* \psi}$  is smooth and strictly positive outside a compact subset of  $Y$  and*

- (i) *if  $\theta$  is a psh function on  $X$  such that  $\eta^* \theta$  is a Ricci compensator for  $\eta^* \psi$ , then  $\theta$  is a Ricci compensator for  $\psi$ ;*

(ii) if  $\eta^*\psi$  induces an integral estimate for holomorphic functions on  $Y$  then so does  $\psi$  on  $X$ .

*Proof.* Since  $\psi$  is an exhaustion function and  $\eta$  has finite fibers (which means that it is proper)  $\eta^*\psi$  is also an exhaustion function.

(i) Suppose that  $\eta^*\theta$  is a Ricci compensator for  $\eta^*\psi$ . Then by (3.7) there exist constants  $A, B$  such that  $|\theta(\eta(z))| \leq A\psi^+(\eta(z)) + B$  outside a compact subset of  $Y$ . Since  $\eta$  is surjective this implies that  $|\theta| \leq A\psi^+ + B$  outside a compact subset of  $X$ . Moreover, by assumption we have

$$\eta^* \left( \frac{1}{4} dd^c \theta + \text{Ricci} \left( \frac{1}{4} dd^c e^\psi \right) \right) = \frac{1}{4} dd^c (\eta^* \theta) + \text{Ricci} \left( \frac{1}{4} dd^c e^{\eta^* \psi} \right) \geq 0 \quad (3.16)$$

outside a compact subset of  $Y$ . Since  $\eta$  is locally a biholomorphism (3.16) implies that

$$\frac{1}{4} dd^c \theta + \text{Ricci} \left( \frac{1}{4} dd^c e^\psi \right) \geq 0$$

outside a compact subset of  $X$  so  $\theta$  is a Ricci compensator for  $\psi$ .

(ii) Assume that  $\eta^*\psi$  induces an integral estimate for holomorphic functions on  $Y$ . This means that there exists a compact set  $S' \subset Y$  such that for every  $\delta > 0$  there exist constants  $A, B$  such that for every  $\zeta \in Y \setminus S'$  and every  $F \in \mathcal{O}(\overline{B}(\zeta, \delta, \frac{1}{4} dd^c e^{\eta^* \psi}))$  inequality (3.8) is true with  $S$  replaced by  $S'$ ,  $\psi$  replaced by  $\eta^*\psi$  and  $z$  replaced by  $\zeta$ . Now let  $S = \eta(S')$ ,  $z \in X \setminus S$ ,  $\zeta \in \eta^{-1}(z)$ , and  $F \in \mathcal{O}(\overline{B}(\zeta, \delta, \frac{1}{4} dd^c e^\psi))$ . Since  $\eta$  has finite fibers, say of degree  $k$ , then integrals change at most up to a factor of  $k$  through the pullback of  $\eta$ . Therefore we have

$$\begin{aligned} |F(z)| &= |F \circ \eta(\zeta)|^2 \leq e^{A\eta^*\psi^+(\zeta)+B} \int_{B(\zeta, \delta, \frac{1}{4} dd^c e^{\eta^* \psi}) \setminus S'} |\eta^* F|^2 \left( \frac{1}{4} dd^c e^{\eta^* \psi} \right)^n \\ &\leq k e^{A\psi^+(z)+B} \int_{B(z, \delta, \frac{1}{4} dd^c e^\psi) \setminus S} |F|^2 \left( \frac{1}{4} dd^c e^\psi \right)^n. \end{aligned}$$

Therefore  $\psi$  induces an integral estimate for holomorphic functions on  $X$ .  $\square$

**Lemma 3.3.4** *Let  $f : I \rightarrow \mathbb{R}$  be a positive  $C^1$  function on an interval  $I \subset \mathbb{R}$  around 0, satisfying  $f'(x) \leq ax^b f^c(x)$  for some constants  $a, b > 0, c \geq 1$  and assume  $f(0) \geq 1$ . Then the domain of definition of  $f$  contains the interval ranging from 0 to*

$$\left( \frac{b+1}{a(c-1)f^{c-1}(0)} \right)^{\frac{1}{b+1}},$$

in the sense that  $f$  does not blow up to  $\infty$  on that interval.

*Proof.* This is a variation of Grönwall's inequality. Write

$$g(x) = f(0)(1 - \alpha x^{b+1})^{\frac{-1}{c-1}}, \quad \text{where} \quad \alpha = \frac{a(c-1)f^{c-1}(0)}{b+1}.$$

Then  $g$  is the solution to the initial value problem  $g'(x) = ax^b(g(x))^c$ ,  $g(0) = f(0)$ . The function  $g$  goes to infinity at  $\alpha^{\frac{-1}{b+1}}$  and clearly  $f \leq g$ .  $\square$

Now we consider several examples of Stein manifolds and corresponding psh exhaustion functions.

**Polynomials in  $\mathbb{C}^n$ .** Let  $X = \mathbb{C}^n$  and let  $g_1, \dots, g_m$  be polynomials on  $\mathbb{C}^n$  such that

$$\psi(z) := \log(|g_1(z)|^2 + \dots + |g_m(z)|^2), \quad z \in \mathbb{C}^n$$

is an exhaustion function. Further assume that the Jacobian of the map  $z \rightarrow (g_1(z), \dots, g_m(z))$  has full rank on  $\mathbb{C}^n$ . Then  $\psi$  is Ricci compensable, induces an integral estimate for holomorphic functions and satisfies inequality (3.11) from Theorem 3.2.5. Moreover it is easy to see that  $\mathcal{L}_\psi$  is an abstract Lelong class so the  $\psi$ -polynomial spaces have finite dimension. We do not prove these claims here since we prove the general case of an algebraic manifold in the next example.

**Affine algebraic manifolds.** Let  $X \subset \mathbb{C}^N$  be a non-singular algebraic manifold of dimension  $n$  and let  $g_1, \dots, g_m$  be polynomials on  $X$  (i.e. each  $g_j$  is the restriction of a polynomial on  $\mathbb{C}^N$  to  $X$ ). Assume that the function

$$\psi(z) = \log(|g_1(z)|^2 + \dots + |g_m(z)|^2), \quad z \in X$$

is an exhaustion function, and further assume that the Jacobian of the map  $X \rightarrow \mathbb{C}^m$ ,  $z \rightarrow (g_1(z), \dots, g_m(z))$  has full rank on  $X$ . Then  $\psi$  is Ricci compensable and induces an integral estimate for holomorphic functions. By Rudin [63], after a linear change of variables, we can assume that  $X$  is a subset of

$$\{z = (z_1, \dots, z_n, z_{n+1}, \dots, z_N) = (z', z'') \in \mathbb{C}^N; |z''| \leq A(1 + |z'|)^B\}$$

for some positive constants  $A, B$ . This implies that the function  $\tau := \log |z'|$  is a parabolic potential on  $X$ . Since  $\psi \in \mathcal{L}_{C\tau}$  for  $C > 0$  large enough, we see that the polynomial spaces  $\mathcal{P}_t^\psi$  have finite dimension.

We now prove that  $\psi$  is Ricci compensable and induces an integral estimate for

holomorphic functions. Our method is based on Demailly's calculations from the proof of [33, Proposition 10.1]. Indeed, we generalize this result by calculating the Ricci curvature of  $\frac{1}{4}dd^c e^\psi$ .

Let  $P_1, \dots, P_r$  be generators of the ideal  $I(X)$  of polynomials in  $\mathbb{C}^N$  vanishing on  $X$  and let  $s = N - n$  be the codimension of  $X$ . For each  $K = \{k_1 < \dots < k_s\} \subset \{1, \dots, r\}$  and each  $L = \{l_1 < \dots < l_n\} \subset \{1, \dots, m\}$  denote by  $J_{K,L}$  the determinant of the Jacobian of the functions  $g_{l_1}, \dots, g_{l_n}, P_{k_1}, \dots, P_{k_s}$  on  $\mathbb{C}^N$ . Further write

$$U_K := X \cap \{z \in \mathbb{C}^N; dP_{k_1} \wedge \dots \wedge dP_{k_s}(z) \neq 0\}.$$

The sets  $U_K$  form an open cover of  $X$  since it is non-singular. Denote by  $(z_1, \dots, z_N)$  the standard coordinates on  $\mathbb{C}^N$ , write  $T_0 := \{n+1, n+2, \dots, N\}$  and let  $w \in U_K$  be fixed. Without loss of generality we can assume that  $X$  can be parameterized in the variables  $(z_1, \dots, z_n)$  in a neighborhood of  $w$ . That means that in a neighborhood of  $w$  in  $\mathbb{C}^N$  we have

$$|D_{K,T_0}|^2 := \left| \text{Det} \left( \frac{\partial P_k}{\partial z_t} \right)_{k \in K, t \in T_0} \right|^2 \neq 0$$

and therefore

$$\begin{aligned} & dg_{l_1} \wedge d\bar{g}_{l_1} \wedge \dots \wedge dg_{l_n} \wedge d\bar{g}_{l_n} \wedge dP_{k_1} \wedge d\bar{P}_{k_1} \wedge \dots \wedge dP_{k_s} \wedge d\bar{P}_{k_s} \\ &= |J_{K,L}|^2 dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_n \\ &= \frac{|J_{K,L}|^2}{|D_{K,T_0}|^2} dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_n \wedge dP_{k_1} \wedge d\bar{P}_{k_1} \wedge \dots \wedge dP_{k_s} \wedge d\bar{P}_{k_s}. \end{aligned} \quad (3.17)$$

Since the gradients  $\nabla P_k$  are orthogonal to the tangent space of  $X$  we see that when we restrict the forms from equation (3.17) to the submanifold  $X$  we have

$$dg_{l_1} \wedge d\bar{g}_{l_1} \wedge \dots \wedge dg_{l_n} \wedge d\bar{g}_{l_n} = \frac{|J_{K,L}|^2}{|D_{K,T_0}|^2} dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge dz_n \wedge d\bar{z}_n.$$

Now by applying Proposition 3.3.2 (i), and by noticing that the function  $\log |D_{K,T_0}|^2$  is pluriharmonic in a neighborhood of  $w$ , we see that

$$\text{Ricci} \left( \frac{1}{4} dd^c e^\psi \right) = -\frac{1}{2} dd^c \log \sum_{|L|=n} |J_{K,L}|^2, \quad \text{on } U_K. \quad (3.18)$$

Now let  $K_0$  be fixed. For any  $K \neq K_0$  the function

$$a_{K,K_0} := \log \sum_{|L|=n} |J_{K,L}|^2 - \log \sum_{|L|=n} |J_{K_0,L}|^2$$

is pluriharmonic on  $U_K \cap U_{K_0}$  since it is the difference of two local potentials of the Ricci curvature. Moreover, this function is locally bounded from above on  $U_{K_0}$  and since  $U_{K_0} \setminus U_K$  is an analytic subset of  $U_{K_0}$  the function  $a_{K,K_0}$  is psh on  $U_{K_0}$ . This is true for all  $K$  so the function

$$\log \sum_{|K|=s} e^{a_{K,K_0}} = \log \sum_{|K|=s, |L|=n} |J_{K,L}|^2 - \log \sum_{|L|=n} |J_{K_0,L}|^2 \quad (3.19)$$

is plurisubharmonic on  $U_{K_0}$ . Now define the function

$$\theta = \log \sum_{|K|=s, |L|=n} |J_{K,L}|^2, \quad \text{on } X. \quad (3.20)$$

By equations (3.18) and (3.19) we see that  $\frac{1}{2}dd^c\theta + \text{Ricci}(\frac{1}{4}dd^c e^\psi) \geq 0$  on  $X$ . Moreover, since the Jacobian of  $g_1, \dots, g_m$  has full rank on  $X$ , the functions  $|J_{K,L}|^2$  never vanish at the same time on  $X$ , i.e. we have  $\theta > -\infty$  on  $X$ . By a simple application of Hilbert's Nullstellensatz we can see that there exist constants  $A$  and  $B$  such that  $|\theta| \leq A\psi^+ + B$  on  $X$  and therefore  $\theta$  is a Ricci compensator for  $\psi$ .

By applying Hilbert's Nullstellensatz again we can find constants  $A_1, B_1$  such that for each  $z \in X$  we can find  $K = \{k_1, \dots, k_s\}$  and  $L = \{l_1, \dots, l_n\}$  such that

$$|J_{K,L}(z)|^2 \geq e^{-(A_1\psi^+(z)+B_1)}.$$

Since  $(g_{l_1}, \dots, g_{l_n}, P_{k_1}, \dots, P_{k_s})$  is a polynomial map it can be seen that it maps a neighborhood of  $z$  in  $\mathbb{C}^N$  bijectively to an open ball in  $\mathbb{C}^N$  of radius  $\varepsilon(z) \geq e^{-(A_2\psi^+(z)+B_2)}$  where  $A_2, B_2$  are constants independent of  $z$ . Since the functions  $P_{k_1}, \dots, P_{k_s}$  vanish on  $X$  we see that  $(g_{l_1}, \dots, g_{l_n})$  maps a neighborhood of  $z$  in  $X$  to an open ball in  $\mathbb{C}^n$  of radius  $\varepsilon(z)$ . Since the functions  $g_j$  are polynomials it is easy to see that inequality (3.15) is satisfied for some  $A, B$  so  $\psi$  induces an integral estimate for holomorphic functions.

**The complement of an algebraic manifold in the projective space.** Let  $\mathbb{P}^n(\mathbb{C})$  be the complex projective space of dimension  $n$  and denote by  $\pi : \mathbb{C}^{n+1} \rightarrow \mathbb{P}^n(\mathbb{C})$  the canonical projection  $(z_0, \dots, z_n) \rightarrow [z_0 : \dots : z_n]$ . Let  $Q$  be a homogeneous

polynomial on  $\mathbb{C}^{n+1}$ , write  $X = \mathbb{P}^n(\mathbb{C}) \setminus \{z \in \mathbb{P}^n(\mathbb{C}); Q(\pi^{-1}(z)) = 0\}$  and define

$$\psi(z) = \log(|g_1(z)|^2 + \cdots + |g_m(z)|^2) - \frac{2r}{\deg(Q)} \log |Q(z)|, \quad z \in X,$$

where  $g_1, \dots, g_m$  are homogeneous polynomials on  $\mathbb{C}^{n+1}$  of same degree  $r$  and assume that  $\psi$  is an exhaustion function. Write  $M = \{z \in \mathbb{C}^{n+1}; Q(z) = 1\}$  and denote by  $\pi_M$  the restriction of  $\pi$  to  $M$ ; then  $\pi_M$  is a  $\deg(Q)$ -fold covering of  $X$ . If the map  $z \rightarrow (g_1(z), \dots, g_m(z))$  has full rank on  $M$  then, by the previous example we see that

$$\pi_M^* \psi = \log(|g_1|^2 + \cdots + |g_m|^2)$$

induces an integral estimate for holomorphic functions and is Ricci compensable with Ricci compensator  $\theta$  as described in equation (3.20). In this specific case, the manifold  $M$  has codimension  $s = 1$  and the ideal  $I(M)$  is generated by the polynomial  $Q - 1$  so, using the notation from the previous example, we have

$$\theta = \log \sum_{|K|=s, |L|=n} |J_{K,L}|^2 = \log \sum_{\{l_1 < \dots < l_n\} \subset \{1, \dots, m\}} |\text{Jac}(Q, g_{l_1}, \dots, g_{l_n})|^2.$$

Since the polynomials  $Q, g_1, \dots, g_m$  are homogeneous and  $g_1, \dots, g_m$  have degree  $r$  we see that each of the functions  $\text{Jac}(Q, g_{l_1}, \dots, g_{l_n})$  is either identically equal to 0 or a homogeneous polynomial of degree  $(r-1)n + \deg(Q) - 1$ . We conclude that we can write  $\theta = \pi_M^* \tilde{\theta}$  where

$$\tilde{\theta} = \log \sum_{\{l_1 < \dots < l_n\} \subset \{1, \dots, m\}} |\text{Jac}(Q, g_{l_1}, \dots, g_{l_n})|^2 - \frac{2((r-1)n + \deg(Q) - 1)}{\deg(Q)} \log |Q|.$$

By Proposition 3.3.3 we see that  $\tilde{\theta}$  is a Ricci compensator for  $\psi$  and  $\psi$  induces an integral estimate for holomorphic functions on  $X$ .

**The complex torus.** Let  $X = \mathbb{C}^n / \mathbb{Z}^n$  be the complex torus and

$$\psi(z) = |\text{Im}(z)| = |y|, \quad z = x + iy \in X.$$

Then  $\psi$  is Ricci compensable, induces an integral estimate for holomorphic functions and satisfies (3.11). The function  $\psi$  is itself a parabolic potential so the polynomial spaces  $\mathcal{P}_t^\psi$  have finite dimension. Indeed the functions

$$\xi_a(z) := e^{2\pi i \langle z, a \rangle}, \quad a \in \mathbb{Z}^n, \quad |a| \leq \frac{t}{2\pi}, \quad z \in X, \quad (3.21)$$

form a basis for  $\mathcal{P}_t^\psi$  for every  $t$ . In this case we can apply our main theorems to prove classical results from Fourier analysis.

We now prove these statements. It is simple to show that  $(dd^c|y|)^n = 0$  if  $y \neq 0$  so  $\psi$  is a parabolic potential. Now suppose  $p \in \mathcal{P}_t^\psi$  for some  $t$ . Since  $p$  is periodic it is of the form  $p(z) = \sum_{a \in \mathbb{Z}^n} c_a e^{2\pi i \langle z, a \rangle}$  for some constants  $c_a$ . By the Paley-Wiener theorem we see that  $p$  is the Fourier-transform of a distribution with support on  $B(0, t) \subset \mathbb{R}^n$ . Therefore we have  $c_a = 0$  if  $|a| > \frac{t}{2\pi}$  and the functions from (3.21) form a basis for  $\mathcal{P}_t^\psi$ .

If  $|y| > 1$ , then the largest eigenvalue of the metric

$$\frac{1}{4} dd^c e^\psi = \frac{ie^\psi}{8} \sum_{1 \leq j, k \leq n} \left( \frac{\delta_{j,k}}{|y|} + \frac{y_j y_k (|y| - 1)}{|y|^3} \right) dz_j \wedge d\bar{z}_k$$

is  $\lambda_1 = e^\psi/4$  and corresponds to the eigenvector  $y$ . Therefore we can apply Proposition 3.3.1 with  $S = \{|y| \leq 1\}$ ,  $\varepsilon(z) = \frac{1}{2}$  (we can map the torus  $X$  to a strip in  $\mathbb{C}^n$  of width 1 centered at  $z$ ) and  $M(z) = \frac{1}{4} e^{\psi(z) + \frac{1}{2}}$  so  $\psi$  induces an integral estimate for holomorphic functions.

The metric  $\frac{1}{4} dd^c e^\psi$  has one more eigenvalue  $\lambda_2 = \frac{e^\psi}{4|y|}$  which corresponds to the  $(n-1)$ -dimensional eigenspace of vectors perpendicular to  $y$ . Therefore we have

$$\text{Ricci} \left( \frac{1}{4} dd^c e^\psi \right) = -\frac{1}{2} dd^c \log(\lambda_1 \lambda_2^{n-1}) = -\frac{n}{2} dd^c \psi + \frac{n-1}{2} dd^c \log \psi \quad \text{on } X \setminus S.$$

As a Ricci compensator we can take the function

$$\theta = v + \begin{cases} n\psi - (n-1) \log \psi, & \text{if } \psi \geq 2 \\ 2n - (n-1) \log 2, & \text{if } \psi < 2 \end{cases}$$

where  $v \in \mathcal{L}_\psi$  is any function which is strictly psh on  $\{\psi < 2\}$ . It is a simple exercise to show that  $\theta$  is indeed psh.

Finally we check (3.11). We have

$$\int_{\{\psi < \log(L)\} \setminus S} \left( \frac{1}{4} dd^c e^\psi \right)^n = \int_{\substack{1 < |y| \leq \log(L) \\ |x| \leq 1}} \frac{e^{n|y|}}{4^n |y|^{n-1}} \omega_0^n \leq \frac{1}{4^n} L^n \log^n(L).$$

**Complement of the zero set of a holomorphic function.** Let  $X = \mathbb{C}^n \setminus \{f = 0\}$  for some function  $f \in \mathcal{O}(X)$ . We assume that  $f$  is not a polynomial since we have

already considered algebraic manifolds. Assume there exist constants  $A, B$  such that

$$\log |\nabla f| \leq A \log^+ (|z|^2 + |f|^2 + |f|^{-2}) + B \quad \text{on } X. \quad (3.22)$$

Then  $\psi := \log^+ (|z|^2 + |f|^2 + |f|^{-2})$  is Ricci compensable, induces an integral estimate for holomorphic functions and satisfies inequality (3.11). There exist a lot of functions such that inequality (3.22) is satisfied; for instance if  $f(z) = e^{p_1(z)} + p_2(z)$  for some polynomials  $p_1$  and  $p_2$  then

$$\begin{aligned} \log |\nabla f| &\leq \log (|\nabla p_1| e^{p_1} + |\nabla p_2|) \\ &\leq \log (\max\{|\nabla p_1|, |\nabla p_2|\}) + \log(|e^{p_1}| + 1) \\ &\leq (m \log^+ |z| + B_1) + \log(|f| + |p_2| + 1) \\ &\leq 2m \log^+ (|z| + |f| + |f|^{-1}) + B_2 \leq m\psi^+ + B_3 \end{aligned}$$

where  $m = \max\{\deg p_1, \deg p_2\}$  and  $B_1, B_2, B_3 > 0$  are some constants. The class  $\mathcal{L}_\psi$  is generally not an abstract Lelong class, so the polynomial spaces are in general large in this case.

We now prove our claims. By Proposition 3.3.2 we have

$$\text{Ricci}(\frac{1}{4}dd^c e^\psi) = -\frac{1}{2}dd^c \log \left( 1 + (1 + |f|^{-4})|\nabla f|^2 \right)$$

and therefore we consider  $\theta = \log \left( 1 + (1 + |f|^{-4})|\nabla f|^2 \right)$  as a Ricci compensator, we just have to check that it satisfies (3.7). Indeed we have

$$\begin{aligned} |\theta| &= \theta \leq \log(1 + |f|^{-4}) + \log(1 + |\nabla f|^2) \\ &\leq (\log^+ |f|^{-4} + 1) + (\log^+ |\nabla f|^2 + 1) \leq (2A + 2)\psi^+ + (2B + 2) \end{aligned} \quad (3.23)$$

where  $A, B$  are the constant from assumption (3.22) so  $\theta$  is a Ricci compensator and  $\psi$  is Ricci compensable.

Next we show that (3.11) is satisfied. Using (3.23) we have

$$\begin{aligned} \int_{\{\psi < \log(L)\}} \left(\frac{1}{4}dd^c e^\psi\right)^n &= \int_{\psi < \log(L)} e^\theta \omega_0^n \\ &\leq L^{2A+2} e^{2B+2} \int_{|z|^2 < L} \omega_0^n = L^{2(A+n+1)} e^{2B+2} v_{2n} \end{aligned}$$

where  $v_{2n}$  is the volume of the unit ball in  $\mathbb{R}^{2n}$ .

To see that  $\psi$  induces an integral estimate for holomorphic functions we apply Proposition 3.3.2 (ii) with  $\varepsilon(z)$  defined as the Euclidean distance from  $z \in \mathbb{C}^n$  to  $\{f = 0\}$ . It is then clear that for each  $z^0 \in X$  there is a neighborhood  $V \subset X$  which is mapped by  $(z_1, \dots, z_n)$  to a ball of radius  $\varepsilon(z^0)$  in  $\mathbb{C}^n$ . By assumption (3.22) we clearly have

$$\frac{1}{4}dd^c|f|^2 \leq e^{A\psi^+ + B}\omega_0, \quad \frac{1}{4}dd^c|f|^{-2} \leq e^{A\psi^+ + B}\omega_0$$

on  $V$  for some constants  $A, B$  (independent of  $z^0$ ) so inequality (3.15) is satisfied. We only have to show that  $\varepsilon$  has the required growth properties. Let  $v$  be a unit vector pointing from  $z^0$  to a point on  $\{f = 0\}$  of least distance to  $z^0$  and consider the single variable function  $t \rightarrow r(t) := |f(z^0 + tv)|^{-1}$ . Then, by definition,  $r$  blows up to infinity at  $\varepsilon(z^0)$ . We can assume that  $|f(z^0 + tv)| \leq 1$  for all  $t \in (0, \varepsilon(z^0))$  because otherwise we just replace  $z^0$  with  $\tilde{z}^0 := z^0 + \tau v$  where  $\tau$  is the largest number in  $(0, \varepsilon(z^0))$  such that  $|f(\tilde{z}^0)| = 1$  and then we have  $\varepsilon(z^0) \geq \varepsilon(\tilde{z}^0)$ . We can also assume that  $|z^0 + tv|^2 \leq 2(|z^0|^2 + 1)$  for all  $t \in (0, \varepsilon(z^0))$  because otherwise  $\varepsilon(z^0) \geq 1$  and there is nothing to prove. Now, using (3.22), we have

$$\begin{aligned} r'(t) &= r^2(t) \frac{\partial}{\partial t} |f(z^0 + tv)| \leq 2r^2(t) |\nabla f(z^0 + tv)| \\ &\leq 2e^B r^2(t) (|z^0 + tv|^2 + |f|^2 + r^2(t) + 1)^A \\ &\leq 2e^B r^2(t) (4 + 2|z^0|^2 + r^2(t))^A \leq 2e^B (4 + 2|z^0|^2)^A r^{2(1+A)}(t). \end{aligned}$$

Applying Lemma 3.3.4 we see that

$$\varepsilon(z^0) \geq \frac{|f(z^0)|^{2A+1}}{2e^B(4 + 2|z^0|^2)^A}$$

which clearly satisfies the required growth condition.

## 3.4 Proofs

In this section we prove all results from Section 3.2. Recall that  $X$  is always a Stein manifold and  $\psi$  is a psh exhaustion function on  $X$ . We start with auxiliary propositions, extending the  $(1, 1)$  form  $dd^c e^\psi$  to a Kähler metric  $\omega$  defined on all of  $X$ .

**Proposition 3.4.1** *Assume that  $\frac{1}{4}dd^c e^\psi$  is smooth and strictly positive outside a*

compact subset  $S$  of  $X$ . Then there exists a Kähler form  $\omega$  on  $X$  such that  $\omega = \frac{1}{4}dd^c e^\psi$  outside a compact subset of  $X$ . Moreover, the metric  $\omega$  is complete.

*Proof.* By adding a constant to  $\psi$  we can assume that  $\psi|_S \leq -1$ . We define

$$\omega := \frac{1}{4}dd^c (e^{\Gamma \circ \psi} + \varepsilon \chi u)$$

where  $\Gamma : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth, increasing and convex function satisfying  $\Gamma(x) = -1/2$  for  $x \leq -1$  and  $\Gamma(x) = x$  for  $x > 0$ ,  $u$  is any strictly psh function defined in a neighborhood of  $\{\psi \leq 0\}$ ,  $\chi : X \rightarrow [0, 1]$  is a smooth function with  $\chi = 1$  on  $\{\psi \leq -1/2\}$  and  $\chi = 0$  on  $\{\psi \geq -1/4\}$  and  $\varepsilon$  is a small constant. We clearly have  $\omega = \frac{1}{4}dd^c e^\psi$  on  $\{\psi > 0\}$  and  $\omega$  is strictly positive on  $\{\psi \leq -1/2\} \cup \{\psi \geq -1/4\}$ . The form  $\frac{1}{4}dd^c e^{\Gamma \circ \psi}$  is strictly positive on  $\{-1/2 \leq \psi \leq -1/4\}$  so by choosing  $\varepsilon$  small enough we can make  $\omega$  strictly positive on  $X$ .

On  $X \setminus \{\psi \leq 0\}$  we have  $\omega = \frac{i}{2}e^\psi (\partial\psi \wedge \bar{\partial}\psi + \partial\bar{\partial}\psi) \geq \frac{i}{2}\partial\psi \wedge \bar{\partial}\psi$  and therefore  $|\partial\psi|_\omega = |\partial\psi + \bar{\partial}\psi|_\omega \leq 2|\partial\psi|_\omega \leq 2|\partial\psi|_{\partial\psi \wedge \bar{\partial}\psi} = 2$ . By [32, Lemma VIII 2.4] the metric is complete.  $\square$

For the rest of the section  $\omega$  is a Kähler form as described in Proposition 3.4.1. By adding a constant to  $\psi$  we can always assume that  $S = \{\psi \leq 0\}$  and  $\omega = \frac{1}{4}dd^c e^\psi$  on  $X \setminus S$ . If  $v$  is a tangent vector in  $X$  we denote by  $\omega(v)$  the length of  $v$  with respect to the metric  $\omega$ . Whenever we work in local coordinates  $(z_1, \dots, z_n)$  we write

$$\nabla g = \left( \frac{\partial g}{\partial z_1}, \dots, \frac{\partial g}{\partial z_n} \right) \quad \text{and} \quad \langle v, w \rangle = \sum_{j=1}^n v_j w_j$$

for every function  $g \in C^1$  and tangent vectors  $v, w$ .

**Lemma 3.4.2** *If  $B(z, r, \omega) \cap S = \emptyset$  then  $|e^{\psi(\zeta)/2} - e^{\psi(z)/2}| \leq \frac{r}{2}$  for every  $\zeta \in B(z, r, \omega)$ .*

*Proof.* For every tangent vector  $v$  we have

$$\omega(v) = \frac{i}{2}e^\psi (\partial\bar{\partial}\psi + \partial\psi \wedge \bar{\partial}\psi)(v) \geq e^\psi |\langle \nabla\psi, v \rangle|^2 \quad (3.24)$$

Let  $\gamma : [0, 1] \rightarrow X$  be a geodesic from  $z$  to  $\zeta$  of length  $r$ . By the fundamental theorem of calculus, and by (3.24) we have

$$\begin{aligned} e^{\psi(\zeta)/2} - e^{\psi(z)/2} &= \int_0^1 \frac{\partial}{\partial t} e^{(\psi \circ \gamma)/2} dt \\ &\leq \frac{1}{2} \int_0^1 e^{(\psi \circ \gamma)/2} |\langle \nabla\psi, \gamma' \rangle| dt \leq \frac{1}{2} \int_\gamma \omega^{\frac{1}{2}} = \frac{r}{2}. \end{aligned}$$

□

The following theorem is a special case of a famous result by Skoda [74].

**Theorem 3.4.3** *Let  $\varphi$  be a continuous psh function on  $X$  satisfying  $\frac{1}{4}dd^c\varphi + \text{Ricci}(\omega) \geq 0$  on  $X$ . Then, for any  $(0, q)$ -form  $f$  with  $C^\infty$  (resp.  $L^2_{\text{loc}}$ ) coefficients satisfying  $\bar{\partial}f = 0$  and  $\int_X |f|_\omega^2 e^{-\varphi} \omega^n < \infty$  there is a  $(0, q-1)$ -form  $u$  with  $C^\infty$  (resp.  $L^2_{\text{loc}}$ ) coefficients such that  $\bar{\partial}u = f$  and*

$$\int_X |u|_\omega^2 (e^\psi + 1)^{-2} e^{-2\varphi} \omega^n \leq \frac{1}{2n} \int_X s |f|_\omega^2 e^{-2\varphi} \omega^n,$$

where  $s$  is a non-negative function on  $X$  which equals 1 on  $X \setminus S$ .

*Proof.* First assume that  $\varphi$  is  $C^\infty$ . Consider the line bundle  $E := X \times \mathbb{C}$  over  $X$  with the trivial projection. On the fibers of  $E$  we define the Hermitian product

$$\langle \zeta_1, \zeta_2 \rangle_z := (1 + e^{\psi(z)})^{-2} \zeta_1 \bar{\zeta}_2$$

and denote by  $|\cdot|_E^2 = (1 + e^{\psi(z)})^{-2} |\cdot|^2$  the corresponding norm. Denote by  $i\Theta(E) = dd^c \log(1 + e^\psi)$  the Chern curvature tensor on  $E$  with respect to this metric (see for example comment (12.6) Chapter V in [32]). On  $X \setminus S$  we have

$$i(1 + e^\psi)^2 \Theta(E) = 4\omega + 2e^{2\psi} dd^c \psi \geq 4\omega$$

so, by assumption on  $\varphi$ , we have

$$i\Theta(E) + \text{Ricci}(\omega) + \frac{1}{4} dd^c \varphi \geq \frac{4\omega}{(1 + e^\psi)^2} \quad (3.25)$$

on  $X \setminus S$ . Therefore the sum of the eigenvalues of the  $(1, 1)$ -form on the left hand side of (3.25) with respect to the metric  $\omega$  is larger than or equal to  $4n(1 + e^\psi)^{-2}$ .

Now, if we consider  $f$  as a section of the cotangent bundle of  $E$ , by [32, Theorem VIII 6.5] we can find a  $(0, q-1)$  form  $u$  such that  $\bar{\partial}u = f$  and

$$\int_X |u|_E^2 e^{-2\varphi} \omega^n \leq \frac{1}{4n} \int_X s (1 + e^\psi)^2 |f|_E^2 e^{-2\varphi} \omega^n,$$

where  $s$  is a positive function equal to 1 on  $X \setminus S$ . If we replace  $|\cdot|_E^2$  by  $(1 + e^{\psi(z)})^{-2} |\cdot|^2$  the result follows.

If  $\varphi$  is not  $C^\infty$  we get the result by finding a decreasing sequence  $(\varphi_m)_{m \in \mathbb{N}}$  of  $C^\infty$  psh functions such that  $\varphi_m \searrow \varphi$  in the  $L^1_{\text{loc}}$  topology and by taking the limit. Since

$X$  is Stein, such a sequence exists.  $\square$

*Proof of Theorem 3.2.6.* By assumption (i) and compactness of  $S$  we can find a constant  $T_0$  such that  $\frac{1}{4}T_0 dd^c\varphi + \text{Ricci}(\omega) \geq 0$ . Let  $\chi : X \rightarrow \mathbb{R}$  be a  $C^\infty$  cutoff function with  $\chi = 1$  on  $\{\varphi < \log(L - \varepsilon)\}$  and  $\chi = 0$  on  $\{\varphi > \log(L - \varepsilon/2)\}$ . For any  $t > T_0$  we can apply Theorem 3.4.3 to find a function  $u_t$  solving the  $\bar{\partial}$ -equation  $\bar{\partial}u_t = \bar{\partial}(f\chi) = f\bar{\partial}\chi$  on  $X$  and satisfying

$$\int_X |u_t|^2 (e^\psi + 1)^{-2} e^{-2t\varphi} \omega^n \leq \frac{1}{2n} \int_X s |f\bar{\partial}\chi|_\omega^2 e^{-2t\varphi} \omega^n. \quad (3.26)$$

Now we define the function  $p_t := \chi f - u_t$ . It is clear that  $p_t$  is an entire function. Let  $z \in X$  be such that  $B(z, 1, \omega)$  does not intersect  $S$  or the support of  $\chi$ . Then  $u_t = -p_t$  on  $B(z, 1, \omega)$ .

Since  $\psi$  induces an integral estimate for holomorphic functions (Def. 3.2.1 (ii)) we can find constants  $A$  and  $B$  (not depending on  $z$  or  $t$ ) such that

$$\begin{aligned} |p_t(z)|^2 &\leq e^{A\psi^+(z)+B} \int_{B(z,1,\omega)} |u_t|^2 \omega^n, \\ &\leq e^{A\psi^+(z)+B} \sup_{\zeta \in B(z,1,\omega)} \left( e^{2t\varphi(\zeta)} (e^{\psi(\zeta)} + 1)^2 \right) \int_X |u_t|^2 (e^\psi + 1)^{-2} e^{-2t\varphi} \omega^n. \end{aligned}$$

Since  $\varphi \in \mathcal{L}_\psi$  we can now use inequality (3.26) and Lemma 3.4.2 to show that  $p_t \in \mathcal{P}_{l+t}^\psi$  where  $l = \frac{A}{2} + 1$ .

Let  $r > 0$  be small enough such that  $B(z, r, \omega) \subset \subset \{\varphi < \log(1 + \varepsilon)\} \subset \{\chi = 1\}$  for all  $z \in K$ . Then  $u_t \in \mathcal{O}(\bar{B}(z, r, \omega))$  for  $z \in K$ . Since  $\psi$  induces an integral estimate for holomorphic functions, and by the compactness of the sets  $S$  and  $K$  we can find a positive constant  $C$  such that for every  $z \in K$  we have

$$\begin{aligned} |f(z) - p_t(z)|^2 &= |u_t(z)|^2 \leq C \left( \sup_{\zeta \in B(z,r,\omega)} 1 + e^{\psi(\zeta)} \right)^{-2} \int_{B(z,r,\omega)} |u_t|^2 \omega^n \\ &\leq C \left( \sup_{\zeta \in B(z,r,\omega)} e^{2t\varphi(\zeta)} \right) \int_X |u_t|^2 (e^\psi + 1)^{-2} e^{-2t\varphi} \omega^n. \end{aligned} \quad (3.27)$$

By the definition of  $r$  we can estimate  $\sup_{\zeta \in B(z,r,\omega)} e^{2t\varphi(\zeta)}$  with  $(1 + \varepsilon)^{2t}$  and recall that we have  $\log(L - \varepsilon/2) > \varphi > \log(L - \varepsilon)$  on the support of  $\bar{\partial}\chi$ . By inequalities

(3.26) and (3.27) we therefore have

$$\begin{aligned} |f(z) - p_t(z)|^2 &\leq C(1 + \varepsilon)^{2t} \int_X s |f \bar{\partial} \chi|_\omega^2 e^{-2t\varphi} \omega^n \\ &\leq C \|s\|_X \left( \frac{1 + \varepsilon}{L - \varepsilon} \right)^{2t} \|f\|_{\{\varphi < \log(L - \varepsilon/2)\}}^2 \|\bar{\partial} \chi\|_{L^2(X)}^2, \quad z \in K \end{aligned}$$

where  $\|s\|_X$  is the sup-norm of  $s$  on  $X$  and  $\|\bar{\partial} \chi\|_{L^2(X)}^2$  is the  $L^2$  norm of  $\bar{\partial} \chi$  with respect to the metric  $\omega$  and measure  $\omega^n$ . We have already seen that  $p_t \in \mathcal{P}_{l+t}^\psi$  and by replacing  $t$  with  $t - l$  we have the result with  $M = C \|s\|_X \|\bar{\partial} \chi\|_{L^2(X)}$ .  $\square$

We need the following technical result for future reference.

**Lemma 3.4.4** *If  $L > 1$  is large enough we can replace  $M$  from inequality (3.14) with  $M_0 \|\bar{\partial} \chi\|_{L^2(X \setminus S)}$  where  $M_0$  is a constant neither depending on  $f$  nor  $L$  and  $\chi : X \rightarrow \mathbb{R}$  is any  $C^\infty$  cutoff function with  $\chi = 1$  on  $\{\varphi < \log(L - \varepsilon)\}$  and  $\chi = 0$  on  $\{\varphi > \log(L - \varepsilon/2)\}$ .*

*Proof.* This is apparent from the proof of Theorem 3.2.6. The only point we need to make is that if  $L$  is large enough then the support of  $\bar{\partial} \chi$  does not intersect  $S$  and  $\|\bar{\partial} \chi\|_{L^2(X)} = \|\bar{\partial} \chi\|_{L^2(X \setminus S)}$ .  $\square$

**Lemma 3.4.5** *Let  $L, \varepsilon > 0$ ,  $\varphi \in \mathcal{L}_\psi^+$  be continuous and  $\theta$  be a continuous function satisfying the growth condition*

$$|\theta(z)| \leq A\psi^+(z) + B, \quad z \in X \tag{3.28}$$

for some constants  $A, B$ . Write  $\tilde{\varphi}_t := (1 - t^{-1})\varphi + t^{-1}\theta$ . Then there exists  $T$  such that for  $t > T$  we have

$$\{z \in X; \tilde{\varphi}_t(z) < L - \varepsilon\} \subset \{z \in X; \varphi(z) < L\}.$$

and the function  $\tilde{\varphi}_t$  is an exhaustion function.

*Proof.* It is trivial to check that

$$\{z \in X; \theta(z) \geq \varphi(z)\} \cap \{z \in X; \tilde{\varphi}_t(z) < L - \varepsilon\} \subset \{z \in X; \varphi(z) < L\}$$

for every  $t > 0$ . Therefore we only need to show that

$$\{z \in X; \theta(z) < \varphi(z)\} \cap \{z \in X; \tilde{\varphi}_t(z) < L - \varepsilon\} \subset \{z \in X; \varphi(z) < L\}$$

for large enough  $t$ .

Since  $\varphi \in \mathcal{L}_\psi^+$  and by (3.28) we can find positive constants  $C_1$  and  $C_2$  such that

$$|\theta(z)| \leq C_1\varphi^+(z) + C_2, \quad z \in X.$$

If  $t > 2$  and  $z$  is such that  $\varphi(z) > 1$ , then we have

$$\begin{aligned} \tilde{\varphi}_t(z) &= (1 - t^{-1})\varphi(z) + t^{-1}\theta(z) > \frac{1}{2} \left( \varphi(z) - \frac{2}{t}|\theta(z)| \right) \\ &\geq \frac{\varphi(z)}{2} \left( 1 - \frac{2C_1}{t} \right) - \frac{C_2}{t}. \end{aligned} \quad (3.29)$$

Since  $\varphi$  is an exhaustion, inequality (3.29) implies that  $\tilde{\varphi}_t$  is also an exhaustion for  $t \geq T_0 := \max\{2C_1 + 1, 2\}$ . In particular the set  $\{\tilde{\varphi}_{T_0} \leq L - \varepsilon\}$  is compact so the continuous function  $\theta - \varphi$  has a lower bound  $-M < 0$  on it. We define  $T := \max\{T_0, \frac{M}{\varepsilon}\}$ . Now let  $t > T$  and  $z \in \{\theta < \varphi\} \cap \{\tilde{\varphi}_t < L - \varepsilon\}$ . Since  $\theta(z) - \varphi(z) < 0$  and  $t > T_0$  we have  $\tilde{\varphi}_{T_0}(z) < \tilde{\varphi}_t(z) \leq L - \varepsilon$  so  $\theta(z) - \varphi(z) \geq -M$  and

$$\varphi(z) = \tilde{\varphi}_t(z) - t^{-1}(\theta(z) - \varphi(z)) < L - \varepsilon - \frac{\varepsilon}{M}(-M) = L.$$

□

*Proof of Theorem 3.2.2.* Let  $\varepsilon \in ]0, (L - 1)/2[$  and let  $\theta$  be a Ricci compensator for  $\psi$ . By adding a constant to  $\theta$  we can assume  $\theta < 0$  on  $K$ . Now apply Lemma 3.4.5 to find  $T > 0$  such that  $\{\tilde{\varphi}_T < \log(L - \varepsilon)\} \subset \{\varphi < \log(L)\}$  where  $\tilde{\varphi}_T := (1 - T^{-1})\varphi + T^{-1}\theta$ . We can now apply Theorem 3.2.6 with  $\varphi$  replaced by  $\tilde{\varphi}_T$  and with  $L$  replaced by  $L - \varepsilon$ . For  $t$  large enough we have

$$d_K(f, \mathcal{P}_t^\psi) \leq M \|f\|_{\{\tilde{\varphi}_T \leq \log(L - 3\varepsilon/2)\}} \left( \frac{1 + \varepsilon}{L - 2\varepsilon} \right)^{t-l}.$$

and

$$\limsup_{t \rightarrow \infty} (d_K(f, \mathcal{P}_t^\psi))^{1/t} \leq \limsup_{t \rightarrow \infty} \left( \frac{1 + \varepsilon}{L - 2\varepsilon} \right)^{\frac{t-l}{t}} = \frac{1 + \varepsilon}{L - 2\varepsilon}.$$

Since  $\varepsilon > 0$  was arbitrary the result follows. □

*Proof of Proposition 3.2.3.* By assumption there is a sequence  $(p_m)_{m \in \mathbb{N}}$  of functions on  $X$  such that  $p_m \in \mathcal{P}_m^\psi$  for all  $m \in \mathbb{N}$  and  $\|p_m - f\|_K \leq (L - \varepsilon(m))^{-m}$  where  $\varepsilon : \mathbb{N} \rightarrow \mathbb{R}_+$  is a decreasing function satisfying  $\lim_{m \rightarrow \infty} \varepsilon(m) = 0$ . We claim that the

sum  $p_1 + \sum_{m=1}^{\infty} (p_{m+1} - p_m)$  is uniformly convergent on compact subsets of  $\{V_{K,\psi} < \log(L)\}$ . Indeed for  $l < L$  we have

$$\begin{aligned} \|p_m(z) - p_{m-1}(z)\|_{\{V_{K,\psi} \leq l\}} &\leq \|p_m - p_{m-1}\|_K \|e^{V_{K,\psi}}\|_{\{V_{K,\psi} \leq l\}} \\ &\leq (\|p_m - f\|_K + \|f - p_{m-1}\|_K) l^m \leq 2l \left( \frac{l}{L - \varepsilon(m-1)} \right)^{m-1}. \end{aligned}$$

Since  $l < L$  and  $\varepsilon(m)$  converges to 0 the series converges. It is obviously equal to  $f$  on  $K$ .  $\square$

*Proof of Proposition 3.2.4.* The inequality  $\log(\Phi_{K,\psi}^*) \leq V_{K,\psi}^*$  is trivial. Let  $z_0 \in X \setminus K$  be fixed. To derive a contradiction, assume that  $V_{K,\psi}^*(z_0) > \log(\Phi_{K,\psi}^*(z_0))$  and write  $L := e^{V_{K,\psi}^*(z_0)}$ . Define the open sets

$$\begin{aligned} \Omega &:= \{z \in X; V_{K,\psi}^*(z) < \log(L)\}, \\ \Omega' &:= \{z \in X; \log(\Phi_{K,\psi}^*)(z) < \log(L)\}. \end{aligned}$$

Then  $z_0 \in \Omega'$ ,  $z_0 \notin \Omega$  and we clearly have  $\Omega \subset \Omega'$ . Denote by  $\Omega'_0$  the connected component of  $\Omega'$  containing  $z_0$ . By assumption (ii)  $\Omega'_0$  contains a connected component of  $K$  and since  $K \subset \Omega$  there is a component of  $\Omega$ , call it  $\Omega_0$ , with  $\Omega_0 \subset \Omega'_0$ . Since  $\Omega_0$  is hyperconvex, i.e. defined as a component of a sublevel set of a psh function, it is a domain of holomorphy and therefore there exists a function  $h$  holomorphic on  $\Omega_0$  with no holomorphic extension to any larger domain. Extend  $h$  to all of  $\Omega$  by defining  $h$  to be 0 on  $\Omega \setminus \Omega_0$ . By assumption (i) there exists a sequence of functions  $(p_m)_{m \in \mathbb{N}}$  with  $p_m \in \mathcal{P}_{m\psi}$  and  $\|p_m - h\|_K \leq (L - \varepsilon(m))^{-m}$  where  $\varepsilon : \mathbb{N} \rightarrow \mathbb{R}_+$  is a function with  $\lim_{m \rightarrow \infty} \varepsilon(m) = 0$ . By the same argument as in the last part of the proof of Theorem 3.2.3, the series

$$p_1 + \sum_{n=1}^{\infty} (p_{n+1} - p_n)$$

converges uniformly on compact subsets of  $\Omega'$ . Clearly it equals  $h$  on  $K$  and since  $K$  is non-pluripolar (because  $V_{K,\psi} < \infty$ ) it equals  $h$  on  $\Omega$ . In particular the series converges uniformly on compact subsets of  $\Omega'_0$  which is a contradiction to the choice of  $h$  because  $\Omega'_0$  is a domain which is strictly larger than  $\Omega_0$ . We conclude that  $V_{K,\psi}^* = \log(\Phi_{K,\psi}^*)$  on  $X \setminus K$ . Clearly  $V_{K,\psi}^* = \log(\Phi_{K,\psi}^*) = 0$  on the interior of  $K$ . Since the functions are equal on  $X \setminus K$  and on the interior of  $K$  we have  $V_{K,\psi}^* = \log(\Phi_{K,\psi}^*)$  on  $\partial K$  as well.  $\square$

**Lemma 3.4.6** For  $a, \rho > 0$  we have

$$\sum_{n=1}^{\infty} \left(\frac{a}{n}\right)^{n/\rho} \leq 1 + 2^\rho a e^{\frac{a}{e^\rho}}.$$

*Proof.* Since  $(a/n)^{n/\rho} < 2^{-n}$  for  $n > \lfloor 2^\rho a \rfloor$  we have  $\sum_{n=\lfloor 2^\rho a \rfloor + 1}^{\infty} \left(\frac{a}{n}\right)^{n/\rho} \leq 1$ . The function  $x \rightarrow (a/x)^{x/\rho}$  is maximized when  $x = a/e$ . Therefore

$$\sum_{n=1}^{\lfloor 2^\rho a \rfloor} \left(\frac{a}{n}\right)^{n/\rho} \leq \lfloor 2^\rho a \rfloor \left(\frac{a}{a/e}\right)^{\frac{a/\rho}{e}} \leq 2^\rho a e^{\frac{a}{e^\rho}}.$$

□

**Lemma 3.4.7** Let  $\varphi \in \mathcal{L}_\psi^+$  and  $L_0 > 1$  be large enough such that  $\{\psi < 1\} \subset \{\varphi < \log(L_0)\}$ . Then for any constants  $L_1, L_2$  with  $L_2 > L_1 > L_0$  there exists a function  $\chi \in C^\infty(X)$  with  $\chi = 1$  on  $\{\varphi < \log(L_1)\}$ ,  $\chi = 0$  on  $\{\varphi > \log(L_2)\}$  and  $\|\bar{\partial}\chi\|_{L^2(X)}^2 \leq \frac{M_1 L_2^2}{L_2 - L_1} \int_{\{\psi \leq \log(L_2) + M_2\}} \omega^n$  where  $M_1$  and  $M_2$  are constants independent of  $L_1$  and  $L_2$ .

*Proof.* Let  $\chi_0 \in C^\infty(\mathbb{R})$  be such that  $\chi_0(x) = 1$  if  $x \leq 1$  and  $\chi_0(x) = 0$  if  $x \geq L_2/L_1$ . We can choose such a function such that  $\|\chi_0'\|_{\mathbb{R}} \leq 2(L_2/L_1 - 1)^{-1}$ . Now define

$$\chi(z) = \chi_0\left(\frac{e^{\varphi(z)}}{L_1}\right), \quad z \in X.$$

We clearly have  $\chi = 1$  on  $\{\varphi < \log(L_1)\}$  and  $\chi = 0$  on  $\{\varphi > \log(L_2)\}$  so we only have to prove the estimate for  $\|\bar{\partial}\chi\|_{L^2(X)}^2$ . First notice that

$$\begin{aligned} |\bar{\partial}\chi|_\omega^2 \omega^n &\leq \frac{\|\chi_0'\|_{\mathbb{R}}}{L_1} |\bar{\partial}e^\varphi|_\omega^2 \omega^n \leq \frac{2}{L_2 - L_1} |\bar{\partial}e^\varphi|_\omega^2 \omega^n \\ &= \frac{2i}{L_2 - L_1} \partial e^\varphi \wedge \bar{\partial} e^\varphi \wedge \omega^{n-1} \leq \frac{1}{2(L_2 - L_1)} (dd^c e^{2\varphi}) \wedge \omega^{n-1}. \end{aligned} \quad (3.30)$$

By assumption there exists a constant  $C$  such that  $\varphi^+ - C \leq \psi^+ \leq \varphi^+ + C$  on  $X$ . Let  $\Gamma_0 \in C^\infty(\mathbb{R})$  be such that  $\Gamma_0(x) = 1$  if  $1 \leq x \leq \log(L_2) + C$  and  $\Gamma_0 = 0$  if  $x \leq 0$  or  $x \geq \log(L_2) + C + 1$ . The function  $\Gamma_0$  can be chosen such that  $\max\{\|\Gamma_0'\|_{\mathbb{R}}, \|\Gamma_0''\|_{\mathbb{R}}\} \leq 4$ . Now define  $\Gamma := \Gamma_0 \circ \psi$ . Then  $\Gamma$  equals 1 on the support of the  $(0, 1)$ -form  $\bar{\partial}\chi$  and the support of  $\Gamma$  is a subset of

$$F := \{0 \leq \psi(z) \leq \log(L_2) + C + 1\} \subset X.$$

Recall that we have  $\omega = \frac{1}{4}dd^c e^\psi$  on  $F$ . By (3.30) we have

$$\begin{aligned} \|\bar{\partial}\chi\|_{L^2(X)}^2 &= \int_X \Gamma|\bar{\partial}\chi|_\omega^2 \omega^n \leq \frac{1}{L_2 - L_1} \int_F \Gamma(\frac{1}{2}dd^c e^{2\varphi}) \wedge \omega^{n-1} \\ &= \frac{1}{L_2 - L_1} \int_F e^{2\varphi} (\frac{1}{2}dd^c \Gamma) \wedge \omega^{n-1} \\ &\leq \frac{1}{L_2 - L_1} \int_F 4e^{2(\psi+C)} i(\partial\psi \wedge \bar{\partial}\psi + \partial\bar{\partial}\psi) \wedge \omega^{n-1} \\ &\leq \frac{8L_2^2 e^{2C+1}}{L_2 - L_1} \int_F e^{-\psi} \omega^n \leq \frac{8L_2^2 e^{2C+1}}{L_2 - L_1} \int_{\{\psi \leq \log(L_2) + C + 1\}} \omega^n. \end{aligned}$$

□

*Proof of Theorem 3.2.5.* Assume inequalities (3.12) are true and let  $\varepsilon > 0$ . We first consider the case when  $\varphi$  satisfies (i) from Theorem 3.2.6. By assumption there is a constant  $C$  such that

$$\|f\|_{\{\varphi \leq \log(r)\}} \leq C \exp((\sigma + \varepsilon)r^\varrho), \quad r \geq 1. \quad (3.31)$$

Since  $f$  is entire, inequality (3.14) from Theorem 3.2.6 is true for all  $L > 1$  and all large  $t$ . In particular if we take  $L = L(t) := (t/(\varrho\sigma))e^{-1} + \varepsilon/2$  then by (3.14), Lemma 3.4.4 and (3.31) we have

$$d_K(f, \mathcal{P}_t^\psi) \leq M_0 C \exp\left(\frac{(\sigma + \varepsilon)t}{\varrho\sigma}\right) \|\bar{\partial}\chi_t\|_{L^2(X)} \left(\frac{1 + \varepsilon}{L(t) - \varepsilon}\right)^{t-l} \quad (3.32)$$

for every  $t$  large enough. Here  $\chi_t$  is a cut-off function with  $\chi_t = 1$  on  $\{\varphi < \log(L(t) - \varepsilon)\}$  and  $\chi_t = 0$  on  $\{\varphi > \log(L(t) - \varepsilon/2)\}$ . By Lemma 3.4.7 and assumption (3.11) (taking  $r = \varrho/2$ ) there are constants  $M_1, M_2, A, B$  such that

$$\begin{aligned} \|\bar{\partial}\chi_t\|_{L^2(X)}^2 &\leq \frac{M_1(L(t) - \varepsilon/2)^2}{\varepsilon/2} \int_{\{\psi \leq \log(L(t) - \varepsilon/2) + M_2\}} \omega^n \\ &\leq \frac{M_1(L(t) - \varepsilon/2)^2}{\varepsilon/2} \exp\left(Ae^{\frac{M_2\varrho}{2}}(L(t) - \varepsilon/2)^{\frac{\varrho}{2}} + B\right) \\ &= \frac{2M_1}{\varepsilon} \left(\frac{t}{\varrho\sigma}\right)^{2\varrho^{-1}} \exp\left(Ae^{\frac{M_2\varrho}{2}} \left(\frac{t}{\varrho\sigma}\right)^{\frac{1}{2}} + B\right). \end{aligned} \quad (3.33)$$

Now, combining (3.32) and (3.33), we get

$$\limsup_{t \rightarrow \infty} t(d_K(f, \mathcal{P}_t^\psi))^{e/t} \leq \exp\left(\frac{\sigma + \varepsilon}{\sigma}\right) (1 + \varepsilon)^\varrho \varrho\sigma. \quad (3.34)$$

Since (3.34) is true for all  $\varepsilon > 0$  the result follows.

Now consider the case when  $\varphi$  does not satisfy (i) from Theorem 3.2.6. Let  $\theta$  be a Ricci compensator for  $\psi$  satisfying  $\theta|_K \leq 0$ , and let  $T$  be large enough such that  $\tilde{\varphi}_T$  (as defined in Lemma 3.4.5) is an exhaustion function. Then clearly the function  $\varphi_\varepsilon := (1 - \varepsilon)\varphi + \varepsilon\tilde{\varphi}_T$  satisfies (i) – (iii) from Theorem 3.2.6. Moreover, since  $\tilde{\varphi}_T$  is an exhaustion, we have

$$\begin{aligned} \{\varphi_\varepsilon \leq \log(r)\} &\subset \{(1 - \varepsilon)\varphi \leq \log(r)\}, \\ \text{and } \|f\|_{\{\varphi_\varepsilon \leq \log(r)\}} &\leq \|f\|_{\{\varphi \leq \log(r^{1/(1-\varepsilon)})\}} \end{aligned} \quad (3.35)$$

for  $r > 1$  large enough. By assumption (3.12) and by (3.35) we have

$$\limsup_{r \rightarrow \infty} \frac{\log^+ \log \|f\|_{\{\varphi_\varepsilon \leq \log(r)\}}}{\log(r)} \leq \limsup_{r \rightarrow \infty} \frac{\log^+ \log \|f\|_{\{\varphi \leq \log(r^{1/(1-\varepsilon)})\}}}{(1 - \varepsilon) \log(r^{1/(1-\varepsilon)})} \leq \frac{\varrho}{1 - \varepsilon}$$

and

$$\limsup_{r \rightarrow \infty} \frac{\log \|f\|_{\{\varphi_\varepsilon \leq \log(r)\}}}{r^{\varrho/(1-\varepsilon)}} \leq \limsup_{r \rightarrow \infty} \frac{\log \|f\|_{\{\varphi \leq \log(r^{1/(1-\varepsilon)})\}}}{(r^{1/(1-\varepsilon)})^\varrho} \leq \sigma.$$

We can now apply our previous conclusion with  $\varphi$  replaced by  $\varphi_\varepsilon$  and  $\varrho$  replaced by  $\varrho/(1 - \varepsilon)$  and we have

$$\limsup_{t \rightarrow \infty} t(d_K(f, \mathcal{P}_t^\psi))^{\varrho/((1-\varepsilon)t)} \leq e \frac{\varrho}{1 - \varepsilon} \sigma.$$

Since this is true for every  $\varepsilon > 0$  the result follows.

To prove the converse, assume inequality (3.13) is true with  $\varphi$  replaced by  $V_{K,\psi}$ . Then for  $\varepsilon > 0$  we can find  $M$  such that for any  $n \geq M$  there is a function  $p_n \in \mathcal{P}_n^\psi$  such that

$$\|f - p_n\|_K \leq \left( \frac{e\varrho\sigma(1 + \varepsilon)}{n} \right)^{n/\varrho}. \quad (3.36)$$

Consider the function  $G := p_M + \sum_{n=M}^{\infty} (p_{n+1} - p_n)$ . Clearly we have  $G = f$  on  $K$ . Moreover, by (3.36) and Lemma 3.4.6, we have

$$|G| \leq |p_M| + \sum_{n=M}^{\infty} |p_{n+1} - p_n| \leq \|p_M\|_K e^{MV_{K,\psi}} + \sum_{n=M}^{\infty} \|p_{n+1} - p_n\|_K e^{(n+1)V_{K,\psi}}$$

$$\begin{aligned}
&\leq \|p_M\|_K e^{MV_{K,\psi}} + \sum_{n=M}^{\infty} (\|p_{n+1} - f\|_K + \|p_n - f\|_K) e^{(n+1)V_{K,\psi}} \\
&\leq \|p_M\|_K e^{MV_{K,\psi}} + 2e^{V_{K,\psi}} \sum_{n=M}^{\infty} \left( \frac{e\rho\sigma(1+\varepsilon)e^{\rho V_{K,\psi}}}{n} \right)^{n/\rho} \\
&\leq \|p_M\|_K e^{MV_{K,\psi}} + 2e^{V_{K,\psi}} (1 + 2^\rho e\rho\sigma(1+\varepsilon) \exp(\rho V_{K,\psi} + \sigma(1+\varepsilon)e^{\rho V_{K,\psi}})).
\end{aligned}$$

In particular we have

$$\|G\|_{\{V_{K,\psi} \leq \log(r)\}} \leq \|p_M\|_K r^M + 2r(1 + 2^\rho e\rho\sigma(1+\varepsilon)r^\rho e^{\sigma(1+\varepsilon)r^\rho})$$

and now it is easy to see that

$$\limsup_{r \rightarrow \infty} \frac{\log \|G\|_{\{\varphi \leq \log(r)\}}}{r^\rho} \leq \sigma(1+\varepsilon)$$

and the second inequality of (3.12) follows by letting  $\varepsilon \rightarrow 0$ . By simple calculus we can show that the first inequality of (3.12) follows from the second one.  $\square$



# 4

## Monge-Ampère measures of plurisubharmonic exhaustions

The importance of S-parabolic manifolds and parabolic potentials in the study of polynomial approximation should be apparent from the discussion in the previous chapters. In this context, we turn our attention to the study of the Monge-Ampère operator.

The *Lie norm*  $|\cdot|_c : \mathbb{C}^N \rightarrow \mathbb{R}$  is the largest norm on  $\mathbb{C}^N$  which agrees with the Euclidean norm on  $\mathbb{R}^N$ . In other words, if we denote the Euclidean norm by  $|\cdot|$ , then

$$|z|_c = \sup\{\|z\|; \|\cdot\| \text{ is a norm on } \mathbb{C}^N \text{ and } \|x\| = |x|, x \in \mathbb{R}^N\}, \quad z \in \mathbb{C}^N. \quad (4.1)$$

It is not immediately clear that the right hand side of (4.1) is actually a norm, but this will be apparent from the results of Section 4.1. In the same section we also establish an explicit formula for  $|\cdot|_c$  and we see that it is a  $C^\infty$  function on  $\mathbb{C}^N \setminus \mathbb{C}\mathbb{R}^N$ , where

$$\mathbb{C}\mathbb{R}^N := \{\alpha x \in \mathbb{C}^N; \alpha \in \mathbb{C}, x \in \mathbb{R}^N\}.$$

The main result of this chapter is an explicit formula for the Monge-Ampère measures of psh functions of the form  $\log|\Phi|_c$  where  $\Phi$  is a holomorphic map. In Section 4.3 we apply this result to calculate the extremal function for various compact subsets of manifolds and the corresponding equilibrium measures.

**Theorem 4.0.1** *Let  $\Phi = (\Phi_0, \dots, \Phi_n): X \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$  be a holomorphic map on a complex manifold of dimension  $n$ . The function  $\log |\Phi|_c$  is psh on  $X$  and a maximal  $C^\infty$  function on  $\Phi^{-1}(\mathbb{C}^{n+1} \setminus \mathbb{C}\mathbb{R}^{n+1})$ . The Monge-Ampère measure  $(dd^c(\log |\Phi|_c))^n$  has no mass on the set  $\mathcal{A}_\Phi = \{z \in X; \text{rank } d\Phi(z) < n \text{ or } \Phi(z) \in \text{range } d\Phi(z)\}$ . If  $M = \Phi^{-1}(\mathbb{C}\mathbb{R}^{n+1}) \setminus \mathcal{A}_\Phi$  is non-empty, then it is an  $n$ -dimensional real analytic manifold and  $(dd^c(\log |\Phi|_c))^n$  is the current of integration along  $M$  of the  $n$ -form*

$$C_n(\Phi_0^2 + \dots + \Phi_n^2)^{-\frac{(n+1)}{2}} \sum_{j=0}^n (-1)^{j+1} \Phi_j d\Phi_0 \wedge \dots \wedge \widehat{d\Phi_j} \wedge \dots \wedge d\Phi_n \quad (4.2)$$

where  $C_n = (-1)^{\frac{n(n-1)}{2}} n! \Omega_n$ ,  $\Omega_n$  is the volume of the unit-ball in  $\mathbb{R}^n$ , and the holomorphic square root of the function  $\Phi_0^2 + \dots + \Phi_n^2$  is chosen with the same argument as the vector  $\Phi$  in  $\mathbb{C}\mathbb{R}^{n+1}$ .

Theorem 4.0.1 gives us a promising method of constructing explicit examples of  $S$ -parabolic manifolds. Indeed, let  $\Phi: X \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$  be as in Theorem 4.0.1. If we assume that  $\Phi$  is proper then  $\log |\Phi|_c$  is a psh exhaustion function on  $X$ . If we further assume that  $M$  is a relatively compact subset of  $X$  then Theorem 4.0.1 implies that  $\log |\Phi|_c$  is a parabolic potential on  $X$  and we reach the following conclusion.

**Corollary 4.0.2** *Let  $X$  be a complex manifold of dimension  $n$ . If there exists a proper holomorphic map  $\Phi: X \rightarrow \mathbb{C}^{n+1}$  such that  $M \subset X$  is relatively compact, then  $X$  is  $S^*$ -parabolic.*

We split the proof of Theorem 4.0.1 into two parts. The first part can be found in Section 4.5 where we regularize the Lie norm from above with a family of smooth functions  $v_\varepsilon$  and we analyze the Levi form of  $\log v_\varepsilon$ . In Section 4.6 we finish the proof of Theorem 4.0.1 by calculating the Monge-Ampère measure  $(dd^c \log |\Phi|_c)^n$  as the weak limit of  $(dd^c \log(v_\varepsilon \circ \Phi))^n$  as  $\varepsilon \rightarrow 0$ .

## 4.1 The Lie norm

The Lie norm on  $\mathbb{C}^N$  is a special case of an extension of a norm on a real vector space  $V$  to a complex norm on its complexification  $V_{\mathbb{C}} = \mathbb{C} \otimes_{\mathbb{R}} V$ , i.e., the extension satisfies  $\|\alpha\zeta\| = |\alpha|\|\zeta\|$  for  $\alpha \in \mathbb{C}$  and  $\zeta \in V_{\mathbb{C}}$ . There are a few different ways of introducing the complexification, see Munoz, Sarantopoulos, and Tonge [58]. One is to define  $V_{\mathbb{C}}$  as  $V \times V$  with usual addition and complex multiplication  $(\alpha + i\beta)(\xi, \eta) = (\alpha\xi - \beta\eta, \beta\xi + \alpha\eta)$  where  $\alpha, \beta \in \mathbb{R}$  and  $\xi, \eta \in V$ . We have  $(0, \eta) = i(\eta, 0)$ , so if

we identify the real vector  $\xi$  in  $V$  with  $(\xi, 0)$  in  $V_{\mathbb{C}}$ , then we can write every vector  $\zeta = (\xi, \eta)$  as  $\zeta = \xi + i\eta$ . Thus  $V$  is a real subspace of  $V_{\mathbb{C}}$  and  $V_{\mathbb{C}} = V \oplus iV$ . The vectors  $\xi$  and  $\eta$  are called the *real* and *imaginary* parts of  $\zeta$ , denoted by  $\operatorname{Re} \zeta$  and  $\operatorname{Im} \zeta$ .

**Proposition 4.1.1** *Let  $\|\cdot\|$  be a norm on  $V$  and define the function  $\|\cdot\|_c : V_{\mathbb{C}} \rightarrow \mathbb{R}$  as*

$$\|\zeta\|_c = \inf \left\{ \sum_{j=1}^k |\alpha_j| \|\xi_j\| ; \zeta = \sum_{j=1}^k \alpha_j \xi_j, \alpha_j \in \mathbb{C}, \xi_j \in V, k \in \mathbb{N}^* \right\}. \quad (4.3)$$

*Then  $\|\cdot\|_c$  is a norm on  $V_{\mathbb{C}}$  and moreover it is the largest norm on  $V_{\mathbb{C}}$  which agrees with  $\|\cdot\|$  on  $V$ .*

*Proof.* Let  $\rho$  be any norm on  $V_{\mathbb{C}}$  which agrees with  $\|\cdot\|$  on  $V$  and let  $\zeta \in V_{\mathbb{C}}$  be a vector. For any representation  $\zeta = \sum_{j=1}^k \alpha_j \xi_j$  of  $\zeta$  with  $\alpha_j \in \mathbb{C}$  and  $\xi_j \in V$  we have

$$\rho(\zeta) \leq \sum_{j=1}^k \rho(\alpha_j \xi_j) = \sum_{j=1}^k |\alpha_j| \|\xi_j\|$$

and therefore  $\rho(\zeta) \leq \|\zeta\|_c$ .

Now we show that  $\|\cdot\|_c$  is a norm on  $V_{\mathbb{C}}$ . By [58] there exists at least one norm  $\rho$  on  $V_{\mathbb{C}}$  which extends  $\|\cdot\|$  on  $V$ . By the preceding argument we have  $\|z\|_c \geq \rho(z) > 0$  for  $z \neq 0$  and we clearly have  $\|0\|_c = 0$ . For the triangle inequality, let  $\zeta_1, \zeta_2 \in V_{\mathbb{C}}$  and let  $\zeta_1 = \sum_{j=1}^{k_1} \alpha_j \xi_j$ ,  $\zeta_2 = \sum_{j=1}^{k_2} \beta_j \eta_j$  be representations of  $\zeta_1$  and  $\zeta_2$  satisfying

$$\sum_{j=1}^{k_1} |\alpha_j| \|\xi_j\| \leq \|\zeta_1\|_c + \varepsilon, \quad \sum_{j=1}^{k_2} |\beta_j| \|\eta_j\| \leq \|\zeta_2\|_c + \varepsilon$$

for some  $\varepsilon > 0$ . Then  $\zeta_1 + \zeta_2 = \sum_{j=1}^{k_1} \alpha_j \xi_j + \sum_{j=1}^{k_2} \beta_j \eta_j$  and therefore

$$\|\zeta_1 + \zeta_2\|_c \leq \sum_{j=1}^{k_1} |\alpha_j| \|\xi_j\| + \sum_{j=1}^{k_2} |\beta_j| \|\eta_j\| \leq \|\zeta_1\|_c + \|\zeta_2\|_c + 2\varepsilon.$$

Since  $\varepsilon > 0$  was arbitrary, the triangle inequality holds. It is simple to show that  $\|\alpha\zeta\|_c = |\alpha| \|\zeta\|_c$  for all  $\alpha \in \mathbb{C}$  and  $\zeta \in V_{\mathbb{C}}$ .  $\square$

The set

$$\mathbb{C}V = \{\alpha\xi \in V_{\mathbb{C}} ; \alpha \in \mathbb{C}, \xi \in V\} = \{e^{i\theta}\xi \in V_{\mathbb{C}} ; \theta \in \mathbb{R}, \xi \in V\}$$

consisting of all vectors in  $V_{\mathbb{C}}$  with parallel real and imaginary parts plays a particularly important role in our calculations. If  $\zeta = e^{i\theta}\xi \in \mathbb{C}V$  and  $\|\cdot\|$  extends from  $V$  to a complex norm on  $V_{\mathbb{C}}$ , also denoted by  $\|\cdot\|$ , then clearly  $\|\zeta\| = \|\xi\|$ .

Assume from now on that  $V$  is an inner product space with norm  $\xi \mapsto |\xi| = \langle \xi, \xi \rangle^{\frac{1}{2}}$ , where  $(x, \xi) \mapsto \langle x, \xi \rangle$  is an inner product on  $V$ . The bilinear form on  $V$  has a unique extension to a symmetric  $\mathbb{C}$ -bilinear form on  $V_{\mathbb{C}}$  by the formula

$$\langle x + iy, \xi + i\eta \rangle = \langle x, \xi \rangle - \langle y, \eta \rangle + i\langle x, \eta \rangle + i\langle y, \xi \rangle, \quad (4.4)$$

and the Hermitian form  $(z, \zeta) \mapsto \langle z, \bar{\zeta} \rangle$  generates an extension of the norm by the formula

$$|\zeta| = \langle \zeta, \bar{\zeta} \rangle^{\frac{1}{2}} = (|\xi|^2 + |\eta|^2)^{\frac{1}{2}}, \quad \zeta = \xi + i\eta \in V_{\mathbb{C}}. \quad (4.5)$$

We use this norm to define the distance  $d(\zeta, \mathbb{C}V) = \inf\{|\zeta - e^{i\theta}\xi|; \theta \in \mathbb{R}, \xi \in V\}$  from  $\zeta \in V_{\mathbb{C}}$  to  $\mathbb{C}V$ .

When  $V$  is an inner product space there exists a representation of every  $\zeta \in V_{\mathbb{C}}$ ,  $\zeta = e^{i\theta}(a + ib)$  with  $\theta \in \mathbb{R}$  and  $a, b \in V$  such that the infimum (4.3) is reached, i.e. we have  $|\zeta|_c = |a| + |b|$ . Indeed, for  $\zeta \in V_{\mathbb{C}}$  let  $\theta \in \mathbb{R}$  be such that  $e^{-2i\theta}\langle \zeta, \zeta \rangle$  is a positive real number and define  $a = \operatorname{Re}(e^{-i\theta}\zeta)$  and  $b = \operatorname{Im}(e^{-i\theta}\zeta)$  such that  $\zeta = e^{i\theta}(a + ib)$ . Then  $e^{-2i\theta}\langle \zeta, \zeta \rangle = |a|^2 - |b|^2 + 2i\langle a, b \rangle \geq 0$ , so we conclude that  $\langle a, b \rangle = 0$  and  $|b| \leq |a|$ . Moreover we have the following result.

**Proposition 4.1.2** *If  $\zeta = \xi + i\eta = e^{i\theta}(a + ib)$ ,  $\theta \in \mathbb{R}$ ,  $\xi, \eta, a, b \in V$ ,  $\langle a, b \rangle = 0$ , and  $|b| \leq |a|$ , then*

$$|\zeta|^2 = |a|^2 + |b|^2, \quad |\langle \zeta, \zeta \rangle| = |a|^2 - |b|^2, \quad (4.6)$$

$$|a|^2 = \frac{1}{2}(|\zeta|^2 + |\langle \zeta, \zeta \rangle|) \quad \text{and} \quad |b|^2 = \frac{1}{2}(|\zeta|^2 - |\langle \zeta, \zeta \rangle|). \quad (4.7)$$

The Lie norm is given by the formula

$$\begin{aligned} |\zeta|_c &= |a| + |b| = (|\zeta|^2 - d(\zeta, \mathbb{C}V)^2)^{\frac{1}{2}} + d(\zeta, \mathbb{C}V) \\ &= (|\zeta|^2 + (|\zeta|^4 - |\langle \zeta, \zeta \rangle|^2)^{\frac{1}{2}})^{\frac{1}{2}} \\ &= (|\zeta|^2 + 2(|\xi|^2|\eta|^2 - \langle \xi, \eta \rangle^2)^{\frac{1}{2}})^{\frac{1}{2}}. \end{aligned} \quad (4.8)$$

The  $|\cdot|$ -distance from  $\zeta$  to  $\mathbb{C}V$  is given by  $d(\zeta, \mathbb{C}V) = |b|$ . If  $V$  is a Hilbert space and  $\langle \zeta, \zeta \rangle \neq 0$ , then  $e^{i\theta}a$  is a unique point in  $\mathbb{C}V$  with minimal distance to  $\zeta$ . If  $\langle \zeta, \zeta \rangle = 0$ , then  $\{e^{i\theta} \operatorname{Re}(e^{-i\theta}\zeta); \theta \in \mathbb{R}\} = \{\frac{1}{2}(\zeta + e^{2i\theta}\bar{\zeta}); \theta \in \mathbb{R}\}$  is a circle consisting of all points

in  $\mathbb{C}V$  with minimal distance to  $\zeta$ .

*Proof.* Equations (4.6) are clear from the orthogonality of  $a$  and  $b$  and equations (4.7) then follow.

For proving the first equality in (4.8) we observe that  $\zeta = e^{i\theta}a + e^{i\theta}ib$  is one of the linear combinations in (4.3), so we have  $|\zeta|_c \leq |a| + |b|$  and equality holds if  $b = 0$ . Assume that  $b \neq 0$  and take  $e_a = a/|a|$  and  $e_b = b/|b|$ . If  $\zeta = \sum_{j=1}^k \alpha_j \xi_j$  is some other representation of  $\zeta$  as in (4.3), then  $a = \sum_{j=1}^k \operatorname{Re}(e^{-i\theta} \alpha_j) \xi_j$ ,  $b = \sum_{j=1}^k \operatorname{Im}(e^{-i\theta} \alpha_j) \xi_j$ , so the Cauchy-Schwarz inequality and Pythagoras' theorem give

$$\begin{aligned} |a| + |b| &= \langle a, e_a \rangle + \langle b, e_b \rangle = \sum_{j=1}^k \langle \operatorname{Re}(e^{-i\theta} \alpha_j) e_a + \operatorname{Im}(e^{-i\theta} \alpha_j) e_b, \xi_j \rangle \\ &\leq \sum_{j=1}^k |\operatorname{Re}(e^{-i\theta} \alpha_j) e_a + \operatorname{Im}(e^{-i\theta} \alpha_j) e_b| |\xi_j| = \sum_{j=1}^k |\alpha_j| |\xi_j|. \end{aligned}$$

This proves the first equality in (4.8) and the others follow from (4.7).

For the remaining statements, let  $\alpha \in \mathbb{R}$  and  $v \in V$ . Since  $a, b$  are orthogonal we have

$$\begin{aligned} |v|^2 + |a|^2 \cos^2(\alpha) + |b|^2 \sin^2(\alpha) &= |v|^2 + |a \cos(\alpha) + b \sin(\alpha)|^2 \\ &\geq 2\langle v, a \cos(\alpha) + b \sin(\alpha) \rangle = 2\langle a, v \rangle \cos(\alpha) + 2\langle b, v \rangle \sin(\alpha) \end{aligned} \quad (4.9)$$

and therefore

$$\begin{aligned} |\zeta - e^{i(\theta+\alpha)}v|^2 &= |\zeta|^2 + |v|^2 - 2\operatorname{Re}\langle \zeta, e^{-i(\theta+\alpha)}v \rangle \\ &= |\zeta|^2 + |v|^2 - 2\langle a, v \rangle \cos(\alpha) - 2\langle b, v \rangle \sin(\alpha) \\ &\geq |\zeta|^2 - |a|^2 \cos^2(\alpha) - |b|^2 \sin^2(\alpha) \\ &= |a|^2 \sin^2(\alpha) + |b|^2 \cos^2(\alpha) \geq |b|^2. \end{aligned} \quad (4.10)$$

Moreover, if  $|b| < |a|$ , then it can be checked that an equality in (4.9) and (4.10) is obtained if and only if  $\alpha = 0$  and  $v = a$  in which case we have  $|\zeta - e^{i(\theta+\alpha)}v|^2 = |b|^2$ . If  $|b| = |a|$  then an equality is obtained with  $v = a \cos(\alpha) + b \sin(\alpha)$  and  $\alpha \in \mathbb{R}$ .  $\square$

Observe that the last two expressions for the Lie norm in (4.8) were first proved by Drużkowski [36] and the formula  $d(\zeta, \mathbb{C}V) = |b|$  is proven in [47].

From now on we restrict our attention to the case when  $V = \mathbb{R}^N$ . We let  $\langle \cdot, \cdot \rangle$  denote the natural bilinear form on  $\mathbb{C}^N$ ,  $\langle z, \zeta \rangle = \sum_{j=1}^N z_j \zeta_j$ , which means that the

Hermitian form is  $(z, \zeta) \mapsto \langle z, \bar{\zeta} \rangle$ . We let  $|\cdot|$  denote the Hermitian norm on  $\mathbb{C}^N$  and  $|\cdot|_c$  denote the Lie norm. We prove the following formula for future reference.

**Lemma 4.1.3** *Let  $\zeta \in \mathbb{C}^N \setminus \{0\}$  and write  $\zeta = e^{i\theta}(a + ib)$  with  $a, b \in \mathbb{R}^N$ ,  $\langle a, b \rangle = 0$  and  $|a| \geq |b|$ . Then*

$$|\zeta + \lambda e^{2i\theta} \bar{\zeta}|_c = |(1 + \lambda)|a| + (1 - \lambda)|b||, \quad \lambda \in \mathbb{D}.$$

*Proof.* Write  $\eta = \zeta + \lambda e^{2i\theta} \bar{\zeta} = e^{i\theta}((1 + \lambda)a + (1 - \lambda)ib)$ . Then we have

$$|\eta|^2 = |1 + \lambda|^2|a|^2 + |1 - \lambda|^2|b|^2 \quad \text{and} \quad \langle \eta, \eta \rangle = e^{2i\theta}((1 + \lambda)^2|a|^2 - (1 - \lambda)^2|b|^2).$$

Observe that for all  $z \in \mathbb{C}$  we have  $|z|^2 + \operatorname{Re}(z^2) = 2(\operatorname{Re}(z))^2$  so

$$|\eta|^4 - |\langle \eta, \eta \rangle|^2 = 2(|1 - \lambda^2|^2 + \operatorname{Re}((1 - \lambda^2)^2))|a|^2|b|^2 = (2\operatorname{Re}(1 - \lambda^2))^2|a|^2|b|^2.$$

Since  $\lambda \in \mathbb{D}$  then  $\operatorname{Re}(1 - \lambda^2) > 0$  and by Proposition 4.1.2 we have

$$\begin{aligned} |\eta|_c^2 &= |\eta|^2 + (|\eta|^4 - |\langle \eta, \eta \rangle|^2)^{1/2} \\ &= |1 + \lambda|^2|a|^2 + |1 - \lambda|^2|b|^2 + 2\operatorname{Re}(1 - \lambda^2)|a||b| \\ &= |(1 + \lambda)|a| + (1 - \lambda)|b||^2. \end{aligned}$$

□

Now consider the function  $h : \mathbb{C}^N \setminus \{0\} \rightarrow \mathbb{R}$  given by  $h(\zeta) = \log |\zeta|_c$ . Since  $|\cdot|_c$  is a norm, the function

$$\lambda \rightarrow h(\zeta + \lambda\zeta) = \log |1 + \lambda| + \log |\zeta|_c, \quad \lambda \in \mathbb{C}$$

is clearly harmonic for each  $\zeta \in \mathbb{C}^N \setminus \{0\}$ . In other words,  $\zeta$  is a member of the kernel of the Levi form of  $h$  at  $\zeta$ . By Lemma 4.1.3 the function

$$\lambda \rightarrow h(\zeta + \lambda e^{2i\theta} \bar{\zeta}) = \log |(1 + \lambda)|a| + (1 - \lambda)|b||, \quad \lambda \in \mathbb{D}$$

is also harmonic, which means that  $\bar{\zeta}$  is also a member of the kernel of the Levi form of  $h$  at  $\zeta$ . Now let  $\Phi : X \rightarrow \mathbb{C}^N$  be a non-vanishing holomorphic function on a complex manifold  $X$  of dimension  $n$ . By the chain rule we have

$$\mathcal{L}_{h \circ \Phi}(z, w) = \mathcal{L}_h(\Phi(z), d\Phi(z)(w)), \quad z \in X, w \in T_z X.$$

We therefore see that if  $U$  is an open subset of  $X$  with the property that for every  $z \in U$  there exists a tangent vector  $w \neq 0$  at  $z$  which is mapped by  $d\Phi(z)$  to the set  $\text{Span}\{\Phi(z), \bar{\Phi}(z)\}$ , then the Levi form of  $h \circ \Phi$  is degenerate at every point of  $U$ , which implies that  $h \circ \Phi$  is a maximal psh function on  $U$ . In particular, if  $N = n + 1$ , it follows from the rank-nullity theorem that if the vectors  $\Phi(z)$  and  $\bar{\Phi}(z)$  are linearly independent, then  $h \circ \Phi$  is maximal in a neighborhood of  $z$ . The vectors  $\Phi(z)$  and  $\bar{\Phi}(z)$  are linearly independent if and only if  $\Phi(z) \in \mathbb{C}^N \setminus \mathbb{C}\mathbb{R}^N$ , we have therefore reached the following conclusion.

**Proposition 4.1.4** *Let  $\Phi : X \rightarrow \mathbb{C}^{n+1}$  be a non-vanishing holomorphic map on a manifold  $X$  of dimension  $n$ . Then the function  $\log |\Phi|_c$  is a maximal psh function on  $\Phi^{-1}(\mathbb{C}^{n+1} \setminus \mathbb{C}\mathbb{R}^{n+1})$ .*

## 4.2 Geometry of $\mathbb{C}\mathbb{R}^N$

The set  $\mathbb{C}\mathbb{R}^N$  consists of all vectors in  $\mathbb{C}^N$  with parallel real and imaginary parts. Since  $\mathbb{C}\mathbb{R}^N \setminus \{0\}$  is the inverse image of the real projective space  $\mathbb{P}^{N-1}(\mathbb{R})$  under the natural map from  $\mathbb{C}^N \setminus \{0\}$  it is a real analytic manifold of dimension  $N + 1$ . Every  $\zeta$  in  $\mathbb{C}\mathbb{R}^N$  can be written as  $\zeta = e^{i\theta}a$  and we say that  $\theta$  is the argument of  $\zeta$ . The map

$$\mathbb{R} \times (\mathbb{R}^N \setminus \{0\}) \ni (\theta, a) \mapsto e^{i\theta}a \in \mathbb{C}^N \setminus \{0\} \quad (4.11)$$

has an injective differential at every point. We have  $e^{i\theta}a = e^{i\varphi}b$  if and only if  $\theta - \varphi = k\pi$  and  $a = (-1)^k b$  for some integer  $k$ . This proves again that  $\mathbb{C}\mathbb{R}^N \setminus \{0\}$  is a real analytic manifold of dimension  $N + 1$ , and that it is a fiber bundle over  $\mathbb{S}^1$  with the projection

$$\mathbb{C}\mathbb{R}^N \setminus \{0\} \ni \zeta = e^{i\theta}a \mapsto \frac{\langle \zeta, \zeta \rangle}{|\zeta|^2} = e^{2i\theta} \in \mathbb{S}^1 \quad (4.12)$$

and fiber  $\mathbb{R}^N \setminus \{0\}$ . We have already noted that  $\mathbb{C}\mathbb{R}^N$  is the set of all vectors  $\zeta = \xi + i\eta$  with parallel real and imaginary parts  $\xi$  and  $\eta$ . We have  $|\zeta|^2 - |\langle \zeta, \zeta \rangle| = 4(|\xi|^2|\eta|^2 - \langle \xi, \eta \rangle^2)$  which gives that  $\mathbb{C}\mathbb{R}^N$  is a real algebraic variety,

$$\mathbb{C}\mathbb{R}^N = \{\xi + i\eta \in \mathbb{C}^N ; |\xi||\eta| = |\langle \xi, \eta \rangle|\} = \{\zeta \in \mathbb{C}^N ; |\zeta|^2 = |\langle \zeta, \zeta \rangle|\}. \quad (4.13)$$

In Section 4.1 we have seen how useful the  $(\theta, a, b)$  coordinates are for the calculation of the Lie norm. We are going to use these coordinates to parameterize the set

$\{\zeta \in \mathbb{C}^N; \langle \zeta, \zeta \rangle \neq 0\}$  and to express the standard volume form on  $\mathbb{C}^N$ ,

$$dV = \left(\frac{i}{2}\right)^N \bigwedge_{j=1}^N dz_j \wedge d\bar{z}_j \quad (4.14)$$

with respect to these coordinates.

**Proposition 4.2.1** *The pullback of the volume form  $dV$  on  $\mathbb{C}^N$ , under the map  $(\theta, a, b) \mapsto e^{i\theta}(a + ib)$  to the  $2N$  dimensional real manifold*

$$L = \{(\theta, a, b); \theta \in \mathbb{R}, a, b \in \mathbb{R}^N, \langle a, b \rangle = 0, |b| < |a|\} \subset \mathbb{R}^{2N+1} \quad (4.15)$$

is given by the formula

$$dV = -d\theta \wedge \sum_{j=1}^N (a_j \Lambda_{b_j} + b_j \Lambda_{a_j}) \quad (4.16)$$

where the  $2N - 1$  forms  $\Lambda_{a_j}$  and  $\Lambda_{b_j}$  are given by

$$\Lambda_{a_j} = da_1 \wedge db_1 \wedge \cdots \wedge \widehat{da_j} \wedge db_j \wedge \cdots \wedge da_N \wedge db_N \quad (4.17)$$

$$\Lambda_{b_j} = da_1 \wedge db_1 \wedge \cdots \wedge da_j \wedge \widehat{db_j} \wedge \cdots \wedge da_N \wedge db_N, \quad (4.18)$$

and  $\widehat{da_j}$  and  $\widehat{db_j}$  is the standard notation for omitted factors. On the open subset  $L_k := \{(\theta, a, b) \in L; a_k \neq 0\}$  of  $L$  we can express the volume form as

$$dV = \frac{|b|^2}{a_k} d\theta \wedge \Lambda_{b_k} - d\theta \wedge \sum_{j=1}^N a_j \Lambda_{b_j}. \quad (4.19)$$

*Proof.* We have  $z_j = e^{i\theta}(a_j + ib_j)$  and  $\bar{z}_j = e^{-i\theta}(a_j - ib_j)$ , so

$$dz_j = ie^{i\theta}((a_j + ib_j)d\theta - ida_j + db_j) \quad \text{and}$$

$$d\bar{z}_j = -ie^{-i\theta}((a_j - ib_j)d\theta + ida_j + db_j)$$

and we get

$$dz_j \wedge d\bar{z}_j = -2i(-a_j d\theta \wedge da_j - b_j d\theta \wedge db_j + da_j \wedge db_j). \quad (4.20)$$

Now equation (4.16) follows by wedging equation (4.20) over all  $j \in \{1, \dots, N\}$ , and noticing that  $\bigwedge_{j=1}^N da_j \wedge db_j = 0$ , which follows from the equation  $a_1 b_1 + \cdots + a_N b_N = 0$ .

Furthermore, this equation implies that  $b_j \Lambda_{a_j} = -\frac{b_j^2}{a_k} \Lambda_{b_k}$  on  $L_k$  for every  $j$  and  $k$ . If we combine this with (4.16), then (4.19) follows.  $\square$

In some cases the variables  $\theta, a_1, \dots, a_N, b_1, \dots, b_N$  on  $L$  are inconvenient to work with because they are dependent through the equality  $\langle a, b \rangle = 0$ . Therefore we define the set

$$\tilde{L} = \{(\theta, a, \beta) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^{N-1}; |\beta| < |a|\}$$

and present the following proposition.

**Proposition 4.2.2** *For each  $m \in \{1, \dots, N\}$  there exists a change of variables  $L_m := \{(\theta, a, b) \in L; a_m \neq 0\} \rightarrow \tilde{L}$ ,  $(\theta, a, b) \mapsto (\theta, a, \beta)$  such that  $|\beta| = |b|$  and*

$$\begin{aligned} dV &= -(-1)^{\frac{N(N+1)}{2}} \left( |a| - \frac{|\beta|^2}{|a|} \right) d\theta \wedge dV_a \wedge dV_\beta, \quad \text{where} \\ dV_a &= da_1 \wedge \dots \wedge da_N \quad \text{and} \quad dV_\beta = d\beta_1 \wedge \dots \wedge d\beta_{N-1}. \end{aligned}$$

*Proof.* For each  $a$  with  $a_m \neq 0$  we can define an orthogonal matrix  $U_a = (u_{j,k}(a))$  whose last column is parallel to  $a$  and the entries of the matrix  $U_a$  are smooth as functions of  $a$ . This can be done by applying the Gram-Schmidt method on the basis  $\{a, e_1, \dots, e_N\} \setminus \{e_m\}$ , where  $\{e_1, \dots, e_N\}$  is the standard basis. Now we define the change of variables  $L_m \rightarrow \tilde{L}$ ,  $(\theta, a, b) \mapsto (\theta, a, \beta)$  by

$$(\beta_1, \dots, \beta_{N-1}, 0)^t = U_a^{-1}(b_1, \dots, b_N)^t.$$

Since  $U_a$  is orthogonal its cofactors are given by the equation  $C_{j,k} = u_{j,k}$ , in particular we have  $C_{j,N} = a_j/|a|$ . Therefore, for any  $s \in \{1, \dots, N\}$ , we have

$$\begin{aligned} (-1)^{\frac{N(N-1)}{2}} \Lambda_{b_s} &= (-1)^{s+N} da_1 \wedge \dots \wedge da_N \wedge db_1 \wedge \dots \wedge \widehat{db_s} \wedge \dots \wedge db_N \\ &= (-1)^{s+N} dV_a \bigwedge_{j=1, j \neq s}^N \left( \sum_{k=1}^{N-1} (u_{j,k} d\beta_k + \beta_k du_{j,k}) \right) \\ &= (-1)^{s+N} dV_a \bigwedge_{j=1, j \neq s}^N \left( \sum_{k=1}^{N-1} u_{j,k} d\beta_k \right) = C_{s,N} dV_a \wedge dV_\beta \\ &= \frac{a_s}{|a|} dV_a \wedge dV_\beta. \end{aligned} \tag{4.21}$$

The result follows by combining equation (4.21) with equation (4.19).  $\square$

### 4.3 Applications

In this section we apply Theorem 4.0.1 to give explicit formulas for the Siciak-Zakharyuta extremal functions and the corresponding equilibrium measures for a few examples of compact sets. We have grouped the examples into three cases; namely we consider compact subsets of the Euclidean space  $\mathbb{C}^n$ , the projective space  $\mathbb{P}^n(\mathbb{C})$  and the complex torus  $\mathbb{C}^n/\mathbb{Z}^n$ . For the rest of the section we write  $\mathbb{R}_+ := [0, \infty)$ .

#### 4.3.1 Compact subsets of $\mathbb{R}^n \subset \mathbb{C}^n$

Although Theorem 4.0.1 can be applied to calculate the extremal function for a variety of compact subsets of  $\mathbb{C}^n$  we restrict our attention to compact subsets of  $\mathbb{R}^n$  simply because authors in the past have shown particular interest in this case.

**Lemma 4.3.1** *Let  $f = (f_0, \dots, f_n)$  be a holomorphic map on  $U \subset \mathbb{C}^n$  satisfying  $f_0 + \dots + f_n = 1$  and write*

$$K = \{z \in U; f_j(z) \in \mathbb{R}_+ \text{ for all } j\}.$$

*Then the function*

$$\begin{aligned} \psi &:= 2 \log |(\sqrt{f_0}, \dots, \sqrt{f_n})|_c \\ &= \log \left( |f_0| + \dots + |f_n| + (|f_0| + \dots + |f_n|)^2 - 1 \right)^{1/2} \end{aligned}$$

*equals 0 on  $K$  and is maximal on  $U \setminus K$ . We have*

$$(dd^c \psi)^n|_K = \pm \frac{n! \Omega_n}{\sqrt{f_0 \cdots f_n}} df_1 \wedge \cdots \wedge df_n. \quad (4.22)$$

*Proof.* It is easy to see that  $\psi = 0$  on  $K$ . First we show that  $K = f^{-1}(\mathbb{C}\mathbb{R}_+^{n+1})$ . We clearly have  $K \subset f^{-1}(\mathbb{C}\mathbb{R}_+^{n+1})$ . Conversely, if  $z \in f^{-1}(\mathbb{C}\mathbb{R}_+^{n+1})$  there exist positive real numbers  $x_0, \dots, x_n$  and  $\theta \in [0, 2\pi]$  such that

$$f_0(z) = x_0 e^{i\theta}, f_1(z) = x_1 e^{i\theta}, \dots, f_n(z) = x_n e^{i\theta}.$$

Adding all the equations gives

$$1 = f_0(z) + \dots + f_n(z) = e^{i\theta}(x_0 + \dots + x_n)$$

so  $e^{i\theta} = 1$  which means that  $f(z) \in \mathbb{R}_+^{n+1}$  so  $z \in K$ . Now we apply Theorem 4.0.1 with  $\Phi = (\sqrt{f_0}, \dots, \sqrt{f_n})$ . We clearly have  $\Phi^{-1}(\mathbb{C}\mathbb{R}_+^{n+1}) = f^{-1}(\mathbb{C}\mathbb{R}_+^{n+1}) = K$  and since  $\Phi$  is a holomorphic map in a neighborhood of any point outside the set  $S := \{z \in U; f_0(z) \cdots f_n(z) = 0\}$  we see that the function  $\psi = \log |\Phi|_c^2$  is psh and maximal on  $U \setminus (K \cup S)$ . But since  $S$  is pluripolar we can conclude that  $\psi$  is psh and maximal on  $U \setminus K$ . By simply substituting  $\Phi_j$  with  $\sqrt{f_j}$  in equation (4.2) from Theorem 4.0.1 we get the equation

$$(dd^c\psi)^n|_K = \pm \frac{n!\Omega_n}{\sqrt{f_0} \cdots \sqrt{f_n}} \left( \sum_{j=0}^n (-1)^{j+1} f_j \Lambda_{f_j} \right).$$

Using the assumption  $f_0 + \cdots + f_n = 1$  it is easy to show that

$$\sum_{j=0}^n (-1)^{j+1} f_j \Lambda_{f_j} = df_1 \wedge \cdots \wedge df_n$$

and equation (4.22) follows.  $\square$

At this point we should point out the similarities to the Joukovski transformation  $J: \mathbb{C} \rightarrow \mathbb{C}$  defined by the formula  $J(z) = \frac{1}{2}(z + \frac{1}{z})$  and discussed in Section 2.3.4. The function  $\psi$  from Lemma 4.3.1 can be expressed as

$$\psi(z) = \log J^{-1}(|f_0| + \cdots + |f_n|).$$

We therefore see that our results, along with the extra condition  $f_0 + \cdots + f_n = 1$  brings us to a familiar territory, as the Joukovski map has been extensively studied for a long time. We include three examples which show the simplicity of this application of Theorem 4.0.1. Two of them can be found in Klimek [49], Section 5.4.

**Example 4.3.2 (The simplex).** It is easy to see that

$$K := \{z \in \mathbb{C}^n; z_j \in \mathbb{R}_+, \text{ for all } j \text{ and } 1 - z_1 - \cdots - z_n \in \mathbb{R}_+\}$$

is the simplex in  $\mathbb{R}^n \subset \mathbb{C}^n$ . By applying Lemma 4.3.1 we see that the extremal function for  $K$  is  $V_K(z) = \log J^{-1}(|f_0| + \cdots + |f_n|)$  where

$$f_0(z) = 1 - z_1 - \cdots - z_n, \quad \text{and} \quad f_j(z) = z_j, \quad 1 \leq j \leq n.$$

Using equation (4.22) we see that the equilibrium measure for  $K$  is

$$(dd^c V_K)^n|_K = \frac{n! \Omega_n}{\sqrt{x_1 \cdots x_n \cdot (1 - x_1 - \cdots - x_n)}} dx_1 \wedge \cdots \wedge dx_n.$$

**Example 4.3.3 (The unit ball).** Let

$$\begin{aligned} K &= \{x \in \mathbb{R}^n; 1 - x_1^2 - \cdots - x_n^2 \geq 0\} \\ &= \{z \in \mathbb{C}^n; 1 - z_1^2 - \cdots - z_n^2 \in \mathbb{R}_+, z_j^2 \in \mathbb{R}_+\} \end{aligned}$$

be the unit ball in  $\mathbb{R}^n \subset \mathbb{C}^n$ . By Lemma 4.3.1 we see that the extremal function for  $K$  is  $V_K(z) = \frac{1}{2} \log J^{-1}(|f_0| + \cdots + |f_n|)$  where

$$f_0(z) = 1 - z_1^2 - \cdots - z_n^2, \quad \text{and} \quad f_j(z) = z_j^2, \quad 1 \leq j \leq n.$$

We have  $df_j = 2x_j dx_j$  for all  $j \neq 0$  and therefore, by equation (4.22) the equilibrium measure for  $K$  is

$$\begin{aligned} (dd^c V_K)^n|_K &= \frac{n! \Omega_n |x_1 \cdots x_n|}{\sqrt{x_1^2 \cdots x_n^2 (1 - x_1^2 - \cdots - x_n^2)}} dx_1 \wedge \cdots \wedge dx_n \\ &= \frac{n! \Omega_n}{\sqrt{1 - x_1^2 - \cdots - x_n^2}} dx_1 \wedge \cdots \wedge dx_n. \end{aligned}$$

**Example 4.3.4 (The first and third quadrant of a disk).** Let

$$\begin{aligned} K &= \{(x_1, x_2) \in \mathbb{R}^2; 1 - x_1^2 - x_2^2 \geq 0, x_1 \cdot x_2 \geq 0\} \\ &= \{z \in \mathbb{C}^2; f_0 := 1 - z_1^2 - z_2^2 \in \mathbb{R}_+, f_1 := (z_1 - z_2)^2 \in \mathbb{R}_+, f_2 := 2z_1 z_2 \in \mathbb{R}_+\} \end{aligned}$$

be the union of the first and third quadrant of the unit disk in  $\mathbb{R}^2 \subset \mathbb{C}^2$ . To see that the sets are actually equal, notice that if  $f_1 \in \mathbb{R}_+$  then  $(z_1 - z_2) \in \mathbb{R}$  which implies that  $\text{Im}(z_1) = \text{Im}(z_2) =: y$ . Since  $f_2 \in \mathbb{R}_+$  we have

$$y(x_1 + x_2) = 0 \quad \text{and} \quad x_1 x_2 - y^2 > 0. \quad (4.23)$$

From (4.23) we see that  $y = 0$  so  $z_1, z_2 \in \mathbb{R}$  and the rest of the argument is trivial.

By Lemma 4.3.1 we have  $V_K(z) = \frac{1}{2} \log J^{-1}(|f_0| + |f_1| + |f_2|)$  and by equation

(4.22) we have

$$\begin{aligned} (dd^c V_K)^2|_K &= \frac{2\pi d(x_1 - x_2)^2 \wedge d(2x_1 x_2)}{4\sqrt{(1 - x_1^2 - x_2^2)}(x_1 - x_2)^2 2x_1 x_2} \\ &= \frac{\sqrt{2}\pi|x_1 + x_2|}{\sqrt{x_1 x_2(1 - x_1^2 - x_2^2)}} dx_1 \wedge dx_2. \end{aligned}$$

### 4.3.2 Compact subsets of the complex projective space $\mathbb{P}^n(\mathbb{C})$

Denote by  $\omega$  the Fubini-Study metric on  $\mathbb{P}^n(\mathbb{C})$ . If we imbedd  $\mathbb{C}^n$  into  $\mathbb{P}^n(\mathbb{C})$  in the canonical way  $(z_1, \dots, z_n) \rightarrow [1: z_1: \dots: z_n]$  then  $\omega$  has a psh potential on  $\mathbb{C}^n$ , namely

$$\omega = \frac{1}{2} dd^c \log(1 + |z|^2), \quad z \in \mathbb{C}^n.$$

We define the class of  $\omega$ -psh functions on  $\mathbb{P}^n(\mathbb{C})$ , denoted by  $\text{PSH}(\mathbb{P}^n(\mathbb{C}), \omega)$ , as the set of functions  $\varphi \in L^1(\mathbb{P}^n(\mathbb{C}))$  which can locally be written as the sum of a smooth and a psh function, and such that  $dd^c \varphi + \omega \geq 0$  in the sense of currents.

For a compact set  $K \subset \mathbb{P}^n(\mathbb{C})$  we define the  $\omega$ -extremal function for  $K$  to be

$$V_{K, \omega}(z) = \sup\{\varphi(z); \varphi \in \text{PSH}(\mathbb{P}^n(\mathbb{C}), \omega), \varphi|_K \leq 0\}$$

and the  $\omega$ -equilibrium measure for  $K$  is  $(dd^c V_{K, \omega})^n$ . For a detailed survey on  $\omega$ -psh functions we refer to Guedj and Zeriahi [43, 44]. See also Magnússon [55, 56].

We denote by  $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n(\mathbb{C})$  the usual projection  $\pi(z_0, \dots, z_n) = [z_0: \dots: z_n]$ . We start with a lemma very similar to Lemma 4.3.1.

**Lemma 4.3.5** *Let  $f_0, \dots, f_n$  be homogeneous polynomials in  $\mathbb{C}^{n+1}$  of degree  $2k$  satisfying the equation  $f_0 + \dots + f_n = (z_0^2 + \dots + z_n^2)^k$  and write*

$$K = \pi(\{z \in \mathbb{C}^{n+1} \setminus \{0\}; (f_0(z), \dots, f_n(z)) \in \mathbb{R}_+^{n+1}\}) \subset \mathbb{P}^n(\mathbb{C}).$$

Then the function

$$\psi := \log |(\sqrt{f_0}, \dots, \sqrt{f_n})|_c^{1/k} - \log |(z_0, \dots, z_n)|$$

equals 0 on  $K$  and  $(dd^c \psi + \omega)^n = 0$  on  $\mathbb{P}^n(\mathbb{C}) \setminus K$ . We have

$$(dd^c \psi + \omega)^n|_K = \frac{n! \Omega_n}{(2k)^n |z_0^2 + \dots + z_n^2|^{\frac{k(n+1)}{2}} \sqrt{f_0 \cdots f_n}} \left( \sum_{j=0}^n (-1)^{j+1} f_j \Lambda_{f_j} \right).$$

*Proof.* It is simple to show that

$$\begin{aligned} \mathbb{C} \cdot \{z \in \mathbb{C}^{n+1}; (f_0(z), \dots, f_n(z)) \in \mathbb{R}_+^{n+1}\} \\ = \{z \in \mathbb{C}^{n+1}; (f_0(z), \dots, f_n(z)) \in \mathbb{C}\mathbb{R}_+^{n+1}\} \end{aligned}$$

and therefore

$$K = \pi(\{z \in \mathbb{C}^{n+1} \setminus \{0\}; (f_0(z), \dots, f_n(z)) \in \mathbb{C}\mathbb{R}_+^{n+1}\}).$$

Since the result is local, and by symmetry, it is sufficient to prove it in the local coordinate chart  $\mathbb{C}^n = \{[1 : z_1 : \dots : z_n] \in \mathbb{P}^n(\mathbb{C})\}$ . In this chart we have

$$(dd^c\psi + \omega)^n = \frac{1}{k^n} (dd^c \log |(\sqrt{f_0}, \dots, \sqrt{f_n})|_c)^n$$

so we get the result by a similar argument as in the proof of Lemma 4.3.1.  $\square$

We consider two examples, the first one is studied in [21].

**Example 4.3.6 (The real projective space).** Let

$$K = \mathbb{P}^n(\mathbb{R}) = \{[x_0 : \dots : x_n]; x_j \in \mathbb{R} \text{ for all } j\} \subset \mathbb{P}^n(\mathbb{C})$$

be the real projective space. It is easy to see that

$$K = \pi(\{z \in \mathbb{C}^{n+1} \setminus \{0\}; z_j^2 \in \mathbb{R}_+ \text{ for all } j\})$$

and therefore

$$V_{K,\omega}(z) = \log |(z_0, \dots, z_n)|_c - \log |(z_0, \dots, z_n)|, \quad z = [z_0 : \dots : z_n],$$

is the  $\omega$ -extremal function for  $K$ . We clearly have  $K \cap \mathbb{C}^n = \mathbb{R}^n$  and

$$\begin{aligned} (dd^c V_{K,\omega} + \omega)^n|_{\mathbb{R}^n} &= \frac{n! \Omega_n}{2^n (1 + x_1^2 + \dots + x_n^2)^{\frac{(n+1)}{2}} \sqrt{x_1^2 \cdots x_n^2}} dx_1^2 \wedge \dots \wedge dx_n^2 \\ &= \frac{n! \Omega_n}{(1 + x_1^2 + \dots + x_n^2)^{\frac{(n+1)}{2}}} dx_1 \wedge \dots \wedge dx_n. \end{aligned}$$

**Example 4.3.7 (The first and third quadrant of the plane).** Let

$$K = \pi(\{(x_0, x_1, x_2) \in \mathbb{R}^3 \setminus \{0\}; x_1 \cdot x_2 \geq 0\}) \subset \mathbb{P}^2(\mathbb{C}).$$

When restricted to  $\mathbb{C}^2$  the set  $K$  becomes the union of the first and third quadrant of the plane  $\mathbb{R}^2$ , i.e.

$$K \cap \mathbb{C}^2 = \{(x_1, x_2) \in \mathbb{R}^2; x_1 \cdot x_2 \geq 0\}.$$

Using an argument very similar to the one in Example 4.3.4 we can see that

$$K = \pi(\{(z_0, z_1, z_2) \in \mathbb{C}^3 \setminus \{0\}; z_0^2 \in \mathbb{R}_+, (z_1 - z_2)^2 \in \mathbb{R}_+, 2z_1z_2 \in \mathbb{R}_+\})$$

and therefore the  $\omega$ -extremal function for  $K$  is

$$V_{K,\omega}(z) = \log |(z_0, (z_1 - z_2), \sqrt{2z_1z_2})|_c - \log |(z_0, z_1, z_2)|, \quad z = [z_0, z_1, z_2].$$

We have

$$(dd^c V_{K,\omega} + \omega)^2|_{K \cap \mathbb{C}^2} = \frac{\sqrt{2}\pi(x_1 + x_2)}{(1 + x_1^2 + x_2^2)^{\frac{3}{2}}\sqrt{x_1x_2}} dx_1 \wedge dx_2.$$

### 4.3.3 A compact subset of the complex torus

Finally we consider one compact subset of the complex torus  $X = \mathbb{C}^n / \mathbb{Z}^n$ . Define the functions  $f_0, \dots, f_n$  on  $X$  by

$$f_0(z) = \frac{1}{n} \sum_{j=1}^n \cos^2(\pi z_j), \quad z = (z_1, \dots, z_n) \in X,$$

and

$$f_j(z) = \frac{1}{n} \sin^2(\pi z_j), \quad j \in \{1, \dots, n\}, z \in X,$$

so we clearly have  $f_0 + \dots + f_n = 1$  on  $X$ . Then the set

$$\begin{aligned} K &= \{z \in X; f_j(z) \in \mathbb{R}_+ \text{ for all } j\} \\ &= \{z \in X; y_j = 0 \text{ or } x_j = 1/2 \text{ for all } j \text{ and } \sum_{j=1}^n \cos^2(\pi z_j) \geq 0\} \end{aligned}$$

is easily seen to be compact. By Lemma 4.3.1 the function

$$\psi = \log J^{-1}(|f_0| + \dots + |f_n|)$$

is maximal on  $X \setminus K$ . For  $J \subset \{1, \dots, n\}$  write

$$K_J = K \cap \{z \in X; y_j = 0 \text{ if } j \in J \text{ and } x_j = 1/2 \text{ if } j \notin J\}.$$

Then we have  $K = \cup_J K_J$  and notice that  $K_{\{1, \dots, n\}} = \mathbb{R}^n / \mathbb{Z}^n$  and  $K_\emptyset = \emptyset$ . To simplify notation we only calculate  $(dd^c \psi)^n$  on  $K_J$  when  $J$  is of the type  $J = \{1, 2, \dots, s\}$  for some  $1 \leq s \leq n$  since every other  $K_J$  is practically identical to some  $K_J$  of this type. Indeed if  $J = \{1, 2, \dots, s\}$ , then

$$f_0|_{K_J} = \frac{1}{n} \left( \sum_{j=0}^s \cos^2(\pi x_j) - \sum_{j=s+1}^n \sinh^2(\pi y_j) \right)$$

and

$$f_j|_{K_J} = \frac{1}{n} \sin^2(\pi x_j) \text{ if } j \leq s, \quad f_j|_{K_J} = \frac{1}{n} \cosh^2(\pi y_j) \text{ if } j > s.$$

By Lemma 4.3.1 we have

$$(dd^c \psi)^n|_{K_J} = \frac{2^n n! \pi^n \Omega_n}{n^{n/2} \sqrt{f_0}} |\cos(\pi x_1) \cdots \cos(\pi x_s) \sinh(\pi y_{s+1}) \cdots \sinh(\pi y_n)| \\ dx_1 \wedge \cdots \wedge dx_s \wedge dy_{s+1} \wedge \cdots \wedge dy_n.$$

## 4.4 Polynomial maps

Let  $X$  be an  $n$  dimensional complex manifold and  $q_0, \dots, q_{n-1}$  be holomorphic functions on  $X$ . For every  $z \in X$  denote by  $P_z : \mathbb{C} \rightarrow \mathbb{C}$  the polynomial defined by the equation

$$P_z(\zeta) = \zeta^{n+1} - \zeta^n + q_{n-1}(z)\zeta^{n-1} - \cdots + (-1)^n q_1(z)\zeta + (-1)^{n+1} q_0(z) \\ = \zeta^{n+1} - \zeta^n + \sum_{k=0}^{n-1} (-1)^{n+1-k} q_k(z)\zeta^k.$$

For every  $z$  denote by  $f_0(z), \dots, f_n(z)$  the roots of the polynomial  $P_z$  and define the set

$$K := \{z \in \mathbb{C}^n; \text{ all the roots of } P_z \text{ are positive real numbers}\} \\ = \{z \in \mathbb{C}^n; f_j(z) \in \mathbb{R}_+, 0 \leq j \leq n\}.$$

Notice that we have  $f_0 + \dots + f_n = 1$  so Lemma 4.3.1 applies in this situation.

**Theorem 4.4.1** *Define  $\psi: \mathbb{C}^n \rightarrow \mathbb{R}$  by  $\psi(z) = \log J^{-1}(|f_0| + \dots + |f_n|)$ . The function  $\psi$  equals 0 on  $K$ , it is maximal on  $\mathbb{C}^n \setminus K$  and*

$$(dd^c \psi)^n|_K = \pm \frac{n! \Omega_n}{\sqrt{q_0(z) \Delta(z)}} dq_0 \wedge \dots \wedge dq_{n-1}. \quad (4.24)$$

where  $\Delta(z)$  is the discriminant of the polynomial  $P_z$ .

Recall that the discriminant of a polynomial  $p$  of degree  $n$  with roots  $s_1, \dots, s_n$  and leading coefficient  $a_n$  is defined as

$$\Delta := a_n^{2n-2} \prod_{j < k} (s_j - s_k)^2 = (-1)^{\frac{n(n-1)}{2}} a_n^{2n-2} \prod_{j \neq k} (s_j - s_k).$$

The discriminant can also be represented in terms of the derivative of  $p$ , indeed we have

$$\prod_{j=1}^n p'(s_j) = \prod_{j=1}^n \left( a_n \prod_{\substack{k=1 \\ k \neq j}}^n (s_k - s_j) \right) = \frac{(-1)^{\frac{n(n-1)}{2}} \Delta}{a_n^{n-2}}. \quad (4.25)$$

It follows easily from definition of the discriminant that for every monic polynomial  $p$  of degree  $n$  with discriminant  $\Delta$  and roots  $s_1, \dots, s_n$  we have

$$\tilde{\Delta}_j \cdot (p'(s_j))^2 = \Delta, \quad j = 1, \dots, n \quad (4.26)$$

where  $\tilde{\Delta}_j$  is the discriminant of  $\frac{p(z)}{z-s_j}$ . Before we prove Theorem 4.4.1 we consider a lemma.

**Lemma 4.4.2** *Let  $P_z, q_0, \dots, q_{n-1}, f_0, \dots, f_n$  and  $\Delta$  be as in Theorem 4.4.1. Then we have*

$$df_1 \wedge \dots \wedge df_n = \pm \frac{dq_0 \wedge \dots \wedge dq_{n-1}}{\sqrt{\Delta(z)}}. \quad (4.27)$$

*Proof.* For every  $j = 0, \dots, n$  we have

$$0 = P_z(f_j(z)) = f_j^{n+1}(z) - f_j^n(z) + \sum_{k=0}^{n-1} (-1)^{n+1-k} q_k(z) f_j^k(z). \quad (4.28)$$

Differentiating equation (4.28) gives

$$P'_z(f_j(z))df_j = \sum_{k=0}^{n-1} (-1)^{n-k} f_j^k(z) dq_k. \quad (4.29)$$

Wedging equation (4.29) over  $1 \leq j \leq n$  (notice we do not include  $j = 0$  in the wedge product) we get

$$\left( \prod_{j=1}^n P'_z(f_j(z)) \right) df_1 \wedge \cdots \wedge df_n = (-1)^{\frac{n(n-1)}{2}} \det(A) dq_0 \wedge \cdots \wedge dq_{n-1} \quad (4.30)$$

where  $A$  is the matrix with coefficients  $A_{j,k} = f_j^{k-1}(z)$ . But  $A$  is a Vandermonde matrix and therefore we see that  $(\det(A))^2$  equals the discriminant of the polynomial  $\frac{P_z(\zeta)}{\zeta - f_0(z)}$ . Now the result follows by multiplying  $P'_z(f_0(z))$  on both sides of equation (4.30) and then applying equations (4.25) and (4.26).  $\square$

*Proof of Theorem 4.4.1.* The functions  $f_0, \dots, f_n$  are holomorphic in a neighborhood of any point  $z \in \mathbb{C}^n$  such that  $\Delta(z) \neq 0$ . Therefore, by Lemma 4.3.1 we see that  $\psi$  is maximal on

$$\mathbb{C}^n \setminus (K \cup \{\Delta(z) = 0\}).$$

It is well known that the discriminant is a holomorphic function so the set  $\{\Delta(z) = 0\}$  is pluripolar and we conclude that  $\psi$  is maximal on  $\mathbb{C}^n \setminus K$ . Equation (4.24) follows from equation (4.22) by applying Lemma 4.4.2 and noticing that  $q_0 = f_0 \cdots f_n$ .  $\square$

Before we calculate an explicit example we prove the following lemma.

**Lemma 4.4.3** *The roots of the polynomial  $p(z) = z^3 - z^2 + az - b$  are non-negative real numbers if and only if  $a, b, \Delta$  are non-negative real numbers.*

*Proof.* Let  $s_0, s_1, s_2$  be the roots of  $p$ . First suppose all the roots are non-negative real numbers. Then we have

$$z^3 - z^2 + az - b = (z - s_0)(z - s_1)(z - s_2)$$

and by comparing coefficients we see that  $a, b \geq 0$ . Also by definition of the discriminant we have  $\Delta \geq 0$ . Conversely suppose  $a, b, \Delta \geq 0$ . Since the discriminant is non-negative all the roots  $s_0, s_1, s_2$  are real numbers. We have  $p(x) < 0$  for all  $x < 0$  and therefore the roots can't be negative.  $\square$

**Remark:** A similar statement for higher order polynomials is not true. As a counter-example we can consider the polynomial

$$p(z) = z^4 - z^3 + \frac{9}{4}z^2 - z + \frac{5}{4}$$

which has discriminant  $\Delta = \frac{289}{16}$ . The polynomial  $p$  would satisfy the higher order analogue of Lemma 4.4.3 but it has 4 complex roots, namely

$$s_0 = i, s_1 = -i, s_2 = \frac{1}{2} - i, s_3 = \frac{1}{2} + i.$$

□

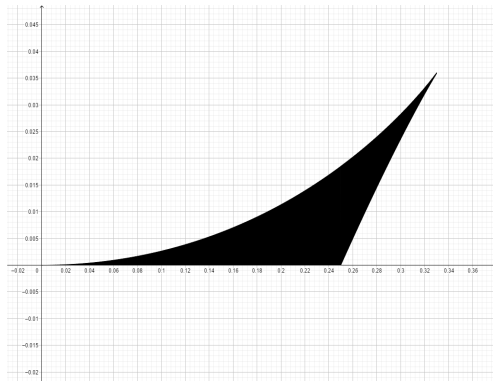
**Example 4.4.4** We apply Theorem 4.4.1 with  $X = \mathbb{C}^2$ ,  $q_0(z) = z_2$  and  $q_1(z) = z_1$  so  $P_z(\zeta) = \zeta^3 - \zeta^2 + z_1\zeta - z_2$ . Then, using the equation for the discriminant for third order polynomials, we have

$$\Delta(z) = z_1^2 - 4z_1^3 - 4z_2 - 27z_2^2 + 18z_1z_2.$$

Using a program we find that

$$\begin{aligned} K &= \{(z_1, z_2) \in \mathbb{C}^2; \text{all the roots of } P_z \text{ are real and positive}\} \\ &= \{(x_1, x_2) \in \mathbb{R}_+^2; \Delta(x_1, x_2) \geq 0\} \end{aligned}$$

looks like this:



The Monge-Ampère measure of  $\psi$  defined as in Theorem 4.4.1 is

$$(dd^c\psi)^2|_K = \frac{2\pi dx_1 \wedge dx_2}{\sqrt{x_2(x_1^2 - 4x_1^3 - 4x_2 - 27x_2^2 + 18x_1x_2)}}.$$

Notice that  $\psi$  is not the actual extremal function for  $K$  because its growth at infinity differs from the logarithm. This means that we have not calculated the actual equilibrium measure for  $K$ .

## 4.5 Regularization of the Lie norm

In this section we study the Levi form of the function  $h_\varepsilon = \log v_\varepsilon$  where  $(v_\varepsilon)_{\varepsilon \geq 0}$  is the regularization of  $v = |\cdot|_c$  given for  $\zeta \in \mathbb{C}^N$  by

$$v_\varepsilon(\zeta) = (|a|^2 + \varepsilon|b|^2)^{\frac{1}{2}} + (|b|^2 + \varepsilon|a|^2)^{\frac{1}{2}} \quad (4.31)$$

with the same notation as in the previous sections. We see that  $v_\varepsilon$  is complex homogeneous of degree 1,  $v_\varepsilon(\zeta) \searrow v(\zeta) = |a| + |b| = |\zeta|_c$  as  $\varepsilon \searrow 0$ . The Levi form of  $h_\varepsilon$  is quite involved, so in order to simplify our calculations we define the functions  $A$  and  $B$  on  $\mathbb{C}^N$  by

$$A(\zeta) = |a|^2 = \frac{1}{2}(|\zeta|^2 + |\langle \zeta, \zeta \rangle|) \quad \text{and} \quad B(\zeta) = |b|^2 = \frac{1}{2}(|\zeta|^2 - |\langle \zeta, \zeta \rangle|)$$

and for every  $\varepsilon \geq 0$ ,  $A_\varepsilon = A + \varepsilon B$  and  $B_\varepsilon = B + \varepsilon A$ . We observe that

$$\begin{aligned} A_\varepsilon + B_\varepsilon &= (1 + \varepsilon)(A + B) = (1 + \varepsilon)|\zeta|^2, \\ A_\varepsilon - B_\varepsilon &= (1 - \varepsilon)(A - B) = (1 - \varepsilon)|\langle \zeta, \zeta \rangle|, \quad \text{and} \\ 4A_\varepsilon B_\varepsilon &= (A_\varepsilon + B_\varepsilon)^2 - (A_\varepsilon - B_\varepsilon)^2 = (1 + \varepsilon)^2|\zeta|^4 - (1 - \varepsilon)^2|\langle \zeta, \zeta \rangle|^2. \end{aligned}$$

We define the function  $\varphi_\varepsilon$  on  $\mathbb{C}^N$  by

$$\varphi_\varepsilon(\zeta) = 2(A_\varepsilon(\zeta)B_\varepsilon(\zeta))^{\frac{1}{2}} = ((1 + \varepsilon)^2|\zeta|^4 - (1 - \varepsilon)^2|\langle \zeta, \zeta \rangle|^2)^{\frac{1}{2}} \quad (4.32)$$

$$= 2((1 + \varepsilon^2)|a|^2|b|^2 + \varepsilon(|a|^4 + |b|^4))^{\frac{1}{2}}. \quad (4.33)$$

The right hand side of (4.32) shows that  $\varphi_\varepsilon$  is a  $C^\infty$  function on  $\mathbb{C}^N \setminus \{0\}$  and the formula for  $v_\varepsilon$  becomes

$$v_\varepsilon(\zeta) = A_\varepsilon^{\frac{1}{2}} + B_\varepsilon^{\frac{1}{2}} = (A_\varepsilon + B_\varepsilon + 2(A_\varepsilon B_\varepsilon)^{\frac{1}{2}})^{\frac{1}{2}} = ((1 + \varepsilon)|\zeta|^2 + \varphi_\varepsilon(\zeta))^{\frac{1}{2}}. \quad (4.34)$$

**Theorem 4.5.1** *The function  $h_\varepsilon = \log v_\varepsilon$  is psh and maximal on  $\mathbb{C}^N$ . If  $\zeta \in \mathbb{C}^N \setminus$*

$\mathbb{C}\mathbb{R}^N$  then the Levi-form of  $h_\varepsilon$  at  $\zeta$  is

$$\mathcal{L}_{h_\varepsilon}(\zeta, w) = \frac{(1 + \varepsilon)}{2\varphi_\varepsilon(\zeta)} \left( \frac{4\varepsilon|\zeta|^4}{\varphi_\varepsilon(\zeta)^2} |\langle w, \xi \rangle|^2 + |w'|^2 \right)$$

where  $w'$  denotes the component of  $w$  orthogonal to  $\zeta$  and  $\bar{\zeta}$ , and

$$\xi = (|b|e_a - i|a|e_b)/|\zeta| \quad (4.35)$$

is a unit vector in the plane  $\text{span}\{\zeta, \bar{\zeta}\} = \text{span}\{a, b\}$  which is perpendicular to  $\zeta$ , and  $e_a$  and  $e_b$  are unit vectors in the direction of  $a$  and  $b$ . In other words, the Levi-matrix of  $h_\varepsilon$  has three distinct eigenvalues

$$\lambda_0 = 0, \quad \lambda_1 = \frac{2\varepsilon(1 + \varepsilon)|\zeta|^4}{\varphi_\varepsilon(\zeta)^3}, \quad \lambda_2 = \frac{1 + \varepsilon}{2\varphi_\varepsilon(\zeta)},$$

and the corresponding eigenspaces are

$$V_0 = \text{span}\{\zeta\}, \quad V_1 = \text{span}\{\xi\}, \quad V_2 = \{\zeta, \bar{\zeta}\}^\perp = \{a, b\}^\perp.$$

*Proof.* The Levi form of  $h_\varepsilon$  at  $\zeta \in \mathbb{C}^N \setminus \{0\}$  is given by

$$\mathcal{L}_{h_\varepsilon}(\zeta; w) = \frac{1}{v_\varepsilon(\zeta)} \left( \mathcal{L}_{v_\varepsilon}(\zeta; w) - \frac{|\langle \nabla v_\varepsilon(\zeta), w \rangle|^2}{v_\varepsilon(\zeta)} \right), \quad (4.36)$$

where  $\nabla v_\varepsilon = (\partial v_\varepsilon / \zeta_1, \dots, \partial v_\varepsilon / \zeta_N)$ . Since  $v_\varepsilon = A_\varepsilon^{\frac{1}{2}} + B_\varepsilon^{\frac{1}{2}}$ , we have

$$\langle \nabla v_\varepsilon(\zeta), w \rangle = \frac{1}{2} A_\varepsilon^{-\frac{1}{2}} \langle \nabla A_\varepsilon(\zeta), w \rangle + \frac{1}{2} B_\varepsilon^{-\frac{1}{2}} \langle \nabla B_\varepsilon(\zeta), w \rangle, \quad (4.37)$$

$$\begin{aligned} \mathcal{L}_{v_\varepsilon}(\zeta; w) &= \frac{1}{2} A_\varepsilon^{-\frac{1}{2}} \mathcal{L}_{A_\varepsilon}(\zeta; w) + \frac{1}{2} B_\varepsilon^{-\frac{1}{2}} \mathcal{L}_{B_\varepsilon}(\zeta; w) \\ &\quad - \frac{1}{4} A_\varepsilon^{-\frac{3}{2}} |\langle \nabla A_\varepsilon(\zeta), w \rangle|^2 - \frac{1}{4} B_\varepsilon^{-\frac{3}{2}} |\langle \nabla B_\varepsilon(\zeta), w \rangle|^2. \end{aligned} \quad (4.38)$$

Assume now that  $\langle \zeta, \zeta \rangle \neq 0$ . Then

$$\frac{\partial |\langle \zeta, \zeta \rangle|}{\partial \zeta_j} = \frac{\overline{\langle \zeta, \zeta \rangle}}{|\langle \zeta, \zeta \rangle|} \zeta_j = e^{-2i\theta} \zeta_j \quad \text{and} \quad \frac{\partial^2 |\langle \zeta, \zeta \rangle|}{\partial \zeta_j \partial \bar{\zeta}_k} = \frac{\zeta_j \bar{\zeta}_k}{|\langle \zeta, \zeta \rangle|},$$

so we get

$$\begin{aligned} \nabla A(\zeta) &= \frac{1}{2} (\bar{\zeta} + e^{-2i\theta} \zeta) = e^{-i\theta} a, & \nabla B(\zeta) &= \frac{1}{2} (\bar{\zeta} - e^{-2i\theta} \zeta) = -ie^{-i\theta} b, \\ \mathcal{L}_A(\zeta; w) &= \frac{1}{2} (|w|^2 + \frac{|\langle \zeta, w \rangle|^2}{A-B}), & \mathcal{L}_B(\zeta; w) &= \frac{1}{2} (|w|^2 - \frac{|\langle \zeta, w \rangle|^2}{A-B}), \end{aligned}$$

and consequently

$$\langle \nabla A_\varepsilon(\zeta), w \rangle = e^{-i\theta} \langle a - i\varepsilon b, w \rangle, \quad (4.39)$$

$$\langle \nabla B_\varepsilon(\zeta), w \rangle = -ie^{-i\theta} \langle b + i\varepsilon a, w \rangle, \quad (4.40)$$

$$\mathcal{L}_{A_\varepsilon}(\zeta, w) = \frac{1}{2} \left( (1 + \varepsilon) |w|^2 + (1 - \varepsilon) \frac{|\langle \zeta, w \rangle|^2}{A - B} \right), \quad (4.41)$$

$$\mathcal{L}_{B_\varepsilon}(\zeta, w) = \frac{1}{2} \left( (1 + \varepsilon) |w|^2 - (1 - \varepsilon) \frac{|\langle \zeta, w \rangle|^2}{A - B} \right). \quad (4.42)$$

The function  $h_\varepsilon$  is logarithmically homogeneous, which implies  $\mathcal{L}_{h_\varepsilon}(\zeta; \zeta) = 0$  and that the Levi matrix of  $h_\varepsilon$  has 0 eigenvalue with eigenvector  $\zeta$ .

If we take  $w$  orthogonal to both  $\zeta$  and  $\bar{\zeta}$ , i.e.,  $\langle \zeta, w \rangle = \langle \bar{\zeta}, w \rangle = 0$ , then  $\langle \nabla A_\varepsilon(\zeta), w \rangle = \langle \nabla B_\varepsilon(\zeta), w \rangle = 0$ , and the formulas (4.36)-(4.42) give

$$\mathcal{L}_{h_\varepsilon}(\zeta, w) = \frac{(1 + \varepsilon)}{4v_\varepsilon(\zeta)} \left( \frac{1}{A_\varepsilon(\zeta)^{\frac{1}{2}}} + \frac{1}{B_\varepsilon(\zeta)^{\frac{1}{2}}} \right) |w|^2 = \frac{(1 + \varepsilon)}{2\varphi_\varepsilon(\zeta)} |w|^2.$$

From this formula it follows that  $\frac{1}{2}(1 + \varepsilon)/\varphi_\varepsilon(\zeta)$  is an eigenvalue of the Levi matrix and that the eigenspace contains  $\{\zeta, \bar{\zeta}\}^\perp$ , which is of dimension  $N - 1$  if  $\zeta \in \mathbb{C}\mathbb{R}^N$  and of dimension  $N - 2$  if  $\zeta \in \mathbb{C}^N \setminus \mathbb{C}\mathbb{R}^N$ .

Now we assume that  $\zeta \in \mathbb{C}^N \setminus \mathbb{C}\mathbb{R}^N$  and let  $\xi = (|b|e_a - i|a|e_b)/|\zeta|$  be a unit vector in the span of  $\zeta$  and  $\bar{\zeta}$  orthogonal to  $\zeta$ . We know that  $\xi$  is an eigenvector and the corresponding eigenvalue is  $\mathcal{L}_{h_\varepsilon}(\zeta, \xi)$ . In order to calculate  $\mathcal{L}_{h_\varepsilon}(\zeta, \xi)$  we first observe that by (4.39)-(4.42) we have

$$\begin{aligned} \langle \zeta, \xi \rangle &= \frac{e^{i\theta}}{|\zeta|} \langle a + ib, |b|e_a - i|a|e_b \rangle = \frac{2e^{i\theta}|a||b|}{|\zeta|} \\ \langle \nabla A_\varepsilon(\zeta), \xi \rangle &= \frac{e^{-i\theta}(1 - \varepsilon)|a||b|}{|\zeta|} = -\langle \nabla B_\varepsilon(\zeta), \xi \rangle \\ \mathcal{L}_{A_\varepsilon}(\zeta, \xi) &= \frac{1}{2} (1 + \varepsilon + (1 - \varepsilon) \frac{4AB}{A^2 - B^2}) \\ \mathcal{L}_{B_\varepsilon}(\zeta, \xi) &= \frac{1}{2} (1 + \varepsilon - (1 - \varepsilon) \frac{4AB}{A^2 - B^2}). \end{aligned}$$

By these equations, (4.37), and (4.38) we get

$$\begin{aligned} |\langle \nabla v_\varepsilon, \xi \rangle|^2 &= \frac{AB(1 - \varepsilon)^2(A_\varepsilon^{1/2} - B_\varepsilon^{1/2})^2}{4(A + B)A_\varepsilon B_\varepsilon}, \\ \mathcal{L}_{v_\varepsilon}(\zeta, \xi) &= \frac{(1 + \varepsilon)v_\varepsilon}{4A_\varepsilon^{1/2}B_\varepsilon^{1/2}} - \frac{(1 - \varepsilon)AB(A_\varepsilon^{1/2} - B_\varepsilon^{1/2})}{(A^2 - B^2)A_\varepsilon^{1/2}B_\varepsilon^{1/2}} - \frac{(1 - \varepsilon)^2 AB(A_\varepsilon^{3/2} + B_\varepsilon^{3/2})}{4(A + B)A_\varepsilon^{3/2}B_\varepsilon^{3/2}}. \end{aligned}$$

By equation (4.36) we have

$$\begin{aligned} \mathcal{L}_{h_\varepsilon}(\zeta, \xi) &= \frac{(1 + \varepsilon)}{4A_\varepsilon^{1/2}B_\varepsilon^{1/2}} \\ &\quad - \frac{AB(1 - \varepsilon)^2}{4(A + B)v_\varepsilon A_\varepsilon^{1/2}B_\varepsilon^{1/2}} \left( \frac{4(A_\varepsilon^{1/2} - B_\varepsilon^{1/2})}{A_\varepsilon - B_\varepsilon} + \frac{A_\varepsilon^{3/2} + B_\varepsilon^{3/2}}{A_\varepsilon B_\varepsilon} + \frac{(A_\varepsilon^{1/2} - B_\varepsilon^{1/2})^2}{A_\varepsilon^{1/2}B_\varepsilon^{1/2}v_\varepsilon} \right). \end{aligned}$$

The last parenthesis equals  $(A_\varepsilon + B_\varepsilon)(A_\varepsilon^{\frac{1}{2}} + B_\varepsilon^{\frac{1}{2}})/(A_\varepsilon B_\varepsilon)$ . Hence

$$\begin{aligned} \mathcal{L}_{h_\varepsilon}(\zeta, \xi) &= \frac{(1 + \varepsilon)}{4A_\varepsilon^{1/2}B_\varepsilon^{1/2}} - \frac{AB(1 - \varepsilon)^2}{4(A + B)v_\varepsilon A_\varepsilon^{1/2}B_\varepsilon^{1/2}} \cdot \frac{(A_\varepsilon + B_\varepsilon)v_\varepsilon}{A_\varepsilon B_\varepsilon} \\ &= \frac{(1 + \varepsilon)}{4A_\varepsilon^{1/2}B_\varepsilon^{1/2}} - \frac{AB(1 - \varepsilon)^2(1 + \varepsilon)}{4A_\varepsilon^{3/2}B_\varepsilon^{3/2}} \\ &= \frac{(1 + \varepsilon)}{4(A_\varepsilon B_\varepsilon)^{3/2}} (A_\varepsilon B_\varepsilon - AB(1 - \varepsilon)^2) \\ &= \frac{\varepsilon(1 + \varepsilon)(A + B)^2}{4(A_\varepsilon B_\varepsilon)^{3/2}} = \frac{2\varepsilon(1 + \varepsilon)|\zeta|^4}{\varphi_\varepsilon(\zeta)^3}. \end{aligned}$$

□

## 4.6 Proof of Theorem 4.0.1

The statement of Theorem 4.0.1 is local, so without loss of generality we may from now on assume that  $X = \mathbb{D}^n$ , where  $\mathbb{D}$  is the unit disc in  $\mathbb{C}$ . In order to calculate the Monge-Ampère measure of the function  $h_\varepsilon \circ \Phi$  we need a simple result from linear algebra.

**Lemma 4.6.1** *Let  $D$  be an  $(n+1) \times (n+1)$  Hermitian matrix and  $A$  be an  $(n+1) \times n$  matrix. Assume that  $v$  is a unit eigenvector for  $D$  with eigenvalue 0 and denote by  $\lambda_1, \dots, \lambda_n$  the remaining eigenvalues of  $D$ . Then*

$$\text{Det}(A^*DA) = \lambda_1 \cdots \lambda_n \cdot |\text{Det}[A|v]|^2 \quad (4.43)$$

where  $[A|v]$  is the  $(n+1) \times (n+1)$  matrix obtained by adding  $v$  as a column-vector to the right of the matrix  $A$ .

*Proof.* By a change of coordinate systems we can assume that  $D$  is a diagonal matrix with entries  $\lambda_1, \dots, \lambda_n$  and 0, and  $v = [0, \dots, 0, 1]^t$ . Let  $\tilde{A}$  be the  $n \times n$  matrix

obtained by removing the bottom row of  $A$  and  $\tilde{D}$  be the  $n \times n$  diagonal matrix with diagonal  $\lambda_1, \dots, \lambda_n$ . Then  $A^*DA = \tilde{A}^*\tilde{D}\tilde{A}$  and

$$\text{Det}(A^*DA) = \text{Det}(\tilde{A}^*\tilde{D}\tilde{A}) = \lambda_1 \cdots \lambda_n |\text{Det}(\tilde{A})|^2 = \lambda_1 \cdots \lambda_n |\text{Det}([A|v])|^2$$

by the product formula for determinants.  $\square$

We have  $L_{h_\varepsilon \circ \Phi}(z) = J_\Phi(z)^* L_{h_\varepsilon}(\Phi(z)) J_\Phi(z)$ , where  $L_{h_\varepsilon \circ \Phi}$  and  $L_{h_\varepsilon}$  denote the Levi matrices of  $h_\varepsilon \circ \Phi$  and  $h_\varepsilon$  respectively. By Theorem 4.5.1  $L_{h_\varepsilon}(\zeta)$  has eigenvalue 0 with eigenvector  $\zeta$ , and the others are  $2\varepsilon(1 + \varepsilon)|\zeta|^4/\varphi_\varepsilon(\zeta)^3$  with multiplicity 1 and  $(1 + \varepsilon)/2\varphi_\varepsilon(\zeta)$  with multiplicity  $n - 1$ . Lemma 4.6.1 gives:

**Theorem 4.6.2** *If  $\Phi : \mathbb{D}^n \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$  is a holomorphic map, then*

$$(dd^c(h_\varepsilon \circ \Phi))^n = \frac{2^{n+2}n!(1 + \varepsilon)^n \varepsilon |\Phi(z)|^2 |\text{Det}[J_\Phi(z)|\Phi(z)]|^2}{\varphi_\varepsilon(\Phi(z))^{n+2}} dV$$

where  $J_\Phi$  is the Jacobian of  $\Phi$  and  $[J_\Phi|\Phi]$  is the  $(n + 1) \times (n + 1)$  matrix obtained by adding  $\Phi$  as a column vector to the right of  $J_\Phi$ .

It is clear from the definition of  $\mathcal{A}_\Phi$  in Theorem 4.0.1 that

$$\mathcal{A}_\Phi = \{z \in \mathbb{D}^n ; \text{Det}[J_\Phi(z)|\Phi(z)] = 0\}.$$

The function  $z \rightarrow \text{Det}[J_\Phi(z)|\Phi(z)]$  is holomorphic so either  $\mathcal{A}_\Phi$  is pluripolar or it equals  $\mathbb{D}^n$ . If  $\mathcal{A}_\Phi$  is pluripolar then  $\int_{\mathcal{A}_\Phi} (dd^c(\log|\Phi|_c))^n = 0$  by [49, Proposition 4.6.4.]. If  $\mathcal{A}_\Phi = \mathbb{D}^n$  then Theorem 4.6.2 implies

$$(dd^c(\log|\Phi|_c))^n = \lim_{\varepsilon \rightarrow 0} (dd^c(h_\varepsilon \circ \Phi))^n \equiv 0.$$

From now on we assume that  $M = \Phi^{-1}(\mathbb{C}\mathbb{R}^{n+1}) \setminus \mathcal{A}_\Phi$  is non-empty. In order to prove that it is a real analytic manifold of dimension  $n$  we introduce the auxiliary map  $\tilde{\Phi} : \mathbb{D}^{n+1} \rightarrow \mathbb{C}^{n+1}$  defined by the equation

$$\tilde{\Phi}(z, z_{n+1}) := (1 + z_{n+1})\Phi(z), \quad z \in \mathbb{D}^n, z_{n+1} \in \mathbb{D}.$$

Notice that the Jacobians of  $\tilde{\Phi}$  and  $\Phi$  are related by the equalities

$$J_{\tilde{\Phi}} = [(1 + z_{n+1})J_\Phi|\Phi], \quad \text{Det } J_{\tilde{\Phi}} = (1 + z_{n+1})^n \text{Det}[J_\Phi|\Phi] \quad (4.44)$$

and

$$\tilde{\Phi}^{-1}(\mathbb{C}\mathbb{R}^{n+1}) = \Phi^{-1}(\mathbb{C}\mathbb{R}^{n+1}) \times \mathbb{D}. \quad (4.45)$$

Let  $z \in M$  and write  $\tilde{z} = (z, 0) \in \mathbb{D}^n \times \mathbb{D}$ . By equation (4.44) the map  $\tilde{\Phi}$  is biholomorphic in a neighborhood of  $\tilde{z}$  and the variety  $\tilde{\Phi}^{-1}(\mathbb{C}\mathbb{R}^{n+1})$  has  $n + 2$  real dimensions in a neighborhood of  $\tilde{z}$ . By equation (4.45) this means that  $M$  has  $n$  real dimensions in a neighborhood of  $z$ .

Denote by

$$W := \{z \in \mathbb{D}^{n+1}; \text{Det } J_{\tilde{\Phi}}(z) \neq 0\} = (\mathbb{D}^n \setminus \mathcal{A}_{\Phi}) \times \mathbb{D}$$

the set on which  $\tilde{\Phi}$  is locally biholomorphic and thus a submersion. If  $z$  is a member of  $W$  then the pullback of currents by  $\tilde{\Phi}$  at  $z$  is well defined and we have the following result.

**Theorem 4.6.3** *The  $(2n + 2)$ -current*

$$(dd^c(h \circ \Phi))^n \wedge \frac{idz_{n+1} \wedge d\bar{z}_{n+1}}{2|1 + z_{n+1}|^2} := \lim_{\varepsilon \rightarrow 0} (dd^c(h_\varepsilon \circ \Phi))^n \wedge \frac{idz_{n+1} \wedge d\bar{z}_{n+1}}{2|1 + z_{n+1}|^2}$$

on  $W \subset \mathbb{D}^{n+1}$  equals the pullback by  $\tilde{\Phi}: \mathbb{D}^{n+1} \rightarrow \mathbb{C}^{n+1}$  of the current of integration along  $\mathbb{C}\mathbb{R}^{n+1}$  of the  $(n + 2)$ -form  $-C_n |a|^{-(n+1)} d\theta \wedge dV_a$ , where  $C_n = (-1)^{\frac{n(n-1)}{2}} n! \Omega_n$  and  $\Omega_n$  is the volume of the unit ball in  $\mathbb{R}^n$ .

*Proof.* To distinguish between them, we now denote by  $dV_n$  and  $dV_{n+1}$  the Euclidean volume forms on  $\mathbb{C}^n$  and  $\mathbb{C}^{n+1}$ , respectively. Since the result is local it is sufficient to prove it on  $\tilde{\Phi}^{-1}(L_m)$  for some  $m \in \{1, \dots, n + 1\}$  where  $L_m$  is defined in Proposition 4.2.2. Indeed by Theorem 4.6.2, homogeneity of the function  $\varphi_\varepsilon$ , and by equation (4.44) we have

$$\begin{aligned} (dd^c(h_\varepsilon \circ \Phi))^n \wedge \frac{idz_{n+1} \wedge d\bar{z}_{n+1}}{2|1 + z_{n+1}|^2} &= \frac{2^{n+2} n! (1 + \varepsilon)^n \varepsilon |\Phi|^2 |\text{Det}[J_{\tilde{\Phi}}|\Phi]|^2}{|1 + z_{n+1}|^2 (\varphi_\varepsilon \circ \Phi)^{n+2}} dV_{n+1} \\ &= \frac{2^{n+2} n! (1 + \varepsilon)^n \varepsilon |\tilde{\Phi}|^2 |\text{Det } J_{\tilde{\Phi}}|^2}{(\varphi_\varepsilon \circ \tilde{\Phi})^{n+2}} dV_{n+1} \\ &= \tilde{\Phi}^* \left( \frac{2^{n+2} n! (1 + \varepsilon)^n \varepsilon |\zeta|^2}{(\varphi_\varepsilon(\zeta))^{n+2}} dV_{n+1} \right) \\ &= (-1)^{\frac{n(n-1)}{2}} n! (1 + \varepsilon)^n \tilde{\Phi}^* (\lambda_\varepsilon d\theta \wedge dV_a \wedge dV_\beta), \end{aligned}$$

where  $\lambda_\varepsilon$  is the function defined by the last equality and  $\theta, a, \beta$  are variables introduced in Proposition 4.2.2. Since the pullback is a continuous operation under weak limits it suffices to show that

$$\lim_{\varepsilon \rightarrow 0} (\lambda_\varepsilon d\theta \wedge dV_a \wedge dV_\beta) = -\Omega_n |a|^{-(n+1)} d\theta \wedge dV_a.$$

By Proposition 4.2.2 and by equation (4.33) we have

$$\lambda_\varepsilon(\theta, a, \beta) = \frac{-\varepsilon(|a|^2 + |\beta|^2) \left(|a| - \frac{|\beta|^2}{|a|}\right)}{\left((1 + \varepsilon^2)|a|^2|\beta|^2 + \varepsilon(|a|^4 + |\beta|^4)\right)^{\frac{n+2}{2}}}, \quad (\theta, a, \beta) \in \tilde{L}.$$

Now let  $\chi \in C_0^\infty(\mathbb{C}^{n+1} \setminus \{0\})$  be a test function. We want to show that

$$\lim_{\varepsilon \rightarrow 0} \int_{|\beta| < |a|} \chi(\theta, a, \beta) \lambda_\varepsilon(\theta, a, \beta) dV_\beta = -\chi(\theta, a, 0) \Omega_n |a|^{-(n+1)}.$$

Then the result follows by a simple application of Fubini's theorem. We calculate this limit directly by switching into spherical coordinates scaled by a factor of  $\varepsilon^{1/2}$ . We write  $r = \varepsilon^{-1/2}|\beta|$ , we denote by  $v$  the unit vector in the direction of  $\beta$  and by  $d\sigma$  the Euclidean measure on the  $(n-1)$ -dimensional unit sphere  $S^{n-1}$ . Hence  $\beta = \varepsilon^{1/2}rv$  and  $dV_\beta = \varepsilon^{n/2}r^{n-1}drd\sigma$ . By the dominated convergence theorem we have

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{|\beta| < |a|} \chi(\theta, a, \beta) \lambda_\varepsilon(\theta, a, \beta) dV_\beta \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\substack{r \in (0, |a|\varepsilon^{-1/2}) \\ v \in S^{n-1}}} \frac{-\varepsilon(|a|^2 + \varepsilon r^2) \left(|a| - \frac{\varepsilon r^2}{|a|}\right) \varepsilon^{\frac{n}{2}} r^{n-1}}{\left(\varepsilon(1 + \varepsilon^2)|a|^2 r^2 + \varepsilon(|a|^4 + \varepsilon^2 r^4)\right)^{\frac{n+2}{2}}} \chi(\theta, a, \varepsilon^{1/2}rv) dr d\sigma \\ &= \int_{\substack{r \in (0, \infty) \\ v \in S^{n-1}}} \frac{-|a|^3 r^{n-1}}{(|a|^2 r^2 + |a|^4)^{1+n/2}} \chi(\theta, a, 0) dr d\sigma \\ &= -\text{Vol}(S^{n-1}) \chi(\theta, a, 0) |a|^{-(n-1)} \int_0^\infty \frac{r^{n-1}}{(r^2 + |a|^2)^{1+n/2}} dr \\ &= -\text{Vol}(S^{n-1}) \chi(\theta, a, 0) |a|^{-(n-1)} \left[ \frac{r^n}{n|a|^2(|a|^2 + r^2)^{n/2}} \right]_0^\infty \\ &= -\Omega_n \chi(\theta, a, 0) |a|^{-(n+1)}. \end{aligned}$$

□

We need one more result before we prove Theorem 4.0.1. To simplify notation we write  $\Lambda_{\Phi_j} := d\Phi_0 \wedge \cdots \wedge \widehat{d\Phi_j} \wedge \cdots \wedge d\Phi_n$  for  $j \in \{0, \dots, n\}$  and  $V := \tilde{\Phi}^{-1}(\mathbb{C}\mathbb{R}^{n+1}) =$

$\Phi^{-1}(\mathbb{C}\mathbb{R}^{n+1}) \times \mathbb{D}$ . Abusing notation a little, for  $z = (z', z_{n+1}) \in \mathbb{D}^n \times \mathbb{D}$  we interpret  $\Phi(z)$  as  $\Phi(z')$ , i.e. we use the same symbol  $\Phi$  to denote the trivial extension of  $\Phi$  to  $\mathbb{D}^{n+1}$ .

**Proposition 4.6.4** *The restrictions of the  $(n+2)$ -forms  $\tilde{\Phi}^*(d\theta \wedge dV_a)$  and*

$$\frac{(1+z_{n+1})^{n+1}}{|1+z_{n+1}|^2} e^{-i(n+1)\theta} \left( \sum_{j=0}^n (-1)^{j+1} \Phi_j \Lambda_{\Phi_j} \right) \wedge \frac{i}{2} dz_{n+1} \wedge d\bar{z}_{n+1}$$

to  $V \cap W$  are equal.

*Proof.* On  $V$  the relation between  $\Phi_j$ ,  $a_j = a_j \circ \tilde{\Phi}$  and  $\theta = \theta \circ \tilde{\Phi}$  is given by

$$(1+z_{n+1})\Phi_j = e^{i\theta} a_j, \quad 0 \leq j \leq n,$$

so

$$\Phi_j dz_{n+1} + (1+z_{n+1})d\Phi_j|_V = ie^{i\theta} a_j d\theta + e^{i\theta} da_j|_V. \quad (4.46)$$

The Jacobian of the map  $\Phi|_V: V \rightarrow \mathbb{C}\mathbb{R}^{n+1}$  does not have full rank because  $\Phi$  is independent of the variable  $z_{n+1}$ . Therefore  $d\Phi_0 \wedge \cdots \wedge d\Phi_n|_V = 0$ . Wedging equation (4.46) over all possible  $j$  gives

$$\begin{aligned} (1+z_{n+1})^n \left( \sum_{j=0}^n (-1)^{n-j} \Phi_j \Lambda_{\Phi_j} \right) \wedge dz_{n+1}|_V \\ = e^{i(n+1)\theta} \left( dV_a + id\theta \wedge \sum_{j=0}^n (-1)^j a_j \Lambda_{a_j} \right)|_V. \end{aligned} \quad (4.47)$$

Let  $0 \leq j_0 \leq n$  be any fixed number. Similarly as before we have

$$\bar{\Phi}_{j_0} d\bar{z}_{n+1} + (1+\bar{z}_{n+1})d\bar{\Phi}_{j_0}|_V = -ie^{-i\theta} a_{j_0} d\theta + e^{-i\theta} da_{j_0}|_V. \quad (4.48)$$

Wedging equations (4.47) and (4.48) gives

$$\begin{aligned} \bar{\Phi}_{j_0} (1+z_{n+1})^n \left( \sum_{j=0}^n (-1)^{n-j} \Phi_j \Lambda_{\Phi_j} \right) \wedge dz_{n+1} \wedge d\bar{z}_{n+1}|_V \\ = ia_{j_0} e^{in\theta} \left( -dV_a \wedge d\theta + (-1)^{j_0} d\theta \wedge \Lambda_{a_{j_0}} \wedge da_{j_0} \right)|_V \\ = 2i(-1)^n a_{j_0} e^{in\theta} d\theta \wedge dV_a|_V. \end{aligned} \quad (4.49)$$

After canceling  $(1 + \bar{z}_{n+1})\bar{\Phi}_{j_0} = a_{j_0} e^{-i\theta}$  the result follows.  $\square$

*Proof of Theorem 4.0.1.* As we already noted after Theorem 4.6.2, the Monge-Ampère measure has no mass on  $\mathcal{A}_\Phi$  and  $M$  is an  $n$  dimensional real analytic manifold, if it is non-empty. The map  $\tilde{\Phi}$  is a submersion on  $W$ . For  $\tilde{\Phi} \in \mathbb{C}\mathbb{R}^{n+1}$  we have

$$|a \circ \tilde{\Phi}| e^{i\theta} = |\tilde{\Phi}| e^{i\theta} = \langle \tilde{\Phi}, \tilde{\Phi} \rangle^{\frac{1}{2}} = (1 + z_{n+1}) \langle \Phi, \Phi \rangle^{\frac{1}{2}} \quad (4.50)$$

where the complex square root is taken such that  $\langle \tilde{\Phi}, \tilde{\Phi} \rangle^{\frac{1}{2}}$  has the same argument  $\theta$  as  $\tilde{\Phi} \in \mathbb{C}\mathbb{R}^{n+1}$ . By combining Theorem 4.6.3, Proposition 4.6.4 and equation (4.50) we see that the  $(2n+2)$ -current

$$(dd^c(h \circ \Phi))^n \wedge \frac{idz_{n+1} \wedge d\bar{z}_{n+1}}{2|1 + z_{n+1}|^2} \quad (4.51)$$

on  $W$  equals the current of integration along  $V$  of the  $(n+2)$ -form

$$C_n \langle \Phi, \Phi \rangle^{-\frac{n+1}{2}} \left( \sum_{j=0}^n (-1)^j \Phi_j \Lambda_{\Phi_j} \right) \wedge \frac{idz_{n+1} \wedge d\bar{z}_{n+1}}{2|1 + z_{n+1}|^2}. \quad (4.52)$$

Let  $\omega_1 := (dd^c(h \circ \Phi))^n$  and define  $\omega_2$  as the current of integration along  $\Phi^{-1}(\mathbb{C}\mathbb{R}^{n+1})$  of the  $n$ -form  $\sum_{j=0}^n (-1)^j \Phi_j \Lambda_{\Phi_j}$ . Then

$$\int_{\mathbb{D}^{n+1}} \chi \omega_j \wedge dz_{n+1} \wedge d\bar{z}_{n+1} = \left( \int_{\mathbb{D}^n} \chi_1 \omega_j \right) \left( \int_{\mathbb{D}} \chi_2 dz_{n+1} \wedge d\bar{z}_{n+1} \right),$$

for  $j = 1, 2$  and  $\chi \in C_0^\infty(W)$  of the form  $\chi(z) = \chi_1(z') \chi_2(z_{n+1})$ . This means that we can simply cancel out a factor of  $\frac{i}{2} dz_{n+1} \wedge d\bar{z}_{n+1}$  from equations (4.51) and (4.52) when taking the restriction to  $W \cap \{z_{n+1} = 0\} = \mathbb{D}^n \setminus \mathcal{A}_\Phi$ .  $\square$

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