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PREFACE

Textbooks are essential curriculum resources for teaching and learning mathematics in classrooms worldwide. They shape the nature of school mathematics, influence the learning opportunities teachers create for their students, and support teachers' professional growth. With ongoing digitalization, the range of curriculum resources available to teachers has expanded, providing materials that can complement—or even replace—traditional textbooks.

Textbook content, development, and use have long been central themes in mathematics education research. In recent years, however, international attention has increasingly turned toward digital curriculum resources, particularly e-textbooks and resources related to Large Language Models (LLMs, such as ChatGPT). This shift includes a growing interest in the theoretically driven and empirically grounded processes involved in designing and using such resources for mathematics teaching and learning. Understanding the nature and development of curriculum resources—both print and digital—and examining how they are integrated into teachers' professional work, how they contribute to curricular reforms, and how they support student learning have become key priorities in the field. Furthermore, research on curriculum resources provides valuable insights into the relationship between culture and educational practice within specific educational systems.

In view of these developments, this volume documents recent issues and the latest advances in research on mathematics curriculum resources and textbooks presented at the Fifth International Conference on Mathematics Textbook Research and Development (ICMT5), held at the Norwegian University of Science and Technology (NTNU) in Trondheim, Norway, from 17th to 20th June 2025. ICMT5 continued the successful series that began in 2014 with the first ICMT in Southampton (UK), followed by ICMT2 in Rio de Janeiro (Brazil, 2017), ICMT3 in Paderborn (Germany, 2019), and ICMT4 in Beijing (China, 2022).

The conference addressed all issues related to the development, design, content, history, and use of mathematics curriculum resources in print or digital formats. Its scope included curriculum resources for settings from pre-K to university, as well as resources used in home or out-of-school contexts. Contributions related to curriculum resources (including, but not limited to, textbooks) in any medium (including, but not limited to, digital and print) and addressed the following themes:

1. Theoretical and/or methodological issues in the design of curriculum resources
2. Theoretical and/or methodological issues in research on curriculum resources
3. Theoretical and/or methodological issues in the evaluation of curriculum resources
4. Studies of the use of—and interaction with—curriculum resources by students and/or teachers (including student teachers and teacher educators)
5. Studies of the production, selection, acquisition, and distribution of curriculum resources within educational systems and institutions
6. Studies of the relationships between educational policy and curriculum resources
7. Comparative studies of curriculum resources (including, but not limited to, international or transcultural perspectives)
8. Historical perspectives on curriculum resources
9. Perspectives on the implications of new technological developments (such as artificial intelligence) for curriculum resources

The international programme included five plenary lectures, one symposium, one panel discussion, 55 oral communications, and a poster session presenting 12 posters. Participants came from Europe, Asia, Oceania, North America, and South America, and the scholarly contributions additionally included work from Africa, reflecting the richness and global scope of research on mathematics curriculum resources and textbooks.

ICMT5 was organized by an International Programme Committee:

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- Siri-Malen Høyne
- Marit Buset Langfeldt
- Anders Sanne
- Solomon Abedom Tesfamichael

These proceedings are published after the conference. Accordingly, the contributions reflect the work prepared before ICMT5, based on a two-round pre-conference peer-review process among the conference contributors. This was followed by an additional post-conference review round conducted by the proceeding editors. The volume includes the five plenary lectures, the symposium, the panel discussion, and 50 papers and 11 posters that successfully completed the full review process.

We would like to thank all reviewers for their valuable work, which has helped enhance the quality of the papers in this volume.

We also thank the Norwegian University of Science and Technology, and especially the Department of Teacher Education, for its support of the conference.

December 2025

Birgit Pepin, Iveta Kohanová, and Marit Buset Langfeldt

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Part I

Plenaries

THE USE OF DIGITAL RESOURCES FOR LEARNING AND METACOGNITION: EXPLORING STUDENTS' PERSPECTIVES

Annalisa Cusi

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This paper explores how digital resources can promote metacognition in mathematics education by drawing on a series of studies conducted at university and secondary school level. The first set of studies examines how students interpret and utilise the scaffolding and meta-scaffolding elements provided by these resources. The second set of studies investigates the potential of generative AI, specifically ChatGPT, to act as a metacognitive tutor. The findings highlight both opportunities and challenges. Theoretical implications are discussed in terms of a reconceptualization of the instrumental genesis framework to capture the novel dynamics of human–GenAI interactions.

INTRODUCTION

The growing incorporation of digital resources into mathematics education has highlighted their potential as not only tools that support procedures, but also mediators of teaching and learning processes. An emerging body of research is investigating how carefully designed digital environments can support students' cognitive and metacognitive activities, facilitating fundamental mathematical processes, such as problem solving, and the reflection on these processes on which formative assessment is based. The advent of generative artificial intelligence (GenAI) systems, such as ChatGPT, presents new possibilities and profound challenges for mathematics education research.

This paper outlines the steps of research conducted over the last five years. Starting with studies focused on designing and implementing digital scaffolding to support cognitive and metacognitive processes, it then shifts to more recent studies examining ChatGPT's potential as a metacognitive tutor in mathematics. The adopted perspective is that of the user. The central concern is how students perceive, interpret and utilise these resources in their problem-solving processes and how this can be analysed and theorised in the context of the use of GenAI in education. Beyond presenting empirical evidence, this paper seeks to encourage theoretical reflection on how the need to interpret and analyse the use of GenAI in mathematics education reshapes our conceptual approach to the subject.

THEORETICAL BACKGROUND

The Instrumental Genesis Framework

The instrumental approach (Rabardel, 2002; Trouche, 2020a, 2020b) offers a conceptualisation of tools and learning processes in which instruments are understood

not as static artefacts but as evolving entities that combine the characteristics of the artefacts with the utilisation schemes developed when these artefacts are used to face tasks.

Two main elements of this framework are the dialectics:

- between artefacts and instruments, as instruments are conceived as living mixed entities constituted by an artefact and one or more utilisation schemes;
- between the two processes that jointly contribute to the emergence and evolution of instruments, that is instrumentation (the process by which the tool shapes the user's actions) and instrumentalization (the process through which the user adapts and shapes the tool to serve specific needs).

The processes of instrumentation and instrumentalization are interdependent and dynamic, and both are crucial to understanding how learners appropriate digital tools in mathematics.

Hegedus and Moreno-Armella (2010) extended the notion of instrumental genesis to take into account the context of dynamic technological environments. This extension includes:

- the simultaneous co-actions between a user's use of a tool and a software environment's use of the same tool,
- the feedback and reaction of a user, interpreted as certain processes of utilisation, and
- the internalisation of how the tool is manipulated, used by the environment, and then re-used by the user.

In light of this new conceptualisation of instrumental genesis, Hegedus and Moreno-Armella (2010) have also extended the two notions of instrumentation and instrumentalization: "For instrumentation, we additionally define it as how co-actions with a tool shape the user's actions and understanding of the use of such a tool within, and with respect to, an environment. Instrumentalization is extended to how the tool is shaped by the user (user's knowledge) and the environment, i.e., when the tool is manipulated by environmental factors following a user-input." (p.30).

Metacognition and Scaffolding

Metacognition is central to students' ability to regulate their own learning processes. Holton and Clarke (2007) describe acts of metacognition as "any thinking act that operates on a cognitive thought in order to assist in the process of learning or the solution of a problem" (Holton & Clarke 2007, p. 133).

Consistent with the idea of acts of metacognition, Meijer et al. (2006) introduce the notion of metacognitive activity, which they define as "the strategic application of metacognitive knowledge to achieve cognitive goals" (Meijer et al. 2006, p. 209). These authors identified five interrelated categories of metacognitive activities (Veenman & Spaans 2005, Meijer et al. 2006): orientating (focusing on relevant information within the text of a task or a problem in order to construct an adequate

representation), planning (setting explicit goals and designing a detailed action plan), monitoring (controlling the process effectively while carrying out the tasks), evaluation (following the progress made and detecting errors or incorrect procedures finally), and elaboration (drawing conclusions, recapitulating and reflecting on the learning experience). These categories capture the complexity of self-regulation as students shift between engaging with content and reflecting on their approach.

Digital scaffolding can support such activities, but learners often need additional support, in the form of a scaffold for the scaffold (Pea, 2004). This means that support should both direct students to correct strategies and help them develop awareness of when and how to activate metacognitive processes. Digital meta-scaffolding elements (DMSEs) have been introduced in recent studies to address this issue (Cusi et al., 2022) and offer a promising way to encourage students to use digital resources more reflectively.

Research on technology in mathematics education has consistently highlighted the tension between providing instrumental support and fostering users' autonomy (Artigue, 2002; Drijvers & Trouche, 2008). Digital resources can either amplify cognitive processes or substitute for them, depending on how students engage with them. Similarly, research on AI in education has raised concerns about transparency, trust, and the balance between guidance and autonomy (Luckin, 2018; Holmes et al., 2022). The studies presented in this paper extend this discourse by focusing specifically on students' perspectives and how they construct utilisation schemes in response to both traditional digital scaffolds and GenAI tools.

PART 1: DIGITAL RESOURCE DESIGN AND THE USERS' AWARE INTERACTION WITH SCAFFOLDING ELEMENTS

The first part of the discussion focuses on the main results of three studies involving university students. The studies had the common educational aim of designing digital resources to promote metacognition in mathematical problem solving.

The artefacts under scrutiny were the scaffolding elements provided within GeoGebra-based applets. Data from video recordings of interviews and open-ended questionnaires were analysed to investigate users' perspectives on the use of specific scaffolding elements.

Study A: Utilisation Schemes in the use of digital scaffolding elements

Cusi and Telloni (2020) investigated the interactions of 15 first-year engineering students with GeoGebra applets containing sliders designed to support the description of plane regions in polar coordinates. Data were collected through video-recorded classroom sessions in which the students worked individually on tasks that required them to use the sliders to visualise polar regions. Transcripts of video recordings of the students' screens and their thinking-aloud protocols were analysed to highlight the emergence of utilisation schemes in students' interaction with the sliders.

Three main utilisation schemes were identified. Each scheme was defined by associating it with a specific function of the scaffolding element:

- Schemes related to the *replacement* function: students rely uncritically on the sliders to obtain answers, without referring to theoretical knowledge. For example, some students manipulated the sliders until the desired shape appeared and recorded the result without reflecting on how the representation could have been deduced.
- Schemes related to the *diagnostic* function: students use the sliders to verify or correct their own results, alternating between digital inputs and the reference to their theoretical knowledge. A typical case involved students first attempting an analytical solution, then using the sliders to confirm its accuracy.
- Schemes related to the *elaboration* function: students use the digital resource to facilitate a more profound understanding of the theoretical knowledge underlying the task that is solved using it. For example, some students discussed why certain slider manipulations produced specific regions, linking the dynamic feedback they received to their knowledge.

This classification highlights the risks of users thoughtlessly relying on digital tools without exercising metacognitive control over their use. These results suggest the need to provide students with meta-scaffolding, intended as the “scaffolds for the scaffolds” (Pea, 2004), and therefore to focus on designing metacognitive feedback.

Study B: Students’ perspectives on the meta-scaffolding elements provided by a digital resource

Subsequent studies (Cusi & Telloni, 2024) involved enriching GeoGebra applets with DMSEs, which were designed to foster students’ metacognitive awareness. The study involved 146 first-year students attending a degree course in “Chemistry and pharmaceutical technologies”. The students were asked to use applets designed to support the resolution of a problem that could be modelled through linear differential equations. The DMSEs were embedded as prompts within the applet interface to encourage students to reflect on the strategies promoted by the applet and by the metacognitive scaffolding provided.

Data were collected through open-ended questionnaires and semi-structured interviews to understand how students interpreted the role of the DMSEs’ support in their learning. Analysis revealed that, while some students engaged deeply with the DMSEs, using them to monitor and evaluate their reasoning, others interpreted them narrowly as procedural hints. This variety of interpretations highlights the importance of students’ perspectives in shaping utilisation schemes.

Study C: Investigating the instrumentation process in the use of meta-scaffolding elements

Study B sheds light on the connections between the different themes in students' reflections on their use of specific DMSEs and the possible utilisation schemes that characterise their use. A further investigation (Cusi & Telloni, 2025) focused on the use of another GeoGebra applet, aimed at supporting the resolution of a problem that could be modelled using a Gaussian distribution. Again, involving the same 146 students from study B, this study examined the instrumentation process from the users' perspective, investigating how they interpretation the characteristics of the DMSEs and the utilisation schemes that could be developed.

The results indicated a shift compared to earlier studies, with a larger proportion of students activating elaboration schemes. Of the 125 coded excerpts, 83 reflected the emergence of schemes related to the elaboration function of the digital resource. Moreover, a key insight of this study was the distinction between two levels of schemes related to the elaboration function, depending on the goal declared by students:

- *schemes aimed at cognitive elaboration*, related to goals pertaining to the cognitive level, such as deepening understanding and promoting a more profound comprehension of the theoretical knowledge behind the problem at hand;
- *schemes aimed at metacognitive elaboration*, related to goals pertaining to the metacognitive level, such as encouraging reflection on the students' own learning, supporting self-assessment processes and developing awareness of students' difficulties.

PART 2: CHATGPT AS A METACOGNITIVE TUTOR AND THE NEED FOR A RECONCEPTUALISATION OF INSTRUMENTAL GENESIS

The three studies presented in the previous sections collectively demonstrate both the potential and the limitations of a digital resource design aimed at scaffolding and meta-scaffolding. While DMSEs enabled more sophisticated elaboration schemes, many students still struggled to identify the purpose of the support provided. This suggests the need for a design approach that explicitly promotes awareness, where the goals of the scaffolds are transparent, guiding students to recognise and activate metacognitive processes. Furthermore, the DMSEs designed still seemed too rigid and not tailored according to students' specific needs. For this reason, we decided to explore the potential use of GenAI as a “metacognitive tutor” to support students in their problem solving and self-assessment processes.

Two studies were conducted with upper secondary school students (grades 9 and 10). Both studies aimed to investigate users' perspectives on using ChatGPT (GPT-4 in study D and GPT-4o in study E) as a metacognitive tutor by analysing data from video-recorded interviews.

Study D: GPT-4 as a tutor that support metacognitive processes

Contel and Cusi (2025) studied the interaction between nine secondary students' (grades 9-10) and GPT-4. The students, who had little or no prior experience of using ChatGPT, worked individually on tasks involving numerical exploration, conjecturing and proving, while GPT-4 was prompted with custom instructions to act as a tutor by asking concise stimulating questions and not providing direct answers. Data sources included video and audio recordings of the sessions, the students' written notes, and transcripts of the chats. A think-aloud protocol was employed, whereby the students were encouraged to verbalise their reasoning during the interaction. Observing the students' interaction with GPT-4 led us to adapt Meijer et al.'s (2006) categories of metacognitive activities by splitting the monitoring activities into two subcategories: action-oriented monitoring, which focuses on the product of an action or reasoning performed, or the process followed to find such a product, and conversation-oriented monitoring, which focuses on the content of the conversation. Furthermore, our analysis involved identifying breaks as relevant features that highlight the (mis)alignment between GPT-4's support and the user's needs. Two types of breaks were identified: deliberate breaks, which students take intentionally to reflect on their actions, and induced breaks, which are imposed by the interaction.

The results of the data analysis were summarised in diagrams representative of the typical interactions observed during the study in terms of both the behaviours that characterise GPT-4's approach and the characteristics of the initial utilisation schemes developed by the students.

Two main utilisation schemes emerged from the analysis, enabling us to adapt the categories of schemes introduced by Cusi and Telloni (2020) to the context of human-GenAI interaction. Schemes related to GPT-4's *replacement* function were characterised by long sections of the chats with no continuity between the metacognitive activities promoted by GPT-4 and its interventions' effects in terms of enhanced metacognitive activities. Moreover, the majority of activities observed were execution-oriented, alongside a predominance of conversation-oriented monitoring requests and only induced breaks, indicating a significant loss of focus on the task goal. Schemes related to the *diagnostic* function were characterised by; shorter interactions between the students and GPT-4; the presence of both induced and deliberate breaks, which were almost always preceded by planning; fewer interventions related to execution; and a prevalence of process-oriented monitoring activities. This testifies to the students' attempt to gain control over the resource. Schemes related to the *elaboration* function, involving the reflective integration of GPT-4's support to achieve a deeper understanding, were largely absent.

Study E: The use of GPT-4o to support self-assessment practices

A large-scale study (Cusi & Contel, in press) focused on the potential role of GPT-4o to promote self-assessment in mathematics. The study involved 19 secondary school students, most of whom with little or no experience of using the chatbot. The students

were instructed to input problem statements and then engage with GPT-4o, guided by prompts designed to avoid providing direct answers.

Analysis of the collected data was developed to investigate the influence of human-GPT-4o interaction characteristics on instrumentation and instrumentalization processes. The findings emphasised the dual nature of GenAI in education, highlighting its potential to encourage reflective monitoring and planning while also introducing instability and anthropomorphisation, which can undermine students' self-assessment.

Toward a reconceptualization of the instrumental genesis framework

The main theoretical insight from studies D and E was that the dynamics of discourse and mathematical meaning construction that occur in human-GenAI interactions are only partially captured by the notion of co-action. This led us to develop a reconceptualization of the instrumental genesis framework in order to consider the completely new context brought about by that the advent of GenAI in the educational and research scene.

In an attempt to capture the essence of the complexity involved in instrumental genesis within the context of human-GenAI interaction, we introduce a metaphor inspired by the approaches developed in physics for modelling complex systems (Cusi & Contel, in press). According to this metaphor, the instrumental genesis process is interpreted *as noise reduction*. In our interpretation of this metaphor, the instrumental genesis process is influenced by various factors acting as noise, which prevent users from developing stable schemes. These include:

- *external factors*, such as the uncertainty and randomness introduced into users' processes supported by GenAI due to its structural limitations;
- *internal factors*, such as the anthropomorphising GenAI tools and holding specific beliefs that lead users to blindly trust its capabilities, due to perceiving GenAI as autonomous social actors with whom they interact, and developing pseudo-theories about the "logic behind the scene";
- the characteristic of *indeterminacy of the co-actions* between users and the GenAI environment, which is a result of the interaction between the above-described external and internal factors.

In line with the metaphor, a drift component is needed to reduce the instability caused by this noise. This corresponds to the ways in which a user might try to dominate, or at least control, the complexity involved in instrumental genesis within the context of human-GenAI interaction. The inability to control the external factors influencing the co-action process indicates the necessity of focusing on internal factors to assist users in directing the instrumental genesis process towards more stable schemes. In particular, the interactions observed in our studies suggest promoting metacognitive skills to help users navigate GenAI's stochastic behaviour and direct the instrumental genesis process towards more stable schemes.

DISCUSSION

A common thread emerges across the studies presented: the integration of digital scaffolding and GenAI is not just a matter of technological enhancement, but also a catalyst for reconsidering the processes by which mathematical meaning is created and reflected upon. From a theoretical perspective, reconceptualising instrumental genesis as a process of noise reduction offers a way to interpret the instability and indeterminacy inherent in human–GenAI interactions. This metaphor highlights the tension between randomness and control, and between external unpredictability and internal regulation. It suggests that metacognition acts as a means of creating coherence — a way for learners to navigate and reshape the flux of interactions that AI environments inevitably produce. This view broadens the instrumental approach by placing the learner’s reflective activity at the heart of the meaning-making process, where agency is continually negotiated between human and artificial participants.

In practice, these studies emphasise the importance of designing learning environments that are transparent about their scaffolding logic and encourage students to interpret it. Digital resources, whether structured through predesigned scaffolds or animated by conversational AI, offer educational value not only through the support they provide, but also through the awareness they elicit. When learners are prompted to reflect on the reasons for and methods of support, metacognitive growth and autonomy can emerge in parallel. Designing for such awareness implies a shift from control to dialogue, and from providing assistance to cultivating discernment in the face of uncertainty.

Taken together, these findings invite dual reflection. Firstly, they advocate for the theoretical expansion of the instrumental genesis framework to encompass the unique characteristics of GenAI-supported learning. On the other hand, they emphasise the need for a renewed commitment to designing educational technologies that empower students, encourage self-regulation, and enhance their ability to navigate uncertainty. The challenge ahead is to transform the stochastic and conversational nature of GenAI into an opportunity to cultivate metacognitive awareness, and ultimately to reimagine what it means to learn mathematics with and through intelligent tools.

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THE EVOLUTION IN CURRICULUM DESIGN OF A PROBLEM-BASED MATHEMATICS CURRICULUM: NEW DIRECTIONS FOR THE AI WORLD

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This paper examines the evolution of Connected Mathematics, the problem-based curriculum developed by the Connected Mathematics Project (CMP). Through four editions, the curriculum has used iterative design, field-testing, and evaluation to support students and teachers in building mathematical knowledge, skills, and connections within and beyond mathematics. More recently, CMP has extended this work with a digital platform that enhances the teaching and learning of problem-based mathematics by integrating artificial intelligence. These AI tools are designed to foster engagement, support diverse learners, and provide teachers with timely insights into learning. The paper highlights how AI integration to support proportional reasoning offers new directions for mathematics classrooms and future educational practice.

INTRODUCTION

For over forty years, the Connected Mathematics Project (CMP) at Michigan State University has developed, field-tested, and researched *Connected Mathematics*, a problem-based middle grades mathematics curriculum (student ages 11-14). Since its inception, the project has been driven by a single vision or overarching goal (Lappan *et al.*, 2004, p.1):

All students should be able to reason and communicate proficiently in mathematics. They should have knowledge of and skill in the use of the vocabulary, forms of representation, materials, tools, techniques, and intellectual methods of the discipline of mathematics, including the ability to define and solve problems with reason, insight, inventiveness, and technical proficiency.

The underlying belief is that mathematics is learned when students engage directly in the practices of the discipline, exploring meaningful problems, making sense of contextual situations, and building connections across ideas. Problem-based curricular approaches, where the mathematics concepts and skills are embedded or encoded within a contextualized problem, have shown significant promise for promoting learning and teaching (e.g., Wirkala & Kuhn, 2011). Problem-based mathematics classrooms matter because they reflect both how humans naturally learn and how mathematics as a discipline develops over time. Humans come into the world as problem solvers, constantly making sense of our environment and seeking solutions to challenges. Similarly, mathematics has evolved historically through the activity of solving problems. In this sense, problem solving is not only the goal of learning

mathematics but also how mathematical understanding is constructed. Teaching mathematics is therefore *through* problem solving, not merely *about*, *for*, or *with* problem solving. This problem-based orientation has guided CMP's curriculum design across four decades, shaping both the structure of materials and the supports provided for teachers.

Guided by this vision, CMP has established a set of enduring design principles: the curriculum is problem-based and coherent, intertwines conceptual and procedural knowledge, develops skills and concepts as needed, promotes inquiry-based instruction, makes effective use of technology, sets high expectations for all students, and organizes around big mathematical ideas pursued in depth. These principles have anchored curriculum development across four decades and four editions of *Connected Mathematics*, while also providing a lens for future innovations in digital and AI-enhanced environments.

EVOLUTION OF CURRICULUM DESIGN

Problem-based mathematics curriculum design does not occur in isolation. Instead, it is shaped by the complex interplay between curricular goals, instructional materials, and classroom enactment. Remillard and Heck (2014) describe curriculum as operating across multiple layers: the designated curriculum (standards and aims), the instructional materials (curriculum resources), and the operational curriculum (the enacted curriculum in classrooms). Their framework emphasizes the influence of instructional materials on teachers' practices, with the operational curriculum reflecting how teachers interpret and implement those materials given their local contexts and student and teacher needs.

Building on this perspective, Edson and colleagues (2025) argue that the relationship between instructional materials and the operational curriculum is bi-directional. Rather than a one-way influence of materials on enactment as described in Remillard and Heck (2014), Edson and colleagues report evidence of a dynamic, reciprocal interaction in which teachers' adaptations and enactments directly inform curriculum design. Field-test structures within CMP exemplify this process: teacher feedback during design research cycles provides iterative insights that shape ongoing revisions of problems, tasks, and frameworks before publication. This cyclical feedback loop reflects a design research approach (The Design-Based Research Collective, 2003; Edelson, 2002; Richey et al., 2004), where instructional materials both guide and are reshaped by classroom enactment.

Within this dynamic, CMP has consistently grappled with enduring curriculum design tensions. As Lappan and Phillips (2009) note, curriculum developers face the challenge of designing sequences of tasks that help students progress from informal to formal reasoning while ensuring that embedded mathematical concepts are visible. They must balance the development of procedural skills and conceptual understanding, attend to the diverse needs of learners, and provide effective transitions between tasks. These

tensions highlight the difficulty of creating materials that are coherent, mathematically rigorous, and responsive to varied classroom contexts.

Teachers, in turn, encounter enactment tensions when bringing problem-based materials to life in classrooms. Edson and colleagues (2019) report that teachers often struggle to recognize the embedded or encoded mathematics within contextual problems and to guide students from informal strategies toward more formal reasoning. Because many teachers did not encounter mathematics through problem-based approaches themselves, sustained support is critical for helping them enact the curriculum effectively and foster deep student learning.

It is within this landscape of curriculum design and teacher enactment tensions that CMP's iterative development cycles are situated. Each edition of *Connected Mathematics* has represented not only a response to new standards and technological contexts but also an effort to bridge the gap between design principles and classroom realities through sustained collaboration with teachers and careful attention to their enactment of the materials.

Technology Evolution Across Editions

From its origins, CMP has treated technology as an essential resource for supporting students' mathematical reasoning rather than as an add-on or enrichment. Each edition has incorporated available technologies in ways that align with the project's problem-based philosophy, emphasizing that tools should deepen understanding, broaden access, and support inquiry.

The Middle Grades Mathematics Project (MGMP) (Lappan et al., 1985) integrated calculators into classroom instruction at a time when their role in schools was often contested. MGMP was developed with the belief that calculators should always be available to students, and that students should know when and how to use them. The availability of four-function calculators enabled students to focus on reasoning, representation, and problem solving, rather than becoming bogged down in tedious computation.

Connected Mathematics (CMP1) (Lappan et al., 1998) continued the MGMP commitment to calculators as essential mathematical tools. Four-function calculators were used to simplify complex calculations and to help students explore patterns in numerical data. Graphing calculators were introduced as powerful instruments for investigating functions and solving problems. In addition to symbolic methods, students were encouraged to solve equations using tables and graphs generated by calculators. Optional computer activities, such as probability coin-toss simulations and the geo-logo programming environment for Turtle Math, extended opportunities for exploration and visualization.

CMP2 (Lappan et al., 2006) expanded digital supports while retaining calculators as core tools. In addition to four-function and graphing calculators, students and teachers were provided with CD-ROM resources. These included 25 digital problem activities,

specialized tools for data, algebra, transformations, and statistics, and multimedia supports such as CD-ROMs for student activities, teacher supports, and assessments. These resources reflected both a growing emphasis on technology integration and the early stages of digital curriculum supplements.

With CMP3 (Lappan et al., 2014), the integration of digital resources accelerated. Alongside calculators, students and teachers accessed the DASH/Realize platform, which hosted 12 digital math tools and 22 student activities. Digital practice and exam tasks provided additional assessment and planning supports. CMP3 also introduced a comprehensive video library: 37 mathematical background videos, 93 teacher connection videos, 21 classroom connection videos, and 126 problem launch videos. These were designed to connect pedagogy and content knowledge, illustrate classroom enactment, and provide just-in-time support for both teachers and students.

The CMP4 (Phillips et al., 2025a, 2025b, 2025c) builds on this trajectory, with technology fully integrated into the publisher-released Math Portal. Alongside calculators, CMP4 provides digital math tools, math activities, editable assessments, and slide decks to support classroom instruction. A hallmark of CMP4 is its attention to accessibility: features include translation into 44 languages, text-to-speech in 29 languages, and customizable text enlargement, highlighting, and spacing. These innovations expand CMP's commitment to equity by making digital curriculum resources more accessible to multilingual learners and students with diverse learning needs.

Across these stages, CMP has consistently positioned technology to support and extend problem-based learning. The progression from calculators to comprehensive digital platforms reflects both advances in educational technology and CMP's ongoing commitment to ensuring that tools are purposeful, connected to mathematical reasoning, and supportive of diverse learners.

CMP DIGITAL COLLABORATIVE PLATFORM

As part of its design research efforts, CMP, in partnership with Concord Consortium, has developed a digital collaborative platform to extend the enactment of *Connected Mathematics* in face-to-face classrooms. Importantly, this platform is project-based and distinct from the publisher's released digital products. It was designed and tested in research contexts, with the goal of enhancing classroom learning rather than replacing in-person instruction.

The digital collaborative platform embeds the CMP4 curriculum materials (Phillips et al., 2025a, 2025b, 2025c) to support face-to-face instruction. Each student can access the platform with an Internet-connected laptop. Teachers typically use a computer connected to a projection system and a tablet to access the digital dashboard and classroom workspace. A guiding principle in development has been to share functionality between teachers and students wherever possible, ensuring both groups engage with similar resources and tools.

In terms of student features, students engage with mathematics through the new CMP STEM Problem format embedded within the platform. The workspace allows students to explore mathematics collaboratively using graphs, tables, text, drawings, and images. A “four-up” view enables each student to view and integrate groupmates’ work in real time, creating a dynamic environment of shared exploration. Students can document, refine, publish, curate, and retrieve their work, moving seamlessly between individual and group spaces. This design allows them to create digital artifacts that showcase their mathematical reasoning, supporting their development as doers, knowers, creators, and communicators of mathematics.

In terms of teacher features, teachers have access to curriculum resources, including student guides, teacher notes, and problem solutions, embedded within the platform. They can monitor student work in real time or asynchronously, filtering by individual, group, or whole class. Work can also be sorted by problem, investigation, unit, or course. Teachers can select, highlight, edit, and incorporate student work into their own workspace, which can then be projected to the class. These features support formative assessment, classroom discourse, and reflection on student strategies.

Through these student and teacher features, the platform supports CMP’s longstanding design principle that mathematics is best learned through problem solving. It creates a collaborative environment where students’ strategies and representations are visible and sharable, while also providing teachers with orchestration tools that enable them to plan, monitor, and reflect on classroom learning in new ways.

Integrating Artificial Intelligence into the CMP Digital Platform

Building on the collaborative digital platform, CMP has begun to explore how artificial intelligence (AI) can enhance the teaching and learning of problem-based mathematics. The aim is not simply to mark answers right or wrong, but to surface the ways students reason mathematically and provide teachers with timely insights into their thinking.

Over the past three years, we have developed a rubric to identify, categorize, and track evidence of students’ proportional reasoning, with the goal of training AI to interpret student work meaningfully and support students and teachers in real time. (see Table 1) Proportional reasoning is a cornerstone of middle-grades mathematics, particularly in seventh grade, where it underpins concepts such as ratios, scale, percent, linear relationships, probability, and reasoning from samples to populations (Cai & Sun, 2002; Cramer et al., 1993; Hoyles et al., 2001; Lamon, 1994). Research on students’ proportional reasoning tend to show students reliance on rote algorithms to get correct answers without explaining the reasoning behind these strategies (Ayan-Civak et al., 2023).

Machine Learning Rubric for Proportional Reasoning

Student Proportional Reasoning Approach to Contextual Problem	The student approaches the contextual problem using reasoning about (1) Part-to-Part Relationship, (2) Part-to-Whole Relationship, or (3) Neither.

Student Representation	Student representations based on their inscriptional resource used in the digital collaborative platform: Text – Fraction form, Text – Decimal form, Text – Percentages, Text –Equation, Geometric figure, Drawing, Graph, Table, Equation, Uploaded image.
Student Strategy	Proportional Strategies include: (1) Building Up Additively; (2) Scaling Up Multiplicatively; (3) Using the Multiplicative Relationship; (4) Other – Proportional; and (5) Non-Proportional Strategies (Adapted from Ben-Chaim et al., 1998)

Table 1. Machine Learning Rubric for Students’ Proportional Reasoning.

Rather than creating separate rubrics for each problem, we designed a single, adaptable framework that captures three core dimensions of student thinking: approach, representation, and strategy. For example, one student might compare ratios by building up iteratively, while another may use the multiplicative relationship such as a unit rate from a table. Both are productive strategies, but they reveal different developmental trajectories. By tagging student work consistently along these dimensions, we generate the high-quality training data needed for machine learning models to recognize patterns in reasoning. Figure 1 highlights three examples of student work from the *Stretching and Shrinking* curriculum unit (Phillips et al., 2025a). While researchers reach consensus on each code, either focusing a teacher’s input or independently, they also document disagreements and patterns in the consensus process as important qualitative data to improve rubric clarity and support future machine learning models.

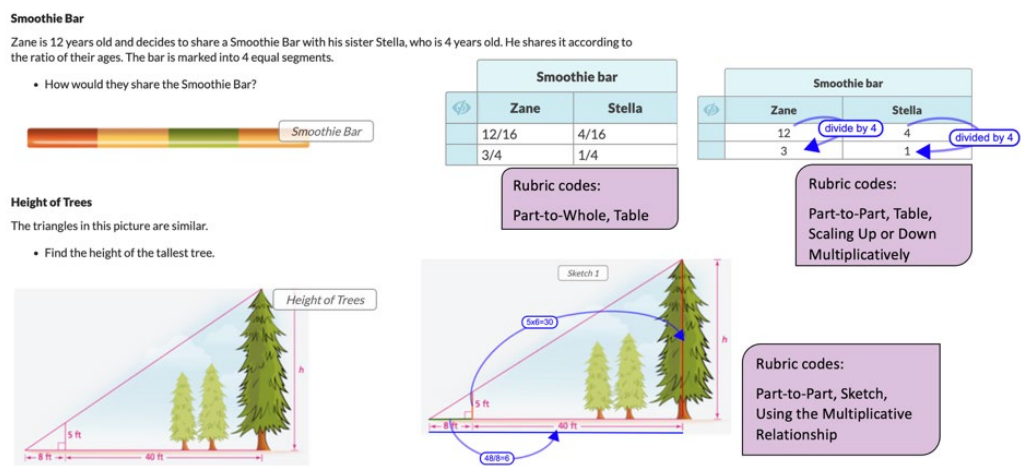


Figure 1. Sample student work manually tagged by the proportional reasoning rubric. This work highlights an important principle: human-coded rubrics provide a necessary foundation for integrating AI into mathematics education. The rubric, co-developed with teachers and researchers, was designed to accommodate multimodal student work and offer insights that are useful for classroom practice. At the same time, it supplies

the structured data needed to train AI models that, over time, may be able to return feedback aligned with the goals of a problem-based mathematics classroom. Still, much remains to be done. Future work will focus on refining the models with larger datasets and preparing ways to deploy feedback within the platform, making clear that this work is ongoing and evolving.

FUTURE DIRECTIONS: OPPORTUNITIES AND CHALLENGES IN THE AI WORLD

The integration of artificial intelligence into a problem-based curriculum like *Connected Mathematics* raises fundamental questions about the nature of curriculum design. Traditionally, the design of curriculum resources has been guided by theoretical commitments to coherence, depth, and inquiry, as well as by methodological approaches that emphasize iterative development, teacher feedback, and classroom-based research. As new technologies reshape what is possible, these same commitments must be re-examined and extended.

Theoretically, the challenge lies in ensuring that AI-enhanced tools are aligned with the principles of problem-based mathematics. If curriculum is to remain a “thinking device” rather than a delivery mechanism (Choppin et al., 2022; Choppin et al., 2015), then AI must support students in making sense of problems, building connections, and engaging in mathematical discourse. This raises questions about what aspects of student work AI should attend to and how AI feedback can amplify rather than replace teacher judgment. In the context of CMP, this means designing AI not simply to score answers, but to illuminate strategies, highlight patterns of reasoning, and provide openings for rich classroom discussion.

Methodologically, integrating AI into curriculum design demands new forms of design research. The long-standing CMP model of iterative development, informed by extensive teacher field testing, provides a strong foundation. Yet AI introduces additional layers: the need for large datasets, the development of rubrics to capture student reasoning in usable ways, and the creation of annotation systems that link multimodal student work with curriculum goals. The proportional reasoning rubric developed in recent years exemplifies this methodological shift. It represents an effort to translate classroom evidence of student thinking into structured data that can train AI systems, while still preserving the complexity and richness of mathematical reasoning.

In our current project, these theoretical and methodological commitments are being carried out in our ongoing work through the systematic collection and human coding of student work in the digital platform. During 2023–2024, students and teachers engaged with 57 problems across six curriculum units, and during 2024–2025 this expanded to 74 problems across seven units. In total, the work spans 18 class sections and includes contributions from 395 students. From this dataset, we have currently curated a substantial amount of student work for training the machine, with over 1,500 tags for part-to-part reasoning, 300 for part-to-whole reasoning, and more than 1,500

strategy instances, including additive build-up, multiplicative scaling, and identifying multiplicative relationships. The dataset is also designed to capture proportional reasoning as it emerges across multiple strands of mathematics, including number, algebra, geometry, statistics, and probability, reflecting the wide-ranging role of proportional reasoning in the curriculum. Importantly, this work draws on student contributions from classrooms that are diverse in terms of racial identity, English language proficiency, socioeconomic status, and disability. In this way, the project illustrates how theoretical commitments to equity and student-centred learning are realized through methodological choices that prioritize diverse participation, careful coding, and the development of resources that can prepare AI systems to recognize the complexity of proportional reasoning.

Looking ahead, these theoretical and methodological considerations highlight both opportunities and risks. On one hand, AI holds the promise of providing teachers with timely insights into student learning, enabling more responsive and equitable instruction. On the other, it raises concerns about narrowing mathematics to what machines can easily recognize or about reducing teaching to a set of automated interventions. For curriculum developers, the central task is to navigate these tensions: to harness AI in ways that extend the possibilities of problem-based learning while keeping teachers and students at the centre of the enterprise.

The evolution of CMP, across four editions, and now into a digital and AI-enhanced era, offers a case study in how curriculum design can adapt to new contexts without losing sight of its core commitments. By grappling with the theoretical and methodological issues of integrating AI, the project continues its long-standing work of reimagining what it means to design curriculum resources that are both ambitious in vision and responsive to the realities of classroom practice.

In conclusion, the evolution of *Connected Mathematics* demonstrates how curriculum design can adapt to new contexts while remaining grounded in a vision of problem-based learning for all students. The integration of AI into the digital collaborative platform represents both a continuation of this trajectory and a new frontier, raising theoretical and methodological questions about how curriculum resources are designed, enacted, and studied. As we move forward, several pressing questions guide our work: How can AI be leveraged to diagnose and track proportional reasoning in real time, and what benefits might such a continuous feedback loop bring to both teachers and students? What strategies are needed to ensure that AI tools are accessible and beneficial for all learners? In what ways can AI supports help teachers foster deeper mathematical understanding while also managing diverse student needs? And what challenges and ethical considerations must be addressed when using AI to support student reasoning? These questions highlight the opportunities and complexities of designing curriculum resources in the AI world, pointing toward future research and development that will shape the next chapter of problem-based mathematics education.

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DIGITALIZING MATHEMATICS TEXTBOOKS IN THE AGE OF AI: WHAT HAVE WE LEARNED FROM A LARGE RESEARCH PROJECT IN CHINESE EDUCATIONAL SETTINGS?

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This article is based on a large research project entitled “A study of developing digital mathematics textbooks and its effect on mathematics teaching and learning”, conducted in Chinese educational settings. Since 2019, the project has developed a comprehensive digital mathematics curricular resource platform for teaching and learning, with a core part being the digitalization of a complete series of mathematics textbooks at the junior secondary level. The digital resources developed have been used in more than ten schools in the East China region. The article is intended to introduce the rationale and research focus, illustrate the process and methods, and report the relevant findings of the project. In addition, it discusses the problems and challenges encountered in the designing, developing, and use of the digital mathematics textbooks.

INTRODUCTION AND BACKGROUND

This article presents a study of digitalizing mathematics textbooks, which is based on two premises. The first premise is that mathematics textbooks, or more broadly, purposefully and systematically designed curricular resources, remain one of the most important instruments for supporting the curriculum development and delivery, teaching and learning of mathematics at various school levels and in many different countries (also see Adler, 1998; Rezat, et al., 2021; Fan, et al., 2025). The second is that, over the last two decades or so, the fast development of information and communication technology, or digital technology, and particularly more recently in the area of Artificial Intelligence (AI), has transformed almost every aspect of education and mathematics education, including the development and design of modern mathematics textbooks. More specifically, the growth of digital formats and the emergence of artificial intelligence (AI) have created new and great possibilities and challenges for how textbooks and more broadly curricular resources are designed, produced, and used.

It should be noted that, while digital versions of printed textbooks and other curriculum materials have become increasingly common nowadays, compared to about a decade ago, as noted by researchers (Fan et al., 2013), they often remain static reproductions rather than fully-integrated digital resources. This leaves an important gap in both research and practice on what digital textbooks can be and how they may truly reshape mathematics teaching and learning. In terms of research, particularly empirical studies in this area on issues about what and how digital technologies (including more recently

AI) can be integrated into the development of mathematics textbooks, are still relatively few and at an early stage.

As part of our response to this gap, a multiple-year large-scale project entitled “A Study of Developing Digital Mathematics Textbooks and its Effect on Mathematics Teaching and Learning”, of which I served as the principal investigator, or PI, was launched in 2019. As the project is jointly sponsored by the East China Normal University or ECNU and Zhejiang Education Publishing House, it is also called ECNU-Zhejiang School Mathematics Project, or EZSMP for short.

In a sense, EZSMP can be considered as a research, development and innovation (RD&I) project. The project has two main aims: first, to develop and implement a comprehensive digital mathematics textbook and curricular resource platform at the junior secondary school level, and second, to investigate the effective ways, through experiment, analysis and improvement, and the effects of its use on the teaching and learning of mathematics in schools in the East China region.

By digitalizing a complete junior secondary mathematics textbook series and examining its use in more than ten schools in Shanghai, Zhejiang, and Jiangsu, the project created an innovative and unique opportunity to study the design, implementation, and impact of digital textbooks and related curricular resources and to explore effective ways for the integration of digital technology and mathematics instruction. This paper presents an overview of the project, highlights major findings, and discusses the challenges and issues encountered in the development and use of the digital resources.

DESIGN, PROCESS AND METHODS

Following the joint effort and collaboration between East China Normal University and the Zhejiang Education Publishing House, the EZSMP project aimed to advance the development and use of digital mathematics textbooks and curricular resources (for convenience, we shall simply call them textbooks hereafter) on a large scale. Since its inception, the whole process of the project has been organized into different phases.

The first phase (2019–2023) was planned to design and develop the digital platform and introduce digital resources, which were mainly based on existing printed materials, to selected schools. The existing printed materials included, most importantly, a complete series of mathematics textbooks for the junior secondary school level (Grades 7 to 9), which were approved by the National Textbook Review Committee for Primary and Secondary Schools and published by the Zhejiang Education Publishing House. In this sense, the approach of the development is a textbook transformation (digital transformation) as discussed by Fan (2024). Moreover, the existing materials also include teacher manuals, lesson plans, assessment books, and other available instructional materials, when they were deemed appropriate, relevant, and helpful for teaching and learning of different topics of mathematics presented in the textbooks.

The key design features or principles of the digital platform are as follows: (1) integrating the four functions of teaching, learning, assessment and management into one platform; in other words, the platform can serve the purposes of teaching, learning, assessment and class management; (2) integrating and reflecting advances in modern information, communication and intelligence technologies, including AI, video explanations, animated demonstrations, experimental operations, gap-filling exercises, information feedback, resource search, resource uploading, and online communication; (3) improving the scientific, digital and intelligent level of mathematics textbooks and supporting resources; (4) providing school teachers and students with a set of smart teaching and learning services that are innovatively complete, systematic, efficient and intelligent; (5) enabling both traditional offline teaching and modern online teaching (when in need or special times such as the Covid-19 period); and (6) enabling full play of teaching materials as the key supporting instrument for teaching, and effectively promoting the development of mathematics teaching activities and the improvement of mathematics teaching quality. It is clear that it is because of the modern digital technologies, including AI, that made all the features possible in the design and development of the digital platform.

The second phase (2023–2025) was to collect and analyse data from participating schools using the platform while refining the materials, based on feedback from the participating schools. The refinements were reflected both in the specific designs, e.g., the layout of a particular lesson unit, and in the contents, e.g., assessment tasks (see more information below). The third phase, which was scheduled to begin in July 2025, aims to institutionalize the project and further explore sustainable models for integrating digital textbooks into classroom practice. This article focused on the first two phases of our work, which have been completed so far.

A central component of the initiatives or innovations is the development of the EZSMP software platform (<https://www.liwamath.com>, as of 20 June 2025), designed as an integrated digital environment for teaching and learning. On the same platform, there are two separate sub-platforms: a teacher edition and a student edition. Figure 1 shows the first page of the online Grade 7 mathematics textbook for the first semester in the teacher edition. Figure 2 is an English translation of it.



Figure 1: First content page of the Grade 7 mathematics textbook for the first semester in the teacher edition in the EZSMP platform.

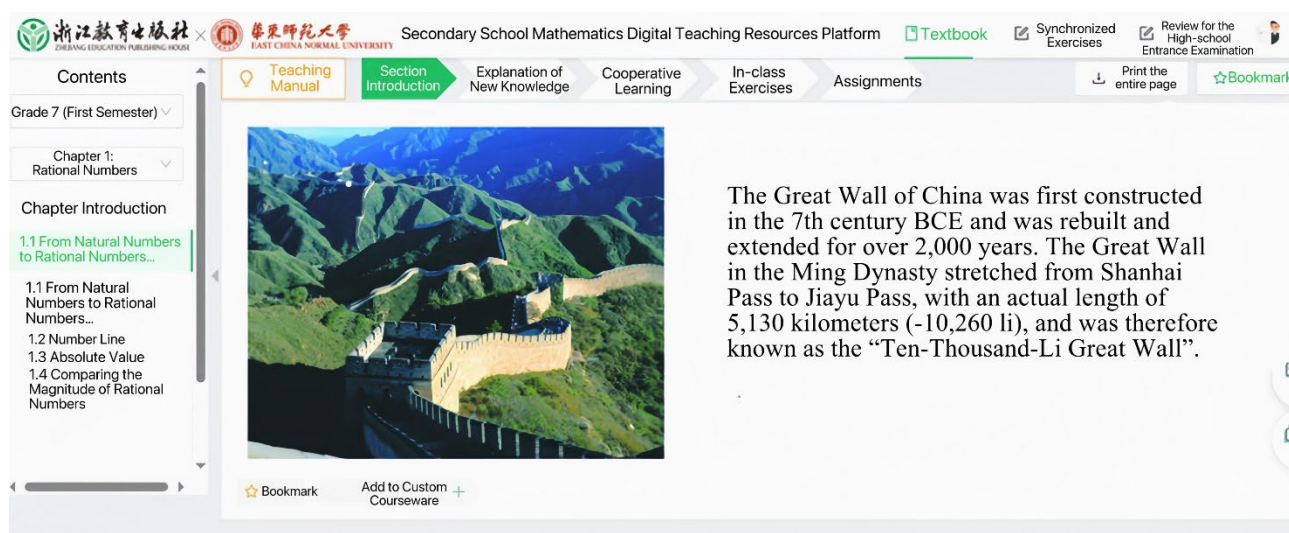


Figure 2: First content page of the Grade 7 mathematics textbook for the first semester in the teacher edition in the EZSMP platform (English translation).

The platform is designed to support four key functions: Teaching, Learning, Assessment, and (Instructional) Management. Its most significant achievement is the digitalization of a complete junior secondary mathematics textbook series and related instructional materials. This includes not only the core traditional paper-based textbooks but also supplementary resources such as worked examples, practice problems, dynamic visualizations, assessment tasks of varying complexity, and analytic tools.

A key feature of the project is the interdisciplinary and inter-institutional collaboration. The development of the digital platform has involved researchers (in mathematics, mathematics education, computer scientists, and educational technologists), curriculum developers, school teachers, and students, working closely together.

After the digital platform was initially established, pilot testing of the digitalized textbooks was conducted in a number of schools in Shanghai, Zhejiang, and Jiangsu, which provided effective opportunities to refine the design and development of the platform, and to verify and improve its effectiveness and usability. In a sense, combining the large-scale design with the field-based study, EZSMP stands as one of the most ambitious research efforts on digitalizing mathematics textbooks in the field of mathematics education in China to date.

Rather than simply a development initiative, the EZSMP project was also conceived as a large-scale empirical study, to which substantial research efforts were devoted. The main objective in terms of research was to examine the feasibility of incorporating digital instructional materials into traditional classroom practice and to evaluate the impact of the use of the digital mathematics textbook platform on teachers' teaching and students' learning of mathematics. To address these questions, multiple research methods were employed.

The project was first implemented in 11 participating schools in Shanghai, Zhejiang, and Jiangsu to obtain preliminary results, and was later expanded to 19 schools in those three provincial places. Data were collected from a variety of sources. At the classroom level, 29 lessons were observed and analysed. Student perspectives were investigated through questionnaires with 84 participants, followed by in-depth interviews with 12 students. Teachers' views were studied using questionnaire surveys with 14 participants, three one-on-one interviews, and three focus group discussions. In addition, four feedback conferences and meetings with school leaders and administrators were organized to supplement and validate the findings. Table 1 provides an overview of the main data sources used in the EZSMP project.

Source type	Methods of collection	Quantity / Participants
Classroom observations	Lessons observed and analysed	29 lessons
Student data	Questionnaires + in-depth interviews	84 questionnaires, 12 interviews
Teacher data	Questionnaires, one-on-one interviews, focus groups	14 questionnaires, 3 interviews, 3 focus groups
Feedback activities	Feedback conferences with teachers and administrators	4 conferences/meetings
Platform analytics	Usage logs, performance records, error reports	Continuous data

Table 1: Data collection in the EZSMP project.

To triangulate the data collected as described above, logs recording the usage, student performance, and error reports were also studied. These were useful for tracing the progress of student learning and teacher preparation. The information was then complemented by cross-referencing lesson observations with interview and survey data, alongside the platform records. The use of these different methods, including both quantitative and qualitative ones, allowed the researchers to examine both the educational effectiveness and the practical relevance of digital textbooks and curricular resources.

FINDINGS

The data gathered from participating schools provided an overview of both student and teacher experiences with the EZSMP package, how the platform was generally used, and what impact it had on teaching and learning.

Most students positively accepted all the offerings. They generally had a positive attitude towards the use of the digital platform and were also willing to use digital textbooks in the future. The vast majority of the students reported that the platform was easy to use and very helpful in cultivating mathematics learning interest, mathematical thinking ability, problem solving ability, and learning mathematical knowledge and skills, and they generally thought that the platform had a positive effect on their learning of mathematics. A relatively detailed report based on a case study of the project can be found at Jin et al. (2024).

However, not all students were equally interested. The data available on the use of the platform also revealed that while some had accessed the platform nearly a hundred times, others had accessed it much less frequently in a semester. This indicates that participation by the students very much depended on how the teachers incorporated the platform into their day-to-day teaching.

The data collected from the teachers revealed that teachers could generally make good use of the platform in the lesson introduction, explaining new knowledge, and example and assigning exercises. They thought that the platform provided teachers with mobile and rich question banks and lesson preparation resources, which also had a positive effect on teachers' lesson preparation and online teaching. In the case of explaining examples and giving assignments, teachers reported that the analysis of the learning situation (learning analytics) provided by the platform was very useful, and in the classroom using the platform, students had better opportunities to learn from the dynamic representation of mathematical content and concepts. Overall, the teachers thought that the digital textbooks and curricular resources provided in the digital platform were also very useful in helping students to understand abstract concepts, and also offered drills and exercises aligned with the mathematics curriculum. In addition, it is believed that the online assessment system of the platform provided effective feedback reports and learning analytics, including evidence and percentage feedback on student work, especially after corrections.

The usage patterns showed differences among the users; some teachers exhibited very high activity during the study—one teacher registered over 3,000 visits and another over 1,500—while others used the system less often. This suggests that the use of digital resources was determined by various factors such as a teacher’s personal style, school support, and familiarity with digital tools, etc.

Monthly statistics further supported the specified tendencies. Figure 2 shows the monthly usage of the EZSMP platform from September 2024 to May 2025.

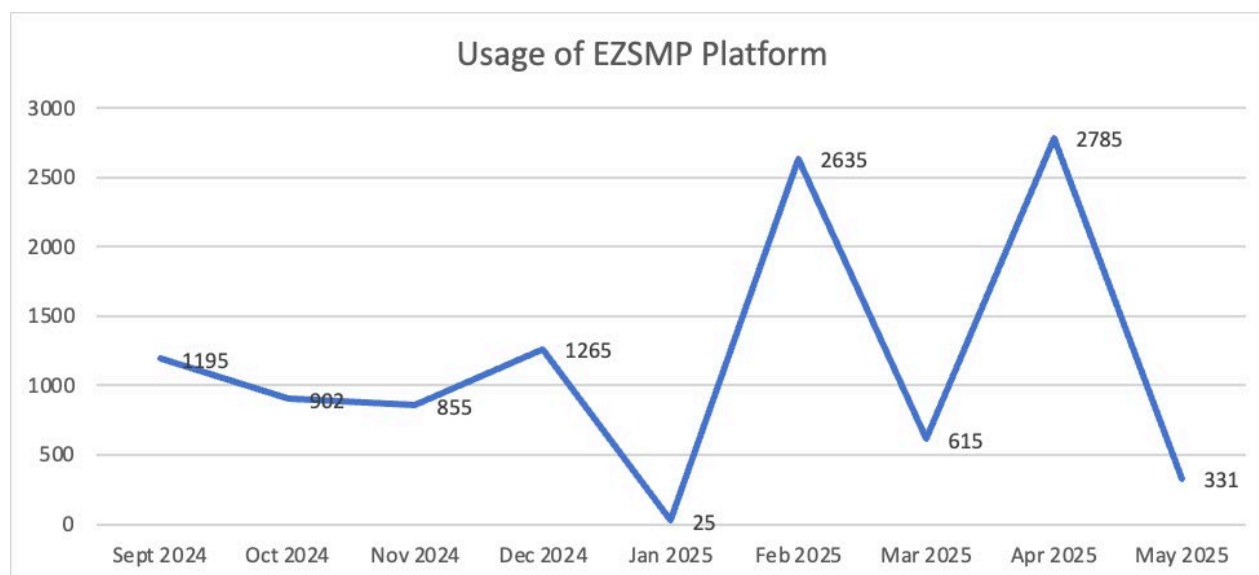


Figure 2: Monthly usage of the EZSMP platform (September 2024–May 2025).

In general, the period in the academic year recorded some slight rises and falls in the use of the platform, with notable marks being in February 2025 (2,635 visits) and April 2025 (2,785 visits). The other months, like January 2025, attracted fewer users, with only 25 visits, mainly due to the festivities that happened during and after this month. On top of that, similar activity patterns were also seen in May and June 2025, for the reason that it was both the end of the term as well as the examination period. Consequently, analysing the digital-textbook usage data, it is reasonable to conclude that the e-textbooks were used in the teaching–learning process not only because teachers promoted them, but also because school activities shaped the timing of their use.

In conclusion, the outcomes indicate that the digital textbooks and curriculum resources provided in the platform developed in the study were a valuable teaching and learning resource and were perceived positively by both students and teachers. However, students’ interest in the subject and the usage depended on various factors, including particularly how it was incorporated in teachers’ teaching and on timing within the school calendar.

CHALLENGES AND ISSUES

While EZSMP did advance, many new challenges and issues came into being during the digital textbooks' design, development, and use in classrooms.

There was one fairly common misunderstanding that digital books are merely scanned PDF versions of the printed books. While a very narrow view, it limited some teachers and even decision makers from seeing what a digital resource could really be used for: interactions, visual animations, built-in assessment tools, just to name a few.

Then there is the rapid evolution of technology. In a sense, new functions and tools emerge almost every day, and so the team found itself constantly updating the system. This was also questioned from the standpoint of long-term sustainability; the continuous evolution, after all, requires not only that researchers, including the developers, keep abreast with the development of related digital technology, nowadays particularly AI, and knowledge, but also both frequent technical updates and sustainable financial support.

This was similarly true for the readiness of the teachers. Some teachers readily adjusted and integrated the platform into their daily lessons, whereas others found it hard to change old routines. The situation emphasized the need for effective teacher training and professional development. The project team organized a number of training seminars and symposiums for school teachers to get familiar with and use the digital platform, but teachers' continuous self-learning and school-based professional development about the use of the digital platform are essential in the long run.

For students, independent use of the platform was not always easy. Many relied on the teacher's guidance to get the best out of the digital textbooks and resources. This prevented the platform from unlocking its full potential for learning outside the classroom. To a large extent, while the new digital resources provide new opportunities for students to learn, the platform also presents challenges, and in particular, it requires a change of students' learning style in the traditional textbook-based curricular environment.

The evaluation of the overall effectiveness of digital textbooks also remains a challenge. While the platform data, surveys, and observations gave useful clues, measuring long-term learning outcomes and classroom impact is a far more complicated affair. A further issue is that sustainable funding and institutional support are needed for the project to continue and succeed in the long run.

CONCLUDING REMARKS

In the EZSMP project, through the design of a full series of junior secondary mathematics textbooks in digital form and their use in schools, new light was shed on how digital technology can support the teaching and learning of mathematics. Overall, it was concluded that the digital textbooks and curriculum resources provided were perceived positively by both students and especially teachers, and had a positive impact on teaching and learning of mathematics. Meanwhile, it also suggests that the

development and use of digital textbooks and curricular resources were not merely a technical issue but also one with educational and cultural implications (Remillard, 2019).

The project experience and findings suggest that digital textbooks cannot be simply static digital copies of print materials. They must be born in a digital and interactive environment. Interaction, visualization, and analytics should be treated as core design considerations instead of afterthoughts.

Another important reflection is about teachers. It turned out that digital textbooks are most useful when they assist teachers as supporting tools rather than replacing them. Teachers used the platform for lesson preparation, introducing topics, and explaining abstract ideas; therefore, it encourages strengthening teacher agency in this digital age. For this reason, continual training and professional development are important if teaching with digital textbooks is to be a long-term practice in their daily teaching.

There are also other factors that need to be considered. For example, times for holidays and examination periods, as well as teachers' or students' schedules, can somehow dictate platform use. This serves as a reminder that any educational technology always operates within broader institutional and cultural settings, and its success depends upon ways of accommodating this.

Furthermore, the project highlighted the importance of collaboration and ongoing support. The overall digital textbook system involves not just technical design but also funding and cooperation between universities, schools, and educational authorities. Without such collaboration, long-term support becomes difficult, if not impossible, for sustained innovation. Finally, with the rapid development of AI, how it can be effectively integrated in the design, development and use of digital textbooks brings new opportunities and challenges and merits increasing attention and research effort in modern mathematics education.

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EVALUATION OF DIGITAL CURRICULAR RESOURCES: AN ISSUE FOR RESEARCH AND TEACHER EDUCATION

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Recent research on the use of digital resources in mathematics education highlights that understanding their affordances and limitations is essential for making informed choices when designing and enacting digitally enhanced teaching. Consequently, the ability to critically evaluate such resources has emerged as a central dimension of mathematics teachers' digital competence. In this talk, related to Conference Theme 3 - Theoretical and/or methodological issues in evaluation of curriculum resources - we explore diverse approaches to the evaluation of digital resources and shed light on the complexity of defining robust evaluation frameworks. We conclude by discussing implications for mathematics teacher education, with a particular focus on how future teachers can be supported in developing their evaluative judgment.

INTRODUCTION

Over the past two decades, the accessibility and diversity of digital resources for mathematics education have increased significantly. Digital textbooks, dynamic geometry software, collaborative platforms, online repositories of tasks, educational applications, and, more recently, generative artificial intelligence tools, offer teachers an unprecedented range of possibilities for designing and enacting their lessons. While this expansion of the digital landscape is often praised for its innovative potential and its capacity to enrich classroom practices, it simultaneously raises fundamental questions about the quality, reliability, and didactic and pedagogical value of these resources, as well as their alignment with curricular goals, institutional constraints and students' needs.

In mathematics education research, the interactions between teachers and resources are now considered a core component of professional practice (Adler, 2000; Gueudet & Trouche, 2009). Teachers' engagement with resources extends far beyond passive use: it involves complex processes of design, adaptation, and recontextualization, through which the teachers interpret and transform the curriculum. From this perspective, teachers can be viewed as active curriculum designers who construct personal and collective systems of resources to support their teaching.

Within this context, the evaluation of digital curricular resources emerges as a key competence. It is essential both for researchers, who seek to conceptualize and understand the dynamics of teachers' interactions with resources, and for teacher

educators, who must prepare future professionals to exercise informed and critical judgement in navigating the abundance of digital material now available.

This plenary paper pursues a dual objective. First, it reviews existing approaches to the evaluation of digital curricular resources, highlighting the main criteria and frameworks developed in recent research. In doing so, it aims to provide an overview of how evaluation has been conceptualized across different traditions and educational contexts. Second, it explores how teacher education can foster teachers' capacity not only to evaluate but also to design and adapt digital resources. This discussion is grounded in the analysis of a teacher education program that explicitly integrates resource evaluation and design as part of the professional development process. The paper concludes by identifying emerging perspectives and challenges.

THEORETICAL BACKGROUND

Mathematics teaching can increasingly be understood as a design activity (Brown, 2009). In this view, teachers are not merely implementers of prescribed curricula but active designers who adapt, modify, and combine a wide variety of resources to construct the curriculum as it is enacted in their classrooms. This design-oriented perspective highlights the creative and reflective dimensions of teaching, emphasizing that teachers continually make didactic, pedagogical and epistemological choices as they interpret curricular guidelines, select tasks, and orchestrate learning experiences. It resonates with Jones and Pepin (2016), who describe teachers as curriculum designers rather than mere users of ready-made materials, thereby acknowledging their central role in shaping the enacted curriculum and mediating students' access to mathematical knowledge.

The appropriation of resources is a complex process that goes beyond initial adoption. As Hoyles et al. (2013) underline, adaptation plays a crucial role: resources only become meaningful when teachers transform them in light of their own educational goals, their students' prior knowledge, classroom dynamics and broader institutional contexts. This process of transformation often involves iterative experimentation, reflection, and re-design. Over time, teachers develop personal and collective systems of resources, shaped by their professional experiences, beliefs about mathematics teaching and learning, and interactions with colleagues and digital environments. Teachers' engagement with resources thus constitutes both a lever and an indicator of professional development (Gueudet & Trouche, 2009): as teachers refine and expand their resource repertoires, they simultaneously build professional knowledge and agency.

In this context, the evaluation of resources emerges as an integral part of teachers' professional activity. Evaluation is not limited to assessing intrinsic quality of a resource but entails a situated and interpretive judgement of its relevance, adaptability, and added value for specific teaching situations. Consequently, research in mathematics education has progressively shifted its focus from the study of resources

themselves to an examination of teachers' evaluative practices, resource selection strategies, and processes of design and adaptation. This shift reflects a broader reconceptualization of teaching as an epistemic and design-oriented profession, where critical engagement with resources is central to teachers' agency and professional growth. Understanding how teachers evaluate and adapt digital curricular resources is thus key to supporting both effective classroom practice and the development of robust frameworks for teacher education. Building on this understanding of teachers' resource work, the next section examines how recent research has conceptualized the quality and evaluation of digital curricular resources.

QUALITY AND EVALUATION OF DIGITAL CURRICULAR RESOURCES

The quality of digital curricular resources is a central concern in mathematics education research, particularly as the diversity of available resources continues to grow. In the literature, two main conceptualizations can be identified, each offering a distinct perspective on how digital resources should be analysed and evaluated.

The first conceptualization views quality as (a confluence of) of features of curricular resources. From this perspective, quality is embedded in the resource itself, and can therefore be assessed by examining its design, structure, and affordances. Among the most frequently cited features are:

- *Accuracy*: the mathematical correctness of the content (Camilleri et al., 2014),
- *Relevance*: the degree to which the resource aligns with specific learning objectives, instructional approaches, and classroom contexts (Pepin, 2020),
- *Coherence*: internal consistency and the capacity of the resource to connect meaningfully with curriculum documents and other instructional materials (Pepin, 2020),
- *Practicality* or *availability*: ease of use, accessibility, technical availability, and ergonomic design (Camilleri et al., 2014; Pepin, 2020), which strongly influence adoption in real classroom settings,
- *Effectiveness*: the potential of the resource to promote the intended learning outcomes, often conceptualized through notions such as learning affordances or expected impact (Camilleri et al., 2014; Pepin, 2020),
- *Interactivity*: opportunities for learners to explore, manipulate, and receive feedback from the digital environment, which are frequently highlighted as key added values of digital resources (Bokhove, 2017),
- *Adaptivity* or *flexibility*: the extent to which the resource can be customized to accommodate diverse teaching approaches or learner needs (Glasnović Gracin & Krišto, 2022; Mariotti et al., 2024),
- *Connectivity*, the degree to which the resource integrates with other resources or systems (Pepin et al., 2016).

This first approach thus foregrounds the *designed features and affordances* of digital resources, often without immediate reference to the user or context. It reflects a tradition rooted in instructional design and technology-enhanced learning.

The second approach conceptualizes the quality of a resource as arising from specific evaluative criteria linked to various dimensions of a resource, thereby offering a more systematic and multidimensional view. A prominent example is the TIPS framework (Kawachi, 2014), which distinguishes four interconnected dimensions: (1) *teaching and learning processes* including pedagogical quality, interactivity, motivation, and assessment opportunities, (2) *information and material content* encompassing accuracy, relevance, and appropriateness of content load, (3) *presentation and format*, such as multimedia integration, design coherence, and openness, and (4) *system and technical aspects* including metadata quality, interoperability, and technological robustness. Similarly, the Intergeo project (Trgalová et al., 2011), which focused on dynamic geometry (DG) resources, proposed an evaluation process along eight dimensions: *metadata* (accuracy and clarity of the resource description), *content* (mathematical correctness and relevance), *technical dimension* (exploitability, and absence of bugs), *instrumental dimension* (coherence between DG files and mathematical content), *didactical dimension* (guidance for classroom enactment), *pedagogical dimension* (classroom management considerations), *added value* (specific contributions of DG to mathematics learning), and *ergonomic dimension* (usability, user friendliness, and adaptivity). These frameworks illustrate a move toward more nuanced, structured, and context-sensitive conceptualizations of quality.

Importantly, research increasingly recognizes that the evaluation of resources is not determined solely by their characteristics but also by who evaluates them, and for what purposes (Trgalová et al., 2024). Governmental bodies may prioritize conformity with curriculum standards and rely on formal procedures for resource approval (e.g., de Carvalho, 2018; Rezat, 2025). Researchers may be interested in conceptual or empirical analyses of digital curriculum programs (Choppin et al., 2014) or e-textbooks (Pepin et al., 2016). Peer reviewers in open educational resource repositories ensure compliance with shared quality norms (e.g., MERLOT repository). Teacher communities develop collective evaluation practices to help colleagues make informed choices, as seen in platforms such as TeachersPayTeachers (<https://www.teacherspayteachers.com/>) or i2geo (<http://i2geo.net>). At the individual level, teachers evaluate resources in terms of their own teaching goals, constraints, and classroom contexts. Such evaluation is often iterative and intertwined with processes of adaptation and redesign. As teachers incorporate resources into their personal or collective systems, they generate situated judgments of quality that may differ substantially from designers' or policymakers' perspectives.

Taken together, these approaches underscore that quality of digital curricular resources is a **multifaceted, context-dependent, and relational construct**, shaped not only by

design features but also by evaluative practices and purposes within diverse educational communities.

These multifaceted understandings of quality highlight that evaluating digital resources is not solely a matter of applying criteria, but a demanding professional task. The following section therefore examines teachers' digital resource design capacity and the competencies that underpin effective evaluation and use.

SUPPORTING THE DEVELOPMENT OF PRE-SERVICE TEACHERS' DESIGN CAPACITY

The growing diversity of digital curricular resources places new demands on teacher education. Teachers must not only be able to use digital tools but also to critically evaluate their relevance, quality, and potential for mathematics learning. This competence has been conceptualized as *Digital Resource Design Capacity* (DRDC) (Trgalová & Tabach, 2024), which encompasses teachers' ability to design, evaluate, and redesign digital resources in ways that serve specific learning goals.

Digital resource design capacity

The DRDC framework, that extends the concepts of pedagogical design capacity (Brown & Edelson, 2003) and teacher design capacity (Pepin et al., 2017), highlights three interconnected phases of teachers' design activity:

1. *Design for teaching (planning)*: Teachers set a mathematical learning goal, select appropriate digital technologies, and design tasks that contribute to achieving this goal. This involves ensuring alignment with the curriculum, leveraging prior mathematical and instrumental knowledge, and articulating the added value of digital technology. Teachers also plan classroom enactment, anticipating students' strategies and difficulties, and preparing interventions.
2. *Design-in-use (enactment)*: During classroom implementation, teachers adapt their plans to students' responses, technical constraints, and unexpected events. They must make on-the-spot decisions—what questions to ask, how to respond to student input, how to manage classroom interactions—while maintaining the coherence of mathematical and instrumental objectives.
3. *Redesign (reflection and adaptation)*: After classroom use, teachers reflect on what worked well and what could be improved. They adapt resources based on observations, student feedback, and peer input, thereby engaging in iterative cycles of design and evaluation.

Across these phases, evaluation plays a central role: teachers constantly judge the relevance, coherence, effectiveness, and worthwhileness of their design choices.

Example of a teacher education program

A concrete example of how teacher education can foster mathematics teachers' DRDC is found in the two-year Master program for secondary mathematics teachers at the

University of Lyon 1 (France). This program includes two courses on digital technologies, structured as follows:

- *Year 1* focuses on the affordances of digital tools for mathematics teaching and learning. Technologies are approached as *objects*: student-teachers explore their functionalities and potential applications.
- *Year 2* emphasizes mathematics topics and the integration of digital technologies as *tools* for enhancing teaching and learning.

The second-year course, lasting 15 hours over 7 sessions, provides opportunities for student-teachers—who are simultaneously half-time secondary school teachers and half-time students—to engage in resource design and evaluation activities.

The structure of the course includes:

Exploration of mathematics topics with digital technologies (Sessions 1–3): Student-teachers solve mathematical tasks using different digital tools, compare solutions, and discuss contributions of the digital tools to mathematics teaching and learning. This develops their ability to evaluate the relevance of digital technology for specific goals and subsequently to choose relevant digital technology.

Collaborative resource design (Sessions 4–5): Working in design teams, student-teachers formulate a learning goal, choose a digital technology, design a technology-based task, justify their design choices, and plan classroom enactment. Instructors provide pro-active support through templates (Figure 1) and theoretical input including:

- old/new dialectics (Assude & Gélis, 2002): when integrating a new digital technology, teachers need to pay attention to students' mathematical and instrumental prerequisites so that no new mathematical knowledge is introduced with a new technology,
- SAMR framework (Puentedura, 2006), which is a conceptual tool to reflect about technology integration. The acronym SAMR stands for *Substitution*—technology acts as a direct tool substitute, with no functional change; *Augmentation*—technology acts as a direct tool substitute, with functional improvement; *Modification*—technology allows for significant task redesign; and *Redefinition*—technology allows for the creation of new tasks, previously inconceivable,
- instrumental orchestration (Trouche, 2004), the notion describing teachers' practice aimed at supporting their students while working on technology-enhanced mathematical tasks.

The instructors provide a re-active support as well in the form of feedback on the intermediate versions of the designed resources.

Peer evaluation and redesign (Session 6): Design teams exchange resources and provide feedback using an evaluation grid that addresses task description, relevance of the technology, student activity, and the teacher's role. Peers also offer an overall

evaluation and suggestions for improvement. This feedback is leveraged by the design teams to redesign their resources before their enactment in classrooms. This develops student-teachers' ability to evaluate design for teaching.

Resource presentation and reflection (Session 7): Student-teachers present their resources and report on classroom implementation. They highlight strengths and weaknesses of their designs, reflect on student engagement, and propose redesigns based on their experiences. This contributes to the development of student-teachers' ability to redesign a resource taking in consideration outcomes of its enactment.

[Title of the technology-supported activity]					
1. Identity card					
Professional question					
Mathematical topic					
School level					
Teaching goal					
Technology					
Author(s)					
2. Text of the activity (as it will be proposed to students)					
3. A priori analysis of the activity					
- What is the student's task (what should they do)? - What do students need to know (in mathematics and with respect to the use of technology) in order to be able to engage in the activity? - What strategies the students can use? What is the expected strategy, if there is one? - What difficulties the students can encounter while solving the activity? What hints and feedback can be foreseen to face them? - What is the role of the technology in the activity? What is its contribution? - What can be put forward during the synthesis (institutionalization), in terms of mathematical and instrumental knowledge?					
4. Implementation of the activity (pedagogical scenario)					
In the table below, describe the main phases of the planned enactment of the activity during a mathematics session. Please feel free to modify you wish.					
Phase / duration	Goal	Modality/instrumental orchestration	Role of the student(s)	Role of the teacher	Material support
Phase 1 5 min	Presentation of the activity	Whole class	Listen	Explain the tasks, asks students' reformulation	Projection on a whiteboard

Figure 1: Resource template used to scaffold the design process.

Findings from the program implementation

This course, that was implemented for several years in pre-service mathematics teacher education, revealed both opportunities and challenges in developing teachers' evaluative and design capacities.

For many participants, the course provided a first authentic experience of integrating digital technologies into mathematics teaching. Collaborative resource design fostered

peer learning, and the possibility of enacting their designs in real classrooms created valuable feedback loops between planning, implementation, and redesign.

Nevertheless, student-teachers often struggled to prioritize mathematical learning goals, focusing instead on introducing the technology itself. Classroom management was another difficulty: they underestimated students' heterogeneous technical skills and the time required for managing digital tasks. Finally, participants found it challenging to articulate the specific mathematical contribution of digital technologies, often limiting their analysis to motivational aspects.

Despite these challenges, crucial aspects of the course design supported the development of pre-service teachers' DRDC: templates guided reflections on resource design and evaluation, collaborative design teams encouraged peer learning, and enactment opportunities highlighted the dynamic interplay between planning, use, and redesign.

CONCLUDING REMARKS

The evaluation of quality of digital curricular resources in mathematics education is a complex and multifaceted endeavour. As this plenary paper has shown, quality cannot be understood solely in terms of intrinsic features such as accuracy, coherence, or interactivity. Instead, it must be considered in relation to users' needs, teaching contexts, and educational goals. This relational view of quality foregrounds the evaluative agency of teachers and highlights the situated nature of resource use.

Multiple frameworks—such as TIPS or Intergeo—offer valuable lenses for assessing digital resources, yet their applicability varies depending on resource type, intended use, and institutional context. Importantly, most digital resources fall outside formal approval processes (Rezat, 2025), leaving schools and individual teachers to shoulder the responsibility of evaluation. This decentralization underscores the need for teacher education programs to cultivate evaluative literacy: the ability to critically judge relevance, adaptability, and pedagogical value across diverse resource formats.

The emergence of generative artificial intelligence (GenAI) adds a new layer of complexity. GenAI tools are rapidly reshaping mathematics education, offering novel forms of content generation, personalization, and feedback. Yet they also raise pressing concerns about content quality, pedagogical coherence, and ethical implications. Preparing teachers to engage critically and responsibly with AI requires a rethinking of professional competencies. Beyond technical fluency, teachers need frameworks for ethical evaluation, strategies for integrating AI meaningfully into instruction, and opportunities to reflect on its impact. Teacher education must therefore expand its scope to include AI literacy as part of broader design capacity, ensuring that future educators can navigate these tools with discernment and pedagogical integrity.

In sum, the evaluation of digital curricular resources is not a peripheral task but a central dimension of mathematics teaching. It is where pedagogical vision,

technological literacy, and ethical judgment converge. By foregrounding this dimension, we can better support teachers in shaping meaningful, responsive, and high-quality learning experiences for their students.

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MATHEMATICS AS (AN INVISIBLE) THEORY IN THE WORLD OF CURRICULUM RESOURCES

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I draw on the history of mathematics and mathematics education, and on analyses of current national curriculum documents from Australia and Mexico to show that some of the curricular changes introduced when mathematics as an academic discipline was reconceptualised, remain consequential for mathematics education today. These once tentative changes removed key sources of meaning and explanation from school mathematics. In the analysed curricula, quantitative phenomena that once provided meaning for number are no longer present. Changes in mathematics theory resulted in mathematical incoherencies in classrooms, but these connections are rarely examined. I call for examining how mathematics shapes (and could shape) curricula today so that curricula could provide sound guidance to textbook designers and teachers.

CURRICULA, TEXTBOOKS, AND MATHEMATICS

The shape and form of mathematics education in classrooms results from many intricately related influences, including those that are pedagogical, educational, and mathematical. Within the ICMT community, we gather around artefacts that provide a window to these influences: textbooks. We position textbooks—that is, print and digital resources intended to support teaching and learning in schools—as the means to understand what is, and to shape what could be (Rezat et al., 2021). We refer to textbooks as curriculum resources because they uphold the broader curricular mandates and communicate the intended curriculum to textbook users.

My aim in this paper is to bring attention to mathematics, and specifically to exploring how mathematics is currently *organised* for teaching in top level curricular documents. This is of relevance to textbook researchers and developers because the organisation of mathematics in official curricula tends to be binding for the design and use of teaching resources. It thus tends to be consequential: how mathematics is organised for teaching shapes what mathematics can (and cannot) be taught and learned; and it shapes what mathematics comes to be for students. I refer to mathematics in this context as “mathematics for teaching”, or “mathematics-to-be-taught”, or “school mathematics”. Another clarification is essential here. When asking how mathematics is organised, I evoke questions about how we conceptualise mathematics which we aim to teach, not just how we select and order mathematical content. Alongside attending to the key mathematical notions that should be taught, we need to attend to their onto-epistemological grounding and the possible paths of their meaning creation.

I aim to convince the reader that, rather than the current organisation being logical, or stable, or the only one possible, there are legitimate options to consider when organising mathematics for teaching in curricula. I illustrate how the organisation that many curricula inherit today carries the legacy of the New Math era. This organisation leads to experiencing mathematics as a body of knowledge that lacks coherence from the perspective of the learner (Višňovská & Cortina, 2025), turning engagement into a struggle. Teaching students who doubt mathematics, or worse, who doubt themselves, makes the job of teaching mathematics *with and for understanding* prohibitively difficult. Yet, given that the organisation of mathematics for teaching is now habitual, it is rarely considered as a possible culprit in the well-documented difficulties.

Presenting an argument that weaves elements of current difficulties in teaching and learning, history of mathematics education, history of mathematics, current curricular analyses, and possible alternatives is a challenge (if not outright a bad idea). I attempt this by sharing several loosely related scenes: glimpses of studies and analyses. I tie these glimpses together into a hopefully plausible story I would like readers to consider. The context of this story is how number broadly, and fractions specifically, are conceptualised and taught, with the multitude of textbooks we encounter today.

ONCE UPON A TIME...

Scene 1: Mathematics as a science of quantity

Up to the eighteenth century, Aristotelian Realism—a view of mathematics as a “science of quantity”—was the dominant philosophical position (Franklin, 2009). Mathematics was considered a disciplined study strongly linked to the quantitative aspects of things “around us”, a study of something real. Mathematical objects (e.g., numbers) had their source and reference in reality from which they were *abstracted*, or which they *idealised*. Definitions of mathematical objects were constructive, describing how a new object can be devised from the previously existing objects (e.g., Otte, 2007). Magnitudes, amounts, and sizes were central to mathematics.

To a mathematics educator, the above description of mathematics may seem commonsensical, at minimum in how reality was a source of or motivation for mathematics, and in the arrival of a new mathematical idea from the organisation of the previous ones. These notions are well-aligned with what mathematics educators came to appreciate as the key tenets of ambitious mathematics teaching (Franke et al., 2007).

When a need emerges to describe what numbers are, such as for the sake of their understanding, mathematicians recognise the importance of their origin. For Davydov (1990, p. 196), number is “a particular case of the representation of a general relationship of quantities, where one of them is taken as a measure and is computing the other”. Bass (2018, p. 466), similarly, clarifies that number “arises from measuring one quantity by another, taken to be the ‘unit’: How ‘much’ (or many) of the unit is needed to constitute the given quantity?” In these statements, “quantities” exist in the physical, material world, and they are something that can be compared. As a result of

such comparison, a quantity can be the same as, or bigger than, or smaller than that to which it is compared. By treating one quantity as a reference unit, number emerges as a tool that makes further organisation of comparisons among multiple quantities possible. I will return to several points from these definitions: centrality of quantities and their multiplicative comparisons, and how these allow us to construe whole numbers and fractions as numbers of the same kind.

Mathematics of this kind, where quantities, measurement, and units play the central role in the mathematical development of number, served as a scholarly reference knowledge for teaching until fairly recently. In France, for instance, Reynaud's (1816/1821) treatise was used in teacher education as a reference for the meaning of mathematical objects and operations, and how these were to be understood and derived, until 1945 (Chambris & Visnovska, 2022; Harlé, 1987; Neyret, 1995). For instance, it guided teachers on multiplication as follows:

The purpose of the MULTIPLICATION is to calculate a number which is called PRODUCT. The latter is formed with a known number, called MULTIPLICAND, in a similar way to how a given number—called MULTIPLIER—is formed with the unit. Thus, to obtain the product, one has to operate on the multiplicand the same operations one would have operated on the unit to form the multiplier. (p. 10, stress in original)

The reader will note that (unlike the Cartesian product explanation) this explanation of multiplication holds for whole numbers and fractions alike.

Scene 2: Length measurement as a source of number

One way of deriving fractions from whole numbers while treating them as numbers of the same kind is by relying on linear measurement as one of the sources of number (another such source, much more common today, is counting). Specifically, learners would be brought to recognise whole numbers as expressing a comparison between a (suitable) measured length and a (measurement) unit, such as through the process of unit iteration (e.g., Cortina et al., 2014; Visnovska et al., 2025). The need for fractions then emerges when learners measure lengths that are not a whole-number multiple of the unit length. In response to such a need, the unit becomes the *reference unit* within the system, and multiplicative sub-units of measure are constructed in relation to this reference unit. For instance, a sub-unit of measure that corresponds to a half would be defined as being of such a length that when the reference unit is measured with it, it measures exactly two (see Figure 1, cf. Vale et al., 2019). Central to the meaning of a unit fraction construed in this way is its size relative to the size of the reference unit. This construction relies on a quantitative property: the same iteration of a greater (smaller) unit renders a greater (smaller) length (see e.g., Chambris in Watanabe et al., 2025).

Any common fraction length is construed as an iteration of the relevant sub-unit of measure. For example, length $7/6$ corresponds to seven iterations of a sub-unit that is $1/6$ as long as the reference unit. This multiple-step, multiplicative construction allows

one to reason about comparisons of a broad range of fractions, and pursue fraction equivalencies quantitatively, as expressing *neither greater, nor smaller*.

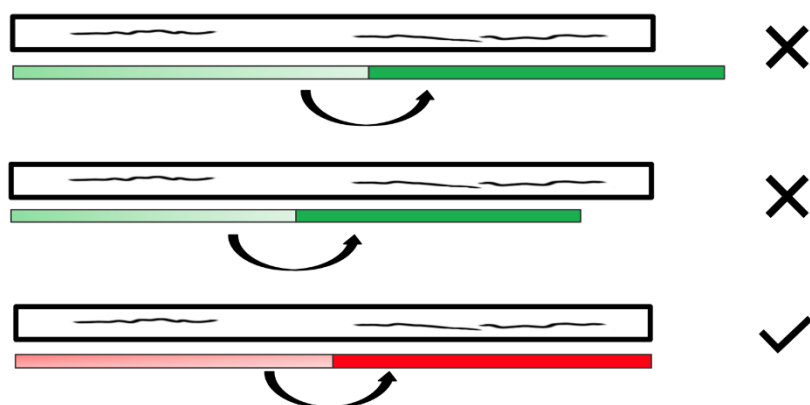


Figure 1: Construction of a half. Trial and error with an adjustment of the size of the iterated rod, building a sub-unit of measure that fulfils the measurement condition.

While trialling this approach to teaching fractions in different contexts with cautiously encouraging results (e.g., Cortina et al., 2014; Vale & Graven, 2018), we came to see fractions as being reasonably easy to teach and learn. The instructional sequence requires further development of extensions and links to other multiplicative and proportional notions, but approaching these notions through quantity is encouraging. The next scene queries “why” the introduction of fractions as a means to quantify continuous magnitudes might appear more seamless than its current alternatives.

Scene 3: Didactic transposition before and since modern mathematics

I draw on a Theory of Didactic Transposition (TDT, Chevallard, 2019) to highlight links between academic mathematics and mathematics-to-be-taught in schools. TDT posits that academic mathematics is taken as a reference knowledge in curricular decisions (about what mathematics is to be taught, and how this is to be organised) and that this in turn shapes what mathematics is taught and learned (Figure 2).



Figure 2: The process of didactic transposition.

In the quantitative approach to teaching fractions described above, the knowledge to be taught is derived from the “old” or “classical” scholarly mathematics theory, such as that described in Scene 1. The focus on quantity underpins the teaching where “the development of meaning, of the ‘existence’ of mathematical objects” (Thom, 1973, p. 202 cited in Otte, 2007) is in the fore. Alongside the constructive nature of definitions, mathematics conceived as a science of quantities provides an organisation of mathematical foundations that support the teaching and learning with understanding. With this scholarly reference knowledge, the school mathematics can mirror the scholarly processes of mathematical creation and use the forms of justification that are

recognisably mathematical. But the scholarly reference knowledge of today is not a “classical” mathematics theory.

During the 19th century, the theoretical ideas about the nature of mathematics and its objects changed, and as a result, mathematics became “more abstract and less ‘explanatory’” (Otte, 2007, p. 245). We know of these changes as a transformation of Euclidean axiomatics into set theory and formal axiomatics—an arrival of modern mathematics. In the pursuit of separating mathematics from reality, quantity lost its position as a foundational mathematical idea. Constructive definitions that previously provided explanations were replaced by definitions by abstraction.

With the change of scholarly reference knowledge, school mathematics lost a source of explanations and could no longer mirror the processes of mathematical creation, as these were now directly grounded in abstractions (Chambris & Visnovska, 2022). Within knowledge to be taught, following the reference knowledge, numbers were reimagined as sets. Given that sets have discrete elements, numbers were “transposed” into mathematics to be taught as counting. Accumulation of continuous quantity no longer represented number in scholarly mathematics. This rendered *counting* the most relevant—if not the sole—source of whole and later natural numbers. Inheriting a set theory definition of rational numbers, a fraction became a pair of whole numbers.

Where scholarly mathematics definitions and justifications could not be used due to their complexity, they were simplified, frequently to the point of becoming mathematically restricted and inadequate (e.g., the numerator-denominator definition of fraction not accounting for fractions greater than one; the Cartesian product definition not accounting for fraction multiplication). Some mathematical justifications were replaced by calculational procedures, and new mathematical objects had to be “parachuted onto the learner from above” (Sfard, 2003, p. 367), as they were not possible to build from already established ideas (e.g., fractions could not be derived from counting).

The change in the affordances that the new scholarly knowledge has for supporting school mathematics, such as in terms of the provision of explanations and mathematical meaning, has been considerable. As we document elsewhere (Chambris & Visnovska, 2022), multiple changes were introduced in curricula and in textbooks subsequently, aiming to overcome some of the difficulties that resulted from the lack of meaning available for some mathematical objects the students are expected to “understand”. However, the reference mathematical theory is rarely analysed as a possible source of these difficulties, and the remedies have not been systematic.

Scene 4: Curricula today are derived from Modern mathematics

The problematic nature of the changes introduced with the didactic transposition from modern mathematics, known as the New Math reform changes, is well acknowledged, but mostly it is believed that the problems introduced at the time are no longer present in current curricula (Kilpatrick, 2012). As we sought viable ways to introduce fractions in schools so that students come to see fractions and whole numbers as numbers of the

same kind, remnants of the New Math changes became visible to us. We thus set out to document quantitative resources that remained available in current primary national curricula (aimed at 5 to 12 years old students) of Australia and Mexico (see Višňovská & Cortina, 2025), such as interpretations of whole numbers in relation to the measurement of continuous quantities. We asked: To what extent is measurement present *in the development of number*? What quantities are implicated in this development? We developed a coding scheme to categorise curricular descriptions with respect to the extent that they ventured beyond counting in the meaning of number, tasks, examples, and representational tools used in relation to numbers (i.e., counting number, counting number plus, potentially continuous, continuous, and measurement literacy, see Table 1).

Code	Description	Example
Counting Number	Overt references to order, counting, discrete quantity	Naming discrete things to be counted or used in tasks
Counting Number Plus	Instances of focus on counting, which could inadvertently facilitate a quantitative sense of discrete collections	Size of the arrangement with dots in subitising activities, ten frames, frequency graphs in statistics, calendars as a means to compare duration in days, months
Potentially Continuous	Potential to position numbers as accounting for quantitative differences	Claims/tools where interpretation of continuous quantity can be made, even though it is not suggested (e.g., number line)
Continuous	Overt references to continuous quantity in relation to number	Quantification of comparisons of relative size
Measurement Literacy	Measurement without a link to the meaning of number	Named attributes to be measured, “uniform units” (sic)

Table 1: Coding scheme.

While Australia and Mexico have more differences than similarities in their histories, systems, or socio-economic set up, the results of the analyses were eerily similar. They confirmed that the knowledge to be taught can indeed be well understood when interpreted as being a result of the didactic transposition from modern mathematics: it exhibited the varied types of limitations sketched in Scene 3.

In both curricula, Measurement and Number are separate areas of teaching (strands). Measurement curriculum is oriented towards measurement literacy, such as the introduction of the measurement systems, standardised units, and tools, including how the tools are to be used to measure accurately, or how to convert units. Measurement

curriculum does not include what we refer to as *mathematics of measurement*, that is, how qualitative attributes of things, such as length, can be quantified, or how an accumulation of a continuous magnitude can serve as a source of number.

Both curricula conceptualise number in the early years solely as a counting number: For all practical purposes, they equate number with the cardinality of discrete collections of objects. In Australia, early years curriculum lists many specific examples of objects that can be counted (e.g., counters, dots, dice, pencils, tigers, bolts, windows in a train carriage). In contrast, there is one description with a reference to a continuous quantity in early years Number strand in Australia: a lump of playdough and a cup of water suggested to indicate half when fractions are introduced in Year 2 (there are two more examples in the Measurement strand and none in the Mexican curriculum). The word “number” is used to refer to whole numbers, even after fractions and decimals are introduced. Given that fractions cannot be meaningfully conceptualised as numbers through counting when they are introduced (Bass, 2018), neither curriculum provides a path towards the emergence of fractions as a mathematically coherent extension of students’ prior notion of number.

In general, mathematics is not introduced as something that students can produce by organising and mathematising previously encountered mathematical and non-mathematical situations and relations. It is presented as something to be given to students ‘ready-made’, something that can later be applied. Students are then envisioned to model the world with mathematics: “mathematics can be used to investigate things they [students] are curious about, to solve practical problems and model everyday situations” (Year 2). Both curricula encourage teachers to guide students towards recognising mathematics as a tool for understanding their everyday experiences, which would, presumably, not be understandable without mathematics. Neither curriculum clarifies where this kind of essential, ready-made mathematics came from.

To summarise, the analysed curricula do not offer mathematical resources from which it would be possible to build fractions as a coherent—from the students’ point of view—mathematical extension of whole numbers. The curricular organisation committed to presenting ‘ready-made’, abstract mathematical objects does not guide teachers and textbook designers towards instruction that students could view as mathematically coherent, where ideas emerge from previous ideas, and are connected to one another. While we have only analysed curricula of two countries, the identified ‘symptoms’ are likely to resonate more broadly. Given that modern mathematics is frequently taken as the scholarly reference in the construction of current curricula, the story outlined here could be plausible beyond Australia and Mexico.

MATHEMATICS AS (AN INVISIBLE) THEORY: WHICH MATHEMATICS?

Let me now connect the issues illustrated in the four scenes above. My intent for the scenes was to:

- Evoke a sense in which whole numbers and fractions are numbers of the same kind (an example of numbers as accounts of an accumulation of length).
- Suggest that fractions can be introduced with coherence from the students' point of view, and thus easy (if mathematics is organised well for teaching).
- Provide empirical data to show that number is currently not introduced with coherence in some curricula (examples of Australia and Mexico).
- Create an awareness of why fractions are currently hard to teach and learn (modern mathematics is not an adequate reference theory for school mathematics) and that this status quo has a historical reason.

The fact that the story I wanted to bring for consideration required this kind of build-up is perhaps a good testament to the durability and invisibility of the effects that the changes introduced through New Math reform continue to have in current curricula.

The change in mathematics towards set theoretical abstraction benefited the discipline. But thanks to the intentional separation from reality, modern mathematics can only ground school mathematics in abstraction (Freudenthal, 1979). The path of parachuting abstractions onto students and subsequently applying these to problems to demonstrate the utility of those abstractions has now been travelled for many decades. We should all be familiar with the problems this 'abstraction and application' approach introduces in terms of students' lack of enthusiasm to engage with mathematics.

Procedures and definitions that lack meaningful explanations and have been simplified to the point of losing their mathematical correctness (e.g., part-whole or numerator-denominator definition of fraction according to a strict application of which $7/5$ is not a fraction) are consequential for whether students come to perceive mathematics as a logical network of connected ideas. They are also consequential for students' perceptions of themselves as people who possess (or lack) what it takes to be good at mathematics. Taking incorrect definitions to their logical consequences (e.g., protesting that it is impossible to take 7 parts of a pizza that was cut into only 5 parts) often results in positioning the student—not the definition—as being incorrect in the classroom. Classroom interactions of this kind invite students to perceive their own mathematical reasoning as being inexplicably deficient. Such interactions might bring students to resent mathematics and disengage from it (Boaler, 2016). Rancière (1987/1991) describes situations of this kind in terms of students being *stultified*, or made stupid, by the system of schooling.

Undoing the systemic stultification of students requires a curriculum where mathematics content is organised for coherence from the learner's point of view, and where mathematical definitions are taken seriously. Perhaps it is time to consider whether and how a mathematical theory in which quantity is central could once again guide classroom mathematical developments in primary and middle school mathematics. The need for mathematics of abstractions would need to be gradually built in classrooms before a reorganisation toward such mathematics would be introduced.

The turn towards modern mathematics created, for the first time, an educational space in which the foundations of mathematicians' mathematics could no longer provide adequate foundations for school mathematics. It is understandable that designers of national curricular documents are hesitant to depart from the current academic mathematics as a reference for mathematics to be taught in school. But our schools urgently need curricula (and textbooks) that would allow students to encounter themselves as mathematically competent, especially when they engage with mathematical definitions they were given, in a logical way. It is my hope that the ideas in this paper help elicit discussions about a more overtly visible mathematics theory capable of supporting curricular designs.

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Part II
Panel discussion and Symposium

THEORIZING DIGITAL CURRICULUM RESOURCES: RECENT CASES FROM RESEARCH IN MATHEMATICS EDUCATION

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The use of digital tools has been common in schools for around 50 years and has affected many aspects of mathematics teaching and learning. The development and use of digital textbooks and similar forms of digital material is more recent. As the tools, digital curriculum resources come in many forms, for instance, text and tools may be integrated. Theorizing is needed for research to capture, with precision, the essential features of digital text for mathematics education, as well as their impact on teaching and learning. In this report of our panel at ICMT5, we present the examples of recent research and the corresponding theorizing that was the starting point for the panel, along with reflections on some of the comments.

INTRODUCTION

The panel set out to address the following questions:

According to current research, what are the new potentials of digital curriculum resources for mathematics teaching and learning? How can such potential be theorized?

What appears here as two questions, is really one. In fact, the “new potentials for mathematics teaching and learning” could only be subject to research if “theorized”, that is, specified with reference to a theoretical framework on mathematics teaching and learning, so that what is new (and perhaps not so new) could be described in precise and shared forms, and recognized in actual observations or experimentation based on a corresponding methodology. At the same time, the rapid and ongoing technological development often appears to call for new or at least revised theoretical notions in order to capture its “newness”.

In an effort to delimit what is meant by “digital curriculum resource”, as opposed to “instructional technology”, Pepin and Gueudet (2020) pointed to the implied “attention ... to cover (all or part of) a curriculum specification”. The digital textbook is, of course, a prime example, but it should be acknowledged that the clear boundaries between “computer tools” and “texts” are thoroughly blurred in many cases. One example is when applets and other tools are integrated into an online textbook, drawing directly on some online facility for computation, visualization, or generation of text.

Another is when a software package (like the computer algebra system “Maple”) includes extensive packages and user manuals designed for use in teaching at certain institutions or even courses. While the notion of digital curriculum resource remains meaningful, the attention to coverage of curricula can lie both in the design, selection, and use of a given resource, and is not immanent to the resource as an isolated object.

The attention to teaching and learning, according to some curricula, lies more in the actual *use* of the resource than in the resource itself.

One can’t spend much time in technology-in-education circles without running into phrases like “the effects of calculators on children’s learning of arithmetic” or “the impact of computer algebra systems on the traditional calculus curriculum”. But no technological innovation, on its own, has any “effects”, still less any “impact”. Things don’t have effects: actions do. What matters isn’t what technology there is, but the way that technology is used. (Ramsden, 1997, p. 419).

This also applies to “new potentials” for teaching and learning that we could associate with the resource.

The same can be said about digital resources. New potential for teaching and learning lies not in the resource (e.g., a digital textbook) itself, but in what use is made of it in the teaching and learning of mathematics; realized potential can be more exactly located in the learning which, in some sense, results from the use. So, to investigate the potential of resource use beyond theoretical possibilities requires a thorough specification of the potential in relation to teaching and learning mathematics – a framework for mathematics education (mathematics didactics) research which allows us to formulate and investigate hypotheses about the potentials.

A simple classification of didactical functions of technology was proposed by Winsløw et al. (2024), based on the media-milieu dialectic and other elements of the anthropological theory of the didactic (ATD) (cf. Chevallard, 2019). A media is here understood as an object meant to instruct, to convey information or answers directly; a book is indeed a standard example. A milieu is a system of objects used to experiment or inquire about something; in teaching, these are often set up for students by the teacher. Certain technological environments (like a GeoGebra sheet or a calculator) are sometimes used as a milieu in itself, to investigate some mathematical problem. Both milieus and media may be *produced* by learners, rather than merely used passively; for instance, internet forums like *mathematics stack exchange* allow students to produce media (answers or instructions) corresponding to other students’ questions.

	Passive use	Productive use
Technology as milieu		
Technology as media		

Figure 1: Uses of technology in mathematics education, adapted from Winsløw et al. (2024).

This classification – summarized in Figure 1 - was set up to analyse the recent research literature on technology use in university mathematics education, which revealed a dominant focus on the diagonal combinations (marked grey in the figure). There is comparably little research on university mathematics students’ production of digital milieus (like student coding of a procedure to investigate some algorithm) and students’ passive use of media (like consulting online resources) in the course of their university study of mathematics.

This first and relatively simple example of a theoretical framework could be used to propose and investigate specific potentials of digital objects for teaching and learning. In these terms, it is clear that students’ uses of digital curriculum resources need not to be confined to the category of passive media use. As we shall now see, the contributions presented in the panel emphasize more “active” and “inquiry” oriented uses which, in various ways, extend beyond the lower left corner of Figure 1, and thus beyond the default use of classical textbooks, videos, or search engines. More specifically related to the second part of the research questions, each of the following four contributions introduces more sensitive and specialized theoretical tools, depending on the objects considered.

RADMEHR: THEORIZING LEARNING OPPORTUNITIES IN YOUTUBE EDUCATIONAL RESOURCES

Previous research has identified YouTube as a frequently utilized resource by undergraduate students to support their mathematics learning for a variety of purposes, including the recall and understanding of mathematical concepts and problem-solving strategies (e.g., Aguilar & Esparza Puga, 2020; Kanwal, 2020). Nevertheless, relatively few studies have examined the specific learning opportunities afforded by such resources. Within this limited body of work, various theoretical perspectives have been applied: some grounded in constructivist traditions, such as APOS theory and Tall’s three worlds of mathematics (Radmehr, 2024a), and others informed by socio-cultural perspectives, including commognition and positioning theory (Radmehr & Turgut, 2024; Radmehr et al., 2024). In addition, ATD, which is situated between these two strands, has also been utilized (Radmehr, 2024b). This section focuses on two illustrative frameworks—ATD and FAMT (Framework of Advanced Mathematical Thinking, a combination of APOS theory and Tall’s three worlds of mathematics)—to theorize how educational YouTube resources can be conceptualized and to highlight the kinds of learning opportunities they may afford for the teaching and learning of mathematics.

The contributions of ATD

The theory of didactic transposition, a sub-theory within ATD, provides a useful starting point for situating YouTube educational resources. In brief, this theory describes how a body of knowledge is transposed from its original creation by scholars to its subsequent teaching and learning within educational institutions, as illustrated in Figure 2 (Chevallard & Bosch, 2020).

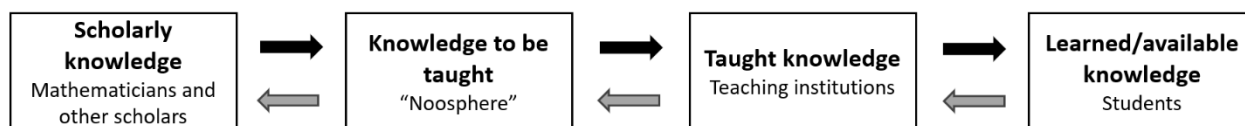


Figure 2: A Diagram of the didactic transposition process (adapted from Chevallard & Bosch, 2020, p. 214).

When scholarly knowledge is transposed from its original context into an educational setting, it undergoes a process of adaptation aimed at reconstructing an environment in which the knowledge can be made “teachable, meaningful, and useful” (Chevallard & Bosch, 2020, p. 214). This process involves multiple actors, collectively referred to as the noosphere—“those who ‘think’ about teaching”—including mathematicians, teachers, lecturers, and curriculum designers (Chevallard & Bosch, 2020, p. 214). Their task is to preserve the essential features of the scholarly knowledge while simultaneously negotiating and responding to the demands that society places on the educational system (Chevallard & Bosch, 2020). When investigating the didactic transposition of a body of knowledge, textbooks—whether physical or digital—are regarded as constituting the “knowledge to be taught.” In contrast, YouTube educational channels can be conceptualized as teaching institutions, with the knowledge disseminated through their resources classified as “taught knowledge.” Consequently, such resources may be situated within the broader framework of the didactic transposition process and compared with other stages (Radmehr, 2024b). For instance, Radmehr (2024b) investigated the opportunities for learning the Fundamental Theorem of Calculus offered by a prominent calculus textbook and compared them with two highly viewed YouTube videos on the same topic.

The comparison of educational resources in ATD is typically conducted through praxeological analysis, a tool within this theory that is used to model knowledge. A praxeology models human activity as a quadruplet $[T/\tau/\theta/\Theta]$. It comprises a type of task (T) to be carried out, a technique (τ) that enables task completion, a rationale (θ) that justifies and explains the technique, and a theory (Θ) that supports and frames the rationale. The first pair $[T/\tau]$ forms the praxis block, commonly referred to as know-how, while the latter pair $[\theta/\Theta]$ makes up the logos block, which accounts for and legitimizes the actions undertaken (Strømskag & Chevallard, 2022). Therefore, in analysing educational resources, including YouTube videos, one can examine what types of tasks are introduced and how those tasks are solved, identifying the corresponding techniques, and also investigate how the techniques are justified—i.e., identifying the underlying logos. For instance, Radmehr (2024b), in his praxeological analysis of YouTube resources on the Fundamental Theorem of Calculus – Part 1, found that these resources vary in focus: some facilitate the learning of techniques, while others emphasize the logos block.

The contributions of APOS theory and Tall’s three worlds of Mathematics

APOS theory and Tall’s theory of three worlds of mathematics offer another set of frameworks that can be utilized to investigate learning opportunities in educational

YouTube resources (Radmehr, 2024a). In short, Tall’s theory of the three worlds of mathematics explains how mathematical thinking develops across three worlds: the embodied, the proceptual-symbolic, and the formal. The embodied world is grounded in human sensory experiences and visualizations of real or imagined objects. The proceptual-symbolic world involves using symbols to carry out and think about mathematical operations, allowing flexibility between processes and concepts. The formal world builds on axioms, definitions, and proofs, emphasizing abstract structures beyond immediate contexts. Together, these worlds can enable learners to progress from concrete experiences to abstract reasoning in mathematics (Tall, 2013). Complementing this perspective, APOS theory (Action–Process–Object–Schema) provides a constructivist framework that models how individuals learn mathematical concepts by describing the mental structures they build. Learning begins with actions, which are explicit, step-by-step transformations of known objects often guided by external instructions. Through repetition and reflection, these actions are interiorized into processes, allowing learners to imagine and manipulate steps mentally. When a process is understood as a whole and treated as an entity that can itself be transformed, it becomes an object. Finally, actions, processes, and objects are organized into broader schemas, which integrate knowledge for problem solving and further abstraction (Arnon et al., 2014).

FAMT combines APOS theory and Tall’s three worlds of mathematics, based on the assumption that “it is possible to develop action, process, and object within each embodied, proceptual-symbolic, and formal world” (Radmehr, 2024a, p. 5). A common way of applying the FAMT is through a table that aligns the APOS constructs—action, process, and object—with the three worlds, as shown in Figure 3.

	Embodied	Preceptual-symbolic	Formal
Action			
Process			
Object			

Figure 3: A table commonly constructed in studies utilizing the Framework of Advanced Mathematical Thinking (FAMT).

Mathematics educators who have utilized the FAMT framework investigate, among other things, the extent to which tasks or teaching more broadly facilitate or encourage students to move between the three mathematical worlds (e.g., Stewart, 2018). Such analyses can also be conducted on educational YouTube resources. Recently, Radmehr (2024a) employed FAMT to examine how YouTube videos might support students’ learning of eigentheory in linear algebra. A rubric grounded in FAMT was developed to operationalize which acts of a content creator could be considered actions, processes, or objects within the embodied, proceptual-symbolic, and formal worlds in general, and then specific examples were provided from eigentheory. This rubric was used to analyze the content of two widely viewed YouTube videos. The findings suggest that

the two videos emphasize different worlds: one focuses more on the embodied world, while the other centers on the proceptual-symbolic world, thereby addressing students with varying levels of linear algebra understanding. These resources provide opportunities to engage with eigentheory in both the embodied and proceptual-symbolic worlds, with a particular emphasis on action, but they devote limited attention to the formal world.

Possible directions for future research

Research on YouTube educational resources is still in its early stages, and further studies are needed to theorize the learning opportunities they offer, particularly from socio-cultural perspectives. Moreover, given their widespread use among undergraduate students in universities, mathematics educators must identify key design principles for developing YouTube resources that foster meaningful mathematical understanding and communicate these principles to stakeholders involved in creating such resources. Another important direction is to investigate how YouTube resources can be integrated into mathematics teaching and teacher education. For instance, how might they be incorporated into pre-service teacher training or in-service professional development programs? In one modeling textbook for teachers (Maaß et al., 2018), for example, a YouTube link (<https://www.youtube.com/watch?v=BlvKWEvKSi8>) discusses how to transform a standard textbook problem into a more realistic one. I have found this resource valuable for discussion with my pre-service teacher students, yet the central question remains: how can such resources be integrated effectively—beyond simply playing them in class or sharing them as supplementary material? Further systematic studies are needed to examine how digital resources can be meaningfully embedded in both mathematics teaching and teacher education.

LI: GENERATIVE AI-SUPPORTED INTERDISCIPLINARY RESOURCES

Interdisciplinary or STEM/STEAM content has been emphasized as important in mathematics education, both in research (e.g., Li et al., 2025) and in recent curriculum reforms (e.g., Ministry of Education of the People's Republic of China [MOE], 2022). A comparative analysis of textbooks from China and the United States indicates that current STEM content in mathematics textbooks remains relatively limited and largely confined to basic levels—contexts drawn from science, technology, or engineering are used merely as backgrounds, with only mathematics content knowledge (Li et al., 2025). The scarcity of enriched interdisciplinary content in textbooks poses continuing challenges for teachers—especially when the curriculum standards, such as those of the MOE (2022), now require approximately 10% of instructional time to be devoted to interdisciplinary learning. Yet, designing high-quality interdisciplinary curriculum resources remains a persistent challenge.

With the rapid advancement of Generative Artificial Intelligence (Gen-AI), new possibilities have emerged for addressing these challenges. This section presents two exploratory cases demonstrating how a Gen-AI tool (Kimi AI, <https://kimi.moonshot.cn/>) was employed to design and implement (1) a mathematics–

arts interdisciplinary module (Zhao & Li, 2025a), and (2) a mathematics-engineering interdisciplinary module (Zhao & Li, 2025b), and also highlights theoretical considerations and future research directions in Gen-AI-supported curriculum design.

Mathematics–arts (M-A) interdisciplinary resources

Zhao and Li (2025a) report a case involving one instructor and two seventh-grade students collaborating on the design of a mathematics–arts module, “The Golden Ratio: A Harmonious Chapter between Mathematics and Music”. Based on analyses of human–Gen-AI interactions and participants’ reflections, the authors identified seven roles that Gen-AI played throughout the design process, mainly positive and neutral. The instructor employed Gen-AI tools for (a) selecting topics, (b) formulating teaching objectives, (c) pre-structuring activities, and (d) generating tests. The two students engaged with Gen-AI to (a) collect musical scores, (b) refine their research plans, and (c) generate PowerPoint presentation slides.

The study illuminates both the affordances and limitations of integrating Gen-AI into the design of digital interdisciplinary curriculum resources. On the positive side, Gen-AI functioned as a productive “idea bank” and an efficient information-retrieval assistant, accelerating preparation and enriching the design process. However, consistent challenges arose: Gen-AI-generated materials often contained mathematical inaccuracies and repetitive content with insufficiently individualized elements, which often failed to address students’ specific learning needs (see also Korkmaz Guler et al., 2024). These issues reaffirm the irreplaceable role of human educators in adapting and refining Gen-AI-generated content to ensure alignment with students’ prior knowledge, unique needs, and curricular goals.

It should also be noted that the generalizability of this case study is limited by its small sample—one instructor and two students. Important questions, therefore, remain: How might Gen-AI-supported interdisciplinary resources unfold in larger student groups? Would outcomes differ across domains—for example, between mathematics–arts, mathematics–engineering, or mathematics–science modules? The following exploration addresses these questions.

Mathematics–engineering (M-E) interdisciplinary resources

In a recent exploration (Zhao & Li, 2025b), twelve eighth-grade students—organized into six pairs—participated in a Gen-AI-supported mathematics–engineering interdisciplinary module titled “Fuel Consumption: The Application of Mathematics in Energy-Efficient Driving,” followed by engagement with the aforementioned mathematics–arts module. In contrast to the previous study, the instructional team in this case comprised one lead instructor and three teaching assistants, who provided continuous real-time guidance and demonstrations of auxiliary operations throughout the module.

Students highlighted the positive roles of Gen-AI in (1) information retrieval and (2) slide generation, while offering a neutral evaluation of Gen-AI’s role in research plan

generation and refinement. Participants universally praised the richness and extensiveness of interdisciplinary knowledge provided by Gen-AI. However, their interactions with Gen-AI were often limited to superficial inquiries about basic concepts—without deeper exploration of underlying principles or meaningful real-world connections. Thus, the potential of Gen-AI to facilitate students' learning of interdisciplinary knowledge might be limited. Furthermore, some students expressed diminished ownership over their work and a lack of personal achievement, for example, “the written report doesn’t feel like my own achievement—it all belongs to Gen-AI.”

Interestingly, students reported higher trust in Gen-AI within the M-E module than in the M-A module. They attributed this difference to the nature of disciplines: Gen-AI was viewed as a reliable provider of objective, factual information suitable for science-oriented inquiry (e.g., M-E module), whereas humanities-based explorations—such as music—require creativity, emotion, and interpretive imagination that current Gen-AI systems struggle to emulate (M-A module). This contrast suggests that students' trust in Gen-AI varies across disciplinary domains (M-E and M-A), reflecting differing expectations for digital interdisciplinary resources.

Potentials to be theorized and future directions

Building on the model of reform implementation mediated through curriculum resources into practice (Rezat et al., 2021), the integration of Gen-AI-supported digital curriculum resources (DCR) introduces new dimensions to be examined and theorized (see Figure 4).

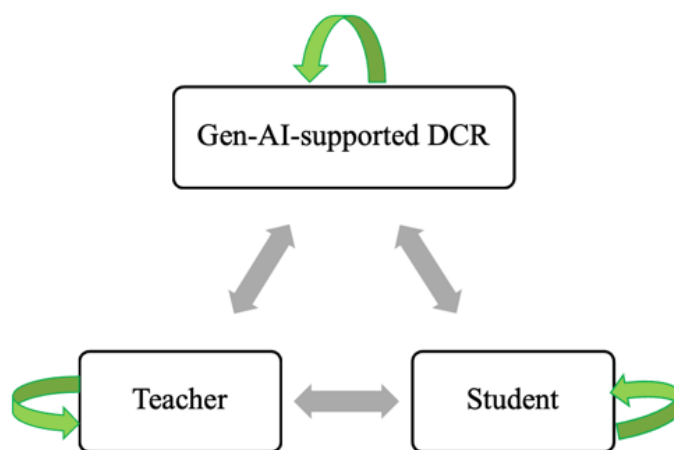


Figure 4: Model of human-Gen-AI interaction.

Focusing on the Gen-AI-supported DCR perspective, the Gen-AI tool employed in both case studies was a general-purpose model rather than one specifically designed for interdisciplinary or mathematics education. A similar pattern is observed in recent research, where most studies exploring Gen-AI in mathematics education have used generic large language models such as ChatGPT (e.g., Pepin et al., 2025; Walkington, 2025). The development and adoption of domain-specific tools, tailored explicitly for mathematics or STEAM education (e.g., ChatGPT-4 integrated with the

Wolfram Alpha plugin), could mitigate common issues such as mathematical inaccuracies, ensure computational correctness, enhance conceptual precision, and provide more reliable support for assessment design. More specialized large language models could produce more efficient and effective outputs for mathematics teachers.

From the student's perspective, students' ways of learning in Gen-AI-supported interdisciplinary environments appear to change substantially. While some scholars have argued that students' ways to learn mathematics never change, the two cases reported here suggest otherwise. Students increasingly position Gen-AI as their primary source of knowledge, with teachers and textbooks traditionally playing important roles. This shift reduced traditional student-teacher interaction while intensifying student-Gen-AI interaction, thereby transforming classroom dynamics. Such a transformation underscores the urgent need for students to cultivate critical engagement with Gen-AI-generated outputs and for teachers (or teaching assistants) to provide explicit scaffolding and ethical guidance throughout the process. Moreover, several students described a diminished sense of personal accomplishment, implying that Gen-AI may influence how learners experience and value their own achievements. Addressing these concerns requires sustained efforts to design pedagogical strategies that foster reflective learning and enhance the intrinsic satisfaction of learning in Gen-AI-supported interdisciplinary environments.

Regarding the teacher's perspective, teachers' professional expertise remains indispensable in adapting Gen-AI-supported interdisciplinary resources (also DCR) to align with students' prior knowledge, learning needs, and curricular goals (see also Trgalová & Tabach, 2024). However, designing high-quality learning objectives and activity structures typically requires multiple iterative interactions with Gen-AI, highlighting the need for effective prompting strategies and targeted professional development to help teachers engage productively in the design of Gen-AI-supported interdisciplinary resources. Additionally, the teacher's professional role was somewhat diminished during the implementation of Gen-AI-supported interdisciplinary modules. As class sizes increase, new questions also arise regarding when and how teachers should provide explicit scaffolding and ethical guidance across different phases of Gen-AI-supported modules—an area that warrants further exploration.

Beyond the individual perspectives of student, teacher, and Gen-AI, there are student-student, student-teacher, teacher-teacher, student-Gen-AI, and teacher-Gen-AI aspects that all deserve further analysis. Each of these interactions undergoes fundamental shifts in a Gen-AI-supported learning environment. Analysing these shifting interactional patterns is essential to fully understand and realize the pedagogical potential of Gen-AI-supported interdisciplinary curriculum resources.

Looking ahead, while research has begun to explore AI literacy and users' Gen-AI perceptions (e.g., Shoufan, 2023), further studies are needed to understand how such perceptions shape actual adoption and effective use of Gen-AI, particularly in DCR design. Furthermore, some key questions remain for mathematics educators and

mathematics education researchers to explore: How could teachers prepare for shifts in students' learning behaviours in Gen-AI-supported learning environments? And how could research methodologies be adapted to the rapid evolution of AI technologies? Addressing these questions will require sustained, collaborative inquiry across educational contexts and research communities.

SWIDAN: INSTRUMENTAL POTENTIAL OF DIGITAL TEXTBOOKS

The rise of digital curriculum resources has brought significant challenges and opportunities to the way mathematics education researchers conceptualize the textbook (Pepin et al., 2017). For decades, the printed textbook has been seen as the central artifact of classroom activity, providing a stable structure for teaching practices and guiding students' engagement with mathematical ideas. When textbooks become digital, however, their role changes profoundly. A digital textbook is not simply a digitized version of a printed book. Rather, it is a dynamic platform that integrates multiple tools: interactive applets, embedded videos, artificial intelligence-based feedback, assessment systems, and collaborative spaces. In this sense, the digital textbook occupies a dual role. On the one hand, it is a digital tool that teachers and students can use directly in the classroom. On the other hand, it is a platform for orchestration, where multiple resources are arranged, sequenced, and mobilized. This hybridity requires us to rethink the theoretical frameworks we use to understand the instrumental potential of textbooks in the digital age.

In this section, I argue that two well-established frameworks, Instrumental Genesis (Trouche, 2003) and Instrumental Orchestration (Drijvers et al., 2010), can be adapted to theorize digital textbooks and their use. Traditionally, these frameworks have been used to study the use of single tools, such as graphing calculators, dynamic geometry systems, or the design and use of (digital) textbooks. When applied to digital textbooks, however, they allow us to highlight not only the interaction between teachers, students, and digital resources, but also the wider ecology in which these interactions are situated. My argument unfolds in two steps. First, I explain how the classical triad model of teacher, student, and tool can be adapted to capture the instrumental potential of digital textbooks. Second, I extend this model into a pyramidal model that incorporates publishers and researchers as additional actors in the orchestration of digital textbooks.

The Triad Model: Teacher, Student, and Digital textbook

Instrumental Orchestration, as developed by Drijvers et al. (2010), offers a useful starting point for theorizing the role of digital textbooks. It emphasizes that teachers do not simply “use” digital tools but orchestrate them: they configure didactical environments, exploit them for pedagogical purposes, and perform with them in classroom interaction. At the same time, Instrumental Genesis highlights the student perspective, focusing on the ways students adapt to the constraints of a tool (instrumentation) and repurpose it for their own goals (instrumentalization).

Together, these frameworks form the basis of the triad model. In this model, the teacher, the student, and the digital textbook form three vertices of a triangle. Each relation in the triad captures a distinct theoretical process. The teacher–digital textbook relation can be conceptualized through Instrumental Orchestration, as the teacher configures, adapts, and enacts orchestrations with the digital textbook. The student–digital textbook relation can be conceptualized through Instrumental Genesis, as the student develops ways of using the resource, appropriating its features for inquiry, exploration, and problem solving. Finally, the teacher–student relation remains central, as orchestrations are performed in interaction and students’ instrumental activity prompts new instructional responses.

The strength of the triad model lies in its ability to highlight the relational nature of classroom activity. No element operates in isolation: the teacher influences how students engage with the digital textbook, students’ actions shape teachers’ orchestrations, and the digital textbook mediates the possibilities available to both. This interplay is visualized in Figure 5, which places teacher, student, and digital textbook at the vertices of a triangle, with arrows indicating bidirectional influences. The figure underscores the equal importance of each element and the dynamic interactions that occur between them.



Figure 5: The triad model capturing the instrumental potential of digital textbooks.

Within this triad, digital textbooks reveal distinct instrumental potentials. For teachers, they provide opportunities to configure flexible digital environments, to exploit them for diverse pedagogical purposes, and to perform dynamically during lessons while drawing on student interaction data. For students, they create conditions for personalized trajectories, inquiry-based exploration, collaborative engagement, and strategic use of feedback. These dual perspectives capture how digital textbooks function as mediating environments for both teaching and learning.

Extending the Model: From Triad to Pyramid

While the triad model captures much of the instrumental potential of digital textbooks, it is limited to the classroom level. In practice, digital textbooks are shaped by actors outside the classroom, most notably publishers (here, meant to include authors) and researchers.

Publishers play a central role as *meta-orchestrators*. They design the architecture of the digital textbook, determine how content can be customized, decide what types of feedback are provided, and establish the ways in which data is collected and shared. These design choices create the very conditions under which orchestration and genesis can occur. Teachers and students may adapt the digital textbook in practice, but the possibilities available to them are structured by the publisher's design. Researchers, meanwhile, contribute by developing theoretical frameworks such as Instrumental Genesis and Instrumental Orchestration, by studying how digital textbooks are used in classrooms, and by providing feedback that informs design decisions. Their role is not to orchestrate directly but to analyze, interpret, and enrich the ecosystem in which orchestration occurs.

To represent this expanded ecology, I propose moving from a triangular to a *pyramidal model*. At the base of the pyramid are teachers and students, engaged with the digital textbook through orchestration and genesis. The digital textbook itself remains central at this base, mediating their interaction. At the apex of the pyramid are publishers, whose design decisions orchestrate the environment at a systemic level. Surrounding the pyramid are researchers, whose theoretical and empirical contributions provide a framework for understanding and improving the system. This structure is visualized in Figure 6, where arrows run vertically between the base and the apex, indicating the ways publishers shape classroom activity and the ways feedback from practice can inform design. Arrows around the pyramid represent the contributions of researchers, who offer perspectives that cut across levels.

The pyramidal model highlights that the study of the instrumental potential of digital textbooks cannot be reduced to classroom dynamics alone. Instead, it is distributed across a broader system in which design, practice, and research are interwoven. Teachers orchestrate within the possibilities created by publishers, students engage in genesis processes within the constraints and affordances of the platform, publishers respond to feedback from the field, and researchers interpret and theorize these interactions.

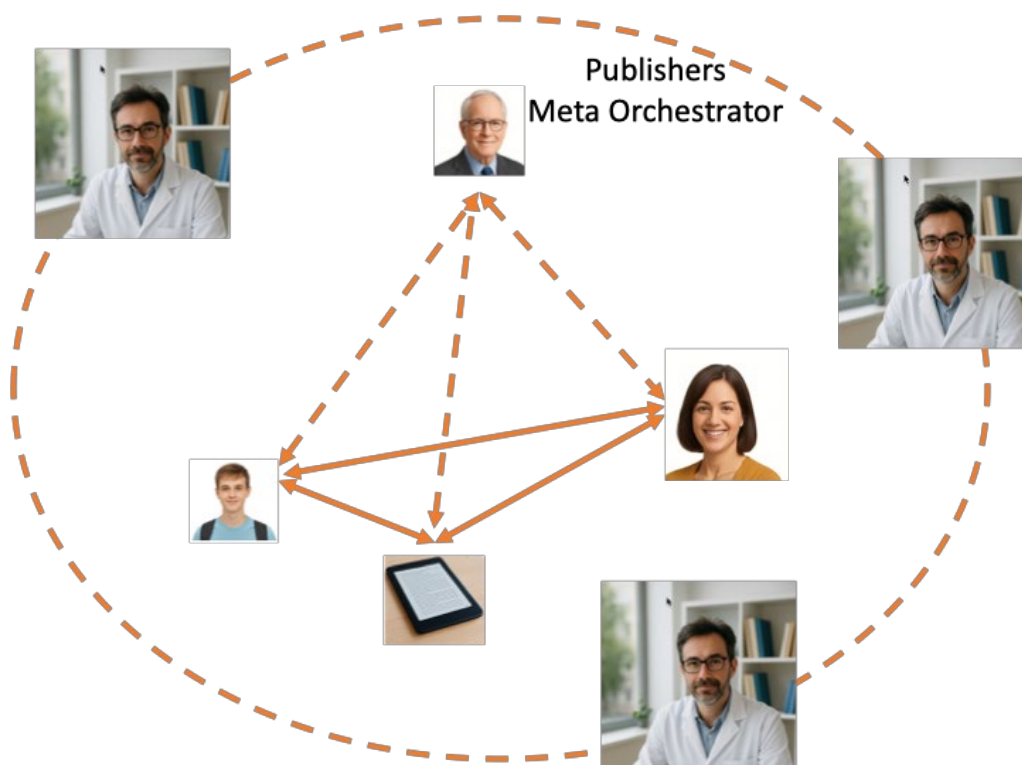


Figure 6: The pyramidal model related to the orchestration of digital textbooks.

Implications and Conclusion

Adopting this expanded perspective has several implications. For research, it requires an extension beyond the classroom of the study of Instrumental Genesis and Instrumental Orchestration. These frameworks must be refined to capture orchestration and genesis not only as local practices but also as processes influenced by systemic and design-level actors. For teaching, it reframes the digital textbook as a flexible environment that can be configured, adapted, and enacted creatively. Teachers are not passive users but active orchestrators, capable of shaping how mathematical ideas are presented and explored. For publishing, it underscores the responsibility of publishers as orchestrators of possibility, whose design choices shape what kinds of learning trajectories are available. For the broader field of mathematics education, it challenges us to view digital textbooks not as static repositories of content but as dynamic orchestrational platforms that mediate meaning-making across multiple levels.

In conclusion, theorizing digital textbooks through adapted and extended versions of Instrumental Genesis and Instrumental Orchestration allows us to recognize their instrumental potential for learning and teaching mathematics. Digital textbooks are hybrid artifacts: at once tools and platforms, configured and configuring, orchestrated and orchestrating. Their potential lies precisely in this hybridity, which makes them powerful resources for shaping mathematics education in the digital age. The triad model provides a foundation for understanding the classroom-level interactions of teacher, student, and digital textbook, while the pyramidal model extends this view to include publishers and researchers. Taken together, these models offer a richer and

more comprehensive account of the evolving role of digital textbooks and open new avenues for research, design, and practice.

GREEFRATH: DYNAMIC AND INTERACTIVE FEATURES IN DIGITAL TEXTBOOKS

Digital curriculum resources open up new possibilities for teaching and learning, particularly in mathematics. Key developments can be identified in contemporary educational approaches. First, the importance of dynamic and interactive representations is becoming clear, as they help to make learning content more accessible and flexible. Second, there is a growing emphasis on personalized and adaptive learning, which takes into account different learning requirements through individualized feedback. Thirdly, collaborative and networked learning is emerging, which focuses on social interaction and cooperative knowledge construction. Fourthly, there is a stronger orientation towards data-based lesson planning, which enables evidence-based control and further development of the learning process based on empirical information. In addition, there is a growing focus on equal opportunities and accessibility, which aims to ensure more inclusive educational practices. Finally, the integration of real-world and STEM-related contexts is emphasized, linking learning content more closely to practical applications and underscoring its relevance to the learners' everyday lives. Some of these developments are exemplified below, using examples from current studies.

Dynamic visualizations

A current research focus is on how dynamic visualizations can promote the understanding of mathematical concepts. The study by Nguyen and Greefrath (2024) addresses the question of whether key basic concepts of differentiation—namely, the local rate of change and the slope of the tangent—can be developed more effectively through dynamic or static visualizations (see Figure 7). The study is based on the assumption that basic ideas as mental models help learners to interpret mathematical concepts meaningfully in different contexts. For the study, a lesson sequence introducing differential calculus was conducted in the 11th grade, with a total of 218 students. The students were divided into two groups: one worked with dynamic GeoGebra applets, the other with nearly identical static representations in the form of screenshots. After six lessons, the students completed a questionnaire to assess the extent to which they had developed the two basic concepts. The results show that both basic concepts were developed in all students. While no difference between the dynamic and static groups could be detected for the local rate of change, there was a significant advantage in favor of dynamic visualizations for the tangent slope, albeit with a small effect size. Overall, the study suggests that dynamic and static visualizations differ in their effectiveness: Dynamic tools are clearly advantageous for a deeper understanding of the derivative as the slope of the tangent, while both approaches appear to be similarly suitable for the local rate of change (Nguyen & Greefrath, 2024).

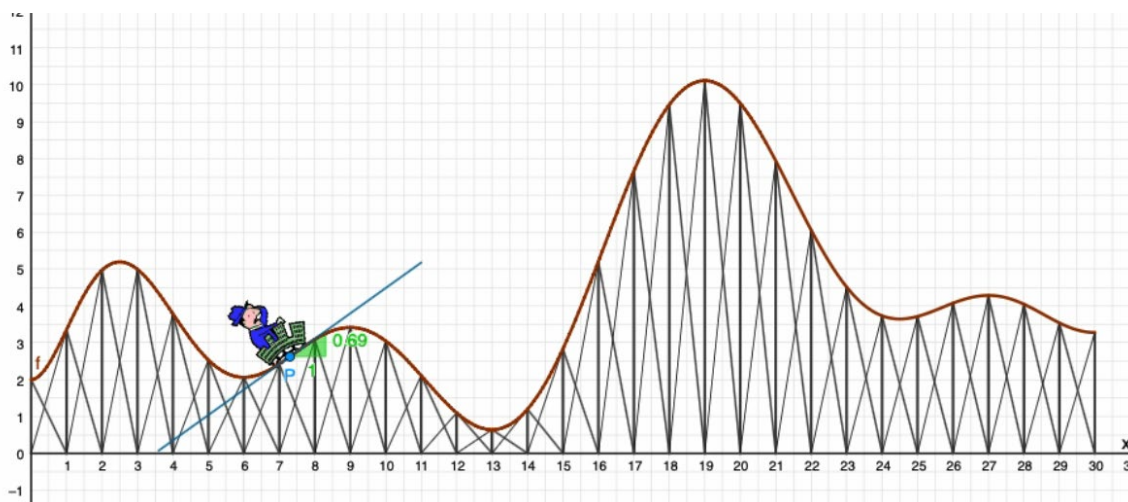


Figure 7: Screenshot of a dynamic visualization for the derivative (Nguyen & Greefrath, 2024)

Combination of visualizations, digital tools, and feedback

Digital textbooks offer a wide range of features that go beyond traditional print materials. These include dynamic visualizations, digital tools, and computer-assisted feedback. The study by Brnic et al. (2024) compares the use of a digital mathematics textbook with dynamic visualizations, digital tools, and feedback functions with a printed version on the topic of conditional probability. 314 students in grades 10 and 11 participated in this study. Both groups improved their performance, but the digital group benefited more. Girls in particular made significant learning progress only with the digital textbook. The authors explain this by the advantages of immediate feedback and interactive representations, which facilitate complex probability tasks. Overall, the study shows that digital textbooks effectively support learning.

Learning process data and computer-based learning environments (CBLE)

Another area of research concerns the use of process data in CBLEs. Analyses show different processing methods among students. Frenken's (2023) study examines how students work on mathematical modeling tasks in a computer-based learning environment during distance learning. Based on process data (logins, processing time, use of tutorials, use of tools) and the quality of the solutions, five processing types were identified: highly self-regulated and successful learners, frequent clickers with average performance, rather self-confident users with moderate performance, intensive tutorial users with limited performance, and weakly interacting learners with low performance. The results show that success in modeling depends less on math grades and more on self-regulation, motivation, and digital competence. This highlights both the potential and the challenges of digital learning environments (Frenken, 2023).

Cognitive load and design issues

The use of dynamic visualizations can be both conducive to learning and stressful. Cognitive load theory is relevant here in the context of dynamic visualizations. The starting point is the assumption of limited working memory capacity and the distinction

between intrinsic, extrinsic, and learning-related cognitive load (Sweller, 2020). Dynamic visualizations aim to support cognitive comprehension processes, with a key challenge being the reduction of extrinsic load through a purposefully designed, distraction-free layout. Various principles can be identified for the design: The segmentation principle aims to divide animations into meaningful units, while the pre-training principle involves introducing key terms and concepts in advance. The modality principle emphasizes the advantages of combining auditory and visual information channels instead of using text and image elements exclusively. Nevertheless, dynamic visualizations carry the risk of increased cognitive load, for example, due to fast movements and the fleeting nature of animations. At the same time, the dynamic linking of different visualizations can also facilitate cognitive processes. The quality of the design is therefore crucial for its effectiveness (Sweller, 2020).

Inclusion and outlook

Digital educational media must not only be effective but also inclusive in design. As part of the Digital Education Material Erasmus+ project, guidelines for inclusive digital textbooks were developed. These include prototypes that enable the use of Desmos in Braille mode, for example (see www.dem-project.eu).

Despite promising results, key research questions remain: Which features of digital textbooks are most effective? How can a fair comparison between digital and analogue materials be made? And to what extent should inclusive aspects take priority, even if this could limit certain digital possibilities? These questions point out important directions for future research and development.

IN GUISE OF CONCLUSION

The contributions in this panel, as outlined above, offer four glimpses into current research on digital curriculum resources and their uses, focusing on the theoretical work related to the research. In some cases, the resource itself can be said to be “new” (like generative AI tools), in other cases, such as YouTube videos, it is more the uses by students that have caught the attention of researchers in view of their actual and potential influence on (higher) mathematics education.

In each contribution, the particular features of the considered types of resources (YouTube videos, generative AI, and Digital textbooks) and the specific contexts considered (university teaching, course design, etc.) led researchers to adapt and combine familiar theoretical frameworks, to enable them to capture mechanisms, relationships and objects which they hypothesize to be essential or at least important to the use of the resource and its impact on teaching and learning. In particular, the contributions point out ways in which teachers’ roles and responsibilities change when the resource is used – conceivably, in ways that initial teacher education and subsequent practical experiences have not fully prepared them for.

Naturally, these hypotheses are generated by specific research interests and by initial observations and examples of a given resource and its usage. The theorization of hypotheses is critical to go beyond those “first impressions” and investigate them in a researchable form. The contributions also illustrate how such investigations – based on the framework – may in return lead to new and more refined hypotheses, and to a more nuanced and critical view on both the resource, its use, and the consequences for student learning of mathematics.

The panel discussion also illustrated the important difference between research and initial impressions (related, often, to personal experience of resource use in teaching or personal mathematical work). As teachers and citizens, we are constantly spanned between various institutional interests, opinions, and needs. Research in this area struggles to identify reproducible results, valid beyond a particular institutional context and instances of some resource (like a specific digital textbook or a specific YouTube video). Refining our theoretical handles on the complex objects and interactions involved is an essential step in this struggle.

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THE QUALITY OF PRINT AND DIGITAL MATHEMATICS CURRICULUM RESOURCES

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The symposium approaches the issue of the quality of print and digital curriculum resources (CR) for the teaching and learning of mathematics from different perspectives. This multifaceted approach aims to better understand how the quality of CR can be defined, investigated, measured, and controlled.

INTRODUCTION

Printed curriculum resources (CR) such as textbooks are considered to be major influences on classroom practice and students' learning of mathematics. As the potentially implemented curriculum, they are considered to put policy into practice by translating goals and performance expectations in official curricula and standards into opportunities to learn to be used in the classroom (Valverde et al., 2002). From a governance perspective, textbooks are important instruments for steering education (Schmidt & Prawat, 2006). However, they also convey the hidden curriculum, that is, the norms, values, and beliefs conveyed in both the classroom and social environment in which students learn unintentionally (Pinto, 2007).

Due to their special role in the educational system, their contents, structure, use, effects, and nature (in particular, the move from print to digital), CR have long been the subject of research (Fan et al., 2013; Rezat, 2024). This research has contributed to a nuanced understanding of CR and various aspects that influence their use. However, there is no shared clear vision of the quality of print and digital CR. In terms of textbooks, this is mirrored in the multitude of different catalogues with textbook evaluation criteria.

This symposium aimed to contribute to the development of a clearer vision of the quality of print and digital CR, the factors that influence and contribute to it, and how the quality of CR can be investigated. Due to the wide availability of several types of print and digital CR we want to broaden the focus and include a broad array of print and digital CR defined as “all kinds of analogous or digital materials mediating curriculum understood as structured and sequenced progression of OTL [opportunities to learn] over time, i.e. for a particular mathematical domain, age, or grade level related to the aims and intentions of educational authorities” (Rezat, 2024, p. 224).

THE QUALITY OF CR

Fey and Matthes (2018) contextualize the discussion about the quality of textbooks within the broader discussion around educational quality that large-scale assessments of the effectiveness of educational systems have triggered. The notion of quality implies taking an evaluative stance towards textbooks or CR. As different stakeholders have different intentions toward CR and thus different conceptualizations of quality, Trgalová et al. (2023) argue that the discussion about quality cannot be separated from two questions: 1) who is evaluating and 2) for what purpose.

Rezat (2025) synthesizes four dimensions of quality from the literature:

- Fit for purpose (Trgalová et al., 2023), “relevance” (Pepin, 2020, p. 25), or “efficacy” (Camilleri et al., 2014, p. 13) depend on the specific intention that a user assigns to a resource within a given situation.
- “Effectiveness” (Pepin, 2020, p. 25) or “impact” (Camilleri et al., 2014, p. 13) refer to the extent to which a resource is effective, meaning it ‘does’ what it claims to ‘do’.
- “Availability” (Camilleri et al., 2014, p. 13) and “practicality” (Pepin, 2020), refer to the accessibility and usability of resources.
- “Accuracy” refers to the correctness of the mathematical content (Camilleri et al., 2014, p. 13).

Taking this list as a starting point to conceptualize the quality of CR, it becomes clear that quality can be conceived of in two different ways: Either quality can be attributed to a CR itself, or it can be seen as a relation between the CR and its use

Regarded as a characteristic of the CR itself, quality generally means that there are more and less desirable ways of presenting the content. This refers especially to efficacy and to accuracy. Considering the sequencing of opportunities to learn in order to implement the official curriculum as the purpose of CR, the fit to the official curriculum is a quality measure that can be attributed to the CR itself. The same accounts for accuracy as this can be seen as a characteristic of the content of a CR.

However, the content of CR has many more aspects than fit to the official curriculum and accuracy. In their comparison of different catalogues of quality criteria for textbooks, Fey and Matthes (2018) identify several additional criteria that stem from very different disciplines and contexts. Very broadly, these can be divided into aspects related to design and layout, aspects related to mathematics, and aspects related to didactics and pedagogy. Depending on who evaluates for what purpose, some criteria may be considered more important than others.

The difficulty with this unilateral perspective on quality is that it does not consider users’ interactions with CR. Regarding the prevalent perspective on the use of CR as participation with the text (Remillard, 2005), a CR that is considered to have high-quality from a normative perspective does not necessarily need to be the CR that

support teachers to implement quality instruction in the best possible way or that has the highest impact on students' learning of mathematics or learning outcomes. Therefore, a perspective that attributes quality not only to the CR itself but considers quality to be a function of the relation between the CR and the users is appropriate.

The relational perspective is especially relevant to the concepts of impact and effectiveness mentioned above. If quality refers to a relation between the CR and its users, CR with a higher effect on classroom practice, students' learning of mathematics, or their learning outcomes are regarded as of higher quality.

The relational perspective on the quality of CR also comprises socio-cultural aspects that Adler (2021) has addressed in her notion of "relational transparency". She introduces this notion to draw attention to the fact that understanding the cultural practices that are inherent in CR requires the user's participation with the design principles. This participation is always culturally situated; therefore, transparency cannot be attributed to the CR alone.

Quality of CR conceived as a characteristic of the relation between CR and their users requires investigating users' participation with the characteristics of the CR itself in a particular socio-cultural setting. This poses methodological challenges.

In order to deepen the understanding of the quality of CR, the symposium addresses the following questions that represent only an excerpt of the range of relevant questions related on CR quality:

1. How can the quality of print and digital CR be conceptualized, operationalized, and measured? This also includes how quality features are related to the use of curriculum resources.
2. What factors and features are related to the quality of print and digital CR?
3. What does "the quality of digital CR" mean, and how does it differ from the notion of "quality of printed CR"?
4. What features of quality are more generic in nature, and which features seem to be more context-specific?

These questions were addressed from different perspectives in four presentations within the symposium:

1. **Quality of digital curriculum resources surfaced in mathematics education research** (S. Rezat, D. Glasnović Gracin, & H. Van Steenbrugge)
2. **Curriculum resources descriptive structure: Difference as a measure of quality** (S. Olsher, M. Yerushalmy, & A. Weizman)
3. **Determining textbook quality in primary school's topic data and chance** (A. Wunsch, R. von Hering, & H. Sievert)
4. **Curriculum material quality from the perspective of teacher enactment** (J. Remillard, T. Koljonen, H. Krzywacki, L. Condon, & R. Sayuj)

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QUALITY OF DIGITAL CURRICULUM RESOURCES SURFACED IN MATHEMATICS EDUCATION RESEARCH

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This paper presents a systematic literature review of studies on digital curriculum resources (DCR) in mathematics education, aiming to identify digital features related to quality enhancement. Thirty-three peer-reviewed studies were analysed through qualitative content analysis, resulting in the identification of twelve recurrent features, including interactivity, providing feedback, personalization, and connectivity. Findings reveal that while DCR features often are implicitly related to quality enhancement, explicit mentioning of quality remains scarce. Implications highlight the need for a stronger conceptualization of DCR quality and further empirical research on how such features affect mathematics teaching and learning.

INTRODUCTION

Digital curriculum resources (DCR) are often considered to enhance the quality of mathematics education, as they can bring new digital features that drive innovation, efficiency, and value. These features, such as dynamics and interactivity, have the potential to transform the teaching and learning of mathematics (Choppin et al., 2014) and bring “new educational dynamics” (Pepin et al., 2017, p. 646). In this growing research area, the features and uses of DCR are related to both possibilities (Pepin et al., 2017) and challenges (Ruthven, 2018) in mathematics teaching and learning. For instance, a feature emphasized in research on DCR is connectivity (Gueudet et al., 2018; Pepin, 2021), which is related to coherence within a DCR (O’Halloran et al., 2018) but also to connecting DCR with other resources and connecting users of DCR (Gueudet et al., 2018). These multi-layered opportunities deliver the new values with DCR to classrooms. At the same time, DCR have also been characterized as more opaque than their printed counterparts (Usiskin, 2013), which has been related to growing demands on the teacher to get insight into how learning opportunities in these resources are structured (Remillard, 2016).

These complexities underscore the urgent need for a more robust conceptualization of DCR quality. Moreover, given the rapid evolution of this field, it is essential to stay current with the latest research on this topic.

This paper aims to take stock of the features identified in research on DCR in mathematics education that are considered to contribute to improving the quality of mathematics education and specify their quality. Based on this idea, we posed the

following research questions: (1) What features of DCR are mentioned in the papers as quality-related? (2) How are these features distributed through the papers?

METHOD

A systematic literature review was conducted to identify studies that address the quality and features of DCR in mathematics education. The following search terms were applied across the ERIC database:

(“digital curriculum resources” OR “digital curriculum materials” OR “digital textbooks” OR “e-textbook” OR “digital instruction resources”) AND mathematics.

This search resulted in 56 papers. After applying the inclusion criteria (peer-reviewed status, explicit focus on DCRs, and availability), 33 studies were retained for analysis, comprising 21 predominantly empirical and 12 conceptual papers.

The analytical procedure included qualitative content analysis and consisted of two steps: (1) Identifying digital features that promote quality of mathematics education, which was followed by organizing the identified digital features (e.g., merging, nesting, etc.); and (2) Finding and analysing their prevalence and co-occurrence across studies using MAXQDA.

RESULTS

Quality-related features of DCR

Twelve features of DCR were identified as quality-related across the reviewed studies. These are: interactivity, feedback, assessment, adaptivity, connectivity, multimodality, customization, multimedia, personalization, authorability, socialization, and user data.

An important finding is that quality was rarely described or explained explicitly. Instead, it was rather implicitly included through references to *digital potentials*, *transforming teaching*, *new features*, or *a shift in mathematics education*. For example, Brnic et al. (2024) reported that “students can benefit from these technology-based features” (p. 559), while Choppin and Borys (2017) emphasized that the “digital materials have potentially transformative features” (p. 663). Alternatively, quality was sometimes mentioned without regard to its meaning, e.g., *high-quality materials*, *high-quality content*, *high-quality instructional resources*, etc.

Feature distribution through the papers

Figure 1 presents the distribution of the 12 identified features across the 33 examined papers. The results show that interactivity and feedback are the two most involved features connected to the quality of DCR (present in more than twenty papers). Personalization, connectivity, user data, and assessment are digital features addressed in 15-19 analysed papers. The analysis also revealed that: (a) some papers focus on one or very few features, e.g., multimodality in Norberg (2019); (b) others consider several features and focus on one feature in particular, e.g., connectivity in Gueudet et al. (2018); (c) some consider several features and do so in a more even way, e.g.,

authorability, user data, and connectivity in O'Halloran et al. (2018); and (d) some papers consider almost all of these 12 features, e.g., empirical study involving multiple feature analysis in Glasnović Gracin and Krišto (2022) or a conceptual review paper by Pepin et al. (2017).

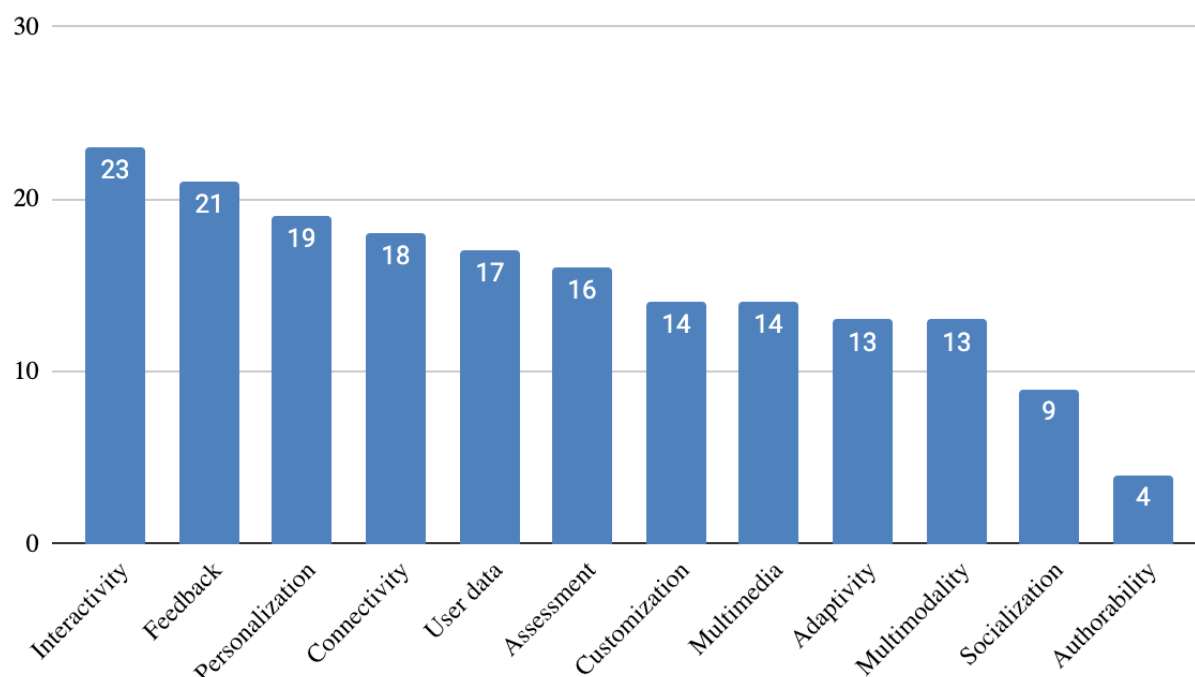


Figure 1: Features of DCR identified in papers and their frequency.

CONCLUSIONS AND DISCUSSION

This review identified twelve features of DCR considered to enhance the quality of mathematics education, with interactivity and feedback being the most commonly mentioned across studies. Despite frequent claims of the transformative potential of DCR (e.g., Choppin et al., 2014; Pepin et al., 2017), explicit mentioning or description of quality remains scarce.

This finding shows that the potentials of DCR to enhance the quality of mathematics education have been acknowledged and connected to specific features of DCR. However, how these features should be designed and used in classrooms to effectively improve mathematics education remains unclear. Therefore, further research is needed on the quality of DCR features, which would include more explicit conceptualizations of DCR quality and further empirical research on how such features affect mathematics teaching and learning and which design features contribute to this.

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CURRICULUM RESOURCES DESCRIPTIVE STRUCTURE: DISTANCE AS A MEASURE OF QUALITY

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We propose a framework using praxeological theory to identify key features—from content to classroom dynamics—measuring quality of digital curriculum resources (DCR) as similarity between desired and actual features via a distance function. Applied to a DCR set, it enables meaningful, context-sensitive evaluation across diverse settings.

INTRODUCTION

When discussing quality, it is essential to first establish the means for such a discussion. Digital curriculum resources (DCR) features can be described by designers, researchers, users, or even large language models and connect diverse practitioners. We suggest that measuring quality involves defining a subset of desired features for a given context. Quality can then be evaluated by assessing the similarity between these desired features and those present in the actual DCR, using a distance function. This contextual approach ensures that the quality measurement focuses on the relevant aspects within each context.

Existing frameworks that evaluate the quality of curricular materials and mathematical tasks typically concentrate on a particular dimension. Silver and Stein (1996), for instance, highlight the cognitive demand of tasks, whereas the QUASAR framework emphasizes their potential to support high-level student thinking and productive classroom discussion (Stein et al., 2000). Other approaches examine alignment with teaching practices and curricular standards (Pepin et al., 2013) or focus on the mathematical quality of instruction (Hill et al., 2008). Other studies propose coherence (in use and design), connectivity, opportunities to learn, and other measures to describe DCR. These varying goals emphasize that no single approach can address the diverse needs of teachers, students, and teaching-learning contexts. It calls to advance a multi-perspectival framework that integrates a range of features, thereby providing a more comprehensive means of describing and assessing quality in ways that are sensitive to the needs and contexts of different users.

DESCRIBING CURRICULUM RESOURCES

To capture key aspects of DCR we propose a structure akin to praxeology. This structure incorporates a vector of features and factors that describe DCR comprehensively. Some features—such as domain, topic, or mathematical objects—

are universally relevant, while others are more context-specific, tailored to a particular group (e.g., grade level in a specific educational system). Some characteristics may even be unique to an individual teacher's context, such as class arrangement, instructional roles, or task duration.

We adopt the praxeology framework proposed by Cooper et al. (2020), which builds on characteristics identified in the Balanced Assessment model (Schwartz et al., 1995) and integrates additional elements from teachers (Table 1).

Category	Possible values	Relevant context
Mathematical Domain	Pattern/Function, Shape/Space, Number/Quantity, Chance/Data	Pattern/Function
Mathematical Object	Polygons, Functions, Circle, whole numbers...	Quadratic function
Representation	Verbal, Graphical, Symbolic, Numeric	Verbal, Graphical, Symbolic
Mathematical Operations	Represent, modify, compare, analyse ...	-
Mathematical Actions	Modelling / Formulating, Transforming / Manipulating, Inferring / Drawing Conclusions	Inferring / Drawing Conclusions
Class Arrangement	Individual, pairs, groups, whole class	Pairs
Role	Opening, practice, assessment	Practice
Task Duration	0-15, 15-45, 45-90, multiple sessions	15-45

Table 1: The praxeology framework.

Prior research using this framework has yielded valuable insights into the features of DCR and their users. It aids teachers in selecting resources tailored to pedagogical goals (Cooper et al., 2020) and offers insights into the teacher-textbook relationship. Tagging can shed light on how teachers use and understand educational materials, which is vital for aligning resources with educational objectives (Olsher et al., 2018).

QUALITY AS DISTANCE FROM RELEVANT CONTEXT

To exemplify our notion of quality we start by defining the desired task profile, as it is elaborated in Figure 1. To identify how closely a task from our repository matches the desired task profile, we use a set of pre-tagged tasks; three (out of 206) tasks are presented in Figure 1 (tasks 1–3), with the tagging values of the relevant fields.

To find out how closely the match is we calculate the distance; the lower the distance is the closer the task is from the desired task. Features compared were Domain, Object, Representation, Actions, Role, Arrangement, and Duration. For each attribute a value

was assigned. For categorical values 0 was assigned for the value, or 1 for a different one. For Representation/Operations we defined two sets, A and B such that $A = \{\text{desired representations}\}$, $B = \{\text{actual representations}\}$, and the containment distance value was defined as $1 - \frac{|A \cap B|}{|A|}$, describing how much of the “desired” set is in the task.

	Domain	Object	Rep.	Actions	Role	Arr.	Duration
Desired task	Pattern/Function	Quadratic function	Verbal, Graphic, Symbolic	Inferring/drawing conclusions	Practice	Pairs	15–45
Task 1 (123)	Pattern/Function	Quadratic function	Verbal, Graphic, Symbolic	Modelling/Formulating, Inferring/Drawing conclusions	Practice	Pairs	15–45
Task 2 (684)	Pattern/Function	Quadratic function	Graphic	Inferring/Drawing conclusions	Assessment	Whole class	0–15
Task 3 (233)	Pattern/Function	General function	Verbal, Graphic, Symbolic	Modelling/Formulating, Inferring/Drawing conclusions	Enrichment	Pairs	0–15

Figure 1: Four tagged tasks.

For Duration, a normalized difference (e.g., $|x-y|/45$) was calculated, and then we used the average of attribute distances as the total distance, or the quality measure.

Table 1 presents the calculated distances of the three tasks under seven features. Lower distance means that the task is closer to the desired task.

	Domain	Object	Rep.	Actions	Role	Arr.	Dur.
Desired task	Pattern/Function	Quadratic function	Verbal, Graphic, Symbolic	Inferring/drawing conclusions	Practice	pairs	15–45
Task 1	0	0	0	0	0	0	0
Task 2	0	0	0.67	0	1	1	0.5
Task 3	0	1	0	0	1	0	0.5

Table 1: The calculated distances.

How would we choose the better task according to that measure? Apart from task 1 which is a perfect fit, tasks 2 & 3 fit under “domain” and “actions” but differ from the desired task on other features. A decision can be made by choosing specific features that are important at that stage: if all I care about is the topic of quadratic function then tasks 2, 3 would fit. But if other features are the focus and some attributes matter more than others then a weighted measure could follow other formula. For example, a focus on Domain & Object will result in assigning the following weights: Domain: 10; Object: 10; Representation: 5; Actions: 5; Role: 1; Arrangement: 1; Duration: 2. The formula for calculating the distance, i. e. the “quality”, now becomes a weighted average of attribute distances.

SUMMARY AND DISCUSSION

Didactic metadata assigned to digital curricular resources enhances coherence by helping educators align content with teaching goals and improve learning outcomes. Tagging tasks with detailed attributes supports teachers in making informed curricular decisions and encourages reflective practice. Our study assumes a multi-perspective framework that integrates a range of features, offering a more comprehensive means of describing and assessing quality in ways that are sensitive to the needs and contexts of different users.

We have demonstrated how measuring distance can inform the coherence of a given task and thereby guide the design of DCR. While our current approach relies on manual tagging, which has proven beneficial, it remains labour-intensive, non-scalable, and dependent on domain experts. We therefore view our proposal as a step towards more efficient alternatives, where AI-based processes can support the description and evaluation of DCR quality.

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DETERMINING TEXTBOOK QUALITY IN PRIMARY SCHOOL'S TOPIC DATA AND CHANCE

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Since 2004, the topic data and chance has been mandatory in mathematics lessons at German primary schools. However, stochastic competences in primary schools are rather unexplored, as are the opportunities to learn in textbooks. This study aims to determine textbook quality for the topic data and chance by analysing the learning opportunities given in two textbook series. The results show significant differences between both analysed textbook series as well as between Grades 1 to 4.

STUDENTS' COMPETENCES AND TEXTBOOK QUALITY ON *DATA AND CHANCE*

Data and chance is an obligatory content area in the German curriculum for primary grades. In Germany, 82.4% of mathematics teachers report using textbooks as a basis for instruction (Wendt et al., 2017), compared to an international average of 75% (Mullis et al., 2012). Textbooks used as the foundation for teaching might affect the achievement of curriculum goals, as has already been demonstrated for arithmetic topics (e.g., Sievert et al., 2021). Looking at achievement in primary schools, deficient competences are reported, as difficulties with data interpretation (Frischemeier & Schnell, 2023), technical terms (Fischbein et al., 1991), and combinatorial tasks (Zapata-Cardona, 2018). At the same time, researchers suggest that it is possible to promote stochastic competence from primary school onwards (Falk et al., 2012; Frischemeier, 2020). Textbooks are the most popular subject of research in the field of curricular resource (CR) in mathematics education, while only a few studies focus on the impact on teacher or student variables (Rezat, 2024). Current findings on the link between opportunities to learn (OTLs) and achievement remain inconsistent: while some studies have shown effects of textbooks on students' learning (Singh et al., 2020; van den Ham & Heinze, 2018), others report no such impact (Blazar et al., 2019; van Steenbrugge et al., 2013).

RESEARCH QUESTIONS AND METHODOLOGY

Following the approach of textbook quality as a relation between the OTLs and students' learning outcomes (Sievert et al., 2021), this study aims to examine whether textbook quality can similarly be conceptualized for stochastic OTLs in primary school textbooks. We analysed the textbooks, addressing the following research questions:

- (1) What stochastic learning opportunities do textbooks offer?

- (2) What differences can be identified between textbooks and across the grades?
- (3) How can the quality of opportunities to achieve stochastic competences in primary school mathematics textbooks be assessed theoretically?

First, we analysed the subtopics *data*, *combinatorics*, and *probability* in textbooks theoretically using qualitative content analysis by Mayring (2022), and then assessed the given OTLs. *Flex und Flo* and *MiniMax* are among the most popular textbooks in Northern Germany. Both textbook series are considered from Grades 1 to 4, resulting in 8 books. The analysed textbooks are structured in thematic booklets for each grade. We define an OTL as a problem with all corresponding subtasks and illustrations. An illustration that cannot be linked to a problem is also considered an OTL.

FINDINGS AND DISCUSSION

The deductive categories follow the German mathematics curriculum and are supplemented inductively by those identified in the textbook analysis (Table 1).

Data	Combinatorics	Probability
Collecting and visualising data	Cartesian product	Running random experiments
Reading the data	Variation without repetition	Comparing events of single-stage experiments
Reading between the data	Variation with repetition	Comparing events of multi-stage experiments
Reading beyond the data	Combination without repetition	Questioning game rules
Changing statistical representations	Combination with repetition	Illustrating random generators
	Combinatorial reasoning	Results of multi-stage random experiments
		Determining partial quantity with an urn-model
		Evaluating statements

Table 1: Inductive-deductive category system for analysing stochastic OTLs.

Interrater reliability shows a percentage agreement of 0.88 and a moderate Cohen's kappa of 0.54, indicating that the category system requires revision and remains a work in progress. A descriptive analysis allows for a comparison of subtopics, revealing that Grades 3 and 4 offer the most OTLs, with *data* being the most prominent one (Table 2). *Combinatorics* appear more often in *Flex und Flo* than in *MiniMax* across all grades. There is a significant difference between the distribution of the subtopics in both

textbook series ($\chi^2(df = 2) = 24.69, p < .001$) and a small significant difference in Grades 1 to 4 ($\chi^2(df = 6) = 16.50, p = .011$).

Grade	<i>Flex und Flo</i>			<i>MiniMax</i>		
	Data	Combinatorics	Probability	Data	Combinatorics	Probability
1	98	13	1	79	5	5
2	109	23	6	117	11	9
3	180	33	31	170	12	12
4	146	21	21	203	10	15
Σ	533	90	59	569	38	41

Table 2: Absolute frequencies of OTLs in subtopics by grade and textbook series.

Grades 3 and 4 provide the most OTLs, while combinatorics and probability are underrepresented. Differences exist both between and within textbook series. For example, in the category *Running random experiments*, *Flex und Flo* offers four OTLs, involving everyday random situations and experiments, whereas *MiniMax* provides three, focusing solely on experiments. Within series, variation also occurs. OTLs in this category are not offered every year.

To theoretically assess textbook quality, we used a partial credit scoring, which we present as an example for the subcategory *Running random experiments* (Probability). This topic is addressed rarely, with fewer than five OTLs per grade in both textbook series from Grades 1 to 4. Textbooks were scored 0 if they don't offer any OTL for this subcategory (e.g., *MiniMax 1*); 1 if they include at least one OTL (e.g., *MiniMax 2*); and 2 if they incorporate random situations from everyday life and experiments (e.g., *Flex und Flo 1*). Final assessments for all subcategories are ongoing.

Limitations include potential overcounting due to multiple allocations, with their treatment still unresolved. The focus on two textbooks also limits generalizability.

OUTLOOK

To validate the theoretical textbook assessment, we will use a longitudinal dataset examining students' stochastic outcomes over one year (2024/2025). Student data were collected at two points using a paper-pencil test: at the beginning of Grade 4 and at the end of Grade 4. Interviews with participating teachers were conducted to gain information on the mathematics lessons taught in Grade 4. Once data analysis is complete, it will be possible to conclude the impact of the textbooks on students' achievement and empirically validate the textbook quality assessed.

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CURRICULUM MATERIAL QUALITY FROM THE PERSPECTIVE OF TEACHER ENACTMENT

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We conceptualize mathematics curriculum material quality relationally, focusing on how these resources support teacher enactment within the cultural and ecological context in which they are used. Through an analysis of mathematics teacher's guides from Finland, Sweden, and the U.S., we illustrate how different levels of flexibility and guidance align with the way curriculum materials are configured within a system.

INTRODUCTION AND GUIDING PERSPECTIVES

We present an approach to examining the quality of mathematics curriculum materials (CMs) from the perspective of how these resources structure and support teachers' use of them. Many assessments of curriculum materials assume that quality resides in the materials themselves and can be determined using "objective" measures, such as efficacy, impact, alignment to given learning objectives or design principles, or mathematical accuracy (Rezat, 2025). We adopt a relational approach that considers the alignment between the materials themselves, the teacher using them, and the cultural context in which they are being used (Adler, 2021). The teacher is seen as a key agent in students' mathematics learning. We conceptualize CM use by teachers as a two-way, participatory process where teachers engage with them as resources when enacting curriculum (Remillard, 2005). From this perspective, CM quality depends in part on their capacity to support teachers to make enactment decisions. Examining how CMs communicate teachers' roles can reveal assumptions about mathematics teaching and learning encoded in their design and in the cultural context.

Curriculum materials are one aspect of the broader professional landscape teachers navigate, which is shaped by policy and organizational structures. Thus, we approach our analysis using an ecological perspective, which views teachers as active agents in a system, navigating the tensions between their professional goals and organizational structures, such as policies and curriculum provided by official governing entities (Oolbekkink-Marchand et al., 2017; Wermke & Höstfält, 2014). From this perspective, curriculum materials are viewed as both representations of a system *and* as material resources that teachers leverage when enacting professional agency within it. Furthermore, as artefacts of culture, CMs hold different positions within teachers' organizational systems, and their quality must be understood in relation to the

particular culture and system of their use – a perspective we explore through a cross-cultural analysis of elementary materials from Finland, Sweden, and the U.S.

STUDY DESIGN, CONTEXT AND ANALYTICAL FRAMEWORK

This study is part of the Math 3C project, which examines elementary teachers' use of print and digital mathematics curriculum materials in four cultural contexts (e.g., Condon et al., 2023). We use the term *curriculum materials* to refer to teacher's guides, student textbooks, and other teacher-facing resources, both print and digital. For this paper, we analysed teacher's guides from Finland, Sweden, and the U.S. Finland and Sweden are known for valuing teacher autonomy, but they have differing conceptions of the teacher's role. The U.S. is known for emphasis on accountability and control.

The analysis focused on how and what the teacher's guides communicated with teachers. We analysed the five most frequently mentioned materials on a survey administered in 2022 (four for Sweden) and compared findings with how CMs are situated with respect to teachers' work in each context (Remillard et al., 2025).

The analytical framework included two dimensions. *Degree of flexibility*, informed by research on teacher positioning and framing (e.g., Sayuj et al., 2024), examined a) the directive versus suggestive nature of the communicative voice, b) the level of scripted versus descriptive guidance, and c) the extent to which teacher decision-making is supported. *Content and nature of guidance*, following and extending earlier approaches (Remillard et al., 2016), considers the quantity of guidance corresponding with different time frames. We averaged program ratings and placed them on a two-dimensional matrix (Figure 2), offering a visual comparison of their relative positions.

FINDINGS

Figure 1 shows placement of the CMs in terms of degree of flexibility and guidance, revealing patterns within and across contexts. Three of the Finnish CMs were largely similar, providing moderate guidance. They offer a highly flexible structure that allows teachers to incorporate their own pedagogical approaches while maintaining educational goals. DB (a game-based platform) provided relatively less guidance, while VN (a Hungarian program) offered less flexibility than the others.

Swedish CMs varied the most. More traditional programs (MD, KP) offered moderate guidance and low flexibility. The two newer programs provided more guidance and were written more flexibly, reflecting Swedish teachers' autonomy and decision-making. Interestingly, the fourth most common response from Swedish teachers on the survey was, "*I don't use a specific CM*", reflecting a preference for using a mix of resources and the freedom to do so. Notably, the FM program was adapted from the Finnish TT; the Swedish versions include more guidance than the Finnish versions.

CMs from the U.S. also varied in relation to the degree of flexibility, ranging from low to moderate flexibility, but they were comprehensive and explicitly aligned with state

standards. They provided considerable, detailed guidance on lesson components, pedagogical actions, and how to cater to different students' needs.



Figure 1: Two-dimensional matrix showing relative positions of each program with respect to the flexibility and guidance continua.

IMPLICATIONS FOR CURRICULUM MATERIAL QUALITY

We interpret our findings from the curriculum analysis in relation to what is known about beliefs, values, and practices in each context. We know, for example, that CMs are configured differently in each of the three educational systems (Remillard et al., 2025). These configurations reflect beliefs about the focus and control of teaching, corresponding to the level of guidance and flexibility in CMs. Such differences shape how we might think about curriculum quality. Our findings underscore that curriculum material quality is not a fixed attribute, but rather emerges through the dynamic interaction between materials, teachers, and their educational contexts.

In Finland, teacher professional autonomy along with local decision making are highly valued. In Finland, this results in a strong pathway for teachers to leverage their expertise to shape curricular decisions. Accordingly, Finnish CMs provide only moderate guidance and a high degree of flexibility. Quality curriculum materials in the Finnish context would need to be designed to promote professional judgment and support the continued development of pedagogical expertise. In Sweden, teacher autonomy is similarly emphasized, but a central expectation is that teachers address students' diverse learning needs. The role of CMs in instruction is more varied, as Swedish teachers use them selectively to adapt their instruction to different groups of learners. Consequently, quality materials should not only support teacher expertise but

also assist teachers in identifying and responding to these varied learning needs. In contrast, the educational system in the U.S. emphasizes top-down control of teaching and curriculum and high levels of accountability. CMs, which provide strong guidance and less flexibility, are positioned prominently as regulating mechanisms in this system. These highly controlled elements of the system leave the teacher with a narrow set of options when making curricular decisions. Quality materials could contribute to teachers' decision making through how they inform enactment.

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Part III
Oral communications

USE OF RESOURCES AND MATHEMATICS TEACHERS' SCHEME IN THE CASE OF AN E-TEXTBOOK DESIGN

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This paper reports on a study that identifies the types of resources teachers choose and how they use these resources when designing a chapter in an e-textbook for learning geometry. The study was guided by the Documentational Approach to Didactics. A qualitative research method was used to analyse the data, which were collected through observation and semi-structured interviews. The findings indicate that teachers utilize a range of resources, including mathematics textbooks, websites, and GeoGebra, during their design process. They develop various utilization schemes for using these resources, which include teachers' rules of action related to incorporating GeoGebra and websites in their e-textbook design.

INTRODUCTION

In the last two decades, along with the development of digitalization, mathematics e-textbooks have been developed and published in many countries (Rezat et al., 2021; Pepin et al., 2017). E-textbooks, as part of digital resources, can offer features like dynamic interaction, customization, and ready-made structures. These elements provide valuable opportunities for mathematics teachers, allowing them to create and modify content independently. This flexibility can enhance the convenience of teachers during the design process. No matter how effective the e-textbooks are, their effectiveness depends on how teachers use them (Gu et al., 2015). Consequently, growing opinions stressing the advantage of positioning teachers at the centre of the design process (Trouche et al., 2020).

While teachers have always been designers of lesson plans and teaching, in this mode, the development of digitalization, the teachers may be designers of e-textbooks. However, the design process is not trivial (Pepin et al., 2017). Teacher design involves creating something new by combining existing and novel elements to achieve a specific educational goal (Pepin et al., 2017, p. 801). When designing e-textbooks, teachers should consider their students as the primary users of the e-textbook. It is assumed that teachers will select resources that help them meet their design objectives when developing a chapter in an e-textbook.

In their previous study, Almahdi and Swidan (2022) provided an overview of the types of resources teachers use to design e-textbooks for teaching mathematics. However, they did not explore how teachers actually utilize these resources. The ways teachers use resources is a crucial aspect for understanding the design process and teaching-learning processes. In this contribution, we aim to specifically report how teachers use

these resources, the factors they consider when selecting them, and the utilization schemes they employ when designing e-textbooks. To achieve this, we will be building on the framework of the Documentational Approach to Didactics (Gueudet & Trouche, 2009), a comprehensive approach to understanding teachers' work, in order to identify how teachers choose and utilize resources.

THEORETICAL FRAMEWORK

The Documentational Approach to Didactics (DAD) focuses on teachers adopting resources to support their work (Gueudet & Trouche, 2009). It aims to analyse teachers' activities through their interactions with resources and schemes of utilization (Trouche et al., 2020). In this paper, resources are intended in a holistic way as anything (e.g., digital, interactive, non-digital/traditional text) that are developed and used by teachers in their interaction with mathematics for teaching, inside and outside the classroom." (Pepin et al., 2013, p. 929).

According to the DAD, the resource is a special concept more than a tool; for instance, teachers' discussions with colleagues can be a resource. The fundamental concept on which DAD is based is that of document, a resources and related utilization schemes. The notion of a scheme was defined by Vergnaud (1998) as a teacher's stable organization of activities for specific situations. Scheme is a central concept in DAD (Trouche et al., 2020). Hence, teachers use the resource differently in their utilization schemes. The utilization schemes for resources are constituted by the goal of the teachers, rules of action, operational invariants, and possibilities of inferences (Gueudet & Trouche, 2009). The rules of action are regular ways of acting for the same goal. Operational invariants are associated with two kinds of components: theorems in action -propositions considered as true by the subject, and concepts in action -concepts considered as relevant by the subject (Vergnaud, 1998). In this context, we focus on the design's goals, operational invariants, and rules of action.

The DAD also focuses on the interaction between teachers and their resources, this interaction involves two processes:(1) instrumentation, where the affordances of resources influence teachers practice and knowledge; (2) instrumentalization, the knowledge and aims of the teachers may lead them to adopt resources and adjust them to their design goals. In this study, we used the DAD framework to identify the types of resources that the teachers chose and to examine their utilization schemas during the design process. Using DAD as an analytical tool might allow us to identify teachers' utilization schemes for adopting resources to support their design of e-textbooks.

This study is guided by the following research questions:

1. What types of resources do middle school mathematics teachers draw upon when designing a chapter of an e-textbook?
2. What are the middle school mathematics teachers' utilization schemes of these resources when designing a chapter of an e-textbook?

METHODOLOGY

This report is about a small part of a larger research project that aims to explore the role of teachers as designers of e-textbooks. The project involved a 30-hour professional development course, held once a week for two months that guided the teachers on how to design e-textbooks. Seven middle school teachers of mathematics holding a BA or MA in mathematics education participated in this study. The teachers' use of e-textbooks was relatively rare compared to the use of printed textbooks. It was the first time they were involved in the process of designing an e-textbook. In this case, we focus on the resources that the teachers chose to design an e-textbook and the utilization scheme that influenced the teacher's design process.

Data Generation and Analysis

The data was generated by observing the design process of e-textbooks and semi-structured interviews with the teachers. The observations and interviews were conducted to generate data about the resources the teachers used and their utilization scheme during the design of a chapter of an e-textbook. During the interviews, the teachers were asked to describe their experience of the design process and explain the reasoning behind selecting specific resources and certain utilization schemes. The data from the observation was recorded, transcribed and coded using DAD framework to identify the utilization scheme that influenced the teachers design process. The observations were focused on the teachers design of e-textbooks.

To analyse the data, we based our methodology on some of the elements of reflective investigation, a methodology associated with the documentary approach; its starting point is to follow up on the teacher's documentation work during their interactions with all the resources, using several elements (Trouche et al., 2020). In the first element, we have an extensive collection of the resources used and produced in the design process; we have collected the documents, and the e-textbook developed throughout the design process. The second element was confronting the teachers' views on their considerations during the design process. We identified teachers' goals declared by the teacher during the interview and from their comments; sometimes, they stated a specific outcome they wanted to achieve in their e-textbook design. We identified from the teachers' declarations during the interviews the discourse about the rules of action when they mentioned things such as "I do that because I think, I believe." Finally, we inferred the teachers' operational invariants to make sense of the rationale behind design decisions of the teachers of e-textbook across different contexts for a particular design purpose.

FINDINGS

The results presented here include a) a description of the types of resources used by teachers and the considerations that guide their selection; b) an identification of the teachers' utilization schemes in the term; teachers' goals, rules of action, and

operational invariants. These results are based on data analysis within the theoretical framework of DAD (Gueudet & Trouche, 2009).

Types of resources

Mathematics textbooks- a source that is available and reliable for them. In the main, teachers rely on textbooks for their teaching, and these also include examples and questions designed for students. The teachers stated that textbooks are a valuable resource that they can use when designing tasks in an e-textbook. When the first author asked about the resources they used for designing, one of the teachers (T1) said, “I rely on the mathematics textbook because I can find tasks that are suitable for the level of students in my class, so I use the textbook as a source for designing tasks that I can incorporate into the e-textbook.” Teachers indicated they could modify textbook tasks to fit their students' educational needs. For instance, a teacher (T2) said, “I can choose tasks from the mathematics textbook. There are tasks in almost every chapter, and sometimes, I adjust them in the e-textbook to make them more interactive.” The teachers' use of the textbook as a resource indicates that it is a prominent source, and the teacher trusts its content.

Dynamic software (GeoGebra)- Teachers find GeoGebra to be a valuable resource for designing e-textbooks. When the first author asked them about the use of GeoGebra resources in the design process, a teacher (T3) stated, “I need GeoGebra in that case; first of all, it is advisable to include it in e-textbook because it allows me... and it's a great resource for tasks that ask students to inquiry properties of a rectangle.” Another teacher stated that he uses GeoGebra for tasks designed to develop mathematical ideas (T2): “The use of GeoGebra here has a role, in terms of accuracy in geometry, it allows students to draw the rectangle more accurately, and it is good for the student to try using it.”. Teachers utilize GeoGebra as a resource to design e-textbooks that engage students in active, inquiry-based learning and help them develop mathematical concepts.

Websites- The seminar had access to computers and an internet connection. The teachers took advantage of the free access to computers. Teachers access websites as resources to get new ideas for designing e-textbooks. One teacher (T5) stated, “When I find a task on a website, if it's interesting, I'll check it out to see if it suits my students. I could change the same task and then design it in an e-textbook.”. The websites here have provided the teacher with a source for searching for tasks from which he can select those that he considers appropriate. Some teachers use mathematics websites to find ideas for tasks. For instance, (T7) stated that she uses such a mathematics website for finding activities to teach quadrilaterals with a focus on their properties.

Sharing materials and collective discussion- The participants collaboratively shared materials that could be embedded in the e-textbook and discussed their suitability for their classrooms. Throughout the professional development course, we observed teachers exchanging resources and discussing their students' mathematical needs, abilities, and challenges on several occasions. Following these discussions, participants

shared tasks they believed should be assigned to their students. In an interview, one teacher (T4) mentioned, "In my e-textbook, there are some tasks that I borrowed from other teachers and modified. Through conversations with colleagues from other schools, I've learned the importance of adjusting tasks to better fit my students' mathematical needs. Specifically, I have many students who are low achievers."

To summarize the results of the first question, the most popular resources that teachers typically use for designing a chapter of an e-textbook include mathematics textbooks, mathematics websites, GeoGebra, and shared materials along with collective discussions. These findings highlight both tangible resources, such as tasks from textbooks, GeoGebra applets, and websites, as well as intangible resources, like collaborative discussions. Additionally, the results illustrate how the affordances of the e-textbook assist teachers in adapting tasks, and they show how teachers' pedagogical strategies come to life during the e-textbook design process.

Teachers' utilization schemes

To identify teachers' utilization schemas while designing a chapter of an e-textbook, we focused on three key elements: teachers' goals, rules of action, and operational invariants. This approach was particularly important during the initial steps of e-textbook design. We have chosen to present selected findings in the table below.

Teachers' goals	Rules of action implemented in the design process.	Operational invariants
G1. To Prepare a task that will be suitable for the students (T3).	R1. Preparing tasks for my class in an e-textbook.	O1. E-textbook helps to develop students' understanding and promotes learning.
G2. To find out what the students know.	R2. Reading the instructions of the tasks.	O2. I am designing these tasks because my colleagues told me that these look-like suitable tasks.
G3. To check understanding of mathematical content (T1).	R3. Discuss with colleagues before designing the chapter of the e-textbook.	O3. The use of the dragging tool of GeoGebra helps students more accurately construct the rectangle and identify its properties.
G4. To make the geometry lesson interesting.	R4. We are checking the tasks that we intend to design.	O4. Most students should be able to complete the tasks in this
G5. Get the students to deepen their geometric knowledge (T6).	R5. Taking notes from colleagues for the design process.	
G6. To get the students to use the GeoGebra tool	R6. I will try to intervene to provide the necessary	

to construct a parallelogram (T2).	assistance to students in any difficulty.	chapter of the e-textbook and identify the mathematical knowledge Included in them.
G7. Find out what should be taught from the curriculum(T1).	R7. I am taking notes from colleagues for the design process after designing the tasks in the e-textbook.	O5. We must make sure that what we design is mathematically correct and suitable for the class level.
G8. Provide tasks to calculate the area. (T5).	R8. Re-reading the tasks of an e-textbook.	O6. We noticed that the -textbook contains many design options in different ways, allowing us to integrate a video.
	R9. I am comparing tasks I designed and tasks designed by another teachers.	
	R10. The teacher reveals the tasks he designed and explains them to the other participants in the design.	

Table 1: Selected findings regarding the teachers' scheme during the design of e-textbook

During the design of the e-textbook, the teachers indicated that they use these resources in their e-textbook design. Teachers are using GeoGebra to design e-textbooks. In this utilization scheme, teachers designed tasks with a twofold goal. Teachers want first, the students to use it as a tool to construct a parallelogram to enhance their level of accuracy in drawing (T2), and second, to make the geometry lesson interesting for students when they are using it to inquire about the properties of a rectangle (T3). To accomplish these goals, several rules of action are put in motion: teachers prepare suitable tasks for their class in an e-textbook, often check the tasks they intend to design to adjust their class; they use notes from colleagues to adapt the e-textbook design to their class (e.g., "We go over the checking the tasks together. If the students have difficulty with them or if it takes the students a long time to answer them in the e-textbook, we spend more time discussing and checking the tasks before we design them in the e-textbook." T6). This response makes evident that the teacher relied on the resource- sharing materials and collective discussion- to discuss and gain ideas about the expected difficulties of students tied to a particular task that they wanted to design in their e-textbook.

The teachers' utilization scheme for mathematics textbooks is expressed in the following way: a primary goal in this scheme is to give students tasks similar to those of their textbook (T2). Another goal is to provide examples with explanations. For instance, (T5) emphasizes that she starts the design of her e-textbook by "giving an explanation or clarifications for the students to read." The rules of action associated with this scheme include designing in the e-textbook tasks to ensure that the students

will receive assignments as they learned in class ("I look for three tasks that I can assign as practice" T2). We inferred two operational invariants associated with this scheme; first, teachers trust the textbook's content in their design process. Second, teachers believe they are responsible for ensuring their design is similar to what the students learn.

During the design of the e-textbook, the teachers shared their designs with each other. They revealed the tasks they designed, received feedback from their colleagues, and searched together for tasks from different resources. They also shared their personal utilization schemes for these resources.

FINAL REMARKS

In this study, we examined the role of mathematics teachers as designers of an e-textbook. Specifically, we focused on how teachers select and utilize resources to achieve their design goals.

Referring to Research Question 1 (RQ1), the initial results indicate that mathematics teachers tend to combine various types of resources, such as textbooks, mathematics websites, and shared materials through collective discussions. Notably, they often incorporate GeoGebra, as the features of e-textbooks encourage this integration. Teachers consider various factors when designing a chapter in an e-textbook to ensure adequate support.

The results related to RQ2 indicate that mathematics teachers utilize various resources in designing their e-textbooks. This includes their goals for selecting mathematical content and the rules of action they employ to make their designs appropriate for the classroom. The rule of action that directs teachers' utilization schemes to e-textbook design demonstrates that each teacher utilizes these resources differently, likely due to their individual design goals. Interactions between teachers and curriculum resources gradually enhance their ability to adapt these resources to meet their goals (Choppin, 2019). According to the DAD framework (Trouche et al., 2020), teachers acting as designers combine resources through specific utilization schemes, which makes the process of instrumentation applicable in this context. Essentially, the process of instrumentalization is connected to the use of resources, as teachers select materials that serve the intended purpose of their designs.

The findings presented in this study provide insight into teachers' motivations for using various resources and how they use these resources in e-textbook design. These findings could contribute to the existing research on e-textbook design and the considerations teachers have as e-textbook creators.

The study presented in this paper is qualitative, involving a small number of participants and not intended for generalization. However, future research with a larger number of participants could provide deeper insights into how resources and teachers' utilization schemas can influence e-textbook design.

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EXPLORING ELEMENTARY SCHOOL STUDENTS' INVOLVEMENT IN MODIFYING LEARNING RESOURCES

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This research contributes to the development of student-centred digital learning resources, fostering 21st-century skills by prompting students to engage critically and actively in their learning. It involves students in the design and improvement of digital resources, shifting their role from users to active participants. Drawing on suggestions from 4th- and 6th-grade students regarding assessment, practice, and game-based questionnaires, this qualitative study identifies key themes in their suggestions, including clarity, task difficulty, and real-world relevance. The findings reveal age-related differences in how students engage with digital tasks and provide insights into how student input can inform curricular design.

LITERATURE REVIEW

Typically, experts develop digital learning resources with occasional teacher input, whereas students mainly use these resources. However, both teachers and students may have valuable insights into the learning process that can enhance task development. Although it may be impractical to expect students to design content from scratch, their understanding of learners' challenges and needs is crucial for refining learning resources. This study explores how involving students in suggesting changes to the learning resources they use can bridge the gap between content experts and users, creating a bidirectional development process. Experts are responsible for designing digital tasks, a process guided by principles aimed at ensuring that students deepen their understanding and actively engage in learning by creating activities that are adapted to different levels (Borba et al., 2016). High-quality tasks not only prompt students to present knowledge but also challenge them to explore, apply, and actively engage with content (Bray & Tangney, 2017). The medium and format of digital tasks, including multimedia elements, interactive videos, and educational games, are crucial for enhancing engagement. Research shows that multimedia improves learning by integrating visual and auditory channels (Mayer, 2014) and combining digital texts with videos enhances learning outcomes by engaging multiple channels (Mason et al., 2022). Digital games further boost motivation in supportive environments (Li et al., 2024), indicating that task design must consider the medium to optimize engagement and learning (Lackmann et al., 2021).

While effective task design focuses on engaging students with pre-constructed activities, an alternative approach, much like problem-posing, encourages students to create problems themselves. Problem-posing emphasizes the learning process over the final product, involving students in creating problems to develop a deeper understanding of concepts. This approach fosters creative and critical thinking, focusing on the learning that occurs through formulating problems rather than solving predefined ones (Silver, 1994). Unlike traditional problem solving, problem posing encourages students to reformulate, explore, and create, leading to insights through the process itself. While task design and problem-posing engage students differently, this research focuses on students' involvement in evaluating and improving existing tasks through reflection, with the aim of enhancing educational processes. Research on design-based methodologies and maker projects shows that engaging students as active creators fosters critical thinking and concept exploration (Peppler & Bender, 2013). However, the current research emphasizes student reflection in refining tasks rather than focusing on creation.

In contrast to approaches such as problem-posing, in which students create new problems, this study focuses on how students can contribute by providing reflections to refine and improve existing learning resources, emphasizing their role in enhancing the learning resources they use. Unlike previous research, this study emphasizes student evaluation and suggestions for improvement, making the design process more responsive to their needs. Although research on participatory design is still limited, especially when it comes to involving students in the creation of educational content, this study adopts a unique approach. Instead of starting from scratch, students engage in refining and improving pre-existing tasks (Vaughn & Jacquez, 2020). By engaging students in refining pre-existing tasks, this research highlights a shift from a static model to a dynamic system, in which students actively contribute to improving learning tools. Students' reflections and suggestions, collected through questionnaires, are essential for iterative task development.

METHODOLOGY

This study aims to enhance students' active role in the educational process by including them in the design process of digital learning resources. To achieve this, we engaged 4th and 6th-grade students to suggest changes to their learning resources. This approach aims to cultivate essential 21st-century skills, such as critical thinking, creativity, and active engagement, by empowering students to contribute to their learning environments. The goal is to promote a more student-centred and participatory approach to curricular design, where students are co-creators of the learning resources they engage with.

To achieve this goal, we posed the following research questions, which focus on the changes students suggest to digital learning resources and the differences between the age groups' suggestions, aiming to understand how their involvement can inform the

design process: (1) What changes do 6th and 4th grade students propose to digital learning resources in mathematics they used? (2) What are the differences between the changes that students from different grade levels (4th and 6th grades) propose in digital learning resources in mathematics they used?

Participants and Research tools

The participants were two groups of student volunteers: seven 4th-grade students and five 6th-grade students. The participants, although of the same age, came from different schools and had no prior acquaintances. All participants attended a summer vacation activity conducted by an educational technology company.

This study employed several tools to gather data from students:

Online mathematics questionnaires: The questionnaires used in this study were sourced from a dataset collected by the Centre for Educational Technology (CET) in 2023. This dataset consisted of responses from thousands of students who rated digital tasks using three feedback icons: a smiley face (positive), neutral face (neutral), and sad face (negative). For the purpose of this research, only questionnaires that had been rated with the highest negative ratings (indicated by a sad face) were selected. This ensured that the tasks the students engaged with were those that had received the most critical feedback in previous studies. By focusing on these questionnaires, the researchers aimed to understand why they were rated poorly and gather students' suggestions for potential improvements.

Student Questionnaires: After completing the tasks, the students answered a questionnaire to reflect on their experiences.

Semi-Structured Group Discussion: An open, semi-structured discussion in which students share their thoughts and reflections on the tasks.

Final Semi-Structured Questionnaire: After discussion, students completed another semi-structured questionnaire to reflect on the entire process.

Procedure

Each group participated in a structured session following the established research protocol, beginning with the completion of three types of digital questionnaire: assessment, practice, and game-based tasks. These tasks were selected based on prior negative ratings from large-scale student feedback data, ensuring that only the most critically evaluated tasks were examined. After completing the tasks, students filled out semi-structured reflection forms, reflecting on key aspects such as task difficulty, clarity, and enjoyment. This was followed by an open group discussion in which students shared their insights and suggestions for improving the tasks, providing deeper qualitative data. The session concluded with students completing a summary reflection form, allowing them to evaluate the tasks using predefined criteria and proposing their own evaluation metrics.

Data Collection and Analysis

All the sessions were recorded and transcribed for qualitative analysis. The process began by transcribing the students' handwritten answers while simultaneously reviewing and transcribing key statements from audio recordings of group discussions. A categorical coding method was used for data analysis. Recurring themes from both the written reflection and recorded discussions were highlighted and categorized using colour coding to differentiate between themes such as task difficulty, clarity, engagement, and real-world relevance. The frequency of each theme was quantified to summarize how often specific issues were raised by the students. These categories were then organized into broader themes and sub-themes for a more structured analysis, ensuring comprehensive integration of both written and verbal data. Several key categories have emerged, offering insights into how students interact with digital tasks. Some of these categories are rooted in the existing literature. However, new categories emerged directly from students' responses. To address the second research question, the coding process was conducted separately for the 4th and 6th grade groups. Upon completion, summary reports were generated: one comparing the data between the groups to identify differences in categories raised by each group and another summarizing the responses to assess the prevalence and significance of each category across all students.

FINDINGS

The analysis of student reflections provided answers to the research questions, revealing key themes in three areas: pedagogical, learning experience, and technical. These themes address the first research question, highlighting the changes that students propose for digital tasks. To answer the second research question, the analysis revealed differences between the two groups in the types of changes that they suggested. Each category was supported by student quotes, illustrating the broader insights gathered from the sessions and highlighting these differences.

Pedagogical aspects

This category focuses on suggestions that students made for various instructional elements, including instructions, feedback, work process, content complexity, and the tasks' contribution to learning. The suggestions in this category were explanations, examples, hints, and additional feedback. The students raised the need for clearer and more detailed explanations. They mentioned that many tasks assumed knowledge they had not yet mastered, and the instructions were not always intuitive. Video or audio explanations have been suggested as alternatives to reading long text-based instructions, which some students found overwhelming. As one student noted, *"Sometimes the teacher plays videos that explain things better, so maybe this could have something like that—someone explaining it instead of me reading"* (Student 3, 4th grade, summary conversation). The students expressed appreciation when the tasks provided examples or hints to guide them. The lack of worked examples in some

questionnaires made it difficult for the students to understand the expected steps. One student commented, *"It would help to see a solved example, just like in the workbook. It makes everything clearer"* (Student 2, 4th grade, summary conversation).

Additional Students appreciated the immediate feedback on their answers. They particularly appreciated tasks that provided explanations for incorrect answers, helping them to understand their mistakes. "I like the feedback when it shows me if my answer is correct or not" (student 5, 4th grade, first questionnaire), one student said, while another mentioned the feedback you get when you press the test button... "that it tests you because there is a test button... students can come back and check what they did wrong, that's really good" (Student 2, 6th grade, summary conversation).

Learning Experience

This category highlighted students' reflections on their overall experience, covering themes such as task difficulty, interest level, real-world relevance, and enjoyment.

Many students commented that some tasks were too easy, particularly for game-based tasks. They expressed a desire for more challenging content, which would further push their understanding. As one student explained, "The basketball task was really easy" (Student 1, 4th grade, summary conversation) while another said, "This one was really easy, I finished it in two minutes" (Student 4, 4th grade, first questionnaire). However, in the other tasks, some students appreciated the challenge, noting that it offered a refreshing change. "I liked that it was more challenging, I'm not used to that," (Student 3, 6th grade, summary conversation) remarked one student, while another added, "The quiz was fun and had a variety of tasks, but some questions were quite difficult" (Student 4, 6th grade, summary conversation).



Figure 1: *"The Starting Five"* - a basketball-themed game for 4th-grade students.

Students generally expressed interest in tasks that had a gaming element but emphasized that the tasks needed to be more engaging, varied, and connected to their learning goals. When the tasks became repetitive, the students quickly lost interest. One student reflected that although the game was enjoyable - "it didn't feel like I was learning anything new" (Student 3, 6th grade, summary conversation). While on another questionnaire it was said "in my opinion the content of the questionnaire is very interesting

and challenging enough" (Student 1, 6th grade, first questionnaire). Another emerging theme is the relevance of tasks to real-world situations. The students preferred tasks related to familiar everyday scenarios. This was particularly highlighted in discussions on how learning could be made more meaningful by linking it to practical situations. One student shared, 'I liked the question about dividing food because it's something we do in real life, and it made sense' (Student 2, 6th grade, summary conversation). Students preferred interactive and enjoyable learning methods. Many students described how using digital tasks made learning more fun compared to traditional methods, with one student saying, "Learning this way is more fun and enjoyable" (Student 2, 4th grade, first questionnaire).

Technical Aspects

This category addresses students' feedback on the technological and design aspects of tasks, including technological usability, layout, and flexibility.

Technological Usability: Students frequently encounter technical difficulties when interacting with digital tasks. Issues such as unresponsive buttons or overly complex interfaces frustrate the learning experience. "There were a lot of problems with the buttons not working," (Student 3, 6th grade, first questionnaire) shared one student, highlighting the need for a smoother user interface.

The design of the tasks, particularly the length of the text and overall layout, was highlighted as an area for improvement. Long passages of text or too many questions in one go could overwhelm students, detracting them from the learning experience. One student noted, "It's stressful when there are too many words on the screen," (Student 3, 4th grade, first questionnaire) while another suggested, "It would be more fun if there were more animations" (Student 7, 4th grade, first questionnaire). Simplifying the layout and adding more dynamic elements could help maintain student engagement.

Moreover, students expressed a desire for more flexibility within the tasks, such as the ability to divide problems into smaller steps or customize certain aspects of the game elements. This customization could help make tasks more engaging and tailored to individual learning styles. One student suggested, "It should be possible to divide into more than 20 parts," (Student 5, 4th grade, first questionnaire).

Differences Between 4th and 6th Grade Students

Based on the data analysis and comparison between the categories raised by 4th and 6th-grade students, in response to the second research question, one key observation was the difference in categories introduced by the two groups. Younger students (4th grade) created categories such as rationale, game elements, technological usability, and real-world relevance, which were not present in the feedback from 6th graders.

This distinction may reflect developmental differences. Younger students with less experience in digital tasks are still exploring and critiquing, perceiving digital tasks to be flexible and open to change. As one 4th-grade student suggested, "It would be more

fun if there were more animations” (student 1, 4th grade, first questionnaire), highlighting their view of digital tasks as adaptable. In contrast, 6th-grade students, who might have more experience with digital tools, may have become accustomed to certain elements and are less likely to critique them. Game elements and technological features are viewed as fixed, resulting in fewer suggestions for improvements. This may indicate a form of "fixation," where older students are more set in their expectations, viewing digital tasks as stable and less adaptable. This is further supported by the lack of suggestions from the older group regarding game elements or technical features.

An additional distinction between the classes lies in the way 4th and 6th-grade students perceive task difficulty. 4th-grade students generally preferred easier tasks and expressed frustration with more challenging ones, which aligns with their developmental stage where foundational knowledge is still being formed, reflecting their desire for tasks within their comfort zone, with a focus on minimizing difficulty.

In contrast, 6th-grade students showed a clear preference for more challenging tasks that provided a sense of accomplishment, as stated above by Students 3, 6th grade. Their feedback indicates that as students grow older, they begin to appreciate the intrinsic reward of overcoming complex challenges. This shift from preferring ease to seeking challenges reflects a natural progression, as students develop confidence and recognize that more difficult tasks contribute significantly to their growth compared with simpler ones.

DISCUSSION

The findings from this study provide insights into how 4th and 6th-grade students perceive and interact with digital learning tasks, particularly regarding the changes they propose and the differences in their reflections. In addressing the first research question, key categories emerged, including the medium and format of digital tasks (Mayer, 2014), clarity of instructions, task difficulty, variety, etc. Additionally, new categories have appeared, such as phrasing, technical issues, and variety, which have not been prominently discussed in previous literature. These categories highlight specific areas in which students see opportunities to improve task design.

Regarding the second research question, differences were observed between the reflections of younger and older students. While 4th graders primarily focused on interface usability and the need for clearer instructions, 6th graders reflected on more complex aspects such as task difficulty and real-world relevance. These findings align with research emphasizing developmental differences in students' engagement, encouraging critical and creative thinking, fostering deeper understanding, and promoting active involvement in the learning process (Silver, 1994). Moreover, the study reveals that students are not only critically reflecting on the content itself, but are also increasingly aware of how the structure and design of tasks impact their learning experience (Li et al., 2024; Mason et al., 2022).

IMPLICATIONS AND FUTURE DIRECTIONS

This study emphasizes the importance of involving students in the design and evaluation of digital tasks, shifting their role from passive recipients of information to active participants in the learning process. This transition allows students to contribute directly to refining and improving tasks, ensuring that they are not only users, but also key players in shaping their learning tools. The findings also suggest that further research is needed to explore how student involvement can enhance both critical thinking among younger students and the complexity that older students seek. The discovery of new feedback layers highlights the value of continually incorporating student input to ensure that educational tools are both relevant and effective.

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GEOMETRIC CONSTRUCTIONS IN TEXTBOOKS BEFORE AND AFTER BRAZIL'S HIGH SCHOOL REFORM

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Brazilian high school has recently undergone a curricular reform, impacting textbooks that are purchased by the government and distributed free of charge to public schools. The research we present compares the 2018 and 2021 editions of two collections of Mathematics textbooks, focusing on geometric constructions content. The 2018 call for proposals was the last before the reform, and the 2021 one, the first after the new guidelines. It was observed that, although the essence of the content remained the same, there were changes in layout and updates to university entrance exam questions. The inclusion of tasks involving computational thinking and dynamic geometry software was the most significant change. However, most tasks still require students to follow instructions, with little room for creative exploration.

INTRODUCTION

The textbook plays a central role in the Brazilian educational system, establishing itself as the main teaching resource in the country (Amaral et al., 2022). In this context, the National Book and Didactic Material Program (PNLD) stands out by annually distributing tens of millions of copies to public elementary and high schools across the country. Recently, basic education in Brazil has undergone a significant reform. In high school, the approval of the National Common Curricular Base (BNCC, Brazilian Standards) in 2018 and the implementation of the New High School (NHS) in 2017 introduced a minimum curriculum structure nationwide, redefined mandatory subjects and included optional subjects. The PNLD 2021 was the first to evaluate textbooks for the NHS, taking into account these curricular changes. The specific collections of Mathematics went from three sequential volumes, one for each year of high school, to six self-contained volumes per collection, one per semester, which can be used in any order.

The research project “Challenges of the New High School: A Look at the Changes in Geometry and Technologies in Mathematics Textbooks”ⁱ investigates the changes in Mathematics textbooks resulting from this curricular reform and the NHS. At this event, the results of this project are presented, focusing on changes related to geometric constructions, in two collections approved in the PNLD 2021, compared to their respective editions in the PNLD 2018, the last acquisition of books for Brazilian high schools before the implementation of the NHS. We know that this context is particular

to Brazil, but we consider that this event, whose theme involves textbook analysis, is a fruitful space for discussions with research that may have resonant natures and for contributions from researchers with views external to our reality.

GEOMETRIC CONSTRUCTIONS

Geometric constructions are part of the Basic Education curriculum and allow students to explore mathematical principles in practice, without the use of numerical measurements, and with the use of resources such as rulers without markings and compasses and digital technologies, such as dynamic geometry software. Also present as a *geometric construction*, it is characterized by “a graphic expression of form, which is embodied through precise constructions, which in turn are governed by principles drawn from Geometry itself” (Carvalho, 2008, p.01).

Constructions play an important role in shaping deductive reasoning and spatial visualization skills. By following rules and applying geometric concepts, students practice problem-solving in a methodical and reasoned manner. In addition to reinforcing the understanding of theorems, the constructions enable the practical demonstration of these principles, expanding the perception of geometric shapes. There is also an aesthetic dimension to this practice, as the constructions can involve harmonic and symmetrical figures, combining mathematical rigor and artistic sensitivity.

In recent decades, the use of dynamic geometry software, such as GeoGebra, has enriched the teaching of geometric constructions. Tasks previously performed with a ruler and compass can now be done interactively in digital environments. These tools allow students to visualize and manipulate figures in a more dynamic way, making it easier to understand complex concepts (Zulatto, 2002). However, Nacarato and Passos (2003) observe that students’ difficulties with geometry persist, even with the use of these technologies, suggesting that technology, by itself, does not solve the challenges of geometric learning.

In addition to deductive reasoning, geometric constructions offer opportunities for the development of abductive reasoning, which involves “thinking backwards”. As highlighted by Mazzi and Amaral (2020), this type of reasoning, still rare in mathematics textbooks, allows students to formulate hypotheses and conjectures based on initial observations, testing possibilities before reaching a definitive conclusion. This broadens students’ capacity for exploration, encouraging them to experiment with different approaches and creative solutions to geometric problems (Gravina, 2001).

It should be recognized that, although dynamic geometry software facilitates constructions, there are complex mathematical codes behind these tools, which follow the same geometric principles as manual constructions (Zulatto, 2002). Discussing this reality in the classroom can provide students with a more complete understanding of geometry, both in the digital and traditional environments, strengthening their ability to think critically and in an integrated way.

METHODOLOGY

The present research follows a qualitative approach. As Bogdan and Biklen (1994, p.209) state, studies of this nature “should reveal greater concern for the process and meaning and not for their causes and effects”. Thus, when considering the objective of the study, we understand that the interpretative approach of qualitative research becomes more pertinent, based on documentary analysis (Alves-Mazzotti, 2001), since it aims to understand the changes in the treatment of geometric constructions in high school books, and not in quantifications and/or products. This methodological perspective is in line with the central characteristics of qualitative research, according to Denzin and Lincoln (2000). For this event we present results based on the two collections of textbooks with translated titles: “Connections with Mathematics” (Leonardo, 2016) and “Connections” (Leonardo, 2020) and “Mathematics - Context and Applications” (Dante, 2016) and “Mathematics in context” (Dante & Viana, 2020). For the comparative analysis, it is necessary to choose works that have been approved in the PNLD 2018 and also in the PNLD 2021. Of a total of 8 and 10 collections approved, respectively, in the 2018 and 2021 PNLDs, these two were the first that we had access to, and therefore were the ones analysed.

For the analysis of these materials, this study is based on the Theorem System (Amaral et al., 2022), a methodological proposal consolidated over more than 20 research projects in the last decade, focusing on the analysis of mathematics textbooks. This system arose from the need to structure investigations involving textbooks, providing guidelines for the selection, categorization, and analysis of these materials, based on rigorous and systematic research approaches. It was assumed for the view of the textbook, and the data treatment developed assuming the path of stratified analysis, in which the textbook is considered as an object of study in itself and, in this case, the extracts are the panelsⁱⁱ and tasks of Geometry. The books selected for the analysis described here are among the 10 collections approved in the PNLD 2021. Of these, seven were also approved in 2018 and therefore provide an opportunity for comparison. Among these, we focused the analysis on the two collections most distributed among Brazilian public schools, looking at the textbook of students and the Guide for the Teachers.

UPDATES TO NHS TEXTBOOKS EDITIONS

In this section, we compare the treatment of geometric constructions in the 2018 and 2021 editions of the two textbook collections and present some discussions. The analysis reveals that, although the essence of the content has been maintained, the main changes concern the layout, the updating of entrance exam questions, the increase in the use of dynamic geometry software, and the insertion of flowcharts, to meet the demand for Computational Thinking (CT), required in the BNCC. The changes in the layout involve images, colours, reorganization of the figures, and their

enlargements/reductions in size, as illustrated in Figure 1. The text in the image is barely legible, but this is irrelevant, since the focus is on visual changes.

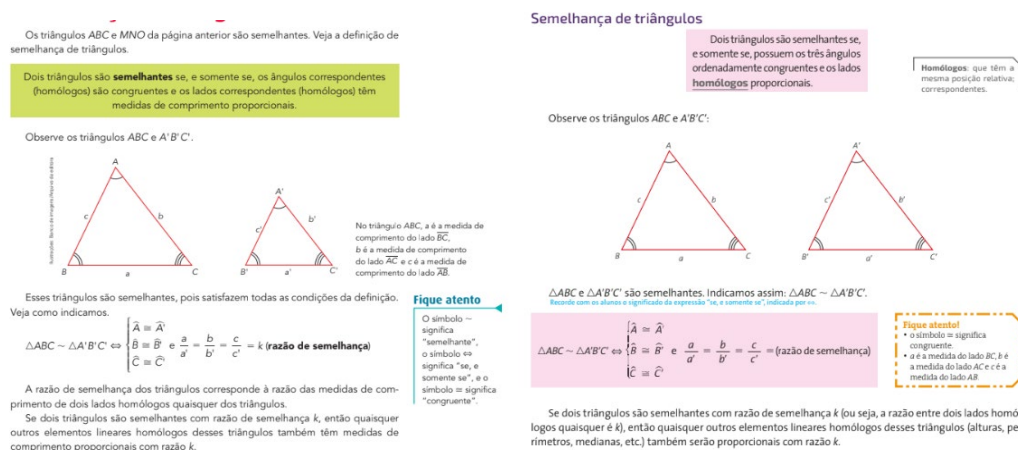


Figure 1: Change in layout between editions. On the left, Dante (2016, p. 240) and, on the right, Dante and Viana (2020, p. 14)

Another update identified concerns about the entrance exam questions. In Brazil, many universities use such tests for admission, and their questions are often included in textbooks for study. In each new edition, it is common to replace questions with more recent ones. This pattern is maintained in the collections analysed.

The presence of digital technologies represents a more significant change in the comparison between editions, indicating alignment with the BNCC, which requires the use of technological resources, such as dynamic geometry software, in teaching. In Dante (2016), the use of software was restricted to trigonometric functions, parabolas and ellipses. In Dante and Viana (2020), this resource is also used in area calculation and tiling, but in solved activities, in which the author presents the GeoGebra software and provides steps for construction, as illustrated in Figure 2.

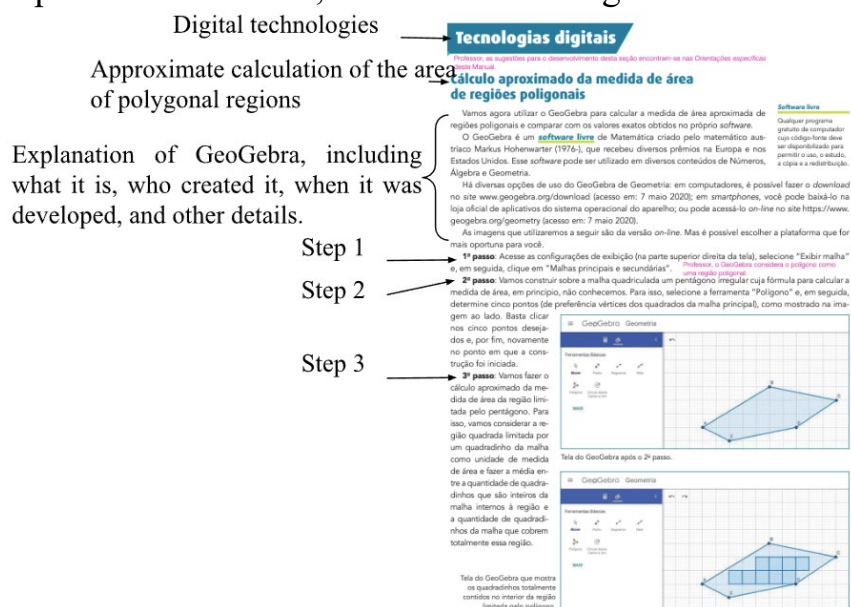


Figure 2: Calculation of the polygonal region area with GeoGebra (Dante and Viana, 2020, p. 37).

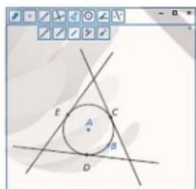
In Leonardo's collection (2020) there are also proposals of this nature: in the image on the left in Figure 3 there is an example with the use of dynamic geometry software, with the step-by-step construction. The teacher and the students are only responsible for the reproduction of the actions for the construction of an equilateral triangle circumscribed to a circle, in a "passive" perspective. The image on the right, on the other hand, illustrates an approach in which an "active" attitude is expected, even fostering an exploration of the "what happens if..." (Skovsmose, 2006).

Solved problem. Using a dynamic Geometry software, build an equilateral triangle circumscribed about a circle.

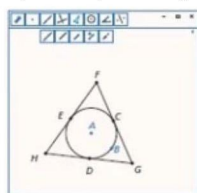
► Resolution: since the apothem of a regular polygon has the same measurement as the radius of the circumference that inscribes it, we can begin the construction by drawing the apothems of the equilateral triangle. Observe the following steps.

1. Let's draw a circle with center A and radius AB around which we will circumscribe the equilateral triangle. Then, we divide the circle into three equal arcs ($360^\circ \div 3 = 120^\circ$), marking the points C, D and E. The apothems of the triangle circumscribed by the circle have a measure equal to that of the radius of the circle that inscribes it. Thus, we construct the apothems AC, AD and AE of the triangle under construction.

2. We hide the auxiliary constructions and construct a perpendicular to each apothem passing through points C, D and E, respectively. These perpendiculars are tangent to the circle with center A and radius AB at points C, D and E.



3. We mark the points F, G and H, intersections between the perpendiculars, and draw the line segments FG, GH and HF, determining a triangle.



Explore

Using dynamic geometry software, follow the steps indicated below.

1. Construct five circles with radii of the same length.
2. In each circle, construct one of the regular polygons inscribed with 3, 4, 6, 8, and 16 sides.
3. Calculate the area of each constructed polygon and compare them. What happens to these areas as the number of sides of the polygon increases?
4. Construct a circle with the same radius and calculate its area.
5. If possible, construct a regular polygon with 32 sides, inscribed in a circle with the same radius as the previous circles. Calculate its area and compare this new measurement with the measurements of the other polygons and with the area of the circle.
6. If we constructed a regular polygon with 64 sides, what would happen to the measure of its area?

Figure 3: Tasks with dynamic geometry, respectively, passive and active (Leonardo, 2020, p. 18 and p. 34, translated by the authors).

We also identified that, despite the inclusion of technologies, tasks with traditional compass and ruler resources are also present. As Borba and Villarreal (2005) point out, we think differently when we use different media, and, therefore, constructions with ruler and compass with paper and pencil must be maintained in the study of geometry.

Also, in the idea of contemplating the BNCC, CT is present in textbooks, and we observed that in Leonardo's collection (2020), there is a task that relates it to geometric constructions (Figure 4). It should be noted that one of the criticisms of the BNCC is that it includes the CT without presenting its concept (Silva, 2023). The only mention in the BNCC of CT refers to a flowchart as an example, and this has left the textbooks pasteurized, including tasks that propose the simple creation of a flowchart so that the requirement to include the CT in the textbook has been fulfilled.

In the example of Figure 2, the procedure performed with the computational resource is the same as that traditionally done with graph paper. As Arcavi and Hadas (2000) observes, technology can "inaugurate a new way of doing and teaching mathematics, instead of a simple tool to present old approaches in new guises" (p. 41). The digital tool can introduce an empirical component and engage students in a variety of mathematical activities, such as the evaluation of arguments, differentiating itself from

conventional practices (Hadas et al., 2000). These ideas should guide textbook authors when designing tasks with technology.

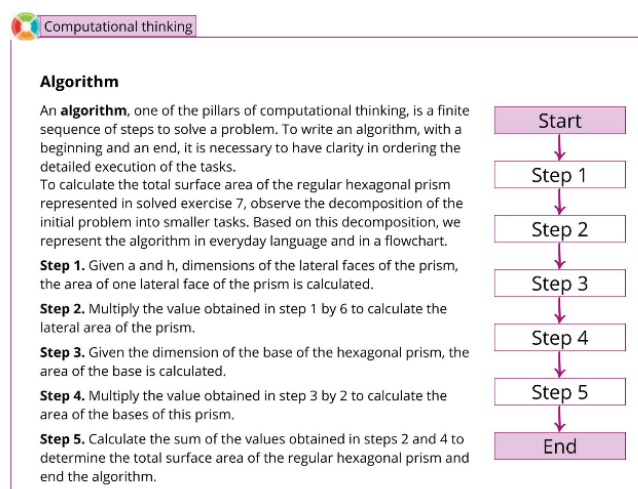


Figure 4: Flowchart on construction that makes it possible to calculate the surface of the regular hexagonal prism (Leonardo, 2020, p.77, translated by the authors).

Thus, there is a need to improve the textbook tasks, as there are few opportunities to reflect on the geometric constructions, as in the active task in Figure 3, and on their justifications, and by reproducing the steps, the student has little space to create their own constructions as a solution to a problem. As he states (Oliveira, 2018, p.61, translated by the authors), the teaching of Geometric Drawing can be insignificant for students when they only develop memorization and reproduction activities, because the “application of the technique of geometric constructions, without proper justification, or contextualization, does not bring significant learning to students regarding geometric knowledge”.

Different solutions could be shared in class, in order to explore the potential of comparing them and understanding why they satisfy the problem and in what ways they differ. Furthermore, the use of dynamic geometry software has its greatest strength in carrying out mathematical investigation, in questions that invite students to think “what happens if...”, and this type of proposal is little explored in the analysed collections.

FINAL CONSIDERATIONS

The results of the comparative analysis between the 2018 and 2021 editions of the mathematics textbook collections, focused on the content of geometric constructions, indicate a few significant changes. Changes are observed in the layout, in the updating of entrance exam questions, in the use of digital technologies, and in the inclusion of flowcharts. The essential content of the geometric constructions was preserved. The demand for the use of dynamic geometry software and the presence of the CT reflects the BNCC guidelines. The integration of digital technologies in the analysed books, such as dynamic geometry software, was a highlight of the Reform. In some activities,

students follow instructions prescriptively, merely reproducing steps without room for exploration. Others only transfer to the digital environment what was already done with pencil and paper, without introducing new pedagogical approaches. However, some tasks place the student in a more active role, as in Figure 3 of Leonardo's collection (2020), where technology fosters investigation and engagement. Even with these specific innovations, many changes relate more to form than to a significant transformation in teaching. Aware of this, teachers can adapt textbook activities to engage students in more meaningful roles, as discussed here.

Another relevant point to highlight is the inclusion of flowcharts as an attempt to insert the CT into the NHS. Apparently, it is the result of the lack of deepening of this concept in the BNCC. Both digital technologies and CT have the potential to enrich geometry education, but in the cases considered, their application lacks an approach that encourages creativity, exploration, and critical thinking.

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Endnotes

- ⁱ Regular Research Project funded by FAPESP (São Paulo Research Foundation).
- ⁱⁱ According to Amaral et al. (2022, p. 187), “panel” refers to “sections of the textbook in which the author addresses a topic, presents their ideas about the concept under study, the context in which these ideas apply, and examples.”

FROM TEXTBOOK TO CLASSROOM: GAPS IN MATHEMATICAL REPRESENTATION AND TRANSLATION IN NEPALI EDUCATION

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Translating real-world problem contexts into mathematical symbols is a key skill in curricula worldwide. This study examines Nepali students' abilities to translate words into symbols and how Nepali textbooks from earlier grades support the development of these skills. To investigate this phenomenon, a sequential explanatory study was conducted with a test for 10th graders and an analysis of Grades 6–9 textbooks. Results show that students struggle to convert real-world contexts into mathematical symbols, while textbooks provide limited opportunities for practice. In particular, students have difficulty translating symbols back into real-world situations, suggesting a link between textbook opportunities and achievement. These translation skills should receive greater emphasis in curriculum and instruction.

INTRODUCTION

Translating real-world problem contexts into mathematical sentences using mathematical symbols—and vice versa—is an essential skill emphasized in curricula. For example, NCTM (2000) emphasizes the ‘use of the language of mathematics to express mathematical ideas precisely’ (p. 60) to convert a real-world problem to a problem in the mathematical world. In doing so, students must use reasoning and thinking to master these communication skills. Researchers argue that this critical representation skill is essential for enhancing understanding and fluency in mathematics, yet students often struggle to learn this concept (Jupri & Drijvers, 2016). In Nepal, formal algebra instruction is included in the middle school curriculum. Therefore, middle school textbooks should help students master text-to-symbol translation skills. However, previous studies of Nepali textbooks indicate that they lack features that promote critical thinking (Basyal et al., 2023; Nepal et al., 2025; Basyal & Mainali, 2025). As a result, it is likely that textbooks have not adequately supported students in developing this vital skill.

Moreover, students' performance on the internal PISA-like National Assessment on Student Achievement (NASA) has demonstrated low performance in algebra, and specific skill gaps may have contributed to this difficulty (Education Review Office, 2018). However, NASA studies do not specify whether students' text-to-symbol translation skills contribute to their difficulty in solving context-based problems. The current study aims to assess students' competence in these essential skills and examine

how these concepts are presented in textbooks. This understanding ultimately enhances our ability to support student learning by aligning the intended and actual curricula. In this regard, we seek to address the following research question:

How well can 10th-grade students in Nepal translate word problems into mathematical symbols and vice versa? Additionally, how do textbooks support the development of these skills?

LITERATURE REVIEW

Word to symbol translation skills and student difficulty

Translating word problems into mathematical expressions using symbols requires specialized skills and is encouraged in curricula worldwide. However, numerous studies have documented that students often struggle with this translation process (Pawley et al., 2005; Jupri & Drijvers, 2016). Researchers have identified several contributing factors, including poor reading comprehension (Duru & Koklu, 2011; Vilenius-Tuohimaa, 2008), limited conceptual understanding (Capraro & Joffrion, 2006), and inadequate instructional strategies (Pawley et al., 2005).

Textbooks and opportunity to learn

Numerous research papers highlight the vital role of textbooks in teaching mathematics, emphasizing that content quality, contextual relevance, and sequencing significantly influence student learning (Van Den Ham & Heinze, 2018). Textbooks are also recognized as potential drivers of education reform and change (Rezat et al., 2021). Emerging research in curriculum resources further underscores the critical role textbooks play in shaping learning experiences (Rezat, 2024). Törnroos (2005) emphasizes that even a straightforward textbook analysis can yield crucial insights into how student achievement is linked to the learning opportunities (OTL) provided by textbooks.

Research in mathematics textbooks in Nepal

A substantial body of research indicates that school mathematics textbooks in Nepal are decontextualized, encourage rote memorization, emphasize computational skills, and are primarily designed to prepare students for examinations. These textbooks not only lack the features widely recognized as essential for effective learning (Basyal & Mainali, 2025), but they also notably lack contextualized problems (Nepal et al., 2025), tasks with high cognitive demand (Basyal et al., 2023), and opportunities for statistical reasoning (Jones & Basyal, 2019).

Moreover, research indicates that textbook authors in Nepal often begin writing without formal training in textbook development (Basyal & Mainali, 2023). As a result, textbooks produced may not effectively support the national curricular goals of fostering competent critical thinkers. Although learning is a complex phenomenon influenced by many factors, textbooks—being essential tools for learning—may have played a role in shaping students' ability to acquire word-to-symbol translation skills.

METHODOLOGY

A sequential explanatory design involves conducting a quantitative study, followed by qualitative research to explain the quantitative findings (Creswell et al., 2003). Accordingly, this study integrates a quantitative approach (student test) with a qualitative approach (textbook analysis) to understand students' ability to translate text into mathematical symbols and the opportunities for such translation provided in textbooks.

Student Data

A six-question test designed to assess students' ability to translate word problems into mathematical symbols and vice versa was administered in classrooms. The test asked students to identify or create transitions between text and symbols, and they were given approximately 15 minutes to complete the tasks. A total of 443 Nepali tenth graders from 17 classes across seven different schools participated in this study during the summer of 2023.

The first three questions were multiple choice, and the next three were open-ended problems. These test questions are adapted from similar studies (Duru & Koklu, 2011; Capraro & Joffrion, 2006), and were translated and contextualized for the Nepali setting.

Textbook Data

We used the 2019 revised edition (originally published in 2015) of textbooks published by the Curriculum Development Centre, a government-run entity. The school textbooks for grades 6 to 9 were analysed to identify opportunities for students to learn how to translate word problems into mathematical symbols. Initially, all algebra chapters were reviewed to examine the extent to which they support this skill. However, we found that many chapters did not include such problems. Therefore, we focused on the chapter on linear equations to explore the types of opportunities reflected in the survey questions.

We used a simple coding schema to count the problems that offer students a chance to translate text into mathematical symbols. Our focus was only on the exercise problems in the linear equations' chapters. This is because the 6-question survey was designed to assess how well students could translate real-world situations into simple linear equations and vice versa, which also aligns with the algebra content standards in the Nepali middle school curriculum.

RESULTS

Results from the quantitative study (student data)

Table 1 shows how students responded to the first three multiple choice questions. For the first question—'Hema has three times more cards than Radha, and altogether, there are 36 cards. Create an equation to represent this'—about 64% of students chose the

correct equation, $x+3x=36$. However, many were distracted by incorrect options such as $3x=36$, $x+3=36$, and $3x+36=x$.

In the second question— ‘In a school assembly, with six students per line, there is a total of n students. Select an expression to determine the total number of lines’—only 27% selected the correct answer $n/6$. About one-third chose $6n$, while the remaining 40% picked $n-6$ or $6/n$.

For the third question, which asked students to identify the correct context for the equation $4x + 1 = 105$, nearly half answered incorrectly.

Options	Question 1	Question 2	Question 3
A	15	24	46*
B	15	16	27
C	64*	32	20
D	6	27*	7

Table 1: Answer choices (%) for the first three problems (*=correct choice).

In the fourth open-ended question, students were asked to translate the statement: ‘Tirtha is exactly one year older than Balendra’ using T for Tirtha’s age and B for Balendra’s age. Only 9% wrote the correct equation: $T=B+1$. About 25% of students forcefully tried to use an x variable, which led to invalid equations. Others either made the incorrect switch between the variables, didn’t write an equation, or left the question blank (see Table 2).

$T=B+1$ *	$B=T+1$	Not an equation	Blank answer	Incorrect answer with x variable	Other Incorrect
9	13	28	22	24	4

Table 2: Percentage distribution of student responses to question 4.

In problem 5, students were asked to create a real-world situation that could be described by the equation $3x-4=11$. Only 7% of students provided a correct context. Another 6% attempted a word problem but didn’t get it right. About 28% left the question blank, and 59% simply solved the equation instead— though 10% of those still got the solution wrong (see Table 3).

Correct context	Incorrect context	Solves equation	Blank answer
7	6	59	28

Table 3: Percentage distribution of student responses to question 5.

In problem 6, students were asked to convert the sentence, ‘One-half of five less than a number is equal to 16’ into an equation. Only 10% did this correctly (Table 4). The

remaining 90% either made errors or did not even attempt it at all, with 9% leaving it blank.

$(x-5)/2=16^*$	All other incorrect equation	Incorrect	Not an equation	Blank answer
10	46	28	7	9

Table 4: Percentage distribution of student responses to question 6.

Results from the qualitative study (textbook analysis)

We examined textbooks from grades 6 to 9 to investigate the opportunity for translating real-world contexts into mathematical symbols. Since the problems included in the student test mostly found in the algebra chapters—particularly in the linear equations section—we initially concentrated on those sections. Other algebra subchapters, such as the rule of indices, factorization, and multiplying expressions, were reviewed, but they did not contain relevant problems. Therefore, we do not report the findings from those sections.

In summary, the chapters on linear equations for grades 7-9 generally begin with a review of concepts learned in the previous year. They then introduce procedures for solving these equations and include a few real-world problems that require translating words into mathematical symbols. For example, Lesson 22.1 in the grade 8 textbook starts by reviewing linear equations like: 'If $x - 3 = 5$, what is the value of x ?' This is followed by five worked examples demonstrating solution methods for equations such as $5x - 6 = 3x + 10$ (Example 3, p. 195). Only Example 5 offers an opportunity to translate word problem into symbols. It states: 'If the length and breadth of a rectangular field are in a 5:3 ratio, and the perimeter is 400, then (a) write the equation for the perimeter, (b) find the length and width, and (c) find the area.' These worked examples are designed to help students become familiar with solving linear equations. The translation skills examined in this study are present, though only to a limited extent. Table 6 presents the distribution of problems that do or do not offer opportunities to translate real-world contexts into mathematical symbols.

	Grade 6	Grade 7	Grade 8	Grade 9
Required	4	10	8	2
Not required	34	31	29	30

Table 6: Word to symbol translation opportunity in textbooks.

These problems, drawn from the exercise sections of the linear equation chapters in grades 6-9, were found to incorporate relevant concepts. However, even within the

linear equations chapters, such problems were limited. Notably, we found no problem that ask students to translate symbolic expressions or equations back into real-world contexts. Although there are more than 1300 algebra problems across grades 6-9, only about 25 provide opportunities for text-to-symbol translation—and none for symbol-to-text translation.

DISCUSSION

Based on our findings, the discussion focuses on two main points.

First, students struggle to translate real-world contexts into mathematical problems using symbols. On average, more than half of the participants were unable to select correct linear expression or equations for real-world problems (Questions 1-3). Fewer than 10% of students could write a linear equation to represent a real-world situation (Questions 4 and 6). The most concerning finding is that only 7% of students were able to construct a real-world context corresponding to a given linear equation (Question 5). We note that the first three problems were multiple-choice and closed-ended, typically requiring recognition rather than generation. In contrast, the last three questions were significantly more challenging, as they require students to produce their own responses. This difference in format may have influenced the results. The skewed performance could be attributed not solely to a lack of understanding, but to the increased cognitive demand of open-ended tasks.

Secondly, the follow-up qualitative study revealed that textbooks from earlier grades provided limited opportunities for students to translate text into mathematical symbols. Moreover, there were no tasks requiring students to convert symbols into real-world contexts. This pattern—students performing relatively better on text to symbol translation tasks but struggling with symbol-to-text translation— suggests a link between student achievement and textbook content, reinforcing Törnroos's (2005) findings.

Reports from NASA studies show that Nepali students struggle with open-ended, contextual, and high-demand problems, with most students not even attempting such questions (Education Review Office, 2018). Additionally, research suggests that school mathematics textbooks in Nepal lack features that promote critical thinking skills (Basyal et al., 2023; Nepal et al., 2025; Yang & Basyal, 2024; Basyal & Mainali, 2025). When reviewing our findings alongside NASA reports and earlier textbook studies in Nepal, we observed that students have limited opportunities to engage with these types of problems. As a result, fewer than one in ten students were able to solve the open-ended problem correctly.

Translating word problems into symbols—and vice versa—remains a challenge for students globally. Egodawatte (2011) found that 11th grade students in Canada struggle with word problems, particularly when converting everyday language into algebraic expression (p. iii). Similarly, middle school students in Turkey have faced difficulties in bridging the natural and mathematical world, with researchers attributing these

challenges to inadequate reading comprehension (Duru & Koklu, 2011). In the United States, Capraro and Joffrion (2006) reported that middle school students were not adequately prepared to translate words into mathematical symbols, largely due to a lack of conceptual understanding.

Pawley et al. (2005) argue that text-to-symbol translation skills are developed and refined through sustained exposure to increasingly rigorous problems, accompanied by balanced instructional support as students advance. Therefore, curricular resources that explicitly teach this skill are essential for helping students build proficiency. Furthermore, introducing focused instruction on both text-to-symbol and symbol-to-text translation could strengthen students' ability to move between real-world contexts and mathematical representations. This, in turn, could significantly enhance the overall quality of mathematics education in Nepal.

This study examined the challenges that 10th graders face when converting text to symbols and vice versa. It also analysed textbooks from grades 6 to 9 to explore opportunities for text-to-symbol translation. We recognize that these challenges may differ across other grade levels, which warrants further investigation.

Additionally, the same textbooks may contain relevant problems in other chapters, such as ratio and proportion and profit and loss, which were not included in this analysis. These sections could offer different insights. Although the government textbooks analysed in this study dominate the Nepali market (over 70% share), some participating students may have been exposed to textbooks from private publishers, which could provide additional opportunities to learn the concepts discussed in this paper.

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TÖLVÍSI - A 19TH CENTURY ICELANDIC TEXTBOOK IN ADVANCED ARITHMETIC AND ALGEBRA

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In 1865, Björn Gunnlaugsson—recently retired after forty years of teaching at Iceland’s sole Latin school—published Tölvísi (“Number Wisdom”), the first Icelandic textbook on advanced arithmetic and algebra. The work closely resembles Danish textbooks by L. Fallesen (1834) and Christian Ramus (1855). It distinguishes itself through a more eloquent style and a particular emphasis on divisibility and the precision of approximations, reflecting Gunnlaugsson’s major scholarly achievement, a geodetic survey that produced the first reliable map of Iceland. Only half of the manuscript was ever printed, and it omitted practical subjects such as ratios, proportions, and interest calculations. The book was never adopted for classroom use. This paper examines the purpose of Tölvísi within its contemporary Icelandic culture.

INTRODUCTION

Tölvísi (Gunnlaugsson, 1865) was the first Icelandic textbook on advanced arithmetic and algebra, and the only one of its kind until 1927 (Daníelsson, 1927). It appeared at a time when Iceland was still primarily an agricultural society with limited infrastructure, no system of organized primary education, and a single Latin school at gymnasium level. This raises a central (research) question: What purpose did *Tölvísi* serve?

According to Niss (1996), mathematics education may serve three broad functions:

- supporting technological and socio-economic development of society,
- contributing to society’s political, ideological, and cultural life, and
- providing individuals with prerequisites for coping with everyday challenges.

To understand why *Tölvísi* was composed, it is essential to consider the life and work of its author, Björn Gunnlaugsson (1788–1876), as well as the broader conditions of eighteenth- and nineteenth-century Iceland. Moreover, to answer the research question, this paper presents findings from an examination of the book and relevant sources and manuscripts, along with a comparison between *Tölvísi* and two Danish textbooks frequently referenced in it.

THE 18TH AND EARLY 19TH CENTURY IN ICELAND

Iceland became a tributary of Denmark under the Kalmar Union of 1397. By the eighteenth century, severe climatic conditions and Enlightenment ideas encouraged

both Danish authorities and the Icelandic elite to pursue modernization. Censuses, reports, and geodetic surveys were undertaken, small-scale industries established, and informative literature on farming, fishing, and health was published.

The volcanic eruption of 1783 and earthquakes of 1784 brought severe setbacks: massive livestock loss, famine, and a population decline of about 20%, leaving fewer than 40,000 inhabitants. Educational institutions also suffered. One of the two cathedral schools closed permanently in 1784, and interruptions in teaching meant that at times—such as 1784–85 and 1804–05—neither school was operational. In 1805, the schools were consolidated into a single Latin School at Bessastadir near Reykjavík, and the same year, the Icelandic Literary Society was founded to promote practical and scientific publications. Meanwhile, the Danish–Norwegian army continued land surveys between 1800 and 1815.

BJÖRN GUNNLAUGSSON

Björn Gunnlaugsson (1788–1876), a gifted yet unconventional youth, was unable to attend school during its closures and later grew too old for admission. Nevertheless, he encountered land surveyors who provided him with mathematics books, sparking his lifelong passion for the subject. In 1808, he graduated from a private school run by the bishop, receiving the highest commendation for his mathematical knowledge (P. + B., 1883). Although considered too eccentric for the priesthood, Gunnlaugsson enrolled at the University of Copenhagen in 1817–1822, where he studied mathematics under Professor Carl Degen. During the summers of 1820 and 1821, he worked on land surveys in Holstein with Professor Heinrich Christian Schumacher (Björnsson, 1997). Concerned about the lack of mathematical instruction at Iceland's Latin School, Gunnlaugsson petitioned the authorities and was eventually appointed to teach mathematics and science there, a position he held from 1822 to 1862.

From 1831 to 1843, he travelled extensively across Iceland conducting surveys that culminated in the first reliable map of the country, published in 1849. For these contributions, he was awarded the Knight's Cross of the Order of Dannebrog and the French Légion d'Honneur.



Fig. 1. Map of Iceland (Gunnlaugsson & Olsen, 1849).

TÖLVÍSI – NUMBER WISDOM

When Björn Gunnlaugsson retired in 1862, at the age of 74, he presented his plan for an arithmetic textbook to the Icelandic Literary Society:

It is the theoretical arithmetic, even though the practical items are mixed in with it, then I valued them less, as the Icelandic arithmetic textbooks [published in 1780, 1785, and 1841] contain them ... I have included many items that are not common and had in mind that it will contain all that is taught at schools and even more (NULI, *Lbs.* 2590 4to).

Published by the Society in 1865, *Tölvísi* covered arithmetic in the decimal system as well as other bases (e.g., 5 and 12). Particular emphasis was placed on “the nature of whole numbers,” especially divisibility. Gunnlaugsson explained this not only through what he termed primitive modules but also with the aid of Fermat’s Little Theorem.

He also treated the use of continued fractions to approximate both rational and irrational numbers—subjects of central importance for land-surveying.

The book ended abruptly on page 400, in the midst of its treatment of continued fractions. The Literary Society had agreed to publish 25 quires (400 pages) in 700 copies and to purchase the second part of the manuscript, but it declined to finance further printing.

Primitive modules

Primitive modules classify all whole numbers into three modules. Then various transverse sums disclose divisibility by primes, of which 3 and 11 are best known, but e.g. also 73:

$$\begin{aligned}
 N &= b + b_1 A^1 + b_2 A^{2t} + b_3 A^{3t} \dots \dots \dots \text{I.} \\
 &= b + b_1 + b_2 + b_3 \dots + b_1(A^t - 1) + b_2(A^{2t} - 1) \\
 &\quad + b_3(A^{3t} - 1) + \dots \dots \dots \text{II.} \\
 &= b - b_1 + b_2 - b_3 + \dots + b_1(A^t + 1) + b_2(A^{2t} - 1) \\
 &\quad + b_3(A^{3t} + 1) + \dots \dots \dots \text{III.}
 \end{aligned}$$

Fig 2. The three-fold classification works in place-value systems with base A.

Fermat’s Little Theorem

If p is a prime number, then for any integer a , not divisible by p , there exists a smallest exponent d such that d divides $p-1$, hence $a^d - 1 \equiv 0 \pmod{p}$.

The theorem gives certainty of divisibility by p , while the number may be very high. The smallest exponent of A may not be $p-1$, but an exponent d that divides $p-1$.

Example: $p = 41$: $p-1 = 40$; $40 = 5 \cdot 8$, lowest exponent is 5, and 41 divides $10^5 - 1 = 99999$. This applies to some numbers in class II, such as 11, 37, 41, 73, ...

In addition, Fermat’s theorem applies in other place-value systems (p. 210).

BJÖRN GUNNLAUGSSON’S INTELLECTUAL SPHERE

Björn Gunnlaugsson’s intellectual life was shaped by two parallel pursuits: on the one hand, his concern with land-surveying and numerical precision; on the other, his

engagement with religious philosophy and cosmology, expressed most fully in his long poem *Njóla, or an Easy Contemplation of the Sky* (Gunnlaugsson, 1842), which was frequently reprinted.

During his student years in Copenhagen, he read works by Euler, Lagrange, and Gauss, though there is no evidence that he engaged with the rapidly developing mathematics of the 19th century after his return to Iceland. His professor, Carl F. Degen, had specialized in continued fractions, an area to which Euler had contributed significantly (Björnsson, 1997).

The Danish authorities supplied Iceland's Latin School with contemporary textbooks, such as L. Fallesen's *Begyndelsesgrunde i den rene Mathematik* (1834) and C. Ramus's *Elementær Algebra* (1855). Gunnlaugsson used Fallesen's textbook in his teaching between 1851 and 1855, and Ramus' during his final years at the school, 1855–1862 (*Skólaskýrslur*, 1851–1862). In *Tölvísi*, he frequently referred to these books, comparing their strengths and weaknesses and explaining his own choices. Ramus in particular was known for his rigor, but his highly concise style made his texts difficult to follow; as Heegaard (2011) noted, Ramus insisted on exactness but may have lacked a sense of what was pedagogically accessible.

In the printed part of *Tölvísi*, Gunnlaugsson did not introduce new subject matter beyond what was covered in Fallesen's and Ramus' books, but he expanded on traditional topics in advanced arithmetic and algebra. He explained how theorems applied across different place-value systems, and he supplied numerous examples of divisibility tests. Some were practical and familiar—such as the tests for 9 and 11, also found in Fallesen's and Ramus' books—while others were more advanced, relying on his concepts of primitive modules and Fermat's Little Theorem.

THE STRUCTURE OF THE THREE TEXTBOOKS

Table 1 compares the length and topical coverage of the three textbooks by L. Fallesen (1834), C. Ramus (1855), and B. Gunnlaugsson (1865) by counting the pages devoted to each subject. The manuscript of *Tölvísi* (NULI, Lbs. 2397, 4to) originally comprised 1,583 pages. Of these, 400 printed pages corresponded to 772 manuscript pages, giving a ratio of 1.93. Applying this ratio, the remaining 812 manuscript pages would correspond to approximately 420 printed pages. Page counts per topic in *Tölvísi* have been estimated on this basis.

Topic/Author & year of publication	L.Fallesen 1834	C.Ramus 1855	B. Gunnlaugs- son 1865
Introduction and counting	11 p.	2 p.	7 p.
The four arithmetic operations in whole numbers, fractions, and polynomials	58 p.	61 p.	120 p.
Properties of whole numbers and polynomials, primitive modules, Fermat's Little Theorem (FLT)	31 p. FLT (4 p.)	41 p. FLT (2 p.)	78 p. (FLT 16 p.)
Decimal fractions, periods	35 p.	37 p. FLT	101 p. FLT
Powers and roots, imaginary numbers	17 p.	21 p.	73 p.
Continued fractions	10 p.	10 p.	12 p.
Subtotal pages	161 p.	172 p.	400 p.
Continued fractions			~ 33 p.
Equations	38	71 p.	~ 251 p.
Ratio & proportions, progressions, series, logarithms, (interests, permutations, combinations)	24	37 p.	~ 136 p.
Total pages	223 p.	279 p.	~ 820 p.

Table 1: Number of pages devoted to various topics in textbooks by L. Fallesen (1834), C. Ramus (1855), and B. Gunnlaugsson (1865).

The three textbooks follow broadly similar structures: their topics largely overlap and appear in roughly the same order. Table 1 presents the topics in the order they occur in *Tölvísi*. Two notable exceptions stand out. Ramus divided the important topic “Properties of whole numbers” into two sections, the latter containing advanced number-theoretical material absent from the other two authors. Fallesen, by contrast, split the treatment of equations into two parts, with quadratic equations forming a separate section. Only Gunnlaugsson included permutations and combinations.

Exact comparison of size is complicated by differences in page format and mathematical layout, which make word counts per page difficult to measure. Using sample pages of continuous prose, rough estimates suggest the following:

- L. Fallesen: ~310 words per page
- C. Ramus: ~207 words per page
- B. Gunnlaugsson: ~391 words per page

These figures indicate that *Tölvísi* contains significantly more text per page than the other two textbooks. Combined with the topic lengths shown in Table 1, this suggests

that Gunnlaugsson's exposition was more elaborate and discursive than that of Fallesen or Ramus.

Gunnlaugsson devoted considerable space to continued fractions. He emphasized their advantages over decimal fractions for approximation, noting, for example, that continued fractions could often be simplified by cancellation without significant loss of accuracy (Gunnlaugsson, 1865, p. 388). This reminder is striking, since in the early 1860s no calculating machines existed to ease the work of human computers and land surveyors.

More than half of Gunnlaugsson's unprinted manuscript was dedicated to advanced equations. These included quadratic, cubic, biquadratic, exponential, and Diophantine equations, systems of linear equations, and even derivatives of functions.

Practical topics, however, such as ratio, proportion, simple and compound interest, annuities, and logarithms, were excluded from the 400 printed pages of *Tölvísi*. Such material would likely have been useful to Iceland's growing class of merchants and craftsmen. By contrast, Gunnlaugsson devoted twelve printed pages to imaginary numbers, which he described metaphorically as "airships in physics, sailing around in the nonexistence of *Ginnungagap*," the primordial void of Norse cosmogony (Gunnlaugsson, 1865, pp. 350–351; Bjarnadóttir, 2009).

THE FATE OF TÖLVÍSI

The 700 copies of *Tölvísi* were distributed to subscribers of the Icelandic Literary Society across a country of roughly 5,000 households. By 1865, only about 250 graduates of the Latin School at Bessastadir or Reykjavík had the background to follow its contents.

A necrology published in the distinguished journal *Andvari* devoted just two lines out of thirteen pages to the book: "*Tölvísi* is one of the books that everyone praises but no one reads. The explanations are said to be very well-defined" (P + B, 1883). The authors of that note are thought to have been Páll Melsted and Björn Jensson. The latter, Gunnlaugsson's grandson, studied at the Polytechnical Institute in Copenhagen and was about to succeed his grandfather as head mathematics teacher at Reykjavík Latin School. He later became a respected and beloved teacher.

Tölvísi was never adopted as a school textbook. Gunnlaugsson's temperament was not unlike that of Ramus (Heegaard, 2011): he may have lacked sensitivity to what was easy or difficult for students, and still more for general readers. His pupils respected and admired him, but chiefly for his achievement in geodetic measurement and his production of Iceland's first reliable map (Melsted, 1912).

GUNNLAUGSSON'S IMPACT ON ICELANDIC MATHEMATICS EDUCATION

Although *Tölvísi* itself had little influence, Gunnlaugsson's teaching shaped the next generation. Two of his former students, neither of whom pursued mathematics beyond the Latin School, went on to publish widely used arithmetic textbooks.

- Lawyer Jón Guðmundsson (1841) authored *Reikningslist, einkum handa leikmönnum* (*The Art of Arithmetic, Mainly for Laymen*), which taught traditional computation in whole numbers and fractions, alongside the use of measurement units and advice on prudent resource allocation. Gunnlaugsson tested the book in his own teaching for one year but did not find it suitable (Skólaskýrslur, 1826–1862).
- Reverend Eiríkur Briem published *Reikningsbók* (*Arithmetic*) in 1869, later expanded into two volumes (1880). His book included compound interest and logarithms and instructed readers on how to maximize agricultural output, avoid unnecessary spending, and invest wisely. As guardian of Iceland's first bank, Briem linked arithmetic directly to financial literacy. His book circulated widely and was reprinted until 1911.

Both Guðmundsson and Briem warned against borrowing for non-revenue purposes. Through these students, Gunnlaugsson indirectly influenced Icelandic mathematics education, as they adapted his ideals to practical contexts. Yet *Tölvísi* itself was little read and scarcely understood.

CONCLUSIONS

In 1865, Iceland had little demand for higher mathematics. The economy remained primarily agricultural, supplemented by open-boat fishing, and the country lacked infrastructure such as harbours, roads, and bridges. Industrialization was still on the horizon, and the need for advanced mathematics or technical education was minimal.

Returning to Niss's (1990) three functions of mathematics education, and to the research question—what purpose did *Tölvísi* serve?—the following may be said:

- *Tölvísi* did not advance Iceland's technological or socio-economic development. Gunnlaugsson demonstrated his mastery of precise computation, but no Iclander in the 19th century carried out surveying or calculation at his level.
- It did not contribute to society's political, ideological, or cultural development. While it sought to establish a higher mathematical tradition, its audience was extremely limited, and it was never adopted in schools.
- It served only a handful of individuals who may have benefited personally from its content. Those seeking advanced knowledge could as readily consult Danish texts.

In short, *Tölvísi* found no real purpose in 19th-century Iceland. Its time had not yet come.

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ANALYSING THE AUTHENTICITY OF A U.S. STATISTICS TEXTBOOK

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We conducted an analysis of over 1200 problems from a textbook for Advanced Placement Statistics using a modified authenticity framework from Paredes et al. (2020). Each task was evaluated based on five components: event, question, data, purpose, and specificity of information. While the textbook problems showed high authenticity in events, data, and specificity of information, they often lacked a clear purpose and rarely involved the student's role. These findings suggest that curriculum designers can enhance authenticity by focusing on purpose and student involvement, thereby improving real-world application and engagement in statistical learning.

INTRODUCTION AND RATIONALE

There has been a call throughout the field of mathematics and statistics education to increase the presence of real-world application and authentic problem-solving in the activities and tasks that students are asked to do during mathematics class (e.g., Depaepe et al., 2010; Palm, 2006, 2008). Mathematics word problems are often ridiculed because they represent a world that only exists in mathematics class (Palm, 2008). Additionally, many mathematical explorations that students do in K-12 mathematics classrooms seem to be contrived and not representative of the work done in the real world (Su, 2020; Wu, 2011).

Some mathematical concepts lend themselves to authenticity more easily than others. In statistics, numbers cannot be considered data without a relevant context, typically based in a real-world phenomenon (Lovett & Lee, 2017; Scheaffer et al., 2008). Learning statistics has become increasingly important for succeeding in 21st century careers (Freeman, 2021; Kuiper & Sturdivant, 2015; Nolan & Temple Lang, 2015). Despite this relevance, students continue to dread statistics, finding the content challenging and confusing, and often failing to see the value in studying it (Kuiper & Sturdivant, 2015; Wathen & Rhew, 2019). Efforts have been made to discover ways to alleviate some of these bad feelings towards statistics, one of which has been to shift the problems that students do in statistics courses to be more authentic in nature (e.g., Blackburn, 2016; Freeman, 2021; Nolan & Temple Lang, 2015). In the age of the internet, where data and information can be accessed with a few keystrokes, it has become more reasonable than ever to expect that statistical tasks in school represent authentic situations from the real world (Lovett & Lee, 2017; Scheaffer et al., 2008; Starnes & Tabor, 2018).

Textbooks are often used to support learning in mathematics and statistics classrooms, so the way a concept is presented can influence how students engage with that concept (Confrey & Stohl, 2004; O’Keeffe, 2013). A textbook analysis helps assess the potential for authentic statistical learning by revealing how its presentation may shape teachers' and students' thinking about statistics (O’Keeffe, 2013). The purpose of this paper is to report findings of a textbook analysis of a popular Advanced Placement (AP) Statistics book to determine how authentic the problems are for students. AP Statistics is an introductory statistics course for high school students in the United States covering topics that include descriptive statistics, distributions, probability, confidence intervals, hypothesis testing, and regression. The authors of this textbook, “deliberately chose contexts and data sets in the examples and exercises to pique [students’] interest” (Starnes & Tabor, 2018, p. xii). This textbook was selected because it is among the most recommended textbooks for teaching AP Statistics, so it is commonly used to teach the course, and it is known for using real-world applications.

This textbook analysis was conducted to answer the research question: To what extent does the textbook attend to the authenticity of the context? The goal of this question is to determine the potential the textbook has in supporting the implementation of authentic, real-world learning of statistics (O’Keeffe, 2013), given how important this is for student learning and interest in statistics (e.g., White, 2019). Moreover, the analysis contributes an adaptation of the theoretical framework described in the next section that is applicable for statistical problems and accounts for some nuance in what counts as authentic.

THEORETICAL FRAMEWORK

Palm (2006) developed an authenticity framework that can be used to analyse mathematics word problems. Paredes et al. (2020) modified Palm’s framework to analyse tasks developed by pre-service teachers. Paredes and colleagues assessed “real-world problems” for their authenticity, or in concordance with how the context actually happens in the real world. The framework describes authenticity across five components: event, question, data, purpose, and specificity. *Event* describes the realistic nature of the context, *question* refers to the problem being posed, *data* concerns whether the data are feasible and how data are collected, *purpose* indicates the reasoning/rationale behind solving the problem, and *specificity* considers the amount of details provided about the given context. In this study, the authenticity framework was adapted and extended to apply to statistics tasks connected to real-world contexts, including word problems, projects, and simulations.

METHODOLOGY

Data Source

The data source for this textbook analysis was *The Practice of Statistics, Sixth Edition* by Starnes and Tabor (2018). Each of the twelve chapters consisted of two to three sections, with main text that was broken up periodically by boxes denoted as either

“Activity,” “Example,” “Check Your Understanding,” or “Technology Corner.” The end of each section has a variety of practice problems, called “Exercises,” and the end of each chapter has extra problems, such as free response questions, review exercises, projects, and AP Statistics practice tests. For the purpose of this analysis, all elements of the textbook that did not typically include a real-world context were excluded from consideration, leaving just the “Example,” “Check Your Understanding,” and “Exercise” elements for analysis. This decision was made because, although a purely theoretical statistics problem could be authentic to how statisticians work, the nature of the theoretical framework only supports analysis of the authenticity of a real-world statistical application. Each analysed element was broken up by sub-question, where applicable (i.e., Example 1a, 1b, 1c), to ensure each question was coded for authenticity. There were over 100 exercises in each chapter, which were repetitive in nature for the purpose of providing students with practice. To make the analysis manageable, a random sample of eight exercises per section (24-32 per chapter) was selected to provide a representative sample of the exercises in each chapter.

Coding the Textbook

The codebook for this study was adapted from that of Paredes et al. (2020), as it provided useful indicators for each of the five components. We modified their framework from a (0, 1) binary scale to a (0, 1, 2) scale to include a grey area of authenticity for each of the components. Table 1 contains the codes we used, with our modifications of indicators from Paredes et al. (2020) displayed in italics.

Component	Code	Indicators
Event	2	The proposed situation is feasible in real life.
	1	<i>The situation appears unexpectedly, perhaps in the midst of another context; or the context makes sense but the details are inconsistent with the context.</i>
	0	The proposed situation is imaginary, although it could be related to a real situation. It could happen in real life but in an unusual way or is purely mathematics. <i>This may only occur in a classroom setting.</i>
Question	2	The question is formulated on a regular basis for the described event, and its answer has practical value or is of interest to others who are not attracted by mathematics.

	1	<i>The question may be formulated on a regular basis, but not explicitly; the answer to the question is relevant, but the question is not asked in a realistic way; or the question corresponds to the event described, even though the event is not a real-world situation.</i>
	0	The question would not be formulated in the real world, or it does not correspond to the event described.
Data	2	The data or data collection processes correspond with the real ones. That is, the data are from a referenced source or are realistic in nature and could have come from real data.
	1	<i>The data are feasible but are harder to confirm and somewhat outside of what would be expected (would require a reference to believe); sources allow for a critique of the data; or the data collection methods are unrealistic; or the data corresponds to the event described, even though the event is not a real-world situation; or the general description of data usage and collection is considered realistic, but there is no data being used and the purpose of the problem relies on data.</i>
	0	The data does not correspond with the real ones or this information is only accessible through competencies that are required in a simulated situation (e.g., means or standard deviations).
Purpose	2	The purpose of the event is explicitly mentioned and it is in accordance with that of the actual situation.
	1	<i>The purpose is clear, but implicit and in accordance with that of the actual situation.</i>
	0	The purpose is not clear enough or the task is described without referring to any specific situation; or the stated purpose is contradictory to the action.
Specificity	2	The characters of the problem have clearly defined roles in accordance with the situation, the objects are defined or familiar, and the places are specific. The problem is formulated in the 1st or 2nd person.

- 1 *Some specificity is provided, but a lot of questions still remain, including, “How were the data collected?”*
- 0 The situation is general without specifying objects and subjects, or the names of the characters are provided but not their role, which means that other aspects such as realism of the data cannot be assessed.

Note: Italics indicate our modifications to Paredes et al. (2020, p. 7)

Table 1: Codes for components of authentic problems

Additionally, during preliminary coding rounds, we noticed that some problems also included a role for the student. To distinguish these problems from the others, we introduced a new code of 2+ for specificity that was applied for any authentically specific problem that also included the student role. Problems that position the students as actors in the problem situation provide students with agency (Herbel-Eisenmann, 2007) similar to what they could experience in that same context outside of the school environment.

The following problem demonstrates an example of a fully authentic problem, with part (a) coded as 2 for specificity and part (b) coded as 2+:

A large study of 3000 Canadian children and their mothers found that the children of mothers who drank diet soda daily during pregnancy were twice as likely to be overweight at age 1 than children of mothers who avoided diet soda during pregnancy.⁴⁹

(a) A newspaper article about this study had the headline “Diet soda, pregnancy: Mix may fuel childhood obesity.” This headline suggests that there is a cause-and-effect relationship between diet soda consumption during pregnancy and the weight of the children 1 year after birth. However, this relationship could be explained by other variables. What other variables might be confounded with a mother’s consumption of diet soda during pregnancy?

(b) You have 300 pregnant mothers who are willing to serve as subjects in an experiment that compares three treatments during pregnancy: no diet soda, one diet soda per day, and two diet sodas per day. Write a few sentences describing a completely randomized design for this experiment. (Starnes & Tabor, 2018, p. 265)

This example problem was scored as authentic because the study described is one that would likely take place in the real world (event), the questions are ones that researchers would consider when designing the study (question), there is a superscript reference showing that the data came from a real source (data), the study is clearly trying to determine whether diet soda during pregnancy impacts childhood obesity (purpose), and there are a number of details that support the understanding of the problem (specificity). However, only part (b) reached the special case of 2+ for specificity because it positioned the student as being a researcher running an experiment to test

this finding, while part (a) does not explain the role that the student plays in relation to the newspaper article.

Achieving Reliability

The first and second authors simultaneously coded three randomly selected chapters, which comprised 25% of the textbook. To build inter-rater reliability (Cohen, 1960; Creswell & Poth, 2018), after each section within a chapter, the coders met to resolve disagreements and solidify the codebook. In the last round of coding, recommended inter-rater reliability of above 80% (Cohen, 1960) was achieved with 91% reliability on event, 83% on question, 87% on data, 87% on purpose, and 85% on specificity. After achieving reliability, the coders evenly divided the remaining textbook chapters (i.e., 4½ randomly assigned chapters), discussing codes as needed. Once the coding was completed, the number and percentages of instances of each level (0, 1, 2) per authenticity component were calculated.

RESULTS

Across the three textbook categories, a total of 1238 problems were coded. The majority of these problems were found to be authentic (see Table 2), with higher rates of authenticity for the components of event (78%), data (76%), and specificity (83%), than those of the components question (55%) and purpose (51%). It should be noted that most coded problems did not specify the student's role, with 4% obtaining the 2+ code for including an authentic student role (not represented in Table 2).

In contrast, a substantial number of problems (18%) have no clear purpose, followed by 11% of coded problems describing unrealistic events/contexts. The question component, on the other hand, stands out as less authentic than other components based on its percentage of problems coded as 2, but only 8% of the questions are considered as not being connected to the real world or the event described.

Code	Event	Question	Data	Purpose	Specificity
2	78%	55%	76%	51%	83%
1	11%	37%	16%	31%	12%
0	11%	8%	8%	18%	5%

Note: Here, the row for "2" encompasses all code assignments of 2 and 2+.

Table 2: Percentage of $n = 1238$ problems at each code level, by component

This analysis revealed that the textbook did a nice job of creating mostly authentic problems to engage students. However, they were not typically up to the level of complete authenticity according to the framework, usually leaving out the purpose behind the action that students were asked to take, and rarely including the student role as a character in the problem.

DISCUSSION

Overall, the common issue of contrived contexts in mathematics word problems (Su, 2020; Wu, 2011) was not present in this popular statistics textbook. Not only did Starnes and Tabor (2018) include contexts that they believed would interest students, they also intentionally incorporated situations and data that they found out in the real world, which created mostly authentic problems throughout the textbook. Thus, this textbook does seem to address the call for more authenticity in the types of problems statistics students engage with in their courses (Blackburn, 2016; Freeman, 2021; Nolan & Temple Lang, 2015).

Conversely, one problem that Herbel-Eisenmann (2007) found in her textbook analysis, namely that textbooks typically use authoritative language rather than incorporating the student in the problem as an actor with agency, persisted in this textbook as well. A small percentage of the coded problems in the book positioned the student as having a role within the problem context but the vast majority did not.

These findings suggest curriculum designers should focus on explicitly incorporating purpose and student roles, as these elements are often unintentionally omitted. Teachers using *The Practice of Statistics* can make problems more authentic with minor adjustments, such as the distinction noted above in parts (a) and (b) of the example problem on the impact of drinking diet soda during pregnancy, as the textbook already shows high levels of authenticity.

This study was a single application of the adapted authenticity framework (Paredes et al., 2020), and should be further investigated with other statistics and mathematics textbooks and curriculum. The findings, thus, are limited to this single textbook and two researchers' interpretation of the problems and are not representative of all statistics textbooks or even those in the United States.

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INTERACTIVE AND ACCESSIBLE? A FIRST INSIGHT INTO THE ANALYSIS OF DIGITAL TEXTBOOKS WITH REGARD TO THEIR CURRENT DEVELOPMENT STATUS

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This study examines the state of the art of digital mathematics textbooks, focusing on the implementation of interactivity and customisation options as key features of modern digital textbooks. This is done in the context of accessibility, which is regulated in the European Union for digital textbooks and can be beneficial for all students in the context of Universal Design for Learning. In a first step, the most up-to-date digital textbooks used in Germany in the 7th grade are examined. The study shows a discrepancy between potential digital improvements and their actual implementation. There are only a few interactive visualisations and task formats in the digital textbooks and hardly any individual customisation options, which are particularly relevant for accessibility.

MOTIVATION

State-of-the-art digital mathematics textbooks integrate new technical possibilities to promote the learning and teaching of mathematics. We understand digital mathematics textbooks of the latest generation (see Pepin et al., 2015) as follows:

“A digital mathematics textbook is a technology-based curriculum resource for learning and teaching mathematics. It provides enhancement features, such as interactive and multimedia elements, feedback options, adaptivity, and customisation features that distinguish it from a printed textbook.” (Brnic et al., 2024)

Initial studies show the benefits and effectiveness of such digital textbooks and comparable digital curriculum resources compared to printed materials (e.g., Brnic et al., 2024). The advantages of digital mathematics textbooks are attributed in particular to their potentials, such as the integration of interactive elements and digital feedback in the sense of scaffolding. However, digital mathematics textbooks offer even more opportunities, such as the inclusion of customisation options, new possibilities for collaborative work or ecological advantages (Choppin et al., 2014). Digital mathematics textbooks thus offer the opportunity to implement innovative technical possibilities and thereby change the way students approach mathematics (Rezat et al., 2021). The question arises as to what extent such enhancement features are implemented in current digital textbooks.

This is where the European project *Digital Education Material* (DEM) comes in. The DEM project pursues three principal objectives. Firstly, digital textbooks are being

tested for their current state of development and for their accessibility. Accessibility means that a digital textbook and its functions can be used by *all* students. For example, students with disabilities, such as visual impairment or mobility issues, should be able to fully access and use the digital mathematics textbook used in class. Subsequently, comprehensive guidelines will be developed for designing a digital textbook that is accessible and includes enhancement features. Finally, based on the guidelines, initial prototypes will be developed. The study presented here can be assigned to the first objective, the textbook analysis. In this analysis, the interactivity of digital textbooks as a central distinguishing feature compared to printed textbooks and customisation options as a decisive aspect of accessibility are investigated in the first step.

INTERACTIVITY IN DIGITAL TEXTBOOKS

In contrast to printed textbooks, which consist of static elements such as images and text, digital textbooks can also integrate interactive elements. Interactive elements are those that are characterised by “responsiveness to the learner’s action during learning” (Moreno & Mayer, 2007, p. 310). Such interactive features in digital textbooks include, for example, task formats and visualisations that react to user input (Usiskin, 2018). In particular, visualisations help to illustrate mathematical concepts and promote the exploration and understanding of mathematics (Tversky et al., 2002). Furthermore, meta-studies have already shown that dynamic visualisations in particular support learning more than static visualisations (Berney & Bétrancourt, 2016). Interactive elements in digital learning environments can be classified into three main types: *interactivity by dialoguing, manipulating and controlling* (Moreno & Mayer, 2007). Interactivity by dialoguing means that learners receive direct feedback or additional information in response to their input. In a digital textbook, this can be realised through interactive task formats, such as drag-and-drop tasks or tasks in which the solution can be entered. A second type of interactivity is manipulating, in which learners can set parameters or move objects. This is typical for so-called interactive dynamic visualisations (Rolfes et al., 2020), in which learners have a certain amount of control over the respective representation. For example, apps can be integrated into a digital textbook in which the visualisation can be changed using sliders or on-screen input. The third type of interactivity is controlling, i.e., the learner can set the speed or order of what is presented. This type of interactivity can be implemented in linear dynamic visualisations, such as films that represent a dynamic process and have pause/play options or can be played at different speeds (Moreno & Mayer, 2007; Rolfes et al., 2020).

CUSTOMISATION AND ACCESSIBILITY OF DIGITAL TEXTBOOKS

Aspects of controlling, such as speed adjustment or on/off functions, but also the possibility of individual settings, not only represent the potential of digital textbooks (Choppin et al., 2014), but also already meet criteria for an accessible digital textbook (UNICEF, 2019). An accessible digital textbook means that “learners who are blind/have low vision, those who are deaf/hard of hearing, those who have

intellectual/developmental disabilities, learning disabilities, or those who prefer to access information in ways different from visual inputs“ (UNICEF, 2019, p. 8) can also access a digital textbook. The technical possibilities of digital textbooks offer particular potential in this respect compared to printed textbooks. The first attempts to develop criteria for accessible digital textbooks refer to the Universal Design for Learning (UDL) framework (CAST, 2024). In terms of digital textbooks, the idea is not to create several parallel versions of a book, e.g., special editions for students with visual impairments, as is often the case at present. Instead, the aim is to use the UDL to create one version of a textbook that can be used by all learners, taking into account multiple means of representation, action & expression and engagement in the design process (CAST, 2024). In terms of digital textbooks, this means that functions are integrated that allow individual settings to be made, such as text adjustments or speed control of interactive features, but also, for example, the implementation of subtitles for videos (for further suggestions, see UNICEF, 2019). Accessibility of digital textbooks is not just a ‘nice-to-have feature’ and a potential of digital textbooks but is now required by the European Accessibility Act (Directive 2019/882). This European directive requires certain services and products to be provided in an accessible way. The rules apply specifically to digital technologies, so that e-books and digital textbooks must also be accessible. The European Accessibility Act is linked to the Convention on the Rights of Persons with Disabilities (United Nations, 2006 Article 24), which has been ratified by almost all states and declares that persons with disabilities should not be excluded from education and should receive appropriate support. Despite the relevance of the accessibility of digital textbooks, there is little research on the current state of development, design or use of digital accessible mathematics textbooks.

THE PRESENT STUDY

The study presented here is part of the multi-stage DEM project. In the first project phase, current digital textbooks are examined in terms of their state of development. The analysis is based on a catalogue of criteria developed in the project (see Digital Educational Material (DEM), 2025). The following two research questions, which provide an initial insight into textbook analysis, are derived from these criteria and from the importance of interactivity and customisation options with regard to the educational potential and accessibility of digital textbooks: To what extent are interactive elements already implemented in digital textbooks (RQ1)? To what extent are customising elements already implemented in digital textbooks (RQ2)?

Method

This study examines digital textbooks for mathematics teaching and learning. Therefore, the latest versions of digital textbooks for the seventh grade in Germany are compared with each other. To ensure that the textbooks are based on the same curriculum, the selection was limited to one grade and the version for the federal state of North Rhine-Westphalia was always selected. Accordingly, these are officially approved textbooks that are freely available on the textbook market. To take into

account different textbook concepts and to get a broad overview, a total of six different textbooks from the three largest textbook publishers (Klett, Cornelsen, Westermann) were selected, which sell a large proportion of textbooks in Germany. First, one textbook was selected from each publisher that emphasises differentiation and heterogeneous learning groups. These are textbooks that are particularly advertised for comprehensive schools (German *Gesamtschule*) where all school-leaving qualifications can be achieved. In addition, one textbook was selected from each publisher that is designed for the highest school type, i.e., the German Gymnasium (grammar school). The textbooks are always the student versions, as the functionalities for these are the focus of the study. The textbook analysis was conducted using qualitative content analysis (Mayring, 2015). The categories were developed and discussed within the DEM project group on the basis of the literature (see category system: Digital Educational Material (DEM), 2025). Regarding RQ1, each of the three types of interactivity – dialoguing, manipulating, and controlling – was examined. Therefore, these types of interactivity were subdivided into specific and distinct *categories* that illustrate the potential of digital textbooks: linear-dynamic visualisations (controlling interactivity) and interactive-dynamic visualisations (manipulating interactivity) as well as interactive task formats (dialoguing interactivity) (see Moreno & Mayer, 2007; Rolfes et al., 2020). In relation to RQ2, a distinction is made between options for customising the interactive elements examined in RQ1 and customising text settings in digital textbooks. The option to adjust the text to individual perceptions is a central UDL guideline (CAST, 2024; UNICEF, 2019). This study specifically examines the extent to which there are setting options for typography, including font, font size, line spacing (leading), character spacing (kerning), line length, text alignment, colour, and text structure (see Digital Educational Material (DEM), 2025).

RESULTS AND DISCUSSION

The textbook analysis is presented in Table 1, sorted by textbook, interactivity and customising. For each textbook, the type of school for which it is mainly designed is also indicated.

The interactive elements are always provided in each textbook as additional materials or as links. This can be attributed to the fact that none of the textbooks was developed as a digital-only textbook. In all cases, the printed textbook was converted into a digital reader, comparable to a PDF file, and then additional material was added. It is also noticeable that although there are some examples of interactive elements, there is no great variation in the interactive elements between textbooks or even within a textbook. This becomes clear, for example, through the integrated videos, which are the only linear dynamic visualisations. These videos are designed to be used as instructional videos in all textbooks, to supplement definitions or sample problems.

Textbook	Interactivity			Customising	
	Linear dynamic visualisations	Interactive dynamic visualisations	Interactive task formats	Text (Typography)	Interactive elements
Lambacher Schweizer (Gymnasium)	58 instructional videos (of which 10 videos have integrated interactive task formats)	None	17 task generators (each generator creates any number of tasks for one specific task type; option to enter a solution number; student receives feedback on correct/incorrect; solution with single steps can be displayed)	None	Play/pause videos; switch full-screen videos on and off; move the video position on the screen; adjust the speed of videos
Schnittpunkt Mathematik (Gesamtschule)	41 instructional videos (of which 10 videos have integrated interactive task formats)	None	82 geometry exercises with a digital tool (student receives feedback on correct/incorrect; students can get hints)	None	Play/pause videos; switch full-screen videos on and off; move the video position on the screen; adjust the speed of videos
Elemente der Mathematik (Gymnasium)	65 instructional videos	10 interactive visualisations (only to the chapter on angles)	36 blocks of 5 tasks (option to enter a solution number; tasks vary when a block is repeated; students receive feedback on correct/incorrect; solution with single steps can be displayed)	None	Play/pause videos; switch full-screen videos on and off; adjust the speed of videos
Sekundo (Gesamtschule)	76 instructional videos	None	12 drag-and-drop tasks; 7 tasks with an interactive coordinate system (for drawing a function); 121 tasks with the option to enter a solution number; 10 tasks in which the solution can be entered as text (for all these tasks, the solution can be checked for correct/incorrect or the solution can be displayed directly.)	None	Play/pause videos; switch full-screen videos on and off; adjust the speed of videos
Fundamente der Mathematik (Gymnasium)	65 instructional videos (only as a YouTube link; not directly integrated into the textbook)	None	None	None	None
Dreifach Mathe (Gesamtschule)	142 instructional videos	None	None	None	Play/pause videos; adjust the speed of videos

Table 1: Results of the textbook analysis.

A few videos in the two textbooks *Lambacher Schweizer* and *Schnittpunkt Mathematik* from Klett stand out here. These videos have integrated interactive task formats, i.e., the learner is asked to enter his or her solution to tasks discussed into the video. Interactive dynamic visualisations can only be found in one book. Such promising visualisations, which offer possibilities for interactivity by manipulation (Moreno & Mayer, 2007), are therefore rarely integrated. Four of the six textbooks also integrate different interactive task formats, which distinguishes them from a printed textbook (Usiskin, 2018). However, most of the formats only require a single number as the solution input. Only the *Sekundo* textbook integrates different types of such dialoguing interactivity. It should be emphasised that students can not only check their solutions, but also often receive help with incorrect solution inputs. However, this is usually a general form of help that does not address the students' misconceptions and thus corresponds to complex, elaborated feedback (Shute, 2008).

The only customisation options for interactive elements relate to the integrated videos. Options such as play/pause or speed control represent the typical features for interactivity by controlling. With regard to the text layout, no global typographic adjustments can be made, such as adjusting font settings. The text can only be zoomed in or out. The text is not responsive in any of the textbooks, and the quality of the font deteriorates when zoomed in. This means that even such simple features that would make a textbook more accessible are missing. A key aspect of an accessible, digital textbook that follows the UDL Guidelines is to adapt the design to the perception of the respective user (see CAST, 2024; UNICEF, 2019).

CONCLUSION AND OUTLOOK

Overall, it can be said that the potential of digital mathematics textbooks in terms of interactivity is hardly used (RQ1) and that digital textbooks hardly meet criteria for accessibility in terms of customisation options (RQ2). The fact that the investigated digital textbooks are characterised as digitalisations of the printed textbook, which are only offered with additional materials, also hardly changes the representation of the content (cf. Rezat et al., 2021). Since publishers often adhere to the classic structure of printed books, technical possibilities for interactive functions and individual adaptations are rarely exploited. Although promising tools or dynamic visualisations can already be found in digital textbooks, they are usually only used as supplementary material and only in limited sections. The design of digital textbooks and the implementation of interactive elements could be consistently developed in line with the principles of UDL (see CAST, 2024). This approach not only aims to make better use of the potential of digital textbooks – such as fostering interest and motivation in learning – but also to address the diverse individual needs of students. This could be achieved by providing alternatives, offering choices, and purposefully varying interactive elements that align with UDL principles, thereby creating learning materials that are accessible to everyone. The results of the study are particularly noteworthy given that the study examined the latest versions of the textbooks that are used in

mathematics classes, and that the accessibility of digital textbooks needs to be implemented by a European directive. Furthermore, it can be assumed that not only students with disabilities benefit from a well-designed textbook, but that the individual needs of all students are addressed. It is important that a textbook does not lose accessibility when its enhancement features are expanded.

In the study presented here, only individual aspects of the accessibility and potential of digital textbooks have been examined so far. In the further textbook analysis, further aspects will be included, such as the integration of feedback or opportunities for cooperation and further accessibility aspects (see Digital Educational Material (DEM), 2025). In addition, textbooks from primary school will be compared with those from secondary school and textbooks from other European countries will be analysed. Based on the textbook analysis, guidelines for accessible digital textbooks will be developed and prototypes will be designed with the help of these guidelines. The project brings together technical and educational perspectives to address the research gap on accessible, advanced digital textbooks.

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TEACHER-TEXTBOOK RELATIONSHIP: INTERPRETING AND USING MATHEMATICS TEXTBOOKS BY TEACHERS

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This article presents the conceptions of doctoral research aimed at comprehending the relationship between teachers and textbooks, focusing on how teachers interpret and use these materials in their practice. The Design Capacity for Enactment framework (Brown, 2009) guides the study, integrating the concepts of agency and affordances to analyse this relationship. This qualitative research, conducted through the Teorema System (Amaral et al., 2022) and a case study, follows four mathematics teachers as they plan and teach classes while participating in interviews and writing reflective narratives about their practice. The research aims to significantly contribute to understanding the teacher-textbook relationship, which still needs to be explored, particularly in the Brazilian context.

WHY STUDY THE RELATIONSHIP BETWEEN TEACHERS AND TEXTBOOKS?

The textbooks and geometry are the focus of a Brazilian research group study named teorEMa – Interlocutions between Geometry and Mathematics Education where both authors take part. This group created the Colloquium of Mathematics Textbooks event in 2022, which occurred at São Paulo State University and was inspired by the International Conference on Mathematics Textbook Research and Development (ICMT) but focused on the Brazilian context. The second edition was in 2024 at the Federal University of Mato Grosso do Sul, and a third one is confirmed for 2026 at the Federal University of Pernambuco.

Participating in this group, the first author conducted a master's research based on textbook analysis to characterize and comprehend the role of dynamic geometry software in activities used in mathematics textbooks (Carvalho, 2022). Now, enrolled in PhD research, the authors decided to focus on teachers and the relationship between them and textbooks since Fan (2013) pointed out the gap in research related to this focus, which is currently maintained.

Choppin (2004) identified an increase in textbook research around the time of his investigation and noted that in the 20 years before this study, there had been intense development of research discussing the use and reception of textbooks, with questions such as: What kind of use is made of them? Do teachers follow them faithfully, step by step, or do they take certain liberties with their proposed organization? And, in that case, which liberties and for what reasons?

Despite this, years later, Lloyd et al. (2009) pointed out that although the research on teachers' use of curricular materials is growing, it still needs to be developed. There has been a recent increase in interest in this field regarding how teachers use curricular materials and whether and how these materials may influence classroom practices and teaching more broadly.

Considering the most used curricular material by teachers (Valverde et al., 2002), in a survey of research involving textbooks in various countries, Fan (2013) noted that research in this area is still very new compared to other areas of Mathematics Education. Due to the lack of studies in this direction, he suggests a broad potential for research, including studies specifically focusing on the relationship between teachers and textbooks. According to him, it would be important to consider textbooks as a variable and, more precisely, as an intermediate variable in education. From this perspective, the fundamental question about textbooks as a research object is not just about what the textbooks are or their characteristics but also about how they affect other factors, such as the teaching process by the teacher, for example.

Focusing on research about curricular materials within the Brazilian context, Januario (2017) mapped 59 dissertations and theses defended between 1989 and 2012 in the fields of Science and Mathematics Education and Teaching that examined the relationship of the teacher-curricular materials as the subject of study. It was notable that only one of the 59 was a doctoral dissertation. Additionally, of these 59 studies, 80% (47) addressed descriptive research questions, focusing on Mathematics curricular materials and describing their characteristics.

In this direction, using the same source as Januario (2017), it was searched for the keywords “teacher”, “use of textbook,” and “mathematics”, resulting in 25 productions. From this, only one research was about the teacher's use of a textbook in mathematics in their practice but focused on their beliefs and its influence on the use of a textbook.

Recognizing a fruitful field for research, the focus was placed on the teacher-textbook relationship. This dyad term is the dynamic interaction between the teacher and the textbook, considering one influences the other. Therefore, the research question (RQ) is: *How do mathematics teachers interpret and use the textbook in their practice?* Thus, this research aims to comprehend the teacher-textbook relationship based on their interpretations and use in practice.

It is important to highlight that in Brazil there exists the National Textbook and Teaching Material Program (PNLD), managed by the federal government, which distributes textbooks to public school students, covering about 75% of the student population. Teachers collectively choose textbooks aligned with their school's political-pedagogical project. However, large-scale procurement often results in textbooks differing from the original selections. The PNLD functions in four-year cycles for each educational segment (kindergarten, elementary school, middle school, and high school). Each year, one segment receives newly purchased textbooks, while

the other segments receive only limited replacement copies, since the books are designed to be reused for four years.

THEORETICAL FRAMEWORK

Focusing on a review of research on the use of mathematics curriculum to examine how key constructs in this body of work are conceptualized, Remillard (2005) identified four different types of curricular material use: i) following or subverting the text, meaning assuming that close fidelity between the written curriculum and the enacted curriculum can be achieved under ideal conditions; ii) drawing on, with an emphasis on teacher agency, where textbooks are seen as one of many resources that teachers use in constructing the enacted curriculum; iii) interpreting, with the teacher acting as the interpreter of the enacted curriculum; and iv) participating with, assuming that teachers and curricular materials are engaged in a dynamic interrelationship that involves both the teacher and the text. The author emphasizes that these contrasting conceptions of curriculum use, based on varying theoretical assumptions, influence what can be learned about how teachers use mathematics curricular materials. In this way, she embassies to her discussion about teacher-curriculum relationships.

Considering the dynamic interaction between teachers and curricular materials, Brown (2009) argues that a teacher might interact with curriculum materials in three ways: offloading, adapting, and improvising. In a single class period, all three may occur dynamically, with none being superior to the others. The author further notes that these modes of interaction characterize the nature of the teacher-tool relationship but do not evaluate the outcomes of this interaction.

Offloading occurs when teachers use literal parts of the material, such as activities, instructions for their execution, or guidance and interventions, to resolve doubts or help students build knowledge. However, during this process, the teacher may intervene in what the didactic material proposes, for example, altering the sequence of activities, suggesting a different classroom organization, or omitting instructions on performing an activity. This is known as adaptation. Improvisation occurs when a teacher mediates/promotes learning by creating new situations and actions based on the student's learning needs (Brown, 2009).

In this way, Brown (2009) proposes a framework called Design Capacity for Enactment (DCE) – Figure 1 – when discussing this interaction, with the various components of the teacher-tool dynamic being captured by this framework. Additionally, it represents the different interactions between teachers and curricular resources as teachers offload, adapt, or improvise with the textbooks. The author adds that, on the one hand, the DCE encompasses the design resources and the embedded knowledge that comprise the curricular materials – procedures, content representations, and representations of physical objects. These aspects reflect the implicit and explicit intentions of curriculum developers. The framework encompasses teachers' knowledge, skills, goals, and beliefs, and how they influence the ways teachers perceive and appropriate different aspects of curricular designs.

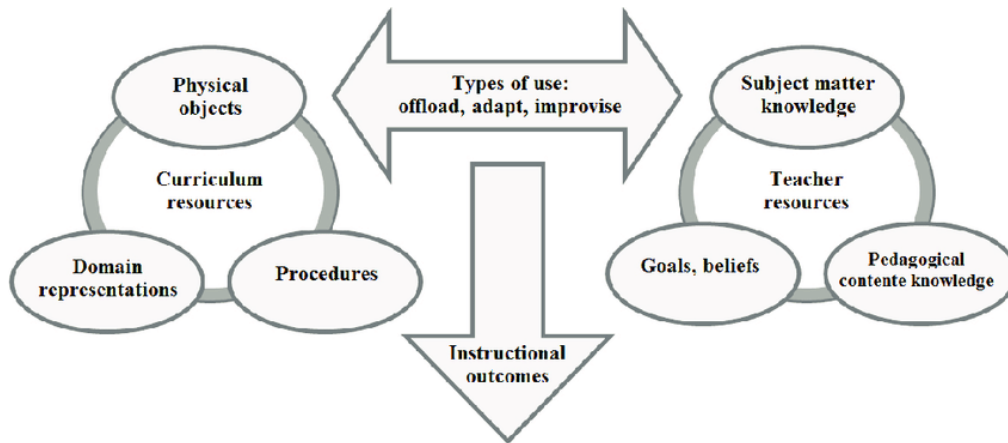


Figure 1: The Design Capacity for Enactment Framework (Brown, 2009).

The DCE framework highlights how teachers interact with curricular materials, balancing design intentions with teacher agency as Januario (2017) expanded. He emphasizes the varying degrees of fidelity in how teachers engage with textbooks – different ways teachers use textbooks occur with varying degrees of fidelity. For example, teachers may use curricular materials as the sole resources for curriculum development, make interventions, or create new learning situations based on their interpretations of the authors' propositions (Januario, 2017). However, beyond observing different uses and applications, it is essential to understand how teachers interact with the textbooks and the reasons for making interventions in the materials.

In this case, with such interventions, it can be said that the teacher has agency, understood as the factor that has authority over mathematics and its teaching. According to Januario (2017), the agent has the power of decision and authority over the mathematics to be taught; therefore, the one who has agency determines the different types of use. Thus, the teacher's agency is characterized by recognizing weaknesses in the material, identifying the relationship between expectations, objectives, and activities, redesigning activities, and understanding mathematics and its teaching, for example. On the other hand, the agency in the materials is characterized by contextualization, mobilizing prior knowledge, connecting learning situations, expectations, and objectives, didactic-methodological guidelines, and clarity regarding underlying didactic-methodological issues.

In addition to the idea of agency, the author discusses affordances, which take on the meaning of certain aspects of the materials that enhance their use by teachers. This means that in curricular materials, teachers can identify the content in these resources and using their knowledge, beliefs, and principles, assume an identity and understand the affordances of these materials. Depending on their resources, each teacher will understand the affordances differently, which can lead to different uses (Brown, 2009).

Thus, Januario (2017), when analysing the relationship between teacher and curriculum material, concludes that agency encourages various types of material to be used. In this case, the type of relationship is determined by the shift of agency, either in the materials or in the teachers, as shown in Figure 2. Therefore, due to the authority

over curriculum choices, curricular materials engage teachers to remain faithful to what they propose – offloading, in line with their role as agents. The teachers shift agency to themselves by adapting or improvising, subverting activity suggestions, creating new ones, or customizing the curricular development on which they are based. Additionally, affordances and constraints shape the materials, enhancing agency and its shift towards the teachers or the materials, thereby imprinting different interactions between these two agents of curricular development in Mathematics.

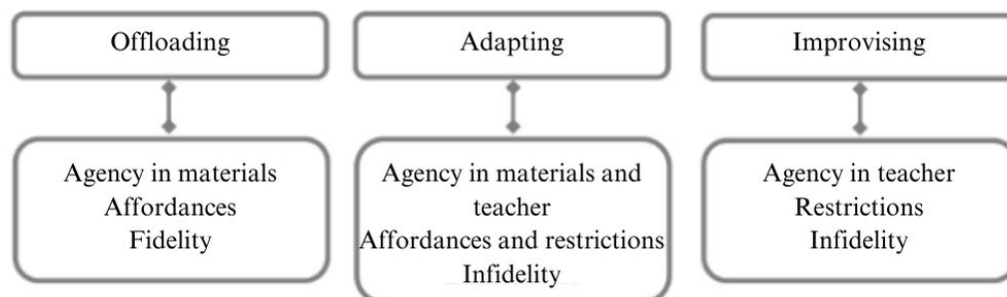


Figure 2: Teacher-curriculum materials relationship: agency, affordances and interventions.

Therefore, based on what the authors have presented, it is possible to see that Figure 2 connects the DCE framework developed by Brown (2009), which is based on the types of curriculum material used by teachers (offloading, adaptation, and improvisation), with the ideas of agency and affordances. Once the textbook is a type of curriculum material, these discussions will guide the analysis of the data produced later in this research.

METHODOLOGY

This qualitative research is based on four case studies, each focusing on a specific mathematics teacher and their interactions with textbooks in their classroom practices. The aim is to grasp the totality of a situation and describe the complexity of a specific case, taking a deep and exhaustive dive into a limited object (Goldenberg, 2011).

This qualitative research applies the Teorema System (Amaral et al., 2022) as a methodological framework for studying textbooks. This System proposes two complementary approaches to research on the field: (i) the textbook view, which treats the textbook itself as the object of study, and (ii) the context view, which investigates how textbooks are used in classrooms, teacher education courses, or other educational settings. With the classroom as a backdrop and investigating the textbook used by teachers, it was considered the better way to follow the second way of the Teorema System – the context view. Data will be collected through multiple techniques, including classroom observations, interviews, and teacher-written narratives, allowing the examination of how textbooks are used within their actual contexts.

For this study, a single public high school in the state of São Paulo, Brazil, was purposefully selected as the research site, allowing for an in-depth examination of teacher practices within a specific educational context. All four mathematics teachers

from this school were invited to participate to provide a comprehensive understanding of textbook use in different classes. An initial individual semi-structured interview will be conducted to understand their interpretation of mathematics textbooks in general, when planning specific content. This interview aims to explore their professional background and training, perception and use of textbooks, strengths and challenges of textbooks, adaptations in textbook use, and the impact of public policies. Therefore, this first phase seeks to understand the teacher's interpretation of a textbook when planning a class. The entire procedure is clarified by Figure 3.

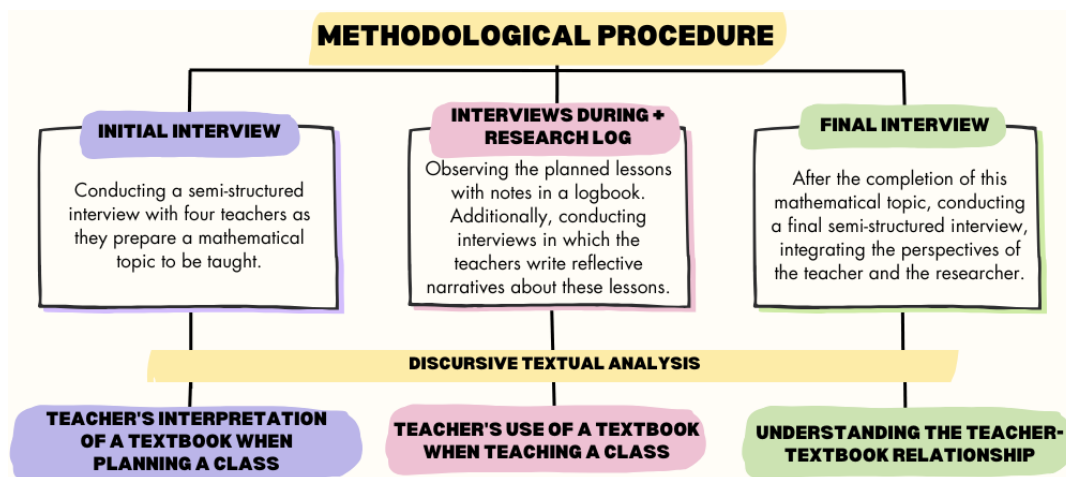


Figure 3: Methodological procedure of this research.

Then, the researcher will accompany mathematics classes for one bimester, while keeping a research log. This second phase aims to understand the teacher's textbook use when teaching a mathematics class. After each week of teaching class, the researcher will conduct an interview asking the teacher to write a narrative about their classes while they reflect on them. There will be a triggering question for this: Write a reflective narrative about your lesson, highlighting how you believe the textbook permeated it. However, the teacher will have the freedom to write what they consider most important for their own reflection. This is important to teacher professional development once they learn and develop professionally through participation in various practices, processes, and contexts that promote their training or improvement of teaching practices (Fiorentini & Crecci, 2013). In this case, it is promoted by narratives written by teachers that the researcher will analyse.

In the third and final phase, a conclusive semi-structured interview will be carried out to merge the viewpoints of both the researcher and the teacher regarding the interpretation and utilization of the textbook, from transcribed interviews. These transcriptions will be transformed into narratives for subsequent analysis, with the goal of comprehending the teacher-textbook relationship in practice. To this, it will be used the MAXQDA software, helping the researchers in this qualitative analysis.

All these three parts will be analysed using discursive textual analysis as the foundation in a three-stage interconnected process that forms a cycle. According to Moraes (2003), the first stage, referred to as the deconstruction of texts or unitarization, consists of

examining the materials in detail, fragmenting them to identify constituent units and statements that are related to the phenomena under investigation. Subsequently, the establishment of relationships, also known as categorization, occurs, in which the identified units are combined and classified, forming more complex sets referred to as categories, which aid in understanding the relationships between the elements. The third stage involves capturing the emergent new, resulting from a thorough immersion in the materials during the preceding stages. This process enables the construction of a new understanding of the whole.

Throughout this process, the researcher aims to employ data triangulation by utilizing various methodologies in data production. This approach involves integrating diverse methodologies to comprehensively describe, explain, and understand the subject of study (Goldenberg, 2011). It is acknowledged that employing multiple methodological procedures and integrating different data sources for final analysis enhances the reliability of the research.

EXPECTED CONTRIBUTIONS

A policy for reorienting the curriculum regarding the creation and evaluation of these materials can be planned based on knowledge of how Mathematics teachers use the textbooks provided to them in their practices, as expressed by Januario (2017). Additionally, this knowledge allows teachers and pre-service teachers to reflect on the curriculum development process and the use of textbooks. This encourages the treatment of content to promote meaningful learning for teachers and students, as there cannot be a dissociation between material resources and teacher resources, given that these resources interact mutually in the relationship between materials and teachers (Remillard, 2005).

Initial observations on this research point to a strong influence of the pedagogical context on the way teachers use the textbook. Additionally, personal conceptions, the prescribed curriculum, and each teacher's education also exert significant influences. Thus, understanding how teachers interpret and use textbooks in their daily work can help textbook developers create materials that better address the teachers' needs, supporting them in utilizing these resources to achieve their pedagogical goals, which is very important (especially in the Brazilian context). It can also serve as a foundation for initial and ongoing teacher education. Therefore, it is hoped that this research will contribute significantly to understanding the teacher-textbook relationship, which still needs to be explored.

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RECONCEPTUALIZING ANALYSIS OF TASKS IN CURRICULUM MATERIALS

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This paper expands perspectives related to the characterization of tasks in curriculum materials by applying the lens of dialogic potential and by specifying the nature of mathematical activity. To specify the dialogic potential of tasks, the paper draws connections between the non-routine nature of tasks and the potential heterogeneity in how students respond to the tasks. The methodological contribution of the paper is that it provides analytic criteria for characterizing tasks in relation to forms of mathematical activity and mathematical tools, allowing for greater specificity when analysing tasks. Theoretically, the paper connects the dialogic potential of tasks to their non-routine nature and provides specifics of what non-routine tasks look like across mathematical domains.

AIM OF THE THEORETICAL OR METHODOLOGICAL CONTRIBUTION

This paper expands perspectives related to the characterization of tasks in curriculum materials by applying the lens of dialogic potential and by specifying the nature of mathematical activity. The framework articulated in this paper builds from and expands an existing task categorization framework (Stein et al., 2009) that has been widely utilized to characterize the nature of tasks, especially in the context of curriculum reform in the US. The paper argues that the Stein et al. framework, while productive, would benefit from additional nuance with respect to the dialogic potential of tasks and the nature of mathematical activity. The lack of nuance in these areas hampers the ability for researchers to make connections between mathematical content and task categories when analysing curriculum materials, especially at the secondary level. In this paper, I outline these limitations and offer a related framework to characterize tasks. The potential contributions of the paper are methodological and theoretical in relation to studies of curriculum design and enactment.

THEORETICAL AND METHODOLOGICAL FRAMEWORK

In prior work I explored the dual communicative functions of curriculum materials in terms of their monologic or dialogic orientation (Choppin et al., 2015; Choppin et al., 2022), emphasizing that one of those functions takes precedence. My colleagues and I analysed these communicative functions with respect to prevalent curriculum programs in lower secondary schools in the US, labelling the programs as having a thinking device (dialogic) or delivery mechanism (monologic) orientation. Furthermore, we showed how these orientations played out in curriculum enactments.

This paper builds from that work to apply the notion of dialogic potential to the study of task design and to expand upon the prior cognitive demand framework for rating tasks (Stein et al., 2009). The contribution of this paper is to provide more nuance to the categorization of tasks in terms of communicative function, cognitive demand, and specification of the mathematical activity on which tasks are focused.

The dialogic potential of tasks builds from the notion of the communicative function of text. Lotman pointed out that texts have two potential functions, monologic and dialogic, referring to the dialogic function of text in terms of its potential as a “thinking device” that affords heterogeneity in how people respond to it (Lotman, p. 37, as cited in Wertsch and Toma, 1995). The monologic function can be viewed in terms of a delivery or conduit metaphor (Reddy, 1979), in which the main communicative functions are encoding, transmission, and decoding. Dialogic functions of texts involve interpretive and subjective processes, while monologic functions of texts involve more objective processes related to transferring information from one entity to another.

The heterogeneity in dialogic texts produces multiple approaches or interpretations that serve to generate meaning (e.g., thinking devices) when their differences and similarities are made explicit. Resolving differences and negotiating a shared understanding generates new knowledge in a community. The heterogeneity and related potential for generating meaning connects to ideas of knowledge building and epistemic resources. Students’ epistemic resources include their intuitive forms of reasoning, their invented representations, and informal language (Hammer et al., 2005; Kapur & Bielaczyc, 2012). The process of collectively developing new resources has been conceptualized as knowledge building (Scardamalia & Bereiter, 2006). Scardamalia and Bereiter explain how knowledge building emphasizes the epistemic value of students’ contributions, so that students not only develop competencies around knowledge building but also come to “see themselves and their work as part of the civilization-wide effort to advance knowledge frontiers” (p. 99), thus impacting their identities and dispositions with respect to a discipline. In sum, we can link the dialogic potential of tasks to the heterogeneous approaches they produce and then to how those approaches are taken up as epistemic resources in classrooms.

ARTICULATING A NEW FRAMEWORK TO CATEGORIZE TASKS

One of the prevalent ways that tasks are categorized in mathematics education is through the framework established by Stein and colleagues (2009). This framework has been useful in defining tasks according to whether they have low or high cognitive demand (cf. Cai et al. 2009; Tekkumru-Kisa et al., 2020). This framework has four levels of task ratings, of which two are considered low cognitive demand and two are considered high cognitive demand. The low cognitive demand categories are *memorization* and *procedures without connections*, while the high cognitive demand categories are *procedures with connections* and *doing mathematics*.

I identify two essential characteristics that distinguish between the low and high cognitive demand categories. The first distinguishing characteristic is the *potential for*

students to make connections. The two primary types of connections are connections between procedures and concepts and connections between representations (Stein et al., 2009). The second distinguishing characteristic pertains to the *ambiguity of the task* in terms of whether an approach is specified. The lack of specificity of an approach invokes the need for strategic flexibility; students must make decisions about what concepts and processes are relevant to a task (Doyle, 1983; Stein et al., 2009). The categorization proposed below emphasizes these two characteristics while providing more specification of their dialogic potential and the two distinguishing characteristics manifest across different forms of mathematical activity and mathematical tools.

I focus on three limitations of the cognitive demand framework outlined by Stein and colleagues (2009). First, the framework is generic with respect to mathematical content, which limits how researchers account for mathematical content when analysing tasks. For example, the topics of proportional reason and algebraic representations differ in terms of the knowledge that is drawn upon and potential approaches. Typically, proportional reasoning invokes a variety of broad concepts (e.g., part to part relationships, part to whole relationships, equivalent ratios, unit rates, scaling), while algebraic representations invoke fewer concepts and more cryptic processes.

Similarly, the Stein et al. (2009) framework is generic with respect to the nature of mathematical activity. I define mathematical activity as a process in which students engage with mathematics. I used the following four categories to characterize mathematical activity: following established processes and recalling terminology, demonstrating flexible use of tools, applying mathematics to contexts, and generating or demonstrating conceptual understanding. The initial formulation of these types of mathematical activity is aligned to conceptions of rigor in US mathematics education policy documents articulated in a prior study (Choppin et al., 2022). Table 1 shows descriptions of each of these forms of mathematical activity.

Mathematical Activity	Routineness	Description
Following established processes and recalling terminology	Routine	Activity that emphasizes accuracy and speed with respect to carrying out established procedures or recalling well-defined terminology
Demonstrating flexible use of tools, including procedures and representation	Non-routine	Activity that emphasizes flexibility regarding selecting tools and justifying the use of the tools

Applying mathematics to contexts	Non-routine	Activity that emphasizes applying mathematics in new situations, invoking strategic reasoning and interpreting connections between mathematical and contextual relationships
Generating or demonstrating conceptual understanding	Non-routine	Activity that emphasizes connecting multiple forms of reasoning or mathematical domains or understanding the nature of mathematical relationships and how they can be expressed in different ways

Table 1: Description of Mathematical Activity Categories

While there is some overlap between the descriptions of mathematical activities in Table 1 and the task rating categories in Stein et al. (2009), there are important differences. The first difference is the distinction between routine and non-routine forms of activity. In Table 1 the first category is classified as routine while the last three categories are classified as non-routine. The non-routine activities are characterized by their ambiguity, with provides opportunities for students to exercise strategic flexibility when working on the task, and the opportunity for students to make connections. For example, in the case of applying mathematics to contexts, the description of the activity includes connections between mathematics and the context in which a task is set. In the case of generating or demonstrating conceptual understanding, the description of the activity includes multiple types of connections, depending on the mathematical topic. These activities are seen as non-routine because students need to make connections and develop approaches that are not specified in the task. The explanation of the category of generating or demonstrating conceptual understanding as a form of mathematical activity addresses another limitation in the Stein et al. (2009) framework, which is the lack of explicit definition of conceptual understanding. The lack of detail leaves researchers to their own devices to define what is meant by conceptual understanding, though it is central to categorizing the mathematical activity afforded by tasks.

The third limitation relates to how the cognitive demand task framework largely ignores the communicative function of curriculum materials, with the potential for heterogeneity and the role of heterogeneity largely implicit in the framework. To address this, I draw connections between the non-routine nature of tasks and the potential heterogeneity in how students respond to the tasks. Non-routine activities are inherently more interpretive and thus have the potential to spawn idiosyncratic and thus varied approaches or ways of reasoning. These different approaches or ways of reasoning in turn afford opportunities for dialogue to reconcile or contrast these approaches or ways of reasoning. In short, non-routine tasks provide opportunities for the collective development of ideas.

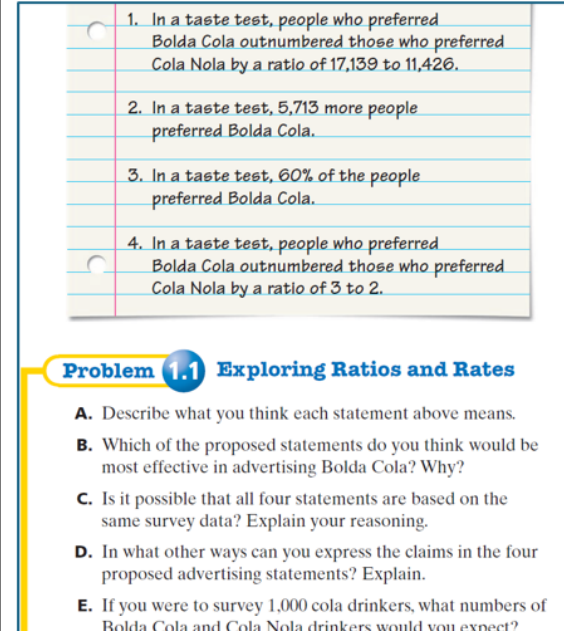
To provide more specification about what these forms of mathematical activity look like in tasks, I relate them to the tools whose use the tasks invoke. The three mathematical tools listed in the table are: mathematical terminology and formulas, procedures and algorithms, and representations. The list of tools could be expanded to include other tools such as deductive reasoning (proof), depending on the scope of content to be analysed. These tools are broadly conceptualized so that they span most mathematical domains in the secondary curriculum. For example, tasks involving the topic of proportional reasoning could include all three types of tools. The same could be said for the topic of linearity. Table 2 elaborates on versions of mathematical activity for each type of tools and makes the case for why mathematical topic and tool use should be considered as distinct dimensions when characterizing tasks.

Mathematical Tools	Mathematical Activity			
	Following established processes and recalling terminology	Demonstrating flexible use of tools	Applying mathematics to contexts	Generating conceptual understanding
Mathematical terminology and formulas	Memorization or recall of basic facts, definitions, or formulas	Using facts, terminology, or formulas flexibly when solving a problem	Applying facts, definitions, and/or formulas in new contexts	Developing facts, definitions and/or formulas.
Procedures and algorithms	Accurately and efficiently carrying out established procedures	Strategically and flexibly select and carry out procedures	Applying procedures in new contexts and interpreting results	Developing or making sense of unfamiliar procedures and/or algorithms
Representations	Generating representations with a well-established method.	Generating new information / interpreting mathematical relationships from different	Interpreting information in representations in relation to disciplinary or real-world context	Creating novel representations and/or using representations in novel ways

representatio
ns.

Table 2: Relating Types of Mathematical Activity to Mathematical Tools

To demonstrate the affordances of this framework, I analyse two tasks, each focused on proportional reasoning. The Bolda Cola task (on the left) is from the Connected Mathematics Project (Lappan et al., 2006), while the Making Bread task (on the right) is a composite task created using examples from existing commercial curricula.



Problem 1.1 Exploring Ratios and Rates

- Describe what you think each statement above means.
- Which of the proposed statements do you think would be most effective in advertising Bolda Cola? Why?
- Is it possible that all four statements are based on the same survey data? Explain your reasoning.
- In what other ways can you express the claims in the four proposed advertising statements? Explain.
- If you were to survey 1,000 cola drinkers, what numbers of Bolda Cola and Cola Nola drinkers would you expect?

Composite ratio problem from commercial curricula

Making Bread		
You are mixing ingredients to make bread. You want to triple the ingredients to make more bread.		
Write the missing ingredient amounts.		
Item	Old Amount	New Amount
Water	2 cups	
Flour	5 cups	
Yeast	1.5 tablespoons	
Oil		6 tablespoons
Salt	2.5 teaspoons	
To make whole wheat bread, the ratio of white flour to whole wheat flour is always in a 2 to 5 ratio.		
Write how much of each you would need to make a recipe that called for 21 cups of flour.		
___ cups of white flour		
___ cups of whole wheat flour		

Figure 1: Two Tasks on Proportional Reasoning (*Bolda Cola* and *Making Bread*)

Figure 2 shows how the framework can be used to analyse tasks. I condensed the analysis into one table for reasons of space. I characterized the Bolda Cola task as providing opportunities for demonstrating flexible use of tools and applying mathematics to contexts with respect to procedures and algorithms and demonstrating flexible use of tools, in this case with respect to representations. The task asks students to strategically and flexibly select and carry out procedures and interpreting results in new contexts. Those categorizations position Bolda Cola as a non-routine task. I characterized the Making Bread task as providing opportunities for following established processes and recalling terminology with respect to procedures and algorithms because the focus is on accurately and efficiently carrying out established procedures. This categorization positions the task as routine. In both cases, the framework provides a means to provide specific descriptions related to the opportunities the tasks provide students opportunities to make connections and to wrestle with ambiguity. When analysing a set of tasks within a curriculum unit, the frequency of occurrences of forms of activity or tool use would provide insights into the overall characteristics of that curriculum unit.

Mathematical Tools	Mathematical Activity			
	Following established processes and recalling terminology	Demonstrating flexible use of tools	Applying mathematics to contexts	Generating conceptual understanding
Mathematical terminology and formulas				
Procedures and algorithms	MB: Accurately and efficiently carrying out established procedures	BC: Strategically and flexibly select and carry out procedures	BC: Applying procedures in new contexts and interpreting results	
Representations	BC: Generating new information or interpreting mathematical relationships			

Figure 2: Analysis of Tasks on Proportional Reasoning

IMPLICATIONS FOR RESEARCH ON CATEGORIZING TASKS

Prior work applied the categorizations in Table 2 to analyse the enacted curriculum, especially regarding to the routine or non-routine nature of mathematical activity in observed lessons (Choppin et al., 2022), which we connected to the design of curriculum programs. This led to claims regarding the impact of the design of curriculum programs on enacted mathematical activity. This analysis of curriculum design was done at a broad level and focused on design considerations broader than the level of task. However, much of the research on curriculum design focuses at the level of task. Consequently, in order to contribute to analysis at the task level, the proposed framework needed to specify the mathematical activity and tools in the analysis of tasks. Doing so would address limitations related to a prevalent framework.

There are methodological and theoretical implications. Methodologically, the analytic categories outlined in Tables 1 and 2 provide criteria for analysing tasks that provide greater nuance to characterizing what *connections* and *ambiguity* look like in various mathematical contexts. The notion of mathematical activity provides a practice-oriented lens on characterizing tasks, in terms of describing the kinds of student practices afforded by tasks and whether they are routine or non-routine. The notion of mathematical tool provides specifics about the kinds of choices students can make while working on tasks. Characterizing the mathematics in terms of activity and tools avoids the need to specify mathematical content into a long list of mathematical topics while capturing essential features of mathematical domains. Theoretically, Table 1 connects the dialogic potential for tasks to their routine or non-routine nature, with implications for heterogeneity and consequent opportunities for dialogue and the collective development of knowledge.

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NAVIGATING STAKEHOLDER DIFFERENCES IN A DESIGN-BASED RESEARCH APPROACH TO COLLABORATIVE DEVELOPMENT OF INTERACTIVE TEXTBOOK QUESTIONS

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In the context of open textbooks that can be authored by multiple people, the question of who can author textbook content becomes paramount. In this project we explore the involvement of multiple stakeholders, Students, Teachers, Authors, and Researchers, as they collaborate to design interactive, short-answer questions for open-source textbooks in calculus and linear algebra, textbooks that also differ in the style of presentation (either definition-theorem-example meant to be presented by the teacher or short texts followed by individual and group explorations and in-class discussion). We present the design-based research methodology we are using and initial differences we perceive as diverse stakeholders negotiate motivations and expectations in the question development process.

INTRODUCTION

The next generation of online, interactive mathematics textbooks must continue to improve support for teaching and learning. Current technical developments facilitate the relatively easy inclusion of interactive content that affords learners opportunities to engage with their textbooks inside or outside of their classrooms. By leveraging textbook models from computer science education that have for years harnessed interactive content, we can enhance online, interactive textbooks for undergraduate mathematics courses. These enhancements can lead to an increase of the use of free, open mathematics textbooks and provide access to high-quality engaging content for more learners in many contexts.

With the aims of promoting enriching student engagement with mathematical content and of providing tools for teachers to make more informed instructional choices, in this paper we share the methodological approach for our study of the development and use of interactive questions in open, online, interactive textbooks in calculus and linear algebra that have been written in PreTeXt and are hosted on the Runestone Academy learning analytics platform (<https://landing.runestone.academy/>). The project, PreTeXt-Runestone: Open Textbooks Engaging Undergraduates in STEM (PROTEUS) follows a design-based research approach (Bakker & Eerde, 2015) involving undergraduate mathematics students, teachers, textbook authors, and researchers in a collaborative development of interactive questions designed to elicit multiple ways of students' thinking. A main goal of the project is to understand how

students and instructors use those interactive textbook questions in their classrooms. In this contribution, we discuss the process that has led to the formation of four question development teams—each tasked with writing interactive questions for one textbook—and some examples of activities in which the teams have engaged. The research question that guides this phase of the project is: What are the strategies and heuristics used by teams of students, teachers, authors, and researchers to create content for interactive short-answer questions for calculus and linear algebra textbooks that differ in how the material is presented?

BACKGROUND

In this work, we are concerned not only with the content of textbooks, but also with how that content is used by learners and instructors. Both curricular and conceptual pedagogical functions of textbooks are achieved through embedded aspects such as “exposition, explanation, questioning, exercises, examples, and tests... [to] achieve a more active reader of the text and to manifest the aims of the mathematical and cognitive progression” (Love & Pimm, 1996, p. 386). We note in particular the key distinction between questioning devices and textbook exercises. While exercises provide a vehicle for active engagement with the flow of content (Love & Pimm, 1996), and indeed a means for students to practice and internalize as content systematically progresses, they exist in a largely post hoc fashion. Questioning devices, on the other hand, may manifest as content precursors and may elicit mathematical exploration more integrated with the textbook’s progressing narrative.

Current interactivity available in textbook development creates a medium for the learner to develop a relationship with the subject matter, and potentially engage with the author’s voice (Weinberg & Wiesner, 2011). We seek to understand in what ways instructors can and do make use of students’ responses when engaging with questioning devices. In an online interactive textbook, student responses to embedded questions can be captured immediately and can thereby be made available to instructors. This window into students’ understanding as they are reading and grappling with the content has the potential to inform instructional choices; aided with analysis of responses across multiple sites, the information can be mined to test learning trajectories for key mathematical notions within the courses. During this first year of the project, we are developing short-answer questions as an entry point into capturing students’ reasoning and heuristics for team-based question development.

The use of Short Answer Questions in mathematics has been promoted to support metacognitive development in relation to problem-solving (Kilpatrick et al., 2001) and proving (Conradie & Frith, 2000), but also as part of consolidation of knowledge and retention (Hopkins et al., 2016). However, these studies investigate the use of these questions after the material has been presented, and not as part of class preparation—reading ahead of time and answering questions prior to exposure to the material in class—which is the intention of the short-answer questions in the PROTEUS

textbooks. Our earlier work with student responses to one set of reading questions on span (Gerami et al., 2024) indicates that these responses are a source of conjectures about students' conceptions (Balacheff & Gaudin, 2010) that can guide instructional activity. Gerami and colleagues' study also highlighted that the questions must be amenable to an analysis of conceptions.

IMPLEMENTING A DBR STUDY WITH DIVERSE STAKEHOLDERS

This work uses design-based research (DBR) methodology, which has: a focus on theory development, an “interventionist nature,” temporally dynamic “prospective and reflective components,” a cyclic nature, and a “humble” domain-sensitive application of theory (Bakker & Eerde, 2015, pp. 437–438). We wish to contribute to theory by developing a body of strategies and heuristics for writing interactive textbook questions that support student understanding and make student thinking clearer to instructors (Yerushalmy et al., 2017) with an intervention that consists of incorporating the developed questions into instructors' textbooks and studying their use through the data analytics provided by the Runestone Academy platform. The questions themselves will be redesigned or discarded in a cyclical fashion by each development team. Figure 1 below illustrates the cycle of question development, where progress from question inception to classroom implementation is informed by iterations of feedback, first within development teams, then from cognitive interviews (CI), and lastly from instructor feedback after use in their classroom.

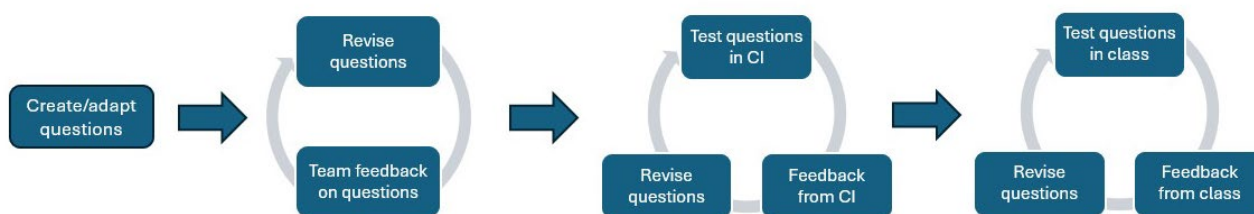


Figure 1: Rapid revision and implementation cycle of question development.

The STAR teams.

Our effort to identify question development heuristics engages students, teachers, textbook authors, and researchers in a team setting. There are four teams, each consisting of two Students, two Teachers, an Author, and either one or two Researchers (constituting a “STAR team”). These 16 stakeholders come from 14 different higher education institutions in the United States and Canada, and bring current and past experience with both four-year universities and community colleges. Two teams work with linear algebra textbooks (ULA: Austin, 2022; FCLA: Beezer, 2022) and two work with calculus textbooks (AC: Boelkins, 2022; APEX: Hartman et al., 2018). Twice per month, each team meets virtually to discuss and give feedback on questions they are developing for their respective textbooks, as well as formulate conjectures about what student understandings and misconceptions the questions will elicit (Bakker & Eerde, 2015). During the initial phase of the project, and between team meetings, the

developed questions are tested in semi-structured cognitive interviews (Saldaña & Omasta, 2018); the information gathered from interviews is provided to STAR teams in order for them to analyse how effectively questions elicited student conceptions and inform question design. In later phases, the information that will be given back to the teams will include the use of the questions in naturalistic classroom environments.

The open and interventionist aspects of DBR (Bakker & Eerde, 2015) permit us to adapt the intended intervention (adding interactive questioning devices to the textbooks) in more or less real time as student interaction unfolds. We capitalize on such adaptability during multiple stages. First, during cognitive interviews with student interviewees to test the effectiveness of developed questions in eliciting thinking, interviewers (researchers) make decisions in real time about aspects of student thinking to probe more deeply. Second, during actual use of developed questions as classroom activities, reading homework, or discussion prompts, instructors make decisions in real time about whether the nature of the question is really providing the teaching and learning support they hoped for, and can choose to adapt in the moment how (or whether) they continue to use the question. In both cases, Researchers and Teachers fulfil the DBR cycle by reflecting and reporting on their experiences back to the question development teams, who use this feedback to inform the cycle of question revision.

In all teams, the Teachers have prior experience using the team's designated textbook in their own instruction and therefore bring perspectives about its strengths and shortcomings in past and current classroom contexts. The textbook Authors obviously bring extensive experience with their own textbook as a part of an evolving open-source project (in PreTeXt and now in Runestone), yet they also speak from experience teaching courses with their textbooks. In their own roles, Researchers and Students also bring widely varying prior experiences with these textbooks, from none at all to quite extensive. Some of the Students have had prior experience as students in a course that used PreTeXt textbooks housed within the Runestone Academy platform; some of the Researchers have encountered the textbooks for the first time in this project, may have added content to other PreTeXt textbooks, or have studied the textbooks in this project in prior studies.

Students were selected to participate in development teams based on four main considerations. First, each Student has had prior experience with the mathematical content; second, as we wanted to make sure that the perspective of a large majority of students of calculus and linear algebra was reflected, we solicited recommendations from colleagues of students who were math majors but not necessarily 'high-achieving', instead looking for students who would be curious and who would persevere with the work; third, we looked to have students who collectively would include those who are not in the main demographic of mathematics students in the United States (male, white or of Asian background, affluent, cis-gender); fourth, we asked for students who would not be intimidated by the perceived authority of the other

team members and who would not be afraid to speak up and contradict assumptions made by Authors, Teachers, and Researchers.

Differing motivations and expectations.

Among these groups of stakeholders, categories of tension exist surrounding the project objectives. These tensions, and their ultimate negotiation, are an important aspect informing the first of our three-year project's overarching questions: *How do groups of textbook users (students, teachers, authors, and researchers) work together to write interactive questions designed to elicit multiple ways of students' thinking in classrooms?* In order to ensure that every team member has the opportunity to contribute equitably to the development process, and to avoid perpetuating existing expert blind spots (Nathan et al., 2001), our design distributes decision-making among all team members without regard to prior mathematical experience, including decisions about for which textbook areas questions are developed. During the first month of the project, and after participating in a workshop on question development (Olsher, 2024), each team engaged in an activity to decide which topics would be the focus of their initial development efforts. During the activity, teams were split into three groups by role: Students, Teachers, and Author/Researcher(s). The groups then independently discussed and determined which topics they wanted to begin with and, importantly, explained their reasoning for each topic choice. The results of this activity provided insights into differing individual stakeholder motivations and expectations.

In all four teams, **Student** reasoning for topic selection featured connectivity between topics, or utility for topics in other contexts, both within and outside the scope of the course. For example, in the linear algebra teams, eigenvalues/vectors were seen as a useful development target due to their connections to system solutions ("solving matrices"), and dot product was seen as useful due to its utility in "many areas of science and math." In the calculus teams, the chain rule was seen by Students as useful due to its "connect[ion] to" composite functions, and the notion of limit was deemed "helpful to connect to finding the derivative." Less pervasive but still present among Students were topic motivations based on learner conceptual difficulties. For example, continuity was proposed based on the difficulty of understanding it "from [a] non-graphical approach," and the nature of the derivative function was cited as a challenge by an AC Student who reported taking Calculus I the prior year.

Teacher reasoning for their chosen topics differed from that of Students, featuring conceptual difficulty as the primary motivation for writing improved interactive questions.

I thought [it] was kind of interesting [...] that the [teachers] kind of focused on span and linear independence, and the students focused on the eigenvalues and I thought that was interesting, and I would suspect some of that is, those of us who have taught this material recognize that the topics that our students seem to understand the least are span and linear independence, which is probably why we sort of thought to go there. They're fundamental but theoretical. [...] So I might suggest that we do at least one of the kind of more theoretical

topics that I always find challenging to get my students to understand. And then maybe one of the topics that the students feel is really like kind of fundamental (FCLA Teacher).

Furthermore, several Teachers in both content areas labelled chosen topics as “foundational” or “fundamental” and frequently reasoned for or against topic inclusion on this basis: “...I would argue against [matrix subsets] being one of the sections we use because it’s not as fundamental” (FCLA Teacher). Other Teacher topic motivations included that a topic was one they were currently teaching in their course, the need for additional material in a textbook section, and, in the case of questions that could be presented computationally, the ease of auto-grading student responses.

Author topic motivations echoed those of Teachers to some extent, citing areas of conceptual difficulty (e.g., linear independence) but Authors also expressed the desire to make technical improvements to the textbook, such as adding interactive features to existing questions or acknowledging that some sections in their textbooks had very poor exercises or activities that could be sent to a different platform (e.g., WeBWorK). Prior to the activity of selecting topics, Authors also expected to “start from the beginning” meaning from the very first unit in their textbook, describing it as a logical way to do “a revision.”

Researcher topic motivations included an interest in using extant research on student difficulties with mathematical notions (e.g., span, limits, derivatives, abstraction) that could inform topic selection and question development.

Differences in content expertise are viewed, in our design, as *valuable* for providing unique insights into how a textbook question functions and what thinking or knowledge it elicits. A delicate navigation is required, however, to ensure that all team members’ contributions are heard and valued despite differing levels of domain expertise. In particular, modes of mathematical communication between Students (content novices relative to the team) and Authors and Teachers (content experts relative to the team) come into sharp relief. For example, in the FCLA team’s topic selection discussion, the Students proposed focusing on eigenvalues/vectors because they are “helpful [for] solving matrices.” In the same team meeting, the Author and Teachers discussed designing questions to support correct use of mathematical language. One Teacher’s assertion, while not directly citing the terminology of the Students’ eigenvalues suggestion, identified such usage as problematic in the classroom: “I spend a lot of time in the first two weeks of class saying, you know, a matrix doesn’t have a solution. Or you know, a linear system isn’t independent” (FCLA Teacher).

An analogous exchange in a ULA linear algebra team highlighted potentially disparate mathematical values between novices and experts. Interestingly, this exchange was also seeded by the Students’ suggestion of focusing on eigenvalues/vectors for question development. One Student initiated the whole-group discussion by advocating for the inclusion of “computational aspect[s]” in team-developed questions, with the reasoning that “computers are becoming a bigger part of math” (ULA Student). Furthermore, the second Student on the team explained that they had chosen

eigenvalues/vectors because the topic “reoccurs in other subjects” (ULA Student). While the ULA Teachers did not initially include eigenvalues/vectors as a potential development topic, they eventually voiced support for its inclusion, though their motivations for doing so differed from that of Students, for example:

[...S]tudents jump to the computational, but I think there are really nice conceptual understandings of what eigenvectors and eigenvalues are doing, that I think I like to kind of highlight with students (ULA Teacher).

CONCLUSION AND IMPLICATIONS FOR FURTHER RESEARCH

Differing stakeholder motivations and expectations must be navigated throughout the collaborative question development process. Authors, confronted with Teachers’ and Students’ experiences, must reconsider what other functions their textbooks fulfil and how they are instrumented by the users (Trouche, 2020). Researchers’ focus on questions designed primarily to elicit thinking may chafe with Teachers’ desires for efficient grading schemes. Furthermore, our design uniquely positions each category of textbook user as an authority, not despite, but *because* of their different relationships to the mathematical content. Researchers need to confront the fact that current literature on student thinking, as well as literature on the distribution of power and mathematical authority in classroom settings (e.g., Sullivan et al., 2020) might be insufficient for framing the distribution of mathematical authority in this context, as we invite Students to inform the process of improving linear algebra and calculus textbook questions as more than just end users. Discoveries that stakeholders make about other textbook users’ experiences will enrich our work with PROTEUS to provide a foundation for further exploration and experimentation with novel interactive questions.

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INFLUENCES ON TEACHERS' ATTENDING TO MATHEMATICS TEXTBOOKS: PATTERNS IN LESSON PLANNING

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This study explores how six high school mathematics teachers attend to curriculum materials during lesson planning and identifies influences on that attention. Staged lesson planning interviews as teachers planned with a diverse set of curriculum materials revealed their attending patterns to certain textbook portions appeared to be influenced by various factors, including the teacher's instructional orientation.

INTRODUCTION

Researchers have increasingly studied how mathematics teachers plan with curriculum materials to understand why the same textbook materials result in different forms of instruction. To explain this phenomenon, Dietiker et al. (2018) theorized that as they interact with curriculum materials, teachers *notice* certain parts of textbooks and disregard others. Studying the eye movements of prospective secondary mathematics teachers as they planned with the same curriculum materials, Males and Setniker (2019) argued that teachers *attend* to different parts of textbooks for different lengths of time, in different sequences, and for different purposes. Furthermore, Roth McDuffie et al. (2018) suggested that when experienced middle-school mathematics teachers plan with unfamiliar curriculum materials, they attend, interpret, and respond to materials in ways that are consistent with their ideas and assumptions on instruction.

Our study expands this work, exploring potential influences on curricular noticing. Specifically, we explore how and why practicing teachers attended to curriculum materials as they each planned with mathematics lessons selected from a range of textbook materials. The lessons were taken from curriculum materials that vary in the ways they approach learning, such as whether they include explicit statements of content with which students are expected to learn new ideas (i.e., delivery mechanism or “DM”) or whether they are designed to inspire student sensemaking from which they are expected to learn new content (i.e., thinking device, or “TD”) (Roth McDuffie et al., 2018). By having different teachers each plan separately with the same diverse set of lessons, we identified potential attending patterns by individual teachers (e.g., one who always looks for worked examples) as well as patterns across teachers by textbook type (e.g., when multiple teachers ignored a part of a textbook for similar reasons). Using this approach, we seek to answer the following questions: *When planning with textbook materials, what do teachers attend to? And what influences that attention?*

THEORETICAL FRAMEWORK

Drawing from Remillard (2005), Dietiker et al. (2018) propose that as teachers interact with curriculum materials, they notice curricular opportunities in three interrelated phases: *attending* (i.e., perceiving parts of curricular materials), *interpreting* (i.e., making sense of those parts), and *deciding to respond* (i.e., determining whether and how those parts would be used for instructional purposes). We assume the relationship between the teacher and curriculum materials is bidirectional—just as the materials may shape the teacher, a teacher’s goals and understandings shape the curriculum materials they perceive (Remillard, 2005). Similarly, although we assume teachers do not interpret what they do not attend to, we also recognize their interpretation of curriculum materials may influence how they shift their attention, such as interpreting a problem in a textbook as a challenge for students and then directing attention to earlier parts of the textbook to learn what prepares students for the challenge. The interrelationships between phases are represented in Figure 1.

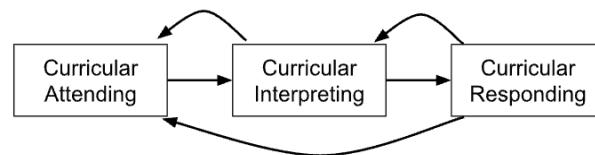


Figure 1: Phases of curricular noticing (based on Dietiker et al., 2018).

Furthermore, we assume teachers hold certain assumptions that can influence their interactions with curriculum materials. For example, Lloyd and Behm (2005) propose that teachers hold *instructional orientations* (i.e., teachers’ assumptions about how mathematics teaching and learning should occur) that can differ remarkably from teacher to teacher. For instance, some teachers might believe students learn best when the teacher makes content explicit to them, while others might think students learn best by constructing their knowledge and engaging in sensemaking. On the other hand, Lloyd (1999) explains that teachers may hold *conceptions of curriculum*, which refer to teachers’ assumptions about the role and function of curriculum. That is, teachers may hold pre-existing notions about what curriculum materials should look like and how they should operate, such as seeing curriculum materials as a reference beyond the classroom, assuming mathematical objects in a task need to be specially selected, or believing curriculum materials should be clear and readable.

METHODS

To explore the connection between different curriculum materials and how teachers engage with them, we recruited six experienced secondary mathematics teachers and observed as they individually planned with four different lessons involving systems of equations from a diverse set of curricular materials. The participating teachers had between 5 and 29 years of experience. Teachers 1 and 2 had primarily worked with DM materials, while Teachers 3, 4, and 6 had worked with a mixture of DM and TD materials (with varying degrees of experience with each), and Teacher 5 primarily drew from online resources.

We interviewed each teacher five times over Zoom, from October 2023 to February 2024, with each interview lasting approximately one hour. The first interview was semi-structured (Weiss, 1994) to learn about teachers' professional backgrounds and views on teaching and curriculum. During this interview, we asked questions such as "Please describe a recent lesson from your algebra class that your students would describe as typical."

The subsequent interviews used the Staged Lesson Planning Interview Protocol (Roth McDuffie et al., 2018), which enabled us to observe each teacher as they planned a lesson. Before each interview, teachers received a PDF of the teacher's version of a selected lesson (one of Lessons A, B, C, and D, see Figure 2) from four different textbooks on the DM-TD continuum so they could read the materials in advance. In the interview, the teacher was asked to plan a lesson without any specific guidelines or expectations (e.g., no lesson plan template was provided). In addition, the teacher was told they could make assumptions about the students for whom they were planning.

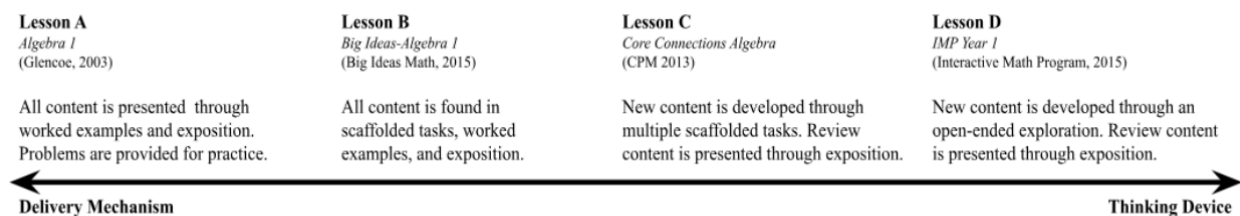


Figure 2: Selected lessons along the DM-TD continuum (from Dietiker et al., 2024).

Because the ways teachers attend to curriculum materials could be influenced by the different parts of a textbook lesson (e.g., problems versus teacher instructions), we began our analysis by identifying components of each of Lessons A, B, C, and D, including teacher-facing notes, student-facing notes, and student-facing activities. We grouped textbook parts into broader categories based on shared features: teacher notes, statements, worked examples, explorations, and exercises (adapted from Smith et al., 2013). *Teacher notes* include long-form expository text (e.g., a description of how a lesson might unfold), logistical information (e.g., time length, materials needed, learning objectives), and short-form recommendations for the teacher. *Statements* include any student-facing content providing explicit information, including definitions and explanations of new content (e.g., how a point of intersection represents a solution of a system of equations). *Worked examples*, found only in Textbooks A and B, include problems for which a worked solution is provided within the student-facing text. *Explorations* are student-facing activities designed to generate new mathematical content, while *exercises* are prompts to practice or review previously learned content, either in class or through homework exercises.

For each transcribed interview, we identified instances where a specific textbook part was referenced and used thematic analysis to describe what directed the teacher's attention. We then identified patterns across all interviews. Visual tools and data summaries from MaxQDA informed our findings.

FINDINGS

Based on our analysis, a teacher's curricular attending to teacher materials can be influenced by educational policies, a desire to understand the scope of the lesson, judgments of their expertise, and teachers' established instructional orientations. Furthermore, what they pay attention to in student materials is guided by their instructional orientations, instructional goals, assumptions about curriculum materials, and textbook design. This section describes each of these influences in relation to specific textbook elements, organized in the approximate order they appear in textbooks and by thematic patterns in attending behaviour.

Curricular Attending of Teacher Notes

The teachers found teacher notes useful when they felt the notes addressed policy concerns or when they answered logistical questions. For example, although most ignored the short-form teacher notes in the margins of Textbooks A and B, Teacher 4 intentionally searched for indications of the intended standards. While attending to one such note, Teacher 4 explained, "I need the Common Core standards for my school; every school I've worked at actually expected those." He similarly looked for this information in Textbook C. Teachers also attended to logistical information (e.g., suggested lesson times) provided by all four texts. For instance, Teachers 3, 4, 5, and 6 used the suggested times to determine if the entire lesson would fit within the class time available to them. Teachers also took note of vocabulary students would need to know, lesson objectives, mathematical background, and what prior knowledge was being leveraged in the lesson.

The teachers in our study were reluctant to read teacher notes unless they supported their goals in some way. There were multiple reasons teachers gave for not wanting to read the teacher notes, including length and the extent to which they felt the notes took into account their professional expertise. For example, Teacher 2, who had little experience with TD texts (which often include long-form teacher notes), expressed dissatisfaction with the length, saying, "I come home after teaching all day..., do you think I want to read all this?... It just is too much work." Teachers 3 and 4, who had experience with TD and DM materials, felt the text did not recognize their expertise as teachers. For example, Teacher 3 shared, "I know this. Okay, the teacher should know that. If the teacher doesn't know that, I think we have a bigger problem on our hands." Teacher 1, who had little experience with TD texts, and Teacher 6, who had significant experience with TD texts, took less critical approaches, with Teacher 1 sharing that she appreciated how the teacher notes "guide you through it," but also noting the suggested lesson activities would be helpful for newer teachers and she likely would not use them.

Despite their initial critiques of the teacher notes, Teachers 1, 3, and 4 searched for information within them when they were confused about the goal of a particular problem or when content was introduced in a way that was unfamiliar to them. When planning with Textbook C, Teacher 4 expressed uncertainty about a method presented, saying, "I was curious about this, why they're adding 16...but let's just read what [the

teacher notes] said, I guess.” Teacher 4 expressed reluctance about the strategy being used but ultimately decided to use the approach recommended in the text.

Curricular Attending to Expository Statements in Student Materials

When interacting with expository statements in student materials (which were mostly found in DM materials), teachers’ attention appeared to be influenced by interpretations of whether their students would find the materials relevant to their lives. For example, Teacher 1 often paid attention to how lessons could help students connect mathematics to the world. While reading a segment in Textbook A entitled “Why is it important?”, she commented, “I like this because kids always ask me, when will I ever use this in real life?” Teachers 3 and 4 also attended to explicit statements with an eye toward whether content was relevant to students, highlighting an explanation that referenced cassettes and CDs as dated and, therefore, not of interest to students.

When student-facing expository text appeared in TD textbooks (only Textbook C), the teachers’ attention was sometimes connected to their instructional orientation. Teacher 2, who is primarily a DM-user and consistently begins her planned lessons with direct instruction, notes, and vocabulary, looked for explicit statements at the start of her planning process, even though they appeared toward the end of the lesson in the textbook. Upon seeing the same explicit statements, Teachers 1, 5, and 6 instead moved on in search of other content.

Curricular Attending to Worked Examples

Teachers’ attention to worked examples also appeared to be shaped by their instructional orientations and assumptions about curriculum. When attending to Textbooks A and B, Teachers 1 and 2, who reported preferring teacher-centred instruction and had experience with DM texts, referenced worked examples more than the other teachers. During their interviews, these teachers clearly expressed a preference for curriculum materials that demonstrate the use of a procedure and provide explicit statements of facts. For instance, while planning with Textbook A, Teacher 2 said, “I am overjoyed with this book. It gives everything that I do... I love how this has the concept summary... I love how it gives [answer keys and worked solutions].” Since Teachers 1 and 2 intended to use the worked examples as presented in the textbook (i.e., their curricular responding), they carefully attended to what content the examples introduced. While reading multiple worked examples in Textbook A, Teacher 1 remarked, “So these are deciding how many solutions. Then example two is just actually solving... Examples one and two are similar. So they just come up with a different result.”

In contrast, Teachers 5 and 6, who reported preferring student-centred instruction and were familiar with both DM and TD texts, were drawn to worked examples in the same textbook as opportunities for student learning through problem-solving and reasoning. This can be tied to their desire to have students develop new understanding through sensemaking and exploration, which led to the teachers’ decision to remove the solutions, demonstrating how curricular responding can influence attending. For

example, when working with Textbook B, Teacher 6 commented: “Just give [a worked example with no solution] to them and don’t tell them anything about it...see what they come up with for a solution.” While making adaptations, neither Teacher 5 nor 6 acknowledged how the textbook expected students to solve these problems.

Curricular Attending to Explorations in Student Materials

Teachers’ attention to explorations appeared to be influenced by their interpretations of tasks and a desire to understand the tasks themselves and the experiences students would have while engaging with them. When they encountered explorations designed for students to learn new content and engage in sensemaking, some teachers read carefully to comprehend these textbook portions. To better understand the affordances of these tasks, Teachers 3 and 4, who were familiar with both DM and TD texts, solved the tasks themselves and considered how students might approach these portions of the text. While attending to an exploration from Textbook D, Teacher 4 expressed, “I am trying to imagine what a kid would do,” and while analysing an exploration from Textbook C, Teacher 3 noted, “But then setting them equal to each other is tricky because you’re going to have two fraction coefficients on x .” Since Teachers 3 and 4 designed lessons to introduce new content through both teacher-centred instruction and student-led problem-solving, it may follow that these teachers invested more focus in understanding these explorations because while they recognized the value of these tasks, they were less familiar with them compared to Teachers 5 and 6 (who favoured new content emerging primarily through student-led explorations) and thus needed more time to get oriented. Moreover, Teachers 3 and 4 jumped between explorations and accompanying teacher notes to make sense of these explorations.

In contrast, Teacher 2, who reported preferring teacher-centred instruction and had experience with DM texts, showed the most striking contrast to the pattern exhibited by Teachers 3 and 4. When she came across student-led explorations, she expressed a need to adapt the activities, which in turn influenced how she attended to these tasks and the accompanying teacher notes. For instance, when planning a lesson with Textbook D, Teacher 2 interpreted an activity designed to generate a new understanding of how to use substitution to solve a system of equations not as an opportunity for new insight, but instead as an opportunity for students to practice prior learning. For this reason, instead of seeking to understand how an understanding of the substitution method can emerge by working with systems of equations in different forms, she used these different systems in an activity she designed for students to practice known techniques of solving systems of equations, deciding quickly that “maybe, [task (a)] would be you had to solve it through elimination, (b) would be substitution, (c) might be elimination, ...” Moreover, due to her interpretation, Teacher 2 never attended to the teacher notes that explained how different parts of this exploration enabled the development of new strategies for solving systems.

Curricular Attending to Exercises for Review and Practice in Student Materials

When attending to exercises, the teachers' attending patterns appeared to be influenced by textbook design. Textbooks A and B featured a considerably higher number of exercises compared to Textbook C, while Textbook D didn't include any exercises. Moreover, unlike Textbook C, the exercises in Textbooks A and B were often organized into clusters on the same page. While working with Textbooks A and B, Teachers 1, 2, and 6 scanned exercises, at times choosing them for their lesson plans arbitrarily as opposed to solving them and carefully examining them for mathematical opportunities. For example, while planning with Textbook A, Teacher 6 explained, "So maybe I would pick a couple of them just in here from 23 to 40. Maybe I'd pick three of those problems..." Similarly, when working with Textbook B, Teacher 1 said, "This is where I'd probably have them try just a couple from here, still at their individual seats, this monitoring progress thing." This tendency to look at superficial features of these exercises without considering their affordances is indicative of the influence of textbook design on curricular attending.

FINAL REMARKS

This study provides insights into the influences on high school mathematics teachers' curricular attending during lesson planning based on our interpretation of the data. Our findings reveal several potential influences that affect how teachers attend to different textbook parts. Teachers' attention can be influenced by external factors, such as school requirements, educational policies, teacher autonomy, and teacher well-being. Alignment of curriculum materials with teachers' instructional orientations and conceptions of the curriculum can also significantly shape their curricular attending, along with their past experiences with these materials and textbook design. Furthermore, the bidirectional relationship among attending, interpreting, and responding showed that interpretation and planned use of materials influenced teachers' attention to parts of the textbook lesson.

This study was limited by the small number of participating teachers, all of whom were experienced teachers with established instructional orientations. Further research is needed to generalize these findings and could focus on teachers whose orientations and conceptions of curriculum are still developing. Additionally, our findings are based on what teachers said and did during staged planning interviews, which may not fully reflect their planning practices or curricular enactments. We also recognize that our own assumptions and analytic decisions influenced how we interpreted the data and that our analysis represents one interpretation among many. Future work might examine how teachers' curricular attending occurs in real-time planning.

Finally, these findings provide a window into why curriculum materials designed to shift instructional practices rarely lead to new forms of instruction. Curriculum designers might consider how to develop materials or other supports that help teachers navigate misalignments between new materials and their existing practices.

Acknowledging these realities can better position future curriculum resources to support teachers as they respond to the complexities of planning.

Acknowledgment

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BILINGUAL MATERIALS FOR MATHEMATICS IN MALTA

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In Malta, mathematics is frequently accessed through translinguaging with English and Maltese. This is certainly the case when the teacher's and students' home language is Maltese, because textbooks and other written materials – including examinations – are in English, a legacy of British colonisation that ended in 1964. However, although much translinguaging occurs in verbal interaction, Maltese is not used for written mathematics. In this paper, I describe classroom materials that I designed, which included Maltese text, as part of a design-based research project carried out in an elementary school. The inclusion of Maltese text was viewed positively by both teacher and students, and I suggest that this approach might be taken on board as an integral part of a translinguaging pedagogy.

INTRODUCTION

During the period of British colonisation of Malta (1800–1964), English gained significance as the language of education and as a means of economic and social advancement. During various points in this era, the Maltese language was afforded varying levels of importance within the educational sphere. Although today, the national language plays a significant role in education, the historical influence of English is still present. This is mainly because several subjects still utilize English-language textbooks, generally published in the U.K. Mathematics is one such subject. This situation often results in frequent switching between English and Maltese in classroom verbal interaction. For example, a problem might be read out in English, and the teacher might elaborate to aid understanding: “The question asks for the perimeter. **Irridu nsibu l-perimetru, id-dawra**” [“The question asks for the perimeter; **We need to find the perimeter, the outline/boundary**”]. It is also common to retain subject-specific terms in English as in the statement “**X'uhud minn dawn ix-shapes huma circles**” [“**Some of these shapes are circles**”]. My awareness of the spoken practice is based on my personal and professional experience and on local research (for example, Camilleri, 1995; Farrugia, 2017; Vella et al., 2020).

There have been but a few attempts to use Maltese for written mathematics texts. T. Vella (1913) wrote a bilingual textbook on arithmetic, wherein explanations and some exercises are presented in Maltese, but many mathematical terms were retained in English. Several decades later, a secondary school textbook was published by Caruana and Muscat (2007). The book has two ‘front’ covers in that the English text starts from one end of the book, while a roughly equivalent Maltese text starts from the other. A more recent initiative was one taken by the *National Literacy Agency*. This was the

piloting of an examination paper for Grade 4, wherein the same questions were provided in English and Maltese on facing pages (Vella et al., 2020). This pilot revealed the importance of having an established bank of Maltese mathematics terminology, and this, in turn, resulted in the compilation of a bilingual glossary to which I contributed (Farrugia et al., 2024). The glossary will be distributed free to schools by the Ministry of Education, to serve as a pedagogic resource. My recent study is a further contribution to using written Maltese for mathematics. The design-based study addressed the question: *How can Maltese be used to support the teaching and learning of mathematics?* While the study focused on both oral and written Maltese, in this paper, I restrict myself to the written aspect. The purpose of this paper is to report on materials that I designed for an elementary classroom. I give examples and explain the approach taken. The materials referred to in this paper have previously been displayed in a gallery walk at the ERME Topic Conference ‘ETC16’, accompanied by a short theoretical write-up (Farrugia, 2024).

BILINGUAL RESOURCES FOR MATHEMATICS

International contexts wherein bilingual materials have been prepared are rather limited. In the context of Wales, Jones (2009) reported that in classrooms consisting of both Welsh-speaking and English-speaking students, teachers prepared materials in both Welsh and English, either as separate worksheets or by using both languages on the same sheet. *Math Essentials* publishes a bilingual English/Spanish textbook set (Fisher, 2020), while *MathAmigos* (n.d.) also provides activities in English and Spanish. Kajoro (2016) describes a project in Tanzania that involved the drafting of a lower secondary textbook in English with Kiswahili translations. Robertson and Graven (2024) report on a project in South Africa where mathematics resources are produced in multiple home languages and in English. Similarly, Venkat and Morrison (2024) describe bilingual materials wherein the texts are produced in a home language (Afrikaans, Sepedi, or IsiXhosa) and in English, on facing pages, and the student’s materials are subtitled in English. In my study, I have taken an innovative approach, interweaving English and Maltese, in a manner that mimics common classroom practice.

‘LANGUAGE AS A RESOURCE’ AND PEDAGOGIC TRANSLANGUAGING

My study was underpinned by an assumption that language is a resource (Ruiz, 1984). Baker and Wright (2021) state that from a ‘language as a resource’ perspective, bilingualism is seen as an asset for individuals and communities; indeed, a home language is considered to be a valuable resource upon which students can build knowledge. The ‘language as a resource’ orientation opens a space for the use of students’ home languages in schools and, consequently, for translanguaging pedagogies. The term ‘translanguaging’ has been used in different ways over the years, and here I subscribe to the *Crosslinguistic Translanguaging Theory* posited by Cummins (2021). This theory views translanguaging as shifts between distinct, identifiable codes. (This is in contrast to a ‘unitary’ view of translanguaging that

considers language as one integrated mental system. See, for example, García & Otheguy, 2014). Cummins (2021) accepts that languages are social constructions and their boundaries are arbitrary, but he concludes that speakers treat them as separate and that they are able to identify each. Despite this, language is “constantly in a state of flux” (Bhatt & Bolonyai, 2022, p. 158), and with potential for variation and change, depending on the function it serves in a given context. Thus, Baker and Wright (2021, p. 297) summarise translanguaging as “the process of making meaning, shaping experiences, understandings and knowledge through the use of two languages”.

With respect to mathematics education, Planas and Setati-Phakeng (2014, p. 887) consider language-as resource as “an organizing principle for classroom practices with the aim of achieving learning opportunities through integration of the foci on mathematics and language”. The practical application of the theoretical assumption aligns with Maltese education policies that encourage language mediation according to context and student needs. Indeed, the policies point to a future availability of bilingual resources and assessment (Ministry for Education and Employment, 2016; Vella et al., 2023).

OVERVIEW OF RESEARCH METHOD

The materials discussed in this paper formed part of a wider collaborative design-based research study that I carried out with two elementary school teachers. Design-based research is described by McKenney and Reeves (2019) as an approach that involves developing and implementing interventions to address problems in practice. In this study, the issue at hand was how written Maltese could be used for meaning-making. Having first observed one of their own lessons, I drew up five or six lessons per class, following their preferred lesson style and structure. However, in written texts, I also included Maltese subject-specific and ‘meaning-making’ vocabulary which, as highlighted by Prediger (2019), may also include everyday language. I discussed the plans with the teachers and made amendments where these were suggested. The teachers then delivered the lessons, while I observed and took notes. The lessons were audio- and video-recorded for later analysis. Daily discussions were held with the teachers, during which we spoke about the lesson and considered any changes to the following day’s lesson. Two lengthier interviews were held with each teacher before and after the series of lessons.

The focus of this paper is the bilingual materials used in one of the classrooms, namely, Grade 6 (children aged 10 to 11 years). The topic was *Fractions* and the lessons formed part of the normal curriculum entitlement. The lessons were designed to be language-rich (Atkins, 2021) and they included several paired discussions and reporting back to the class. The teacher commented that although the content being tackled was not new to the students (except for comparing fractions to $\frac{1}{2}$), the fact that they had to articulate ideas rendered the lessons a bit more challenging. With regard to language, the teacher and students were left free. The teacher tended to use English more, but she switched to Maltese when she felt that this would aid students’ understanding. The students

tended to use Maltese amongst themselves, but often reported back to the class in English. The teacher and students also translanguaged frequently.

THE DESIGN OF THE MATERIALS

There were three types of materials that included written text: interactive panel projections, worksheets and study notes. I will consider each in turn. In the original materials, Maltese text was shown in red, while here it is shown in **bold** font, with a translation provided in square brackets.

Interactive panel projections

Interactive panel projections were used to promote discussion. Figure 1 shows two examples. In the first, ‘Alfred the alien’ is wondering about the meaning of two phrases, and the students were to discuss in pairs how they would explain the terms to him. The second example shows a word problem to be discussed and solved.

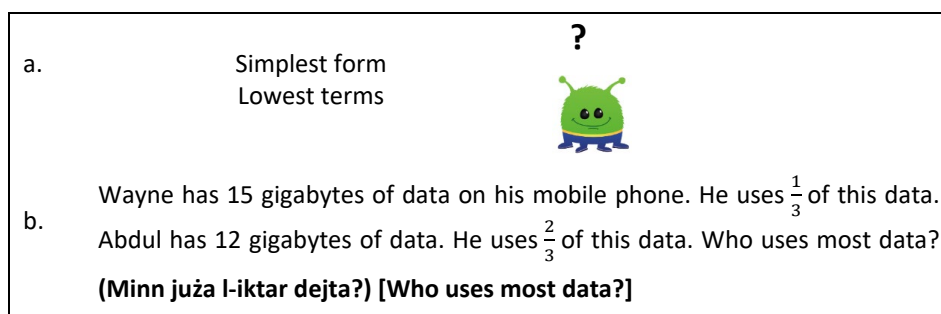


Figure 1a & b: Interactive panel projections to promote discussion. (Image retrieved from <https://dribbble.com/shots/34846-Cute-Little-Alien>. Designed by Kris Black and reproduced with permission).

In the first example, I used two phrases since I assumed that, in the previous Grades, they may have heard both, but it was quickly established by the students that the phrases referred to the same idea. My reason for projecting the *English* phrases, rather than the Maltese one – **il-forma l-iktar sempliċi** [**the form that is most simple**] – was because the students were already familiar with the former, and at this point, I wished them to articulate their understanding of the concept as an introductory task. The Maltese expression was introduced a short while later, both verbally and in writing. With regard to the problem, this was presented in English, since all their school lives, this is how the students would have encountered word problems in textbooks, worksheets, and examinations. However, I also wrote the question in Maltese, repeating what they were being asked to find. Here, Maltese served to support the first step of solving a problem, that is, understanding it (Polya, 1945).

Worksheets

The second type of material I utilised was hard copy worksheets. Although the students had an English language textbook and corresponding ‘practice’ book, it was also customary for the teacher to provide complementary worksheets. It was necessary for me to prepare bespoke worksheets, partly because I wished students to work on

bilingual materials, and also because I wished the students to tackle more word problems than were provided in their textbook. I chose not to duplicate all text in both languages for three reasons. First, I wished to try out a strategy that would potentially be taken up by teachers; producing *two* versions of each worksheet would duplicate preparation and use up a lot of paper. Second, my idea was to mimic common verbal patterns of speech. Third, English is still the language of textbooks and assessments for the foreseeable future, so I considered it appropriate to maintain English as the main language of the texts. Hence, I limited Maltese to that which I believed would help with meaning-making. Figure 3 shows two examples of worksheet problems.

- (a). Alfred says that the fractions $\frac{4}{12}$ and $\frac{12}{36}$ are equivalent because both fractions include the number 12 (**it-tnejn li huma fihom in-numru 12**). Is he right? Explain.
- (b). Alfred knows that $\frac{1}{4}$ and $\frac{2}{8}$ are equivalent fractions. He says that $\frac{1}{4}$ of €20 must be equal to $\frac{2}{8}$ of €32. Is he right? (**Jghid li peress li $\frac{1}{4}$ u $\frac{2}{8}$ huma frazzjonijiet ekwivalenti, mela $\frac{1}{4}$ ta' 20 huwa daqs $\frac{2}{8}$ ta' 32**). Is he right? Explain.

Figure 2: Examples of problems in worksheets.

In example 2a, the Maltese text is a translation of the English, but the expression **it-tnejn li huma** [the two of them] might potentially stress the meaning of *both*. In example 2b, the Maltese is not an exact translation. Rather, the Maltese version includes the logical structure ‘**since ... then**’. This allowed me to express the alien’s difficulty more specifically: Alfred’s (mistaken) reasoning is that *since/because* the fractions are equivalent, *then* they result in equal quantities, irrespective of the amount of money. In Malta, the if/then reasoning in English is generally avoided at the elementary level to keep the English sentence structure simple. Thus, using Maltese allowed for a more complex sentence structure and the use of logical reasoning.

Study notes

The third type of material was soft-copy notes intended for study purposes. The teacher would project the notes in class at the appropriate point in the lesson, and later forward them to the students electronically. Figure 3 shows some examples.

- (a) We can order fractions by comparing them to $\frac{1}{2}$. (**inqabbluhom man-nofs**) [We compare them to half].
- (b) A fraction size depends on the whole. The whole is the total number of things in the set. (**Id-daqs tal-frazzjoni jiddependi fuq il-ħaġa shiħa. Il-ħaġa shiħa hija l-ammont kollu ta' affarijiet li hemm fis-sett**). [The size of the fraction depends on the whole thing. The whole thing is the full amount of things that there are in the set].
- (c) $\frac{1}{4}$ of a set of 40 stickers is MORE than $\frac{3}{4}$ of a set of 12 stickers. This is because $\frac{1}{4}$ of 40 is 10 and $\frac{3}{4}$ of 12 is 9. (**Wieħed minn kull erbgħa ta' 40 stiker huwa IKTAR minn tlieta minn kull erbgħa ta' 12-il stiker**) [one from every four of 40 stickers is MORE than three from every four of 12 stickers].

Figure 3: Examples of ‘notes’ text.

In example (a), I translated the idea since this was new to the students. In example (b) I translated the whole text, introducing the Maltese mathematical phrase **the whole thing** for what is frequently referred to as *the whole* in English. In Maltese, one cannot use the word *whole* as a noun. In example (c), the translation offers support in conceptualising fractions: the symbol $\frac{3}{4}$ is stated as *three quarters* or *three fourths* in English, while in Maltese, one says **three from four** or (as in the text above), **three from every four**. Thus, the Maltese is more explicit.

CONCLUSION

In this study, I tried out something that was innovative for our classrooms, namely, inserting Maltese into written text. Due to the recent compilation of a bilingual glossary (Farrugia et al., 2024), it was possible for both myself and the teacher to feel confident in the Maltese terminology that might be used for both verbal interaction and for written texts. The approach was perceived positively by the teacher. Furthermore, while the teacher and I had anticipated that the students might find the mix of both languages for written mathematics strange, in reality, they took it very much in their stride. They also reported to the teacher that they found it helpful. Admittedly, no formal evaluation of the materials and the approach was carried out. However, the observed engagement of the teacher and students, and the points arising from my daily discussions with the teacher, indicated that the approach has good potential for assisting teachers to reflect on their use of language, and to utilise Maltese with increased purpose for the benefit of student understanding. Future studies might incorporate an evaluation process to identify benefits more specifically.

I must stress that I am not suggesting that English be side-lined. Ultimately, textbooks and examinations continue to be in English, so that students will need access to “higher status forms of language” (Morgan, 2007, p. 241). Finding an appropriate use of both languages is key. As I designed the lessons, I made decisions about what English text to translate. A class teacher would be in a better position than I was to decide what should be translated for her class (which might vary from cohort to cohort). Then, at some point, a teacher would reduce the amount of written Maltese in preparation for national end-of-year examinations, which, to date, are still in English. On the other hand, the possibility of bilingual examinations should not be excluded; recent education policies encourage flexibility of language and, in particular, the language policy for primary level education specifically recommends that “bilingual resources and assessment materials are to be designed to cater for various learner needs” (Vella et al., 2023, p. 6). In the eventuality of examination papers utilising Maltese, the glossary will ensure that the Maltese terms used in classrooms will be consistent with those of assessment. The availability of a bilingual glossary and increased experience with using mathematical Maltese at the grassroots level can also support any future publication of textbooks and digital resources. I believe that my contribution to the glossary and researcher/teacher reflections as part of design-based studies can contribute to strengthening our local translanguaging pedagogy.

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LEARNING GEOMETRY IN TWO CULTURAL CONTEXTS: WHAT DOES EXAMINING TEXTBOOKS TELL US

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This paper reports on a study that examines the presentation of second-grade geometry in two textbooks used in the respective countries - Ethiopia and Norway. It draws on didactical phenomenology, the van Hiele model of geometrical thinking, and adopts suggestions of three geometry strands for second-grade geometry education. The study employs a document analysis method. The results show that the Ethiopian textbook focuses mainly on the descriptive level, while the Norwegian focuses on the visualization level. Though little is available, real-life phenomena are used as starting points in the Ethiopian textbook. While the Norwegian presents real-life phenomena well, there is little use of these phenomena as starting points. Both textbooks focus on constructing and operating with shapes and figures, but hardly address orienting.

INTRODUCTION

This paper reports on a study that examines the textbooks used in two countries - Ethiopia and Norway. It focuses on geometry for second grade. The significance of children's earlier experiences in learning geometry is emphasized in many studies (e.g., Gravemeijer et al., 2016; Clements & Sarama, 2000; Freudenthal, 1983). According to Freudenthal (1983), geometry is an important part of mathematics, which children begin to experience early in their lives. More recent literature suggests that children at the kindergarten level can start to experience diverse strands of geometry (van den Heuvel-Panhuizen et al., 2008). Considering geometry education at lower primary grades as dealing only with figures and shapes is often criticized. Particularly, the orienting aspect is often ignored (cf. van den Heuvel-Panhuizen et al., 2008). Geometry education for kindergarten, for earlier grade children alike, can focus on three strands, namely, "orienting, constructing, [and] operating with shapes and figures" (van den Heuvel-Panhuizen et al., 2008, p. 227). The traditional approach, where the children are taught in isolation from real-life situations, is often criticized (e.g., Doorman et al., 2020; Freudenthal, 1983). These scholars suggest everyday experience as a starting point for teaching geometry.

The importance of textbooks as resources, which have a pivotal influence on mathematics classrooms and teachers' reliance on the textbook for teaching, is explained in earlier studies (e.g., Lepik et al., 2017). Textbooks, among other curriculum resources, are both instruments and objects of change during educational reforms (Rezat et al., 2021).

The study was initiated due to the author's (my) familiarity with the two countries and their education system. My experience as a teacher and teacher educator in the two countries allowed me to follow the developments related to mathematics education and to develop a particular interest in their curricula and textbooks. There were changes in the textbooks around the same time in Norway (e.g., Alseth et al., 2020) and Ethiopia (e.g., Mehammed et al., 2022). These changes followed the change in the Norwegian curriculum (Ministry of Education and Research [KD], 2019) and the new roadmap in Ethiopia (Teferra et al., 2018).

Hans Freudenthal's notion of enabling the child to 'grasp space' has been a well-accepted goal of learning geometry in RME-oriented primary school geometry (e.g., Gravemeijer et al., 2016; van den Heuvel-Panhuizen et al., 2008). In geometry at primary grades, one aspect of enabling the child to grasp space includes learning about rotation and direction (e.g., van den Heuvel-Panhuizen et al., 2008). These scholars suggest three types of orienting for grades 1 and 2, which are localizing, taking a point of view, and working with rotations and directions. Such an early experience with this geometry strand can enhance a better understanding of angles at the letter stage (van den Heuvel-Panhuizen et al., 2008).

The following research questions guide this study: Which levels of geometrical thinking are given emphasis in second-grade textbooks of the two countries? Are real-life situations used as a starting point for presenting geometric concepts? Which geometry topics are prioritized? An earlier study about the old Ethiopian textbook was guided by the first two research questions and exposed limitations both in using real-life phenomena as a starting point and in maintaining the appropriate level of geometric thinking (see Gebremichael, 2025). This earlier study, which focused on Ethiopian middle school, did not address the third research question. The current comparative study is a continuation of the previous study. This paper has five sections, including this section. The remaining sections - Theoretical Issues, Method, Discussion, and Concluding Remarks - are presented in this order.

THEORETICAL ISSUES

The current study draws on a framework which, borrows ideas from didactical phenomenology (Freudenthal, 1983) and van Hiele's model of geometrical thinking (Crowley, 1987). It also borrows ideas from the literature that suggest strands of geometry content for second grade (e.g., van den Heuvel-Panhuizen et al., 2008).

The first aspect of the framework is based on an idea borrowed from Freudenthal. Freudenthal (1983) emphasizes that geometry teaching should build on children's everyday experiences, using them as starting points for developing geometric concepts. The framework's second aspect draws on the van Hiele model (van Hiele, 1959/1984, 1999), which outlines five sequential levels of geometric thinking. These are visualization, analysis/ descriptive, informal deduction, deduction, and rigor. This study uses the first three. Children at the visualization level can recognize shapes based

on appearance. At the analysis/ descriptive level, they can recognize shapes by their properties. At the informal deduction level, they can understand definitions (van Hiele, 1959/1984, 1999). Unlike others (e.g., Crowley, 1987; Doorman et al., 2020), who used the term *analysis*, van Hiele himself used *descriptive* instead. In fact, it was only in his later works that he explicitly named the levels (e.g., van Hiele, 1999). In his original work, he simply labelled them as Level 0 through Level 4 (van Hiele, 1984/1959). I prefer to follow this original usage. Therefore, unlike in my earlier studies (Gebremichael, 2025), I adopt the term descriptive here. The third aspect concerns the three strands of geometry. As set out earlier, the literature that draws on realistic mathematics education suggests three strands of geometry for second grade, which are orienting, constructing, and operating with shapes and figures (e.g., de Moor, 2008; van den Heuvel-Panhuizen et al., 2008). According to van den Heuvel-Panhuizen et al. (2008), an important part of the orienting strand is working with rotation and directions. The analysis uses the framework's three aspects.

METHOD

Data are extracted from the textbook using the document analysis method (e.g., Bowen, 2009). The method comprises three steps: skimming, detailed examination, and interpretation. I skimmed the two textbooks, with extra focus on the geometry chapters. This helped to explore and expose the structures of the textbooks and the structures of the chapters. I also skimmed the mathematics curriculum documents from the two countries, with an extra focus on geometry. The relevant data were extracted and translated from Amharic and Norwegian into English. The framework serves as an analytic lens for the data. The three elements of the framework guided the coding process and the formation of categories. The next section presents sample data. They are labelled with numbers in square brackets, such as [1], which stand at the start of the data segment they are assigned to. Each data segment is accompanied by coding in curly brackets, for example, {descriptive: recognizing properties}. In the Discussion section, these data segments will be referenced by their assigned numbers.

DATA PRESENTATION AND ANALYSIS

In this section, data and analysis are presented. The data is presented in two subsections, namely, The Ethiopian Textbook and The Norwegian Textbook. The presentation starts with some features of the education system and general issues about textbooks in the two countries. Unlike Norway, textbooks in Ethiopia are published by the education authorities. In the Norwegian school system, in addition to the textbook, which the current study has examined, there is a resource called a workbook. It has additional tasks corresponding to each of the textbook's chapters. More resources on the education directorate website (Utdanningsdirektoratet.no) support the teaching of mathematics in general and geometry in particular. The website provides, among others, the national mathematics curriculum for every grade level, which includes the goals and learning outcomes for the topics. In this study, the data is extracted from

textbooks. Some relevant data from the national curriculum are also presented. There is no such support system for the Ethiopian textbooks. As will be seen in the following section, the learning outcomes of each chapter of the Ethiopian textbooks are provided at the start of the chapter. In the Norwegian textbooks, stating the learning outcomes begin at fourth grade. The second-grade Norwegian textbook does not state the learning outcomes. But the directorate's website provides them.

The Ethiopian Textbook

The Ethiopian second-grade textbook has ten chapters. It has only one volume. The eighth chapter is titled “Geometric Shapes.” The chapter has two learning outcomes. [1] It states, “After finishing this chapter: You will categorize plane shapes by their number of sides and angles. You will describe shapes by their sides and angles. {descriptive: Recognizing properties}.” [2] The chapter has four sections: Categorizing plane figures by seeing, {visualization level}; Describing shapes by their sides and angles, {descriptive, understanding properties}; Comparing similar figures based on size {descriptive, understanding properties}; Identifying solid figures (Mehammed et al., 2022, p. 118), {descriptive, understanding properties}. The chapter ends with two additional sections: a summary and exercises. [3] In every section, there are tasks. Some of them are for group work. [4] The chapter provides tasks relating to the children’s real life. For example, the second section starts with an activity with two tasks. The first asks children to list things in their home with shapes of three sides and three angles. The second one asks children to list things at their school with shapes of three sides and three angles {descriptive: recognizing properties; real-life as starting point} {they are used as starting points for building the concept of plane figures}. [5] The Ethiopian textbook provides some illustrations. It illustrates the sides and angles of a triangle, a quadrilateral, and a hexagon. This is accompanied by a note that the number of sides and angles is the same for each plane figure {visualization level}.

[6] It then provides a task that involves plane figures (triangles, quadrilaterals, pentagons, and hexagons). It asks children to categorize them according to their shapes (Mehammed et al., 2022, p. 114) {visualization level}. [7] The second section provides a set of exercises where children are asked to match a list of plane figures with a list of the number of sides (p. 115) {visualization level}. [8] The third section is titled “Comparing similar figures based on their sizes” (Mehammed et al., 2022, p. 115). It starts with notes including the following description/ definition, “Similar figures are those which have the same shapes but can have different sizes.” (Mehammed et al., 2022, p. 115) {informal deduction level: understanding definition}. [9] The last section of the chapter is “Identifying solid figures”. It starts with an activity, where pictures/figures representing various objects (such as a banana, a ball, a triangle, and a square) are provided. It asks children to separate the solid objects from those which are not (i.e., categorize the solid objects in one group and those which are not in another group) {pictures of phenomena as starting point}. [10] The geometry chapter in the Ethiopian textbook uses much text. It has a few illustrations.

The Norwegian Textbook

[11] The Norwegian second-grade textbook has eight chapters. It has two volumes, each containing four chapters. The textbook has a geometry chapter called “shapes and figures” (Alseth et al., 2020). It is the last chapter in the first volume of the student textbook. [12] The geometry chapter has four sections: Recognize two-dimensional figures; Properties of two-dimensional figures, Symmetry and shapes, and Symmetry in Christmas decorations. The chapter ends with a set of tasks titled “Can you do this?” [13] The chapter does not have an introductory text. The first section, which is “Recognize two-dimensional figures,” starts immediately after the title of the chapter {descriptive, recognize by properties}.

[14] Unlike in the later grade levels, the chapters of the second-grade textbook do not state the expected learning outcomes. [15] They are stated on the education directorate's webpage. It states: “The aim of the learning is for the student to be able to ... explore, draw and describe geometric figures from their immediate environment and argue for ways to sort them according to their properties” (KD, 2019, p. 6). [16] The first section of the chapter is “Recognize two-dimensional figures” {visualization}. It starts with an illustration of houses, a signpost of a bicycle, children, a ball, a balloon, and a child in a cart. The illustration also contains objects in circular, triangular, square, and rectangular shapes. [17] Then, there is a task with an illustration where children are asked to recognize geometric shapes in the illustration. They should do this by connecting the given geometric figures (i.e., circle, triangle, square, and rectangle) and the corresponding shapes in the illustrations, which they recognize as such. The task reads as “draw lines to the correct geometric figure” (Alseth et al., 2020, p. 96) {Visualization: recognize, categorize based on appearance}. The terms used in the section’s title, “recognize two-dimensional figures,” made the section seem to demand the descriptive level. However, the tasks demand a visualization level. It also has such tasks, [18] “Make a shape of triangles, squares, and rectangles. Draw lines from dot to dot.” (Alseth et al., 2020, p. 101). {Strand: constructing}.

The third section of the chapter is on symmetry. In this section, the textbook provides tasks such as the following: [19] Ask children to draw a line of symmetry for drawings/pictures of naturally occurring things, such as a leaf {taking real-life phenomena as a starting point}. [20] It has another task which asks the students to complete a design such that the picture will be symmetrical. {descriptive, Property of symmetry}, {not taking real-life phenomena as a starting point} [21] It also has a task about a “symmetric pattern,” with a line of symmetry. In this task, part of the design of a picture on a square paper is given. Then, it asks the second-grade students to complete the picture by drawing the other part of the given design such that the complete picture will have a “symmetric pattern.” (Alseth et al., 2020, p. 112) {descriptive, Property of symmetry}, {not taking real-life phenomena as a starting point}.

DISCUSSION

The Norwegian textbook, which has eight chapters in two volumes, presents the geometry chapter in the first. Whereas in the Ethiopian textbook, which has only one volume, presents a geometry chapter. Using real-life phenomena as starting points for geometry teaching has been advocated for decades (Freudenthal, 1983). However, such an approach to geometry teaching is limited. The Norwegian textbook provides several tasks with real-life context; however, they are rarely used as starting points for teaching the concepts (see 20, 21). The Ethiopian has few real-life phenomena and illustrations; however, some of such tasks served as starting points (see 3, 9).

In the Ethiopian textbook, both learning outcomes of the geometry chapter indicate the demand for the descriptive level (see 1). Among the four sections of the chapter, only the first one is at the visualization level (see 2-4). The chapter has a few illustrations and focuses mostly on the descriptive level rather than visualization (see 6). There is also a demand for an informal deduction level, where second-grade children are expected to understand definitions (see 8). The Norwegian focuses mainly on the visualization level with a few instances at the descriptive level (see 11-17). For example, its tasks involve many more illustrations than texts. This is consistent with van Hiele's model (e.g., Crowley, 1987). As set out in the analysis section, unlike using the terms in the section's title, the tasks in its first section demand visualization level (see 16 & 17).

In terms of content, there are some differences. The Ethiopian textbook focuses mainly on one theme - shapes. An additional theme is solid figures. The Norwegian focuses on symmetry, and figures and shapes. These fall into the two strands of constructing and operating with shapes and figures (van den Heuvel-Panhuizen et al., 2008). Both textbooks give hardly any attention to the orienting strand. This is inconsistent with van den Heuvel-Panhuizen et al. (2008). Ignoring the orienting strand can limit the exposure to experiences that pave the way for a better understanding of the concept of angle (van den Heuvel-Panhuizen et al., 2008). It might also limit the diversity of children's exposure to visual experiences and the consequent analysis of properties.

CONCLUDING REMARKS

This paper was guided by the questions about which levels of geometrical thinking are given emphasis in second-grade textbooks, whether real-life phenomena are used as a starting point for presenting geometric concepts, and which geometry topics are prioritized. This was done in the light of a framework composed of ideas from the notion of didactical phenomenology, the van Hiele model of geometrical thinking, and strands of geometry content for second-grade children suggested by the literature drawing on realistic mathematics education. The results show that though little is available, real-life phenomena are used as starting points in the Ethiopian. The Norwegian presents real-life phenomena well, but using these phenomena as starting points is rare. While the Ethiopian textbook is mainly descriptive, the Norwegian is

visual. Regarding contents, both textbooks focus on the two strands - constructing and operating with shapes and figures - and hardly address the orienting strand.

These results are based on the textbooks and the relevant curriculum documents. It would be interesting to examine the implementation in the classrooms. For example, the Norwegian curriculum document stresses using an exploratory approach. How and if the textbook helps the teacher employ this approach would be interesting research to undertake in the classroom. As mentioned, the Ethiopian textbook uses a few real-life phenomena as a starting point. However, in some of these tasks, there is an underlying assumption that second-grade children recognize plane figures based on their properties. It would be interesting to see the implementation in the classroom. It is particularly interesting to examine how the competence goals in geometry are implemented in different parts of the two countries. Although the curricula are nationally common, the teaching takes place within different cultural and linguistic contexts. In Ethiopia, there are diverse cultures and languages. Though there is much less diversity in Norway, in addition to Norwegian-language schools, there are both Sámi-language and Kven-language schools. Further exploration of how such linguistic and cultural differences can affect students' learning and achievement of the competence goals in geometry can therefore provide valuable insights. The study has implications for geometry education in the two countries and beyond. It is important to follow certain theories for geometry education or a combination of theories. The textbooks need to balance between introductory text and illustrations. The curriculum documents should also be examined for if, and how, they guide the inclusion of fundamental aspects of geometry appropriate to early primary education.

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USING EDUCATIVE CURRICULUM MATERIALS TO SUPPORT ANALYSING MATHEMATICS TEACHING IN DEEP AND SPECIFIC WAYS

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In this paper, we examined how educative curriculum materials can support mathematics teacher educators (MTEs) and teachers to engage in fine-grained analyses of teaching. Our thesis is that conversations focused on educative materials can provide especially rich collaborative learning opportunities for teachers. Through analysing planning conversations within teacher collaborative groups, we found that the teachers and an MTE used curricular materials to examine and prepare rich learning opportunities for students. This analysis provides an additional perspective of what an educative curriculum means in terms of supporting teachers to develop ambitious teaching practices.

FOCUS AND RATIONALE

The goal of this paper is to examine how an educative curriculum¹ can support mathematics teacher educators (MTEs) and teachers to analyse teaching in ways that lead to more ambitious teaching (Lampert et al., 2011). We examine the role of educative curriculum, materials designed to support both teacher and student learning (Davis et al., 2017), MTEs and groups of teachers collaboratively plan a lesson using a structured routine that encourages fine-grained analyses of teaching.

BACKGROUND AND THEORETICAL FRAMEWORK

Our thesis is that when teachers engage in *deep and specific conversations* about the content of individual lessons, it can provide especially rich learning opportunities for teacher learning (Russell et al., 2020; Stein et al., 2022). In the U.S., facilitating these conversations has been challenging as novice teachers are typically trained to teach their subject independently of a specific curriculum (Cohen, 2010), and the abundance of different textbooks makes it difficult to focus on preparing teachers to use a single one. We are interested in understanding how a well-designed, shared textbook lesson can create opportunities for teachers to work together to learn to teach ambitiously.

An educative curriculum supports teacher learning by providing opportunities to deepen content knowledge, anticipate student thinking, and plan effective instructional strategies (Davis et al., 2017). Learning to teach for ambitious learning goals, such as conceptual understanding, increases the importance of educative curriculum because, at least for U.S. teachers, these instructional practices differ significantly from most

current practices (Snow-Renner & Lauer, 2005). Teachers' development of ambitious teaching often requires significant learning that involves revising deeply held beliefs, knowledge, and routines (Thompson & Zeuli, 1999).

Researchers have documented the value of teachers collaborating to improve their teaching (Vangrieken et al., 2017). However, how teachers interact and what work they collectively accomplish can vary (Horn & Little, 2010). With the recent push for curriculum-based professional learning (e.g., Short et al., 2023), it is important to understand how an educative curriculum could support teachers to engage in deep and specific conversations about teaching.

To develop ambitious teaching practices, we agree that teachers need opportunities to make sense of curriculum through focused discussions that examine interactions between teachers, students, and mathematical content (Russell et al., 2020; Stein et al., 2022). It is as if teachers are slicing an individual lesson into small pieces, identifying an especially important piece, and spending significant time examining the key mathematical idea, the way students might think about that idea, and what they might do to support the richest thinking possible. Stigler and Hiebert (1999) describe a debate among a small group of teachers regarding which number combination to use for their students' first experience subtracting a single-digit from a two-digit number. Possibilities included 12-7, 13-7, 15-7, 12-8, and 12-9. Teachers spent considerable time anticipating how their students might solve each problem and deciding which strategies would best position students to see the value of the base-ten number system. Other examples illustrating the richness and specificity with which teachers can analyse content, students, and teaching can be found in Stein et al. (2022), Fernandez (2002), Horn and Little (2010), Russell et al. (2020), and Melville (in press).

This study explores how educative curricula aligned with ambitious student learning support meaningful teaching analysis which in turn impacts lesson enactment. We aim to show how a well-designed, shared textbook lesson can create opportunities for teachers to work together to learn to teach a very specific version of mathematics.

CONTEXT

We report preliminary findings from a four-year project seeking to understand how teachers translate professional learning experiences into mathematics teaching that will ultimately improve student learning. The project, which emerged from a partnership between a statewide mathematics coalition and researchers at a state university, employed a lesson analysis cycle guided by MTEs using educative curricula to engage Grades 4-7 teachers in a video-based PD program.

The analysis cycles provided a highly structured process centred on the curriculum. Each cycle involved: (1) planning, (2) enacting, and (3) reflecting. Planning involved an MTE and 3-5 same-grade level teachers collaboratively analysing and planning a lesson (approx. 2.5 hours). Enacting involved each teacher teaching the lesson in their own classroom with the MTE videotaping the enactment (approx. 45 mins). Reflecting

involved the MTE and each teacher watching video clips from the lesson, discussing student thinking, teacher moves, and reflecting on any changes to inform teaching of the lesson the following year (approx. 45 mins). The cycle occurred three times throughout the school year.

The Grade 4 and 5 teams in the project were teaching *Bridges in Mathematics* curriculum (The Math Learning Centre, 2019). *Bridges* is an educative curriculum, aligned with ambitious mathematical content and practice standards, emphasising mathematical discourse, fluency, and critical thinking (Lunde, 2021). Teachers are provided an extensive overview of each unit, module, and lesson including descriptions of mathematical concepts, models used, algebraic connections, questioning strategies, a trajectory of skills across grades, and observational and written assessments.

METHODS

Participants

This project included 13 MTEs and 36 Grade 4-7 teachers. The data used for this analysis came from shadowing Mae, an expert MTE (20 yrs. exp.) in her second year of working with a 5th grade team of teachers. While we analysed a larger corpus of data for this analysis, we focus on the first lesson analysis cycle Mae conducted with the teachers, Erin, Sami, Diane, Noelle, and Laney. These teachers' years of experience ranged from 4-20 with a mean of 12 years.

Data Collection and Analysis

All data were uploaded, transcribed, and marked using Vosaic software. We were interested in understanding how educative curricula supported MTEs and teachers to analyse teaching with specificity. Therefore, three members of the research team independently listened to and marked segments of the planning process that captured MTEs and teachers engaged in deep and specific conversations about content, pedagogy, and student thinking. We compared selections and agreed on segments that focused on one or more of these aspects. We transcribed these segments and created memos that identified instances of fine-grained descriptions of mathematics content, pedagogy, and/or student thinking.

RESULTS

The planning phase we describe took place over a full day (08:30 to 15:30 pm). In the morning, grade 5 teachers worked together to analyse the unit overview, identifying key mathematical ideas. In the afternoon teachers and MTEs met in smaller teams and went through the structured process to collaboratively analyse the lesson. The educative structure of the curriculum provided information to the teachers about how the lesson fit into the big ideas of the unit, supporting them to understand the learning goals of this lesson. To illuminate the types of in-depth conversations that MTEs and teachers engaged in, and how the curricular materials (Figure 1) supported these conversations, we consider one planning episode between an MTE and a 5th grade

team. The episode demonstrates how the curriculum supported conversations that allowed the team to analyse connections between students, teachers, and content in detail.

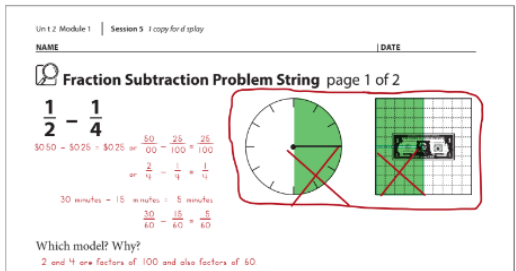
Problems	Sample Strategies & Recording	Connections
$\frac{1}{2} - \frac{1}{4}$	Think about $\frac{1}{2}$ like 50 cents and $\frac{1}{4}$ like 25 cents, so it's 25 cents. Fractions of a dollar: $\frac{50}{100} - \frac{25}{100}$, and the answer is $\frac{25}{100}$. A clock where half an hour is 30 minutes and $\frac{1}{4}$ of an hour is 15 minutes. When you subtract them, you get 15 minutes out of an hour. Fractions with sixtieths for the clocks, so $\frac{30}{60} - \frac{15}{60} = \frac{15}{60}$. $\frac{15}{60}$ is the same as $\frac{25}{100}$ or $\frac{1}{4}$. You can divide a clock into halves and fourths, and you can also do that with money. 	Big Idea The purpose of this string is to allow students additional opportunity to think about which model, clock or money, is more helpful when finding equivalents to add or subtract fractions with unlike denominators. Although both models can be used, along with others, for many of these subtraction problems, students should begin to determine when one model is preferable over another.
$\frac{3}{4} - \frac{1}{5}$		Students will likely choose to use money, rather than the clock face to help solve problems 2 and 3, as these both involve fifths. You may have one or more students who choose the clock, reasoning that there are 5 sets of 12 minutes in an hour, but most will probably conclude that money is quicker and easier to use when dealing with fifths.
$\frac{3}{4} - \frac{2}{5}$		
$\frac{2}{3} - \frac{5}{12}$		There will likely be agreement that the clock is the best model to use in solving the fourth combination, as it involves thirds and twelfths.
$\frac{4}{5} - \frac{1}{10}$		Money is much more easily used than the clock for the last two problems, both of which involve fifths and tenths (or hundredths).
$\frac{4}{5} - \frac{10}{100}$		

Figure 1: Fraction Subtraction Number String².

In this episode, the team was planning a lesson focused on adding and subtracting fractions with unlike denominators using a variety of strategies and models (including money and clocks). The team agreed on the following learning goal for the lesson: “Students will consider which denominators are better suited for solving a subtraction problem with a money model (factors of 100) and which will be better suited to a clock model (factors of 60)”. Because the lesson focused on two different models, and because the authors provided detailed descriptions of the models’ purposes, teachers were able to articulate a very specific learning goal for students, a learning goal that guided their remaining analyses.

The written lesson began with a story problem followed by a number string (Figure 1) that provides opportunities for students to grapple with the appropriateness of two models, the “clock model” and the “money model.” The curriculum writers intended for number strings to develop students’ flexibility, efficiency, and accuracy with numbers (The Math Learning Center, 2019). In the curriculum, the authors identified a series of related math problems that encourage students to recognize patterns, use mathematical properties, and build number sense by explaining and comparing

strategies with their peers. When teachers engaged in the number strings themselves, they were challenged to consider how students would interact with the mathematics and how best to plan for these interactions. As seen below, this resulted in deep and specific conversations about the content of the lesson.

Early in the planning, teachers shared their initial thoughts about the number string. Laney posed a question to the team about the following expression:

I feel a little stuck... I tried my best to do these problems in both ways to anticipate what kids might do or say, because I feel like you always get [a student] that's like, "you can divide this one [$\frac{2}{3} \cdot 5/12$] into 12-minute pieces". But that's not visually fun. And then your answers don't look the same at all. And so, I guess that's where I'm wondering would we...steer kids artfully toward one [preferred model]? I feel like if I get an answer of $33/60$ and then somebody else says, well, it's $55/100$. I guess that's where I'm like, how do I connect those things together so they can see that those are the same?

The team discussed Laney's question, some suggesting that if the students did use different models, it would provide an opportunity to discuss equivalency. The teachers continued to work through the number string. Mae used the *Connections* provided by the curriculum writers (see Figure 1) to prompt discussion. These notes provided fodder for the teachers to consider what students might do and how they would want to respond in the moment. For example, when considering the second equation ($\frac{3}{4} - \frac{1}{5}$), the team agreed that students will likely use a money model, but they return to the question Laney posed earlier:

Mae (MTE): Now we come to the hard one [$\frac{3}{4} - \frac{1}{5}$] ...So I think we're imagining that most people want to do money, right? Students will likely choose money. What are we going to do with the clock people? Anything?

Laney: That's...my question...

Erin: And you know, when I'm looking...for students to share, am I looking for just the equation or am I looking for a model here again?

Mae: What if somebody did an equation? Could you say, "What would that look like on your model?" ...or if they've already done it without a model...You could have two people and compare. Like, "Here's somebody that used the [money] model. Here's somebody that did it in their head. How did that person manage to do it in their head? Why didn't they need the model?" ...I mean, there's so many ways to go here. How deep do you want to go? ...

Laney: I feel like now I would want to walk around and then like, solicit answers from people who use the money model and then ask questions that were like, "Why did you choose that over the clock?" And then they could say [their reason]. Because I feel like I would have kids in my room that I could like, strategically pick, that would probably pick money and have a really good answer...

Erin: I almost wonder if I'm going to end up with a kid who's going to start it on a clock and switch to money, and that person I would want to talk to. And I'd want to show [the class] "Look, I'm seeing he started this and then he

switched to this over here” and I’d want that up on the board. “Why did you do that?”

Diane: That’s kind of what I wrote for the $\frac{2}{3}$ and $\frac{5}{12}$. Like, absolutely no prompting. Let me see if I can find somebody who attempted the opposite and that’s where I wanted to be.

This episode demonstrates how collaboratively analysing the number string and the accompanying notes from the curriculum authors provided opportunities for teachers and the MTE to unpack the mathematical intent of the task, anticipate student responses, and identify pedagogical strategies to support their students to think more deeply about how models can support subtraction of fractions. Because the curricular materials were so detailed, teachers could draw on their past experiences teaching fractions and pose specific questions to one another about how to pedagogically plan for anticipated problems.

Through analysing the number string and descriptions provided by the curriculum authors, the teachers anticipated student thinking, including the thinking of their own students. Anticipating student thinking helped the teachers engage in meaningful discussions, connecting different ideas about the models that might be shared by students during the lesson. For example, Erin anticipated that a student in her class might start with the clock model and then switch to the money model. The teachers identified questions to ask to draw attention to the models, such as “Why did you choose that [money model] over the clock?” (Laney). They planned instructional moves to provide opportunities for students to make connections between different representations. For example, Diane suggested looking for a student who “attempted the opposite” in their work to present to the class. The rich mathematical tasks and information provided by the curriculum allowed the group to identify their work in supporting students to grapple with the big mathematical ideas and for students to participate in the co-construction of mathematical connections.

CONCLUSION

In this analysis, we focused on how educative curriculum materials supported MTEs and teachers to analyse teaching with the specificity that would enable a productive enactment of the lesson with students. Our focus was guided by the understanding that teachers need opportunities to make sense of the curriculum, and this can be achieved through focused discussions that examine interactions between teachers, students, and mathematical content (Russell et al., 2020; Stein et al., 2022). Because the curriculum was educative, the teachers and MTE could use it to guide the conversations about *implementing* the lesson. In other words, the teachers could focus on the details of teaching the lesson instead of designing the lesson.

Through our analysis of the ways MTEs and teachers engaged in collaborative discussions focused on curriculum, we now have an additional perspective of what educative curriculum means. The episode we reviewed demonstrated how collaboratively analysing the curriculum, using its educative supports, provided

opportunities for teachers to have deep and specific conversations about teaching, conversations not possible when working in isolation. We hypothesise that an educative curriculum enables groups of educators to analyse teaching with the depth and specificity that is likely to impact the enactment of the lesson, as well as orient teachers toward principled changes in their future work. If we are right, future research should show important differences in the conversations of teachers using educative curriculum materials compared with those using non-educative curriculum materials.

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Endnotes

- ⁱ In the U.S. curriculum refers to a package of materials that are designed to meet the educational standards selected by individual states, which in 41 states are the Common Core State Standards (CCSS) for mathematics. The word curriculum and textbook are often used interchangeably.
- ⁱⁱ Figure 1 is republished with permission of The Math Learning Center, from ©2019 Bridges® in Mathematics, Second Edition. All Rights Reserved by The Math Learning Center. Used Here under License. Which Model is Better is available here: <https://www.mathlearningcenter.org>. The contents in this article are not provided, approved, or endorsed by The Math Learning Center.

DESIGNING A TOPIC-CENTRED TEACHER GUIDE: A CASE STUDY OF QUADRATIC EQUATIONS

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In this paper, the process of designing a teacher guide for quadratic equations is described. In specific, how educative features together with contextual factors influence the design process of the teacher guide. The results indicate that this is an iterative two-way process between designing educative and contextual features, and that using features to guide the design is a productive way forward. Further, taking the research findings on quadratic equations as a starting point in the design had implications on the process of designing different features. Implications for designs of topic-centred teacher guides are suggested in the paper.

INTRODUCTION

In design research, existing theories and resources are used in designing resources (Bakker, 2018). One important question to ask is how resources can be designed that support teachers in planning, teaching and learning in their specific context. In this paper the design of a special type of curriculum resource, a teacher guide for quadratic equations, is in focus.

A teacher guide is a type of curriculum material that acts as the interface between curriculum designers and teachers, and in China these are studied extensively by teachers to improve teaching knowledge (e.g., Li, 2004). Curriculum materials (or in a wider meaning: curriculum resources), that have the potential to support teachers broadly and lead to teacher learning are called educative, and design principles for features of such resources have been theorized and empirically investigated (e.g., Ball & Cohen, 1996; Davis & Krajcik, 2005; Davis et al., 2017). Nevertheless, the processes of designing such educative resources have not been focused explicitly in research, especially in terms of the influence that context may have. When designing educative curriculum materials, different features need to be balanced (Davis & Krajcik, 2005). The extent to which the resource should be prescriptive or guiding, how to fit different teachers' needs and the level of interactivity are three such features to be balanced.

It is also important to take contextual factors into account in any design process. For instance, teacher resource use in planning (Gustafsson et al., 2024), common lesson practices (Tengberg et al., 2022), and topics where students struggle, such as the upper-secondary school topic of quadratic equations, are all needed to consider in the design.

Combining the need to explicate design processes, this paper aims to give an account of how design principles and ideas of educative curriculum materials can be

implemented into a teacher guide for and within a specific context. This is done by describing a case in which a teacher guide for quadratic equations for upper-secondary school in Sweden was designed, aiming to support teachers in planning, teaching and learning. The actual use by teachers is reported on in a CERME14 paper (Gustafsson, 2025). Teacher guides are often associated with a student textbook, but in this study, to meet the need of addressing a specific topic, a teacher guide was designed as a stand-alone resource that could work alongside different student textbooks. A guiding research question intended to be answered is: How can a teacher guide be designed on educative design principles while taking into account contextual factors?

THE SWEDISH CONTEXT

Swedish upper-secondary school teachers are subject teachers, meaning that they have a university teacher training that contains significant amounts of subject knowledge. Their teaching is by law to be based on research and proven experience (SFS 2010:800). In a previous study, it was found that official documents and the representation of the official curriculum through student textbooks and national exams are important actors for teachers in selecting what content to teach (Gustafsson et al., 2024). Further, the planning of instructional activities to use in class is guided by colleagues and didactical models, and not teacher guides, a process that sometimes is time-consuming. Teachers' views on teaching are that students are to be presented opportunities to experience important aspects of the mathematical concepts, and that the teachers' role is to facilitate these experiences for example through giving tasks, showing examples, and explaining concepts (Gustafsson et al., 2024). Studies investigating classroom mathematics teaching in Sweden have shown that students' own individual work in student textbooks is a common classroom practice (e.g., Tengberg et al., 2022). In contrast to primary and lower secondary school, student textbooks for upper-secondary school in Sweden rarely have printed teacher guides accompanying the textbooks, which might explain why the student textbook is an important resource in both planning and enacting teaching (Gustafsson et al., 2024).

The mathematical topic of quadratic equations is one of the gatekeepers between primary school and university mathematics. In Sweden, this content is taught in an upper-secondary school mathematics course where students are known to struggle. In this course, *quadratic equations* are defined as algebraical single-variable general equations on the form $ax^2 + bx + c = 0$ with real coefficients a , b and c , as described by major common Swedish student textbooks (Sönnerhed, 2021). The official curriculum highlights methods for solving quadratic equations as the content to be taught. In textbooks, solution methods taught mainly involve using different approaches taking the square root (when $b = 0$), using a version of the quadratic formula (when $a, b, c \neq 0$), or using the zero-product property, in the cases where the equation is already factored or can be easily factored out (when $c = 0$) (Sönnerhed, 2021). Factoring in the sense of using Vieta's formulas for finding relations between coefficients and roots, is not a commonly used approach, in contrast to other contexts.

EDUCATIVE CURRICULUM MATERIALS

When it comes to the conceptualization of what a teacher guide that supports teachers in planning, teaching and learning could look like, Ball and Cohen (1996) proposed that teacher guides could include help to prepare teachers in anticipating what students may think or do in response to instructional activities, or to add discussions about different representations of content and connections among them. Further, they argued that including rationale for the use of specific representations, explanations, tasks or instructional activities, could help teachers in deciding how to present content with respect to intended student learning. Davis and Krajcik (2005) proposed design heuristics for what they call *educative* curriculum materials, that can lead to teacher learning, building on ideas on developing teachers' Pedagogical Content Knowledge (PCK) (Shulman, 1986). These design heuristics have later been empirically tested mainly within a North American context, resulting in proposed design principles for educative curriculum materials (Davis et al., 2017). These state that teachers will adapt curriculum materials informed by their concerns about students and time, and therefore the designed material should contain suggestions on how to adapt the materials. Representations of what this may look like through descriptions of classroom practice can support teachers' uptake of ideas in the curriculum material. Designers should also highlight important content to support teachers' subject matter knowledge, and to include different types of support to suit different teachers' needs. Examples of educative features are concept maps, lists of common students' misconceptions, recommendations for how to address these in teaching, sample classroom dialogue, and guidance for reviewing and assessing student work (Davis et al., 2014).

For mathematics, Pepin (2018) suggests that educative materials should clearly describe an intended student learning trajectory and corresponding teacher roles in supporting the learning, that materials should help teachers enact teaching successfully and according to curriculum intentions, and that the educative materials in themselves should help teachers' own learning through support for reflections on teaching. Educative materials should therefore address lesson preparation, subject matter/content, the pedagogical/didactical role of the teacher, and learning outcomes.

THE DESIGN OF THE TEACHER GUIDE

The design of the teacher guide for quadratic equations for upper-secondary school was carried out by the author, who is a former upper-secondary school teacher with 15 years of teaching experience. The process included informal feedback from researcher colleagues with a background in teaching. Below follows a description of the design process. First, the rationale for choosing content features and characteristics of those features to include in the guide is described. Second, the design process is described.

The content features and their characteristics were either contextual or educative. Contextual factors, such that teaching should be based on research and proven experience (SFS 2010:800), or that teachers need to align teaching with textbooks and official documents (Gustafsson et al., 2024), were turned into contextual features.

Features that could be educative were picked from five articles (Ball & Cohen, 1996; Davis & Krajcik, 2005; Davis et al., 2014; Davis et al., 2017; Pepin, 2018), and both contextual and educative features were iteratively implemented into the guide and revised during the design process. A final summarized list of the educative and contextual features is presented in Table 1, numbered but without specific ranking.

Content features included
(1) possible student difficulties including (mis)conceptions (E)
(2) suggestions of instructional activities addressing student difficulties (E)
(3) representations of the mathematical content and how it relates to other content (E)
(4) student example works and teaching examples (E)
(5) guidance for reviewing and assessing student work (E)
Characteristics of content features
(6) based on research (C)
(7) embed features within theoretical or didactical models (C)
(8) align with teacher ideas on teaching and teaching practice (C)
(9) based on proven experience (C)
(10) align the content with official documents and common curriculum materials (C)
(11) provide transparency and rationale (E)
(12) speak <i>to</i> teachers as well as <i>through</i> them (E)
(13) act as a helpful colleague (C)
(14) include a variety of types of content to suit different teachers' need (E)
(15) include easily modifiable features to fit teachers' individual context (E)

Table 1: Contextual (C) and educative (E) content features and their characteristics

The design process started with the context, writing a research summary from a systematic literature review conducted by the author, to be based on research (contextual feature 6 in Table 1). The research summary, in the form of an expository text, included three findings: important pre-requisite knowledge before learning quadratic equations, common student (mis)conceptions and mistakes associated with different aspects of teaching and learning of quadratic equations (1), and implications for teaching. The implications offered guidance for the design of instructional activities to address the identified student difficulties, rather than explicit instructional activities. Thus, a design choice was made to construct instructional activities (2) that targeted

some of the issues from the research summary, adding them to an appendix of the teacher guide. This ended up in four activities in total. Two of these were guided by theoretical ideas of variation theory (e.g., Lo, 2012) as a didactic model (7), in which students in an open task had to sort and group some mathematical objects. In the first activity, a mix of expressions, functions and equations, that were either linear or quadratic had to be sorted. In the second activity, different quadratic equations had to be sorted in ways that corresponded to different solution methods. The remaining two activities were designed more closed to meet the current teaching context (8), more focused on individual student work. All of these activities used content of the common methods for solving quadratic equations aligning with the official curriculum and common student textbooks (10), as well as ideas on teaching (8). The descriptions of all activities were designed to provide transparency in what they were aiming to induce in terms of student learning and what difficulties they addressed (11) as well as to meet different teacher needs by giving suggestions to how activities could be modified (15).

Some further expository texts were constructed to meet more features. In an attempt to build teaching on proven experience (9), a short narrative on the teaching of quadratic equations was provided, based on the authors' experience. For representing the mathematical content and providing how it related to other content (3) a third expository text was written that summarized where quadratic equations are located in the curriculum, what mathematics it builds on, and what mathematics that follows. This section also provided examples of how common curriculum materials described it (10). Further, a fourth expository text was written as a way to increase the number of suggestions for instruction addressing possible difficulties found from the research summary. An intention in the expository texts was to speak *to* teachers (12), but also *through* them, as directive support had previously been potentially educative for teachers (Remillard & Reinke, 2012) and could also help reduce planning time. The text intended to sound and act like a "helpful colleague" (13).

In order to provide a variety of content features (14) a number of tasks were added. The first tasks added was addressing suggestions of instructional activities (2) and student example work (4) by designing fictional student solutions of tasks, one example for any common mistake or misconception found in the research summary. These tasks were organised into a second appendix. In here also some exit ticket tasks were put to give guidance for reviewing and assessing student work (5), challenging tasks to fit teacher needs for advanced students (14), old (freely available) national exam tasks (10), and a quick reference for solution methods for different types of quadratic equations, as proposed by one of the articles in the systematic literature review.

The final design was a 17-page pdf-document containing the expository texts in 4 parts: previous research on teaching and learning of quadratic equations (2 pages), teacher experience from teaching quadratic equations (2 pages), quadratic equations in Swedish curriculum, textbooks and exams (2 pages), and suggestions on how previous research and teacher experience can be implemented in the teaching of quadratic equations (8 pages). In addition, the two appendices were word-documents, to be easily

modified, where the first contained the four proposals for lesson activities (14 pages) and the second contained the different tasks to use (12 pages).

To give an account of explicit teacher guide content, elements from the first proposed lesson activity is presented in Figure 1. The aim of the lesson activity was to make students develop conceptual understanding of quadratic equations in relation to other mathematical objects. The teacher information included a proposed enactment, where students were to be presented with mathematical objects A through O, and sort and group them in a way they saw fit from the properties they identified. The Venn diagram-like image was afterwards discussed (without headers) as to what possible properties the mathematical objects within the boundaries could contain (e.g., equation or expression, quadratic or linear). Students were encouraged to share their views on similarities and differences between their groupings and the proposed grouping. Afterwards, headers were presented, and whole-class discussion guidelines were included, for example to highlight for students that not everything containing x^2 is a quadratic equation, or that coefficients can be different types of numbers.

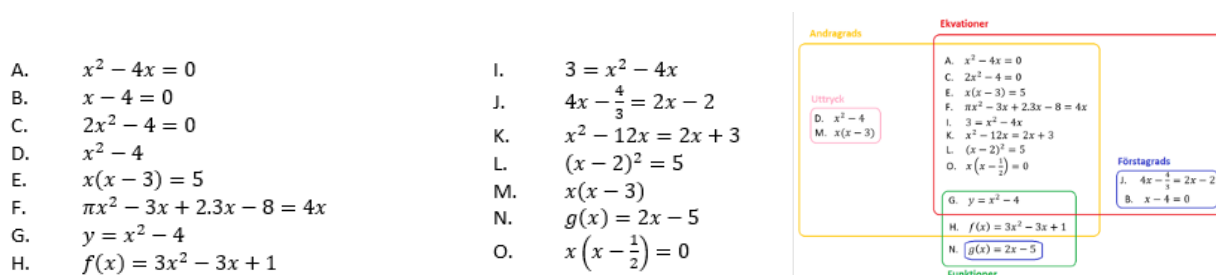


Figure 1: Elements of the first lesson activity, a sorting and grouping task

DISCUSSION

Designing a teacher guide based on educative design principles while taking into account contextual factors was in this case an iterative two-way process between designing educative and contextual features. It became a *research-based* teacher guide due to the contextual feature that teaching should be based on research (SFS 2010:800) and that this seemed like a valid point of departure. Generating a research summary of quadratic equations as a starting point of the design process, gave input to the educative content feature by providing a list of possible student difficulties and (mis)conceptions (e.g., Davis et al., 2014). This created a need to design instructional activities to meet these difficulties, as proposed as an educative feature. When these instructional activities were designed, they needed to take into account contextual factors such as aligning to official documents or teacher teaching preferences (Gustafsson et al., 2024). The use of possible tentative features, both educative and contextual, throughout the design process helped guide the process.

Another important finding is that with this design approach that built on research findings, the combination of how the systematic literature review produced findings (not reported on here in detail), together with the available research done on the topic, sets the stage for how and what content that can be produced. Findings on common

student difficulties, (mis)conceptions, and mistakes were found to directly feed into the corresponding educative feature. The research summary did not suggest instructional activities, but offered guidance through its implications regarding what seemed important to elicit. Thus, the research summary acted as an intermediary design heuristic for how to produce educative content features such as suggested instructional activities addressing these difficulties. In short, the research summary strongly influenced the teacher guide's content features and their characteristics.

Some of the exemplified educative features, from for example Davis et al. (2014), were harder to implement, or did not seem reasonable to choose to. One example of this was sample classroom dialogue, which is harder to design and could better be generated through teachers trying out and enacting activities and reporting on them.

The teacher guide that is described in this paper has been empirically tried out (Gustafsson, 2025) and the findings on teachers' use indicate the time used for reading and using the guide in planning is a constraint. This indicates that the teacher guide in its design as reported on in this paper is too long. Further, the parts of the teacher guide that was used the most were the previous research findings, and the sorting and grouping activities proposed in the first appendix, indicating a need for both research summaries and directive support for teaching more elaborated lesson activities.

The author being also teacher, had both benefits and restrictions in designing the teacher guide. Being an insider, this can have increased the ecological validity, but at the same time exposed the design of the teacher guide to a risk of normative bias from teaching experience. The importance of designing the teacher guide as a researcher and an outsider, with external warrants for choices was therefore acknowledged through the design process. However, this could be improved by expanding the design team.

Implications

When designing topic-centred teacher guides for teaching a specific topic in a specific context, having potential educative as well as contextual features to guide the design process is productive. A reasonable approach is to begin with reviewing the research literature on teaching and learning of the topic, as it feeds into educative features. Available research conducted on the topic is likely to affect the design process.

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HOW PROOF-RELATED WORDS ARE USED IN GERMAN AND JAPANESE MATHEMATICS TEXTBOOKS

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In this paper, we analyse and compare the usage of proof-related words in German and Japanese mathematics textbooks. To achieve this, we first explored the words used in proof-related activities. We then identified specific and common attributes of these words and examined functions attributed to these words in proving contexts. The results show that German textbooks use a wider variety of words than Japanese textbooks; however, in both countries' textbooks, the transition from informal to formal usage of proof-related words, from ordinary to mathematical usage, and the formulation of reasoning structures are linguistically marked.

INTRODUCTION

Diverse conceptualisations of proof and proof-related words (such as proving, reasoning, explanation, argumentation, etc.) in mathematics education have been discussed from different perspectives (e.g. Balacheff, 2008; Reid & Knipping, 2010). Looking at proof taught in schools across different countries, one may notice that what is called proof and what is taught as proof can vary. Additionally, what curricular documents (e.g. textbooks, national curricula) in a given country emphasise as proof or proving activities also differs. From an international perspective, it is crucial for researchers to consider such specificities and commonalities. Otherwise, many scientific studies conducted so far may end up in vain or may be useful only in a specific context at best (Reid et al., 2019). One reason for this diversity is that proof-and-proving is a human and cultural activity that is shaped by language. Some researchers highlighted such cultural and linguistic differences (e.g. Sekiguchi & Miyazaki, 2000).

International comparisons offer one way to investigate the diversity related to proof. For various aspects of mathematics education, international surveys have been productive. However, in the cases of proof and proving there have not yet been extensive international studies, despite the recognised potential (e.g. Reid et al., 2019). The study presented here is part of an international research project on argumentation and proof from a linguistic and cultural perspective, aiming to investigate the usage of proof-related words in curricular documents as well as classroom teaching. In this paper, we focus on a comparison of the usage of proof-related words in textbooks used in two specific countries: Germany and Japan. We selected these two countries because

their teaching styles differ partly due to cultural factors (Stigler & Hiebert, 1999). Additionally, the differences in communication styles between German-speaking and Japanese-speaking cultures, as highlighted by Meyer (2014), justify the comparison of them.

LITERATURE REVIEW AND THEORETICAL PERSPECTIVE

Proof and proving can be characterised differently according to theoretical perspectives or researchers' epistemologies (Balacheff, 2008). Recently, Miyakawa and Shinno (2021) modelled proof and proving as a triplet composed of 'structure', 'language', and 'function', to characterise the proving activities in classrooms from a cultural perspective.

- *Structure* refers to the organisation of reasoning or arguments showing how different statements in a proof are connected (e.g. geometric proofs often require a chain of deductive steps, consisting of given statements, a theorem/axiom/definition, and a conclusion).
- *Language* is the semiotic representation (Duval, 2006), including gestures, oral and/or written discourse, diagrams, etc., used to express the arguments and reasoning structures.
- *Function* denotes the role of proof in different ways (Hanna, 2000). The function attributed to proof is not reserved for those often mentioned in the literature (e.g. verification, explanation, exploration, communication, or systematisation, etc.).

Among these aspects, *language* is often regarded as a critical aspect of proof and proving, reflecting strong cultural influences, because the formulation of proof is concerned with ordinary language. For example, Cabassut (2009) investigated French and German textbooks and found that mathematical arguments can join non-mathematical arguments, such as pragmatic arguments from daily life, to make a validation.

In addition, from an international perspective, as Sfard (2021) states, "mathematical discourses in different languages do not have to be fully isomorphic" (p. 47). The same word might be used differently depending on the cultural and linguistic contexts. *Function*, another aspect of the aforementioned model concerns here. Understanding the function attributed to proof can thus provide insights into the cultural aspects of proof and proving (e.g. Knipping, 2004). For example, while a mathematical proof or proving generally establishes the *logical value* of a statement (true or false), argumentation—aiming at convincing or persuading someone—changes the *epistemic value* of a statement (i.e. the degree of conviction) (Duval, 1991).

This paper explores the three aspects—language, function, and structure—of proof-related words in German and Japanese textbooks. This study focuses on terms describing human actions that change the epistemic or logical value of statements, reflecting our interest in proof as an activity like proving and argumentation. Given the limited scope of analysis, the structural aspect will be only studied in terms of the

formulation of reasoning structure expected with these proof-related words. We aim to answer the following research questions: *What kinds of proof-related words are used in textbooks in each country? How are these words used in the context of proving in the textbooks? How does this usage indicate the different functions and structures of proving?* Addressing these questions will deepen our understanding of cultural diversity in proof and proving expressed in textbooks.

CONTEXT AND METHOD

Context and selection of textbooks for the analysis

In Germany, textbooks must be approved by the Ministry of Education in each province, in accordance with provincial curricula and national standards. Textbooks also reflect reforms, such as the change of the educational system in middle school from three to two tracks over the past decade. In principle, each school is responsible for selecting its textbooks. In Japan, the Ministry of Education approves the textbooks based on the national curriculum, which is revised about every ten years, and teachers are required to use these approved textbooks. In principle, the selection of textbooks is overseen by the local board of education. Although the educational systems of the two countries differ, ‘proof’ is introduced in middle schools in both, and the concepts related to ‘proof’ are commonly taught in geometry and little in algebra. Therefore, textbooks from both countries exhibit contrasting yet comparable resources.

The data were obtained from middle school mathematics textbooks in each country. In Germany, the secondary education in most provinces covers grades 5 to 10, with specific attention to ‘proof’ starting in grades 7 to 9. To represent the varieties within the German school system, we chose five commonly used series of textbooks, two (D1 to D2) from common schools (*Oberschule* (Os)), which most students attend, and three (D3 to D5) from the more selective schools (*Gymnasium* (G)). While all textbooks are expected to adhere to the national standards and cover provincial curricula, each series has its own distinctive style and orientation. In Japanese middle schools, comprising grades 7 to 9, ‘proof’ is taught in grades 8 and 9. We chose five series of textbooks (J1 to J5), which account for most of the textbook series used in Japan. Unlike in Germany, the difference in the organisation of content across the textbooks is rather small.

Method of analysis

The analysis followed a two-step process. First, we identified proof-related words, operationally defined as verbs (and their corresponding nouns) used in contexts that require students to justify the truth of a mathematical statement. The identification focused on three common chapters where proof is typically taught: ‘the sum of consecutive integers’, ‘the sum of three angles in a triangle’, and ‘the Pythagorean theorem’. To handle context-dependent meanings and examine the function of each word, one author conducted an initial interpretation of the relevant tasks. This preliminary analysis was then shared with all authors for thorough discussion. In cases where interpretations differed, the group revisited the original textbook data and

deliberated until reaching a shared understanding. The final analysis presented in this paper reflects the outcome of this collaborative, consensus-building process.

RESULTS

Comparing identified words and their frequencies

As a result of our exploratory study, we identified five proof-related words in the German textbooks: *argumentieren* (argue), *begründen* (justify), *beweisen* (prove), *erklären* (explain) and *zeigen* (show), along with four corresponding nouns: *Argumentation*, *Begründung*, *Erklärung*, and *Beweis*. Some of these words occur occasionally (*argumentieren*, *erklären*), while other words appear frequently (*zeigen*) or systematically (*begründen*, *beweisen*). In Table 1, the ‘number of pages’ indicates how many pages contain the three common chapters. For each proof-related word, the listed frequencies include their occurrences as nouns, verbs, or verbal nouns.

Publisher	Number of pages	<i>Begründen</i>	<i>Erklären</i>	<i>Argumentieren</i>	<i>Beweisen</i>	<i>Zeigen</i>
D1 (Os)	7	3	4	1	14	0
D2 (Os)	8	10	3	5	12	1
D3 (G)	11	7	1	0	14	7
D4 (G)	19	25	1	2	56	3
D5 (G)	6	3	0	0	6	1
Total	51	48	9	8	102	12

Table 1: Frequencies of the proof-related words in the German textbooks.

In the Japanese textbooks, we identified three proof-related words and recorded their frequencies (see Table 2): 説明する *setsumei-suru* (explain), 証明する *shōmei-suru* (prove), and 示す *shimesu* (show) along with two corresponding nouns: *setsumei* (explanation) and *shōmei* (proof).

Publisher	Number of pages	<i>Setsumei-suru</i>	<i>Shōmei-suru</i>	<i>Shimesu</i>
J1	8	8	11	2
J2	7	7	1	0
J3	10	19	4	2
J4	9	14	9	1
J5	9	7	4	0
Total	43	55	29	5

Table 2: Frequencies of the proof-related words in the Japanese textbooks.

Comparing the frequencies in the two tables, German textbooks show a greater variety of words than Japanese textbooks. It appears that there are no Japanese terms corresponding to *begründen* and *argumentieren*; instead, the word *setsumei-suru* is typically used. However, the German word *erkären* occurs less frequently, whereas *begründen* is used more often. Acknowledging this contrast, it is reasonable to focus on the usage of these terms for comparison. It is noted that the verb *zeigen*, in the context of proving in German textbooks, is primarily used as a synonym for *beweisen*. The same applies to the Japanese words *shimesu*, which has a meaning nearly identical to *shōmei-suru*. Therefore, we will not pay much attention to these words in the following analyses.

Comparing the usage of proof-related words

Informal versus formal usage. In both countries' textbooks, before the words *Beweis* and *shōmei* are introduced to describe mathematical proof in a formal way, other words are used informally. For example, in one of the German textbooks (D4-series), there is a clear emphasis on *begründen* and *beweisen* (see Table 1), and one may identify an evolution in the usage of these terms. While in grade 7, *begründen* is mostly used, in grades 8 and 9, *beweisen* is consistently employed. Prior to grade 7 no theorems or properties are proven explicitly, so the term *begründen* is used as an introductory term for proving. Consequently, both terms are connotated with the meaning of providing mathematical reasons for a statement.

In Japanese textbooks, similar evolution occurs, from *setsumei* to *shōmei*. In grade 8 textbooks, the word *shōmei* is formally introduced as a mathematical proof in geometry. Unlike German textbooks, prior to the geometry chapter, *setsumei* is used for proving certain algebraic properties, such as 'the sum of three consecutive integers is a multiple of 3'. Figure 1, under the heading '*Setsumei-suru* (with algebraic expressions)', illustrates what is expected by *setsumei* in the J3-series textbook. In this context, the meaning of *setsumei* is identical to that of *shōmei*, although *shōmei* is not used. This is due to a curriculum constraint in Japan, which prohibits the use of terms and symbols before they are introduced in the designated chapter outlined by the national curriculum.

If we let the smallest number be n , the 3 consecutive numbers are expressed as n , $n+1$, $n+2$. Their sum is

$$\begin{aligned} & n + (n+1) + (n+2) \\ &= 3n + 3 \\ &= 3(n+1) \end{aligned}$$

Since $n+1$ is an integer, $3(n+1)$ is a multiple of 3.
Therefore, the sum of 3 consecutive integers is a multiple of 3.

Figure 1: *Setsumei* for proving algebraic properties (J3-series textbook).

Ordinary versus mathematical usage. In German textbooks, some words have meanings that are more commonly found in everyday usage. For example, *erklären* typically means to clarify, comment, or illustrate something so that another person can understand the interconnection among facts and/or properties. The words *argumentieren* and *begründen* are used in ordinary contexts to ask for a reason why a

statement is true. In the context of proving, students are expected to provide mathematical reasons for why a mathematical statement is true. The following excerpt from the D1-series for grade 7 is a typical example of this usage of *begründen*: “*Skizziere ein rechtwinkliges Dreieck. Welche Summe haben die zwei spitzen Winkel im rechtwinkligen Dreieck? Begründe.* (Sketch a right triangle. What is the sum of the two acute angles in the right triangle? Justify.)” The word *erklären* is also used in the context of proving, which we find occasionally in most German textbooks; however, it is mostly associated with the understanding why and how statements are interconnected and the resulting explanation.

In Japan, *setsumei-suru*, ranging from ordinary to mathematical usage, applies to different kinds of *direct objects*. For example, explain ‘your thinking’, ‘your methods’, ‘your reasons’, ‘why this is true’, and so forth. It seems that *setsumei-suru* has a broad sense, which may encompass an ordinary usage of *erklären* and a mathematical usage of *begründen*. The word *setsumei-suru* sometimes means to communicate one’s thinking with other students, but at other times it refers to providing a reason why a mathematical statement is true. The following excerpt from the J2-series for grade 8 illustrates the latter case: “The point D is on a line formed by extending side BC of $\triangle ABC$. **Explain** (*Setsumei-suru*) why $\angle A + \angle B = \angle ACD$.”

Formulation of reasoning structures. There are various formulations of proof and proving under the terms *begründen* and *beweisen* in the German textbooks. *Begründen* serves as an introductory activity to proof and proving often asking for mathematical reasons (e.g. why all angles in an equilateral triangle are 60°). In such activities, *begründen* also allows and includes visual and example-based reasoning. *Begründen*, particularly in grade 7, asks for reasons, but no specific form of reasoning is expected. However, in the upper grades, when *beweisen* or *Beweise* become an explicit topic of study, deducing a statement mathematically and logically is required and emphasised. For example, in the D4-series textbook for grade 9, a complete two-column proof of the so-called Thales’s theorem is required. The heading *Begründungen* in the right column asks for mathematical grounds (reasons) for deriving or deducing the corresponding statements in the left column.

In contrast, asking for *setsumei* in the context of proving, as shown in Figure 1, requires students to demonstrate a specific deductive form of reasoning (such as ‘if we let...’, ‘since...’, ‘therefore,...’). Until *shōmei* is introduced in grade 8, *setsumei* is consistently used in this sense. A distinction between *setsumei* and *shōmei* only appears in the Japanese textbooks once *shōmei* is introduced. *Setsumei* is then used as a less formal way of reasoning, not requiring a specific deductive form.

DISCUSSION AND CONCLUSION

After identifying proof-related words in German and Japanese textbooks, we compared different usage of the words in terms of ‘function’ and ‘structure’ (Miyakawa & Shinno, 2021). The results reveal that different functions can be attributed to different words in the textbooks (see Hanna, 2000). For example, both *beweisen* and *shōmei*-

suru emphasise the function of verification, providing the logical value of a statement (Duval, 1991). Other words (*erklären*, *begründen*, and *setsumei-suru*) are associated with the function of explanation, understanding why a statement is true. Additionally, both *erklären* and *setsumei-suru*, in their ordinary usage, point to the function of communication or illumination, as they are used for presenting or expressing one's thinking in a social context. In this regard, some activities in German textbooks are also linked to increasing the epistemic value of a statement. For example, when students are asked to provide an argument or to argue (*argumentieren*) in the context of proving, they are sometimes encouraged to choose one argument from a group of given arguments to convince others. In Japan, however, there is almost no precedent for convincing or persuading someone (cf. Sekiguchi & Miyazaki, 2000). This may explain why the Japanese textbooks do not have a term related to argumentation.

In both countries' textbooks, different usage of proof-related words is also associated with different formulation levels of reasoning structures. In German textbooks, *begründen* can indicate an informal level, where a *short* proof (such as 'one step proof') without any specific form of reasoning is expected (although there are some exceptions in upper grades). Conversely, *beweisen* indicates students should demonstrate a proof by a chain of reasons without any gaps and with reference to known mathematical properties. Although the usage of *beweisen* is almost identical to that of *shōmei-suru*, what is required by *setsumei-suru* is evidently different, as indicated by its context.

The international comparison in our study provides valuable insights into educational practices related to proof and fostered reflection on our own culture. For example, German researchers found that *begründen* and *beweisen* often involve different logical structures. This highlights how languages differentiate actions related to changes in logical and epistemic values. While both *begründen* and *setsumei-suru* are everyday terms used in the early stages of learning proof, the former implies a positive shift in these values, whereas the latter does not. This finding opens new avenues for curriculum research, such as deciding which terms to use at different stages of students' participation in the practice of proving, from the introductory phase to advanced stages.

While our study reveals linguistic specificities and commonalities, further comparative research is needed. In particular, an in-depth examination of the interplay among 'structure', 'language', and 'function' of proofs in textbooks seems promising. Analysing the usage of proof-related words and their function in classrooms compared to textbooks and other curricular documents is another future research direction. Based on these studies and their linguistic and cultural perspectives, a theoretical framework for understanding proof and proving in mathematics classrooms may be established. Such a framework could be beneficial for further comparative studies.

Additional information

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NORWEGIAN MATHEMATICS TEACHERS' USE OF RESOURCES IN REASONING AND PROVING

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Learning reasoning and proving (R&P) is an important part of learning mathematics. In this paper, we present the resources that 161 Norwegian teachers reported using when preparing and teaching lessons on R&P. Follow-up interviews with ten of these teachers provided additional insights into how these resources are used. We applied the curricular noticing framework (Dietiker et al., 2018) to identify how teachers attend to, interpret, and respond to different curricular materials. Findings reveal varied approaches to engaging with resources based on teachers' instructional goals, as well as challenges in interpreting the new curriculum and navigating available resources. These findings underscore the need for clearer curricular guidance and more accessible resources to support the curriculum's focus on R&P.

INTRODUCTION

In mathematics education, reasoning and proving (R&P) are recognized as vital processes that foster deeper understanding and critical thinking among students (NCTM, 2009). However, teaching R&P effectively across grade levels and content areas remains challenging, as it depends on teachers' knowledge (Buchbinder & McCrone, 2022) and involves difficulties such as selecting tasks that encourage reasoning or finding appropriate resources for assessment (Clarke et al., 2012; Sari et al., 2020). Some researchers (e.g., Thompson et al., 2012) also highlight the lack of examples in curricular materials for effectively engaging students in R&P.

Despite the accessibility of various resources, the integration of these resources in lesson planning and enactment, particularly for R&P-focused instruction, varies across countries. The most-used resources are consistent, however, their order of use differs slightly (Çakiroğlu et al., 2023). As part of the European MaTeK project, which includes partners from Slovakia, the Czech Republic, Italy, Norway, and Turkey, this study investigates the resources Norwegian mathematics teachers draw upon and the way they engage with these resources when designing and implementing R&P lessons. Norwegian curriculum changes also provide a timely context for examining R&P instruction. The 2020 reform, known as LK20, introduced a competency-based curriculum, which, compared to prior curricula, places significant weight on reasoning, as reflected in the new core element of Reasoning and Argumentation (Ministry of Education and Research, 2019). However, studies suggest that competence aims in LK20 are not clear, leaving teachers to interpret broad curriculum guidelines with limited detailed exemplification (e.g., Lada et al., 2025). Consequently, Norwegian

teachers may experience challenges in aligning resources with these curricular goals, which underscores the need to understand their resource choices and instructional practices within the evolving educational landscape. Thus, this paper aims to address this gap by exploring the following research question: *What resources do Norwegian mathematics teachers attend to, and how do they interpret and respond to them when planning and enacting lessons on reasoning and proving?*

THEORETICAL FRAMING

Reasoning and proving

In the context of school mathematics, R&P is often conceptualized as a unified construct (e.g., Cai & Cirillo, 2014; Stylianides, 2008). This perspective suggests that activities essential to understanding and establishing mathematical knowledge, such as identifying patterns, generalizing, formulating conjectures, and exploring examples and counterexamples, naturally integrate with processes of justification and proving. Rather than viewing reasoning and proving as discrete, in our study, we consider reasoning activities as foundational steps that lead to proof and validation, supporting the idea of R&P as a unified construct.

Resources and the curricular noticing framework

In the educational context, a "resource" encompasses any asset teachers use to design and implement instruction. Pepin and Gueudet (2018) differentiate between material and nonmaterial resources. Nonmaterial resources include *social resources*, such as interactions with colleagues, and *cognitive resources*, such as mathematical and pedagogical concepts, routines, and frameworks that teachers use or develop. Material resources, specifically *curriculum resources*, are defined as “all the material resources that are developed and used by teachers and students in their interaction with mathematics in/for teaching and learning, both inside and outside the classroom” (p. 1). When teachers design lessons, resources assist them in making decisions regarding mathematical content (what to teach and in what sequence), instructional strategies (how and why to teach it), and lesson enactment (Gustafsson et al., 2024).

To better understand teachers' interactions with curriculum materials, Dietiker et al. (2018) introduced the *curricular noticing* framework, inspired by the concept of teacher noticing. While they define curriculum materials as “written and digital textbooks” (p. 522), in our study, we extend this term to encompass any resource that teachers participate with. This framework divides noticing into three skills: (1) *curricular attending*, skills used when knowing how and where to search in resources when planning a particular lesson; (2) *curricular interpreting*, skills used when making sense of this information and linking it to the knowledge the teacher has about the class where the lesson will be taught and about mathematics teaching in general; and (3) *curricular responding*, where teachers decide on “what they want to do and how” (p. 527), based on their interpretations. This might include selecting or adapting a task, deciding how to present content, or choosing not to use the task. Hence, this framework

aids in identifying what teachers notice and how their noticing shapes resource use. For instance, McDuffie et al. (2018) found that middle school mathematics teachers attended to similar features of resources but interpreted and planned differently depending on their teaching orientations and the type of curriculum.

METHOD

Data for this study were collected through an online questionnaire developed as part of the MaTeK Project (Çakıroğlu et al., 2023). In Norway, the questionnaire was piloted with 20 respondents in November 2021, followed by the main study (with minor adjustments) conducted in March–April 2022 with 141 respondents. The same questionnaire (with language adaptations) was used across all five participating countries. It asked mathematics teachers in grades 1–13 about the resources they use for various teaching purposes, including lesson preparation, assessments, personal knowledge refreshment, and specifically for planning lessons focused on R&P. The resource list, adapted from Kock and Pepin (2019), included examples tailored to the Norwegian context and an open field where teachers could specify additional resources they used beyond the listed options. In Norway, teachers were recruited via emails sent to principals nationwide, requesting that they forward the questionnaire to their mathematics teachers. Additionally, the authors directly invited teachers they knew, resulting in a diverse sample in terms of gender, experience, education, and school level. Respondents willing to participate in a follow-up interview provided their email addresses. These Zoom interviews, lasting 46 to 90 minutes, explored teachers' resource use across teaching purposes and their approaches to R&P-focused lesson planning and instruction. Teachers also reflected on the adequacy of mathematics textbooks for supporting R&P and whether the new curriculum sufficiently emphasizes pupils' R&P skills. Interviews were conducted with all teachers who responded: one from the pilot (December 2021) and nine from the main study (May–August 2022), referred to as T0–T9. Of these, T3, T5, T6, T7, and T9 taught at the primary level (grades 1–7), T0, T2, T4, and T8 at lower secondary (grades 8–10), and T1 at upper secondary (grades 11–13). The analysis focused on the second half of each interview, where teachers were specifically asked about preparing for and teaching R&P-focused lessons. Both authors independently reviewed all ten interview transcripts to identify segments where teachers discussed which resources they use and how they use them. These identified segments were then further analysed using the curricular noticing framework.

FINDINGS

Although this paper focuses mainly on the interview data, we first present questionnaire findings on the resources teachers usually use for planning and teaching R&P, followed by an analysis of how interviewed teachers engage with these resources.

Resources used

Teachers vary in the number of resources they use, with some relying on only one (6.8%), always a textbook or teacher guide. Most use multiple resources, commonly three (20.5%) or four (19.9%), suggesting a preference for combining resources. The graph in Figure 1 shows that textbooks emerged as the most commonly used resource (67%), often alongside a teacher guide (43%). About 46% of respondents utilize self-created materials, and 43% of teachers indicated they use professional books on mathematics or mathematics teaching. In the follow-up interviews, teachers clarified that by “other professional books,” they refer to books from their teacher education, covering both mathematics and mathematics teaching, as well as popularization books on mathematics teaching. About 40% of respondents selected the *social resource* of consultation with colleagues, and 40% indicated using digital *curriculum resource* - online professional platforms. Interviews confirmed that teachers visit educational websites such as matematikk.org (a collaboration between several Norwegian universities) or the Norwegian Centre for Mathematics Education’s website, both of which were also listed as examples in the questionnaire.

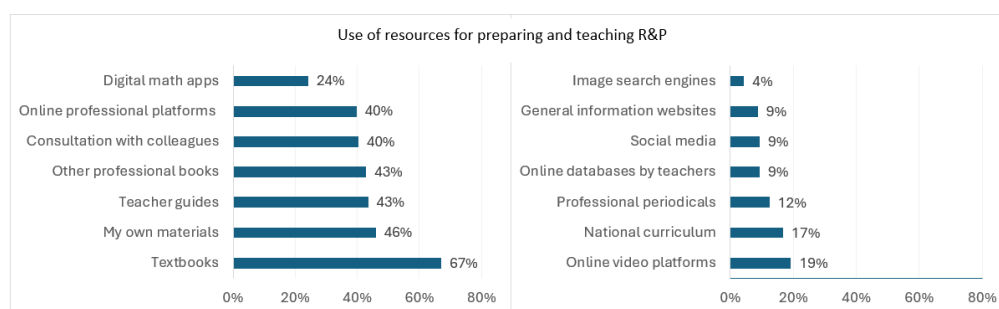


Figure 1: Percentage of respondents in the questionnaire (n=161) using particular resources for preparing and teaching R&P.

Resource use based on search aims

Interview data revealed that teachers’ engagement with resources depends on their needs. Some seek written-out proofs/solutions or “complete” R&P tasks, while others prioritize discussion-based learning or seek pedagogical guidance to improve their teaching of R&P. These differences shape how they interact with resources.

a) Teachers Seeking written-out Proofs/Solutions

Some teachers, particularly those in the higher grades, are *looking for written-out proofs or problem solutions*, either as a way of refreshing their own memory before teaching or because they want their students to engage in verifying the proof or solution step-by-step and convince themselves that each step is logical. When attending to resources with this aim, these teachers typically start with textbooks or, as T6 mentions, searching online, identifying easily a written-out proof or a worked solution.

T6: how was... the binomial formula [...]. You forget the math you haven’t had in 28 years. Then I need to find, yes that’s how it was. [...] Then I simply search different websites.

Interpreting involves verifying whether the material aligns with the topic and is manageable for students. As T1 explains, “seeing if I can find a proof that is manageable within the current topic”. Responding then entails selecting proofs or worked solutions and deciding how to engage students in checking each step.

b) Teachers Seeking Discussion-Driven Tasks

Many interviewed teachers are *looking for tasks to use as a starting point for a discussion*, focusing on R&P through the discussion itself. When attending to curricular materials with this aim, they often search across a range of sources: textbooks, self-created materials, social media (like Facebook teacher groups), and general websites. When interpreting, they assess topic relevance and discussion potential. While most teachers are not explicit about their criteria when interpreting and deciding how to respond, they emphasize relying on their own expertise:

T7: I feel that you need a bit more knowledge yourself to really focus on what’s important, what’s interesting, and what aligns with the lesson goals for a deeper understanding and to highlight it so that more students can also gain the same understanding.

T9: So, a task can really be worked with for a whole lesson if you use it in a creative way.

Some teachers, like T5, follow guidance from teacher guides when deciding how to respond, and T2 is looking for tasks without an answer key available so that students have to compare their answers and discuss discrepancies.

T5: They [teacher guides] provide explanations and even tips on almost exactly what you should say. So, I have to say, if I only used the textbook and not the teacher guide, then it would be a bit... You’d need to bring a bit more of your own knowledge into the lesson, in your head, to ensure reasoning and proof.

c) Teachers Seeking Complete R&P Tasks and Lesson Plans

When preparing a lesson on R&P, some teachers are *looking for a “complete” R&P-task*, meaning a task where the prompt for students to start reasoning and/or proving is part of the task. Here the most common strategy when attending to curricular material, looking in the textbook (digital or paper-based), fails. When T2 is asked if textbooks are good enough at providing help with planning R&P-lessons, he replies: “No! No, textbooks that I know, haven’t really given any .. any input on or help to practice tasks like that, where they [pupils] should discuss this.”

One teacher (T7) is looking for *suggestions for a lesson plan* focusing on R&P or a sequence of lesson plans. When attending to this, the teacher is using ‘other professional books’ and, in interpreting these materials, primarily assesses whether the content level is appropriate for his class; if so, he uses the lesson plan as it is.

d) Teachers Seeking Pedagogical Guidance

Some teachers in our study express uncertainty about how best to foster their students’ R&P skills and thus they sometimes look for *teaching advice on teaching R&P*. However, most of these teachers report limited options for finding such resources.

T1: I don't know. [...] How to get them to understand the proof. One thing is to go through the proof and work yourself through it, but.. To get the formal proof writing in a way that you could explain ... that I would want some more resources on.

One teacher, T4, has adopted Peter Liljedahl's 'Thinking Classroom' method as a way to develop R&P skills with his students:

T4: I would say every time we have «thinking classroom». Then we do it [work with students' R&P-skills], and we try to have it two times per month.

T4 mentioned first learning about this method at a conference organized by the Norwegian Centre for Mathematics Education, which also provides information about it on its website.

Challenges in resource use

Teachers face several challenges when they engage with resources to teach R&P, including difficulties interpreting the curriculum, limited textbook support, and challenges navigating and accessing resources.

The shift to the new LK20 curriculum disrupted established attending strategies, like using old exam tasks. With no completed LK20 exam at the time of the interviews, teachers relied on sample exams (with examples of low, medium, and high-achieving responses) provided by the Norwegian Directorate for Education and Training (Udir). These examples, however, were often difficult to interpret. T2 describes the confusion:

T2: [...] some students write three pages and some write three lines. And this student who has written three lines is high attaining and the one who has written three pages is medium. [...] I think this is hard to explain to the students what we are expecting.

When T2 struggles with the interpretation, he also finds it challenging to know how to respond. Interpreting the suitability of these tasks requires careful alignment with the new curriculum, which many teachers find challenging. As T6 notes, "a lot of work before we get that [curriculum] under our skin [...] I think the curriculum, reading and interpreting it, I think it is really hard".

Many interviewed teachers noted that textbooks provide limited support for R&P, often focusing on procedural tasks rather than exploring underlying reasoning. T9 commented: "Too few 'why' questions are asked. The newest editions might be a bit better at that, but I think they move too quickly." Additionally, teachers mentioned challenges in navigating and accessing available resources. For example, T5 explained: "It certainly does [exist] if you search for it, but I can't say it's something that pops up in my face." Similarly, T8 echoed this frustration: "there probably are, but how easy it is to find them, I don't know. But you have to be a bit used to it, searching both online and in books to find good ways to do this."

DISCUSSION AND CONCLUSIONS

Our analysis reveals that teachers use a variety of resources to plan and teach R&P-focused lessons, with distinct patterns emerging among interviewed teachers based on

their specific aims. Consistent with McDuffie et al. (2018), we observed that in the curricular attending phase, these teachers either look for tasks or whole lesson plans. However, our study extends these two categories by identifying an additional type of engagement: some teachers also seek *meta-resources*—guidance on conducting R&P lessons and improving their teaching of R&P—highlighting the need for resources that provide not only content but also pedagogical strategies, supporting mathematics teachers’ planning of teaching and learning activities (Gueudet et al., 2023). Furthermore, our findings highlight that many interviewed teachers face difficulties navigating a diverse set of available resources. In our analysis, we incorporated this navigation skill as part of the curricular attending process, but it could be recognized as a distinct preliminary step within Dietiker et al.’s (2018) framework, particularly if the framework is to encompass all resources beyond traditional textbooks.

A significant finding from our study is the challenge some teachers encounter in interpreting the updated curriculum, particularly the new emphasis on reasoning and argumentation. These teachers would benefit from targeted support, which aligns with what McDuffie et al. (2018) concluded. Stakeholders such as National Centre, Udir, and teacher educator institutions, could collaboratively operationalize what it means to enact this curriculum as intended, for example by developing exemplary curricular materials or professional development programs that model R&P-focused instruction across various grade levels. For further studies, analysing the entire interview to study each teacher’s noticing skills more in depth and in relation to their conceptualization of R&P in school mathematics could provide valuable insights. Additionally, now that some time has passed since the introduction of LK20, conducting new interviews could shed light on the development of teachers’ noticing skills and the resources they need.

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FIDELITY OF INFORMAL INFERENTIAL REASONING TASKS IN MIDDLE GRADE TEXTBOOKS

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With the recent emphasis on statistics as a core component of the secondary mathematics curriculum, inferential tasks are becoming prevalent in curricular materials. Recommendations from statistics educators advocate for inferential reasoning tasks with informal approaches during middle school and the introduction of procedures in later years (Bargagliotti et al., 2020). In this study, we selected five grade 7 mathematics textbooks from the three major textbook publishers: Pearson, McGraw-Hill, and Houghton Mifflin Harcourt. The associated informal inferential reasoning tasks were analysed using a framework developed to assess informal inferential reasoning opportunities (Huey & Jackson, 2015). We share results and discuss implications for students' opportunity to engage in meaningful learning.

BACKGROUND

For decades, inferential reasoning has served as a principal learning goal for introductory statistics courses at the college level (Konold & Pollatsek, 2002). More recently, reasoning about data in order to generate inferences has become a curricular focus for secondary mathematics internationally (Senior Secondary Assessment Board of South Australia, 2002; Ministerio de Educación y Ciencia, 2007; National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA Center & CCSSO, 2010]. The Pre-K-12 Guidelines for Assessment and Instruction in Statistics Education [GAISE II] report advocates a scaffolded approach with informal inferential reasoning at the middle school level followed by formalization during secondary and tertiary years (Bargagliotti et al., 2020). The Common Core State Standards for Mathematics (CCSS-M) largely adopted these recommendations in the United States (NGA Center & CCSSO, 2010), and mathematics textbook publishers produced curricular materials that claimed to be aligned as well.

Since informal approaches to inference are new in nature and not yet agreed upon universally (Harradine et al., 2011), this topic poses unique challenges and opportunities for middle school teachers, who likely have neither experienced teaching

nor learning informal inferential reasoning previously. We argue that guidance is needed on how to assess the fidelity of informal inferential reasoning tasks contained within curricular materials so that teachers can understand the purpose of such tasks and provide their students the intended opportunities to learn. While this may seem a narrow focus, inferential reasoning is a key learning goal of statistical education that requires relational reasoning between core statistical concepts, as well as application of statistical thought processes. In addition, inferential tasks provide a venue for developing students' statistical reasoning and sense-making skills often missing from instruction in school settings (Garfield & Gal, 1999).

CURRICULAR STUDY AND WHAT THE TASK ANALYSIS CAN TELL US

Curriculum materials play an integral role in the design of mathematics instruction. In fact, over 80% of K-12 United States classes utilize a textbook or curricular program for mathematics instruction (Banilower et al., 2013). Moreover, not only are curriculum materials prevalent in classrooms, but curriculum materials may have the most direct influence on teachers' planned and enacted tasks with students (Brown & Edelson, 2003). Although this influence is not straightforward, research indicates that curriculum unfolds in a series of temporal phases.

Before students have the opportunity to learn from a written curriculum, the curriculum goes through a series of transformations or changes. These phases are mediated by a variety of factors, such as teachers' knowledge and beliefs, orientation towards curriculum, classroom structures and norms, and organizational and policy contexts. Regardless of these transformations, we know is that the influence of curriculum on student learning begins with the written curriculum itself. The written curriculum that teachers have access to often serves as a dominant source of tasks for students and knowledge for teachers (Grouws et al., 2004; Schmidt et al., 1997; Stein et al., 2007).

TEACHERS' KNOWLEDGE OF STATISTICAL INFERENCE

From the few empirical research studies involving K-12 teachers' knowledge of statistical inference, findings indicate that teachers exhibit difficulty in applying formal knowledge and procedures to real-world scenarios. These results underscore challenges that teachers have, generating data-based arguments and the associated challenges related to coordinating statistical concepts. In one study, Watson (2008) reported on a classroom design experiment conducted with a grade 7 teacher over the duration of three lessons. Watson designed classroom activities with the teacher to introduce the topic of data distribution, including concepts of shape, skew, centre, spread, and outlier. In alignment with prior research results, Watson found that the teacher was comfortable leading all activities with the exception of those related to inference and generating justifications. In general, in-service and preservice teachers struggle to effectively generate inferences and tend to reason deterministically. Garfield and Ben-Zvi (2004, 2007) attribute a portion of the problem to the historically

procedural approach to teaching statistics and note that many tertiary teachers hope that inferential reasoning skills will develop as a by-product of learning concepts and procedures rather than attending to the reasoning process directly.

While research related to the statistical knowledge needed to teach inferential reasoning is scarce, from the few studies that have been conducted, teachers tend to exhibit similar struggles as students in generating inferences and are likely ill-equipped to teach inferential reasoning without professional development or further opportunities to learn. In summary, in-service middle school teachers are likely not well prepared to teach statistics, thus increasing the importance of providing sound curricular materials, as these materials are likely to serve as the basis for both students' and teachers' learning of statistics during middle school.

DEFINING INFORMAL INFERENTIAL REASONING

Two concepts form the basis for inferential reasoning: statistical inference and statistical reasoning. *Statistical inference* involves generating a prediction that extends beyond the data available while considering implications of variation and uncertainty (Moore, 2004). *Statistical reasoning* is the sense-making process involving statistical ideas and characteristics of the data (Garfield & Ben-Zvi, 2004, p. 7). Blending these concepts together, *inferential reasoning* is the process of generating a predictive claim through coordinating statistical ideas and information.

Zieffler et al. (2008) identify three inferential reasoning problem types: (a) generalizing from a sample or samples about characteristics of an unknown population, and (b) comparing two samples in order to reason about possible significant differences between two populations, and (c) comparing a sample statistic to an expected value of a population statistic to determine if the sample is likely to be from the population. While these problems are often addressed through set procedures and methods, students can also formulate responses based on informal approaches and reasoning. Students can coordinate statistical concepts, apply prior knowledge specific to the context, and generate an inference supported by the data. Similar to Zieffler et al.'s definition, we define *informal inferential reasoning* as reasoning about inferential tasks involving coordination of statistical concepts such as, centre, variation, skew, distribution, uncertainty, and variation, as well as the problem context.

In order to generate an inference that is grounded versus hypothetical in nature, students must consider and attend to all elements of the statistical investigation process, consisting of: question formulation, data collection, analysis and interpretation (Bargagliotti et al., 2020). Unfortunately, many textbook tasks do not provide opportunities for students to experience a full investigation cycle but rather focus upon one or two steps at most in this process. Therefore, in these cases, the extent of students' reasoning is limited to the scope of the tasks provided.

In summary, the importance of developing students' statistical reasoning abilities has

been acknowledged both by recent mathematics education and statistics education recommendations. In statistics education, inferential reasoning plays a crucial role in uncovering how well students understand core statistical concepts, the relationships between these concepts, and how to draw inferences based on these comparisons and understandings. Researchers have determined that students can reason inferentially at a young age given appropriate tasks and supports, and current standards endorse reasoning inferentially with informal methods during middle school and progressing to formal strategies later (Bargagliotti et al., 2020).

METHODOLOGY

We purposefully selected five grade 7 textbooks from the three major education textbook publishers: Pearson, McGraw-Hill, and Houghton Mifflin Harcourt, and identified the lesson(s) in each textbook that focused on informal inferential reasoning (see Table 1). The textbooks were selected based on their inclusion of informal inferential reasoning tasks, claims to be aligned with current curricular standards, and utilization within United States classrooms. In this study, we focused specifically on informal inferential reasoning tasks involving the comparison of two data distributions. According to the standards (NGA Center & CCSSO, 2010), students in grade 7 need to “draw informal comparative inferences about two populations.” More specifically, students need to “informally assess the degree of visual overlap of two numerical data distributions with similar variabilities, measuring the difference between the centres by expressing it as a multiple of a measure of variability” (7.SP.B.3) and “use measures of centre and measures of variability for numerical data from random samples to draw informal comparative inferences about two populations” (7.SP.B.4).

Textbook	Year	Publisher	Lessons	Labs	Tasks
College Preparatory Mathematics: Core Connections Course 2	2012	CPM Educational Program	1	0	13
Glencoe Math Course 2	2013	McGraw-Hill	1	2	13
Holt McDougal Mathematics Grade 7: Common Core Edition	2012	Houghton Mifflin Harcourt	1	3	8
Connected Mathematics Project 3: Samples and Populations	2014	Pearson	4	0	18

Go Math! Teacher	2014	Houghton	3	0	9
Edition Grade 7		Mifflin Harcourt			

Table 1: Textbooks analysed in the study.

All problems from each textbook involving inferential reasoning were analysed using the Informal Inferential Task Framework (Huey & Jackson, 2015), which assesses tasks according to five dimensions: 1) generating an inference beyond the data at hand, 2) promoting ill-structured reasoning related to core statistical concepts, 3) open-ended versus deterministic, 4) need for considering the context of the data, and 5) access to raw data organized in visual representations. In this study, two authors collectively analysed each task along a continuum of low, medium, and high for each category of the informal inferential reasoning framework. After assigning a rating for each category, each task was given an overall rating based the individual components. During the task analysis, the researchers wrote theoretical memos about the ratings they assigned and notable characteristics of the tasks. The memos were analysed, and overall summaries of the textbook tasks were written. The researchers read and discussed any discrepancies within the summaries, and all discrepancies were resolved. We specifically coded all tasks that students were asked to complete. Because textbooks often referenced examples for students to follow, we analysed approaches to completing tasks in alignment with examples when they were expressly cited. We acknowledge that the tasks could be solved in a variety of ways; however, the examples often implied a prescribed approach. If the answer key required only a numerical answer, the task was classified as close-ended (deterministic). Finally, if the task could be completed without considering the context, we coded the task low in this category.

RESULTS

We first provide an overview of the tasks with respect to each of the dimensions in Figure 1. We chose to separate tasks that appear in lessons from those that appear in labs and projects. Only two texts, Glencoe and Holt, provided separate activities categorized as labs or projects. We chose to separate these tasks for two reasons. First, these tasks were quite different in character with respect to the task dimensions. Second, research indicates that portions of textbooks that are not perceived to be essential by teachers, but rather supplementary in nature, such as activities, labs, and projects, are more likely to be skipped or omitted during instruction.

Across the five textbooks, we identified a total of 61 inferential reasoning (54 tasks found in lessons and 7 in labs and projects). Figure 1 illustrates the percentages of tasks within lessons and projects that were categorized as low, medium, and high for each of the task dimensions. Tasks were more often coded as high in all dimensions in labs and projects than in lessons, except for the dimension of visual representation, which had the same percentage of high ratings, 43% for both lessons and labs and projects. This means that compared to tasks in the lessons, tasks in labs and projects had fewer

trimming attributes that lowered the cognitive demand of the tasks. It is important to note that there were far fewer labs and projects than lessons and these labs and projects were included in only two texts, Glencoe and Holt.

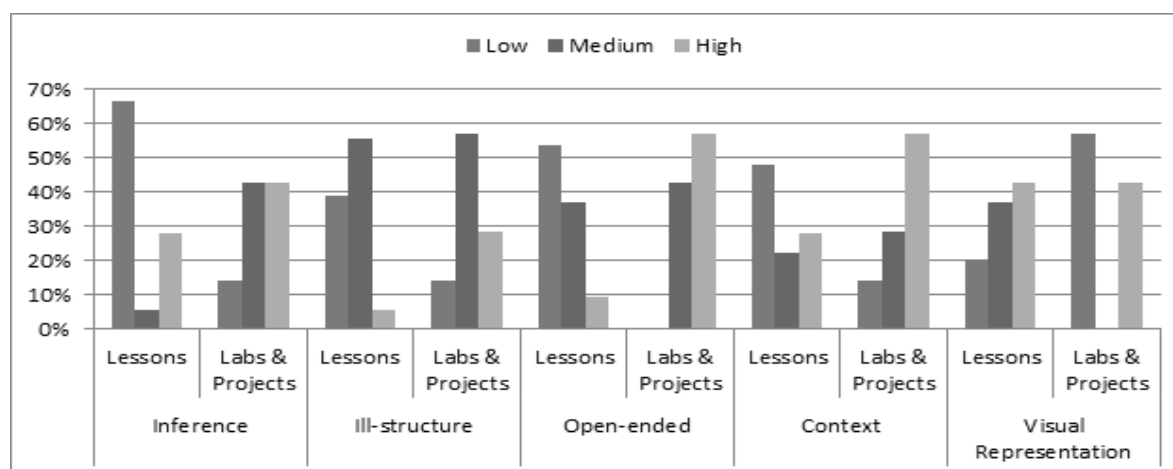


Figure 1. Task Characterization within Lessons and Labs/Projects.

Specific to the tasks found in lessons, the percentages of low, medium, and high codes differed across dimensions. However, all dimensions were similar in that they had more ratings in the low or medium categories than in high. More than 50% of lesson tasks were coded as low with respect to inference and open-endedness, with 48% and 39% coded as low in the context and ill-structure dimensions, respectively. Visual representation was the only dimension with more high than medium or low ratings, with 43% of lesson tasks coded as high.

Across all texts, two data distributions were compared in tasks. Students were often asked to compute a measure of centre as well as a measure of variation. The types of measures were usually specified, but not in all cases. Students were also asked to compare the measures of centre and variation between the two data distributions. The inferential questions tended to follow and ask which population had a higher result than the other and which data set was more predictable based on the computed measures. These questions did not tend to rely upon the context of the data and were formulaic in nature and deterministic. Upon occasion, students were asked to make predictions about differences in data sets without prescribed computations and comparisons, but these were less common. In all cases, data was provided to students, typically 10 to 20 values per data distribution, versus being collected, and the source of the data was rarely shared or described. Given this lack of information, the implication was that the data provided was a population versus a sample.

DISCUSSION

While many if not all mathematics textbooks claim to be aligned with national standards and international research-based recommendations, studies such as this one that assess the fidelity of tasks are critical. The results of this study indicate that the intent of the standards and research-based recommendations have not been realized in

the first generation of textbooks attempting to address new content standards specific to informal inferential reasoning. Rather, the recommendations have been repackaged into formats that resemble other mathematics problems within the curricular materials in ways that do not require teachers to embark upon new types of reasoning and sensemaking. Implications of these findings suggest that students will have access to measures of centre and variation in grade 7 and experience visual representations of data distributions. However, students are not likely to experience statistical investigations that involve data collection, use technology tools such as CODAP, or reason about predictions beyond the data at hand if teachers rely upon their school adopted curricular materials. Projects and labs seem to provide access to richer and more meaningful experiences and incorporate more facets recommended by research. While access to educational technology tools and authentic data sets has never been greater, the challenge of how to integrate these resources into school routines remains. Additional studies are needed to assess the enacted curriculum in grade 7 specific to informal inferential reasoning as well as teachers use of supplementary materials whether that be labs and projects or external resources endorsed by national and international statistics education organizations.

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FRACTIONS AND FUNCTIONS IN LITHUANIAN MATHEMATICS TEXTBOOKS

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Following the revision of the General Mathematics Curriculum in 2022, the preparation of new math textbooks has commenced in Lithuania. We present the results of the analysis of the topics of fractions and functions in the new books. Some problems are related to the lack of conceptualization of numbers and ambiguity in the concept of function. Inconsistencies in the treatment of mathematical ideas and insufficient conceptual grounding may be linked to the way the textbook development system operates, in which private publishers bear full responsibility for content quality and external evaluation is limited. We have discussed the possible cultural causes of problems with school mathematics content.

INTRODUCTION

In Lithuania, the goals, content, and student assessment criteria of mathematics education are formulated by the General Mathematics Curriculum (hereafter referred to as the Curriculum). The Ministry of Education, Science, and Sport is responsible for developing and implementing the Curriculum. Meanwhile, private publishers are accountable for textbook development and their quality. In 2022, the Curriculum was updated, and in 2023, a plan for implementing and monitoring measures to improve mathematics teaching and learning for the period 2023-2027 was adopted. Both documents focus on the development of students' mathematical thinking. Currently, new mathematics textbooks are being prepared in accordance with the Curriculum. We are interested in how the content of textbooks reflects plans to develop students' mathematical thinking.

We conducted a content analysis of new mathematics textbooks to determine the mathematical validity of the content suggested by the Curriculum, ensure compliance with the above-mentioned mathematics teaching goals, and infer information about the textbook development system in Lithuania. This paper presents some results of empirical studies related (1) to the definitions of fractions and their arithmetic operations, and (2) to the definitions of functions and their properties. We have chosen to discuss these two topics of school mathematics content due to their importance.

Research evidence shows that understanding fractions predicts later achievement in mathematics (e.g., Torbeyns et al., 2015). Empirical research also shows that, among the possible representations of fractions, the number line enables the most efficient results (e.g., Hamdan & Gunderson, 2017). For logical reasoning, we need a precise

definition of a fraction (e.g., Wu, 2018, p. 60). It is known that a mathematically correct understanding of functions is no less important and complex in school mathematics. Here, we will mention only the literature related to the problem of ambiguity in the concept of function, which, as we will see, is also found in Lithuanian textbooks (e.g., Paoletti & Moore, 2018).

In this study, we aim to answer the following *research question*: How are fraction and function concepts defined, and how are statements about them justified in Lithuanian mathematics textbooks? Before discussing the theoretical framework and research methods, we will present the Curriculum structure and textbook development system in more detail.

DEVELOPMENT OF MATHEMATICS CURRICULUM AND TEXTBOOKS

In 2018, the Ministry of Education, Science, and Sport began updating the curriculum for all educational subjects, collaborating with experts, teachers, and scientists. On September 30, 2022, the Minister of Education, Science, and Sport approved all the revised general education curricula, including mathematics (Ministry of Education, Science, and Sport, 2022). Starting from September 1, 2023, pupils in grades 1, 3, 5, 7, 9, and 11 began learning according to the updated curricula, while students in grades 2, 4, 6, 8, 10, and 12 started from September 1, 2024.

The Curriculum outlines the purpose, goals, and objectives of mathematics education, the competencies developed through the subject, areas of achievement, progression in learning outcomes, the content of the subject, and the criteria for assessing students' performance. In teaching mathematics, the goal is not only to achieve subject-specific objectives but also to promote general educational goals, particularly in the areas of metacognitive thinking, communication, and collaboration skills. Students' involvement in the learning process and assessment fosters a sense of responsibility, helping them understand the value of self-development. *The primary goal of mathematics education is to empower every student to develop mathematical and statistical literacy, which is understood as **the ability to reason mathematically and apply acquired competencies** in solving real-world, relevant, and understandable problems.* Additionally, the Curriculum promotes competencies such as cognitive, communication, digital, creativity, cultural, citizenship, social, emotional, and healthy living skills.

The learning content, which outlines the topics pupils must learn at each grade level and the corresponding achievement criteria, is presented in the Curriculum. Based on the Curriculum, textbooks are developed. The guidelines for evaluating and ensuring the compliance of general education textbooks and teaching materials with legal requirements were approved on February 15, 2024 (Ministry of Education, Science, and Sport, 2024). These guidelines define a general education textbook as a reusable printed or digital resource designed to support the implementation of the Curriculum and help achieve the set learning objectives.

The quality of textbooks is the publishers' responsibility. The publisher assures that the textbook is assessed by at least three evaluators who are practicing teachers or researchers in the field. Additionally, textbooks must be reviewed by the State Commission for the Lithuanian Language to ensure linguistic accuracy and be registered in the national textbook database by the National Education Agency. External evaluation of textbooks is conducted only in exceptional cases, such as when complaints arise or changes to the program occur. Therefore, while the national curriculum is developed at the state level, the creation of textbooks is not part of the curriculum itself, and the responsibility for their quality lies with the textbook authors and publishers, with external reviews being rare.

THEORETICAL FRAMEWORK

We used a theoretical framework that conceptualizes textbooks as instruments for instructional change. To this end, we employ a two-level theoretical framework developed by Rezat and Sträßer (2012) and Rezat et al. (2021). The first level consists of the relationships between the four components of the didactical situation – student, teacher, mathematics, and textbook – described by the tetrahedron model (Fig. 3 in Rezat & Sträßer, 2012). This model suggests that the content of textbooks influences the classroom work by shaping certain attitudes towards mathematics. The second level of the theoretical framework involves the textbooks' mediating role between the intended and enacted curriculum, conceptualized as the potentially implemented curriculum (Fig. 1 in Rezat et al., 2021). Historical analysis of the role of textbooks reveals that they are perceived and utilized as mediators of educational reform and change. This role of textbooks creates opportunities to present declared reform ideas to the enacted curriculum. Implementing change through the use of textbooks also depends on the system of organizing their writing, publication, and distribution, which is influenced by educational leaders and policy-makers. A key component of the current theoretical framework is the requirements for textbook structure, which facilitate the implementation of reform ideas. The selection of specific tasks and worked examples is part of the formation of such a textbook structure.

We are interested in whether Lithuanian mathematics textbooks play a mediating role as instruments for implementing the Curriculum goals in classroom work. Since we aim to enable every student to reason mathematically, our textbooks could become mediators in the implementation of this goal if they provided opportunities to develop the ability to prove. A necessary condition for achieving this goal is precise definitions of basic concepts. As will be shown in the following sections, this aspect appears to be insufficiently emphasized in the analyzed topics in Lithuanian textbooks.

RESEARCH METHODS

We have two publishing houses – “Šviesa” (referred to as A) and TEV (referred to as B) – that publish mathematics textbooks. The topic of fractions is treated in grades 5 and 6, and the topic of functions is introduced in grade 9. The study's data were

collected by examining every page of the textbooks; a total of 268 pages of textbook content were analyzed in this manner.

To answer the research question, we analyzed the text by categorizing the claims according to their role in describing fractions and functions. Parts of the text were coded as a mathematical definition, a rule, a description of the symbol, an example, an illustration, tasks, and worked examples. Definitions, rules, and descriptions of concepts were analyzed for mathematical correctness.

We will illustrate the analysis with the following example, which identifies the treatment of a concept with two different definitions. We coded the statements S_1 and S_2 as definitions of the same concept, where S_1 = “The zero of a function is the x value of those points where the function’s graph crosses the Ox -axis,” and S_2 = “The zero of a function is a solution of the equation $f(x) = 0$.” A mathematically correct alternative to this situation is to treat S_2 as a definition and to prove the statement S_3 , where S_3 = “The graph of a function f crosses the Ox -axis at a point $(x,0)$ if and only if x is a zero of f .”

Another example of textbook analysis is the study of task complexity. Their diversity in terms of complexity enables a better agreement between the goals of the intended curriculum and the learning opportunities provided by the textbooks. Examples, exercises, and tasks were examined for complexity by separating them into three categories. The first category contains worked examples, including a rule and its application to find an answer. We refer to this task, which includes a given solution template and code, as “task T+S.” Similar tasks, based on the same templates, are presented in the textbook for the reader to find an answer and comprise the second category. We refer to this as a solvable task using a template and code as “task T.” Tasks not belonging to these two categories are labeled as non-routine and coded as “task NR.” The following table presents the number of tasks in each of the three categories for the three sections in the 9th-grade textbook A.

	functions	linear function	quadratic functions
T+S	21	18	20
T	234	223	351
NR	11	16	6

Table 1: Number of tasks

The small proportion of non-routine tasks, which is less than 2 percent in some cases, raises doubts about the alignment between the goals of the intended curriculum and the learning opportunities provided by the textbooks.

RESULTS: THE TOPIC OF FRACTIONS

Textbook A describes a fraction explaining the symbol a/b . Namely, “the numerator a shows how many parts of a divided thing or number have been taken, and the denominator b shows how many equal parts the thing or number is divided into.”

Textbook B describes a fraction as follows. “Number a/b is called a common fraction. The denominator b of a common fraction shows how many equal parts the unit is divided into, and the numerator a shows how many of those parts are taken.”

Each description of the fraction may be interpreted differently. First, both may suggest that a fraction is a pair of two counting numbers called the numerator and the denominator. The only meaning of the number concept in the two textbooks is counting numbers. Neither of them revises the concept of numbers. So, the word number possibly attached to the fraction has no other meaning than counting numbers.

Second, textbook A may suggest that the fraction is a part of a thing or a quotient. However, the quotient of two natural numbers was explicitly defined in the case when one number is a multiple of the other, i.e., when the quotient is a natural number. Using the quotient to describe fractions without an explicit change of meaning has no mathematical sense. A mathematical definition does not create a new meaning. It just gives a new name to an already established property of a mathematical object.

Third, textbook B mentions the word unit in the description of the fraction. However, how this word should be understood is not explained. It may be interpreted as the number 1. In that case, we have once again the problem of the meaning of the quotient when dividing the number 1. The word “unit” may be interpreted as the point on a number line corresponding to the number 1. Then the fractions could be defined as the endpoints of suitable intervals. However, textbook B uses fractions to compare intervals on a number line, stating, for example, that “one interval equals $1/3$ of another interval,” which is unclear. Note that the fraction is not yet defined.

Another very obscure topic in the textbook is mixed numbers. They are defined as fractions, but not as the sum of a natural number and a fraction. Next, we look at how the fundamental property of fractions (FPF) is justified in our textbooks. The FPF states that the value of a fraction does not change if both its numerator and denominator are multiplied or divided by the same non-zero number.

Textbook A justifies the FPF using two illustrations. First, the concept of the property is explained by dividing an equal part of a cake into equal parts again and noting that the total size of the cake remains unchanged after the division. Second, two number lines are drawn side-by-side, and the points at the same distance from zero are marked with corresponding fractions, such as $1/3$ and $2/6$. Textbook A cannot explain why the two fractions measure the exact distance since there is no definition of a fraction in terms of constructing suitable points on a number line, and there is no explicit identification of numbers with points (e.g., Hamdan & Gunderson, 2017).

Textbook B justifies the FPF as follows. “We know that a fraction is a quotient obtained by dividing the numerator by the denominator, i.e., $a/b = a \div b$. We also know that when the dividend and divisor are multiplied or divided by the same nonzero number, the value of the quotient does not change (the fundamental property of the quotient). Hence, this property also applies to common fractions. It is called the fundamental property of a common fraction.” Then the FPF is stated.

The cited justification ignores the essential property of mathematical definition again. The fundamental property of the quotient stated in textbook B, assuming that one natural number is a multiple of another natural number, is applied when this assumption is not satisfied.

We turn to descriptions of the product of fractions in our textbooks. Textbook A justifies the product of two fractions, a/b , and n/m , by interpreting the phrase “ a/b of n/m .” This is done by calculating the $\frac{3}{4}$ of $5/8$ of the pizza. In this process, dividing the part $5/8$ of a pizza into 4 equal parts is interpreted as dividing the fraction $5/8$ by the number 4, namely $5/8 \div 4$. However, such an interpretation makes no mathematical sense since the division of a fraction by a number is not yet defined in the textbook A. This is a missed opportunity to demonstrate the usefulness of the number line and FPF.

Textbook B presents the product of two fractions as a rule. Namely, given a/b and n/m , their product is the fraction $(an)/(bm)$. This rule is motivated by reference to the calculation of the area of a rectangle.

In response to the fraction part of the research question, we suggest the following conclusions. A part of the complexity of the topic of fractions is related to the need to expand the concept of counting numbers and to generalize the arithmetic of natural numbers to fractions (Wu, 2018). These issues are not addressed as the textbook excerpts presented above show. In addition, mathematical problems come from misunderstandings of the essence of mathematical definitions. The excerpts show that the descriptions of fractions focus on describing the symbol a/b rather than formulating the basic properties needed to justify basic operations with fractions.

RESULTS: THE TOPIC OF FUNCTIONS

We quote the textbook A for 9th grade. “Suppose X and Y are number sets. The function is a rule that assigns to each number $x \in X$ exactly one number $y \in Y$.” According to this definition, a function is defined when three objects are given: a rule f , its domain of definition X , and its codomain of values Y , i.e., when a triple (f, X, Y) is given. We refer to this conception as the function *triple*.

After moving on to the examination of functions f , the domain X and the codomain Y are treated as properties that can be determined, for example, from the function's graph or the formula of its values $f(x)$. Another feature of the function concept in the Lithuanian textbook is identifying its codomain with its image, i.e., the set of numbers $y \in Y$ such that $y=f(x)$ for some $x \in X$. This is a different concept of a function, which we call *the function as expression*.

The identity of a function's codomain and image means the function is always surjective. In such cases, the function is invertible if and only if it is injective, while the function as a triple is invertible if and only if it is injective and surjective. This shows that a concept with two non-equivalent definitions coexists in the same text.

School mathematics in Lithuania from the period after World War II until the country regained its independence in 1991 was heavily influenced by Russian school mathematics. As a result, Russian math textbooks translated into the Lithuanian language were widely used several decades ago. Some such textbooks treat functions differently. To illustrate, we take the following passage from Teliakovskis (1999, p. 3):

A function is called a dependence of the variable y on the variable x when each value of the variable x corresponds to a single value of the variable y . The variable x is referred to as the independent variable or argument, and all its values comprise the domain of the function's definition. Variable y is called the dependent variable, and all its values form the domain of values of the function. When a function is expressed in a formula, and its domain of definition is not specified, it means that the domain of the definition of the function consists of all values of the argument for which the formula makes sense.

This is an example of the function as an expression. Many mathematics teachers in Lithuania are accustomed to concepts from Russian textbooks, which are reflected in the new textbooks. At the same time, some mathematicians deepened their education under the influence of Western universities. The ambiguity of function concepts in the same textbook may be attributed to the coexistence of Eastern and Western cultural differences within the same country (Malaty, 1998).

We note only two things. First, the basic properties of functions are presented as rules illustrated by their graphs in textbook A. Second, in response to the function part of the research question, despite presenting the function as triple, it is treated as an algebraic expression defined only when it has a meaning.

CONCLUSIONS

The idea and goal of the Curriculum is to enable every student to reason mathematically. For the textbook to serve as a mediator between the idea of reform and its users, it should provide opportunities to develop mathematical reasoning. We found that textbooks A and B contain examples of incorrect mathematics. Since definitions are given near the beginning of each section and low-level tasks dominate the content of textbooks, we conclude that textbooks A and B promote “teacher-centered forms of teaching.” Therefore, the textbooks have restricted opportunities to mediate the needed changes through the theoretical framework.

It should be noted that this study on the mediating role of textbooks is only meaningful when the textbooks are used by teachers in their classroom work. According to the preliminary results of the national survey we are currently conducting and analysing, 96,1 percent of the teachers surveyed are using the new textbooks.

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MATHEMATICAL PROOFS IN JAPANESE AND NORWEGIAN GRADE 5–9 TEXTBOOKS

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This study compares the presentation of proofs in Norwegian and Japanese Grade 5–9 mathematics textbooks, focusing on logical, mathematical, and didactical structures. Proof and proving have distinct roles in the curricula of these two countries, with Norway recently emphasizing reasoning and argumentation, while Japan has long incorporated proof into mathematics education. The research focuses on specific theorems such as the sum of interior angles in a triangle, and the Pythagorean theorem. The results reveal differences in how proofs are introduced, structured, and integrated into the learning process across these educational systems.

BACKGROUND AND GOAL

Proof and proving are central to mathematics and essential for understanding how mathematical principles are established; however, research internationally shows that understanding proof and proving is difficult for students (Mariotti et al., 2018; Reid & Knipping, 2010). Without an understanding of the education systems of different countries, it is not possible to make sense of differences in students' performance between them (Pepin & Haggerty, 2001), and it is known that the positioning of proof and proving varies across the curricula, textbooks, and classroom practices of different countries (Miyakawa & Shinno, 2021). The most recent Norwegian mathematics curriculum (Utdanningsdirektoratet, 2019) has given a new prominence to reasoning and argumentation, which is interpreted to include proof (Valenta & Enge, 2020). In contrast, proof has been an important part of the Japanese mathematics curriculum for decades. This suggests that a comparison of how proof is presented in these two contexts, with distinct cultures and education histories, can reveal important differences in the approach taken to teaching proof, which will in turn give insight into differences in student performance.

Here we explore the following research question: *How do proofs presented in Norwegian and Japanese Grade 5–9 mathematics textbooks differ, in terms of their logical structure, mathematical structure, and didactical structure?*

To focus this comparison, we compared proofs related to: The sum of two even/odd numbers; The sum of three consecutive numbers; The sum of the interior angles in a triangle; The Pythagorean theorem.

These topics were chosen as part of a larger research project that identified theorems that are included in both Japanese and European textbooks.

THEORETICAL FRAMEWORK

Based on different theoretical perspectives and conceptualisations of argumentation and proof in mathematics education (Mariotti et al., 2018; Reid & Knipping, 2010), Miyakawa and Shinno (2021) proposed characterising proof and proving in a given context by three aspects: *structure*, *language*, and *function*. They conceptualise these aspects as a triplet that can characterise cultural specificities of proof and proving.

Structure describes how different statements in a proof are connected. *Language* refers to the representations used in the proof, including symbols, gestures, diagrams, oral and/or written discourse, and concrete objects. *Function* includes verification, explanation, exploration, communication, and systematisation, but may go beyond these functions often mentioned in the literature (e.g., Hanna, 2000).

Knipping et al. (2025) elaborate on these distinctions, specifically by describing four kinds of structure:

The *logical* structure of a proof refers to the deductive steps in it, leading from given statements to conclusions, warranted by theorems, axioms, and definitions. Analysing the logical structure could include reconstructing the type of logic (e.g., propositional, predicate) which is employed. Non-proof arguments do not have a purely deductive logical structure, but may involve some combination of inductive, deductive, abductive, and analogical reasoning (as described by Reid & Knipping, 2010).

Reconstructing *argumentation* structures requires a conceptualisation of rationality related to the context of practice (Knipping & Reid, 2019). Toulmin's (1958) functional model of argumentation is elaborated into complex networks capturing the interconnections between statements. This is most relevant to classroom argumentation.

The *mathematical* structure of a proof refers to choices made about the meaning of variables, the theorems, axioms, and definitions employed, and the order of steps. For example, in proving that the sum of three consecutive numbers is a multiple of three, one might use a variable n to stand for the first number, making the sum $3n + 3$, which can be shown to be a multiple of three by applying the distributive law. Alternatively, one might use a variable n to stand for the middle number, making the sum $3n$, which can be seen to be a multiple of three without reference to the distributive law.

The *didactical* structure of a proof refers way proofs fit into the flow of a lesson, or a textbook chapter. This can involve whether the proof is given by the teacher or a task for the students, or whether a single proof or multiple proofs are used.

METHODS

We surveyed textbooks from three publishers in each country. The textbooks chosen are currently in use in schools and widely distributed. In the case of the Japanese books, the availability of English translations was also a selection factor. The publishers and

textbooks are listed in Table 1. Note that Japanese junior high school grades 1, 2 and 3 correspond to Norwegian grades 7, 8 and 9.

Country	Publisher	Titles
Japan	Keirinkan	Math 5A/5B/6A/6B for elementary school Math 1/2/3 for junior high school
Japan	Gakko Tosho	Mathematics for elementary school 5 th /6 th volume 1/2 Mathematics 1/2/3 for junior high school
Japan	Tokyo Shoseki	Mathematics 7/8/9
Norway	Gyldendal	Multi 5A/5B/6A/6B/7A/7B Maximum 8/9
Norway	Cappelen Damm	Matematikk 5/6/7/8/9 fra Cappelen Damm
Norway	Aschehoug	Matemagisk 5A/5B/6A/6B/7A/7B/8/9

Table 1. Publisher and textbooks sampled.

Both physical and digital versions of the textbooks were used. Keywords (e.g., *even*, *odd*, *consecutive*, *triangle*, *angle*, *Pythagoras*, *Pythagorean*, *prove*, *proof*, etc.) were used to identify pages that could contain proofs of the target theorems. Details of this process are given in Jåtun (2024). Once the relevant pages were identified, the proofs on them were classified according to their logical, mathematical, and didactical structure. Argumentation structure was not analysed as it is more applicable to multivocal, dynamic classroom argumentation than to text analysis.

Logical structure was coded only as Deductive or Empirical. Arguments using specific examples were coded as Empirical. If the examples were explicitly generic (with indications that they stood for the general class), and the argument was otherwise deductive in structure, the structure was coded as Deductive.

Codes for mathematical structures were generated by noting the objects used in each argument and the actions on those objects. If the objects and actions were the same, then the arguments were considered to have the same structure. If there were small differences the arguments were coded using different but related codes. For example, two proofs of the Pythagorean theorem use somewhat different diagrams but are otherwise the same. These were coded as Behold and Behold+.

The didactical structure depended on where proofs occurred in the chapter related to the theorem, whether the students were expected to read and comprehend a given proof, or produce a proof themselves, and whether multiple proofs were given or expected.

The coding of the logical structure was based on that carried out in Jåtun (2024), but was redone, resulting in slightly different results. The analysis of the mathematical structures and didactic structures is reported here for the first time.

RESULTS AND DISCUSSION

We will describe our results according to the three types of structure investigated, Logical structure, mathematical structure and didactical structure.

Logical Structure

Table 2 shows the number of arguments with Deductive and Empirical logical structures found for the four theorems we focused on. Note that these numbers may not represent the distributions of empirical and deductive arguments in the texts overall. As we did not have access to the Tokyo Shoseki textbooks for Grades 5–7, the number of arguments found in those grades for Japanese textbooks is likely to be too low.

		Grade 5	Grade 6	Grade 7	Grade 8	Grade 9	Total
Deductive	Norway					9	9
	Japan				21	10	31
Empirical	Norway		3			3	6
	Japan	5				1	6

Table 2. Number of deductive or empirical arguments, by grade and country.

Overall, in both countries empirical arguments are used in grades 5–7, and deductive arguments appear in Grade 8 or 9. Proof is introduced as a topic in Grade 8 in Japan, which accounts for the high number of deductive arguments in that grade.

Mathematical structure

In the textbooks, we observed seven distinct mathematical structures of arguments that the sum of the interior angles of a triangle is 180 degrees. Five are empirical and two deductive. We have coded these as follows:

- E-measure: Measuring the angles in several triangles and generalising.
- E-corners: Tearing the corners from a triangle and reassembling them to lie on a straight line.
- E-patternL: Placing congruent triangles in a linear pattern so that the interior angles of three triangles lie on a line.
- E-patternP: Tiling the plane with congruent triangles so that the interior angles of six triangles meet at a point.
- E-folding: Folding the corners inwards so that they lie on one side.
- P-parallelB: Constructing a line parallel to the base of the triangle, and using alternate and corresponding angles to prove the theorem
- P-parallelS: Constructing a line parallel to one side of the triangle, and using alternate and corresponding angles to prove the theorem.

Table 3 shows how these mathematical structures are distributed among the textbooks. The Tokyo Shoseki textbooks for Grades 5–7 were not available and so are not included. The letters C and P indicate whether the student is expected to comprehend (C) a given argument or produce (P) an argument (guided by the text).

Among the empirical arguments, tearing off corners of a triangle was the most common structure, and was found in both countries. The Japanese textbooks present two or three different empirical arguments in the Grade 5 textbooks, and one or two different deductive arguments in the Grade 8 textbooks. The Norwegian textbooks present at most one argument in a grade.

	Norway Gyldendal	Cappelen Damm	Aschehoug	Japan Keirinkan	Gakko Tosho	Tokyo Shoseki
E-measure	6-P			5-P		
E-corners		9-C	6-P	5-C	5-C	
E-patternL					5-C	
E-patternP					8-P	
E-folding					5-C	
P-parallelB			9-P		8-P	8-P
P-parallelS				8-C	8-C	8-C

Table 3. Distribution of mathematical structures for the angle sum of a triangle.

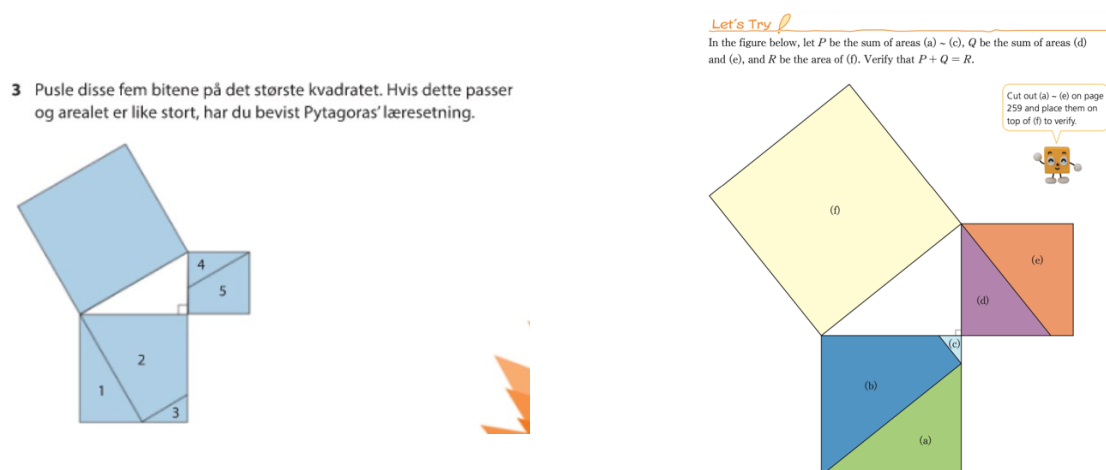


Figure 1: Diagram for Puzzle argument. Left: Gyldendal; Right: Tokyo Shoseki. The text on the left reads “Puzzle together the five pieces on the largest square. If they fit and the areas are the same size, you have proved the Pythagorean Theorem”.

We observed eight distinct mathematical structures of arguments for the Pythagorean Theorem. Two are empirical and six are deductive. We have coded these as follows:

- E-345: argument for the special case of sides measuring 3, 4 and 5.
- E-Puzzle: puzzle argument using a diagram (see Figure 1).
- D-Behold: proof sometimes ascribed to Bhāskara, but also found in the *Chou Pei Suan Ching*, using a square of side length $a+b$, containing four congruent triangles bounding a square of side c .
- D-Behold+: the Behold proof with extra squares drawn outside main square.
- D-Euclid: Euclid’s proof from *Elements*.
- D-Garfield: Garfield’s trapezoid proof.
- D-Similar: proof using the division of the triangle into two similar triangles divided by the altitude to the hypotenuse.

- D-Square: proof using a square of side c , containing four congruent triangles bounding a square of side $a-b$, also sometimes ascribed to Bhāskara.

Table 4 shows how these mathematical structures are distributed among the textbooks. Only Grade 9 textbooks included the Pythagorean Theorem. The letters C and P indicate whether the student is expected to comprehend (C) a given argument, fill in the blanks (F) of a partial argument or produce (P) an argument (guided by the text).

	Norway Gyldendal	Cappelen Damm	Japan Aschehoug Keirinkan	Gakko Tosho	Tokyo Shoseki
E-345		C			
E-Puzzle	P				P
D-Behold	P		P		C
D-Behold+			C	C	
D-Euclid			P	C	P
D-Garfield			P		
D-Similar			P	F	P
D-Square	P		P	P	P

Table 4. Distribution of mathematical structures for the Pythagorean Theorem.

All the textbooks, except one Norwegian book, include several arguments. The Behold and Behold+ proofs are included in five of the six books, and these are the proofs the Japanese books present first. The Square proof and the Similar proof are included in four of the six textbooks, including all three Japanese textbooks. The Norwegian textbook that included only a single empirical argument has an associated exercise book that includes proof production tasks for D-Behold and D-Garfield, but these are not included as we considered only the main textbook. Overall, the sets of proofs included in the Japanese textbooks are quite similar to each other in terms of mathematical structure. The Norwegian books, however, have very little overlap.

The empirical Puzzle argument is interesting, as it is included in both a Japanese book and a Norwegian book and two of the three empirical arguments included have this structure. In both cases a diagram similar to Figure 1 is given and the students are instructed to cut the pieces of the smaller squares out of a provided worksheet and to arrange them to cover the largest square. The Norwegian book claims “If they fit and the areas are the same size, you have proved the Pythagorean Theorem” (p. 191, our translation). The Japanese book says that the task is to “Verify that $P + Q = R$ ” where P , Q , and R are the areas of the three squares. Neither comments on the special properties of the dissections of the smaller squares, nor on the possibility of seeing this particular diagram as a generic example.

Didactical Structure

The sequencing of the proofs of the angle sum of a triangle and the Pythagorean Theorem is consistent in the Japanese books. All three books begin by offering the parallelS proof for students to comprehend. Later students are to produce the ParallelB proof as an exercise. Similarly, all three first offer the Behold+ or the Behold proof for students to comprehend, and later students are expected to produce other proofs. In contrast, the Norwegian books never offer proofs to be comprehended. The angle sum of a triangle is verified empirically and applied in exercises. Only in one textbook it is later proved, and in that case the students are expected to produce the ParallelB proof. The Pythagorean Theorem is stated in the three Norwegian textbooks, and then it is applied in exercises. In two of the textbooks, students are expected to produce proofs of the theorem at the end of the chapter. In the third, they are shown the 345 special case before the theorem is stated, and no deductive proof is ever mentioned.

Overall, the didactic structure of these proofs in the Japanese books is two-fold. They are first used to explain and verify the theorem, and later proving becomes an activity students are expected to perform. In the Norwegian books the didactic structure is not so consistent, but in two of the books the theorems are established empirically, or simply stated without justification, and proving theorems is the task of the students, after they have applied the theorems in exercises.

CONCLUSIONS

We have shown that there are noteworthy differences in how proofs are presented, used and integrated into Norwegian and Japanese Grade 5–9 mathematics textbooks. In terms of logical structure, both countries gradually shift from empirical to deductive reasoning in later grades. However, the Japanese textbooks emphasize deductive reasoning earlier and more consistently than the Norwegian textbooks, where empirical approaches are more prevalent.

Japanese textbooks provide multiple proofs for the same theorem that differ in their mathematical structure. In contrast, Norwegian textbooks tend to focus on a single argument per grade level, or offer no argument, limiting the exposure students have to alternative methods. The exception is the Pythagorean theorem, for which Norwegian textbooks offer multiple arguments.

Regarding the didactical structure, Japanese textbooks follow a clear progression: proofs are first presented to explain and verify theorems, with students later expected to construct other proofs independently. In contrast, Norwegian textbooks emphasize student-led proof construction after empirical explorations, offering no completed proofs to be understood beforehand.

This study reveals important cultural and pedagogical differences in how proofs are introduced and taught in Norwegian and Japanese mathematics textbooks. These differences might impact students' engagement with proof, influencing their development of logical and mathematical reasoning skills. Further research is needed

on whether these differences are reflected in classrooms practices, and whether they account for differences in student performance. They also suggest ways that proof and proving could be better integrated into textbooks, especially Norwegian textbooks, but also Japanese textbooks for grade before Grade 8.

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THE ROLE OF TEXTBOOKS IN PRIMARY MATHEMATICS EDUCATION IN CROATIA: SELECTION AND USE

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This study explores how primary school teachers in Croatia choose and utilize mathematics textbooks, both print and digital formats. Data were collected through a questionnaire that examined textbook utilization in lesson planning and implementation, exercise selection, and the integration of digital textbooks. The results show that teachers predominantly rely on printed textbooks to structure lessons and select exercises, whereas the adoption of digital textbooks remains limited. The findings highlight a tension between the structured use of textbooks and the potential for promoting student autonomy.

INTRODUCTION

Curriculum materials provide structured pathways for teaching and learning mathematics, guiding both lesson planning and classroom enactment (Remillard & Kim, 2017). These resources include traditional printed textbooks, teacher-created content, and increasingly, digital resources such as e-textbooks and interactive platforms (Trouche et al., 2023). Among them, textbooks hold a special place: they link educational policy with classroom practice, support lesson sequencing, and serve as central tools for organizing student tasks and assessment (e.g., Fan et al., 2021; Gustafsson et al., 2024; O’Keeffe & White, 2017). With the development of digital formats, textbooks are no longer static resources; instead, they may include interactive tasks, visualizations, and feedback mechanisms that potentially reshape teaching practices (Brnic et al., 2024; Pepin et al., 2017). Yet, despite technological advances, research continues to show that printed textbooks remain dominant in many classrooms worldwide (Rezat, 2024).

While research has increasingly examined how secondary mathematics teachers adopt and adapt digital textbooks, considerably less is known about the practices of primary school teachers. Studies of secondary education point to heavy reliance on textbooks for structuring lessons, providing exercises, and supporting both formative and summative assessment, with limited integration of digital alternatives (e.g., Glasnović Gracin, 2011; O’Keeffe & White, 2017). However, whether these patterns extend to the primary school level, where teachers are generalists rather than subject specialists, is not yet well understood. The aim of this study is therefore to investigate how primary school mathematics teachers in Croatia select and use textbooks in their teaching. Specifically, the study explores two research questions: (a) What criteria do teachers

prioritize when selecting mathematics textbooks? (b) To what extent do primary teachers use printed and digital textbooks in their teaching practices?

CONTEXT OF THE STUDY

Previous research about Croatian mathematics education has shown that secondary teachers rely heavily on textbooks for lesson planning, practice, and assessment, with limited use of alternative or digital resources (Glasnović Gracin, 2011; Glasnović Gracin & Jukić Matić, 2021). These findings suggest that textbooks remain central to teaching practices even when curricular reforms encourage more flexible or inquiry-based approaches (Jukić Matić, 2024). Building on this work, the present study turns to the primary level, where evidence on how teachers engage with both print and digital textbooks is still lacking.

In Croatia, textbooks are mandatory at all school levels and must align with the national mathematics curriculum, as well as meet scientific, pedagogical, psychological, and methodological standards defined by national legislation (Hrvatski sabor, 2018). Although textbooks require approval from the Ministry of Science, Education, and Youth (MSEY), teachers are free to select from a list of approved titles. Once selected, textbooks are generally used in the same school for a period of at least four years. In 2019, Croatia implemented a comprehensive educational reform that introduced a revised national mathematics curriculum, significantly transforming teaching approaches. The reform aimed to strengthen students' problem-solving skills by promoting active learning strategies supported by contemporary resources, while also seeking to boost student engagement and teacher motivation (Divjak & Pažur Aničić, 2019). A key goal of the reform was to provide greater professional autonomy and freedom for teachers, including the ability to independently plan and organize lessons, choose methods, materials, and content within defined learning outcomes, and apply more flexible approaches to assessment and evaluation. As part of these changes, new textbooks were released between 2019 and 2021.

THEORETICAL FRAMEWORK

Textbooks occupy a central position among curriculum resources, serving as a bridge between official curriculum policies and teachers' enacted classroom practices (Valverde et al., 2002; Fan et al., 2021). To examine how Croatian primary teachers engage with textbooks, this study adopts the model of curriculum enactment proposed by Remillard and Heck (2014). This model conceptualizes curriculum as consisting of both official and operational dimensions. The official curriculum includes curriculum goals and objectives, national assessments, and designated curricula such as textbooks. The operational curriculum refers to what occurs in the classroom: the teacher's intended curriculum, enacted curriculum, and resulting student learning outcomes. The interaction between the official and operational curriculum reflects a dynamic process which is influenced by the resources available, the teacher's beliefs and experience, and the educational environment. The model emphasizes that teachers are not passive users of curriculum materials. Instead, they act as curriculum designers, making

informed decisions about how to interpret, adapt, and implement materials based on their professional judgment, student needs, and classroom context.

METHODOLOGY

The study included 104 primary school teachers from eastern Croatia who participated in a professional development event organized by the national Education and Teacher Training Agency in autumn 2023. Participation in the study was voluntary and anonymous, ensuring reliable responses. The Teachers' Opinion Questionnaire on the Use of Textbooks in Mathematics Instruction (Glasnović Gracin, 2011) was used, with modifications made to include items on digital interactive textbooks. The questionnaire had three parts: demographic information, items on a Likert scale (1 = never, 4 = almost always) about using textbooks, and multiple-choice questions about choosing textbooks and using digital textbook elements. The original questionnaire had a four-factor structure, and the addition of new items prompted an exploratory factor analysis. Some items from the original questionnaire were excluded during this process. Principal axis factoring with Varimax rotation revealed a four-factor structure, explaining 40.48% of the variance:

- Following Textbook (5 items) – creating an annual teaching plan according to the textbook; lesson planning with the teacher guide and textbook; following the structure of the textbook; emphasizing details in the textbook
- Use of Textbook Tasks (5 items) – direct teaching of new topics; relying on textbook tasks for practice and homework; reviewing before assessments and designing summative assessments.
- Flexible Activity Selection (4 items) – lesson planning using additional textbooks and online materials; differentiating tasks by students' ability; students' autonomy in choosing tasks from the textbook
- Digital Resources (4 items) – using digital interactive textbooks and platforms for motivation, teaching new content, assigning homework, and creating assessments in mathematics.

Cronbach's alpha indicated strong reliability for Factors 1 (0.7945) and 2 (0.7760) and moderate reliability for Factors 3 (0.6068) and 4 (0.5409). Descriptive analysis and correlations were also conducted to interpret the findings.

Other questions in the questionnaire were multiple choice: Which criteria do you find the most important in choosing a textbook? (Select three criteria.) (see Figure 1 for criteria), and Select all elements from the digital interactive version of the textbook/digital platform you use in teaching mathematics (see Figure 2).

RESULTS

Table 1 presents data on average values and distributions for individual factors (referred to as subscales) regarding the role of mathematics textbooks in primary education. Teachers often rely on textbooks to structure their teaching practices based on the chosen textbook (Subscale 1) and to assign tasks for homework, practice, and

summative assessment (Subscale 2). Given the prominence of textbooks and their accompanying guides in teaching, participating teachers seldom use alternative materials, offer students the choice of practice tasks, or differentiate tasks based on student abilities (Subscale 3). Additionally, teachers rarely use the digital version of textbooks or the associated digital platform (Subscale 4).

Subscale	M	SD	1	2	3	4
1. Following Textbook	3.03	0.5081	1	0.472*	-0.221*	0.103*
2. Use of Textbook Tasks	2.92	0.4737	–	1	-0.184*	0.051*
3. Flexible Activity Selection	2.51	0.4634	–	–	1	0.174*
4. Digital Resources	2.26	0.4603	–	–	–	1

* $p < 0.05$

Table 1: Correlations between scales.

The findings show a positive correlation between teachers who rely heavily on textbooks for structuring their teaching and those who use textbooks as the main source of exercises (Subscale 1 and Subscale 2). This reliance on textbooks can simplify lesson planning but may limit teaching flexibility. A weak positive correlation between teachers promoting student autonomy and those using digital resources (Subscale 3 and 4) indicates that while some teachers integrate both approaches, this practice is not common. Additionally, a weak negative correlation between textbook reliance and flexible activity selection (Subscales 1 and 3) suggests that teachers who depend on textbooks may be less likely to encourage independent student task selection.

The bar chart in Figure 1 shows primary teachers' selection criteria for textbooks. The top three criteria for selecting a textbook are compliance with the curriculum, didactical design, and the appropriateness of content for the age of the students. These factors are considered more important than worked examples, mathematical accuracy, or graphic design.

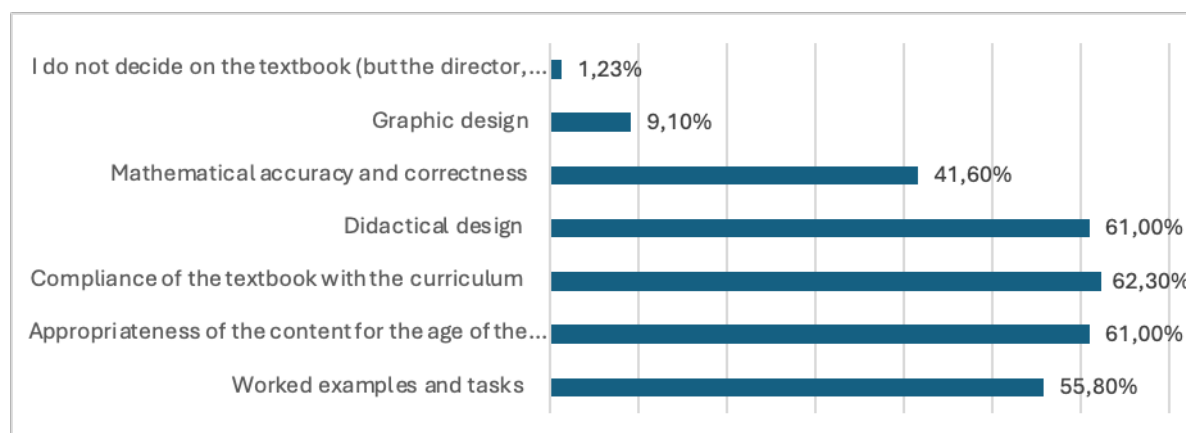


Figure 1: Selection criteria for textbook.

The bar chart in Figure 2 shows that the most commonly used elements in digital textbooks are Knowledge test quizzes, Presentations, and Interactive tasks that provide feedback, while elements like Interactive applets/simulations and Links to content on the Internet are used less frequently.

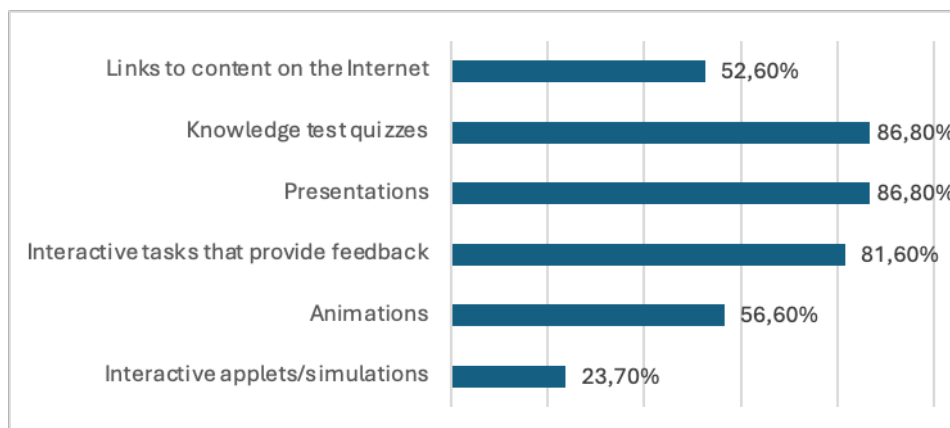


Figure 2: Most commonly used elements of digital textbooks.

DISCUSSION

The aim of this study was to investigate how Croatian primary teachers select and use mathematics textbooks, both print and digital.

Criteria Croatian primary school teachers prioritize when selecting mathematics textbooks

The results of the study show that Croatian primary school teachers prioritize curriculum alignment, didactical design, and age-appropriateness when selecting mathematics textbooks. This indicates that teachers regard textbooks as foundational resources that structure and legitimize their instructional decisions. Such a perspective can be understood within the Croatian educational context, where textbooks must be formally approved by the MSEY, a process that reinforces their authority. These findings are consistent with earlier research showing that textbooks often serve as a bridge between educational policy and classroom practice (Valverde et al., 2002) and that teachers value them for providing security, clarity, and coherence in instructional planning (Fan et al., 2021).

Use of printed and digital textbooks in everyday practice

Despite reforms that have granted teachers more autonomy in instructional design, the results of this study indicate that printed textbooks hold a dominant role in primary mathematics instruction in Croatia. Teachers rely on these officially approved resources to sequence content across the school year, plan individual lessons, and for practice, homework, and assessment. This suggests a strong alignment between the designated and operational (i.e., intended and enacted) curriculum.

The study also reveals a relatively low emphasis on differentiation and student autonomy, as captured in the low scores on the subscale measuring flexible

instructional planning. Only a minority of teachers report differentiating tasks or incorporating alternative resources, and such practices remain isolated rather than systemic. Teachers who closely follow textbooks are less likely to adapt materials or involve students in selecting tasks, suggesting a trade-off between instructional control and flexibility. These findings reflect what international research has also shown: the efforts to promote student-centred learning often run into challenges because of long-standing teaching habits and practices (Gouédard et al., 2020). In Croatia, where textbooks have a long-standing role (Glasnović Gracin, 2011), changing this dynamic requires more than policy. It calls for professional and cultural transformation.

Furthermore, the results show that digital textbooks or associated platforms are rarely used. Wijaya et al. (2022) reported that teachers' intention to adopt digital mathematics textbooks is strongly shaped by performance expectancy. In line with this, the findings suggest that primary teachers are more inclined to use digital textbooks when they believe these resources will enhance instructional quality. This is reflected in their adoption of features such as interactive tasks and quizzes that provide feedback, though these largely replicate the functions of print-based materials. In contrast, more advanced tools, such as simulations or links to external resources, are seldom employed. A weak but positive association between digital tool use and student autonomy points to untapped potential. Digital platforms could support differentiated instruction and exploratory learning, yet current practices indicate they are treated more as supplementary add-ons than as integral teaching components. However, the caution is also needed here because the choice of digital textbooks and their associated teaching features can significantly affect primary school students' performance (Shalgimbekova et al., 2024), what shows the importance of selecting and using these resources thoughtfully in the classroom. According to this, primary teachers require both technical and pedagogical support to integrate digital tools more meaningfully into classroom practice.

Moving away from textbook-driven teaching toward more flexible approaches will only be possible if teachers receive the right support. The participating teachers could benefit from strategies that combine using textbooks and encouraging students to be independent. Using digital resources and promoting independence can help create more personalized and flexible learning experiences. When students are given more freedom to choose tasks and direct their learning, they can develop critical thinking skills, creativity, and a sense of ownership over their education (Kressin, 2020). The challenge, however, lies in finding a balance between these approaches. It is possible that teachers who prioritize structure may feel that granting too much autonomy undermines their ability to manage the classroom or ensure content coverage, while those who favour student-centred approaches may struggle with the lack of clear guidelines provided by textbooks. This points to a certain paradox: autonomy is granted in policy, but constrained in practice by institutional expectations because the textbook is a mandatory curricular resource. Moreover, future research should explore how

schools and policy environments shape teachers' decisions, and how teachers perceive their autonomy in relation to the textbook as a mandatory resource.

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STUDENTS VOTING WITH THEIR FEET? THE EFFECTS OF CURRICULUM PACING GUIDES

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This paper reports on the effects of a curriculum pacing regime on teachers' instructional decisions in elementary mathematics classrooms. Twenty elementary teachers in a large urban school district in the US were interviewed to understand their choices about what they taught each day, what students needed and were learning, as well as teachers' perceptions of pacing guides and other directives related to curriculum use. Our analyses indicate that years of experience, teaching assignment, and building leadership influence teachers' decisions about what to teach when, even when school district policy directs teachers to base such decisions on the pacing guide alone. The paper concludes with questions about the effects of curriculum choices on teachers' and students' experiences in one large urban school district.

INTRODUCTION

Mathematics textbooks and other curriculum materials have long been an integral element for ensuring the quality of mathematics instruction at all levels of schooling. Textbooks are a representation of the mathematical content to be covered in a course or grade and provide a shared roadmap for instruction. In the United States, mathematics textbook content is increasingly aligned with curriculum standards, which also form the basis of standardized tests (Bauml, 2015). With the rise of accountability measures in public education in the US in the last two decades, textbooks are viewed as an essential resource for preparing students to learn the material specified by local, state, and national standards and performing well on standardized tests that are linked to those standards.

This paper is situated in an area that Remillard (2018) refers to as “teacher-resource interactions.” Most enumerations of curriculum resources, however, do not regularly include the sub-genre of *pacing guides*. Pacing guides are meant to supplement textbooks and teachers' guides, requiring additional layers of teacher interpretation and implementation. Pacing guides specifying content “coverage” for each day of instruction are increasingly present, especially in large urban school districts for children of colour living in poverty. Reasons for such strict pacing guides include curriculum cohesion and articulation, especially in large districts with high degrees of student and teacher turnover; an assumption that teachers in such districts may not have the pedagogical and content knowledge to make warranted instructional decisions for their students; and the expectation that curricular resources align with standards-based high-stakes assessments and will therefore lead to increased student performance (cf.

achievethecore.org). We investigate the effects of curriculum pacing directives on teachers' decision-making related to their pedagogical design capacity, or PDC (Brown, 2006) in the largest school system in one Midwestern state in the US. The district serves an urban population, 89% Black, with 84% of youth living in poverty and 39% living in concentrated poverty. The district had adopted new textbooks in mathematics for all grades three years prior to the conduct of this study and created strict pacing manuals to guide the use of these resources. Central office personnel made frequent visits to elementary mathematics classrooms to enforce the curriculum pacing, and a large part of the role of mathematics lead teachers at each school was to encourage teachers to follow the pacing guides.

Ensuring Equity or “Deskilling”?

Curriculum pacing guidelines have been framed by Good and colleagues (1978) as an effort to serve students more equitably. Without such guidance, they argued, teachers could underestimate students' abilities, slow the instructional pace, and fail to reach grade-level benchmarks. Alternatively, pacing guides are viewed as “deskilling” teachers, that is, taking a complex job requiring professional knowledge, and “breaking [it] down into specified actions with specified results so that less skilled and costly personnel can be used or so that the control of work pace and outcome is enhanced” (Apple, 1982, p. 110). Stoller (2015) characterized these efforts as a modern-day Taylorization of the teaching profession.

Effects on Teachers and Teaching

Teacher agency around curricular mandates is not uniform across contexts (Bower & Parsons, 2016). Bauml (2015) and Certo (2006), for example, found that pacing mandates were the top source of anxiety among beginning teachers. In addition, these pacing guides can alter what, how, and for how long curriculum topics are taught. Pacing guides are often paired with standardized curriculum-based assessments, and teachers engage in “selective skipping” of content (Good et al., 1978), “curricular privileging” of tested topics (Bauml, 2015), and other compromises as they balance the needs of their students and the demands of the pacing guides. When teachers feel as if they have freedom to do so, they augment, adapt, and extend the curriculum to fit within the mandated pacing. Such moves can come with significant professional risk (Bauml, 2015).

METHOD

Our work is situated in the interstices of two theoretical frameworks: one framework is specific to instruction and the other looks at the use of routines in organizations of all kinds. To keep instruction front and centre, we rely on Cohen et al.'s (2003) “instructional triangle” model (see Figure 1).

The instructional triangle, with its naming of teacher, students, content, and milieu, formed the basis of our research instruments, including the interview protocol and coding manual. Layered on that framework, we make use of Feldman and Pentland's

(2003) distinction between ostensive and performative routines. They argue that “while routines can be a source of inertia and inflexibility, they can also be an important source of flexibility and change” (p. 94). We view the use of curriculum pacing guides as a key routine in curriculum implementation in the district we studied.

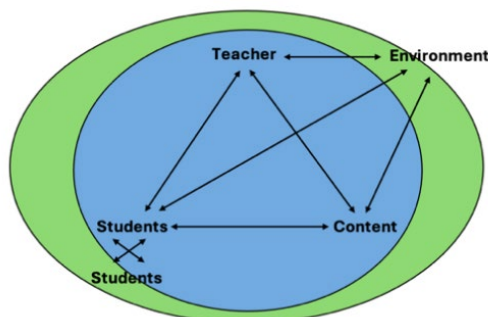


Figure 1: The instructional triangle (Cohen et al., 2003).

To investigate the research question of how teachers perceive and make use of textbook pacing guidelines in making instructional decisions, the research team designed a study drawing on two different data sources: interviews with teacher users of a pacing guide for elementary mathematics textbooks; and document analysis of written curriculum resources associated with the district’s elementary textbook pacing guides. Twenty elementary teachers participated in hour-long interviews, focusing on how teachers chose what and how to teach a mathematics lesson. The interview poses questions to the teacher about the particular lesson they taught the day of the interview, and how they decided to teach it. Over a period of six months, district documents pertaining to elementary mathematics curriculum pacing were collected by the research team. These documents include professional development plans focused on curriculum pacing guides; memos and directives sent by administrators and school-based lead teachers; and posters and other publicly displayed written communications about pacing; the pacing guides themselves.

Interview transcripts and documents pertaining to textbook pacing were analysed in a grounded theory approach using a constant comparative method (Strauss & Corbin, 1997). Initial coding was done by the research team together, to establish shared categories for coding. Subsequently, the researchers independently coded the documents in pairs. A final round of coding was then done by the entire team.

RESULTS

Document analysis revealed that the pacing guide written by district personnel is meant to be the sole basis on which teachers are to make decisions about which lessons to teach on any given day, and that teachers have strict deadlines for completing textbook units or “modules”. The pacing guide includes several “flex days” for each module, which allow teachers to catch up, revisit topics, or complete the work from the module as needed. One of the documents included in our analysis, shown in Figure 2, is a sign that teachers are to post on the outside of their classroom each day showing the module

and lesson number that they are teaching. In another school, the signs also included the mathematical topic.

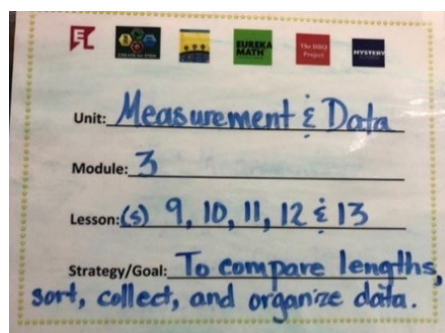


Figure 2. Curriculum pacing sign required to be hung outside of classroom.

The teachers we interviewed named a variety of factors in deciding what to teach on a given day, including time of year, time relative to testing, teacher perception of student knowledge, advice from colleagues, individualized special education plans, teacher memories of student learning from the past, the weather, messages from their building principals, and guidance from central office personnel. The curriculum pacing guide informed but did not dictate instructional decisions. One teacher relayed why he had chosen to teach coordinate graphing that day:

I picked that because it was in module 6 which is our end of the year.... because the way this year went we did not get through all our lessons all the way through the year, and based on what I saw from the kids on the [standardized test] and from their [test preparation materials] end of the year testing a lot of them were struggling with coordinate planes. [We did] a quick review of it before testing because I knew we hadn't covered much [My colleagues] all were discussing what we're ... how we're thinking about ending the year, and we got, you know, good input from each other.

Other teachers expressed frustration over the limitations imposed by the pacing guide. One shared that she felt pressure to stay on pace despite sensing that her students needed more time to fully grasp a specific topic, "Because it [the guide] doesn't give you that time for those students...if there are gaps, it just continues to plow ahead." Another teacher felt that the mandated curriculum and pacing guides, working in concert, limited her ability to bring in supplementary curriculum resources.

They [school officials] were concerned with how this could affect my pacing, and they were basically telling me that I wasn't giving the materials that I'm provided with a chance. I do get frustrated at times...because there are materials that I would much rather use.

A special education teacher made few references to the mandated curriculum or pacing guide when interviewed about her decisions around what to teach on a given day. Her perceptions of student needs drove much of her decision-making. "Well, teacher observation is a lot of it. I can tell by their body movements when they're frustrated."

Professional development in mathematics was provided by a commercial company affiliated with the publisher, and one teacher shared that "the training is about

following the pacing guide... if you go to math training, they might say, ‘Pull up the pacing guide.’”

Teacher leaders and mathematics specialists from the central office also reinforced the singularity of the pacing guide:

We also have master teachers in our building and their job is to help guide and train and make teachers better teachers. So, when they come into your room, they’ll have conversations about the pacing guide and even during like we have what’s called common planning so all the second-grade teachers on Wednesday meet together and we don’t always. But it’s brought up throughout the year, like, “Are you on pace and if you’re not, how can you get there?” Or how like, “Let’s talk.”

We conceive of teachers’ decision-making in reference to mandated curriculum pacing as a point in a matrix of external vs internal warrants along one axis, and student-centred vs document-centred along another. We show this in Figure 3 below. For example, one teacher shared in an interview that she was conflicted about moving on when students did not understand the material, but was persuaded by the district training that it was important for equity’s sake that all students be exposed to grade-level material. This would place her somewhere between the internal warrants (her own judgment of student learning, and belief that this matters) and external warrants (the district’s view about equitable learning). She would also be somewhere between a student-centred view (her noticing that students had not mastered the mathematics in the current lesson) and the curriculum-centred view (the pacing guide determined that it was time to go on to the next lesson.) We contend that teachers’ points on this matrix change over time due to context, time in the school year, with regard to mathematical content, and in response to internal and external pressures.

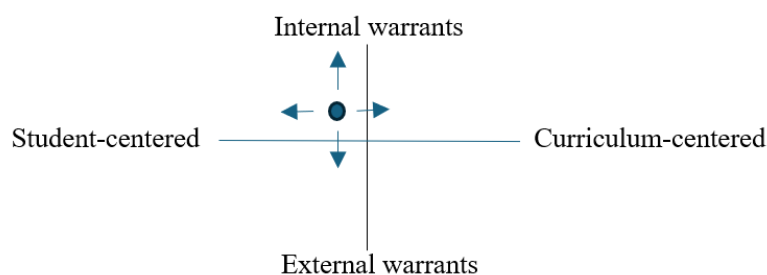


Figure 3. Teachers’ decision-making using curriculum pacing guides.

DISCUSSION

In the school district in this study, central office personnel communicate through multiple formats (professional development, master teacher official guidance, pacing guides, signage, etc.) that teachers are to follow a prescribed curriculum at a pace dictated daily. In our preliminary data analysis, we find that teachers cite myriad factors in deciding what to teach each day, described as “curricular privileging” by Bauml (2015), and that, few teachers are rarely “on pace,” despite messaging from the central administration that the pacing guide should be the sole basis on which to make such decisions. The intersection of district policy and school demographics raises a set

of questions for further investigation: Why are such strict pacing regimes more common in large, urban school districts for children of colour living in poverty? Do pacing guides reflect particular assumptions about the pedagogical design capacity of teachers who work in such districts? To what extent are such policies racialized? To what degree are the voices and experiences of teachers present in the creation of curricular decisions such as pacing, and how does such involvement or lack thereof contribute to teacher efficacy and student learning? While the field assumes that teacher education prepares teachers with the knowledge of how to fish, pacing guides give them fish. In the district we studied, the gap between teacher preparation at the university (assignments writing detailed original lesson plans) varied greatly from what teachers are expected to do once in schools (follow the curriculum scripts, at a predetermined time). Such regimes do not view teachers as “designers,” or partners in the creation of mathematics curriculum (Jones & Pepin, 2016). Our analysis prompts questions about whose knowledge counts, and is valued, in the development of mathematics curriculum. At the same time, in the districts we studied, every teacher we interviewed was using newly adopted curriculum materials, and this is no small feat. The opportunities for student learning of worthwhile mathematics may be greatly expanded when virtually all teachers in a system are using high-quality curriculum materials.

IMPLICATIONS FOR PRACTICE

Teachers look to curriculum pacing guides as one factor among many when making instructional decisions. In this study, teachers interpret such policy directives rather loosely, incorporating other factors including indicators of student performance, the school year calendar, advice from colleagues and their own memories of teaching the same mathematical topics in the past. Perhaps this is as it should be. The creation of strict pacing guides far away from the exigencies of everyday teaching practice may result in a divide between central office expectations and teachers’ implementation of policy that ultimately may not benefit student learning. Pacing guidelines were considered unrealistic and conceived of at some remove from teachers’ everyday knowledge and experience. As Feldman and Pentland (2003) suggest, teachers and district personnel make use of the routines afforded by pacing guides to both mute conflict and to open spaces for agency. “The performance of routines can also involve adapting to contexts that require either idiosyncratic or ongoing changes, and reflecting on the meaning of actions for future realities” (p. 95).

Our analyses suggest that school districts would do well to co-create pacing guides along with teachers, to set realistic timelines for completing textbook modules, to pilot the suggested pacing guide before disseminating it to teachers, to suggest the inclusion of formative and summative assessment data in teachers’ decision-making about pace, and to incorporate rather than set aside teachers’ experience and knowledge of their students in drawing up guidance for instructional decisions. The vertices, the edges, and the field of the instructional triangle (Cohen et al., 2003) could shape the creation

of pacing guides. In an era when the United States is seeing dramatic decreases in the number of people seeking teacher certification, with grave teacher shortages around the country, it is essential to consider the effects of pacing guides on teachers. These concerns must be counterbalanced with the advantages that pacing guides bring: a shared instructional purpose among teachers, supporting collaboration around content and pedagogical decisions; a sense that content is “covered” predictably; and the promise of equitable opportunities to learn, especially for children of colour living in poverty.

Effects on Students

Indicators of student learning were not prominent in the pacing guides’ rationale for instructional decision-making. In fact, in this school district, student learning is measured by a tool that is not keyed to the curriculum; teachers are instructed to use student learning measures for instructional decision-making that are not aligned with the curriculum materials or pacing guide. More than 50% of the teachers in our study did not reference student understanding or student learning when asked what they would be teaching each day and why.

We end this paper posing questions about the possible relationship between curriculum pacing that disregards student learning and the chronic absenteeism rates in this district along with low test scores. 65.8% of students in this district were chronically absent in 2023-2024 (meaning they missed more than 10% of the school year). Chronic absenteeism has been a strong focus in this school district, which district leaders and university researchers have attributed to “transportation, mental health and trauma, policy and legislation, community-based support (defined as multiple actors to help kids get to school across different sectors), student perceptions of educational quality, and systemic/institutional racism” (Lenhoff & Singer, 2022). This ecological perspective regarding chronic absenteeism minimizes the central role of day-to-day instruction. We wonder whether students’ experience in mathematics lessons that march on without regard to their learning has the cumulative effect of dampening enthusiasm for attending school. Poverty is undoubtedly a factor. Lack of reliable transportation makes attendance difficult. Our study suggests that the central routines of schooling—routines for teaching and learning—should garner more attention in discussions of chronic absenteeism, and the wise use of curriculum materials might make a difference.

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IMPACT OF INTEGRATED STEM ON PRE-SERVICE TEACHERS' MATHEMATICS LEARNING

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This preliminary study explores the design, implementation, and impact of an innovative curriculum featuring three integrated STEM modules in a mathematics content course for pre-service elementary teachers at a Hispanic-serving institution in the southern United States. The analysis highlights the modules' impact on students' learning experiences and outcomes, offering insights into how an integrated STEM curriculum enhances conceptual understanding and engagement in mathematics for future elementary educators. Although not centred on traditional textbooks, this study introduces innovative, evidence-based teaching materials that integrate STEM content, providing teacher educators with cutting-edge resources to enhance instructional practice in preservice teachers' content courses.

INTRODUCTION

Incorporating integrated STEM (Science, Technology, Engineering, and Mathematics) modules into pre-service elementary teacher education, particularly in mathematics content courses, is critical to preparing future educators for the interdisciplinary nature of STEM teaching. Research shows that STEM integration enhances pre-service teachers' content knowledge and pedagogical skills, enabling them to approach mathematics education from a broader, more applied perspective (Liang & Wang, 2023). These integrated STEM modules provide opportunities for pre-service teachers to engage in hands-on, inquiry-based learning experiences, fostering critical thinking, problem-solving, and collaboration skills, all of which are essential for effective STEM instruction in elementary classrooms (Ojeogwu & Mumba, 2025). Moreover, studies indicate that integrating STEM into teacher preparation courses improves pre-service teachers' abilities to design lessons that connect mathematics with science and engineering concepts, offering students authentic learning experiences (Becker & Park, 2011; Stohlmann et al., 2012). This alignment with real-world applications is crucial for equipping educators with the skills to engage students in solving interdisciplinary problems that extend beyond traditional classroom boundaries (Honey et al., 2014). Furthermore, integrated STEM modules allow pre-service teachers to explore the pedagogical use of technology and engineering design processes, which are essential for fostering 21st-century skills (Basham et al., 2010).

Despite the promising potential of integrated STEM education, several gaps persist in

the existing literature. First, while many studies highlight the benefits of STEM integration for K-12 students, there is limited research on its impact on pre-service teachers' learning experiences, particularly in mathematics content courses. This gap makes it difficult to identify how integrated STEM modules influence future educators' conceptual understanding and self-efficacy in teaching STEM-related content (Margot & Kettler, 2019). Second, there is a need for more empirical studies that examine the effectiveness of integrated STEM curricula in fostering connections between mathematics and other STEM fields in teacher preparation programs (Becker & Park, 2011). Additionally, few studies have explored how pre-service teachers' attitudes, beliefs, and pedagogical approaches evolve when exposed to integrated STEM modules. This lack of understanding hampers efforts to develop evidence-based strategies for teacher preparation programs to address misconceptions about STEM disciplines and promote interdisciplinary teaching approaches (Rodriguez et al., 2020). Furthermore, while integrated STEM education often emphasizes equity and inclusion, more research is needed to examine how these approaches impact preservice teachers' abilities to design equitable and culturally relevant STEM lessons (Honey et al., 2014; Rodriguez et al., 2020).

By embedding STEM modules into mathematics content courses, teacher preparation programs can address these gaps by providing pre-service teachers with opportunities to engage in active, real-world problem-solving scenarios, enhancing their confidence and self-efficacy in teaching STEM subjects (e.g., Holincheck & Galanti, 2023). This approach encourages pre-service teachers to view mathematics as part of a larger problem-solving process rather than as a set of isolated procedures, preparing them to deliver meaningful, interdisciplinary STEM lessons in their future classrooms.

By addressing these research gaps, this study aims to contribute to the growing body of literature on integrated STEM education, providing insights into its potential for transforming teacher preparation programs and preparing future educators for the demands of 21st-century classrooms. The research question guiding this study is: *How does an innovative curriculum designed with integrated STEM modules influence pre-service elementary teachers' mathematics learning?*

THREE INTEGRATED STEM MODULES AND IMPLEMENTATION

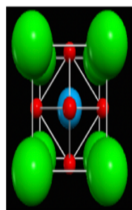
The modules in this project considered as “Integrated Mathematics-Enhanced (IME) STEM Inquiries,” are designed to enhance PSETs' mathematics learning within the contexts of science, technology, and engineering. These modules provided an engaging learning environment where PSETs can connect and apply mathematical knowledge to solve problems in science and engineering using technology.

An inquiry-based learning approach was applied throughout the implementation process. PSETs were divided into small groups, with each group consisting of three or four students working collaboratively to solve the problems. Rather than delivering lectures, the instructor acted as a guide and resource, offering support and facilitating

the learning process. Figure 1 shows some samples of materials related to the class activities for Module 1 – 3D Crystal Structure of Solids.

There are two GUI files for Ideal Perovskite Structure solids. One for Calcium Titanate (CaTiO_3), and one for Strontium Titanate (SrTiO_3)

1. Open the file **CaTiO₃ (Calcium Titanate)** by clicking on <http://cs.boisestate.edu/~mlong/GUI/catio3.html>
2. Click **Run** to see how the fcc structure of NaCl is rotated and **Pause** it to explore the rotation from different viewpoints.
3. Rotate: use **Right Button Drag** or **Ctrl-drag**
4. Zoom: Drag with the **middle Button** or use **Scroll Wheel**
5. Click **Reset** to get it back to its original position.



Explore

Stay on the file **CaTiO₃ (Calcium Titanate)**.

1. Click **Run** to see how the CaTiO_3 unit cell is rotated and **Pause** it to explore the rotation from different viewpoints. Provide two pictures of it from two different viewpoints.
2. Try to **Rotate** the CaTiO_3 unit cell, use **Right Button Drag** or **Ctrl-drag**. What type(s) of symmetry does it have?
3. Compare NaCl version 1 in **B. Cubic Structure Example** and the CaTiO_3 unit cell in **CaTiO₃ (Calcium Titanate)**. Discuss the structure and types of symmetry they have.

4. Change the Parameter for **L** (CaTiO_3 unit cell) and complete the table

L	1	2	3		n + 1?
Total Atoms					

Figure 1: The GUI app (left), Example exploration questions (right).

Since Spring 2022, we have been implementing three distinct STEM modules in our pre-service teacher mathematics content course: *3D Crystal Structure of Solids*, *Make Your Own Windmill*, and *Astronomical Distance & Constellations*. The first two modules were developed by STEM faculty at Boise State University (BSU), while the third was created by a collaborative team of Mathematics and Physics faculty at our university. Each module is grounded in the ACE (Applying, Connecting, Experiencing) model, developed by BSU faculty. This model engages pre-service elementary teachers (PSETs) by having them apply their knowledge and skills to teaching practices, connect crosscutting concepts through STEM inquiries, and experience community-based experiential learning. Ultimately, the goal is to enhance PSETs' attitudes and beliefs about mathematics teaching and learning.

Module 1: 3D Crystal Structure of Solids: Visualization and Printing introduces PSETs to key geometric concepts in science and engineering through the use of innovative digital resources. The module is built around five Graphical Users Interface (GUI) applications, that serve as interactive learning tools, enabling PSETs to explore crystal structures such as primitive cubic (cp), face-centered cubic (fcc), and perovskite, found in materials such as salt (fcc), silver chloride (fcc), Calcium Titanate (perovskite), Strontium Titanate (perovskite) and cutting-edge solar cells. These structures are chosen for their symmetry, relevance to course content, and real-world importance. Complementary lab worksheets further guide PSETs in applying these concepts, designing lattice models, and using 3D printing technology.

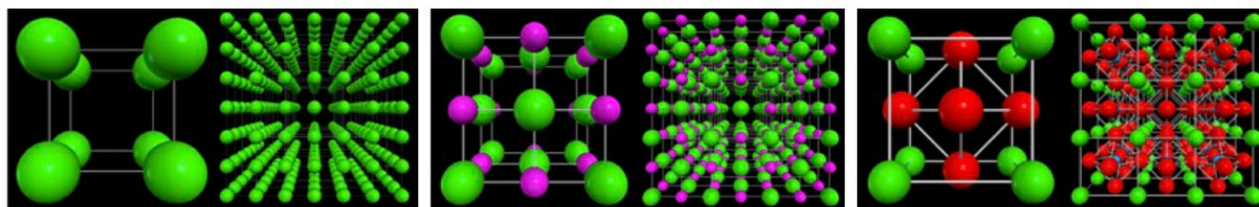


Figure 2: Primitive Cubic (left), Face-Centered Cubic (middle), perovskite (right).

Module 2: *Make Your Own Windmill* engages PSETs in designing and testing windmill blades to explore how variations in blade design influence power output. Guided by structured in-class worksheets, PSETs work collaboratively in small groups to construct blades using materials such as paper, cardboard, and wood. Throughout the activity, they measure blade length, compute surface area, determine the number of blades, and set appropriate angles before testing the power generated by their windmills. When a design fails to spin, students troubleshoot by adjusting blade angle, size, or material to improve performance. The module encourages inquiry through questions such as “*What blade shapes and lengths work best at different wind speeds?*”—allowing PSETs to integrate mathematical measurement, scientific reasoning, and engineering design within an authentic, hands-on STEM learning experience.



Figure 4: Making blades (left), placing blades (middle), testing blades (right).

Module 3: *3D Astronomical Distance* engages PSETs in creating a scaled model of the solar system, exploring planetary sizes and distances in our solar system, and perspective with the stars of constellations. PSETs are guided to use a free, 3D-printing online design resource - Tinkercad to model planetary distances by applying mathematical concepts such as scaling, unit conversion, symmetry, and the x, y, z axes. In addition to the virtual model, physical modelling is conducted both in the classroom and outdoors using butcher paper and coloured foam balls. Through this hands-on process, PSETs explore the spatial relationships between celestial bodies using geometric concepts.

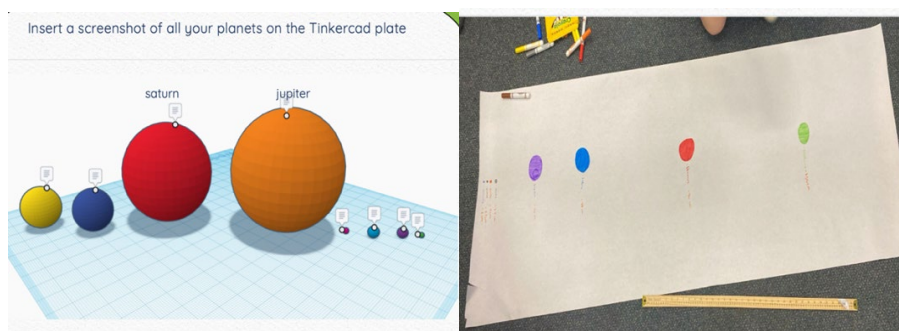


Figure 4: Modelling by Tinkercad (left), modelling on the classroom floor (right).

The three integrated STEM modules have been implemented in different semesters since Spring 2022 in the PSETs content course. See Table 1 below.

Academic Semester	STEM Module	Number of PSETs Enrolment
Spring 2022	3D Crystal Solids	22
Fall 2022	3D Crystal Solids	23
Spring 2023	Windmill	10
Fall 2023	Windmill	11
Spring 2024	Astronomical Distance	33
Fall 2024	Astronomical Distance	12
Total	3 Models	111

Table 1: Overview of STEM modules implementation.

Each module is taught over four 75-minute class periods each semester. After completing the module in the PSETs mathematics content course, the PSETs apply what they have learned to design activities for local elementary school students as part of after-school programs, which run for three 75-minute sessions.

METHODOLOGY

The participants and Data collection

Students enrolled in the PSETs mathematics content course were recruited to participate in this study. Most of them were juniors, while some were sophomores or seniors planning to teach in elementary schools after graduation. Three instructors with PhDs in mathematics education participated in the study. Two of them implemented the STEM modules in their course in Spring 2022 and Spring 2024, respectively, while the third instructor conducted the implementation from Fall 2022 to Fall 2024. All three instructors had experience teaching mathematics at either the high school or elementary school level.

Pre- and post-surveys were administered through *Qualtrics* to assess potential changes in PSETs' perceptions of STEM learning and teaching. The survey instruments were designed to capture shifts in attitudes, confidence, and understanding resulting from participation in the integrated STEM modules. Pre- and post-Cognition-Based Assessments (CBA) were administered in class using paper and pencil, assessing geometric concepts embedded in the modules, asking questions like "Describe the method(s) for describing estimating the area of the blade" (a figure illustrating the design for a windmill blade was provided). Each semester, four PSETs were recruited to participate in interviews at the beginning and end of the semester. The interviews were conducted to assess changes in PSETs' understanding of geometric concepts such as surface area, symmetry, and rotation in three-dimensional space. During the interview, participants were asked questions like: "*The rectangular prisms are similar.*

What is the surface area of the larger prism if the surface area of the smaller prism is 64 square yards?" Two rectangular prisms were provided, with side lengths of 9 yards and 6 yards, respectively. These students also formed a focus group during class discussions, which were observed and video recorded. The instructors participated in 30-minute interviews at the end of each semester. Additionally, documents created to support project activities – such as PSETs' critical reflections, facilitation plans, and course evaluations – were collected and included in the analyses. Instructors' evaluations and interviews provided feedback on the curriculum design, the implementation of the ACE model, triangulating the findings generated from other collected data.

Data Analysis

A mixed-methods approach was employed to address the research question, combining both quantitative and qualitative data. The preliminary analysis drew on various data sources, including pre- and post-surveys, cognition-based assessments (CBA), pre- and post-critical reflections, and instructor interviews. These data were used to assess changes in PSETs' mathematical learning. By triangulation these data, the study aims to provide a comprehensive view of how the integrated STEM model impacts PSETs' learning experiences. Future research will involve a more in-depth and comprehensive analysis of the classroom videos that captured the implementation process of the integrated STEM modules. This analysis will provide richer insights into how PSETs engaged in collaborative problem-solving, applied STEM concepts in real time, and translated theoretical understanding into practice.

RESULTS

The data analysis revealed several converging themes regarding the impact of the integrated STEM modules on PSETs' learning experiences. First, PSETs found that connecting mathematics concepts to engineering and technology helped them contextualize and deepen their understanding of these concepts in a more integrated and applicable way. This aligns with constructivist learning theory (Wibowo et al., 2025), which emphasizes that learners build knowledge more effectively when concepts are embedded in meaningful and authentic contexts. Second, PSETs expressed appreciation for the learning experience. One PSET wrote in the reflection:

...the STEM module was very interesting to me and they even aligned to the eclipse we saw. By doing this, I felt like I was more engaged because it was fun and interesting. Connecting math or science to an area of interest can benefit the students by allowing them to learn while also enjoy learning. This can also spark their interest to expand their thinking and learn more on their own.

They agreed that the modules were highly engaging and gained valuable insights into designing similarly engaging integrated STEM activities for their future classrooms. This reinforces situated learning theory, which holds that learning is most effective when it occurs in authentic, practice-based settings (Hamilton & Margot, 2023).

PSETs noted that the modules were highly engaging and motivating, fostering active involvement in learning. They indicated that the integrated STEM activities supported a deeper understanding of mathematical ideas, including scaling, scale factor, unit conversion, angles, symmetry, rotation, and similarity. Additionally, some PSETs realized that emphasizing teamwork in the modules was very helpful. Quoting one PSET's reflection:

We were always working in groups, and we would help each other with any questions or confusions we had based on the lesson we were learning that day and I found that extremely helpful.

DISCUSSION AND CONCLUSION

This preliminary study indicates that the integrated STEM modules effectively engaged PSETs in learning mathematics within contexts that integrate science, technology, and engineering. The findings suggest that PSETs demonstrated positive learning outcomes, including an improved understanding of three-dimensional geometric concepts (e.g., angles, surface area, rotation, and symmetry), increased awareness of mathematical applications in STEM fields, and enhanced ability and confidence to design integrated STEM lessons for future classrooms. Data analysis is ongoing. The next phase will involve a detailed examination of classroom videos and in-class worksheets to investigate how the integrated STEM modules were enacted and how these enactments facilitated PSETs' engagement in learning mathematical concepts and their related STEM applications—particularly in comparison with traditional textbook-based instruction. This forthcoming analysis will provide a richer account of how integrated STEM instruction supports conceptual understanding through classroom discourse and hands-on exploration. Ultimately, these findings will contribute to the broader conversation on STEM education reform by offering evidence-based insights that inform curriculum design and promote innovative instructional materials for teacher preparation programs.

One of the most significant challenges in STEM education is the lack of appropriate assessment instruments that effectively measure students' understanding and application of STEM concepts (Honey et al., 2014; Gao, et al., 2020). Through the completion of our data analysis, we hope to develop improved assessment items tailored specifically to integrated STEM modules, which will better capture the depth of student learning and engagement in STEM-related content.

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OPPORTUNITIES TO DEVELOP CRITICAL THINKING SKILLS IN CHINESE AND SINGAPOREAN 1ST GRADE MATHEMATICS TEXTBOOKS

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This paper examines opportunities for developing critical thinking (CT) skills in first-grade mathematics textbooks from Singapore and China. Using the Delphi Report framework, we conducted a quantitative content analysis of Targeting Mathematics 1A and New Century Mathematics Grade 1, Part 1. The findings indicate that both textbooks emphasize interpretation but give limited attention to higher-order skills like evaluation and self-regulation. New Century Mathematics incorporates more CT elements in “Statistics and Probability,” while “Graphic and Geometry” is the most CT-rich domain in Targeting Mathematics. Additionally, Targeting Mathematics offers a broader range of CT skills and more frequent analysis opportunities across content areas compared to New Century Mathematics.

INTRODUCTION

Mathematics education plays a central role in cultivating students’ problem-solving abilities and critical thinking (CT), a key 21st-century competency (Monteleone et al., 2023). Textbooks, as the primary teaching resource, strongly influence learning by structuring knowledge, shaping task design, and creating opportunities for reasoning (Chegini et al., 2023). While research on CT in science and language textbooks has expanded, comparative investigations of mathematics textbooks and their potential to foster CT remain limited. Early exposure is particularly important, as research indicates that CT can emerge as early as age three (Hübscher et al., 2016), and the lower grades of primary school mark the formative stage of children’s logical and abstract thinking. Within this context, Grade 1 mathematics textbooks hold special significance, as they represent students’ first sustained encounter with formal mathematical reasoning. When designed to incorporate opportunities for questioning, justification, and critique, these textbooks can lay a solid foundation for children’s long-term cognitive growth and foster resilience in learning.

Singapore and China offer a meaningful context for comparison. Both nations’ most recent curriculum frameworks—the Mathematics Syllabus Primary One to Six (Ministry of Education, Singapore, 2025) and the Compulsory Education Mathematics Curriculum Standards (Ministry of Education of the People’s Republic of China, 2022)—explicitly highlight CT development. Yet, despite similar cultural traditions,

Singaporean students consistently outperform their Chinese peers in problem-solving and mathematical literacy (OECD, 2019). This difference may be linked to variations in curriculum design, learning objectives, and the structure of textbooks (Leong et al., 2019; Toh, 2021). A comparative analysis of Grade 1 mathematics textbooks from the two countries is therefore both timely and essential for improving future textbook design and enhancing opportunities for CT development.

LITERATURE REVIEW

Critical thinking

CT has been defined from philosophical, psychological, and educational perspectives, highlighting its richness and complexity (Lai, 2011). Dewey (1933) described it as “the active, persistent, and careful consideration of a belief or supposed form of knowledge in light of the grounds which support it.” Facione’s Delphi Report (1990) offered a widely accepted definition of CT as “purposeful, self-regulatory judgment” involving interpretation, analysis, evaluation, and inference, which underpins this study. While a fixed definition of CT remains elusive, scholars generally agree that CT includes both skills and dispositions with skills forming the foundation for applying CT to problem-solving. In this context, tasks in Grade 1 mathematics textbooks—such as open-ended problems with multiple solutions—are particularly valuable for fostering students’ CT skills.

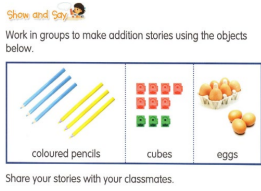
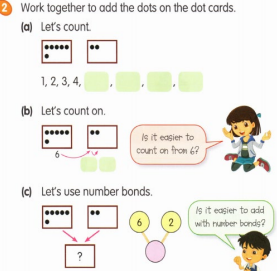
Comparative studies of Singaporean and Chinese mathematics textbooks have examined content knowledge domains, design, difficulty levels, and mathematical literacy (e.g., Son et al., 2017; Yang et al., 2024), but few address CT explicitly. Research in other contexts shows similar trends: fifth-grade textbooks in Iran emphasized questioning over analysis (Samiee Zafarghandi et al., 2020), while Chegini et al. (2023) found CT components accounted for only about a quarter of content in Japanese and Iranian textbooks. Studies in other subjects likewise reveal limited attention to higher-order CT skills (Oikonomidis, 2018). Overall, comparative research on CT in mathematics textbooks—particularly between Singapore and China—remains scarce, with most studies highlighting an overemphasis on questioning and interpretation and insufficient focus on analysis, evaluation, and self-regulation.

Establishment of analytic framework

Building on these findings, it is important to note that researchers such as Ennis (1985), Facione (1990), Ilyas (2015), and Lipman (2003) have proposed various frameworks for CT skills, which have been widely applied in textbook analysis. Collectively, these frameworks emphasize a range of interrelated sub-skills that contribute to effective reasoning. Ennis (1985) focuses on practical reasoning skills, including providing explanations, building foundational skills, drawing conclusions, applying strategies and tactics, questioning, and summarizing information effectively. Facione (1990) identifies key skills such as interpretation, analysis, evaluation, inference, explanation, and self-regulation, providing a developmental perspective from basic comprehension

to advanced reasoning. Ilyas (2015) emphasizes clarification, identifying assumptions, considering perspectives, generating alternatives, and making predictions, which support reflective and forward-looking thinking. Lipman (2003) highlights the importance of questioning, evaluating evidence, reasoning logically, and exercising judgment in analysing issues, as well as the social and reflective dimensions of thinking.

This study applies Facione's Delphi Report (1990) as the framework for CT skills for several reasons. First, it has achieved broad interdisciplinary consensus, making it an objective criterion for analyzing mathematics textbooks. Second, its detailed descriptions and examples make it practical for application. Third, it captures the comprehensive and developmental nature of CT, ranging from basic skills such as interpretation and analysis to advanced skills like inference and self-regulation. Finally, it highlights the interactive, non-linear process of CT, where sub-skills work together to enable flexible reasoning and well-supported conclusions. The analytical framework used in this study is shown in Table 1.

Sub-skills	Sub-Dimensions	Description	Examples
Interpretation	Categorization, Decoding significance, Clarifying meaning.	• Classify elements based on specific rules, symbols, or sequences. • Identify critical elements of the problem-solving process. • Understand and express the meaning of various conditions, data, information, displays, symbols, rules, processes, or standards. • Analyze similarities and differences related to concepts or displays.	
Analysis	Examining ideas, Identifying arguments, Analyzing arguments.	• Judge whether the reasoning supports a given argument. • Find the reasoning supporting a particular conclusion.	
Evaluation	Assessing claims, Assessing arguments.	• Identify the credibility of information sources, evidence, and conclusions. • Assess the reasonableness of an assumption or hypothesis.	

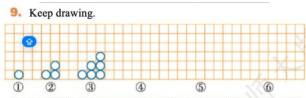
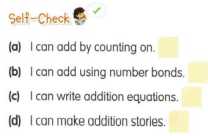
Inference	Querying evidence, Conjecturing alternatives, Drawing conclusions.	• Judge whether the reasoning aligns with the situation, problem, or context. • Evaluate the pros and cons of different solutions. • Reason based on evidence. • Identify key information needed for the reasoning process. • Use appropriate methods to solve problems. • Draw conclusions based on evidence and reasoning. • Explain the process or reasoning that led to a particular conclusion.	
		• Describe or analyze the process of reasoning. • Explain strategies for solving problems. • Provide evidence supporting or opposing a conclusion. • Evaluate one's own viewpoint or reasoning.	
		• When errors are identified through self-examination, make corrections or adjustments and try to identify the cause of the error.	

Table 1: Analytical framework in the study.

Research questions

This study aims to compare the opportunities for developing CT skills in primary school mathematics textbooks in Singapore and China, focusing on both individual sub-skills and their interactive nature within the learning process. Specifically, the study will address the following research questions: (1) What are the similarities and differences between Singapore’s and China’s 1st grade mathematics textbooks in opportunities to develop CT skills? (2) How do these textbooks present opportunities for developing CT skills across different mathematical content areas?

METHODOLOGY

In Singapore, *Targeting Mathematics (TM)* textbooks (Chan Chun & Cole, 2009) are widely used, emphasizing problem-solving and task-driven learning. In China,

following the implementation of new curriculum standards, the latest edition of primary school mathematics textbooks was introduced in the fall semester of 2024. *New Century Mathematics (NCM)* textbooks (Liu, 2024) were updated, significantly impacting primary mathematics education. This study examines *TM 1A* from Singapore (2009) and *NCM Grade 1, Part 1* from China (2024).

In this study, three experts coded the textbooks: a second-year PhD student in mathematics education, a faculty member with a master's degree in mathematics education, and a math teacher with three years of experience, all of whom contributed to the New Century Mathematics (NCM) textbooks. First, tasks and questions were counted page by page, with related sub-items considered as a single task. Using a quantitative content analysis, each task was analyzed for CT skills and sub-skills, coded as 1 if present and 0 if absent. Tasks involving multiple sub-skills were coded for each relevant skill, meaning the total percentage for each skill may exceed 100%. Prior to formal coding, the researchers discussed the framework to ensure consistency, followed by a preliminary independent coding of the first three units, achieving a Kappa coefficient of 0.90, indicating high inter-rater reliability. During formal coding, any discrepancies were resolved through discussion, and a third researcher reviewed these cases to establish a unified coding scheme. SPSS 29.0 was used to analyse the coding results, yielding a Kappa coefficient of 0.86 for CT skills and at least 0.8 for each sub-skill, indicating strong reliability. Discrepancies were discussed until 100% agreement was reached.

RESULTS AND DISCUSSION

Opportunities for developing CT skills of NCM and TM textbook

NCM textbook has more active units (251) than TM (188). As shown in Table 2, of these, the 116 units (46.2%) in NCM and 156 units (83.0%) in TM are coded for CT skills, indicating that TM offers a broader range of CT skill opportunities.

Components	NCM textbook		TM textbook	
	Frequency	Percentage	Frequency	Percentage
CT	116	46.2%	156	83.0%
Interpretation	68	27.1%	124	66.0%
Analysis	21	8.4%	66	35.1%
Evaluation	7	2.8%	13	6.9%
Inference	14	5.6%	15	8.0%
Explanation	41	16.3%	35	18.6%
Self-regulation	27	10.8%	19	10.1%

Table 2: Comparison of opportunities to develop CT skills in NCM and TM textbook.

Both textbooks prioritize interpretation skills, with 68 interpretation-related units (27.1%) in NCM and 124 (66.0%) in TM, while evaluation skills are minimally present, with only 7 cases (2.8%) in NCM and 13 (6.9%) in TM. This suggests a common focus on interpretation and limited emphasis on evaluation skills. The textbooks also differ in their CT skills distribution. NCM assigns a notable proportion to explanation (41 cases, 16.3%), whereas TM emphasizes analysis (66 cases, 35.1%). These patterns align with first graders' cognitive development stage, as they are transitioning from Piaget's "preoperational" to "concrete operational" stages, where interpretation, analysis, and explanation are more accessible. Higher-order skills like evaluation and self-regulation, which require abstract thinking, are more suitable for later grades as cognitive abilities continue to develop.

Opportunities for developing CT skills across mathematical content areas

Table 3 highlights both shared emphases and distinct approaches to CT skills development across content areas in the NCM and TM textbooks. In NCM, the "Statistics and Probability" domain shows the highest CT integration, with 94% of tasks coded as CT-related, emphasizing interpretation (78%) and explanation (44%). This suggests a focus on data comprehension and justification within this area. Additionally, the "Comprehensive Practice" section in NCM has a high CT integration rate (62%), possibly aimed at reinforcing multiple CT dimensions.

In contrast, TM's most CT-intensive domain is "Graphic and Geometry", where 95% of tasks are CT-related, with a strong focus on interpretation (75%) and analysis (39%). This reflects an approach encouraging students to identify patterns and relationships in geometrical contexts. TM also places a greater emphasis on analysis in various areas, particularly "Number and Algebra" (34%) and "Graphic and Geometry" (39%), indicating a focus on developing analytical thinking skills.

	NCM textbook				TM textbook		
	N&A	G&G	S&P	CP	N&A	G&G	S&P
Tasks	186	30	18	21	145	44	5
CT	66 (35%)	21 (70%)	17 (94%)	13 (62%)	114 (79%)	42 (95%)	4 (80%)
Interpre- tation	41(22%)	13(43%)	14(78%)	1(5%)	91(63%)	33(75%)	3(60%)
Analysis	17(9%)	4(13%)	0(0%)	0(0%)	49(34%)	17(39%)	0(0%)
Evalua- tion	2(1%)	3(10%)	0(0%)	2(10%)	10(7%)	3(7%)	0(0%)
Inference	9(5%)	5(17%)	0(0%)	0(0%)	2(1%)	13(30%)	0(0%)
Explana- tion	30(16%)	2(7%)	8(44%)	2(10%)	33(23%)	2(5%)	0(0%)
Self- regulation	12(6%)	4(13%)	1(6%)	10(48%)	15(10%)	4(9%)	1(20%)

Note: N&A=Number and Algebra; G&G=Graphics and Geometry; S&P=Statistics and Probability; CP=Comprehensive Practice.

Table 3: Comparison of opportunities to develop CT skills in first-grade NCM textbook and TM textbook across mathematical content areas.

Overall, TM appears more comprehensive in fostering CT skills, especially in analytical thinking, reflecting a potential focus in the Singaporean curriculum on early analytical development to support complex problem-solving. Meanwhile, NCM's focus on interpretation and explanation in "Statistics and Probability" may indicate an instructional priority in helping students interpret data within real-world contexts.

SUMMARY AND CONCLUSIONS

This comparative analysis of first-grade mathematics textbooks from Singapore and China shows that both emphasize interpretation as a foundational skill while limiting higher-order CT skills like evaluation and self-regulation, which aligns with the findings of previous studies (Samiee Zafarghandi et al., 2020). The NCM textbook exhibits a particularly high integration of CT skills within the "Statistics and Probability" domain, whereas the TM textbook's most CT-intensive area is "Graphic and Geometry." Notably, the TM textbook offers a broader range of CT skills opportunities, with a higher proportion of CT-coded content overall and a stronger emphasis on analysis, especially in the "Graphic and Geometry" and "Number and Algebra" domains. These results highlight the role of curriculum design in shaping early critical thinking skills, suggesting that the Singaporean curriculum may support a more comprehensive foundation for analytical thinking from a young age. Future studies could investigate how these curriculum differences influence students' long-term development of critical thinking skills.

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CRYPTOGRAPHY, TASKS, AND COGNITIVE DEMANDS

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Cryptography serves as an interdisciplinary context in mathematics education, offering opportunities to integrate mathematical concepts with real-world applications. This study analyses the cognitive demand of 24 cryptography-related tasks from two Brazilian textbooks, focusing on their classification as Low or High cognitive demand. Using an established framework for task analysis, we found that only 8% of tasks (2 out of 24) met the criteria for High cognitive demand, while the remaining 92% were classified as Low level. These results suggest that the potential of cryptography to promote critical thinking and problem-solving is underutilized in current textbooks. Our findings highlight the need to redesign cryptography-related tasks to foster deeper mathematical engagement and higher-order thinking skills.

INTRODUCTION

This study investigates the level of cognitive demand of cryptography-related tasks in Brazilian textbooks. The role of textbooks in the teaching and learning process of mathematics is widely recognized in the educational literature, as they are fundamental resources for planning and developing classes. These materials organise content and tasks that guide teaching and learning (Amaral et al., 2022). The planning of a class, as well as its development, involves several moments, among which the work with the tasks stands out. Regardless of the teaching methodologies adopted or the materials used, tasks are always present in the school setting (Jesus & Naggy, 2014).

Tasks, according to Stein et al. (1996), are activities whose objective is to focus students' attention on a given content, which makes them "important vehicles for building [...] mathematical thinking and reasoning" (p. 455). The way tasks are structured and presented in textbooks directly influences both the organization of classes by teachers and the way students perceive and learn mathematics (Smith & Stein, 1998). Textbooks are often the main guides for teacher planning and the course of classroom activities (Amaral et al., 2022; Matic, 2019). These materials, widely used in the school environment, offer tasks that belong to and are part of their structure. Litoldo (2021, p. 101) defines tasks in textbooks as "any and all types of proposals, offered by these materials, to be solved by the student", and these tasks can be analysed in terms of their cognitive demands. The definition of tasks in textbooks does not refer to specific types, such as problem-solving or application tasks, but only characterizes

what is considered a task in this context. In this sense, the analysis of tasks in mathematics textbooks has been a recurring research theme. For example, Henningsen and Stein (1997) proposed a structure that relates tasks to students' learning outcomes, from curricular materials to classroom implementation. One of the central aspects of this process is the cognitive demand of the proposed tasks, classified into Low levels (memorization and procedures without connections) and High levels (procedures with connections and 'doing mathematics').

Cryptography emerges as an interdisciplinary area of relevance in mathematics teaching, involving concepts such as algebra and number theory (Rizvi & Wadhwa, 2010), which can offer a rich field for tasks with high cognitive demand. Singh (2000) points out that cryptography can promote mathematical thinking, as it requires logical and abstract reasoning. Despite this potential, Litoldo and Lazari (2014) highlighted the presence of cryptography in textbooks, but noted that it is often limited to curiosities, superficial examples, or applications in some sections, without exploring its mathematical depth. For example, a cryptographic task may require only the application of a simple substitution cipher (Low cognitive demand) or, at a more complex level, ask students to identify patterns in encrypted messages, recognize the substitution rule, establish a functional relationship, and generalize the function used to perform the cipher (Litoldo, 2016). Thus, cryptography does not inherently guarantee High cognitive demand; it depends on how the tasks are designed and presented in textbooks.

In our literature review, we did not find discussions that connect textbook tasks with cognitive demand levels. This gap reinforces the need to articulate the three pillars – tasks in textbooks, cognitive demand, and cryptography – to understand how educational resources can be optimized to provide learning that addresses themes related to the contemporary world, linked to High levels of cognitive demand. With the growing emphasis on interdisciplinary learning in education and ongoing reflection on digital security, examining the cognitive demands of cryptography-related tasks can provide valuable insights into how textbooks can better support students in mastering mathematical concepts and real-world applications. Therefore, the research question guiding this study is: What is the level of cognitive demand of cryptography-related tasks in Brazilian textbooks?

COGNITIVE DEMAND

According to Doyle (1988, p. 170), the cognitive level of a task “refers to the cognitive processes students are required to use in accomplishing it”. In the context of textbooks, this concept is related to the types of cognitive processes that students are expected to mobilize to solve the proposed tasks. As stated by Litoldo (2021, p. 118), the cognitive demand in these materials corresponds to the “types of cognitive processes expected to be mobilized and, in most cases, revealed by students to solve it”. In this way, when elaborating the tasks in textbooks, the authors presume the mobilization of certain

thought processes, aligned with the mathematical and pedagogical objectives for which the tasks were designed (Litoldo, 2021).

Doyle (1988) differentiates thought processes into two levels: tasks that require simpler processes, such as the reproduction of previously learned content, and those that require more complex thinking, involving the understanding and connection between different contents. Thus, the level of cognitive demand varies from tasks that engage students on a superficial level to those that engage them on a deeper level, requiring interpretation, flexibility, and meaning-making (Stein et al., 1996). In the case of mathematical tasks, this distinction is observed between those that require the direct application of definitions, formulas, or algorithms and those that require more complex thinking, whose solution does not come from already established procedures (Doyle, 1988; Stein et al., 1996). Between these two extremes, there is a multiplicity of tasks that combine thought processes in different ways, with the aim of achieving different educational goals. Through a task-analysis guide, Smith and Stein (1998) sought to structure and classify the levels of cognitive demand. According to the authors, the task-analysis guide

consists of a listing of the characteristics of tasks at each level of cognitive demand. It serves as a judgment template – a kind of scoring rubric – that can be applied to all kinds of mathematical tasks, allowing a rating of the tasks (Smith & Stein, 1998, p. 345).

This guide is structured with the following levels of cognitive demand: Lower-level demands (memorization); Lower-level demands (procedures without connections); Higher-level demands (procedures with connections); and Higher-level demands (doing mathematics) (Smith & Stein, 1998). Research indicates that working with tasks with high cognitive demands is associated with improved student learning (Stein & Lane, 1996). However, Benítez (2016) notes that not all tasks need to be cognitively challenging, since the cognitive demand must be aligned with the mathematical objective to be achieved. Thus, Smith and colleagues (2004) state that different tasks are designed to develop different mathematical objectives, which implies the need for different levels and types of thinking. Low-demand cognitive tasks enable the development of certain types of thinking, while high-demand tasks promote a diverse set of opportunities for cognitive development (e.g., Stein & Smith, 1998; Smith et al., 2004). Carrillo and colleagues (2013, p. 789) highlight that “the resolution of different cognitive demand areas presents opportunities to activate different cognitive processes”. Therefore, the development of classroom activities that contemplate a diversity of tasks, with different levels of cognitive demand, leads to the maturation of students’ conceptions about the nature of mathematics (Stein & Smith, 1998). Thus, it is essential that the teaching of mathematics contemplates a plurality of these levels, allowing for broader and deeper learning. Despite this, we did not find in the literature any relationship between the approach to mathematical concepts through cryptography and their analysis based on cognitive demands. Thus, we understand that there is a research gap in this field, and this work offers a first look at this discussion.

METHODOLOGICAL STRUCTURE

The work presented here is part of a broader study that aims to investigate the presence of cryptography in Brazilian textbooks (Litoldo & Guimarães, 2023). The textbooks selected for this research were those approved by the National Textbook Program (PNLD) in 2021, related to specific textbooks of “Applied Human and Social Sciences in dialogue with Mathematics”. Such textbooks, as required by the PNLD, were expected to include an interdisciplinary approach. It is therefore considered that cryptography, by its very nature, enables connections with different fields, such as history and mathematics, and could thus be explored in these materials designed for interdisciplinarity. These textbooks must be used throughout high school in Brazil. Of the 10 approved textbooks, it was possible to access nine of them, which were analysed for the presence of cryptography in their contents. Of these, only two presented content related to cryptography, becoming the objects of study of this research. Due to the data demand, the study was divided into three investigations.

The first investigation focused exclusively on the analysis of the panel, which refers to the concepts, examples, and solved exercises (Amaral et al., 2022), and brought together the choice of textbooks and their presentations (Litoldo & Guimarães, 2023). The second investigation, which is still in progress, examines various characteristics of the tasks presented in the sections of the textbooks that deal with cryptography. The third investigation, which is presented in this text, has as its research problem the investigation of the level of cognitive demand of cryptography-related tasks found in textbooks. Thus, in line with the two previous investigations, this study adopts a qualitative, documentary approach (Gil, 2002), considering textbooks as documents of analysis. The choice for a qualitative analysis is justified by the research objective and its necessary articulation with the adopted theoretical framework. We will refer to the textbooks as Textbook 1 (T1) and Textbook 2 (T2), corresponding to ‘Dimensões’ (Selke et al., 2020) and ‘Moderna Plus’ (Paiva et al., 2020), respectively. The characteristic of the task related to cognitive demand was analysed through its different levels: Low – Memorization and Procedures without connections, and High – Procedures with connections and Doing mathematics.

It is important to emphasize that all tasks related to cryptography were selected. For data production and analysis, each task was solved, and the solutions were discussed by the authors using the cognitive demand framework proposed by Smith and Stein (1998) to ensure the consistency of the assigned classifications. Thus, all relevant content was carefully examined. As highlighted by Litoldo (2021), this careful approach is essential, since levels of cognitive demand may decline when considering only what the textbook proposes in relation to the presented task. For the organization of classifications and the construction of the graph that systematizes their distribution, Microsoft Excel was used. There was no coding scheme and no divergences regarding their classifications. Finally, regarding ethical considerations, we emphasize that all textbooks are public, and there was no need for anonymity.

PRESENTATION AND INFERENCE OF DATA

Based on the classifications and descriptors proposed by Smith and Stein (1998), the data in this analysis derive from 13 T1 tasks and 11 T2 tasks. It is important to mention that not all the tasks in the analysed sections were included in this analysis (19 in T1 and 17 in T2), since due to the theme being cryptography, other contents do not correspond and/or may not include the area of mathematics. Therefore, these were disregarded in this text. Consequently, the 24 tasks were classified into one of the four types of cognitive demand mentioned. Figure 1 summarizes the quantified data.

The data (Figure 1) in both books present results that show the level of cognitive demand favoured by the authors – low – and, consequently, which types of demand are incorporated in the chapter/section of the books that deal with cryptography. These types are concentrated in those that deal with the memorization and direct application of rules, formulas, basic facts, and procedures without connections that underlie the content studied. In a very small way, the high level of cognitive demand is contemplated in only one of the textbooks (T1), which is of the type of procedures with connections. Thus, the predominance is low cognitive demand, corresponding to 11 tasks in T1 and 11 tasks in T2. With a marked discrepancy, the rest of the tasks, i.e., 2 in T1 and 0 in T2, were classified as having high cognitive demand.

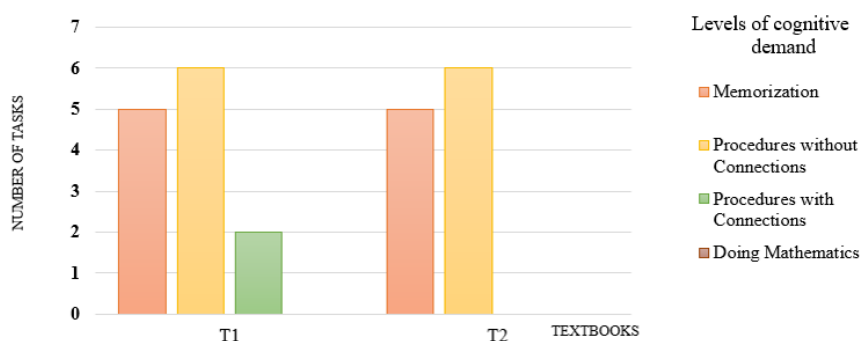


Figure 1: Quantification of the levels of cognitive demand of the tasks.

This distribution could reflect a limited approach to the development of students' critical thinking and mathematical thinking skills, since tasks with low cognitive demand tend not to promote deep interpretation, cognitive flexibility, and the construction of meanings, which are essential aspects for the formation of more complex mathematical skills, as discussed by Smith and Stein (1998) and Stein and Lane (1996). Thus, the predominance of low cognitive demand tasks may impact broader learning objectives in mathematics teaching, especially in an interdisciplinary field such as cryptography, because these skills are being limited.

The results of this analysis point in a direction contrary to what is debated and recommended in the literature regarding diverse work with levels of cognitive demand (Stein & Smith, 1998). It is important to recognize and legitimize the contributions that one has to the learning process when the student is given the opportunity to perform a

plurality of cognitively different tasks, since the level of cognitive complexity of a task is directly concatenated to the proposed mathematical objective (Benítez, 2016). Therefore, tasks with different mathematical objectives require different levels and types of thinking from the student (Smith et al., 2004). Unfortunately, due to the page limit, we will not present examples of the levels of cognitive demand and their respective types identified in our analysis. These examples, as well as the mathematical discussions related to them, will be addressed in an article resulting from the expansion of this communication.

RESULTS AND FINAL CONSIDERATIONS

The results presented in this investigation provide an overview of the level of cognitive demand of cryptography-related tasks found in textbooks, revealing a predominance of tasks classified as low cognitive demand. The analysis of the tasks showed that, in both textbooks analysed, most of the activities required students only to memorize or apply procedures without connections, while a small portion was classified as having high cognitive demand. One of the textbooks did not include any tasks within this classification.

Although cryptography has been developing as an interdisciplinary context in mathematics education, offering opportunities to integrate mathematical concepts with real-world applications, the tasks related to this theme in the analysed textbooks are predominantly of low cognitive demand. This limits opportunities to develop more complex mathematical skills in students, such as critical thinking and problem-solving that involves the application of multiple interconnected concepts.

When compared to similar studies in mathematics education that focused on other topics, the predominance of low-level tasks was also evident (Litoldo, 2021). What distinguishes this study is its focus on cryptography, which we regard as having strong interdisciplinary potential, although previous analyses (Litoldo & Lazari, 2014) have shown that it has been scarcely explored in textbooks. The results reported here may contribute to curriculum design by highlighting what has been presented in these materials and by suggesting that opportunities for high-level cognitive thinking should be explored as much as those of lower demand, as the literature on the plurality of such levels suggests.

It is not possible to determine why low-level tasks predominated in the analysed textbooks. Nevertheless, we consider it relevant to bring these results into the discussion, in order to raise awareness among researchers, authors, teachers, and students, among others, about this theoretical framework and to encourage its consideration in the process of designing tasks in these materials.

Finally, this research opens paths for future investigations. New studies could compare the treatment of cryptography with other interdisciplinary themes in textbooks, analyse how teachers mediate the cognitive demand of these tasks in practice, and explore

whether the integration of cryptography contributes effectively to the development of students' mathematical thinking.

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STRUCTURAL-ORIENTED DESIGN OF MATHEMATICS TEXTBOOK

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Textbooks serve as a bridge connecting the ideal curriculum with the implemented curriculum, acting as a crucial vehicle for transforming curricular ideas into practice. Many countries emphasize big ideas in mathematics curriculum standards. China has proposed a curriculum philosophy that focuses on the structuring of curriculum content. Guided by this philosophy, the mathematics textbook of Beijing Normal University Press for Junior High School (2024 edition) uses big ideas as the logical thread, organizing the content based on students' cognitive structure. The textbooks innovatively achieve a structured design of textbook objectives, subject content, learning activities, and assessment activities. This provides valuable experience for the structured design of other curricular resources and the implementation of big ideas.

INTRODUCTION

As the primary conveyor of the curriculum, textbooks play a dominant role in modern educational contexts across different disciplines (Fan et al., 2013; Rezat et al., 2021). On one hand, textbooks cannot possibly teach students all knowledge, as seemingly complex and diverse knowledge actually possesses a certain structural system. Only by teaching the big ideas that dominate the discipline and embody its fundamental principles and concepts can students develop the competencies needed to adapt to modern advancements. On the other hand, the learning process itself is a structured construction, and students' cognitive outcomes also have a certain structural system. Textbooks should provide opportunities for the development of students' structured learning abilities. This necessitates designing textbooks from a holistic perspective and in a structured manner.

Countries such as Singapore, Australia, the United States, and Canada commonly use big ideas as the fundamental framework for curriculum content across various subjects. They design curriculum and textbook content based on students' learning progressions and trajectories, which reflect the structured construction process of student learning. China's current education reform also emphasizes the structured integration of curriculum content and explores pathways to develop students' core competencies. Textbooks have the potential to contribute significantly to mathematics curriculum reform (Remillard, 1999). Therefore, textbooks should not only embody new curriculum concepts but also translate these concepts into practical advantages that support the transformation of curriculum perspectives.

The authors of this study have all participated in the design and development of curriculum resources. One of the authors has been involved in the development of China's curriculum standards and the writing of the mathematics textbook of Beijing Normal University Press for Junior High School (2024 edition) (BNUP) in China. Having experienced the entire process of textbook development—"planning, implementation, reflection and re-planning"—the author serves as both a developer of the ideal curriculum and a designer of the formal curriculum. The planning section of this textbook series has introduced significant innovations. Planning also represents a crucial aspect of the entire textbook development process. Therefore, this study aims to explore the concepts and characteristics of the structured design of textbooks.

THEORETICAL BACKGROUND

Bruner (1960, p. 18), based on the structuralist view of curriculum, proposed two principles for curriculum design. First, central importance should be given to the universal and powerful ideas and attitudes related to foundational subjects. Second, the materials should be divided into different levels to align with the varying abilities of students across different grades in schools. This approach emphasizes both the structure of the subject content and the cognitive structure of students. The design of textbook content should achieve a balanced integration of the two. Bruner's first principle points to the fundamental structure of knowledge and disciplines, placing it at the core of the curriculum. Wiggins (2005, p. 66) referred to these as "big ideas," highlighting their role as "Velcro" and "linchpins." It can be said that the fundamental structure of a discipline should radiate from big ideas, achieving the effect of "lifting the headrope the meshes spread open". The second principle points to the development of students' cognitive abilities, emphasizing the need to follow students' cognitive structures. Learning involves changes in cognitive structures, and textbook design should align with students' cognitive structures and processes.

Therefore, the structured design of textbooks should be guided by a curriculum perspective that organically integrates the structuring of subject content and students' cognitive structuring. It should use big ideas as the logical anchor and base on students' cognitive structures for designing textbooks.

METHODOLOGY

This study employs the design research method (Bakker, 2018). Its goal is to develop a blueprint for teaching and learning for junior high school mathematics (grades 7-9) and to examine the implementation effectiveness of the textbook through empirical research. The designed BNUP mathematics textbook and its accompanying Teacher's Manual were released and put into use in 2024. The textbook was designed based on Bruner's principles of structuralist curriculum design and development, undergoing an iterative development process that included planning, implementation (expert discussions and classroom experiments), reflection and re-planning.

The textbook adopted in this study is the BNUP mathematics textbooks, one of the most widely used textbooks in China, which has evolved over 30 years. Its new edition was released in 2024. This textbook series has introduced certain innovations in implementing structured design anchored by big ideas, offering valuable experience for the structured design of curriculum resources internationally. Specifically, we will focus on addressing two research questions:

RQ1: In what aspects is the structured design of mathematics textbooks implemented?

RQ2: What are the specific characteristics of the structured design in the BNUP mathematics textbooks?

RESULTS

Dimensions of the Structured Design of Textbooks

Textbooks, as artifacts, inherently possess a certain structure. The components of textbooks include several elements: textbook objectives, textbook content, and learning activity formats. Among these, the objectives of the textbook run through the entire textbook and govern the content and the methods of learning activities. The content of the textbook is an "alloy" that integrates foundational knowledge, skill training content, and moral education content, determined based on the textbook's objectives. The activity formats refer to the procedures, and rules that students master the knowledge. Modern textbooks also include evaluation components such as exercises, aimed at assessing how well students have learned. This reflects the logic of the four fundamental questions of curriculum design proposed by Tyler (1949, p. 1): why to learn, what to learn, how to learn, and how well is the learning achieved. These are the four main components considered in the structured design of textbooks: the textbook's objectives address the question of "why to learn", pointing to the cultivation of students' core competencies; the subject content answers "what to learn", focusing on the knowledge of the discipline; the learning activities respond to "how to learn", emphasizing the design of tasks; and the evaluation activities aim to address "how well is the learning achieved". The structured design of textbooks refers to these four components. Only by effectively structuring these four components can the functions of the textbook be maximized.

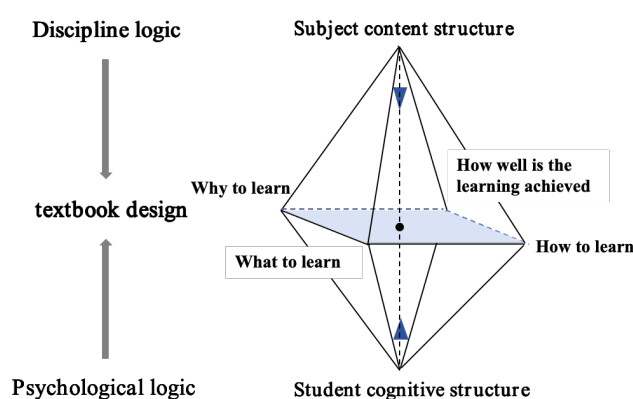


Figure 1: The textbook structure design.

The Characteristics of Textbook Structured Design

The structured design of textbook objectives: pointing to the integrity and progression of core competencies

The BNUP mathematics textbook sets the development of core competencies as its objective. In China, the specific manifestations of core competencies at the junior high school level are divided into abstract abilities, spatial awareness, geometric intuition, reasoning abilities, computational skills, model concepts, and data concepts. Core competencies are formed and developed over long-term learning processes. Therefore, the BNUP mathematics textbook establishes a systematic structural sequence and a cross-unit learning system, designing the textbook structure holistically. The BNUP mathematics textbook reintegrates the arrangement of chapters and adopts the organization of subject knowledge structure of "research object +", ensuring coherence and integrity in curriculum content that shares the same essence. For example, function content is integrated into "object + property + model applications", fostering students' abstract abilities, reasoning skills, and model concepts (Figure 2). This allows students to experience the development of core competencies from concrete to abstract and then using mathematical models to interpret the real world.

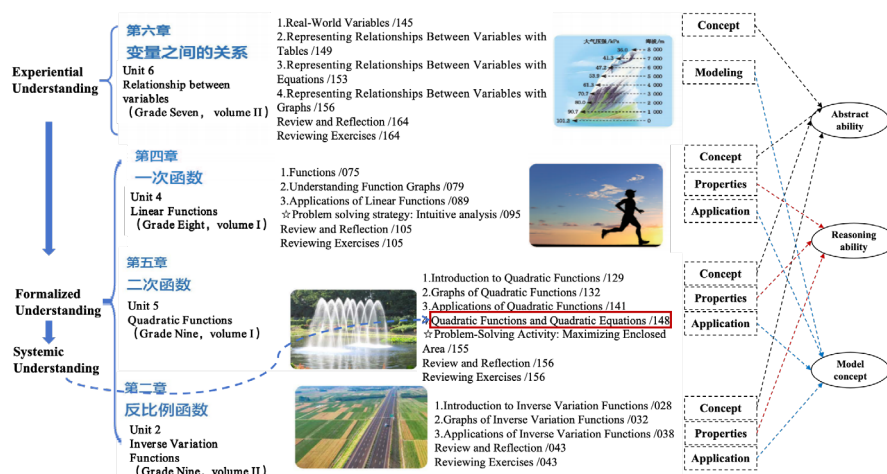


Figure 2: Overall and progressive design of functional content in textbook.

The specific manifestations of core competencies will vary at different stages. The BNUP mathematics textbook adopts a spiral progression approach to design textbook structure. For example, the development of mathematical modelling abilities and mathematical abstract abilities goes through stages of experiential understanding, formal understanding, and structural understanding. In the "function" structure design, a spiral progression is implemented based on students' cognitive processes to achieve gradual development of modelling and abstract abilities (Figure 2). In experiential understanding stage (7th grade). Through the exploration of various phenomena and problems in life and mathematics, the textbook guides students to experience processes of change and relationships between variables. In formal understanding stage (8th - 9th grades). The textbook focuses on the general concept of functions, gradually unfolding the substantive content of the function theme: including understanding the basic

concepts and properties of functions, as well as solving problems using function knowledge and methods. In structural understanding stage (8th - 9th grades). By analysing the substantive connections between functions and other related content (equations, systems of equations, inequalities, etc.), students understand the central role of functions in algebraic content.

The structured design of textbook subject content: focus on big questions

Textbook content design addresses what to learn, that is, the acquisition of mathematical knowledge. How can they be structured and integrated? The BNUP mathematics textbook presents big ideas through chapter-leading big questions, encouraging students to engage in continuous thinking during unit learning. These big questions are refined from knowledge and learning experiences and correspond to the key big ideas of each chapter. Hurst (2014, 2015) divides mathematical big ideas into three types. One is the more overarching big ideas that reveal the essential characteristics of mathematics. One is mathematical big ideas with clear learning trajectories, the third type is process big ideas, such as discovering, reasoning. Correspondingly, the textbook divides chapter-leading big questions into three categories (Figure 3): first, those that adopt a high-level perspective and focus on the essence of the mathematical discipline, similar to Hirst's first type of big ideas; second, those that aim at understanding knowledge, skills, and the underlying thought methods, similar to the second type but more fundamental; and third, those that focus on research methods in knowledge production, pointing to the holistic understanding and transfer of basic activity experiences, similar to the third type of big ideas. The big questions in chapters with similar big ideas will echo each other, forming a transfer of ideas, which is in line with the students' cognitive process, which can be seen in Figure 3.

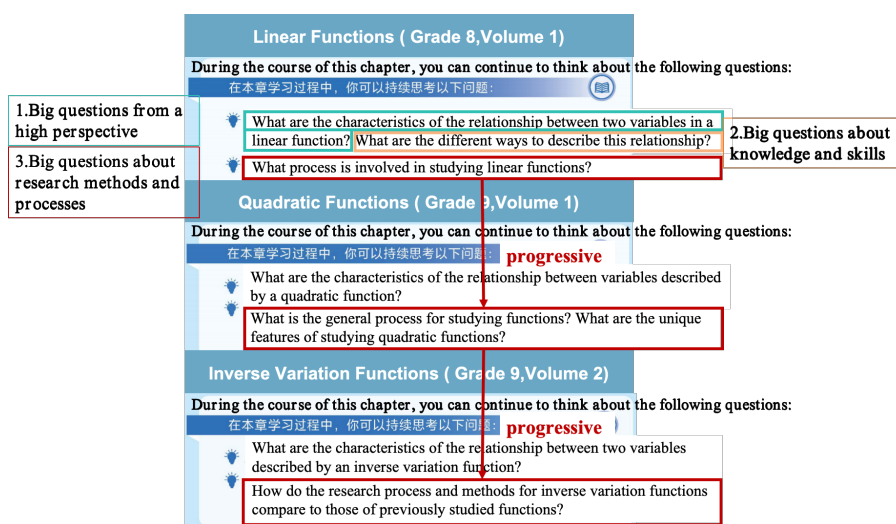


Figure 3: Structural design of big questions of function content in the textbook.

Secondly, the unit theme design of the BNUP mathematics textbook follows the process of chapter-leading big questions-learning process-review and reflection. This helps students explore unit content under the guidance of big questions, analyse problems with curiosity, and ultimately solve them. Finally, through reflection and

review, students summarize the knowledge chain and thought method chain of the unit, reinforcing the knowledge system of the unit. In the textbook, big questions serve as an outline for the entire chapter, acting as learning anchors. The question chains in each lesson are stepping stones, and the reflection and review serve as the stage end of learning, forming a structured closed-loop from beginning to end. For example, in the "Function" unit design for 7th grade (Figure 4), two big questions for continuous thinking are designed in the chapter introduction. The first big question points to the characteristics and representation of functions. The second big question points to the general methods of studying functions. The review and reflection at the end of the unit correspond to these two big questions, and the in-unit reviews also address the second big question. This subtly helps students construct the big concepts of the chapter.

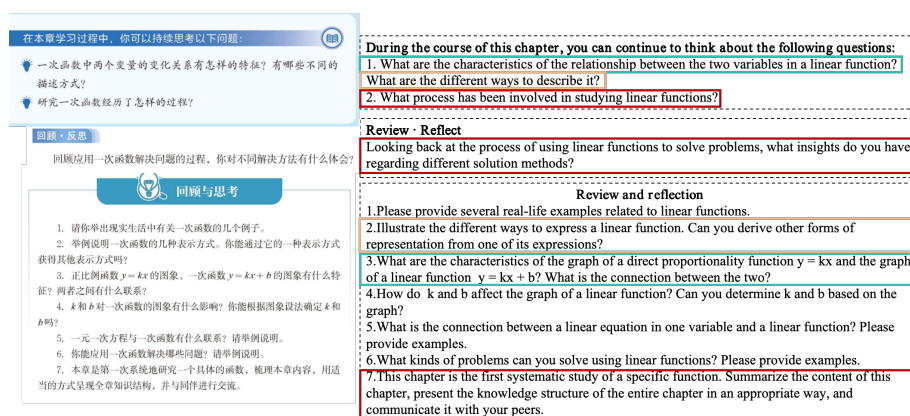


Figure 4: Questions' structure design of textbook.

The structured design of textbook learning activities: consider students' cognitive processes

The BNUP mathematics textbook facilitates students' learning of subject content through the arrangement of activity tasks, designed according to students' cognitive sequences. Bruner (1982) classified students' cognitive representation structures into three types: enactive, iconic, and symbolic. These correspond respectively to representing knowledge and experience through actions, using concrete images or mental representations, and employing symbolic systems to represent knowledge and experience. In structuring the learning process of each chapter, the BNUP Mathematics textbook builds upon students' cognitive structures, ensuring that their cognitive development progresses from concrete experiences to abstract thinking, and then from abstract reasoning to practical application. For example, in the learning process of "function" (Figure 5), students first abstract the concept of function from real-world contexts.

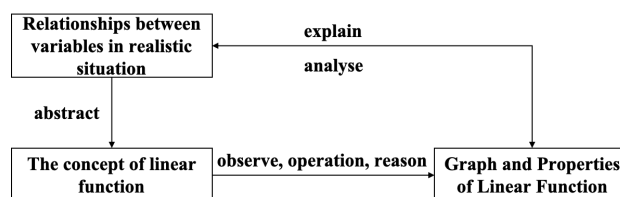


Figure 5: The cognitive process and structure of function learning.

They then engage intuitive observations and graphical explorations to understand the characteristics and properties of a function. Through logical reasoning and other activities, they derive mathematical conclusions. Finally, they apply them to analyse the real world, where the real world can test and explain mathematical models.

The structured design of textbook evaluation activities: response to big questions to design exercise

The assessment of students' learning outcomes in textbooks addresses the question of how well they have learned. The BNUP mathematics textbook designs an exercise system that points to the overall connections of big concepts, emphasizing not only the mastery of knowledge and skills but also students' understanding of big concepts, their grasp of mathematical research methods, and their comprehension of the essence of mathematics. The structural design of exercises includes three levels. First, reflective exercises are designed to make students reflect on the learning process of this chapter, echo the big problems, and summarize their own knowledge structures and thinking methods, as question 19 (Figure 6). Second, forward-thinking exercise aim to help students establish connections with prior experiences, and backward to integrate existing schema structures with current ones, forming logical links, as question 4 (Figure 6). Third, heuristic exercises aim to help students, based on their existing experiences and the transfer of mathematical intuition and research experience, think about future similar learning content and methods, as question 21 (Figure 6).

Exercise to reflect **this lesson's** learning content

19. Through the study of this chapter, what insights have you gained about linear functions and their applications? Please write a short essay on the theme of linear functions.

Exercise to associate the **previous** learning content

4. In the chapter 'Relationships Between Variables' in the seventh-grade textbook, there are the following three questions. Can the relationships between the variables in these questions be considered functions?

- (1) The relationship between reaction time and reaction distance;
- (2) When the height on one side of a triangle is fixed, the relationship between the area y of the triangle and the length x of that side;
- (3) The relationship between water depth and time as determined by the water depth change curve from 0:00 to 12:00 at a certain port on a specific day.

Exercise to imagine the **following** learning content

21. The general form of a primary function is $y=kx+b$ (k, b is constant, $k \neq 0$). Combined with the learning experience in this chapter, imagine: What other forms of functions may be studied in the subsequent study? For specific forms of functions, what might be studied in subsequent learning?

Figure 6: Structured design of textbook exercises.

DISCUSSION AND CONCLUSION

We discuss innovative textbook design methods based on the structuralist view of curriculum, following the fundamental logic of curriculum resource design: "why to learn, what to learn, how to learn, and how well is the learning achieved." From the design of textbook objectives, learning content, learning activities, and evaluation activities, we explore the main characteristics of textbook structural design, opening up new methods for the design of other textbooks and curriculum resources. This textbook series, following the process of "planning, implementation, reflection and re-planning". This study primarily presents the innovative ideas and methods from the design stage. Through the trial teaching of the new textbook, we made encouraging discoveries: the textbook design positively influenced students' learning, as evidenced by their performance (Figure 7). Guided by the textbook, students were able to independently construct their own knowledge systems, clearly articulate their

reasoning and thinking processes, and promote the development of their structured learning abilities.

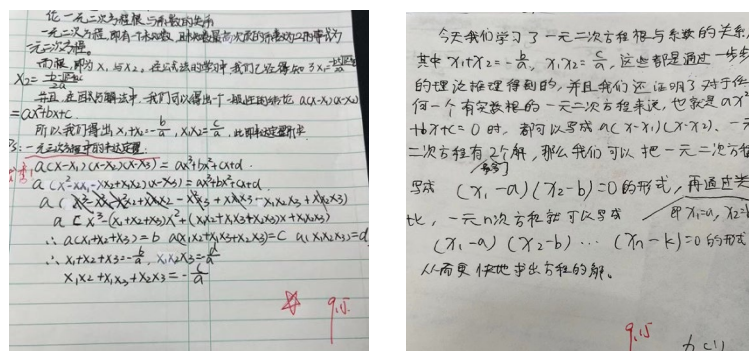


Figure 7: Students' performance in the textbook trial teaching stage.

Additional information

The BNUP textbook and its accompanying curriculum resources can be accessed here: <https://bit.ly/4iOzWFi>

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INVESTIGATING EXEMPLARY INTERDISCIPLINARY CONTENT IN MATHEMATICS TEXTBOOKS: CASES FROM CHINA AND THE UNITED STATES

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This study examines the features of exemplary interdisciplinary content (EIC) in mathematics textbooks from China (PEP) and the United States (BIM), focusing on their interdisciplinarity, cognitive demand, and some additional aspects. We selected four representative EIC (two from each series) for in-depth analysis and comparison. The findings reveal some common features of EIC, including real-world contexts and high cognitive loads, which require various content knowledge from mathematics and other disciplines. Notable differences were also found: PEP adopts more specialized scientific scenarios, requiring higher-level abstract mathematics (e.g., functions, probability), while BIM emphasizes more real-life social contexts, requiring simpler arithmetic but more complex calculations. Implications are discussed at the end.

INTRODUCTION

Interdisciplinarity has become increasingly prominent in the field of mathematics education and has been advocated in mathematics curriculum standards in various countries, such as China (e.g., Ministry of Education [MOE], 2022), Singapore (Fan et al., 2010) and the United States (e.g., Common Core State Standards Initiative [CCSSI], 2010). Meanwhile, it is widely recognized that textbooks play an important role in teaching and learning (e.g., Fan et al., 2013). With the ongoing curriculum reforms calling for interdisciplinarity, EIC from mathematics textbooks can offer much-needed support for teachers to teach interdisciplinary content effectively. Despite this, research concerning how interdisciplinarity is represented and interdisciplinary content is presented in mathematics textbooks remains limited (Li et al., 2024). Furthermore, textbook comparisons of interdisciplinary content, let alone with case studies of exemplary examples, are scarce.

In light of this, we intend to explore the potentials of exemplary interdisciplinary content (EIC) from current mathematics textbooks on how they could support the implementation of interdisciplinarity in mathematics teaching and learning. The research question guiding this study is: What are the features of EIC in mathematics textbooks from the US and China?

ANALYTICAL FRAMEWORK AND SELECTION OF CASES

Analytical framework

This study is part of a larger research project comparing interdisciplinary content in the US and Chinese secondary mathematics textbooks. In this study, we particularly examined some EIC mainly based on the diversity construct of interdisciplinarity (Wagner et al., 2011) and curricula requirements (CCSSI, 2010; MOE, 2022), with an analytical framework as shown in Table 1 (see more details in Li et al., 2025).

Aspect	Description
Essential criteria	<i>Interdisciplinarity</i>: Full integration (Israel & Lash, 2020) of mathematics content knowledge and rich content knowledge from other disciplines; In-depth and a wide range of knowledge <i>Cognitive demand</i>: Aiming at applying mathematics knowledge, and experimentation with mathematics knowledge
Additional aspect	<i>Context</i>: Authentic contexts from other disciplines <i>Task design</i>: Creativity, openness, comprehensiveness & practicality <i>Disciplinary balance</i>: Even distribution of content from different disciplines (Wagner et al., 2011) <i>Pedagogical function</i>: Promoting interdisciplinary competencies

Table 1: Analytical framework for examining EIC.

All the EIC must meet both essential criteria. Interdisciplinarity should show the variety of disparity of disciplinary content (Wagner et al., 2011), and include vertical structuring (Michelsen, 2006) of integrating the knowledge and provide learning opportunities for students to explore various phenomena and patterns in an interdisciplinary environment. Cognitive demands were determined by the role of the content aligned with the descriptions of the curricula (CCSSI, 2010; MOE, 2022). Besides, the content would manifest some additional features derived from curricula (CCSSI, 2010; MOE, 2022) or previous literature (Maass et al., 2019; Wagner et al., 2011) to some degree.

Selection of textbooks and interdisciplinary content

We selected the secondary mathematics textbooks series of the People's Education Press (PEP, Grades 7–9) from China and the *Big Idea Maths: Modelling Real-Life (Common Core 2019 version)* (BIM, Grades 6–8) series from the US. They have been widely used in their own countries over the past decade and reflect the mainstream ideas and practices of mathematics curriculum reform in each country. They have also been the main research objects of the US and Chinese mathematics textbook comparisons in recent years (e.g., Li et al., 2025). In the larger project, all interdisciplinary content in both textbook series was coded by two coders in terms of their related disciplines (including secondary subjects and tertiary majors), grades,

mathematics domains, textbook sections, contexts, and whether they met the criteria for EIC, with reliability checked by percentage of agreement. For this paper, we purposefully selected two cases of each series, respectively, which are representative of the features of interdisciplinary content in each series and differ from each other in terms of the dimensions mentioned above. We analyse the cases using the framework presented in Table 1 and discuss how these cases could be applied to interdisciplinary teaching and learning.

RESULTS

Case 1: Currency exchange between U.S. and Canadian dollars (BIM, Grade 6)

Case 1 is an exercise about currency exchange between U.S. and Canadian dollars (see Figure 1). With the two-way exchange rates, students are first asked to calculate the received amount of Canadian dollars exchanged from 150 U.S. dollars, and then calculate the received amount of U.S. dollars exchange from the remaining Canadian dollars after the trip, which allow students to apply their knowledge of multiplying decimals and economic knowledge on currency exchange, and realize that the inverse exchange rate may not necessarily be equal to the reciprocal of the exchange rate, leading to the gain or loss of possession in back-and-forth currency exchanges.

72. **DIG DEEPER!** You are preparing for a trip to Canada. At the time of your trip, each U.S. dollar is worth 1.293 Canadian dollars and each Canadian dollar is worth 0.773 U.S. dollar.
- You exchange 150 U.S. dollars for Canadian dollars. How many Canadian dollars do you receive?
 - You spend 120 Canadian dollars on the trip. Then you exchange the remaining Canadian dollars for U.S. dollars. How many U.S. dollars do you receive?

Figure 1: Details of Case 1 (Larson & Boswell, 2019a, p. 80).

The disciplinary balance between mathematics and economy is manifested in this case, as the knowledge about currency exchange is needed alongside the knowledge of multiplying decimals. It also resembles a certain level of practicality, as US students are likely to have the opportunity to exchange for Canadian dollars in real life.

Teachers could use this case as an introduction to exchange rates and let students explore the amount of loss in this task, or with a larger amount of money. This will allow us to cultivate not only students' skills in decimal multiplication but also their financial literacy. Meanwhile, teachers outside the US and Canada may change the currencies to their local currencies, use the live exchange rates to build up an authentic context, or enrich it with the scenario of exchange among three or more currencies, which would lead to an economic phenomenon called "Triangular Arbitrage" (2023).

Case 2: Exploring percent of change of salmons (BIM, Grade 7)

Case 2 provides an authentic context of salmon migration and influence of dams in the Columbia River Basin (see Figure 2). It is an "*Exploration*" task serving as an introduction to the unit, which involves the mathematics knowledge of percent of change, as well as knowledge from biology (salmon's reproduction and migration),

geography (reading map), and environmental sciences (how public infrastructures influence local species). Students need to apply them to solve the task and give inquiry on the influence of building dams on young salmon population in the annotation “*Math Practice*”, and consolidate them further by the sub-questions.

EXPLORATION 1 Exploring Percent of Change

Work with a partner.

Each year in the Columbia River Basin, adult salmon swim upstream to streams to lay eggs.

To go up the river, the adult salmon use fish ladders. But to go down the river, the young salmon must pass through several dams.

At one time, there were electric turbines at each of the eight dams on the main stem of the Columbia and Snake Rivers. About 88% of the young salmon pass through a single dam unharmed.

Math Practices

Check Progress
As the number of dams increases, what should be true about the number of young salmon that pass through unharmed?

- One thousand young salmon pass through a dam. How many pass through unharmed?
- One thousand young salmon pass through the river basin. How many pass through all 8 dams unharmed?
- By what percent does the number of young salmon decrease when passing through a single dam?
- Describe a similar real-life situation in which a quantity increases by a constant percent each time an event occurs.

Figure 2: Details of Case 2 (Larson & Boswell, 2019b, p. 253).

In this case, not only is the context a real issue that raised governmental attentions, but it also gives a figure (88%) that is very close to the authentic data of the average estimated juvenile survival rate of salmon species in 2015 (Faulkner et al., 2016). Meanwhile, it shows the balance between mathematics and geography, as students need to understand the sequential distribution of the dams to realize that the answer should be achieved by multiplying the given percentage eight times, while learning the life-cycle of salmons and dams' effect in the process. Moreover, sub-question (d) is an open question for students to explore similar real-life scenarios.

This case is almost ready for direct use in mathematics classrooms. Teachers from other countries may consider revising them to reflect the contexts of local species. Moreover, teachers may also let students find authentic statistical data by themselves via the Internet and estimate the results, so as to cultivate students' data literacy, which is considered an interdisciplinary competency (e.g., Schenck & Duschl, 2024).

Case 3: How scientists measure the age of the rock (PEP, Grade 8)

Case 3 is a post-lesson reading material (see Figure 3) related to half-lives in radiative decay, which requires students to understand the half-lives of radium and uranium (a concept in nuclear physics), use the function concept, and learn how scientists measure the age of the rock using the uranium-lead dating method in geology (Parrish, 2015). Students are also asked to explore the phenomenon of radiative decay through a question about the rebound of a ball, which is an idealized model from mechanic physics, making the context non-authentic.

This case shows the full integration of mathematics content knowledge and abundant content knowledge from physics and geology as well as the balanced relationship between mathematics and natural sciences, as it provides students with opportunities to use the graph of a function to represent the phenomenon of radiative decay, and build

up the analogy of the phenomenon of half-life and the rebound height of a falling ball. Students need to comprehensively use knowledge from multiple disciplines altogether to understand the content and solve the problem.

In classroom teaching and learning, teachers can utilize this case to help students understand how mathematics knowledge could be applied to natural sciences, encouraging them to see the world through a mathematical lens. Additionally, by discussing the half-lives of radioactive materials, teachers can illustrate related physics and geology phenomena in the evolution of the Earth by themselves or by collaborating with science teachers, enhancing students' scientific literacy and improving their interdisciplinary competencies to solve problems in various scientific disciplines using mathematics knowledge and tools.

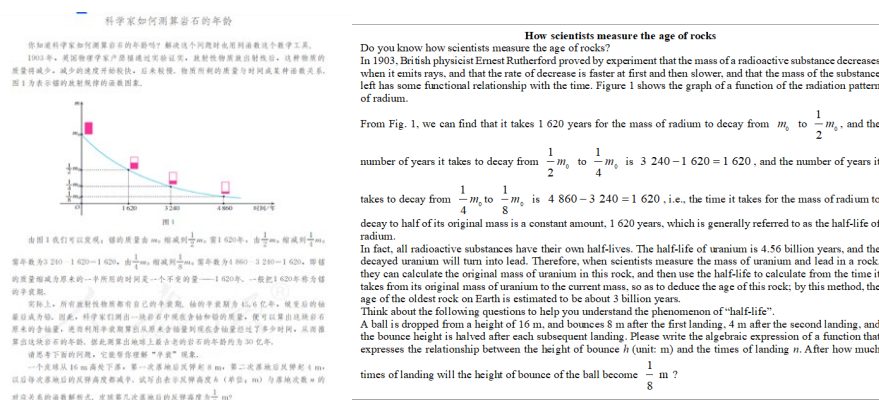


Figure 3: Details of Case 3 with translation (PEP, 2013, p. 85).

Case 4: Estimating the number of birds in a forest park (PEP, Grade 9)

Case 4 is an exercise from the chapter review in "Probability" (see Figure 4). Students are asked to estimate the number of birds in a forest. The case is designed using a non-authentic context as the task assumes that the park is isolated from the external environment over a certain period. It is designed to reinforce and apply mathematics concepts in probability and statistics, such as sampling, estimating probabilities from frequencies, inferential statistics, as well as biology knowledge (the methods of estimating the population of some species), encouraging students to apply such knowledge flexibly to solve this problem, indicating high cognitive demand.

A high degree of openness, creativity, and comprehensiveness are manifested in this case. It presents an open-ended question based on the context of biology and common challenges ornithologists face, and encourages students to explore diverse approaches for estimating species populations. Although the mark-recapture method (which is an estimation method introduced in the secondary biology curriculum of China) is supposed to be the most appropriate method given that birds fly around the forest park, students may also be creative to propose other methods even though the estimation may not be enough accurate. Moreover, students can develop feasible plans through practical activities, reflecting a certain level of practicality.

In classroom teaching and learning, teachers can use this case to conduct interdisciplinary activity, which will not only reinforce student's understanding of probability and statistics, cultivate their ability to apply mathematical knowledge comprehensively and solve problems in diverse ways to enhance their innovative skills, but also help develop their scientific literacy, understand common issues and feasible methods for estimating and inferring species populations in biology. Teachers may also adapt this material as a practical project for extra-curriculum activities, letting students estimate the populations of certain species in local ecology systems, such as fish in a man-made lake or insects in a bush, in which biology teachers may also engage in finding appropriate species for exploration.

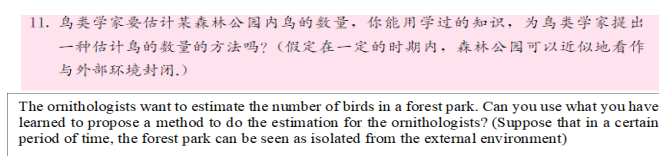


Figure 4: Details of Case 4 with translation (PEP, 2014, p. 153).

CONCLUDING REMARKS

The analysis of the four cases reveals several common characteristics of EIC in the mathematics textbooks from both China and the US. These cases were designed using contexts rooted in real life, mainly the scientific or economic contexts, which require students to use various knowledge from mathematics and other disciplines to solve the problems with high cognitive demands. The knowledge from other disciplines is not just a camouflage for the mathematics problems; instead, students need to understand the related disciplinary knowledge to identify the underlying mathematical relations that are the most accurate or appropriate and use mathematics as a tool to interpret scientific or economic models.

Notable differences can also be observed between the two series of textbooks. The contexts of the PEP cases are relatively more specialized, focusing on scientific research scenarios. Students need to utilize some basic knowledge from secondary science curricula and apply their mathematics knowledge in modelling scientific contexts. The BIM cases, instead, tend to build up the contexts in social life, either public environmental protection issues (Case 2) or daily life (Case 1), which emphasize the connection between mathematics and practical knowledge from other disciplines. Students need to do more complex calculations to solve real-life problems. It might be related to textbook developers and curriculum standards: the PEP textbook writers conceptualised mathematics as “the golden key to the gate of sciences” (PEP, 2013b), which is aligned with the curriculum requirements of selecting content that “reflect modern sciences and technology as well as the needs social development” (MOE, 2022, p. 3); while the BIM textbooks, as stated in the “Modelling the real life”, design problems that “help students apply the mathematics they are learning to everyday life” and “see that math is used across content areas, other disciplines, and in their own experiences” (Larson & Boswell, 2019a, p. vi). The differences may also be explained

by the fact that the BIM cases are from lower-grade mathematics (arithmetic) while the PEP cases are related to more abstract higher-grade mathematics: function and probability. Moreover, multiple disciplines are generally intertwined in the three STEM cases, while the non-STEM case (Case 1) only involves the economy discipline. It should be noted that even though we aimed to select the most representative cases of EIC, the overall design features of interdisciplinary content in mathematics textbooks in the two countries, let alone in other countries, still need further studies. Moreover, the effectiveness of the proposed adaption of the EIC should be assessed by actual classroom teaching practices or interventions.

Finally, we would argue that the above cases of EIC can offer useful insights for textbook authors, teachers, and teacher educators about interdisciplinary content. For textbook authors, we think reasonable attention should be paid to the design and provision of appropriate contexts to integrate knowledge from other disciplines, either deriving from the daily life of students (as BIM does) or from scientific research (as PEP does), and ideally connecting with knowledge in other school level disciplines or other disciplinary knowledge that serves as “common sense” for target student population; in designing such tasks, the scaffolding for moving towards higher cognitive demands can be achieved by setting a series of sub-questions (as BIM does) or involving similar scenarios as an analogy (as PEP does). For teachers and teacher educators, it is advisable to develop sample lesson plans for the EIC, through collaborative teaching research or resource-sharing among the teaching communities, so that the interdisciplinary content can be better implemented in the classroom for enhancing students’ interdisciplinary competencies.

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FROM CURRICULUM TO TEXTBOOKS: ALIGNMENT AND ENRICHMENT

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Given the limited research on the textbook development process, this study analyses how the newly added performance standard on the equal sign and equivalence relationship in the Korean mathematics curriculum has been incorporated into new textbooks. Specifically, we analyse the changes in equation structure in the new textbooks for grades 1-2 compared to previous editions, and how the nine textbooks for grade 4 have organized activities related to the definition of the equal sign, equation structure, and equation solving. Based on these findings, this paper emphasizes the importance of curriculum alignment and enrichment in the textbook development process.

INTRODUCTION

To provide students with a better mathematics education, curricula are being revised. Curriculum standards, or written curricula, involve the selection and organization of content to achieve educational goals. For these intended curricula to be implemented in mathematics instruction, high-quality textbooks must be developed that align with the curriculum's intent (Rezat et al., 2021). Teachers often interpret the curriculum's intent through the changes in textbooks, using them as the primary instructional resource rather than the curriculum itself. Therefore, alignment between curriculum and textbooks is crucial. However, ensuring this alignment is not straightforward (Polikoff, 2015).

Curriculum standards are typically abstract or broad, requiring interpretation and enrichment to transform them into concrete activities in textbooks (Pang, 2024). Thus, studying the textbook development process is necessary. While mathematics textbook research has garnered attention in recent years, the textbook development process remains underexplored. This study examines how the intentions of the Korean national mathematics curriculum are reflected in textbook activities. Specifically, recent analyses and reconstruction of textbooks on elementary students' understanding of the equal sign have led to the addition of a new performance standard for the equal sign and equivalence relation in an already crowded curriculum. This addition is notable, as the topic has not been sufficiently addressed in previous textbooks. Given this context, this study examines how textbooks address the new standard. The research questions are: (a) To what extent are the new textbooks aligned with the standard? (b) What are the characteristics of the development process?

STUDENTS' UNDERSTANDING OF THE EQUAL SIGN AND ITS RELATED TEXTBOOK ANALYSIS

Students' understanding of the equal sign can be examined in terms of its meaning, equation structure, and equation solving (Matthews et al., 2012). It can be categorized as either relational, meaning “equal”, or operational, meaning “add”, “answer”, or “do something” (Kieran, 1981). Students with an operational understanding of the equal sign may consider equations such as $8=8$ or $9=5+4$ incorrect, as they involve either no calculation or a backward calculation (Stephens et al., 2013). In contrast, students with a relational understanding of the equal sign can evaluate not only standard equations like $a+b=c$, but also equations with non-standard structures (Molina & Ambrose, 2008). Furthermore, when they find $\_\_$ in equations such as $7+4=5+\_\_$, they can reason that since 5 is 1 greater than 4, $\_\_$ is 6, which is 1 less than 7. Thus, students' understanding of the equal sign is closely related to their comprehension of its meaning, the equation structures they recognize, and their ability to solve equations.

Recent studies have focused on the notion of “substitution” for the equal sign. For instance, Jones et al. (2013) showed that in calculating $31+67$, 31 can be replaced by $30+1$ and 67 by $60+7$, resulting in $30+1+60+7$. By rearranging and combining terms (i.e., $1+60=60+1$, $30+60=90$, $1+7=8$), the calculation becomes $90+8=98$. Lee and Pang (2021) suggest that full relational understanding involves both the notions of sameness and substitution.

Previous studies analysing the presentation of the equal sign in textbooks have primarily focused on equation structure (i.e., standard versus non-standard). A major finding is that textbooks' overuse of standard equations tends to reinforce an operational understanding of the equal sign (e.g., Li et al., 2008), and this operational view is difficult to overcome as students advance through the grades (Kim & Pang, 2018). Therefore, it is essential to examine how textbooks address the equal sign and equivalence relationship. This study draws on previous research to analyse textbooks in terms of equation structure, the meaning of the equal sign, and activities related to solving equations.

METHODS

Research contexts

The purpose of this study is to examine how the newly introduced standard for the equal sign and equivalence relationship in the most recent revision of the Korean mathematics curriculum has been incorporated into textbooks. The Korean elementary curriculum is organized into grade bands, and the new standard is presented as a subtopic of the “Change and Relationships” strand for grades 3 and 4. As shown in Table 1, the new standard is accompanied by an explanation and considerations for its application.

Note that the Standard Explanation in Table 1 states, “based on the meaning of the equal sign,” and in Korea, the equal sign is first introduced in Grade 1 when addition

is introduced. In this context, the curriculum includes the following new considerations for applying the standard to Grades 1-2: Students are expected to understand that the quantities on both sides of the equal sign ($=$) are equal in addition, subtraction, and multiplication expressions.

Standard	The equal sign can be used to express the relationship between two quantities of the same amount.
Explanation	Based on the meaning of the equal sign ($=$), students understand an equivalence relationship through activities such as determining whether a given equation is correct and finding two quantities of the same amount using concrete objects or drawings.
Considerations	Equivalence relationships are covered within the range of two-digit numbers. Students can use their number sense or intuitive understanding of the properties of operations to compare two quantities when determining whether an equation is correct.

Table 1: The revised curriculum's new standard for the equal sign and equivalence relationship (Ministry of Education, 2022, p. 19).

Overview of analysis

A content analysis of the Korean mathematics textbooks was conducted. Textbooks in Korea are developed and implemented sequentially by grade level in accordance with the national curriculum. Textbooks for Grades 1 and 2, aligned with the latest curriculum, have been used in schools since 2024, while textbooks for Grades 3 and 4 are used in 2025. The textbooks for Grades 1 and 2 are developed by the Ministry of Education, and all elementary schools use the same version. In contrast, the textbooks for Grades 3 and 4 are developed by multiple writing teams and must be approved by the government, resulting in nine different textbook series. Because all nine textbooks organized the new standard into a few sections in the Grade 4 textbooks, our analysis focused on these sections. Additionally, the new curriculum consideration for Grades 1-2 led to an examination of changes in the treatment of equation structure and the meaning of the equal sign in the Grade 1 and 2 textbooks compared to previous editions. Table 2 summarizes the foci of the content analysis.

Textbooks	Main analytic criteria
Grades 1 and 2	How has the structure of equations ($a+b=c$, $a=b+c$, $a=b$, $a+b=c+d=e$) changed compared to previous textbooks? How are activities designed to emphasize the relational meaning of the equal sign?

Grade 4	(Equal sign definition) How is the equal sign defined? How does it express the relationship between two quantities of the same amount?
	(Equation structure) How do activities encourage students to determine whether a given expression is correct and explain why?
	(Open equation solving) How do activities guide students in determining whether the process of finding the value of ___ is correct?

Table 2: Criteria for content analysis of textbooks.

DEVELOPMENT OF NEW TEXTBOOKS ALIGNED TO THE REVISED CURRICULUM

Changes of Grade 1 and 2 textbooks

Table 3 shows the results of the comparison with the previous textbooks, focusing on the equation structures in the Grade 1 and 2 mathematics textbooks.

Equation structure	Previous textbooks		New textbooks	
	Grade 1	Grade 2	Grade 1	Grade 2
$a+b=c$	290 (100%)	207 (87.3%)	229 (97.9%)	161 (82.1%)
$a=b+c$	-	8 (3.4%)	5 (2.1%)	8 (4.1%)
$a=b$	-	20 (8.4%)	-	23 (11.7%)
$a+b=c+d=e$	-	2 (0.8%)	-	4 (2.0%)

Table 3: Frequency of equation structures between previous and new textbooks.

The analysis revealed that while the total number of equations in the new textbooks slightly decreased, there was a significant change in the proportion of each equation structure. Specifically, the previous Grade 1 textbooks did not use any nonstandard equations, whereas the new textbooks introduced equations in the $a=b+c$ structure, such as $10= \_ + \_$, in activities designed to find the sum of two numbers that add up to 10. The teacher's guide emphasizes the understanding of the equal sign by helping students recognize that $\_ + \_ = 10$ and $10 = \_ + \_$ represent the same relationship. It is suggested that if students have difficulty representing addition as $10 = \_ + \_$, they should be taught to understand that the quantities on either side of the equal sign ($=$) are equal, helping them become familiar with this nontraditional representation of addition.

Furthermore, the slight increase in the percentage of nonstandard equations in the new Grade 2 textbooks is a positive development in terms of students' understanding of the equal sign. For example, in the addition and subtraction unit, when students are asked to create expressions with $\_$ and find the value of $\_$, the new Grade 2 textbook not

only provides a visual representation of the problem context, as in previous textbook, but also includes an additional bar diagram, as shown in Figure 1. This diagram uses two equal-length bars to visually demonstrate that both sides are equal in magnitude. The textbook further uses these bars to represent equations, such as $14 = \square + 6$, in subsequent activities.

한끼 이야기 해요 Let's talk about the equation for finding the number of fish brought.

1 Let's find the number of fish brought.

- Express the number of fish brought as $\square$ in an addition equation.

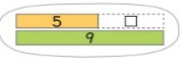


addition equation _____

- Talk with your friend about how you can find the value of $\square$.

I place the Baduk stones so that they add up to 9.

I continue counting from 5 to reach 9.



equation _____

the value of $\square$ _____

3 Using $\square$, create an equation that matches the picture, and find the value of $\square$.



equation _____

the value of $\square$ _____

• How many fish did you bring?

Figure 1: Bar diagrams to represent different equation structures (Ministry of Education, 2024, pp. 84-85).

Development of Grade 4 textbooks: Activity about equal sign definition

In our analysis of the nine textbooks, we found that the equal sign definition activity typically involved finding two quantities of the same value, emphasizing the relational meaning of the equal sign, and expressing them as an equation, as suggested by the curriculum. Specifically, visual models such as a two-armed scale were used to teach students that two quantities of the same value can be represented as an equation using the equal sign. The results of the activity were then used to define the relational meaning of the equal sign. By placing most of these equal sign definition activities at the beginning of the lesson, the textbooks aimed to build upon students' understanding of the relational meaning of the equal sign to develop other types of activities.

In addition, when expressing two quantities of the same value, students were often asked to represent an equation in the structure $a+b=c+d$. For example, using blocks representing 10, 20, 30, 40, and 80, students were asked to create two quantities of the same value, such as $80+10$ and $30+30+30$, and represent them as equations. Examples such as $80=40+40$ were given to encourage students to find equations with different structures.

Development of Grade 4 textbooks: Activity about equation structure

Our analysis of the organization of the equation structure activities revealed two key characteristics. First, most of the activities designed to determine whether a given equation was correct utilized the equation structure of $a+b=c+d$. These activities were presented in accordance with the considerations for applying the standard: Students can use their number sense or intuitive understanding of the properties of operations to

compare two quantities when determining the correctness of an equation. In other words, the expression was presented in a way that allowed students to examine the relationship between the two sides using number sense or properties of operations, without the need to calculate the two sides. For example, textbooks used equations such as $40+11=11+40$, which employs the commutative law; $17+1+49=17+50$, which utilizes number sense; and $2\times 3\times 5=3\times 2\times 5$, which uses the commutative law.

In the activities for determining whether a given equation was correct, the strategies varied. In some cases, students were simply presented with a set of equations and asked to determine whether they were true or false, with prompts like “Let’s see if this equation is true or false.” Other textbooks encouraged students to use relational strategies rather than computational ones, with prompts such as “Let’s figure out how to tell if an equation is correct without calculating.”

Furthermore, some textbooks provided examples of strategic approaches. For example, in some cases, relational strategies were presented visually to aid student understanding. In other cases, three different strategies (i.e., hands-on models, computational strategies, and relational strategies) were presented as examples, allowing students to choose from them, as illustrated in the example on the left in Figure 2. It is also noteworthy that a strategy utilizing the substitution meaning of the equal sign, as proposed by Jones et al. (2013), was presented in the example of $58+26=60+24$.

3 Determine if the given equation is correct.

$58 + 26 = 60 + 24$

- Use base-ten models to compare both sides if they are the same amount.
- Compare the results of calculating $58+26$ and $60+24$ to see if they are the same.
- Compare the magnitude of 58 and 60, and compare the magnitude of 26 and 24. Be specific.

Compare your results with a friend who chose another method.

5 Let's talk about Ha-Yun and Dong-Yeon's opinions.

$17 + 54 = \square + 4$

- Look at the two friends' opinions and choose a traffic light for each.

Ha-Yun: $17+54=71$, so the correct number in $\square$ is 71. Add another 4 to it, so the result is $17+54=71+4=75$.

Dong-Yeon: Since $17+54=71$, I can write 71 instead of $17+54$. $17+54=\square+4$, $71=\square+4$.

Ha-Yun's traffic light: disagree, unsure, agree.

Dong-Yeon's traffic light: disagree, unsure, agree.

Figure 2: Various approaches related to equation structure and equation solving activities (Pang et al., 2025, pp. 134-135).

Development of Grade 4 textbooks: Activity about equation-solving

Analysis of the organization of the equation-solving activities revealed three characteristics. First, most of the activities designed to find the value of $\square$ in a given equation used the $a+b=c+d$ structure, particularly when the relationship between the two sides was relatively clear. It was more efficient to find the value of $\square$ by examining the relationship between the two sides rather than using computational

strategies. For example, textbooks presented $13+35= \_\_+32$ to help students recognize that $\_\_$ is 16, which is 3 greater than 13, rather than calculating $13+35$.

Second, the presentation of strategies in the activity of finding the value of $\_\_$ in a given equation varied across textbooks, with more textbooks not explicitly presenting strategies than in the activity of determining whether a given equation is correct.

Third, some textbooks incorporated typical student errors. For example, it is common for students to answer the value of $\_\_$ in $17+54=\_\_+4$ as 71, which is $17+54$, as shown on the right side of Figure 2. In particular, students often misunderstand $17+54=\_\_+4$ as a sequential calculation and write it as $17+54=71+4=75$. However, as shown on the right side of Figure 2, the textbook presents Dong-Yeon's reasoning by substituting 71 for $17+54$ to obtain $71=\_\_+4$, making it easier for students to understand why the typical error is incorrect.

CONCLUSIONS AND IMPLICATIONS

This study analyzed how Korea's new standard was translated into textbooks. Based on the main findings, we present the following implications for textbook development.

First, the curriculum change emphasizing the relational understanding of the equal sign is effectively embodied in the textbook activities. The content presented in the national curriculum is well aligned with the textbook activities, including not only the performance standard but also its explanation and considerations for application. Although this alignment may be taken for granted, as curriculum alignment is a requirement for textbook approval in Korea, the alignment between the revised curriculum and its corresponding textbooks should be examined. Furthermore, since curriculum revision ultimately aims to improve student learning (Pang, 2024), the successive alignments of curriculum and student learning through new textbooks should be investigated by examining how Korean students' understanding of the equal sign is enhanced through the effective use of the new textbooks.

Second, the organization of the curriculum standard in the textbooks is enriched by a variety of activities. While the activities for defining the equal sign were similar across the nine textbooks, emphasizing the relational meaning of equality with visual models and equations like $a+b=c+d$, the strategies for constructing and solving equations varied considerably. For example, some textbooks used visual models to emphasize relational strategies, while others introduced substitution activities. It is also noteworthy that equation-solving activities reflect common errors and misconceptions that students make. This rich organization of textbook activities is related to various textbook analysis studies, research on students' understanding, or textbook reorganization based on students' understanding of the equal sign. Therefore, it is expected that the curriculum can lead to textbook commonality through alignment, while textbook research can lead to textbook enrichment through diversity.

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META-RESOURCES AS PERCEIVED BY PRE-SERVICE MATHEMATICS TEACHERS IN AN INTERVENTION STUDY ON DESIGNING LESSONS ON REASONING AND PROOF

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In this paper we report on an intervention study (which was part of an EU-funded study including five international country teams), where pre-service mathematics teachers (PMTs) used and experienced particular resources to design lesson plans for 8th grade students in Norway. Recognising that material resources were needed for PMTs to design lessons, we identified particular material resources as important, but also social/human resources. Leaning on the conceptualisation of ‘resources’ (material, social/human, cognitive) in the literature, we categorise the four most important resources reported upon and explain and broaden the understanding of ‘meta-resources’ to include social/human resources. Moreover, we link the notion of meta-resources to the notion of “curricular noticing” (Dietiker et al., 2018).

INTRODUCTION

Research highlights the importance of lesson planning skills for teachers, particularly for prospective mathematics teachers (PMTs) (e.g., Lim et al., 2018). Effective lesson planning involves selecting and adapting mathematical tasks, designing instructional units, and ensuring that lessons are meaningful and beneficial for both teachers and students. Research indicates that PMTs often struggle to integrate mathematical ideas into lesson plans (LPs) and lack the flexibility to modify their approaches to meet diverse learning needs (e.g., Sadak, 2021). They frequently view lessons as disconnected events rather than cohesive units, leading to challenges in developing effective plans. A well-structured LP can serve as a blueprint for guiding student learning and preparing for potential classroom issues (Fitriati et al., 2023). Ultimately, successful lesson planning requires anticipating classroom dynamics and focusing on student understanding of mathematical content.

In the EU-funded MaTeK project (<https://www.projectmatek.eu>), an intervention was designed to help PMTs in five countries design lessons for 8th grade students with a focus on reasoning and proving (R&P). This intervention consisted of several phases where PMTs were guided, with particular resources, through the following cycles of learning: (1) Learning about various R&P methods, (2) Preparing R&P tasks and LPs aligned with national curricula, (3) Reflecting on the created LPs, (4) Implementing one selected LP in a classroom, (5) Reflecting on the lesson to refine the plan.

In this paper, we focus on the resources Norwegian PMTs found most useful during the intervention. We ask the following research questions: (1) Which resources did PMTs identify as most useful and why, and how can they be categorized? (2) In which ways do the resources link to PMTs' curricular noticing?

After this introduction, we explain the theoretical frames we used and our methodological choices. Next, the findings are discussed in terms of the identified resources (and with respect to the research questions), followed by conclusions.

THEORETICAL FRAMEWORKS

Resources and meta-resources

The literature on the *teachers' design landscape* suggests that teachers, especially PMTs at the start of their careers, benefit from support to develop as curriculum designers. Whether selecting tasks, planning learning paths, or adapting lessons, teachers often need support, typically provided by educative curriculum resources in professional development groups (e.g., Pepin, 2019). To become creative and didactically effective designers of teaching, PMTs require more than student-focused curriculum resources (e.g., commonly used textbooks); they also need educative resources that enhance their teaching knowledge and guide their design processes.

These resources are called meta-resources: “material resources or parts of material resources, whose audience are teachers, and which have been explicitly designed for supporting mathematics teachers' planning of teaching and learning activities” (Gueudet et al., 2023, p. 3). At the same time, the literature provides a definition of “resources” for teaching. We lean on the definition by Pepin et al. (2023) which regards curriculum resources as “all the material resources that are developed and used by teachers and students in their interaction with mathematics in/for teaching and learning, inside and outside the classroom” (p. 10). A differentiation is made between “material” (e.g., digital/ICT based curriculum resources like interactive e-textbooks, websites, e-worksheets) and “non-material” resources (e.g., web conversations with tutors, peers, and friends; social and cultural resources; cognitive resources, such as mathematical concepts and techniques) used by students and teachers.

Lesson planning and lesson plans

Lesson planning appears to be difficult for PMTs. Lim et al. (2018) reported that PMTs found it difficult to recognize the mathematical idea within the task and transfer it to an LP. Chen and Zhang (2019) also discovered that PMTs lacked knowledge of how to modify planning aspects to meet the needs of their learning groups. They apparently displayed a teaching style that was less adaptive and more formulaic, hence demonstrated little didactic flexibility. It is typically expected that planning a lesson should result in a document referred to as a “lesson plan” which includes an outline of the important components of the lesson.

Curricular noticing

Dietiker et al. (2018) introduced the Curricular Noticing Framework, intended to help teachers recognize opportunities in curriculum materials, assess their strengths and limitations, and employ strategies to engage with them effectively. The framework views the curriculum as dynamic, positioning teachers as active ‘curriculum designers’ who shape and are shaped by the resources.

Planning with curriculum resources requires teachers to develop curricular reasoning, which involves the thought processes used in planning, implementing, and reflecting on instruction. Dietiker et al. (2018) propose that noticing enhances teachers’ interactions with these resources through a framework comprising three phases: (1) *Curricular Attending* involves searching and surveying materials to understand their components, including tasks and recommendations; (2) *Curricular Interpreting* entails making sense of the materials by connecting them with teachers’ mathematical knowledge, including subject matter and pedagogical content knowledge; (3) *Curricular Responding* involves making decisions based on interpretations, such as selecting tasks and determining how to engage students. This process emphasizes the importance of intentional learning in these skills to enhance teaching practices.

METHODS

Our findings are based on data from the MaTeK project, specifically the intervention conducted in Norway between September and December 2022. The participants were 4th-year PMTs in an integrated master’s program specializing in mathematics, qualifying them to teach grades 5-10. The sample consists of 16 PMTs, referred to as S1-S16. We primarily used data from group interviews conducted after the intervention, aimed at exploring PMTs’ reflections on their learning about lesson planning with respect to R&P, and their key takeaways from the intervention. Additionally, we analysed PMTs’ reflective texts, which encouraged them to critically assess their learning experiences, development and confidence in planning R&P-focused lessons. In terms of analyses, we first categorised data on the most important resources according to what PMTs said (in the interviews and texts) and linked those to the definition of *meta-resource* as defined by Gueudet et al. (2023). Second, we analysed those resources according to how they were used and how they link to the three categories of Dietiker et al. (2018)’s *curricular noticing*.

FINDINGS

Important resources and their categorisation

(1) One of the most important resources mentioned by the PMTs was the *curricular spider web* (Van den Akker, 2003), a conceptual model used to visualize and analyse the complexity of curriculum design. It illustrates the interrelated components of the curriculum, including objectives, content, teaching methods, assessment, and context. Each “strand” of the web represents one of these elements, emphasizing how they influence and depend on each other. This model helps educators understand the

dynamic nature of curriculum, promoting a holistic approach that ensures coherence and alignment across its elements.

Although the Norwegian PMTs had used a similar model before (in their pedagogy courses), some of them emphasised that the curricular spider web had immensely helped them to make a more detailed LP.

I found it very interesting to use the spider web model. Previously, I have used the didactic relational model and the classic what/how/why table in planning, and over time, one might become less aware of the components that are actually included, as it becomes somewhat automatic. Using a new model made me step back and reflect more on my own choices, and I felt that I developed my competence in lesson planning as a result of this. (S9)

Creating an LP using the spider web framework we were given was challenging, as there was a lot of information to include, but it also forced us to consider more details. (S16)

It appeared that the curricular spider web encouraged PMTs' *deeper reflection and critical thinking* regarding the different components of LP and curriculum development. They also realised that they had choices and many details to consider.

(2) Another important resource for PMTs was the research paper by Ellis et al. (2019), which they were given to study. The paper explores effective teacher moves to enhance student reasoning and introduces the Teacher Moves for Supporting Student Reasoning (TMSSR) framework. It outlines strategies like *Prompting Questions, Encouraging Discourse, Building on Student Responses, Modelling Reasoning, Providing Feedback*, emphasizing their importance in fostering a supportive learning environment and enhancing students' mathematical understanding and reasoning abilities.

[...] when we were introduced to the TMSSR framework, and perhaps I had previously, consciously or unconsciously, thought that asking why do you think that or can you explain what you were thinking were the main ways a teacher could promote student reasoning. The introduction of this framework, and the work in lectures and assignments, has given me a deeper understanding not only of what a teacher can do to promote student reasoning but also of the significance of these actions. (S8)

The different categories by Ellis et al. have made me aware of how much a teacher can do, while also showing me that I need to delve even deeper into them before I can say that I have sufficient knowledge of the framework. (S15)

After the enactment of the lesson (they had designed in the intervention), it became even clearer how important and complex the teacher mediation of reasoning tasks was:

Another thing I have learned is how, as a teacher, you can make small verbal moves to promote students' reasoning. These are the moves we read about in Ellis' text [...]. The moves the teacher used during the lesson contributed to more reasoning than I felt the students would have achieved on their own. I therefore think that this knowledge will be very valuable when we start working as teachers. [...] it's very helpful to have some tools to draw on that can promote reasoning in the classroom. (S4)

In the post-interviews, the Ellis et al. (2019) paper was also mentioned as one of the most important 'inspirations' and learning tools.

I wish I could say reasoning and proof because that's where I feel weakest, but what I really feel I'll take with me the most is the Ellis article on teacher moves, because I feel it was very much directed at me as a teacher. That this can work to promote reasoning. So, that table is something I'll probably refer back to when I'm fully qualified, and look at it more because I feel it's more concrete. Yeah. (S6)

Overall, the Ellis et al. (2019) paper was seen as an essential tool for improving PMTs' instructional strategies and understanding the complexities of mediating reasoning.

(3) A third important resource was the ways students worked during the intervention, the group work, in particular in terms of reflection on their own work on mathematical tasks, and also in terms of feeling the support of peers towards a common goal. Group discussions allowed PMTs to challenge their assumptions and refine their reasoning through collaboration. For instance, PMT S11 noted that group work

helped me by allowing me to hear others' perspectives and reasoning, which made me reflect on my own opinions and choices. This was especially rewarding in the work on the planning document, as we discussed many different viewpoints on lesson planning and task selection, which broadened my knowledge in this area. Working in a group gave me greater motivation to attend class and to do well on the group assignments.

Other PMTs also reported that group work fostered a sense of accountability and motivation. This shared responsibility, combined with the opportunity to collaborate on tasks, allowed PMTs to feel more connected to the learning process. Additionally, working with a consistent group throughout the intervention was seen as valuable because it fostered a shared understanding and complementary approaches.

Having this continuity throughout the project, I feel that a shared understanding of the topics is built within the group, and we also come to understand which aspects each member is strongest in. This has made it easier to have discussions within the group [...] For example, when we were developing our LP, I noticed that I tend to be a very visual mathematician, often relying on models and practical tasks, while others in the group are more theoretical mathematicians. I realized that it doesn't necessarily have to be either visual or theoretical; rather, a combination of both can create even more engaging tasks and activities for the students. (S8)

Overall, group work was key to enhancing PMTs' engagement, learning, and reflection on lesson planning, broadening their instructional strategies and deepening their understanding of mathematical content.

(4) A fourth important resource was the summary sheet on R&P on different ways of reasoning, developed by the intervention teacher educators and handed out in intervention session 1. The document discusses various ways of reasoning (appeal to authority, reasoning by analogy, inductive reasoning, empirical reasoning, deductive reasoning), providing examples for each.

PMTs were appreciative of this support for the different types of reasoning. For example, PMT S16 reflected on how it helped deepen her understanding:

In the beginning, I felt that there was a lot of information at once, which was quite overwhelming. Now, looking back on the sessions [...] I found the handout with examples of different ways of reasoning to be very useful, and it was only then that I felt I truly understood the difference.

The summary sheet provided practical examples that helped PMTs differentiate between the different types of reasoning (in addition to seeing it enacted in the video). However, some expressed a desire for even more examples to deepen their understanding. In the post-interviews, PMT S8 emphasized this need, especially given the complexity of classroom situations:

if you analyse classroom situations, you get more fluid, realistic responses, while a lot of what we have, like the handouts is kind of, not perfect, but very much like, this is this, and this is that. But sometimes, when we watched the classroom video, you notice it's not so easy to distinguish when things are a bit mixed up. So maybe more classroom situation examples, which aren't necessarily so... [perfect].

PMT S16 also highlighted the potential of the summary sheet as “a way to help pupils understand by providing examples and letting them see the differences between, yes, what is a valid proof and what is not a valid proof.”

Looking across the four resources, first, it can be claimed that they were *meta-resources*, in the sense that they have been conceptualised and *explicitly designed for supporting mathematics teachers' planning of teaching and learning activities*. However, they were not all of material nature (as Gueudet et al.'s (2023) definition demands). These four ‘most important’ resources (as reported by PMTs) in terms of lesson planning were clearly very different: whereas the curricular spider web was in essence of idealistic nature, for the PMTs it became of material nature, as they were provided with a prototype grid, to be filled in with examples of the different components for their lesson design. Similarly, whilst the Ellis et al. (2019) paper was in essence of idealistic nature, what PMTs took from it was the TMSSR framework (fig. 3 in the paper), which made it material. The group work resource (3) is clearly of social/human nature, whilst the summary sheet was of material nature.

Linking to “curricular noticing”

In terms of how PMTs reasoned about the resources and why they found them useful, we have summarized it in the following way:

(a) Group work facilitated both *curricular interpreting* and *curricular responding*. In the *curricular interpreting* phase, group discussions allowed PMTs to collaboratively make sense of curriculum materials, incorporating their peers' insights into their understanding of lesson tasks and objectives. As PMTs shared different perspectives, they refined their interpretations of the curriculum and of reasoning tasks:

[...] in the feedback from the other groups, we discovered that they had a different perspective on the tasks that we had not considered, which meant the task might not necessarily promote reasoning in the way we thought. (S11)

Group work also played a crucial role in *curricular responding*, as it helped PMTs make joint decisions on how to implement their LPs, leading to more thoughtful and effective responses during enactment.

(b) The curricular spider web was key in the *curricular responding* phase, guiding PMTs in aligning LP elements and encouraging more detailed, thoughtful planning. PMTs used it to guide their decisions when planning lessons, as it provided a comprehensive model for ensuring that LP elements were coherently aligned. PMTs reported that using this framework challenged them to consider more details in their planning, leading to more thoughtful curricular responses. In this phase, the Ellis et al. (2019) paper also contributed directly, offering specific strategies (e.g., the TMSSR framework) to help PMTs decide how to foster reasoning in the classroom. This resource provided PMTs with a practical tool for lesson enactment, which influenced their curricular decisions and helped them adjust their responses to student reasoning.

(c) The summary sheet was essential for the *curricular interpreting* phase. It helped PMTs to understand different types of reasoning, providing clear distinctions that allowed them to interpret how reasoning could be integrated into LPs.

I feel it has become easier to be critical of tasks given in textbooks. [...] I will be better at looking at the tasks I intend to give the students to see if they actually make sense and if they promote what I want to focus on in class. (S2)

This resource directly contributed to PMTs' ability to connect curriculum materials with their mathematical knowledge for teaching, enabling them to plan lessons that promote deeper student reasoning.

CONCLUSIONS

We conclude with two points. (1) In terms of meta-resource, we propose a new developing conceptualisation that takes care of the nature of meta-resources over time and in different contexts. A *meta-resource* refers to a resource (not necessarily material) that provides subjects (in particular teachers) with information about or facilitates access to other resources, often by categorizing or linking to them, or helping subjects to reflect on other resources. In other words, it is a resource about resources. Curricular meta-resources are valuable for navigating information and making it easier to identify, select or find, categorize or reflect on specific resources or materials.

(2) Meta-resources can help PMTs develop deeper curricular noticing by offering several benefits. (a) **Enhanced resource accessibility**: Meta-resources (e.g., digital teaching platforms) allow teachers to easily access a variety of materials and tailor instruction to students' needs, promoting personalized learning. (b) **Improved lesson planning**: By using curricular noticing to identify areas needing reinforcement, teachers can use meta-resources to find the best materials and strategies for effective lesson planning. (c) **Collaboration and sharing**: Meta-resources offer platforms for teachers to share insights and lesson plans, fostering collaboration and saving time by centralizing access to quality resources.

In summary, combining meta-resources with curricular noticing can enhance instructional precision, personalization, collaboration, and efficiency in teaching.

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PROBLEM POSING AS A CURRICULUM RESOURCE FOR TEACHING MATHEMATICS THROUGH INQUIRY-BASED LEARNING: AN ACTIVITY THEORY ANALYSIS OF MATHEMATICS PRE-SERVICE TEACHERS' PRACTICE REPORTS

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The purpose of this study is to determine which problem posing (PP) task can serve as a curriculum resource to support the learning objectives in an inquiry-based learning (IBL) approach. Our theoretical framework is based on activity theory. A total of seventeen mathematics pre-service teachers' practice reports were collected and analysed using content analysis. Findings show that semi-structured PP tasks are the most effective and it can be used as curriculum resources for implementing IBL approach. This study recommends to balance openness and guidance in diverse classrooms by integrating semi-structured PP tasks with contextual prompts for teaching mathematics in IBL approaches. Future research should explore how curriculum materials can integrate PP examples for implementing IBL to foster inquiry and creativity.

INTRODUCTION

This study explores how problem posing (PP) tasks serve as curriculum resources to mediate teaching mathematics through inquiry-based learning (IBL). Using activity theory, I analyse pre-service teachers' (PSTs') reflections and interactions with these tasks, and which PP tasks enhance the implementation IBL.

PP is defined as the process of formulating mathematical problems based on given problems or reformulating existing problems. In mathematics education, PP is not sufficiently integrated into teachers' lessons, and teachers need more teaching cases involving PP to gain insight into how to use it (Zhang & Cai, 2021). Although policy documents acknowledge its importance, PP is neither substantially nor consistently integrated into the school mathematics curriculum (Cai & Hwang, 2021). Stoyanova (1997) identified three categories of PP, which raise students' awareness to generate and solve mathematical problems: 1) Free situations: Students are asked to pose problems for an open mathematical situation and without any restrictions; 2) Semi-structured situations: Contain more detailed information than free posing situations, but students are invited to explore the structure of that situation and to complete it by applying knowledge and concepts from their previous mathematical experience; and

3) Structured situations: Require students to pose problems by reformulating the already-solved problems or by changing the terms or conditions of the given problems.

IBL is a student-centred teaching strategy that encourages students to explore situations, inquire, pose questions, seek explanations, and it fosters confidence, independence, and creative thinking in mathematics education (Artigue & Blomhøj, 2013; Blomhøj et al., 2022). Despite these benefits, planning and implementing inquiry-based lessons remains challenging and teachers need support in planning inquiry-based mathematics lessons (Blomhøj et al., 2022). Integrating PP into IBL lesson design offers a promising approach for supporting teaching mathematics through IBL (Pozhegu-Ermeni et al., 2025b), while also revealing challenges and attitudes PSTs encounter when implementing IBL with PP tasks (Pozhegu-Ermeni et al., 2025a). Based on these findings this study examines which PP tasks works best to achieve the IBL learning objectives. Thus, this paper considers PP tasks as curriculum resource that PSTs used to balance the openness structure of IBL while implementing it during school practice. By analysing how these resources were used and adapted in real classroom contexts, this study aimed to understand their role in supporting teaching mathematics in IBL approach through the lens of activity theory. The research question guiding this study is: Which types of problem posing tasks can serve as effective curriculum resources for teaching mathematics through inquiry-based learning, according to preservice teachers' practice reports?

THEORETICAL FRAMEWORK

Activity theory emphasises that learning occurs through social and cultural interactions with a focus on understanding how people interact with tools, rules and communities (Engeström, 1987). An activity system is often represented by a triangular illustration (see Figure 1) and serves as a unit of analysis for understanding a larger flow of human life (Engeström, 1987). I give a brief description of Engeström's (1987) six components with examples from our study: The subject is the PSTs teaching IBL with PP. The object is the goal of the activity which is exploring effective PP tasks to improve teaching mathematics in IBL approach. The tools are recourses, instruments or strategies used to support the activity, for example the strategies or tasks used for PP (e.g., Stoyanova (1997) categories). A rule is a set of guidelines or norms that shape an activity, for example time limits or PSTs' instructions guiding the process. The community includes the social environment in which the activity is involved, including peers, PSTs and the teacher mentor. The division of labour refers to how roles are distributed among participants in the activity, such as individual vs. group work, teacher's involvement. For example, the PST design the lesson and facilitates in the process, then the students take on active role by creating their own problem and guiding discussion in the classroom. The outcome is the results of the activity, which should reflect the effectiveness in reaching the object (e.g., posed problems' quality).

This study aimed to measure "effectiveness" using activity theory by evaluating how well the components of an activity system contribute to achieving the intended

outcomes of the activity. In the context of PP in IBL approach, this means assessing how well the students achieve the desired learning outcomes of an IBL lesson plan using the tools (PP strategies), with the rules (classroom guidelines), and in interaction with the community (peers and teachers).

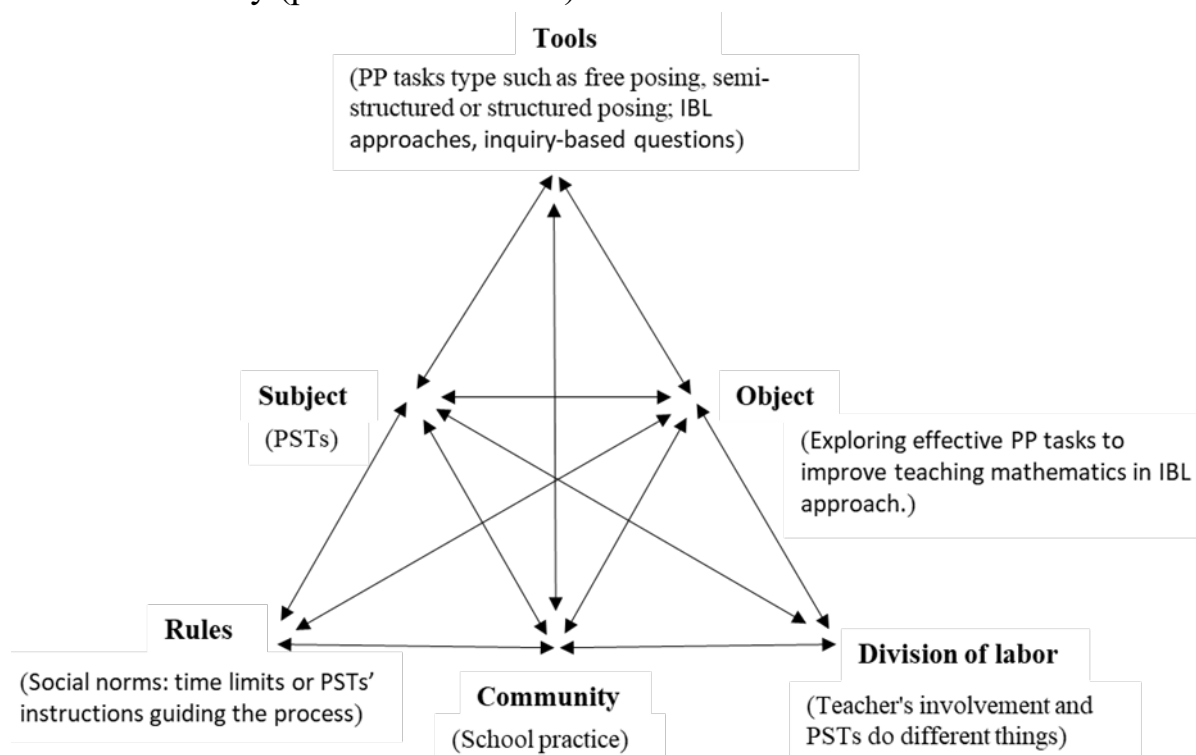


Figure 1: The activity system, adapted from Engeström (1987).

METHODOLOGY

This study analysed practice reports written by Norwegian mathematics PSTs enrolled in the five-year integrated master's program (MGLU 5–10), qualifying them to teach mathematics in grades 5–10. At the time of data collection, they were in the second semester of their first year. The practicum was organized in groups of three to four PSTs, each placed in a different school under a mathematics teacher mentor. The practice report, a mandatory coursework component, included lesson plans, teaching experiences, and reflections. PSTs were required to design lessons aligned with IBL through PP while following the school's agenda. Data consist of 17 reports, one from each group.

The data were analysed using qualitative content analysis through the lens of activity theory. Based on the object and the components of the activity system this study reflects on outcome-based effectiveness of PP tasks as tools for teaching IBL. Thus, in terms of activity theory, this study aimed to see if there were specific PP tasks (tools) that help PSTs (subjects) achieve the goal (object) more effectively. To measure the effectiveness, I look on the PSTs' practice reports by investigating for three situations of PP (Stoyanova, 1997) focusing on the quality of posed problems. This has been done

by analysing how well the students pose problems based on solvability, alignment with the objectives of the IBL lesson plan, and PSTs' reflections about the lesson.

RESULTS

This study explored the effectiveness of PP tasks for teaching an IBL approach within an activity theory analysis. Thus, the results are structured around the key components of the activity system: the subject, object, tools, community, rules, division of labour, and outcome. References in quotes (e.g., "PR 1", "PR 2") indicate the corresponding practice report numbers.

Subject

For students to successfully complete the PP tasks in an IBL lesson plan, the PST's facilitation was essential. The results indicated that teachers play a crucial role in guiding and engaging students in inquiry-based exploration through PP tasks. PSTs guided students in classroom with mathematical hints, open-ended questions, mathematical prompts or by encouraging discussions that clarified the ambiguities of the students during the process. For instance, in a semi-structured PP activity with arithmetic calculations, PST emphasised the importance of providing sufficient information in the problem text:

PST (PR 10): A statement from a student who worked on the assignment is: "There is 1 cake for the birthday. Christine eats 2 pieces. How many pieces are left of the cake?" We overheard a student say: "There is no way to solve the problem since we don't know how many pieces the cake is divided into." We chose to address this at the end of the session. We showed them one example on the board so that everyone gets the importance of giving enough information in one text assignment.

Object

The primary goal of this study was to explore which PP tools can help PSTs teach and achieve IBL learning objectives. The tools found in this study are discussed below.

Tools

Different PP tools were identified for teaching and achieving the IBL learning objectives. Due to the collaborative nature of the IBL approach, all PP tasks were designed to be solved by peers rather than the students themselves. Thus, posing problems for others is a tool that impact the effectiveness of teaching IBL because this encourages discussion. Time was also an important tool for this method of teaching. The results showed that when the PSTs had enough time (more than 45 min), the PSTs discussed with the students the obstacles they encounter during task creation. This allowed PSTs to achieve the learning objectives of the IBL lesson plan. Also, the results showed that teachers must be aware about different elements when they teach an IBL approach with PP tasks: the topic of the lesson need to be appropriate for PP tasks; or there is variability among students' understanding of mathematics because every class

has students who struggle with mathematics. Therefore, it is important that teachers consider balancing these elements:

PST (PR 13): We observed that most groups worked well, but that it was some students who had difficulty doing what they were supposed to. This may be due to lack of understanding within the topic, or problems understanding the task itself.

Tools such as programming languages or PP worksheets provided to be highly effective in facilitating students IBL learning objectives. The use of programming in geometry facilitated the IBL lesson plan during a lesson with angles and degrees. The students were able to create geometric figures through programming (see Figure 2):

PST (PR 13): Here we can see that the fellow student who was supposed to model what the code on the left in figure drew, got it right with his interpretation. Here we can see how this student has cracked the code when it comes to “thinking like a robot”.

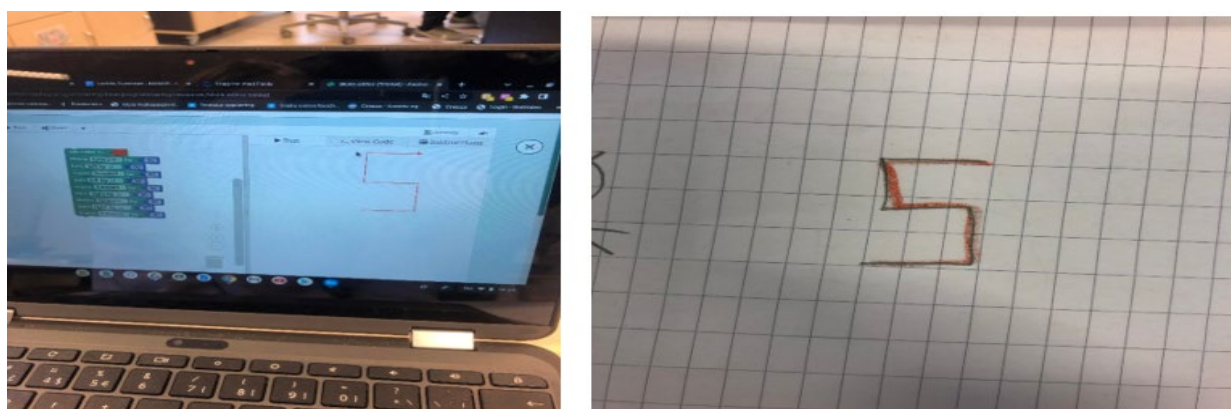


Figure 2: Example from a programming in geometry.

PP worksheets (see Figure 3) provided prompts that guided and supported students through the PP process. But the results showed that different PP strategies play differently. The nature of free situations is more about open task, in which students can end up in similar ways of thinking. During a lesson about measures of area and volume in practical (see Figure 3), the PST used a worksheet for a swimming pool with a question on it “How much water is in a swimming pool?” and the situation ended up like this:

PST (PR 4): We found that despite the open nature of the task, the students proceeded in somewhat similar ways. This may be because the groups were presented with the same question for reflection.



Figure 3: Examples of different worksheets that PSTs used in their teaching practicum.

Based on this, this study found that it is crucial for teachers to structure free situation tasks and introduce guided questions clearly at the start of the lesson. Integrating these elements of PP early can transform the lesson into an IBL approach, while using PP only during the end of lesson plan, limits its impact to a supplementary role. In another lesson with the topic of number systems, PSTs used free situation task but the way the question was introduced allowed the students to be more creative in producing problems and avoid the same thinking of students:

PST (PR 5): In the introduction to the lesson, we told the story of a family that created a t-shirt factory. When the family increased production, it became necessary to organise the T-shirts so that they knew how many they had at any given time. We therefore introduced the pupils to the question of: “How it is wise to organise these t-shirts?”, and they were allowed to explore this for themselves.

Community

It was the collaboration between students and PSTs that enhanced the quality of the problems posed during the activities. For example, when students struggled to create problems, then the PST used prompt question:

PST (PR 10): There were some students who didn’t continue after 10 minutes, we tried to ask open questions that started the thought process such as: “How much soda do you want each person to get?”

Rules

Students were encouraged to create and explore mathematical problems, under the rules of IBL and PP approaches. Withing these norms, students were able to take ownership of their learning by posing problems and seeking solutions earthier independently or with peers. The rules promoted the students’ independence and creativity as one PST mentioned that:

PST (PR 16): The students got started with the task preparation fairly quickly, and many showed interest in the responsibility and the freedom they were given. Their work on designing the task showed that they felt ownership of their learning process and the final product they created.

Division of labour

There was a clear division of labour within the classroom: the teacher facilitated PP tasks, while students created and solved problems. Keeping this balance can sometimes be challenging based on how participants can manage their roles in the activity. Thus, as a result of the appropriate division of labour, students will be able to engage deeply in PP process in an IBL classroom. For example, during a lesson related to programming in mathematics, the PSTs distributed the roles among the students in four different classes:

PST (PR 13): We ran the scheme in 4 different classes. In class A and B of the classes we ran the activity where the students had to draw something the other had to program, and in classes C and D we ran the activity where the students had to look at their partner's program and try to draw what they thought the program was going to draw.

Outcome

Overall, the PP tasks successfully met the learning objectives for an IBL lesson plan, At the end of the lessons, students were able to generate and solve more complex mathematical problems independently:

PST (PR 13): In summary, we are positively surprised by the learning outcome for the students. The students are a heterogeneous group, and they will therefore not have the same prerequisites for creating and solving equations. When the students were allowed to create their own tasks, they were able to adapt the degree of difficulty, and experienced mastery at their level.

DISCUSSION AND CONCLUDING REMARKS

In this study, activity theory provided a useful lens for understanding how PP tasks mediate teaching mathematics through IBL. Findings indicate that semi-structured PP tasks are the most effective type of curriculum resource to mediate the implementation of IBL in mathematics, according to PSTs' practice reports. Semi-structured PP tasks provided clear prompts and scaffolding, enabling students to generate problems that align with lesson objectives and foster deeper engagement in inquiry. In contrast, free posing tasks, while offering flexibility, often led to repetitive or limited thinking unless supported by contextual framing or guiding questions. Structured posing tasks were useful for reinforcing specific concepts but offered less opportunity for open inquiry.

Programming-based PP tasks were particularly effective in geometry lessons, promoting collaboration and interpretation consistent with IBL principles. However, their success depended on sufficient time and teacher facilitation, highlighting the importance of organizational conditions for PP effectiveness. Interaction between PSTs and students also played a decisive role. PSTs' use of open-ended questions and mathematical hints played a decisive role in overcoming challenges during problem creation, therefore reinforcing semi-structured PP tasks as effective mediating tools within the activity system.

By situating PP within an activity theory framework, this study highlights the interplay between tools, rules, community, and division of labour in shaping teaching mathematics through IBL effectiveness. For teacher educators and curriculum designers, these findings underscore the need to develop PP tasks as curriculum resources that balance the openness structure of IBL, integrate contextual prompts, and allow flexible adaptation in diverse classrooms. Overall, semi-structured PP tasks were most effective when combined with PSTs mediation and embedded scaffolding. These insights inform both practical implementation and theoretical understanding of PP tasks as a curriculum resource design for teaching mathematics through IBL. Future research should examine broader samples beyond Norwegian PSTs and investigate the direct classroom implementation of semi-structured PP tasks, as this study relied on analysed practice reports, including how curriculum materials can integrate PP tasks for implementing IBL to foster inquiry and creativity.

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TEXTBOOKS AND DIDACTIC TRANSPOSITION ANALYSES: MATHEMATICAL MODELLING IN NORWAY AND FRANCE

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Textbooks have always been critical empirical material for analysing didactic transposition processes as indicators of the knowledge to be taught. Considering textbooks from a didactic transposition perspective helps locate their proposals within institutional conditions and constraints. This contribution illustrates how textbook analysis operates within didactic transposition by examining mathematical modelling as a case study. The specific characteristics of this newly introduced teaching content in European countries are addressed through cases from Norwegian and French first-year upper secondary courses. Compared to other mathematical content, an intriguing difference in treatment emerges, likely linked to the lack of a common, structured body of knowledge about modelling in the scholarly institution.

INTRODUCTION

Didactic transposition, as outlined by Chevallard (1989b), is the process through which knowledge from the scholarly domain is transformed into knowledge to be taught in educational institutions. This process is necessary because knowledge in its original form is created for *use* in specific professional contexts, not for teaching. Chevallard emphasises that the adaptation from “knowledge as a tool” to “knowledge as something to be taught” is not neutral. It involves decisions, shaped by institutional and societal conditions and demands, on what aspects of the knowledge should be prioritised. This process introduces changes in the way knowledge is presented, resulting for instance in simplifications, restructuring, recontextualisation, or exclusion of certain aspects of the knowledge. While necessary for the teaching process, these changes highlight the tension between preserving the integrity of knowledge and making it accessible in a structured, teachable form.

Textbook analysis plays a critical role in the study of didactic transposition processes since its first developments by Chevallard (1985). As a primary vehicle for conveying the curriculum, mathematics textbooks significantly shape mathematics education globally (Fan et al., 2013). Thus, textbooks serve as appropriate empirical objects for examining the knowledge to be taught, along with the constraints they impose on teaching and learning processes. The research question we address is: “How does didactic transposition analysis of newly introduced textbook content, specifically

mathematical modelling, help identify conditions and constraints that affect its teaching in the school institution?”

The aim of this contribution is to present the methodology used to address the research question by considering two particular case studies of modelling, based on textbooks from Norway and France.

DIDACTIC TRANSPOSITION AND TEXTBOOK ANALYSIS

The process of didactic transposition can be examined through two primary lenses (Strømskag & Chevallard, 2024). The first, *diachronic transposition analysis*, investigates the evolution of a piece of knowledge k over time, tracing its genealogy by studying how its components have changed or disappeared as k has been transposed between institutions. This type of analysis seeks to understand which parts of k come from which ancestor and why certain components may no longer be present. The second approach, *synchronic transposition analysis*, focuses on comparing two or more versions of k across different institutions at a single point in time, without any direct historical connection between them. It involves contrasting how these institutions treat a particular knowledge organisation simultaneously; for example, comparing how different textbooks handle a particular mathematical concept. The following selected studies illustrate varied applications of didactic transposition analyses that are supported by textbooks analyses.

A case of diachronic transposition analysis is presented in Klisinska’s (2009) study, which focuses on how the fundamental theorem of calculus (FTC) has been historically adapted for teaching purposes. Her analysis traces the evolution of the theorem’s proof, showing how its rigor and complexity have been adjusted to meet the needs of various institutions. Another study of the same type has been done by Strømskag and Chevallard (2022), who examined the historical disappearance of parameters in elementary algebra, revealing key events in its genealogy that led to this collapse.

Other studies, focusing on synchronic analysis, examine textbooks to identify results of didactic transposition processes. Wijayanti and Winslów (2017) have analysed how the content on proportion is presented in Indonesian textbooks and found that it focused on a limited set of task types and techniques, with minimal emphasis on the theoretical justifications. Topphol (2023) examined the didactic transposition of the FTC by comparing its organisations in a university textbook and a Norwegian textbook for Grade 13. Lastly, Strømskag and Chevallard (2024) analysed the didactic transposition of concavity of functions by comparing its organisations in university and Grade 12 textbooks, while also examining its broader applications in disciplines such as architecture and economics. These studies highlight the challenges of didactic transposition where theoretical elements are compromised for technical simplicity at the school level, with less emphasis placed on the utility of the piece of knowledge at stake, whether in extramathematical or intramathematical contexts.

MATHEMATICAL MODELLING IN SECONDARY EDUCATION

Our research examines modelling as a core element at upper secondary level in Norway and France, focusing on recent curriculum reforms (2020 in Norway, 2019 in France), though present in the Norwegian curriculum since 1994. Mathematical modelling is being explicitly integrated into curricula of an expanding number of countries (Blum & Pollak, 2018). Some studies have analysed the treatment of modelling in textbooks (e.g., Berget, 2022; Frejd, 2013; Zwaneveld et al., 2017). This research shows that mathematical modelling is underrepresented or poorly implemented in school textbooks. Many tasks focus on procedural skills, such as applying formulas and solving stereotypical problems, rather than engaging students in the full process of modelling, which includes interpreting, validating, and iterating models. There is also a notable disconnection between curriculum goals, which often emphasise the application of mathematics to real-world contexts, and the actual textbook content, which tends to limit students' opportunities to engage in authentic, open-ended problem-solving. Our study proposes an extension of this research line by addressing the phenomena related to the teaching and learning of modelling from a didactic transposition perspective. This perspective is especially relevant given the novelty of modelling as knowledge to be taught, compared to other more established mathematical topics.

MAIN ELEMENTS OF THE METHODOICAL APPROACH

Analysing didactic transposition processes

When considering mathematical modelling as knowledge to be taught, textbook analyses play a double role. On the one side, textbooks are considered as empirical data of a broader unit of analysis that includes evidence from the different institutions that intervene in the didactic transposition process, especially the school and the scholarly institutions. The questioning then focuses on the elements of the knowledge to be taught that are offered by textbooks, their rationale, origin, and legitimacy. On the other side, textbooks are also components of the didactic transposition process, thus participating of the set of conditions and constraints that shape the knowledge to be taught and facilitates its teaching and learning. Three intertwined phases appear in the analysis of textbooks from a didactic transposition perspective. One is the elaboration of an epistemological reference model about modelling that corresponds to the researcher conception about modelling without considering school or scholarly institutional constraints (Barquero et al., 2019). Another phase corresponds to the analysis of the scholarly knowledge about modelling that provides ingredients and legitimacy to the knowledge to be taught but can also limit its structure and evolution. Finally comes the analysis of the textbooks themselves as indicators and shapers of the knowledge to be taught. The three phases are closely related and interact continuously along the research process, the reference model providing a lens to look at the empirical data while being in continuously evolution according to it.

Explanation of the reference model

Our epistemological reference model of mathematical modelling is based on previous works in the ATD by Chevallard (1989a) and Barquero (2023). Mathematical modelling consists of establishing a relationship between a system to be studied and a model. This process typically involves three main stages: defining the system and its parameters, constructing the model by establishing mathematical relations between the parameters, and working on the model to generate knowledge about the system. Therefore, in the modelling process, a system is a “piece of nature” governed by its own laws, about which we pose questions, while a model—also a system—is the constructed to answer those questions. No object is inherently a system or a model; this depends on its role in the modelling process. A model in one context can become a system in another. This shift is called *reversibility* when a system becomes the model of its own model, and *recursivity* when modelling proceeds in successive iterations, with a model becoming the system of a model at a higher level.

The *raison d’être* of modelling processes is to answer specific questions. When a system does not readily provide answers, it becomes necessary to work with a model of that system instead. Thus, two essential criteria for an effective modelling process are the *validity* of the answers obtained and the *easiness* of working within the model.

Analysis of the scholarly knowledge

Puchaczewski et al. (2024) characterise key aspects of the dominant model of modelling in scholarly institutions by analysing nine university textbooks that include “modelling” or “modeling” in their titles. We assumed that knowledge intended for university teaching, as a result of internal didactic transposition, would closely reflect scholarly knowledge in academic settings, since this transposition occurs within the same institutional context (Bosch et al., 2021). Our analysis reveals varied conceptions of modelling and diverse content organisations. Some authors structure their content around modelling steps, while others focus on types of systems or mathematical content used in modelling. Two primary trends emerge from this diversity. The first is an application-oriented view, where existing models are emphasised and applied, aligning with Barquero et al. (2019). The second is an integration-oriented view, presenting modelling as an inquiry process for answering open questions. Despite the proliferation of books titled “Mathematical Modelling,” there is no standardised approach to organising scholarly knowledge on modelling, nor is there a shared discourse on its definition or main components.

Analysis of the knowledge to be taught

The final part of our analysis examines the knowledge to be taught as presented in textbooks and curriculum guidelines in Norway and France with an initial focus on two textbooks, one from each of the countries (Puchaczewski et al., 2025). Modelling, recently introduced as a core component, is approached distinctly in each country.

The Norwegian textbook (Borge et al., 2020) dedicates a chapter to modelling, with sections on “Mathematical models,” “Regression analysis,” and “Modelling in practice.” Modelling is linked to formulas, functions, graphs, and tables. The textbook includes various examples of modelling extramathematical systems, such as estimating fabric requirements for a shirt sleeve using geometric approximations and area formulas, predicting maximum heart rate as a person ages with a linear function, and comparing linear and exponential models for population growth, among others. In the regression section, modelling involves fitting functions to data, using digital tools (e.g., GeoGebra) for regression analysis. For example, a hiker records air pressure at different altitudes, revealing an exponential decrease in pressure as altitude increases. By applying exponential regression, students are asked to model this trend and use the concept of average rate of change to understand how the decrease in pressure diminishes with altitude.

Functions play a central role in the Norwegian textbook, with emphasis on understanding a model’s *range of validity*—recognising that models are simplifications effective within specific conditions and intervals and may need adjustment as new data emerges. Each example in the modelling chapter involves constructing a model of an extramathematical system by identifying relevant parameters and exploring their relationships. The importance of choosing parameters based on empirical observations is highlighted, showing that parameters need to align with observed data to make the model realistic and applicable. However, the *absence of intrinsic system laws* in these models—such as the use of Newtonian mechanics for braking—means that the models focus more on fitting observed data rather than deriving relationships from foundational principles of the system.

By comparison, in the French textbook (Gringoz et al., 2023), modelling is linked to inequalities and functions, described through two main steps: identifying the placeholder and translating the task using it. Modelling examples focus on intramathematical or simplified systems, such as geometrical figures or cost functions. For example, students are asked where to place a point on an already drawn geometrical figure so that one area is greater than another, or what function gives the profit of a product-selling company, when the cost of production is a second-degree function and the price of a unit constant.

Extramathematical systems in these tasks are highly simplified, as all provided information is directly relevant. For instance, in the mentioned exercise on profit, the cost of production is already expressed as a quadratic function without any justification or derivation. Similarly, when dealing with geometrical systems, the algebraic model is derived from straightforward combinations of known geometric formulas for lengths, areas, or volumes. In both cases, this approach renders the modelling process relatively straightforward, focusing primarily on *identifying appropriate algebraic expressions*. There is little scope for exploring alternative models or for extending the modelling process through deeper questioning or additional layers of complexity.

The Norwegian textbook emphasises the applied, empirical nature of modelling, while the French textbook integrates modelling within algebraic contexts. Although both use functions as a modelling tool, their methods diverge: the Norwegian approach emphasises practical applications and model validation through real-world data, encouraging students to explore how models align with or deviate from empirical observations, while the French approach focuses on variable identification and relationships.

From a didactic transposition perspective, these variations reflect the lack of standardisation in modelling discourse and terminology between scholarly knowledge and knowledge to be taught in upper secondary education. Although modelling has been part of the Norwegian upper secondary curriculum since 1994 (Ministry of Education, Research, and Church Affairs, 1999), in which it was a special focus area under “Modelling, experimenting, and exploration,” its designation as a *core* element for Grades 1–13 in the 2020 curriculum highlights its expanding role. This long-standing inclusion may explain the broader conceptualisation of modelling in the Norwegian textbook, as the didactic transposition process of integrating modelling into upper secondary education has been ongoing for many years (see e.g., Berget, 2022).

We hypothesise that the notable diversity in mathematical organisations of modelling between the Norwegian and French textbooks reflects both the more mature integration of modelling in Norway and a general lack of scholarly consensus on modelling. This results in significant variation in mathematical organisations in textbooks—a diversity that is considerably less pronounced in other mathematical topics. This is a research hypothesis that needs to be tested with further evidence.

PRELIMINARY FINDINGS AND DISCUSSION

These textbook analyses draw on a reference model of modelling that helps avoid bias towards either school or scholarly conceptions. The model provides a structured framework and essential terminology—such as *model*, *system*, *recursivity*, and *reversibility*—clarifying what textbooks make visible and what they omit. Viewing these omissions through the lens of didactic transposition situates them institutionally, rather than as isolated issues at the school level, enables us to distinguish between what stems from the field’s relative novelty (e.g., lack of consensus) and what arises from adapting knowledge for teaching. This perspective facilitates analysis across multiple levels, from individual tasks to institutional frameworks.

At the scholarly level, the absence of standard terminology and structure signals an underdeveloped framework for modelling, which in turn constrains what becomes available for teaching. The transposition process nonetheless produces variation in teachable knowledge, as seen when comparing Norwegian and French textbooks. These differing interpretations complicate modelling’s presentation, since—like problem-solving or proof—it is not a mathematical object but a dimension of mathematical activity. While expectations that teachers incorporate modelling in upper

secondary education are increasing, the degree to which current textbooks offer opportunities for genuine modelling rather than pre-structured modelling-like tasks remains uncertain and warrants closer examination in future research.

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BRIDGING BORDERS: UNDERSTANDING UKRAINIAN TEACHERS' INTERACTIONS WITH FOREIGN CURRICULUM RESOURCES

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This study aims to enhance our understanding of how teachers interact with foreign mathematics curriculum resources and the factors influencing these interactions. It analyses two cases of foreign resource adaptation by Ukrainian teachers. This includes examining the original intentions behind resources designed in different cultural contexts, as well as how these resources are viewed and used from the perspective of Ukrainian teachers. The findings emphasize that teachers' interactions with foreign curriculum resources are shaped by multiple factors, particularly the existence of national evaluation and state-certified textbooks provided free of charge to all students in Ukraine, as well as the pedagogical approaches and mathematical praxeologies embedded in these resources.

INTRODUCTION

The globalization of the curriculum publishing industry, along with the widespread availability of online resources to teachers worldwide, regardless of location or language, highlights the need to study curriculum use in cross-cultural contexts (Haggarty & Pepin, 2002). This entails investigating how resources are adapted and applied across different countries and cultural settings, particularly when a resource designed in one country is utilized in another with a distinct cultural background and institutional settings. In this context, a key issue is understanding the influence of cultural and institutional differences on resource adaptation. This study aims to contribute to this discussion by examining how Ukrainian teachers interact with curriculum resources developed in other countries.

In Ukraine, the education system is undergoing significant reform, particularly since gaining independence in 1991. These reforms aim to modernize the educational framework and align it with European standards. The state certification of textbooks and directives for mathematics teaching and learning at various levels coexist with the Ministry's demand for teachers to innovate their practices. As a result, there is a growing trend among Ukrainian teachers to explore and adopt foreign resources. Examining how these resources are integrated into teaching practices within this evolving cultural landscape can, in turn, enrich the broader discourse on curriculum resource adaptation and teacher practices in cross-cultural settings.

This study aims to contribute to answering the following research question:

Which cultural and institutional factors influence the adaptation of foreign resources by teachers in Ukraine, and how do they operate?

THEORETICAL FRAMEWORK

This study draws on the Documentational Approach to Didactics (Trouche et al., 2018), which offers a comprehensive view of teachers' work by considering the wide array of resources available for their use, design, and adaptation. A central concept in this approach is the notion of "resource," understood as a tool that "re-resources" a teacher's work (Adler, 2000). Teachers interact with different types of resources through processes of selection, analysis, interpretation, modification, classroom implementation, and reflection, all forming part of their documentational work. In this study, we focus specifically on teachers' interactions with *curriculum resources* defined as any print or digital artifacts designed to support instructional programs and student learning over time (Remillard, 2019).

As our study focuses on both teachers' interactions with resources and the conditions and constraints that influence these interactions, we also draw on the Anthropological Theory of Didactics (ATD) (Chevallard & Sensevy, 2014; Artigue & Winslow, 2010). This theory considers the institutional and cultural factors that affect knowledge dissemination and teaching practices, organizing them into hierarchical levels of didactic co-determination (Table 1). These levels range from the highest (*civilization*) to the lowest (*subject*) and mutually influence and constrain each other.

Level	Description	Examples
<i>Civilization</i>	Cultural principles and broader societal values regarding education	Values of Western or Eastern culture
<i>Society</i>	Social structures that govern and shape educational systems	State, ministry and their decisions
<i>School</i>	Teaching institutions including their characteristics and conditions	Different types of schools
<i>Pedagogy</i>	Approaches, principles, and methods for teaching and learning	Constructivism
<i>Discipline</i>	The broadest context for the content taught	Mathematics
<i>Domain</i>	A larger part of discipline that brings together several sectors unified by common theories	Algebra
<i>Sector</i>	A specific area within a discipline that shares a common theoretical foundation	Polynomials
<i>Theme</i>	Particular topics within a sector	Equations
<i>Subject</i>	Specific tasks and solving techniques	Linear equations

Table 1: Levels of didactical co-determination according to ATD.

In our study, by analysing teachers' interactions with foreign resources in the context of teaching specific subjects, we aim to trace the levels of the hierarchy of co-determination to identify the interrelated factors influencing these interactions.

METHODOLOGY

This paper presents a case study involves two Ukrainian secondary school mathematics teachers with different levels of experience and backgrounds, working in distinct types of schools in Kiev (capital of Ukraine). The teacher profiles are as follows.

Mila is 58 years old, female with 36 years of experience teaching secondary school mathematics. She holds the title of "teacher-methodist" (equivalent to "senior teacher") and serves as the school leader for the methodical organization of Mathematics, Physics, Astronomy, and Economics teachers. She regularly uses ICT tools in her teaching and extracurricular work with students and shares her experience by talks at methodical organization meetings, publishing papers in professional journals as well as via her personal blog.

Lisa is a 35-year-old female with 10 years of mathematics teaching experience at the secondary school level. She holds a PhD in Educational Sciences and currently teaches at a private school that offers two types of classes: one following the Ukrainian curriculum, taught in Ukrainian, and the other following the Canadian curriculum, taught in English. Lisa teaches in the first type of classes and also works as an assistant in the second type of classes. She also systematically uses ICT in her teaching.

For data collection, we conducted semi-structured interviews (Lichtman, 2012, p.115) with each teacher. The interview consisted of three parts: 1) the teacher's professional profile, including initial education, career trajectory, current teaching context, and collaborative work; 2) the resources they have used throughout their past and current teaching practices; 3) teacher-selected examples of foreign resources use.

For this study, we focus on two examples - one from each teacher. Specifically, we conducted a comparative analysis of the original foreign resource and its modified version following the teacher's adaptations. In particular, we identified the types of tasks and solving techniques intended in both versions (original and teacher-adapted) of each resource. Our analysis was further complemented by interviews with teachers, uncovering the context, motivations, and objectives behind the modifications. Additionally, if the teacher mentioned other resources during the discussion, we included them in our analysis. We ultimately analyzed cultural factors through the lens of didactical co-determination, paying particular attention to the differences highlighted in the comparative analysis of the original and adapted resources.

RESULTS

Case of Mila

Mila's resource is a digital version of the "number slide" used to illustrate the rules for multiplying and dividing by multiples of 10 (such as 10, 100, 1000, etc.). The initial resource is a software adaptation of a tool proposed by the French Ministry of Education. The user can input a number into a place value chart (ensuring the correct position of each digit) and, after selecting the operation (multiplication or division) and the multiple (10, 100, or 1000), observe the results. The digits shift in the place value

chart either to the left or right depending on the operation, moving a number of columns corresponding to the power of 10 in the multiplier. For instance, multiplying 2,537 by 100 (Fig. 1) results in the digits shifting two columns to the left (Fig. 2).

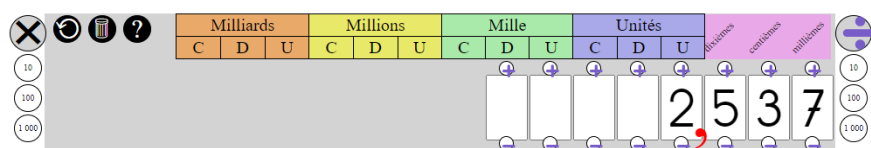


Figure1: Initial state after entering the number in the place value chart.

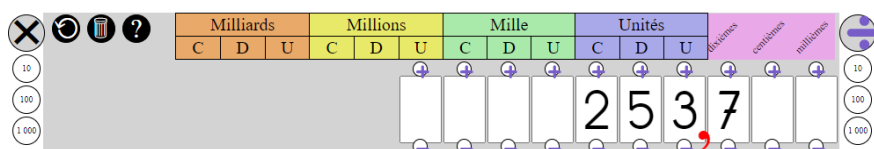


Figure 2: The result of the multiplication.

As specified in the teacher guide provided by the French Ministry of Education, the "number slide" aims to reinforce the concept that when multiplying or dividing by powers of 10, it's not the decimal point that moves, but rather the digits shift position in the place value chart as their value increases or decreases by powers of 10. The justification of this technique is grounded in the properties of the decimal system of numeration.

The adapted resource takes the form of a short instructional video created by Mila, using an application that enables screen recording and voice narration. While the visual elements remain unchanged, Mila's narration introduces the following rule: "To multiply by a power of 10, move the decimal point to the right by the number of digits corresponding to the power. To divide by a power of 10, move the decimal point to the left by the same number of digits." Thus, Mila's audio explanations omit any reference to the movement of digits seen in the video or the resulting change in their values due to multiplication by a power of 10. Consequently, the adapted resource juxtaposes two different techniques: the movement of digits within the place value chart, as demonstrated visually, and the shifting of the decimal point, as explained in the narration.

From the interview with Mila, we learned that the resource was created for 5th-grade students as part of the distance learning program implemented due to the war in Ukraine. Mila's goal was to support students struggling with the textbook rule by providing a visual aid.

I used arrows of different colours to indicate the direction for moving the decimal point, but that didn't help. So, when I discovered this animation, I thought it would illustrate better the direction in which the decimal point moves.

Analysing the Ukrainian textbook used by Mila, we found that Mila's audio explanation given in the resource aligns with the rule provided in the textbook. From the interview, we speculated that the overlap of both techniques in the resulting resource did not concern the teacher. She noted that the animation did not align with

the rule presented in the Ukrainian textbook. However, she believed that the application could still be useful for visually illustrating the movement of the decimal point, thereby making the textbook's rule clearer for students. In addition, Mila explained that she avoided discussing the movement of digits in the place value chart (as originally intended in the resource) for the following reasons:

All my students have access to the same textbook, so they can revisit the rule at any time and eventually memorize it. If I introduce an additional rule, it could only confuse them. "Why did the teacher give us another rule?!" that's what they might say.

It is evident from this extract that the textbooks serve as a common reference for Mila, providing mathematical concepts, rules, and theorems shared with her students. She avoided discussing rules in the manner originally intended by the foreign resource because they were not part of this shared reference. Mila does not see the foreign resource as an opportunity to introduce another approach to teaching the topic or to deepen the explanation of the rule presented in the textbook, instead, she focuses on ensuring that all students can effectively apply the rule given in the textbook. Thus, the intended use of the foreign resources is altered, and their didactical potential is not fully utilized.

We can suppose that the textbook serves as a pivotal resource for Mila, dictating both the content and the method of teaching mathematics in class (*discipline* level and *subdisciplinary* levels), while foreign resources are mostly supplementary, used to support the instructions (with modifications, if necessary). Mila's adherence to the textbook is rooted in her long-standing relationship with this resource, which dates back to the Soviet era, when only one Ministry-certified textbook was available for classroom use, and teachers were required to follow it strictly. Today, textbooks in Ukraine remain state-certified and are provided to students free of charge by the state (*societal level*). Although more state-certified textbooks are now accessible, Ukrainian teachers still often choose a particular textbook authored by the same group and adhere closely to its content and pedagogical approach. The state certification of textbooks, which requires compliance with established curricula, appears to be seen by some teachers as a guarantee of both what mathematics should be taught and how it should be taught. However, this reliance on certified textbooks, as we observe, can limit the potential of foreign resources for teaching and learning mathematics.

Case of Lisa

Due to the specific context of the school where Lisa works, she has access to both Ukrainian and Canadian textbooks. In this study, we analyse a task from the Canadian textbook that Lisa adapted for her ninth-grade lesson taught in Ukrainian. The original task from the textbook is shown in Fig. 3 (a), Lisa's adaptation is presented in Fig. 3 (b), and our English translation of her adaptation is provided in Fig. 3 (c). As illustrated, the teacher opted to limit the number of questions to one and proposed various answer options. An analysis of the task from the initial resource and its placement within the Canadian textbook reveals that the intended technique for solving involves

transforming function graphs. However, this technique is not strictly necessary for the task derived from Lisa's resource. Instead, students can simply calculate the value of N for the five given numbers and select the one that produces the maximum value.

The finance team at an advertising company is using the quadratic function $N(x) = -2.5(x - 36)^2 + 20\,000$ to predict the effectiveness of a TV commercial for a certain product, where N is the predicted number of people who buy the product if the commercial is aired x times per week.

- Explain how you could sketch the graph of the function, and identify its characteristics.
- According to this model, what is the optimum number of times the commercial should be aired?
- What is the maximum number of people that this model predicts will buy the product?

(a)

Фінансова група рекламної компанії використовує квадратичну функцію $N(x) = -2,5(x - 36)^2 + 20000$ для прогнозування ефективності телевізійної реклами для певного товару, де N - передбачувана кількість людей, які купують товар, якщо рекламний ролик виходить у ефір x разів на тиждень. Скільки разів на тиждень має виходити реклама в ефір, щоб передбачувана кількість покупок була найбільшою?

А	Б	В	Г	Д
100	90	36	25	10

(b)

The financial team at an advertising company is using the quadratic function $N(x) = -2,5(x - 36)^2 + 20000$ to predict the effectiveness of a TV commercial for a certain product, where N is the predicted number of people who buy the product if the commercial is aired x times per week. What is the number of times the commercial should be aired by week that the number of people who buy the product will be maximal?

A	B	C	D	E
100	90	36	25	10

(c)

Figure 3: (a) The task found by Lisa; (b) The task modified by Lisa; (c) The translation of the modified task in English.

From the interview with Lisa, we learned that she regularly gives her students tests to prepare them for the Ukrainian External Independent Testing (EIT). The EIT involves tests with single or multiple-choice questions (like in the example on Fig.3). This evaluation occurs at the end of high school (after 11th grade), and the results are critical for university admission. Schools, especially private institutions, are keenly interested in students' performance on this evaluation, as it directly impacts their reputation and enrolment numbers. School's concerns about student performance on the EIT (*school level*) play a significant role in shaping Lisa teaching practices. Although the EIT is not an official evaluation of her work, it serves as an informal indicator of her effectiveness as a teacher. Consequently, her focus on preparing students for the national evaluation and its framework influences how she interacts with foreign resources. Canadian textbook, in particular, serves for Lisa as a source for practical problem-solving tasks, which are common in EIT's tests.

In Canadian textbooks, there are many practical problems, unlike in Ukrainian ones. This type of problem is becoming increasingly common in EIT tests. So, I take the context of the tasks from the Canadian textbook and create tests that I give to my students.

Lisa's modifications to the format of the task provided by the Canadian textbook influenced the range of solving techniques that students could apply. Moreover, framing the task as a test may obscure whether students are employing solving techniques different from those intended by the original resource and/or the teacher. In Lisa's case, it can be concluded that the format and type of tasks included in the national evaluation framework (spanning from *pedagogy* to *subject level* in terms of co-

determination) imposed by the government (*society* level) play a significant role in shaping her interactions with the foreign textbook.

DISCUSSION AND CONCLUSIONS

In this paper, we explored the cultural and institutional factors that influence teachers' interactions with and adaptations of foreign curriculum resources within a specific country, and how these factors shape this process. Analysis of two Ukrainian teacher cases revealed a range of factors operating at different levels of co-determination, which affected how the foreign resources were used. In particular, the national evaluation and state-certified textbooks provided free of charge to all students (corresponding to the *society* level), which, in turn, determine the pedagogical approaches (*pedagogy* level) and mathematical praxeologies to be taught (spanning from the *discipline* to the *subject* level), significantly impacted the documentational work of teachers with foreign resources in the two observed cases. We identified substantial differences between the original resources and their adapted versions, particularly regarding intended goals, solving techniques and underlying theoretical foundations. Drawing on the two cases, we argue that, while teachers adapt foreign resources, there are opportunities for overlapping techniques from both foreign and national curriculum resources in the resulting teachers' documents (as seen in Mila's case), as well as for changes in the range of solving techniques that are either available to students or originally intended by the task (as illustrated in Lisa's case).

When comparing the two teachers' interactions with foreign resources, based on the interviews and beyond the two analysed resources, it appears that factors such as the teachers' professional paths and the type of school they work in can influence how they engage with and adapt foreign resources. This influence occurs indirectly through their established relations with national curriculum materials. In contrast to Mila, who began her career during the Soviet era and has spent her entire career in state schools, where her work is regularly evaluated and inspected, Lisa, being a relatively young teacher, appears to have more flexibility in working with national textbooks. This is likely due to the characteristics of her private school environment and the absence of teacher work evaluations. This autonomy might enable her to explore and adapt various foreign resources with fewer constraints than those typically found in the more regulated setting of state schools. However, the national evaluation seems to play a more significant role for Lisa in shaping her interactions with foreign resources than it does for Mila. This leads us to the hypothesis that teachers' professional paths and type of school (*school* level) may either mitigate or amplify the impact of higher-level cultural and institutional factors on how teachers use foreign resources in their classrooms. To verify the hypothesis, further research involving a broader range of teachers of different ages and school types is needed. This approach could also deepen our understanding of teachers' interactions with foreign resources by uncovering additional cultural and institutional factors not explored in this study.

The findings also suggest that analysing teachers' interactions with a foreign resource can reveal missed opportunities (Remillard, 2019) within national curriculum materials that create openings for integrating such resources. In Lisa's case, the foreign textbook was selected not only for its accessibility through school resources but also because of the limited number of practical problems available in national textbooks. This choice underscores a misalignment between the goals of national evaluations and the content provided in the textbooks, further highlighting why she incorporates external resources into her teaching.

The study also revealed that teachers may differ in their awareness of the distinctions between the original foreign resource and the adapted version they produce through their documental work. While Mila deliberately avoids discussing the techniques outlined in the foreign resource, Lisa appears not to fully recognize how her modifications may influence students' problem solving. These findings highlight the importance of professional development in helping teachers understand how resource adaptations affect students' learning outcomes and can consequently contribute to improving teacher training.

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STUDENT-TEACHER INTERACTIONS DURING GEOMETRY LEARNING WITH E-TEXTBOOKS

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This paper presents a study on interactions when students engage with geometrical topics using an e-textbook. Guided by the Interactionism Approach, the study employed qualitative research methods to analyse data collected through observations. The findings indicate that e-textbooks significantly transform classroom interaction dynamics. Specifically, they shift the teacher's role from being the primary source of knowledge to a strategic facilitator, guiding students through their digital exploration. Furthermore, both teachers and students increasingly ask procedural questions, moving beyond basic recall to achieve a deeper conceptual understanding.

INTRODUCTION

The rapid development of digital technology has led to a significant shift in educational resources, with e-textbooks becoming essential tools in modern classrooms. Unlike printed textbooks, e-textbooks provide an interactive learning experience through features like live hyperlinks, multimedia content, and dynamic simulations. This evolution has attracted the attention of educational systems and teachers, resulting in the increasing incorporation of e-textbooks, at least in Israel, into curricula due to their unique interactive capabilities. These features have the potential to increase student engagement and active participation in the learning process. By offering a more immersive and interactive educational experience, e-textbooks are transforming how students interact with educational materials (Dobler, 2015; Fletcher et al., 2012).

The literature suggests that the use of e-textbooks in classrooms can transform student-teacher interactions (Pepin et al., 2015). Teachers can guide students through hyperlinks, encourage exploration of mathematical relationships in interactive diagrams, and promote collaborative learning using integrated chat tools. These interactions can strengthen the teacher-student relationship, fostering active engagement and ultimately enhancing meaningful learning and understanding (Beutel, 2010).

This study aims to identify the types of interactions that arise when students engage with geometrical topics using an e-textbook. It seeks to characterize the interactions between students and teachers that may emerge in the mathematics classroom as a result of utilizing e-textbooks for learning geometry. Specifically, we aim to explore the nature of the questions posed to students and how teachers articulate their

expectations and requests. To achieve our goal, we systematically observed geometry lessons, focusing on teacher-student interactions while using e-textbooks.

THEORETICAL FRAMEWORK

Interactionism

This study was guided by the symbolic interactionism approach (Williams & Collins, 1995), which analyses communication patterns, interpretation, and adjustment between individuals, groups, and their social environment. Symbolic interactionism suggests that individuals understand their social environments through communication, learning, the use of symbols, and by assigning meanings to objects within their social settings. Unlike theories that focus on how institutions objectively influence individuals, symbolic interactionism highlights subjective viewpoints and personal meaning-making processes (Carter & Fuller, 2015). The key principles of symbolic interactionism (Blumer, 1986) are: a) People act based on the meanings that things hold for them; b) Meanings develop within specific social and cultural contexts through interpretive processes; c) Meanings emerge from interactions with others and society.

The meanings attached to self, others, and interactional situations are central to this theory. These meanings, which guide social behaviour, are dynamically shaped and reshaped through the interaction process. Our interpretations of situations and subsequent interactions are heavily influenced by the individuals we engage with and the surrounding circumstances (Stryker, 2001). Interactionism posits that people derive meaning from their interactions with others and interpret symbols based on shared social contexts. In the case of geometry classes, the framework could uncover how students and teachers collaboratively construct meaning around geometric concepts, not just through verbal communication, but also through their engagement with the symbols, diagrams, and interactive elements in the e-textbook. The technology itself becomes a symbol that both the teacher and students interpret and interact with in ways that shape their learning experiences.

For this purpose, the following *research question* was derived:

What are the characteristics of teacher-student interactions when utilizing e-textbooks in geometry lessons?

METHODOLOGY

We employed a qualitative research method, which is well-suited for addressing our research question. This approach prioritizes the exploration of processes—specifically the interactions between teachers and students—over merely focusing on results or outcomes.

E- textbook

The e-textbook chosen for this study covers a wide range of mathematical topics, including geometry, which is in line with the Israeli math curriculum. Its interactive features enhance the learning experience through live hyperlinks, audio-visual content,

graphic simulations, and manipulable diagrams. Students can actively engage with the geometrical figures by dragging shapes, highlighting sections, adding personal notes, and copying text. Moreover, the e-textbook provides layered, interactive content with links to supplementary resources (e.g., videos, quizzes, and activities), multimedia information, regular updates, and customized activities catering to diverse learner groups. This dynamic approach transforms printed textbook learning into an interactive e-learning experience. (see Figure 1).



Figure 1: Example of an item from the e-textbook. (a) Dynamic diagram-student can drag the blue vertex of the triangle; (b) The angle bisector, the altitude, and the median meet in the isosceles triangle.

Participants, Data Collection, and Analysis

The study involved five senior mathematics teachers from middle schools in northern Israel, each with at least seven years of experience teaching grades 7-9. These teachers used the e-textbook in their geometry lessons approximately two to three times per month. The student participants, also from grades 7-9 in northern Israeli middle schools, were described as high-achieving, motivated learners. They actively participated in classroom discussions and showed a strong desire to learn.

We used observation as a research tool to collect the data. Observations were conducted to explore the nuanced interactions between teachers and students during geometry lessons using an e-textbook. We analysed how teachers introduce questions and facilitate guidance, as well as student performance and participation patterns during e-textbook-based activities and geometric tasks.

The author spent 20 hours observing and documenting e-textbook-based lessons to gain deeper insights. A camera placed at the back of the classroom recorded the teacher's actions and the content on the board, offering a complete view of the learning environment. The focus of these observations was on analysing the teachers' questioning techniques and their assessments of student performance, including the types of questions asked by both teachers and students. All data, such as interview responses and relevant classroom incidents, were carefully transcribed. We then used the Atlas software to thoroughly analyse the exact responses from teachers and students. This enabled us to systematically examine the qualitative data.

The author transcribed recordings and used an inductive approach for data analysis, focusing on complete sentences. The researcher categorized teacher-student interactions into three main categories: questions, requests, and explanations, each with subcategories. Teachers' questions and requests were further divided into types, such

as factual or procedural questions, and requests for experiencing tasks using digital textbooks. Similarly, students' interactions were categorized into questions and explanations, each with subcategories.

FINDINGS

Interactions: Teacher-Students

The study revealed that when using e-textbooks, teachers mainly engaged in three types of interaction: 1) Teachers ask procedural questions; 2) Teachers handle students' requests; 3) Teachers explain the topic to the students.

Teachers ask procedural questions

Our analysis revealed that teachers who used e-textbooks consistently posed questions. These questions are specifically asked to assess students' ability to apply their knowledge in solving practical problems. They encourage students to actively engage in problem-solving rather than merely recalling facts or theories.

During geometry lessons, teachers used e-textbooks to ask students questions that required them to create figures to solve problems, adjust angle measurements, and manipulate triangle vertices.

Teacher: How can you draw a perpendicular in the triangle?

Student: First, I draw the triangle, and second, I choose the help tool here. This means a perpendicular from one vertex in a triangle; press on, [and] the perpendicular [goes] to the opposite side in the triangle.

In this instance, after the teacher's procedural question, a student constructed a perpendicular line within the triangle using the e-textbook's built-in drawing tools, such as those for creating segments and altitudes. The student's response to the teacher's question was procedural, namely, the student described the procedure for creating a perpendicular: "First, I draw the triangle" and "Second, with the help of the tool." The student's second statement suggests that he was partly aware of the perpendicular's meaning, such as "from one vertex" and "going to the opposite side." The e-textbook's features, such as the ability to manipulate shapes and animate components, enhance engagement and accessibility for procedural questions throughout the lessons, but they do not ensure coverage of the geometrical meaning of the perpendicular.

Teacher's requests

The second type of interaction observed between teachers and students involved teacher requests, which typically consisted of instructions for students to complete specific tasks and draw conclusions. Teachers primarily directed students to participate in hands-on activities within the digital learning environment. These performance-based requests included completing assigned activities and exercises using features from the e-textbook, allowing students to gain practical experience with digital learning tools.

Unlike printed textbooks that rely on physical actions, e-textbooks allow for the digital manipulation of geometric shapes. Data shows that teachers' task performance requests were prominent when they used e-textbooks.

Teacher: I request you to change the triangle angle value gradually, use the help tool to change the angle value, and explain in your words the result of this change.

Student: As I gradually increase the angle value, at a certain moment, the triangle disappears because, in this case, the sum of the angles of the triangle becomes greater than 180 degrees.

This excerpt describes a teacher asking students to interact with a dynamic figure in an e-textbook by varying the angles of a triangle and reporting on the results. When the student interacted with the dynamic figure of the triangle, he noticed that the angles disappeared from the triangle. This dynamic interaction – the variation of the angles and the disappearance of the angles – led the student to conclude that the sum of the angles in a triangle is 180° , as any greater sum would result in a shape that is not a triangle (see Figure 2).

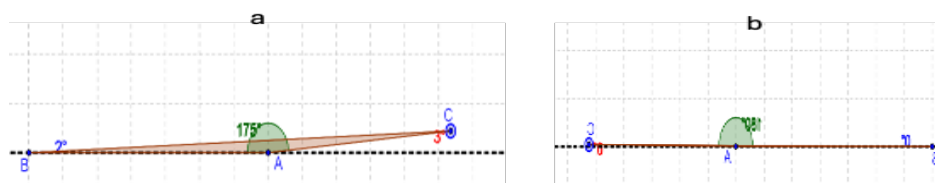


Figure 2: (a) The student drags vertex C; (b) The student creates a figure which is not a triangle.

Teachers' explanations

Teacher explanations, the third observed interaction type, are a crucial component of verbal classroom communication. After careful analysis, we identified several categories of teacher explanations, each serving a distinct pedagogical purpose, such as explaining mathematical material, providing instructional guidance, and solving mathematical problems. However, the predominant category that emerged from this analysis was instructional guidance. Instructional guidance involves providing thorough information that explains the concepts and procedures students need to learn. Key aspects of instructional guidance include:

- Engaging learners through dialogic discourse
- Monitoring student progress
- Providing feedback for advancement

Students progressively develop their mathematical understanding through designed experiments and systematic problem-solving activities, thoughtfully guided by their teachers.

The transcript below is from the dialogue between a teacher called Fuzia, who explained that the median drawn from the vertex angle is congruent with the perpendicular in an isosceles triangle.

Teacher: Given a triangle with one vertex and a median from that vertex, adjust the vertices as you wish and describe what you observe.

Student: When I drag the vertex of the triangle, both the median and the perpendicular move, but I noticed that they are congruent (meeting) together when the triangle is an isosceles triangle.

Teacher: You reached an amazing result; you are a wonderful student.

The teacher instructed the student to use the e-textbook tools to draw a triangle, a median, and a perpendicular line to one of the edges. The student varied the vertices of the triangle until he achieved the desired configuration, resulting in an isosceles triangle where the median and perpendicular are congruent. Tasks such as drawing, manipulating the vertices of geometric shapes, altering the problem data, and observing the outcomes on the screen are accomplished under the teacher's guidance. Teachers directed students and assisted them in problem-solving, provided feedback on their progress, and supported them when they encountered difficulties.

Interactions: Student-Teacher

The findings show that interactions between students and teachers when they teach geometric lessons through an e-textbook, mainly fall into two types: 1) the student asks the teacher questions, and 2) the student explains to the teacher.

Students' questions

The first type of interaction between the student and the observed teacher from the data analysis is student questions, through which the student addresses the teacher to gain information. By analysing the data, we found several types of student questions for the teacher, such as factual questions, procedural questions, and comprehension questions.

Procedural questions emerged as the predominant type of student questions. These questions primarily focused on task implementation, utilization of digital tools, and geometric concept construction.

Students using e-textbooks frequently posed questions related to specific mathematical tasks, such as:

- Drawing geometric shapes with digital tools
- Manipulating angle values in dynamic figures
- Exploring the effects of moving triangle vertices

Student: Why, when I move the vertex in this triangle, does the perpendicular move outside or inside the triangle?

Teacher: Because it is related to an obtuse- or acute-angled triangle.

The student inquired about how to implement a mathematical task involving the movement of a triangle's vertices and observing the corresponding changes in the perpendicular.

Students' explanations

The primary form of student explanation is describing their implementation of geometrical tasks. This involves students explaining how they apply theoretical mathematical concepts to practical applications, such as drawing shapes or manipulating digital tools. E-textbook users frequently explained their use of digital measurement tools, suggesting that interactive features promote more active engagement and articulation of problem-solving processes in geometry lessons.

Teacher: Students, try to move the vertex of the triangle. How does the angle value change?

Student: When I move this vertex in the triangle, the angle of the angle changes but stays between 0° and 180° .

The student explained to the teacher his experience through the geometric task implementation in the e-book lesson, whereby he moved the vertex of the triangle and noticed that the value of the angle changed from 0° to 180° .

FINAL REMARKS

Analysis of our research findings revealed that e-textbooks transform educational interactions by creating a dynamic learning environment where both teachers and students engage in deeper application discourse (such as questions and requests). Through our research, we observed that this digital transformation manifests in three key dimensions. First, the interaction patterns shift as both teachers and students increasingly pose procedural questions, moving beyond basic recall to deeper conceptual understanding. Second, teachers leverage e-textbooks to structure meaningful task performance experiences, requiring students to actively engage with digital content rather than passively consume information. Third, the teacher's role evolves to provide strategic instructional guidance, facilitating students' navigation through digital resources while supporting their independent exploration and problem-solving.

This reconfiguration of educational dynamics creates new shared meanings where learning becomes more interactive and experiential. The combination of procedural questioning, hands-on task performance, and guided instruction establishes a learning environment that promotes active knowledge construction. e-textbooks transcend traditional digital text conversion, emerging as dynamic learning catalysts that fundamentally reshape educational interactions. By enabling more interactive, personalized learning experiences, they require strategic collaboration among teachers, policymakers, and curriculum designers to effectively integrate innovative digital pedagogical approaches.

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DEMOCRATIC COMPETENCE IN THE CONTEXT OF HIGH SCHOOL: REFLECTIONS FROM INTEGRATIVE PROJECTS

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The National Book and Didactic Material Program (PNLD) distributes textbooks of Integrative Projects (IP) to provide situations that promote Curriculum Integration, as stipulated in the New Brazilian Upper Secondary Education. This study discusses the contributions of IP to the critical education of students in the light of Critical Mathematics Education and contrasts the notion of Curriculum Integration with this governmental initiative. We examined the textbook with the highest circulation in PNLD 2021. The results indicate that mathematics is rarely employed to interpret the presented contexts, that relevant discussions and critical reflections are left to the teacher's discretion, and that the notion of Curriculum Integration is adapted in ways that, when compared to theoretical references, weaken its transformative potential.

INTRODUCTION

In Brazil, the federal government has for decades maintained a program that distributes textbooks to all students and teachers in public basic education (about 75% of Brazilian enrolments). More recently, the National Book and Didactic Material Program (PNLD) has expanded its scope: in addition to Mathematics textbooks, it now also includes so-called Integrative Projects (IP) textbooks, intended for upper secondary education.

The focus of this text is to discuss how the IP textbooks sent to schools, as a didactic resource, can contribute to the critical formation of students, in the light of Critical Mathematics Education (Skovsmose, 2001). Seeking to understand the potential of IP books, a study was developed in the search to unveil how Curricular Integration is present in this government initiative. To give concreteness to the discussion, we present examples of projects from the book “+ Ação – na escola e na comunidade [+Action – at school and in the community]” (Bueno, 2020) in the area of Mathematics and its Technologies, which accounts for 38% of the copies distributed nationwide, making it the most frequently chosen by teachers (the second most distributed title represents less than 8%, see FNDE, 2021). The analysis is not intended to be representative of all titles approved in the same call; rather, it seeks to contribute to the discussion on IP in Brazilian Basic Education. For a more detailed version of this research, see Simas et

al. (2025), and for a similar approach from the perspective of Mathematics for Social Justice, see Simas et al. (in press).

THE INTEGRATIVE PROJECTS: PNLD AND BRAZILIAN STANDARDS

In 2021, the PNLD approved fourteen IP works in the area of Mathematics and its Technologies. The public notice required that each book contain six separate projects in *integrative themes* (Brazil, 2019) and with general competencies of the National Common Curricular Base - BNCC (Brazil, 2018). For example, the integrative theme of Project 1 is Science, Technology, Engineering, Art and Mathematics (STEAM) and the general competencies are: argumentation, knowledge, scientific, critical and creative thinking. According to the notice, the integrative themes serve to mobilize knowledge that allows students to develop the general competencies of Basic Education established by the BNCC.

In textbooks, projects usually contain texts to motivate the themes, individual and collective activities to raise reflections and propose the development of final products with different formats. For example, the six projects in the chosen book are (1) Water: how to reuse this resource?, (2) Budget: how to take care of our money?, (3) Research results: how are they obtained and disseminated?, (4) Games: can they help resolve conflicts?, (5) Architecture: how to build sustainably? and (6) Healthy eating: how to grow what you eat?.

CURRICULUM INTEGRATION

Alonso (2002) problematizes the fragmented and static view of the disciplinary curriculum, pointing out that this organization prevents students from dealing with some interdisciplinary and significant social issues, which demand articulation between different areas. She argues that the curriculum should be a collaborative project of the entire teaching team, where knowledge is organized in an integrated way to promote the critical and responsible training of students. In addition, he observes that the lack of integration generates a perception that the disciplines function as isolated drawers, making it difficult to understand the connections between them and the students' ability to develop a global view of reality.

Beane (2003), in turn, also criticizes disciplinary fragmentation and defends the creation of curricular spaces that allow the active participation of students in curriculum planning. In its Curriculum Integration model, knowledge is organized around "thematic units", planned by students based on questions they consider significant about themselves and the world. Beane believes that the curriculum should reflect students' interests and promote democratic skills by organizing around core problems that unify knowledge in a more accessible and relevant way. In addition, it highlights that this process leads to the equivalence of "popular culture" with academic "official knowledge", making the use of traditional disciplines more flexible as tools for understanding the issues raised by students.

Aires (2011) enters the debate by comparing Interdisciplinarity and Curricular Integration. While the latter, based on Beane, focuses on the students' centres of interest and starts from the problems raised to integrate the knowledge necessary for their solution, Interdisciplinarity focuses on articulating the disciplines in a coordinated way, focusing on the structure of academic knowledge. For Aires, both approaches are opposed to disciplinary fragmentation, but have different starting points: Curriculum Integration prioritizes practical and social issues, while Interdisciplinarity seeks to reorganize formal knowledge within a more interconnected disciplinary logic.

CRITICAL MATHEMATICS EDUCATION

Critical Mathematics Education (CME), proposed by Ole Skovsmose in the 1980s, aims to promote critical citizenship. From this perspective, citizens understand how Mathematics supports the resolution of social problems, such as politics, economics and education, enabling them to claim rights in a critical way, with the aim of fighting for a fairer society. Skovsmose (2001) expands the concept of democracy, emphasizing that it goes beyond the distribution of rights and duties, and the active participation of citizens in society is fundamental. It structures CME based on Critical Education, highlighting three main concepts: (a) *critical competence*, which involves the active role of students in learning; (b) *critical distance*, which encourages students and teachers to adopt a critical view of the curriculum; and (c) *critical engagement*, which refers to students' interest in the content studied.

CME's goal is to develop *democratic competence*, which prepares students to exercise critical citizenship and be able to critically evaluate the actions of constituted authorities. This requires mastery of three types of knowledge: *mathematical*, *technological* (or pragmatic), which indicates how to build and use mathematical models, and *reflective*, which makes it possible to question mathematical models and the criteria used in their construction and application (Barwell, 2013). Paiva and Sá (2011) highlight the importance of a reflective teaching of mathematics, which equips students to critically analyse situations and seek solutions. They argue that it is not enough to teach the use of mathematical models, it is necessary to question how, why and when to use them. In this scenario, the development of reflective knowledge is crucial for a critical analysis of the results generated by mathematical and technological knowledge. The school must ensure that these three types of knowledge are explored, promoting democratic competence and forming critical and autonomous citizens, capable of acting in society in a conscious and engaged way.

Although neither the BNCC nor the analysed textbook explicitly reference Critical Mathematics Education, some projects propose social reflections articulated with mathematics, which justifies adopting this theoretical framework in our analysis.

METHODOLOGY

This research adopts a qualitative approach (Creswell & Creswell, 2018). According to the Teorema System (Amaral et al., 2022), we follow the three phases: Planning,

Material Exploration, and Data Processing. We followed the view of the textbook, in which the book is the object of study, and for the treatment of the data we relied on stratified analysis (see Figure 1). We seek to understand Object 1 of the PNLD, which are the IP textbooks for the New Brazilian Upper Secondary Education, to evaluate how they can contribute to the critical education of students, in the light of CME. During the study of the concept of Integrative Project, we confronted the notions of Curriculum Integration with the IP initiative.

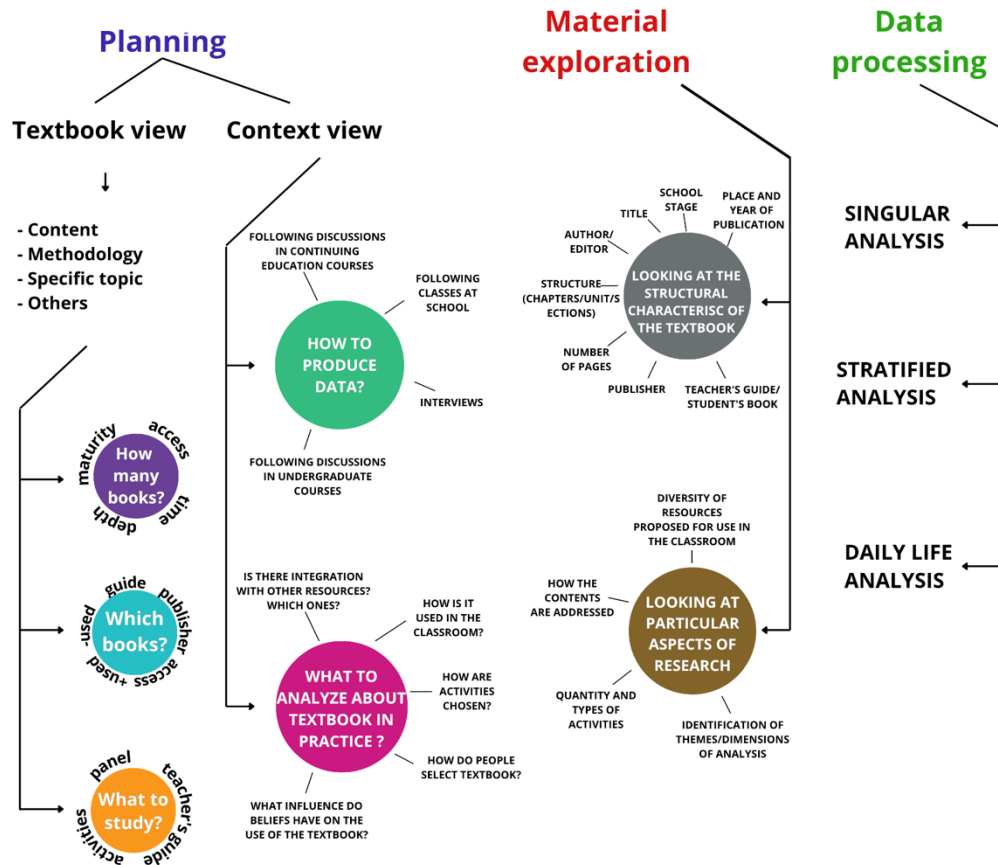


Figure 1: Methodology of Teorema System (adapted from Amaral et al. (2022)).

In the process of producing the data, we tabulated/organized information from all approved works. We analysed all their projects and identified relevant issues (such as the critical training of students, a central theme here) that made us reflect on the contributions (or obstacles) of these projects in teaching practice and student learning, assuming extracts produced based on the tasks or the panel - Amaral et al. (2022, p.187) call *the panel* the sections of the textbook in which the author addresses content, brings his ideas about the concept under study, the context in which these ideas apply, and examples. In addition to the panelⁱ, the textbook is composed of a set of activities.

ANALYSIS OF THE RESULTS

In the study, we considered three dimensions of analysis: (i) the treatment and depth of sociopolitical aspects; (ii) the support given to the teacher; and (iii) the use of Mathematics to read the world. Here, we present an analysis, with a particular focus on dimension (i), based on some projects that illustrate our findings.

“Project 3 - Research results: how are they obtained and disseminated?” has Media Education as an Integrative theme. The textbook discusses the difference between journalistic text and other textual genres and seeks to lead students to reflect on the various moments of conducting research, while leading them to carry out a survey, analysis and dissemination of the data produced. The final product is the creation and communication of the results in a news program.

Seven sample studies help illustrate each of the aspects intended to be taught: problem, methodology, sample and dissemination of data. While some themes of these examples touch on relevant sociopolitical issues, more controversial topics that could better support students’ critical development are avoided. The questions that involve Mathematics, as a rule, are empty of content and purpose, nothing is done with the results obtained. It is not clear why the requested calculations should be performed. Let’s look at the example of the survey “In Brazil, about 11 million young people do not study or work” (Bueno, 2020, p.81). In it there are four paragraphs with the discussion of the term NEET (Not in Education, Employment, or Training), two quantitative tables (Figure 2) and activities (Figure 3).

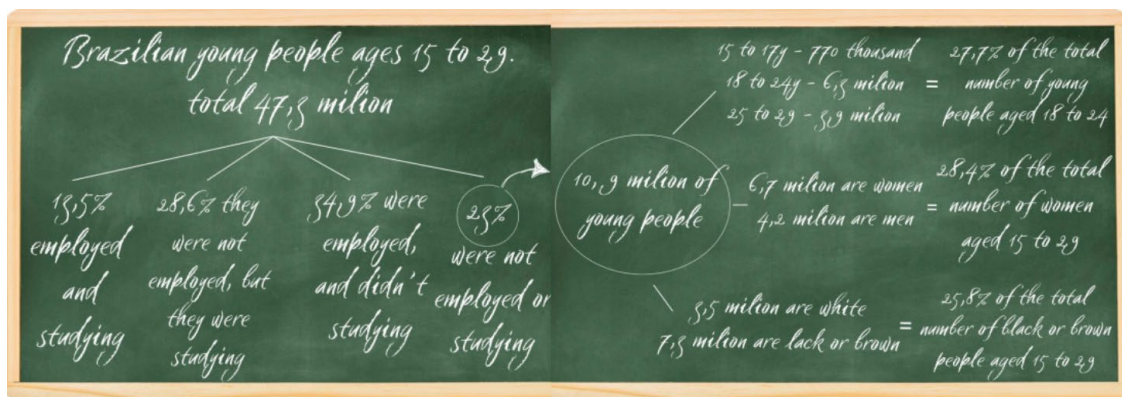


Figure 2: NEET board activity reproduced and adapted from Bueno (2020).

The questions in Activities 2 and 3 (Figure 3) involve mathematics and disconnect from sociopolitical issues. The responses from the cohorts of women, and black and brown individuals in items b) and c) of Activity 3 represent 61.5% and 67.9% of the NEETs, respectively. The data allow for a social discussion, but the teacher does not receive any support from the textbook to facilitate it. Note that the context is merely illustrative for the Mathematical activities of transforming percentages into absolute values and vice versa. The solutions of the items do not contribute to a better understanding of the core theme of the text. The textbook goes against one of the characteristics, pointed out by Skovsmose (2001), present in a “liberating” teaching and learning material, as the author considers that the “goal is to develop an insight into the hypotheses integrated into the model, and thus develop an understanding of the processes (for example, decision process) in society” (Skovsmose, 2001, p. 43, translated by the authors).

Activity 1 opens space for a sociopolitical discussion; the panel indicates that the teacher should encourage students to raise hypotheses about why there are young people in this situation and suggests that they work together with teachers of the Human

Sciences disciplines. In other words, teachers cannot count on the textbook for an effective discussion of the topic.

ACTIVITIES

DO NOT WRITE IN THE BOOK

Suggested answers and comments on the activities are in the **Teacher Guidelines** at the end of the book.

1. What do you understand by neet youth? In your opinion, why are there people in this situation?
2. Team up with a classmate and answer the questions based on the information from the text and the images presented earlier.
 - a) How many young people aged 15 to 29 years were both employed and studying in Brazil?
 - b) How many were not employed but were studying?
 - c) How many were employed but not studying?
 - d) How many were neither employed nor studying?
3. Among the 10.9 million young people who were neither working nor studying in Brazil in 2018, what percentage of them:
 - a) Were between 18 and 24 years old?
 - b) Were men? And how many were women?
 - c) Were Black or Brown? And how many were White?

Figure 3: Activity on young NEETs (Bueno (2020), p.82, translated by the authors).

The example illustrates what was observed throughout the textbook analysed. The resource feeds the perception of mathematics as a neutral tool used to quantify and represent data, but not to infer or analyse to better understand the phenomena studied. At various times in the textbook, the referrals for social problems are individual changes in behaviour or small interventions in the school environment. The discussions do not lead the student to reflect on the role of public power in the face of the issues that are posed, even in situations in which it is the main actor involved, even social criticism is not present. This lack of reflection on aspects that go beyond individual actions hinders the promotion of democratic competence, since one of its roles is to enable the individual to critically evaluate the actions of constituted authority.

All projects begin with a general or “macro” presentation of the problem and include activities that allow for reflection at this stage. For example, Project 1, “Water: how to reuse this resource?”, starts by addressing the scarcity and inequality in access to potable water across Brazil. However, it limits the discussion to individual or small group actions to mitigate the problem, without considering broader societal solutions. In Project 2, “Budget: how to take care of our money?”, one section, “In focus: The financial situation of Brazilian families,” presents a graph showing household indebtedness in two income strata over a year. Discussions of social inequality and the minimum wage are avoided by quickly shifting to family-level financial organization. In this process, it rushes from the general to the particular, adding technical details of little relevance to the final project and neglecting the complexities of the broader discussion.

FINAL CONSIDERATIONS

In analysing the IPs, we observed that curriculum integration does not occur in the textbook as described by Beane (2003) and Alonso (2002). For these authors, the

themes, final products and knowledge mobilized in the IPs are not predefined in the curriculum or didactic material, they are determined in the classroom, considering the interests of the learners. The students' protagonism in the process is fundamental for their engagement in learning and for their critical look at the curriculum (Skovsmose, 2001). However, in a country the size of Brazil, with deep inequalities and diverse classroom realities, the promotion of IP textbooks constitutes a public policy that advances in the direction of Curriculum Integration. It is natural that there are limitations, such as theme selection. What matters is providing opportunities for critical issues that foster rich discussions and empower students with as much freedom as the curriculum allows.

The notion of Curriculum Integration dialogues well with the CME and seems to constitute a real possibility for achieving democratic competence. In the textbook analysed, many gaps were left for the teacher when considering the interest in forming critical citizens and carrying out Curriculum Integration. Thus, this study contributes to reflections encouraging authors and public calls of the PNLD to seek approaches that align with the theoretical frameworks and analyses discussed here.

Although it is not the focus of our study, we identified that interdisciplinarity in textbooks is little explored. Thus, a fruitful field of study is the analysis of IP textbooks from the perspective of interdisciplinarity. In addition, it seems opportune to carry out research involving more in-depth looks or other theoretical references relating Curriculum Integration and the proposals of the IP in the PNLD.

In the PNLD 2026, the Integrative Projects were defined as “Integrative Projects in interface with the World of Work” (Brazil, 2025). This redefinition appears to prioritize workforce training, which calls for further research on its possible impacts from the perspective of Mathematics for Social Justice. It should be noted that we did not find public upper secondary teachers who have carried out any integrative project from the 2021 PNLD textbooks with their students. The government lacks a system to monitor their classroom use. If these books are not being used, around 7.7 million euros were wasted on producing and distributing Mathematics IP books. Further research is needed to determine whether this material is being used and, if not, why; and, if so, how.

Acknowledgment

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Endnotes

- ⁱ Amaral et al. (2022, p.187) call the panel “the sections of the textbook in which the author addresses content, brings his ideas about the concept under study, the context in which these ideas apply, and examples. In addition to the panel, the textbook is composed of a set of activities [...]”

DEFINITIONS OF QUADRILATERALS IN NORWEGIAN MATHEMATICS TEXTBOOKS FROM 1930 TO 1986

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Definitions are important in mathematics. In this paper, we discuss definitions of quadrilaterals in mathematics textbooks in Norway. Hovtun and Dreyer (2024) studied 24 new textbooks from 2006 to 2020. We have, based on their criteria, analysed definitions of quadrilaterals from 380 textbooks in the period from 1930 to 1986. We find a similar lack of consensus on the definitions of quadrilaterals as Hovtun and Dreyer, but a higher proportion of definitions including both necessary and sufficient conditions in the 1960s and 1970s than in the newer textbooks. We also found more minimum definitions. We argue that our method, based on searches of full-text databases, is a valuable supplement to manual reading of textbooks.

INTRODUCTION

Grevholm (2017) argued that apart from teachers, textbooks are the elements that have the most significant impact on students' learning in mathematics. Moreover, in the Nordic countries, textbooks are even more important than in other countries (Grevholm, 2017).

Textbooks include explanations of the mathematical content, definitions, and tasks. According to Morgan (2005), definitions have an important place in mathematics. "The importance of mathematical definitions in teaching and learning mathematics is irreducible" (Zazkis & Leikin, 2008, p. 132).

Four attributes of definitions are often highlighted. A definition should cover exactly the instances that one wants to include. Thus, it should include conditions that are *sufficient* to make sure that no other instances are included. A definition should only include *necessary* conditions – if not, some instances that should be included, are not. A definition is called a *minimum* definition if there exists no subset of the conditions that would include exactly the same instances. Finally, definitions could be *inclusive* or *exclusive* – for instance, a rectangle can be defined to include squares as rectangles (inclusive) or to exclude squares (exclusive).

The "optimal" definition of a mathematical object is one that is both mathematically correct and didactically suitable for those who need to understand it (Winicki-Landman & Leikin, 2000). Battista (2007, p. 865) argues that, in mathematics, "inclusive definitions (and thus hierarchical classification) are preferred, although exclusive definitions and partitional classifications are certainly not mathematically incorrect".

Avcu (2019) studied how seven Turkish and seven US secondary textbooks define various quadrilaterals (trapezium, parallelogram, rectangle, square, kite, and rhombus). There were far more inclusive than exclusive definitions. Most exclusive definitions were of trapeziums and kites. In both countries, trapezium, parallelogram, and kite were mainly defined using quadrilaterals and sides, while rectangle and rhombus were mainly defined using parallelograms and angles. Squares were mainly defined using rectangles and sides in Turkey, and using parallelograms, sides, and angles in the US.

Hovtun and Dreyer (2024) studied the definition of seven types of quadrilaterals (quadrilateral, trapezium, kite, parallelogram, rhombus, rectangle, and square) in 24 textbooks in Norway. The textbooks were written for the curricula LK06 (2006) and LK20 (2020). Grade levels were chosen based on where the researchers expected to find definitions, based on the curriculum documents. Approximately a quarter of the definitions were inadequate. The most common deficiency was a lack of sufficient conditions, so that the definition included more objects than intended. Also, many of the definitions were not minimum definitions according to their way of defining minimum definitions (see below). Hovtun and Dreyer saw a trend towards more minimum definitions in textbooks for LK20 than for LK06, interpreting this as a result of LK20 emphasizing minimum definitions. Another main finding was that there was no clear stance on which quadrilaterals should be defined and no common understanding of how to define them. The type of quadrilateral with the fewest definitions was the kite. It was only defined in 3 out of 24 textbooks. They concluded that “The lack of consensus on the definitions of quadrilaterals can make it challenging for teachers to facilitate teaching that supports students’ understanding connected to quadrilaterals” (Hovtun & Dreyer, 2024, p. 45, our translation).

The study of definitions of quadrilaterals is particularly interesting, as it provides a whole family of concepts that are often defined in textbooks. Hovtun and Dreyer’s findings concerning recent textbooks led us to wonder how these quadrilaterals have been defined in previous textbooks. Thus, our research question is: *How are quadrilaterals defined in Norwegian mathematics textbooks for primary and lower secondary school in Norway from 1930 to 1986?*

The reason for wanting to conduct a similar study is twofold. Comparing definitions of quadrilaterals over time can provide insight into changes in rigor and definition styles in the textbooks. Moreover, it is of methodological interest to compare two studies where one has physically searched through all textbooks for definitions while the other study has used an app to conduct this search from a full-text corpus. In this last respect, the study is comparable to Smestad and Opsal (2025).

METHOD

To collect definitions of quadrilaterals, we use a corpus of all Norwegian mathematics textbooks 1930–1986, which was developed for another project (Opsal & Smestad, 2025). An expansion to cover 1987–2025 as well could be fruitful and may be

undertaken in the future. The National Library of Norway has digitalised all textbooks, and in this corpus, we have links to each digitalised textbook.

To study change over time, we split the corpus into three parts: the first covering textbooks for the curriculum N39, the second covering textbooks for the experimental plan F60 and the temporary plan M71, and the third covering textbooks for the final new curriculum in 1974, which was in operation until 1987. We have not included experimental textbooks developed for the NKMM project. In the corpus, we only included textbooks in bokmål (the most common of the two variants of Norwegian), only the main textbooks (not exercise books, etc), and only the first available edition of each book. In total, 380 textbooks were included in the analysis (see Table 1).

Curriculum	N39	F60 + M71	M74
Number of books	67	184	129

Table 1: Number of textbooks included in the subsets.

For the analyses, we have used an app from the National Library of Norway, which allows full text searches in the textbooks listed in our corpus. The app, Concordances (Konkordanser), returns the context of each occurrence of every search word, including up to 25 words surrounding the search word within a paragraph. It is these concordances that we use to identify definitions of the concepts. We searched for the words “firkant*” (quadrilateral), “trapes*” (trapezium), “drage*”/“drake*” (kite), “parallellogram*”/“parallelogram*” (parallelogram), “rombe*” (rhombus), “rektang*” (rectangle), “kvadrat*” (square).

The Concordances app returned 11,374 concordances. To identify the definitions, we first removed all concordances that showed no sign of being a definition. Then, for the remaining concordances, we accessed the full text of each textbook to see the word in a wider context to decide whether it was a definition or not, in which case we included the definition in our list. After this work, we ended up with 196 definitions. Table 2 shows the distribution of definitions across quadrilateral types and curricula.

	Quad.	Trap.	Kite	Para.	Rhom.	Rect.	Square	SUM
Concordances	2413	663	38	1120	290	2811	4039	11374
Definitions	12	36	1	28	36	37	46	196
Def. N39		4		1		2	4	11
Def. F60 + M71	7	22	1	19	21	24	13	107
Def. M74	5	10		8	15	11	29	78

Table 2: Number of concordances and definitions.

The final stage of the analysis was to code the material. To facilitate comparison with Hovtun and Dreyer (2024), we used their coding scheme, with two codes for minimum definitions (M) or not (NM), three codes for inclusive (ID), exclusive (ED) or

ambiguous (AD) definitions, and four codes for necessary and sufficient conditions: both necessary and sufficient (NS), only sufficient (S), only necessary (N) or neither (F). We also grouped the definitions into types, according to which conditions were included, and which previously defined concepts were invoked.

The concept of minimum definition is debated in the literature (Zaslavsky & Shir, 2005). While the idea of a minimum definition is to avoid superfluous information, a definition such as “a square is a quadrilateral with four equal sides and four right angles” is usually allowed as a minimum definition, even though technically, checking three of the angles is enough. We adopt such an approach. Also, Hovtun and Dreyer do not accept a definition as a minimum definition if it is an exclusive definition. For comparison reasons, we will follow Hovtun and Dreyer in this respect. Moreover, Hovtun and Dreyer do not consider a definition to include both necessary and sufficient conditions if it defines a figure exclusively. We follow this as well.

We have not used the code “apparently inclusive” from Hovtun and Dreyer, and below we include the definitions they coded as “apparently inclusive” in the inclusive definitions. All coding was done by one researcher and checked by the other researcher, and discussed to resolve any disagreements.

We must stress that we cannot know whether all definitions in these textbooks have been found. Most importantly, the digital versions of textbooks have been created by the National Library using OCR (optical character recognition), which may lead to misspellings at times. This may be more of a problem in older books than in newer books, due to factors such as the preservation of the books and the printing technology. Therefore, we should not consider our collection of 196 definitions as the complete set, but rather as a random sample of definitions, with a possible tendency to exclude more of the older definitions than newer ones. However, there is no reason to believe that some kinds of definitions will be more prone to being excluded than others. Thus, the existence of a variety of definitions can be established with our method, but we cannot make claims about which definitions are “missing”, as Hovtun and Dreyer did.

RESULTS AND DISCUSSION

The quadrilaterals studied had many types of different definitions in the period 1930–1986. The number of different definitions varies from 1 (kite) to 11 (quadrilateral). For the other quadrilaterals, the numbers are trapezium (5), parallelogram (4), rhombus (2), rectangle (5), and square (8). We find it interesting that there are some figures that are often defined in our material, but still give rise to a few types of definitions. The rhombus is one example, where the 36 definitions are of just two types. On the other hand, a quadrilateral and a square both give rise to many different types of definitions.

There are only 13 types of definitions that have been included more than once in the material, and only 10 that have been included more than three times (see Table 3).

Type of quadrilateral	Type of definition
Trapezium	A quadrilateral with two parallel sides and two non-parallel sides is called a trapezium. (17) A trapezium is a quadrilateral where two sides are parallel. (15)
Parallel-ogram	A parallelogram is a quadrilateral where two and two opposing sides are parallel. (24)
Rhombus	A rhombus is a parallelogram where all sides are equally long. (19) A rhombus is a quadrilateral where all sides are equally large. (17)
Rectangle	A rectangle is a quadrilateral where all angles are right. (22) If the angles of a parallelogram are right (90°), we call the quadrilateral a rectangle. (8) A rectangle is a quadrilateral where all the angles are 90° . Two and two sides are equally long. (5)
Square	A square is a quadrilateral where all sides are of equal length, and all angles are right angles. (25) A rectangle with all sides equally long is called a square. (13)

Table 3: Types of definitions mentioned more than three times in the material.

As the kite was defined just once in our material, it cannot be found in Table 3. Moreover, among the twelve definitions of quadrilateral, only one was included twice, and none were included more than twice.

Like in Avcu (2019), trapeziums and parallelograms are defined using quadrilaterals and sides. Rhombuses are defined using parallelograms and sides, like in Avcu (2019), but also sometimes using quadrilaterals and sides. In Norway, rectangles are commonly defined using quadrilaterals and angles, while in Avcu (2019), they are defined based on parallelograms and angles. Squares are mostly defined using quadrilaterals, sides, and angles, while in Turkey, they are defined using rectangles and sides, and in the US using parallelograms, sides, and angles. Despite some differences within each country, there seems to be more variation between countries, based on the textbooks studied in Avcu (2019), Hovtun and Dreyer (2024), and our study.

Necessary and sufficient conditions

The criterion that a definition should include both necessary and sufficient conditions for the figure in question is a basic one. Table 4 shows that the proportions of definitions that include both necessary and sufficient conditions were higher in the 1960s and 70s than in the textbooks analysed by Hovtun and Dreyer (2024). In our material, most of the definitions that are not coded as NS, are coded as S (sufficient), and almost all of these are trapeziums that are not defined inclusively. However, the

newer textbooks (LK06 and LK20) have more definitions that do not include necessary conditions, mostly because it is not specified in the definition that the figure is a quadrilateral (giving definitions of parallelograms that could also be a hexagon, for instance).

Curr.	Necessary and sufficient (NS)	S	N	F
N39	7	3	1	
F60+N71	110	11	1	1
M74	56	4	2	
LK06	45	3	13	3
LK20	35	1	5	1

Table 4: Necessary and sufficient conditions in definitions. The LK06 and LK20 numbers are from Hovtun & Dreyer (2024).

Minimum definitions

Hovtun and Dreyer (2024) noted a higher proportion of minimum definitions in the LK20 material than in the LK06 material and argued that this was connected to the explicit mention of minimum definitions in the LK20 curriculum. However, we find an even higher proportion of minimum definitions in textbooks for the curricula from the 1960s and 70s (Table 5), which did not include explicit mentions of minimum definitions. As mentioned in the methods section, we have not considered as a minimum definition a definition that is exclusive. In our material, there are 17 definitions that would be considered minimum definitions if we included exclusive definitions.

Curr.	Minimum definition (M)	Not minimum definition (NM)
N39	4	7
F60+N71	85	38
M74	42	20
LK06	22	42
LK20	22	20

Table 5: Minimum definitions. The LK06 and LK20 numbers are from Hovtun & Dreyer (2024).

We do not suggest that all definitions in textbooks should be minimum definitions. The definition “A rhombus is a parallelogram where all sides are equally long” is not a minimum definition, as the parallelism of the sides is a logical consequence of the sides being equally long. However, it could still be argued that it is a pedagogically more useful definition than “A rhombus is a quadrilateral where all sides are equally long”, since it has the virtue of explicitly showing that the rhombus is a kind of parallelogram.

When choosing a definition, it can also be important which definition is more elegant and which is easier to remember (Zazkis & Leikin, 2008).

Inclusive or exclusive definitions

The proportion of inclusive definitions has been rather stable throughout the period considered. In our material, almost all exclusive definitions are trapeziums that are defined to exclude parallelograms. This partly mirrors Avcu (2019), where most exclusive definitions were trapeziums and kites. While there may be arguments for both inclusive and exclusive definitions, it seems unfortunate that one kind of quadrilateral is singled out for exclusive definitions in this way.

CONCLUDING DISCUSSION

Overall, our results show that the “lack of consensus on the definitions of quadrilaterals” (Hovtun & Dreyer, 2024, p. 45, our translation) is not a new phenomenon. In particular, the quadrilateral and the square have many different definitions. We note a higher proportion of definitions including both necessary and sufficient conditions, and definitions that are minimum definitions, in the 1960s and 1970s than in Hovtun and Dreyer’s material. At this stage, we can only hypothesize that one reason can be that the tradition of Euclid, with its emphasis on definitions, was still strong in the 1960s and 1970s, but waning in later years. If we expand our corpus to cover the period since 1986, we could shed more light on how definitions in textbooks change over time and any connection between such change and curriculum change.

Methodologically, in this paper, we have studied 380 textbooks, while Hovtun and Dreyer (2024) studied 24 textbooks. To make their manual search feasible, they limited their research to textbooks that were likely to include definitions, based on stipulations in the curriculum. We, on the other hand, were able to include all textbooks in our period, and may have included definitions found in unlikely grades. Our analyses (once the corpus had been established) took hours instead of weeks. The disadvantage is that we are unsure how trustworthy the National Library’s full texts are. Therefore, we regard our set of definitions as a random sample, rather than the full set of definitions in textbooks. Nonetheless, we regard the method as useful – in this article, one clear result is the existence of a variety of definitions. In other cases, the method may be only a starting point for more thorough research. In any case, it is a useful supplement to manual searches of textbooks. In connection with future studies of such corpora, it would be useful to combine full-text searches with manual searches in parts of the material, to understand to what degree problems with OCR influence the results of research based on full-text searches.

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STUDENTS' USE OF DIFFERENT MATERIAL RESOURCES FOR SPECIFIC PURPOSES IN THE PROCESS OF LEARNING MATHEMATICS

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The growing importance of digital resources alongside traditional ones raises questions about how students use different resources to learn mathematics outside class. However, existing studies provide little insight into students' specific purposes of resource use. To address this gap, a self-developed questionnaire was administered to 1101 secondary school students. Descriptive statistics and Cochran's Q-tests were used to analyse the data, showing that students' selection of resources varies with the learning purpose, with certain resources clearly preferred over others. Despite the wide range of digital resources, the textbook is still the most widely used resource. For "better understanding", though, educational videos are used more frequently than textbooks.

INTRODUCTION

The textbook used to be the only specialist literature available to students for learning mathematics (Zimmermann, 1992). As society has digitised in recent decades, the importance and availability of digital resources alongside the traditional ones for learning mathematics have increased. Students can choose from a variety of resources to help them learn mathematics. Besides the traditional mathematics textbook, students can refer to their own self-created documents, such as notebooks, or search the internet for helpful explanations, videos, or educational platforms. In Germany, almost all secondary school students (grades 5–13) have access to a smartphone or computer (Feierabend et al., 2022), which means that the hardware providing access to digital learning resources is no longer a limitation. Due to the broader accessibility of various resources, students must select from different options and develop agency in selecting and using different kinds of resources (Pepin et al., 2023).

In the growing range of resources to support mathematics learning, both inside and outside the classroom, research has expanded its focus on the wide range of traditional and digital curriculum resources. The long tradition of research on mathematics textbooks and, in particular, their use by teachers and students is constantly being updated. Research on other curriculum resources, such as educational platforms or digital worksheets, and general resources independently selected by students, such as the internet or educational videos published on YouTube, has also developed (e.g., Cusi et al., 2017; Korntreff & Prediger, 2021). Most research findings concerning the use of

specific resources are qualitative and primarily concentrate on individual resources rather than the broader array available. While extensive research exists on the purposes of textbook use, it is not clear which resources students prefer for particular purposes. There is a lack of insight into which material resources students prefer for specific objectives. To fill this research gap, a large-scale study was conducted with secondary school students to explore their resource usage for learning mathematics. This paper focuses on the purposes for which students use different material resources outside the classroom and investigates the following question: What material resources do students prefer for specific purposes?

Answers to this question can provide researchers and teachers with an overview of students' preferences for material resources in the context of mathematics learning. Teachers could use this knowledge to embed preferred resources more strongly into classroom culture aligned with specific learning objectives and recommend specific resources for mathematics learning outside class. It also offers insights into independent learning with resources from a theoretical perspective.

THEORETICAL FRAMEWORK

Adler (2000) highlights the diversity of resources in the context of mathematics education. In line with her categorisation of resources in school mathematics, resources can be differentiated into material, social, and cognitive resources. While social resources refer to human interactions related to mathematics learning, and cognitive resources include mathematical frameworks and concepts students work with (Pepin & Kock, 2021), the present study focuses exclusively on material resources. These encompass curriculum resources, defined by Pepin and Gueudet (2018) as “all the material resources that are developed and used by teachers and students in their interaction with mathematics in/for teaching and learning, inside and outside the classroom” (p. 1), as well as general resources, that are mobilised by students independently and without teacher’s instruction (Pepin & Kock, 2021).

Research has shown that resources are used for different aims and goals. Anastasakis (2018) investigated the tool use in undergraduate mathematics. The most frequently mentioned immediate goals involved understanding, summarising, refreshing, or practicing a specific mathematical topic or procedure. Remote goals were mostly linked to assessments, such as exams, coursework, or tests. When looking at the overall data without considering the hierarchy of these goals, the most common goals identified were those related to exams, understanding, coursework, and tests (Anastasakis, 2018). While Anastasakis considered all undergraduates’ resources, Weinberg et al. (2012) focused on undergraduate students’ use of mathematics textbooks. As part of a qualitative pilot study for a questionnaire development, the authors highlighted, in addition to nine purposes for textbook use (e.g., read for better understanding, look up definitions and theorems, read/copy homework problems to complete homework assignments), three conditions in which undergraduates primarily use their textbook outside of class: preparing for class, doing homework (or other graded assignments),

and studying for exams (Weinberg et al., 2012, p. 157). Rezat (2011) analysed the use of mathematics textbooks by secondary school students and identified five purposes of students' textbook use: 1) as an aid to work on tasks, 2) consolidating, 3) learning new content, 4) interest-motivated learning, and 5) metacognitive purposes. In the context of "working on tasks", students use the textbook to understand the task, to obtain hints for solutions, or to check their solutions. In the context of "consolidating", both content-related parts of the mathematics textbook and tasks are used. Wang and Fan (2021) also investigated textbook use by students. Referring to the work of Rezat (2011) and Weinberg et al. (2012), they considered seven purposes of textbook use: preview, in-class learning and exercises, revision, looking up definitions, theorems, and formulas, looking up examples, answers, and references, doing homework, and doing extra exercises (Wang & Fan, 2021, p. 679).

A detailed comparison and synthesis of the literature yielded the following six purposes that are most relevant: the use of a specific resource "as an aid for doing homework", "for learning for tests and exams", "for better understanding", "for regular practicing", "to look up a rule, a procedure or a term" and "for learning new content". To expand the theoretical approaches and existing findings, mainly gained through qualitative research, a quantitative research design is chosen to answer the question of which material resources students prefer for specific purposes regarding the process of learning mathematics.

MATERIALS AND METHODS

The study presented in this paper focuses on the purposes of material resource use. It is embedded in a research project investigating secondary school students' use of resources for learning mathematics. Given the limited research findings on students' resource use, hypotheses are generated within an exploratory framework instead of testing hypotheses. A quantitative research design is used to gain an overview of students' resource use and related purposes. A digital questionnaire was developed on the basis of previous qualitative work, for example, how and for which purposes the textbook is used by students (Rezat, 2011), as well as quantitative work on resource use by undergraduates (e.g., Anastasakis, 2018) or textbook use by students (e.g., Wang & Fan, 2021). As part of a validation study with 125 students of the sixth and ninth grades, group interviews using cognitive techniques were conducted at the same time as the questionnaire was answered. The results showed that the questionnaire was found to be easily understood, and the responses were found to be valid. The reliability of the final questionnaire was examined with 112 students of another high school within the scope of a test-retest survey. The results indicate a sufficient to good reliability. The study was conducted in public schools in urban and rural areas with a mix of grammar schools and comprehensive schools to represent a range of academic environments. In total, 1101 students from the sixth, ninth, and upper grades across ten schools participated in the survey. Participation depended on the consent of school principals, the students, and their guardians. Thus, the sample is a convenience sample.

Nevertheless, it reflects the diversity of the student population in terms of school type, gender, and socioeconomic background, with a slight overrepresentation of grammar schools.

Instrument

The questionnaire consists of three content blocks. Aligning with the conceptualisation of resources in mathematics education, the first two content blocks deal with material and social resources. The last block is about the interplay of these two kinds of resources when students face difficulties. For the scope of this paper, only items from the first block are relevant. Therefore, only the first block is described in more detail.

The items in the first block relate to the use of material resources. In line with the definition of material resources and insights about students' use of specific resources, the following resources are considered: text resources such as (analogue or digital) textbooks, workbooks, worksheets, and self-created documents such as rulebooks or (analogue or digital) notebooks, along with web resources like an internet search engine, learning platforms and apps, educational videos, AI applications, and math software like CAS or Dynamic Geometry Systems as well as internal school platforms. It is distinguished between the frequency of use inside and outside the classroom, the extent of self-directed usage, and the purposes of use. The first set of items on the frequency of use is designed to find out which material resources are used most frequently in and outside the classroom. A five-point Likert scale is used for the items with expressions ranging from “every lesson” for inside the classroom or “every time I do mathematics” for outside the classroom to “never” for both scales. While the focus of the study is on self-directed use outside the classroom, information on in-class use is collected to find out whether there are correlations between in-class and out-of-class use. The six essential purposes outlined in the theoretical framework were gathered through multiple-choice items (Figure 1).

Purposes of using learning materials

Now we would like to know what you mainly use the materials for outside of math lessons.

Outside class I use an internet search engine (e.g., Google, Bing, etc.) ...

Select all that apply

- ☐ as an aid for doing homework.
- ☐ for learning for tests and exams.
- ☐ for better understanding.
- ☐ for regular practicing.
- ☐ to look up a rule, a procedure or a term.
- ☐ for learning new content.

Figure 1: Item 'Purposes of using an internet search engine'.

At the end of the first block, students are asked to rate how much the provided material resources assist in learning mathematics using a four-point Likert scale that ranges from “very helpful” to “not helpful”.

Analysis

The data is analysed using descriptive statistics. A total of 1076 students completed the items regarding the purposes of material resource use, forming the basis for the analysis. 46% of the 1076 students are in sixth grade, 32% in ninth grade, and 22% in upper secondary level. For the research question, it is pertinent to begin the analysis based on the purposes rather than resources. To provide an overview of the preferred resources used for specific purposes, the presentation of results focuses on the five most used resources for each purpose. The material resources are ranked according to the proportion of users per purpose. To examine differences between the proportions of the five preferred materials for each purpose, Cochran's Q-Tests were conducted. The Cochran's Q-Test is a non-parametric test used to evaluate the consistency of proportions across multiple groups in a dichotomous dataset. When interpreting the results of these items, it should be noted that this is not about the frequency of resource use, but rather about which resources are used for which purposes. The presentation of the relative frequencies clarifies which purposes generally guide mathematics learning the most.

RESULTS

Figure 2 illustrates the five most preferred resources for each of the six purposes of use, ranked by relative frequency. Overall, the relative frequencies indicate that material resources outside the classroom are most commonly used as aids for doing homework, to study for tests and exams, and for better understanding. For each of the six purposes of use, a Cochran's Q-Test was performed to investigate differences in the proportions of usage among the most preferred resources. The tests revealed significant differences among the material resources for each of the six purposes (Table 1), indicating that students use them selectively, with some strongly favoured over others.

The mathematics textbook is the most used resource for the purposes “as an aid for doing homework” (76%), “for learning for tests and exams” (82%), “to look up a rule, a procedure, or a term” (57%), and “for learning new content” (43%). Only for the purposes “for better understanding” and “for regular practicing”, the textbook does not rank highest. Educational videos are the most popular resource for “better understanding”. In this context, a greater number of students favour educational videos (63%) and general internet research (53%) over the textbook (50%). For regular practice, the notebook is the most used resource (45%).

A grade-specific evaluation reveals differences in the preferred resources used for specific purposes. While the textbook is the most popular resource among sixth graders for nearly all purposes (except for regular practice, where it ranks second), it holds first

place for only four purposes among ninth graders (excluding better understanding and regular practice).

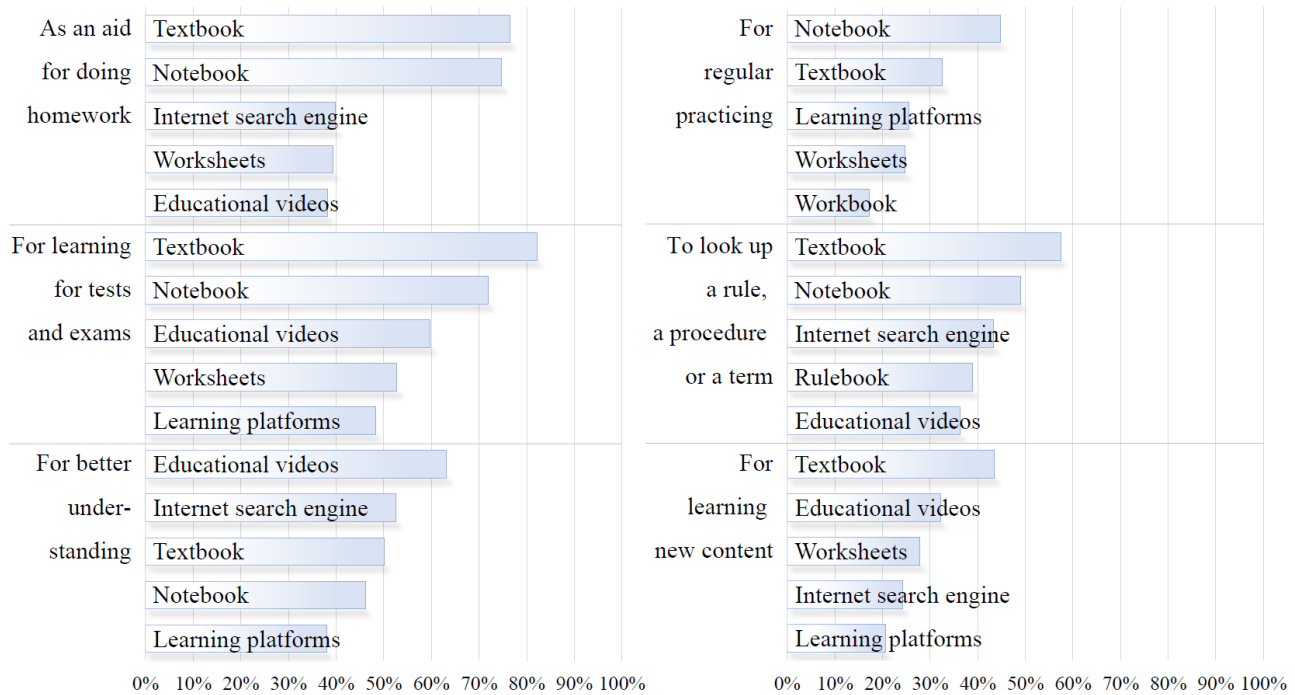


Figure 2: Most popular resources for specific purposes ($N = 1076$).

Purpose of use	Q	df	p
As an aid for doing homework	819.259	4	< .001
For learning for tests and exams	443.623	4	< .001
For better understanding	193.277	4	< .001
For regular practicing	301.856	4	< .001
To look up a rule, a procedure or a term	160.428	4	< .001
For learning new content	208.458	4	< .001

Table 1: Cochran's Q-Test results for differences in the use of resources per purpose.

In upper secondary school, the textbook is not the most popular resource for any of the six purposes. For the older students, their own notebook is most used as “an aid for doing homework”, “for learning for tests and exams”, and “for regular practicing”. Educational videos are the most favoured resource “for better understanding” and “for learning new content”. “To look up a rule, a procedure, or a term”, the internet search engine is used by most students. However, relative frequencies indicate that the textbook does not lose significance with age; rather, other resources such as the students' own notebooks, educational videos, or internet research grow in importance in higher grades. Compared to sixth grade, a significantly greater number of purposes were selected for the resources in the upper secondary school.

DISCUSSION

The textbook has proven to be the most popular resource for students across four out of the six specified purposes, specifically as a help with homework, to prepare for written exams, to look something up, or to learn new content. Thus, the wide range of available resources has not diminished the important role of the mathematics textbook. Therefore, the textbook remains a central material resource for students' learning of mathematics. It is striking that one of the two purposes for which the textbook is not at the top of the list is "for better understanding". If they want to understand content better, more students watch educational videos or explore the internet, which is consistent with the findings of Wolf et al. (2021). In addition to various advantages, such as positive effects on knowledge acquisition through multisensory representations and great potential for individualisation, mathematical educational videos are also criticised for conveying mainly procedural knowledge and turning learners into passive consumers (e.g., Korntreff & Prediger, 2021). When students watch such videos to improve their understanding, there is a risk that they may not develop conceptual comprehension but instead learn only how to perform calculations. Further research should examine the types of educational videos that students select. Are they more focused on procedural methods, or do they offer opportunities for conceptual understanding? This could clarify whether video use supports or hinders the development of deeper mathematical insight. The factors and criteria that influence the selection of resources should be further explored. Since the use of resources by students is still relatively underexplored, this remains a significant research gap. The grade-level-based analysis indicates that with increasing age, students use considerably more resources for specific learning purposes in addition to the mathematics textbook. This could suggest that older students select resources more independently overall. Especially educational videos and internet research seem to gain in importance. Future studies could incorporate more objective measures or triangulate findings with other data sources, as this study relied on self-reports from students in a non-random sample.

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MAKING MATHEMATICS CONNECTIONS: A FRAMEWORK OF "VARIATION PROBLEMS" IN CHINESE TEXTBOOKS

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A central issue of curriculum design is developing a coherent mathematics curriculum and avoiding treating mathematics as a body of isolated concepts and procedures—an important goal of mathematics textbooks. This issue has been analysed in a series of curricula and task design. Moving towards mathematics connections practice in curricula remains largely unseen. One possible reason is that we fail to possess the necessary “framework”. This study aims to develop this framework, identifying this practice and its functions in general. This paper reports on the development of an analytical instrument from variation problems [“one problem multiple solution methods” and “one problem multiple changes”] in Chinese textbooks, which identifies concept /solution connections for a coherent mathematics curriculum.

MAKING KNOWLEDGE CONNECTIONS IN TEXTBOOK DESIGN

The fragmentation of mathematics curriculum and teaching is a major explanation for unsatisfactory mathematics learning in the United States (e.g., Schmidt et al., 2002). Encouraging connection-building is an important goal of mathematics education, rather than treating mathematics as a body of isolated concepts and procedures (e.g., Cockcroft, 1982). Building connections represents a major shift from traditional practice in mathematics instruction in some states in the United States curriculum. From a learning perspective, Duckworth (1979) proposed the notion of “learning with depth and breadth” and argued that intellectual breadth and depth “is a matter of making connections”.

From the perspective of teachers’ knowledge, Ma (1999) further clarified that a teacher with a profound understanding of fundamental mathematics has a general intention to make connections amongst mathematical concepts and procedures and amongst individual pieces of knowledge, in order to reveal the underlying connections amongst different mathematical operations and subdomains. Depth and breadth depend on both the connectedness of different concepts, and thoroughness—the capability of ‘passing through’ all parts of the field—to weave them together. This thoroughness means to ‘glue’ knowledge of mathematics into a coherent whole. From a task design perspective, how to make connections has been considered important (e.g., Schmidt et al., 2002; Sun, 2016). Sun (2011a) illustrates a Chinese approach to organizing a curriculum with an emphasis on connectedness, by discerning relationships through the use of a variation approach from local pedagogical traditions and philosophies, ‘no

clarification, no comparison’ (沒有比較就沒有鑒別), rather than ‘to consolidate one topic or skill, before moving on to another’, a notion broadly used in most textbook development (Rowland, 2008) in Europe and throughout the world. The ‘one-thing-at-a-time’ design, which reflects a hidden conception that making a connection could happen naturally, might miss the chance to discern critical aspects between two or more topics. It would clearly provide fewer opportunities for ‘making connections’ compared to those of contemporaneous variation approaches. More importantly, the variation approach provides a platform to discern and compare the invariant feature of the relationship amongst concepts and solutions that may lead to ‘the general relationship or mathematical abstraction’ (e.g., Schmidt, et al., 2002; Sun, 2018, 2011a). Sun (2019) further clarifies the Chinese approach and its framework for making connections between whole numbers and fractions using problem variations. Although making connections is important, few studies have systematically examined this topic. The purpose of this paper is therefore to explicitly develop the framework to characterise task designs with or without connections.

There is robust literature in textbook comparison in the field of math education. However, few textbook studies reveal the Chinese design. The main direction of textbook comparison focuses on content and problems. For example, Fuson et al. (1988) concentrated on the grade placement of topics, the content topic covered, and page space of each topic. Rowland (2008) pointed out that the use of variation in structured exercises varies considerably from country to country and from text to text. It seems that the research tool is one of the reasons. Could we find a research tool to uncover the skill of variation practice for textbook design and analysis?

To find a suitable tool, the definition of variation practice is a useful window. However, the definition of variation practice is very confusing. For example, Gu et al. (2004) viewed it in terms of “conceptual variation” and “procedural variation” (The differentiation between conceptual and procedural variations can be confusing to teachers since teaching concepts, or conceptual knowledge is often involved in the teaching of processing skills, or procedural knowledge). Developmental variation and multi-faceted variation comprise Park and Leung’s definition of variation (2006). Huang (2002) and Huang et al. (2006) stated that variation has to do with explicit variation and implicit variation. Qingpu Curriculum Reform Experimental Group (1991) defined open-ended variation and closed-ended variation. One of the possible reasons is that “learning is seen as a change in an individual’s way of experiencing the object of learning” (Marton & Booth, 1997). From this perspective, it is difficult to categorize what is not a variation case in learning or teaching, since the pedagogy of every individual is always changing. Marton and Booth (1997) presented a framework to analyse “Variations and invariance”.

Although the studies above could not suggest an approach to define variation, they indicate a direction to the grounded theory approach (Glaser & Strauss, 1987). The idea of using an indigenous “prototype” of variation practices began to emerge. Encouragingly, by translating “indigenous” variation practices, known widely in

Chinese mathematics curricula as "One Problem Multiple Changes " (OPMC, varying presentations), "One Problem, Multiple Solutions" (OPMS—題多解, varying solutions) in China into codes. Cai and Nie (2007) argued that the use of this type of variation problem is widespread in China. In a survey, the teachers were asked to point out how often they use each of these problem-solving activities in the classroom. The results showed that all of the teachers used these problem-solving activities. Over half of the teachers surveyed used OPMS and OPMC very often in their instruction (Cai & Nie, 2007). The studies above indicate this practice seems rather popular in China.

In a three-year project, we found that a closer analysis of the unconnectedness and connectedness of mathematical concepts and solutions would be useful in the field of textbook comparisons. A framework was consequently developed (Sun, 2011a; 2016), presented below.

THE CONCEPTUAL FRAMEWORK

As far as we know, making connections is a new area where little is known. The current framework is based on the grounded theory (Glaser & Strauss, 1987). The textbooks on the topic of fraction division (Sun, 2011a), the topic of division (Sun, 2011b), the topic of addition and subtraction (Sun, 2013), topic of four fraction operations (Sun, 2019) were analysed. From these analyses, the conceptual framework "with/without concept/solution connection" was derived.

A. Problem variations with and without concept connection

Problem variations with concept connection reflect the mathematical structure underlying the problems in this respect. In contrast, problem variation without concept connection illustrates objects in the problems but does not reflect the mathematical structure of problem design.

B. Problem variations with and without solution connection

Problem variations with solution connection (i.e., multiple solution connection features) provide a setting in which learners are required to compare and possibly expand the solution structure and generalization of the solution system. Problem variations without solution connection (i.e., one solution feature) illustrate the surface solution structure of the problems that the curriculum designers intend the student to learn.

Obviously, this framework stresses not only conceptions and solutions connection but also invariant mathematical relations. The invariant mathematical relations repeated as the curriculum core provide much more opportunity to re-abstract process again, which is helpful in making a coherent curriculum.

Due to the problem design as a mathematical curriculum centre, this framework could have strength in deeper uncovering Chinese "indigenous" practice in mathematical curriculum practice (i.e., "mathematics-specific knowledge") than those in traditional variation theory. From practice to its re-constructed rational explanation, this

“indigenous” framework is different from any Western model, which takes into account variations as abstracting and generalizing conditions in mathematical curriculum design. This framework is more pertinent to curriculum experience and mathematical curriculum practice than those developed by Professor Marton and his colleagues (e.g., Marton & Booth, 1997).

The theoretical perspective of the conceptual framework

The “indigenous” variation practice mainly benefits from its cultural grounds, including the goal of examination, the goal of the national curricula, and the goal of pedagogy (Sun, 2011b). The curriculum rationale to extend a lesson or to organize exercises by using variation problems is related to “indigenous” philosophy, which could be traced back mainly to the ancient “general consensus” developed from the system of Daoism (Chinese: 道家) and Confucianism (Chinese: 儒家). For example, the following view from the Chinese I-ching theory (the Book of Changes), one of the oldest classic Chinese texts, subscribes to “discerning unvaried law beyond the varied (in Chinese 變中發現不變)” and “using ‘invariability’ in response to ‘variability’ (以不變應萬變);” the central idea of the evolution of events as a change process and the acceptance of the inevitability of change further reveal the ideologies behind the implementation of variation problems. A simple way to appreciate variation thought is to consider it as being based on the spirit of deep understanding.

USING THE ANALYTICAL FRAMEWORK IN THE TEXTBOOK COMPARISON

In Chinese textbooks, addition and subtraction are almost always connected using the OPMC transformation principle rather than separated into different chapters as in a Portuguese textbook. Here is a “prototype” example of OPMC, in which the concept of subtraction is always introduced in the Chinese textbook below as: $1+2=3$, $2+1=3$, and $3-1=2$. Within the problem set, there are two concepts of addition and subtraction behind three similar problems made with 3, 2, and 1. Clearly, this OPMC provides a setting in which learners can reflect and generalize between the concepts of addition and subtraction in order to concentrate on the relationships involved. In contrast, the concept of subtraction is introduced in one Portuguese textbook by the problem set: $4-1=3$, $5-3=2$, and $3-2=1$. Within this problem set, instead of embedding the two concepts above, only the concept of subtraction is included. Based on this perspective, it is easy to note the difference when we compare Chinese and Western curricula (Sun et al., 2013). For example, different task design features for addition and subtraction are found in Chinese versus Portuguese textbooks, featuring invariant versus variant concept/solution methods embedded in their examples. The problem set hinges on exemplifying connections/relationships rather than objects. Although Chinese textbook authors appear to use multiple concepts for every example, the underlying invariant concept is about part-part-whole relations, and the invariant knowledge is about inversive relations between numbers. In contrast, the addition examples in the Portuguese textbook use multiple underlying concepts, such as counting, combining,

and adding. The subtraction examples also use multiple concepts, such as subtracting, comparing, and identifying inverse operations, but do not connect these simultaneously to addition concepts. Although Chinese textbook authors use multiple solution methods in every example, the particular methods in the Portuguese textbook that depend on counting and doubling are rarely introduced. Only one specific solution method, “make 10,” is explicitly addressed among all the addition/subtraction examples in the first six chapters (Sun et al., 2013). In contrast, the addition examples in the Portuguese textbook suggest multiple solution methods, such as “doubles,” “doubles plus 1,” “compensation,” (e.g., $6 + 8 = 7 + 7 = 14$), and “reference number” (e.g., $6 + 7 = 5 + 1 + 5 + 2 = 10 + 3 = 13$). The subtraction examples use multiple solution methods such as “counting back,” “tables for addition to subtraction,” and “identifying the inverse operation of subtraction as addition.” Thus, the learner might get a temporary sense of these methods from being offered a variety of suitable examples without getting an overall understanding of the whole additive relation. The underlying Portuguese design principle is not made explicit, but we can infer that it is about learning “one thing at a time” and is hence more fragmented and less dependent on laying down basic foundational principles for future work (Sun et al., 2013).

By using this framework, I found that in the topic of fraction division, in a most popular textbook in China (Math textbook developer group for elementary school, 2005), 87.5% of examples are OPMC and 57.5% of examples are OPMS, and in the topic of fraction multiplication, 87.5% examples are OPMS and 28.6% of examples are OPMC (see Table 1). None of these examples appears in the American textbook (Bolster et al., 1996).

	OPMC in the Chinese textbook	OPMS in the Chinese textbook	OPMC in the American textbook	OPMS in the American textbook
examples in the topic of fraction division	87.5%	57.5%	0	0
examples in the topic of fraction multiplication	87.5%	28.6%	0	0

Table 1: The examples distribution in the topic of fraction division/ multiplication in the Chinese / the American textbook.

The Chinese textbook includes six examples with problem sets of concept connection (OPMC) and abbreviated ones, in total, accounting for about 85.7% of all seven examples, and four examples with problem sets of solution connection (OPMS), in total, accounting for about 57.1% of all seven examples in the topic of fraction division. It includes six examples with problem sets of concept connection (OPMC), accounting

for about 85.7% of all seven examples, and two examples with problem sets of solution connection (OPMS), accounting for about 28.6% of all seven examples in the topic of fraction multiplication.

IMPLICATIONS: WHY IS THE FRAMEWORK OF “VARIATION PROBLEMS” IMPORTANT?

As the mainstream of international research is frequently from a ‘Western’ perspective and under a ‘Western’ criterion, to avoid a fragmented view, in this paper, we attempted to develop an “indigenous” pedagogical framework from “indigenous” practice appearing rarely in the West. This framework focuses on a pedagogical phenomenon in organizing a curriculum with an emphasis on discerning connections /relationships through variation approach. This framework further indicates how to develop a coherent curriculum with an emphasis on invariant concepts, using the problem variation approach.

Many readers may argue that the variation approach may be confusing and that a sequential organization with time gaps (“one-thing-at-the-time”) should be preferred. The issue of variations in problem sets directly reflects the old Chinese proverb, “no clarification, no comparison” (沒有比較就沒有鑒別), rather than “to consolidate one topic, or skill, before moving on to another,” a notion broadly used in most textbook development (Rowland, 2008) in Europe and throughout the world. In contrast, this “one-thing-at-the-time” design would clearly provide fewer opportunities for “making connections” compared to those of contemporaneous variation approaches. The “one-thing-at-the-time” design might possibly reflect a hidden conception, making a connection could naturally happen. In this context, the curriculum’s role of making connections could either be relatively neglected or taken for granted. Thus, OPMC and OPMS aim to discern and compare the invariant feature of the relationship among concepts and solutions, and provide opportunities for making connections, since comparison is considered the pre-condition to perceive the structures, dependencies, and relationships that may lead to mathematical abstraction. The emphasis on “general relationships” reflects the soul of variation theory, emanating from the work of Bowden and Marton (1998), who argue that “variation is a necessary condition for effective discernment”. The priority of “contextualization” in the interest of facilitating engagement, motivation, and meaningfulness is regarded as the common curricular trend in the West (Clarke, 2006). In this light, variation problems suggest a way in which Western counterparts could learn from content-orientation curricula in China. This framework further indicates how to develop a coherent curriculum with an emphasis on invariant concepts, using the problem variation approach.

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GAPS IN REASONING AND PROVING OPPORTUNITIES BETWEEN EXPOSITIONS AND EXERCISES IN LOWER SECONDARY TEXTBOOKS RELATED TO THALES' THEOREM

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The study presented in this paper examines reasoning and proving opportunities in textbooks from the Czech Republic, Hungary, Slovakia and Thuringia. Our findings suggest not only a gap between expositions and exercises in terms of general and particular statements, but they also show that Hungarian textbooks favor deductive reasoning more than textbooks from the three other regions under investigation.

INTRODUCTION

In the discipline mathematics, proofs play a unique role in gaining and verifying knowledge (Hanna & Barbeau, 2008) and represent a type of reasoning that distinguishes mathematics from empirical sciences (Healy & Hoyles, 1998). Thus, proofs are seen as the heart of mathematics (Rav, 1999) and proving is presumed as a fundamental mathematical activity (Meyer, 2021). Moreover, students' experiences in school should be consistent with the way mathematics is done (Schoenfeld, 2016). Therefore, mathematics-education researchers consent that proofs should play a central role in mathematics classrooms, as well (e. g. Stylianides, 2014). However, there is also a consensus in mathematics-education research that students have different kinds of difficulties when it comes to proving mathematical statements. To give an example, it is well-known that students often use inductive arguments rather than deductive arguments and see empirical evidence as satisfying reasoning about general cases (Harel & Sowder, 2007; Hong & Choi, 2018). Moreover, Kunimune et al. (2009) found that even those students who can construct formal proofs accept empirical arguments as evidence for the generality of a mathematical statement. Further, students are often not convinced that a mathematical claim always holds when a deductive proof is given (Reid & Knipping, 2010). These difficulties assumably stem from an overreliance on empirical forms of argumentation in school (Harel & Sowder, 2007) as well as from insufficient learning opportunities provided in textbooks (Rezat et al., 2021) as carried out in more detail in the next section.

THEORETICAL FRAMEWORK

In the last few decades, growing attention is paid to the examination of reasoning and proving (R&P) opportunities in textbooks (e. g., Fan et al., 2013). While initial investigations focused on revealing types of exercises that might support an

understanding of proof (Johnson et al., 2010), due to the fact that both tasks and textbook narratives provide students potential engagement in R&P (Otten et al., 2014), recent research on textbooks also often involves the analysis of the latter.

Reasoning and proving in textbook expositions and exercises

When including the textbook narratives into the inquiry, one of the widest used frameworks is that of Thompson et al. (2012). Some key aspects of that framework can also be found earlier by Johnson et al. (2010). To investigate the initial narrative part of each lesson in a textbook, this framework suggests categorising the justifications provided in the textbook by using the following categories: general justification (GJ), specific justification (SJ), left to the student (LT) and no justification (NJ). In addition, the framework includes seven types of tasks in classifying R&P exercises as follows: make a conjecture (1), investigate a conjecture (2), develop an argument (3), evaluate an argument (4), counterexample (5), correct or identify a mistake (6) and principles of proof (7). Note, that the authors distinguish between a general and a specific case within the categories (1), (2), (3), (4) and (6).

However, by adapting the framework of Thompson et al. (2012) when investigating six secondary geometry textbook series from the USA, Otten et al. (2014) used the term 'exposition' instead of 'narrative' and referred to all parts in the lesson which present information for the students including definitions, formulas, theorems, and so on. Based on this description, the textbook exposition does not contain any learning activities like conjecturing, investigating special cases, etc. To give an example, learning activities aiming at the discovery of the theorem are part of the 'narrative', but they do not belong to the 'exposition'. On the other hand, theorems or additional information provided between exercises rank among the 'exposition'. Besides a few additional changes, the authors also involved the examination of the statements in the exposition by using the categories general (g), particular (p) and statement about R&P (s). The results of Otten et al. (2014) suggest that there is a meaningful discrepancy between exposition and exercises regarding the nature of mathematical statements: Whereas general statements are paramount in exposition, particular statements are dominant in the exercises of the textbooks under investigation. These findings are in line with the results of Hong and Choi (2018), who adopted the framework of Otten et al. (2014) for comparing textbooks from the United States and Korea, as well as with the outcomes of the research of Bergwall (2021) on Swedish and Finnish textbooks. In addition, another important result of Otten et al. (2014) is that in exercises, the most common R&P activities are to investigate a statement (30 %-46 % of all R&P tasks) and to develop an argument or to construct a proof (55 %-76 %).

Thales' theorem (angle inscribed in a semicircle theorem)

In geometry, there is an important theorem which states that connecting a point on the circumference of a semicircle with the endpoints of its diameter generates a right angle. In English-language sources, it is usually called angle inscribed in a semicircle theorem, whereas in other sources, it is often called Thales' theorem. This theorem plays

a very prominent and wide-ranging role in elementary geometry and it is a useful tool when solving a broad spectrum of constructing and proving problems in lower secondary geometry classrooms (French, 2008; Roscoe, 2012). For this reason, in this paper, we pay attention to this theorem and will refer to it as Thales' theorem.

RESEARCH QUESTIONS

Because of the unique position proofs have in mathematics as well as in mathematics classrooms, in this article, just like in our previous paper (Szűcs & Slavíčková, 2025), we examine the treatment of R&P in lower secondary mathematics education. The focus lies on four different regions in Europe whose educational systems, curricula, frameworks, teachers, teacher training programs and textbooks are particularly familiar to us. Moreover, because geometry is the traditional home of reasoning and proving (Otten et al., 2011), in this paper, we exclusively refer to R&P in geometry lessons. Finally, due to its special role in lower secondary geometry classrooms, this study focuses on Thales' theorem, in particular. Our work was guided by the following questions: (RQ1) How is Thales' theorem treated in textbooks from four different regions with regard to particular and general statements in the expositions? (RQ2) Which opportunities to improving R&P skills are offered to students in textbook narratives related to Thales' theorem? (RQ3) How does the quality of R&P exercises – including the distinction of general and specific cases – differs across those textbooks?

METHODS

The sample of this study includes a total of 12 textbooks that involve Thales' theorem and are presently intended for the usage in grammar school. We chose three textbooks from each of the following educational administrations: Czech Republic, Hungary, Slovakia, or the state of Thuringia in Germany. Note, that the sample is identical with that in our previous paper (Szűcs & Slavíčková, 2025). The full list of the textbooks under investigation can be viewed here: https://www.uni-erfurt.de/fileadmin/fakultaet/erziehungswissenschaftliche/Mathematikdidaktik/list_of_analysed_textbooks.pdf. The textbooks are tagged by CZ1-CZ3, HU1-HU3, SK1-SK3, TH1-TH3.

In accordance with the framework extended by Otten et al. (2014), we coded each statement in the exposition of each lesson including text boxes with formulas, proofs and theorems, provided worked examples, etc. by using the categories (g), (p) and (s) (Table 1). However, we did not code claims like definitions or agreements about notations, therefore, all coded statements can be viewed as statements related to R&P. Connected to this, a general statement is defined as a claim made about an infinite number of mathematical objects or situations, whereas a particular statement refers to a single or a finite number of mathematical objects or situations (Otten et al., 2014). In addition, according to Thompson et al. (2012), we also coded the justifications of the statements (g) and (p) by using the codes (GJ), (PJ), (LT) and (NJ).

Contrary to the study of Otten et al. (2014), we wanted to examine what kind of R&P activities are offered to students in the narratives, before the textbook moves to a separate training part with exercises. Therefore, we coded the statements in the tasks requiring R&P activities in the initial narrative part of the textbooks using the categories (g), (p) and (s) as well as the type of the exercises using the codes (1)-(7). As in our previous paper (Szűcs & Slavičková, 2025), based on the study by Szűcs (2023), we added the category investigate a premise (8) to our framework. Tasks requiring R&P activities are exercises matching one of the definitions of the categories (1)-(8), fully described in Thompson et al. (2012). Standard examples from geometry are shown in Table 2 as well as in our study (Szűcs & Slavičková, 2025), while trigonometry examples appear in Szűcs (2023).

Category	Standard example
Gen. stat. (g)	All angles over the diameter are right angles (CZ1, p. 20)
Part. stat. (p)	According to the inverse of Thales' theorem, $\gamma=90^\circ$ (TH1, p. 124) Our note: γ labelled an angle in a particular triangle.
Statement about R&P (s)	Every theorem consists of two parts, the premise and the conclusion. If you swap premise and conclusion, you get the inverse of the theorem. (HU2, p. 41)

Table 1: Excerpts from the encoding manual for statements in textbooks' expositions.

Lastly, based on the work of Thompson et al. (2012) and of Szűcs (2023), we categorised the tasks in the separate, training part of each lesson using the codes (1)-(8). In addition, in contrast to the study of Otten et al. (2014) but in accordance with the examination of Thompson et al. (2012), we distinguished between general and specific cases in the tasks. The difference between those two codes is demonstrated with regard to the categories (1) and (3) in Table 2.

Category	General case	Specific case
make a conjecture (1)	Can we construct exactly three tangent lines from a point [to a circle]? (SK3, p. 53)	Draw the points A(-2 1) and B(4 1) in a coordinate system. Where do the points P lie, if the line segments PA and PB are perpendicular? (TH3, p. 101)
develop an argument (3)	Justify, that a circle drawn on the leg of an isosceles triangle has a diameter that bisects its base. (HU1, p. 224, task 1)	Does a point C exist, not touching the [given] circle, such that the angle ACB is a right angle? (CZ1, p. 21)

Table 2: Excerpts from the encoding manual for coding the type of textbooks' exercises.

RESULTS

Statements and their justifications in the textbook expositions (RQ1)

Regarding the type of statements in the expositions, we detected a dominance of general statements (g) in all textbooks under investigation, although with more emphasis in textbooks from the Czech Republic or Hungary than in those from Slovakia or Thuringia (Table 3). This result is in line with Otten et al. (2014), Hong and Choi (2018), and Bergwall (2021). Interestingly, we discovered a slightly inverse situation in Czech, Slovakian and Hungarian textbooks compared to Thuringian textbooks regarding the remaining statements: Whereas few particular statements (p) have been found in textbooks from Thuringia, statements about R&P (s) are prevalent in textbooks from all other regions under investigation.

textbooks	(g)	(p)	(s)	total
CZ1-CZ3	7.7	0	3.0	10.3
HU1-HU3	9.0	0.7	2.3	12.0
SK1-SK3	4.7	0	0.7	5.3
TH1-TH3	4.0	1.7	0	5.7

Table 3: Average number of different types of statements in the textbook expositions.

Similarly, when investigating the justifications of those statements, we discovered that textbooks from the Czech Republic, Slovakia and Hungary differ significantly from textbooks from Thuringia, in terms of the ratio of justified statements (96 %, 79 % and 78 % vs. 56 %). Further, while general justifications (GJ) prevail in all textbooks under investigation, examples for particular justifications (PJ) were sporadically found in CZ1, SK1 and TH2. It is worth noting that we found the explicitly claimed rephrasing of the statement of Thales' theorem in Hungarian textbooks HU2 and HU3 (in addition to an earlier stating of it). In the textbooks TH1 and TH3 from Thuringia as well as CZ3 from the Czech Republic, Thales' theorem was rephrased in an implicit way.

R&P opportunities in the textbook narratives (RQ2)

With respect to the statements in the exercises provided in the textbooks narratives, we found a well-balanced ratio of general (g) and particular (p) statements on the one hand, but no example for a statement about R&P (s) on the other (Table 4). All textbooks from Hungary lack R&P activities in the narrative part completely.

textbooks	(g)	(p)	(s)	total
CZ1-CZ3	2.3	2.7	0	5.0
SK1-SK3	2.7	2.7	0	5.3
TH1-TH3	0.7	1.0	0	1.7

Table 4: Average number of types of statements in tasks in textbook narratives.

However, according to the results of Otten et al. (2014), developing an argument (3) is the main activity students are required to conduct in the narrative in all Czech and Slovakian textbooks (Figure 1). Contrary to these findings, textbooks from Thuringia tend to utilize the making and investigating of conjectures in the narratives, although the former is prevalent in almost all textbooks from the Czech Republic and Slovakia.

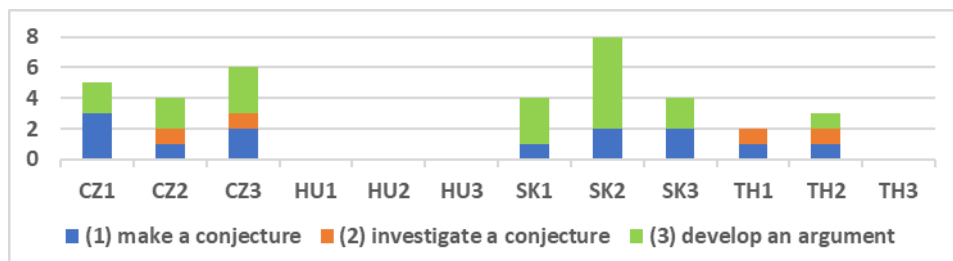


Figure 1: Expected students' activities in tasks proposed in textbook narratives.

R&P opportunities in the textbook exercises (RQ3)

In respect of the textbook exercises, we detected the dominance of specific cases in all textbooks with the exception of those from Hungary (Figure 2). It is worth remarking that this result is consistent with recent studies, such as Otten et al. (2014), Hong and Choi (2018), and Bergwall (2021). Further, we did not find any tasks requiring the construct of a counterexample (5) or the use of principles of proof (7). This outcome is in line with the findings of Bergwall (2021), in particular. In addition, similar to the findings of Otten et al. (2014), our data suggest that the main expected activity in textbook exercises is to develop an argument (3) on the one hand, and to make a conjecture (1) on the other. We also found that investigating a premise (8) as well as a conjecture (2) are mainly emphasised in textbooks from Thuringia.

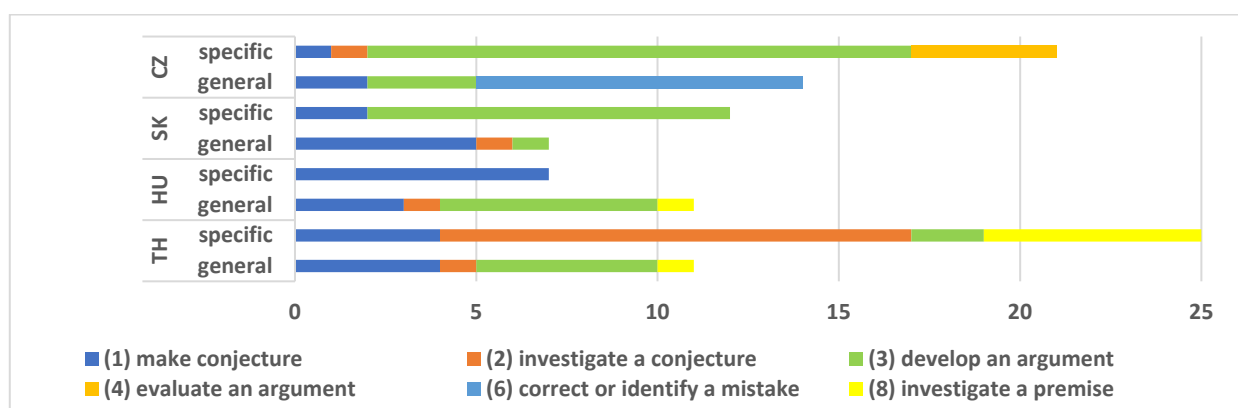


Figure 2: Number and type of expected students' activities in textbook exercises.

Further results about volume and quality of R&P opportunities and the treatment of proof in the same textbooks can be found in Szűcs and Slavičková (2025).

CONCLUSIONS

Our analysis shows that, with the exception of Hungary, textbooks provide fewer opportunities to engage with general statements in training sections than in expositions

and narratives, where such statements and justifications are more common. Exercise sections thus focus mainly on specific cases and procedural practice (e.g., conjecturing) rather than on deductive reasoning, limiting students' development of proof strategies beyond guided exposition. Hungarian textbooks differ markedly: they contain the highest number of general statements and more opportunities to reflect on proving in expositions (Table 3), along with more exercises addressing general cases (Figure 2). They also present several wordings of Thales' theorem and lack narrative-based student activities (Table 4, Figure 1), suggesting a stronger emphasis on deductive reasoning than in Czech, Slovak, or Thuringian textbooks. This aligns with Bergwall's (2021) finding of stronger support for reasoning in Finnish than in Swedish textbooks. We recommend that textbook authors balance R&P between expositions and exercises, particularly by including more tasks requiring deductive reasoning in general cases, and that teacher training highlight strategies for eliciting students' reasoning and connecting conceptual understanding with proving practice. Overall, these results underline how textbook design can either constrain or foster opportunities for reasoning and proving, with significant implications for curriculum development and teacher education.

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TEXTBOOKS IN A DECOLONIZED FUTURE WHERE TWO 1S AND ONE 2S TOTAL ONE 4S, AND NOT THREE 3S

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Meeting existence before essence, counting makes children use bundle-numbers with units, which reverses the operations' order: power bundles bundles, division pushes away bundles, multiplication pushes back to a stack that subtraction pulls away to find the unbundled to place on-top of the stacks. Recounting now roots core math. Between units on a BundleBundleBoard used as division-table, it roots proportionality. Between icons and tens, it roots multiplication-tables and early algebra as well as solving equations. Between physical units, it roots per-numbers and fractions. Between the lines in a stack, it roots trigonometry. And from stacks to squares, it roots square roots and quadratic equations. Once counted and recounted, stacks may finally add, on-top after recounting has made the units like, or next-to as areas rooting integral calculus.

TO BE OR NOT TO BE: IS 1 PLUS 1 REALLY 2, OR IS 1 PLUS 1 STILL 1?

“No, that is not four, that is two twos” said a 3year old child when asked “How many years next time?” and when seeing four fingers held together two by two. As educated, essence is all we see. But as uneducated, the child sees what exists, bundles of 2s in space, and 2 of them when counted in time. In the real world, the number ‘two’ thus exists both in space and in time both falsifying the ‘ $1+1=2$ ’ foundation of mathematics when an open and a closed V-sign shows that two 1s and one 2s total one 4s, and not three 3s. So, we ask: How may a textbook look like if using children’s own bundle-numbers with units?’ To answer, we visit a grade one classroom in a not so far future.

A GRADE ONE CLASS IN A ‘DECOLONIZED’ FUTURE

The teacher: Welcome children, I am your teacher in mathematics, which is about the numbers on this number line, and that is built upon the fact as you can see here that one plus one is two always and everywhere, both in time and in space. So ... Showing a V-sign a child says: Mister teacher, here is one 1s in space, and here is also one 1s. Now we count them in time to see how many 1s we have by saying ‘one, two’. So, we have two 1s. But not when added as a bundle. Then we have one 2s, so 1s plus 1s become 2s, but one plus one is still one when we count it, and not two as you say. And with this neighbour V-sign the total is one 2s plus two 1s which add to one 4s, and not to three 3s. But, if I two times show you three 1s, I have shown you two times three, six, 1s. The counting-numbers two and three can be multiplied, but they cannot add without units. So, don’t only teach us counting-numbers, also teach us bundle-numbers.

And also, please help us to add the bundle-numbers with units we bring to school, as 2 3s and 4 5s, that we can add next-to as eights, or on-top as 3s or 5s as we can see on a pegboard. If we add them next-to, we add tiles, which my uncle calls integral calculus. And if we add them on-top the units must be changed to the same unit, which my uncle calls linearity or proportionality. He says this is taught the first year at college, but we need it here to keep and develop the bundle-numbers with units we bring to school, instead of being colonized by your line-numbers without units.

We know that you want to bundle in tens, and in ten-tens, and in ten-ten-tens, but we like to bundle also in 2s, in 3s, in 4s, in half-tens, etc. We know that you have not been taught this and that the textbook doesn't teach it. But don't worry, we will teach you what we found out in preschool. Or better, instead of you colonizing our ways let us find out together what math may grow from our bundle-numbers with units. My uncle is a philosopher, and he calls it existentialism if we let existence come before essence. And, with existence before essence, we must count the totals before we can add them. You say that $8/2 = 4$, but we say that $8s/2 = 4s$ and that $8s/2s = 4$ so that 8s may be both 2 4s and 4 2s. You cannot split 9 in 2 parts, but we can easily count 9 in 2s as 4 bundles and 1 unbundled that becomes a decimal, $9 = 4B1\ 2s$, or a fraction if we count it in 2s also, $9 = 4\frac{1}{2}\ 2s$. Or, with negative less-numbers we get 5 bundles less 1, $9 = 5B-1\ 2s$. You only write 47 and say '4ten7' when you should write 4B7 tens and say '4 bundles with tens per bundle and 7 unbundled 1s' to show the difference between the counting-numbers in time, 4 and 7, and the bundle-numbers in space, tens and 1s.

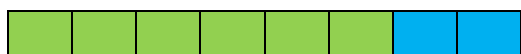
F01. Teachers only see the essence, five, we see all the different ways that five fingers may exist. A total of five fingers may exist as five ones, $T = 5\ 1s$, or as one bundle of fives, $T = 1B0\ 5s$. Also, the fingers may be bundled in 4s as $T = 0B5 = 1B1\ 4s$, or as two bundles less 3, $T = 2B-3\ 4s$. Or in 3s as $T = 0B5 = 1B2 = 2B-1\ 3s$. Or in 2s or pairs as $T = 0B5 = 1B3 = 2B1 = 3B-1\ 2s$. But 2 2s is also one bundle of bundles, 1 bundle-bundle, 1 BB, so we also have that $T = 1BB\ 0B\ 1\ 2s$. Putting two hands together we see that eight is one bundle of bundle-bundles, 1 BBB, so that ten is 1BBB 0BB 1B 0 2s. And, if we count ten fingers in 3s, $T = 3B1\ 3s = 1BB\ 0B\ 1\ 3s$. Likewise, if we count in tens, twelve is 1B2 tens, and forty-seven is 4B7 tens, and 345 is 3BB 4B 5 tens.

F02. Before we counted in space, but we also use bundles as the unit when we now count in time. If we count our fingers in 3s we cannot say '1, 2, ...', since 1 is not 1 3s. Instead, 1 is '0 Bundle 1' 3s, 0B1 3s. So, we count '0B1, 0B2, 0B3 or 1B0, 1B1, 1B2 or 2B-1'. Or we may count with less-numbers as '1B-2, 1B-1, 1B0, 2B-2, 2B-1.'

F03. With sticks we see that 5 1s may be bundled as 1 5s to be rearranged as one icon with 5 sticks. The other digits may also be seen as icons with the number of sticks they represent, with zero as a looking glass finding nothing. We don't need an icon for ten since ten is 1B0 if we count in tens, or 2B0 if we count in 5s, or 2B2 if we count in 4s.

F04. The calculations are icons also. If we remove 2 from 8, we may use a rope as a 'pull-away' icon so that 8-2 in time means 'from 8 pull-away 2', and in space means

‘8 with 2 pulled-away’ or ‘8 less 2’. A calculator may predict the result as $8-2=6$. That ‘8 remains after pulling away and back the same number’ creates a *reunite formula* ‘ $8=(8-2)+2$ ’, or ‘ $T=(T-B)+B$ ’ with T for the total and B for the bundle.



If we recount 8 in 2s, we may use a broom as a ‘push-away’ icon for division so that $8/2$ in time means ‘from 8 push-away 2s’, and in space means ‘8 recounted in 2s’. A calculator may predict the result, $8/2=4$. That ‘8 contains 2s $8/2$ times’ is written as ‘ $8=(8/2)\times 2$ ’, or ‘ $T=(T/B)\times B$ ’. Here a lift is a ‘push-back’ icon for multiplication so that 4×2 in time means ‘4 times push back 2s’, and in space means ‘a stack of 4 2s’. This *recount-formula* is used to change units all over math, economy, and science.

Finally, addition is a ‘two-ways icon’ showing that stacks as 2 3s and 4 5s may unite horizontally next-to as areas, or vertically on-top after recounting makes the units like.

F05. A reversed calculation is called an equation using the letter u for the original unknown number. The reunite and recount formulas may be used to solve equations. The reverse calculation or equation ‘ $u+2=8$ ’ asks ‘8 is reunited by 2 and what?’ The answer, u , is of course found by pulling away what was united, so $u=8-2$. This also follows from using the reunite formula, $u+2=8$, since here $8=(8-2)+2$, so $u=8-2$. The reverse calculation or equation ‘ $u*2=8$ ’ asks ‘8 is how many 2s?’. The answer, u , is of course found by recounting 8 in 2s when pushing away the 2s, so $u=8/2$.

This also follows from the recount formula: $u*2=8$, since $8=(8/2)*2$, so $u=8/2$.

If ‘ $u-2=6$ ’ then $u=(u-2)+2=6+2=8$. And, if ‘ $u/2=6$ ’ then $u=(u/2)\times 2=6\times 2=12$.

Here we see that we find the solution by moving a known number to opposite side with opposite sign. My uncle says that this follows from the official definitions. $8-2$ is the number u that added to 2 gives 8, so if $u+2=8$ then $u=8-2$. And $8/2$ is the number u that multiplied with to 2 gives 8, so if $u*2=8$ then $u=8/2$. And he warned us against a ‘same on both sides’ method you might want to teach us as being truer.

A combined equation as ‘ $u*3+2=14$ ’ may be solved by walking backwards: first pull-away the 2 to $14-2$, then recount this in 3s as $(14-2)/3$. Or solved by a song:

$u*3+2=14$	Equations are the best we know, they’re solved by isolation.
$(u*3)+2=14$	But first the bracket must be placed, around multiplication.
$u*3=14-2$	We change the sign and take away, so only u itself will stay.
$u=(14-2)/3$	We just keep on moving, we never give up.
$u=4$	So feed us equations, we don’t want to stop.

Figure 1: An equation solved by moving numbers to opposite side with opposite sign.

F06. On a ten-by-ten Bundle-Bundle-Board BBBoard the bundle-bundles are squares. We see that 2 2s need 2 more 2s and 1 corner to become 3 3s, which need 2 more 3s and 1 corner to become 4 4s, which need 2 more 4s and 1 to become 5 5s, and so on. So, 1BB 3s = 1BB2B1 2s, and 1BB 4s = 1BB2B1 3s. Or, 1BB 4s = 1BB-2B1 5s where 1 is added since it is pulled away twice. Thus, 1, 4, 9, 16, and 25 are the first 5 squares.

From ten, 1 BB 9s = 1BB-2B1 or 8B1 tens. And 1 BB 8s = 1BB-4B4 or 6B4 tens. And 1 BB 7s = 1BB-6B9 or 4B9 tens. And 1 BB 6s = 1BB-8B16 tens or 2B16 tens or 3B6 tens. So, 1, 4, 9, 16, 25, 36, 49, 64, and 81 are the nine squares. We see that, of course, 1 and 9 have the same last digit, as has 2 and 8, and 3 and 7, and 4 and 6.

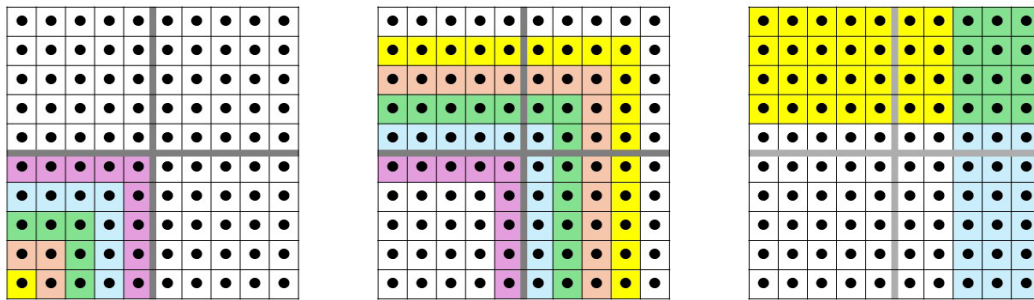


Figure 2: A BBBoard shows that BB-squares grow when adding $2B+1$. And that $6*7$ is what remains after pulling away $3B$ and $4B$ and adding the 4 3s pulled away twice.

And that $6*7$ as $6*\frac{1}{2}B2$ is $3B12$ or $4B2$, also occurring if counting $8\frac{1}{2}B \& 2$.

F07. On the BBBoard we see that totals may be counted with different units, 2s, 3s and tens. So, we may need to recount the total in another unit and ask 2 3s is how many 4s, or 2 tens is how many 7s, or 6 7s is how many tens. We can predict it with calculations using the recount formula. A calculator says that $2*3/4$ is '1.more', and that $2*3 - 1*4$ is '2', so the prediction is that 2 3s is $2B2$ 4s.

F08. Using a BBBoard as a 'division-table' for recounting in icons, we see that 6 7s recount in 5s as $6B12$ or $8B2$ 5s, and in 8s as $6B-6$ or $5B2$ 8s, tested on a calculator as: $(6*7)/8 = 5.some$, and $6*7 - 5*8 = 2$, so $6*7 = 5B2$ 8s. Using it as a 'multiplication-table' for recounting in tens, we may see 6 7s as $(B-4)*(B-3)$, i.e., what is left if we pull-away $3B$ from the side and $4B$ from the top and add the 4 3s that is pulled away twice, which gives: $1BB - (4+3)B (4*3) = 3B12 = 4B2$ tens = 42. This my uncle calls the FOIL-method in algebra: $(B - 4) * (B - 3) = B*B - 3*B - 4*B + 4*3$. With 5 as 1 half-bundle tens, $5 = \frac{1}{2}B0$ tens, $4 = \frac{1}{2}B-1$ tens, and $7 = \frac{1}{2}B2$. So, $6*7 = 6*\frac{1}{2}B2 = 3B12 = 4B2$ tens. A calculator directly shows that $6*7 = 42$ without unit and decimal point.

F09. Splitting a BBBoard in two squares and two stacks solves quadratic equations. If the sides are split in an unknown number, u , plus 3 then the board contains a u -square and a 3-square and two $3*u$ stacks, so if u squared + $2*3*u + 8$ is zero, then the $(u+3)$ square only contains $9-8 = 1$ which is a 1-square, so u can be -2 or -4 since both $(-2+3)$ and $(-4+3)$ give 1 when squared. So, if $u^2 + 2*b*u + c = 0$, then $(u + b)^2 = b^2 - c$.

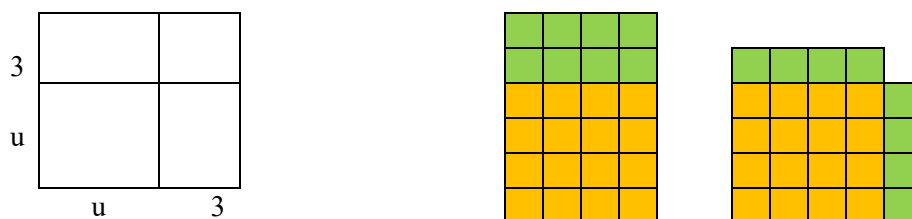


Figure 3: A BBBoard shows the four parts of a $(u+3)$ -square. And shows how a stack may be almost squared by moving half the excess from the top to the side.

F10. A rectangular stack as 6 4s may be changed into a square by moving half the excess from the top to the side to make it close to the so-called square-root: $T = 6 \cdot 4 = (6-1) \cdot (4+1) - 1 = 5 \cdot 5 - 1$. We fill the 1 with a $4 \cdot t$ stack from the side and $t \cdot 4$ stack from the top. This gives $2 \cdot 4 \cdot t = 1$, or $8 \cdot t = 1 = (1/8) \cdot 8$, so $t = 1/8$, making $T = (5-1/8) \cdot (5+1/8) = 4.88 \cdot 4.88$. Here, 4.88 as the square-root is close to what the calculator says, $\sqrt{6 \cdot 4} \approx 4.89$. In this way we can see that $\sqrt{2}$ is a little more than 1.4.

F11. We see that in shops we can exchange apples for money. If the sign says 4\$ per 5kg then we have the per-number 4\$/5kg, or 4\$ for each 5kg. So, if we want to have 20kg we simple recount them in 5s to see how many times we have 4\$: $20\text{kg} = (20/5) \cdot 5\text{kg} = (20/5) \cdot 4\$ = 16\$$. Likewise, 20\$ may be recounted in 4s to see how many times we get 5kg. If my share of a 5\$ bet is 2\$ then I will get 2\$ per 5\$ of a 30\$ gain, that I then recount in 5s. Here 2\$ per 5\$ is a fraction $2/5$. To find $2/5$ of 30, I simply recount 30 in 5s to see how many times I get 2\$: $30\$ = (30/5) \cdot 5\$$ gives me $(30/5) \cdot 2\$ = 12\$$. The fraction $2/5$ can occur with other units as $2/5 = (2 \text{ 3s}) / (5 \text{ 3s}) = (2 \cdot 3) / (5 \cdot 3) = 6/15$. Likewise with the fraction $8/20 = (2 \cdot 4) / (5 \cdot 4) = (2 \text{ 4s}) / (5 \text{ 4s}) = 2/5$.

F12. A stack has an inside line called a diagonal. To find how steep it is we can recount the up-number in the out-number: $\text{up} = (\text{up/out}) \cdot \text{out}$. Here the per-number, up/out, is called the line's slope or gradient or $\tan(\text{Angle})$ -number. So, with a $3/5$ gradient in meters, a total of 20out will give $(20/5) \cdot 5\text{out} = (20/5) \cdot 3\text{up} = 12\text{up}$ meter. And, in a circle with radius 1, half the perimeter is $\pi = n \cdot (\tan(180/n))$ for n large $= 3.141\dots$

F13. Once counted and recounted, totals may be added next-to or on-top. Adding 2 3s and 4 5s next-to as 8s means adding areas, which my uncle calls integral calculus where we multiply before we add. On a BBBoard we see that the sum is 3B2 8s as predicted by recounting on a calculator. Adding 2 3s and 4 5s on-top as 5s, 2 3s must first be recounted to 1B1 5s, so here the sum is 5B1 5s. Or 8B2 3s if we add them as 3s after recounting 4 5s to 6B2 3s. Again, this may be predicted by recounting on a calculator.

We turn the question around if we ask '2 3s needs how many 5s to total 4 8s?'. On a BBBoard or with cubes we see that we must pull away the initial 2 3s before we recount what changed it in 5s. So here subtraction comes before division, which my uncle calls differential calculus used to calculate changes as per-numbers. We know that an n -square almost changes with $2 \cdot n$. Later we learn that they do so with small changes.

F14. With '2kg at 3\$/kg plus 4kg at 5\$/kg' we can add the unit-numbers 2kg and 4kg to 6kg, but we must multiply the per-numbers to unit-numbers so they add as $2 \cdot 3\$$ and $4 \cdot 5\$$, which is the same as adding 2 3s and 4 5s by using integral calculus since we get an area when we multiply. My uncle says that the per-numbers here are piecewise constant, and that we later will learn to use integral calculus to also add locally constant per-numbers as the area under their per-number curve. Which is easy to find with differential calculus if the areas may be written as differences so that the middle terms all cancel out when added even if there are millions of them. Therefore, in textbooks, integral calculus should precede differential calculus to show its magic.

F15. Fractions also occur in mixtures as 2 apples with $1/2$ reds and 3 apples with $2/3$ reds. Again, we add the unit numbers directly to 5 apples, whereas the per-numbers first must be multiplied to $1/2 \cdot 2$ and $2/3 \cdot 3$ that add to 3. So here the sum is 5 apples with $3/5$ reds. My uncle says that you will try to teach us that $1/2$ plus $2/3$ is $7/6$, but I think he jokes because there cannot be 7 reds among 6 apples. But, if you will give my sister and me $1/2$ and $2/3$ of the same 6 pieces in a pizza then she gets 1 per 2 pieces, i.e., 1 piece 3 times since $6 = (6/2) \cdot 2 = 3 \cdot 2 = 3$ 2s. And I get 2 per 3 pieces, i.e., 2 pieces 2 times since $6 = (6/3) \cdot 3 = 2 \cdot 3 = 2$ 3s. This totals $1 \cdot 3 + 2 \cdot 2 = 7$ pieces. So, you need 2 pizzas that leave 5 of the 12 pieces. My uncle says that this is because digits and fractions are not numbers, but operators that need numbers to become numbers.

F16. To add squares, we take four copies of a 2 3s stack and place them as a 5-square. Inside this we now see a 3 2s and a 2 3s stack on top of a 2-square to the left and a 3-square to the right. But we also see a square formed by four diagonals that is surrounded by four half stacks. So, in a 2 3s stack the two side-squares add as the diagonal square.

It thus is the bottom-top line that shows how a blue 4-square grows as a square when a yellow 3-square is added. And again, we see that the change is two 4-sides and a corner.

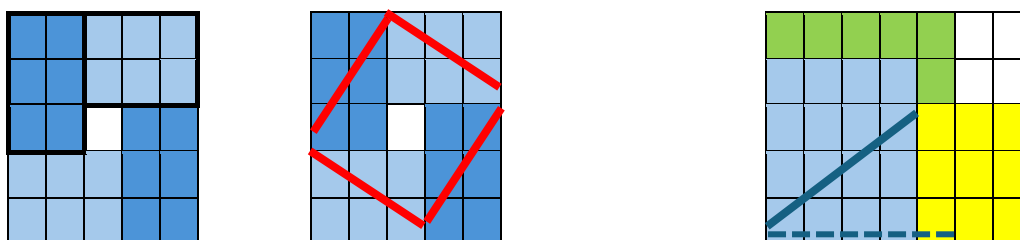
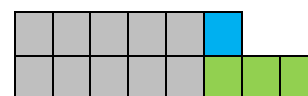


Figure 4: Two squares below and one inside two stacks. Squares adding as a square.

F17. We see that the textbook wants to teach us that $6 + 8$ is 14. But 6 and 8 cannot exist without units. If we count in 6s, then $6 + 8 = 1B0 + 1B2 = 2B2$ 6s or $1B4$ tens or 14. If we count in 8s, then $6 + 8 = 1B-2 + 1B0 = 2B-2$ 8s or $1B6$ 8s or $1B4$ tens or 14. Counting in tens we can use half-bundles, $\frac{1}{2}B$, so here $6 + 8 = \frac{1}{2}B1 + \frac{1}{2}B3 = 1B4$ tens.

On a BBBoard, we place the 6 and 8 on-top instead of next-to to see the sum, $1B4$, and to see that $8 - 6 = 2$, and $6 - 8 = -2$.



F18. We see that the textbook wants to teach us to carry and to borrow. But we don't need that since bundle-numbers are flexible with both overloads and underloads.

Overload	Underload	Overload	Overload
65 + 27	65 - 27	7 x 48	336 /7
6 B 5 + 2 B 7	6 B 5 - 2 B 7	7 x 4 B 8	33 B 6 /7
8 B 12 9 B 2	4 B -2 3 B 8	28 B 56 33 B 6	28 B 56 /7 4 B 8
92	38	336	48

Figure 5: Bundles and over- and underload make place values and carrying unneeded. On a BBBoard, the two-digit numbers 23 and 47 can be multiplied as $23 \cdot 47 = 1081$, and 1081 can be divided by 23 and give $1081/23 = 5B-3 = 4B7 = 47$.

	2B	3			2B	3	
↓	8BB	12B	4B		↑	10BB	15B
↓	14B	21	7		↑	-6B	-9
8BB	26B	21			10BB	9B	-9
10BB	8B	1			10BB	8B	1

Figure 6: Multiplying and dividing Bundles and Bundle-Bundles on a BBBoard.

F19. Counting ten fingers in 3s we get $T = 1BB \ 0B \ 1 \ 3s = 1 \cdot B^2 + 0 \cdot B + 1$. My uncle says that this bundle-formula is a polynomial showing the four ways to unite: addition, multiplication, repeated multiplication or power, and stack-addition or integration. And that, with letters, it roots the constant linear, exponential, and power change formulas.

With units we see there can be only four ways to unite numbers: addition and multiplication unite unlike and like unit-numbers, and integration and power unite unlike and like per-numbers. Going backwards, we split a total into unlike or like unit-numbers or per-numbers. Here subtraction and division split a total into unlike and like unit-numbers. Differentiation splits a total into unlike per-numbers. And finally, a total is split into like factors by the factor-finding root and the factor-counting logarithm. My uncle calls it an ‘Algebra Square’ where the Arabic word Algebra means to reunite.

Calculations unite/ split Totals in	Unlike	Like
Unit-numbers	$T = a + n$	$T = a * n$
m, s, kg, \$	$T - n = a$	$T/n = a$
Per-numbers	$T = \int f \, dx$	$T = a^n$
m/s, \$/100\$ = %	$dT/dx = f$	$\sqrt[n]{T} = a \quad \log_a T = n$

Figure 7: An Algebra Square shows how to add and split our four kinds of numbers.

F20. With the Algebra Square we now have a number-language we can use to predict numbers with calculation sentences that my uncle says contain the same as in the word-language sentences, a subject and a verb and a predicate, e.g., the total is three fours, or $T = 3 \cdot 4$. Now we can number-talk about things in space, geometry, and things in the past, statistics, and things in the future, growth and decay. And we can number-talk about money, meter, and motion. And discuss if what we say is fact or fiction or fake. My uncle says that if we are allowed to keep and develop our counting-numbers with bundle-units we will all be numerate. So, allowed to keep and develop our natural intelligence we don’t need any AI. It is fun to count and add in time and space on our BBBoard, that also may be used as a coordinate system to show how totals change.

F21. Finally, to protect our own number-language and innate number-sense, we now want to slightly reformulate the American Declaration of Independence:

“We, the children, hold these truths to be self-evident, that all humans are created equal, that they are endowed by their Creator with certain unalienable Rights, that among these are Life, Liberty and the pursuit of realizing their individual talents. And that to secure these rights education is instituted, deriving its just power from the consent of the educated. And, that whenever any education becomes destructive of these ends, it is the Right of the learners to alter or to abolish it, and to institute new education, laying its foundation on such principles and organizing its power in such form, as to them shall seem most likely to allow them to realize their individual potentials.”

CONCLUSION

With existence before essence, adding comes after counting and recounting where children use counting-numbers with bundle-units to count the bundles, bundle-bundles, and unbundled singles that become decimals, or fractions if counted also, or negative numbers to tell what is missing in space or pulled away in time. Counting also implies that the operations occur in opposite order: power for bundling bundles, division for pushing away the bundles, multiplication for pushing ‘back to stack’, to be pulled away by subtraction to find the unbundled to place on-top of the stacks that finally may add next-to or on-top. Then changing units leads to a recount-formula, $T = (T/B) \cdot B$, for proportionality, and to solving equations when recounting from tens to icons, and to early algebra when using a BBBoard to recount from icons to tens, and to square-roots when recounting stacks in squares. And to per-numbers when recounting in physical units, and to trigonometry before geometry when recounting the sides and the diagonal in a stack. The totals add next-to as areas using integral calculus, or on-top if the units are changed. Bundle-numbers with units make place-values and carrying unneeded. As to the math core, it springs from counting-before-adding. As does the Algebra Square showing how to unite and split unlike and like unit- and per-numbers. Textbooks with CATS-tasks in Counting & Adding in Time & Space on a BBBoard thus may be used for self-instruction since, in doubt, you just ask the BBBoard instead of an instructor.

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ANALYSIS OF THE AUTOMATED FEEDBACK TYPES EMBEDDED IN VIRTUAL MATHEMATICS TEACHING RESOURCE FOR T-MATTE IN NORWAY

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This paper examines automated feedback integrated into the mathematics course "forsering matematikk 1T," on the platform Digilær.no, aimed at lower secondary school students across Norway. Using frameworks by Hattie and Timperley (2007) and Shute (2008), 1,056 tasks and activities were analysed. The findings revealed that the predominant form of feedback was Knowledge of Results (KCR), which accounted for 68.2% of the feedback at the task level and was characterized by delayed timing. In contrast, Elaborated Feedback (EF), which is typically more detailed and informative, constituted only about 17.0% of the feedback provided, also delayed. While existing literature suggests that automated feedback can enhance learning outcomes, the predominance of KCR over EF indicates a significant area for improvement.

INTRODUCTION

The implementation of formative feedback has become popular in general education and, in particular, in the teaching and learning of mathematics by providing continuous evaluation and feedback during learning (Wiliam, 2018; Hattie & Timperley, 2007; Shute, 2008; Van de Kleij et al., 2015; Menéndez et al., 2019; Rezat, 2021). To improve learning, the Assessment Reform Group in the UK requires, among other things: provide effective feedback to students, actively involve students in their own learning, and ensure that students are able to assess themselves (Wiliam, 2018). In USA, NCTM (2014) claims that "effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning" (p. 53). In the same way, in Norway, the Directorate of Education (KD) recommends that formative assessment be an integral part of the training and must be used to promote learning, adapt training, and increase competence in subjects (Utdanningsdirektoratet, 2024). As a consequence, the revised curriculum (LK2020) has ample emphasis on formative feedback (assessment), and curriculum resources like textbooks (paper-based and digital) are being designed accordingly (Bakken & Andersson-Bakken, 2021).

For instance, KD provides a national platform for online teaching: Digilær.no (<https://digilaer.no/nb>). The teaching materials are computer based and the content on the platform is provided by several providers with whom the KD has an agreement.

The feedback format called automated feedback, a type of feedback in computer-based learning environments, is mostly embedded in the platform. This feedback system provides immediate, personalized responses to a learner's performance on an item-by-item basis (Van de Kleij et al., 2012; Rezat, 2021). Rezat (2021) stresses that, digital textbooks (resources) offer additional features "interactive diagrams, feedback, and formative assessment, and provide teachers and students with possibilities of personalization and individualization" (p. 1433).

In 2023, Professor Rezat visited our institution and talked about "learning mathematics from (digital) textbooks and other curriculum resources" in connection to feedback, specifically automated feedback. That inspired us to look at the feedback features embedded in the digital materials used on the platform. In this particular study, we conducted a preliminary investigation only on the types of automated feedback embedded in the subject called 'forsering matematikk 1T', which we abbreviated as T-matte, provided on the platform. The aim is not to study how effective the feedback is, as recent studies show that findings on the effectiveness of feedback are not conclusive (Van der Kleij et al., 2015; Rezat, 2021). Instead, the aim is to understand the different categories of feedback types embedded in T-matte so that we can contribute to improving the design and implementation of feedback embedded on such platform. To our knowledge, no studies have previously been conducted on the platform, Digilær.no, and we intend to study the effects of such feedback by involving pupils and teachers further. But for now, we ask the research question: Which types of feedback are embedded in the Digilær.no platform in the context of the teaching and learning of the course T-matte? Hence, the study is focused on the available feedback when students interact with the resources and does not include feedback from the teachers.

THEORETICAL FRAMEWORK

In their endeavour to investigate the effects of feedback in Computer-Based Assessments (CBA), Van de Kleij et al. (2015) discuss that one-to-one tutoring is an effective way of teaching and learning since it can provide timely and continuous feedback tailored to the learner. In the same regard, they claim that CBA, like one-to-one tutoring, "can serve to immediately resolve the gap between student's current status in the learning process and intended learning outcomes" (p. 479). In a computer based learning environment, students can benefit from the immediate feedback embedded to each task and activity (Mory, 1994; Rezat, 2021). The question could be what are the variants of feedback types that can be embedded in a given computer-based learning platform? To answer this question, we first look for the different ways of classifying feedback in education, in general.

Hattie and Timperley (2007) and Shute (2008) provided different characterizations of feedback in educational studies. According to them, there are four levels of feedback related to task, process, self, and self-regulation. Feedback at the task level focuses on providing information on the correctness of the learner's response or solution to the given task or activity. For example, it assesses whether the solution provided for a

given task is correct or not. Feedback at the process level focuses on the procedures and steps the learner follows in completing the task, while feedback at the self-level is geared toward the learner itself. The self-regulation level refers to the learners' own adjustments, including "self-assessments and willingness to receive feedback" (Van de Kleij et al., 2015, p. 477).

Feedback can also be categorized as knowledge of results (KR), knowledge of correct response (KCR), and elaborated feedback (EF), according to Shute (2008). KR provides information to the learner about whether the answer is correct or not without providing the correct answer or extra information, while KCR provides information about the correct response, not just whether the answer is correct or not. Van de Kleij et al. (2015) stress that KR aligns with the behaviourist view of learning, while KCR originates from the cognitivist view. EF provides additional instructional information focusing on the process of reaching the correct answer. According to Shute (2008, pp. 158–159), EF has several forms: attribute isolation, topic contingent, response contingent, hints/cues/prompts, bugs/misconceptions, and informative tutoring, in which some provide "elaboration that (a) addresses the topic, (b) addresses the response, (c) discusses the particular error(s), (d) provides worked examples, or (e) gives gentle guidance." Some of these EF forms listed here are more specific and directive, while other types are more general and facilitative (Shute, 2008; Rezat, 2021), hence making EF preferable to the other forms of feedback. In addition, there are two other types of feedback embedded in this study: locations of mistakes (LOM) and repeat-until-correct/multiple try (MT). LOM is "an error flagging highlighting errors in a solution, without giving the correct answer," and MT "informs the learner about an incorrect response and allows the learner one or more attempts to answer it" (Shute, 2008, p. 160).

Furthermore, Shute (2008) has argued for the inclusion of two other forms of feedback in relation to the timing of feedback: immediate feedback (IF) and delayed feedback (DF). According to Shute (2008, p. 163), followed by many other researchers, IF is defined as feedback "right after a student has responded to an item or problem or, in the case of summative feedback, right after a quiz or test has been completed. DF is defined relative to immediate feedback, and such feedback may occur minutes, hours, weeks, or longer after the completion of some task or test." Researchers have reported contradicting results regarding the effectiveness of IF over DF, and vice versa (Shute, 2008; Van de Kleij et al., 2015; Rezat, 2021; Harel et al., 2022). In this study, since mathematical tasks and activities that are enhanced with technologies like GeoGebra and programming with Python are highly embedded in the course, both IF and DF are deemed to be relevant. In relation to the focus of automated feedback in the digital resource on the digilær.no platform, IF is considered feedback given right after the learner provides a response virtually, while DF refers mostly to the feedback type provided at the end of one lesson or chapter. Harel and Yerushalmy (2021) argue that online elaborated feedback, particularly in synchronous learning, can become more effective when students are supported by mediating tools that prompt them to engage

actively with the feedback and revise their work accordingly. However, this study focuses on feedback in offline learning.

METHOD

Qualitative content analysis guided by the interpretive paradigm is used in this study. Clark et al. (2021) stressed that content analysis is a useful technique to explore patterns and trends in relation to social research topics. We interpret the meanings of the automated feedback types embedded in the course found on the platform Digilære.no using predefined types of feedback categories from previous research.

Data Collection: T-matte on Digilær.no platform

The data is collected on the T-matte course on the Digilær.no platform with the knowledge and permission of those who administer it. It is a national platform for online teaching aimed at strengthening subjects like T-matte and bilingual subject training (Utdanningsdirektoratet, n.d.). The platform is an extension of the pilot project called “Den Virtuelle Matematikskolen (DVM)” —meaning The Virtual Mathematics School. DVM is an initiative designed to offer advanced courses to academically high-achieving students at the secondary school level. The project stems from a governmental push to motivate and challenge these students by providing opportunities to take advanced subjects through virtual schooling. Based on the experiences from DVM, the Ministry of Education in Norway decided that a national platform for online teaching should be established (Digilær.no, n.d.). Digilær.no provides opportunities for learners to attend virtual classes from any corner of the country, from Malaga in the south to Longyearbyen in the north (Digilær.no, n.d.). Hence, the platform provides to those students at the lower secondary level (grade 9 or 10) who are assessed by their teachers, school leaders, and parents to take advanced courses than their grade level. Digilær.no generally follows a flipped classroom instruction approach, and in flipped learning, “teachers shift direct learning out of the large group learning space and move it into the individual learning space, with the help of one of several technologies” (Hamdan et al., 2013, p. 4). Inspired by the flipped classroom model, the project delivers content in two main ways: i) Asynchronous Learning (via a Learning Management System, in this case, Moodle): Students receive lessons with objectives, videos, tasks, and summaries, working through these independently; and ii) Synchronous Learning: Students attend online sessions twice a week to interact with peers and teachers, focusing on collaborative problem-solving. This study focuses on the automated feedback embedded in the first method: asynchronous learning. The study of the other part is deferred for further investigation. The core curriculum contents of the course on the platform include key mathematical topics, such as patterns and generalizations, factorizing, functions, logical structuring of equations and order of operations analysis of functions.

Data Analysis

Content analysis provides numerical data from qualitatively interpreted text/word data, proving the opportunity to identify the emphasis of certain features and categories, as

well as the bias of the data source (Cohen et al., 2018). In each chapter and each lesson, the tasks and activities embedded are interpreted and categorized according to the analytical framework built based on the theoretical framework presented above. The analytical tool used in this study is an amalgam of the different categories of feedback presented by Hattie and Timperley (2007) and Shute (2008): types of feedback (KR, KCR, MT, or LOM), level of feedback type (task, process, self, or self-regulation), and timing of feedback (immediate or delayed). For instance, KCR-T-I means the feedback type is knowledge of results, it is at the task level, and it is immediate. The analytical tool has 22 such categories; however, not all the categories were meaningful for automated feedback embedded in asynchronous learning. Content analysis provides the opportunity “for verification through re-analysis and replication,” as the categories and codes are explicit, transparent, and public (Cohen et al., 2018, p. 674). Furthermore, the researchers used the analytical tool to collect and analyse the first chapter independently, and they later compared the results. In 90% of the cases, they reached the same categorization and discussed the remaining 10% difference before proceeding to other chapters. This process enhanced the reliability of the findings by calibrating our interpretation of the feedback categories.

ANALYSIS AND DISCUSSION

The research question posed in this paper is: Which types of feedback are embedded in the Digilær.no platform in the case of the teaching and learning of the course T-matte? To answer this question, in total, 1,056 tasks and activities were analysed across the lessons of the seven chapters in the course using frameworks by Hattie and Timperley (2007) and Shute (2008).

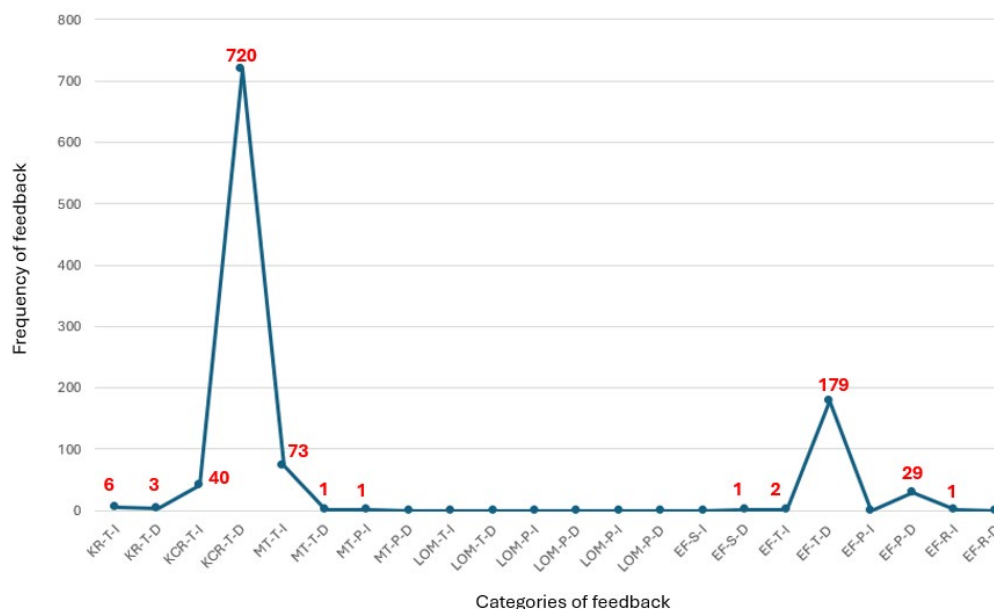


Figure 1: Feedback categories and correspondent frequencies of seven chapters

The first dominating type of feedback embedded in the T-matte course is KCR-T-D (knowledge of correct response – task level – delayed), with 720 occurrences, and the second most frequent feedback type is EF-T-D (elaborated feedback – task level – delayed), with a frequency of 179. Ten feedback categories have zero occurrences (see Figure 1). Some categories related to self and self-regulation could be difficult to observe in asynchronous learning. Studies have reported that, in general, KCR feedback types dominate resources (paper or digital-based) used for learning (Van de Kleij et al., 2015). Figure 2 provides examples of EF-T-I (left) for the incorrect solution and KR-T-I (right) feedback tasks from Chapter 1, section 1.4 and 1.3, respectively.

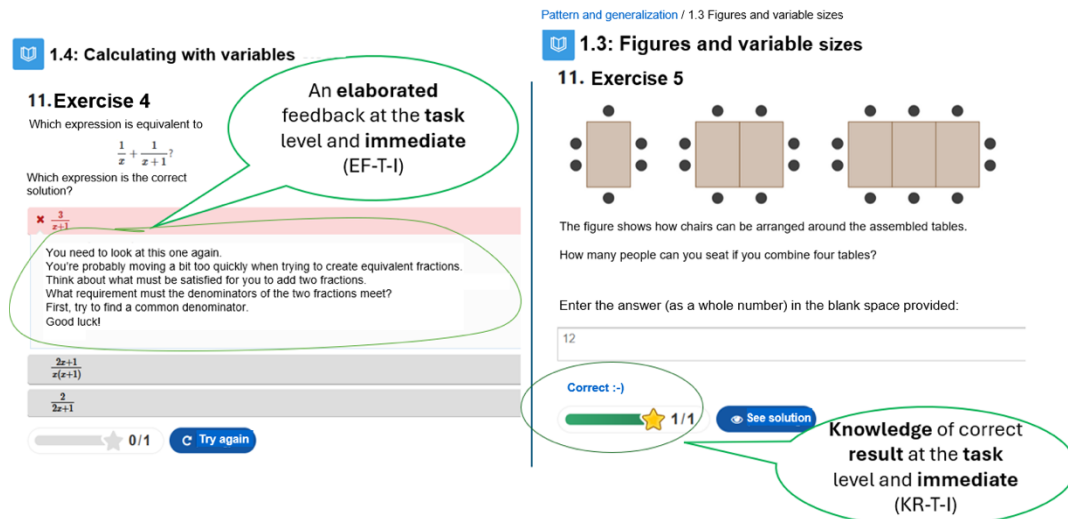


Figure 2: Examples of feedback types: EF-T-I (left) and a KR-T-I (right).

Further, let us take a closer look at each type of feedback the learners in the course received.

Knowledge of Results (KR) and Knowledge of Correct Response (KCR)

As shown in Table 1 and Figure 1, the feedback types KR and KCR are only at the task level, not at the self, self-regulated, and process levels. The total feedback occurrences under the two categories is 769, of which 760 are KCR and 9 are KR. Of these, 723 (94.02%) are DF, while 46 (5.98%) are IF task-related feedback.

Feedback	Chapters							Sum	Percentage
Categories	1	2	3	4	5	6	7		
KR-T-I	0	0	0	6	0	0	0	6	0,78 %
KR-T-D	3	0	0	0	0	0	0	3	0,39 %
KCR-T-I	7	0	0	5	1	20	7	40	5,20 %
KCR-T-D	135	96	124	98	79	119	69	720	93,63 %
Sum	145	96	124	109	80	139	76	769	

Table 1: Knowledge of Results (KR) and Knowledge of Correct Response (KCR) with feedback level and timing of feedback across the seven chapters

There is a catch here: most of the KCR feedback (of both IF and DF) is accumulated in the tasks provided at the end of the chapter. The pattern and generalization chapter, Chapter 1, has more KCR-T-D than the other chapters, followed by the factorization chapter. If this implies that a particular mathematical content might favour a particular feedback category, or vice versa, this should be investigated further.

Elaborated Feedback (EF)

212 tasks, 20% of the 1,056 feedback items related to the seven chapters embedded in the course on the platform, are of EF types, most of which are at the task level and are delayed (see Table 2). EF types are mostly embedded throughout the lessons, and learners receive EF in the form of texts and, at times, videos as they engage with the lessons. Most of the EF-P-D types, 29, are highly connected to virtual lessons, not to the tasks provided at the end. Shute (2008) and Rezat (2021) have reported that EF is preferable to other feedback types. Hence, designing more tasks with such feedback types might be important to boost learning.

Feedback	Chapters							Sum	Percentage
Categories	1	2	3	4	5	6	7		
EF-S-D	0	0	0	0	0	0	1	1	0,47 %
EF-T-I	1	1	0	0	0	0	0	2	0,94 %
EF-T-D	61	31	17	26	12	9	23	179	84,43 %
EF-P-D	4	0	7	0	6	3	9	29	13,68 %
EF-R-I	1	0	0	0	0	0	0	1	0,47 %
EF-S-D	0	0	0	0	0	0	1	1	0,47 %
Sum	67	32	24	26	18	12	33	212	

Table 2: Elaborated Feedback (EF) with level and timing of feedback across chapters

Repeat-until-correct/Multiple Try (MT) and Location of Mistakes (LOM)

There are 75 occurrences of MT feedback types distributed across the chapters, of which almost all are at the task level, with only one occurrence at the process level. While there were no specific places where LOM feedback types were identified for any of the tasks or activities, we anticipated that such feedback could be embedded in tasks that involve programming or other digital tools. We did not track these kinds of feedback, which could be a limitation of the study.

CONCLUSION

The findings of this investigation are limited to the T-Matte course provided by Digilær.no platform. As research informs EF provides more information to the learner, especially automated feedback has the potential to provide feedback tailored to individual learner. Hence, those contributors at Digilær.no platform can use this result

to develop the virtual courses further with sufficient EF embedded in the lessons. As there is no clear consensus on the effectiveness of feedback types across the different studies, it might help to vary the feedback types to optimize learning. For example, embedding LOM feedback types can help learners to get one-to-one feedback.

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LEARNING OPPORTUNITIES IN YOUTUBE RESOURCES: AN INTRINSIC CASE STUDY ON LINEAR TRANSFORMATIONS

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YouTube videos are one of the learning resources many undergraduate students use to reinforce their mathematical learning. However, only a few studies have investigated the learning opportunities provided by such online resources. In this study, through an intrinsic case study, we investigate the learning opportunities offered by these resources for studying linear transformations, a key topic in linear algebra. To achieve this, we draw on commognition theory, focusing particularly on how different realization of this mathematical object are discussed in a YouTube video from a highly viewed channel, 3Blue1Brown. The findings indicate that four out of five main realizations of linear transformation are well discussed and connected in this video, making it a valuable supplementary resource for learning linear transformation.

INTRODUCTION

YouTube videos are recognized as one of the learning resources many undergraduate students utilize to support their mathematics learning (e.g., Esparza Puga & Aguilar, 2023; Pepin & Kock, 2021). Their use has been reported for various purposes, including understanding mathematical concepts and problem-solving processes (Esparza Puga & Aguilar, 2023). Despite YouTube's significant role in supporting the mathematical learning of many university students, the learning opportunities (LOs) of these videos has not been the primary focus of past research in mathematics education (Radmehr, 2024a). Recently, we identified this gap in the literature and explored the LOs available in YouTube videos by applying different theoretical frameworks and focusing on various mathematical topics, such as derivatives (Radmehr & Turgut, 2024), the Fundamental Theorem of Calculus (Radmehr, 2024a), and eigen theory (Radmehr, 2024b).

In this study, we focus on another key topic within the undergraduate mathematics curriculum, namely linear transformations. This topic was selected because it connects several core notions (e.g., functions, linear combinations, and matrix equations) in mathematics and have many applications in STEM fields, such as computer graphics, dynamical systems and economics (Lay et al., 2022; Turgut, 2019). The guiding research question for this study is: *What are the potential learning opportunities offered by YouTube videos for learning linear transformations?* To address this question, we draw on commognition (Sfard, 2008) which we recently found useful for analysing LOs in YouTube videos (Radmehr & Turgut, 2024).

THE IMPORTANCE OF REALIZATIONS IN LEARNING MATHEMATICS

Within the commognitive perspective, mathematical discourse is a “historically established activity” (Sfard, 2008, p. 203) consisting of four distinctive, interrelated characteristics. It includes specific keywords (e.g., *linear transformation*) “used in distinctly mathematical ways”, visual mediators (e.g., *graphs*), routines—“patterned ways in which mathematical tasks are being performed” (e.g., *calculating the transformation of vectors in a 2D space using a transformation matrix*)—and endorsed narratives (e.g., *a linear transformation preserves vector addition and scalar multiplication*) (Sfard, 2017, p. 42). Mathematical objects are the products of human discourse and come into existence through the process of objectification, specifically through *saming*, *encapsulating* and *reifying* (see Sfard, 2023). The term ‘realization’ is introduced in commognition theory as an alternative to ‘representation,’ emphasizing that mathematical objects are, in fact, products of human discourse (Sfard, 2008). These discursive constructs are formed to facilitate communication about the world we inhabit (Sfard, 2017). Within the commognition perspective, we can conceptualize the learning of linear algebra “as the process of becoming familiar with different realizations of algebraic objects (such as matrices, systems of linear equations, vector spaces, etc.) while saming the equivalent realizations and treating them as objects ‘existing’ in the world” (Wallach et al., 2022, p. 4858). More broadly, this theory, and others, emphasise the importance of discussing different realizations of mathematical objects in teaching and learning mathematics. For instance, Wallach et al. (2022) noted that *saming* the different realizations of a mathematical object is an important step towards objectification. Furthermore, Sfard (2017) argued that helping students connect various realizations of mathematical objects “should be the main objective of school learning” (p. 43). Therefore, YouTube videos have the potential to create valuable LOs by encouraging viewers to reflect on and make connections between different realizations of mathematical objects. This reflective process can deepen their learning of abstract mathematical objects (Radmehr & Turgut, 2024).

DIFFERENT REALIZATIONS OF THE LINEAR TRANSFORMATION

Linear algebra, as a mathematical field, primarily utilizes three modes of description (Hillel, 2000): the geometric mode (referring to plane and space geometry), the algebraic mode (involving n -tuples, matrices, and associated algebra), and the abstract mode (focusing on formal theory, such as vector spaces and subspaces). Within linear algebra, Linear transformations (LTs) represent a specific type of transformation, defined by two particular properties (formulated in the following). As a result, LTs involve a certain degree of axiomatic language, along with various realizations and connections to different mathematical objects. For instance, set theory, the concept of functions, and vector spaces are key elements within the topic of LTs (Lay et al., 2022). From the lens of commognition, the following realizations can be unpacked in terms of *five* key realizations from the literature on calculus (e.g., Radmehr & Turgut, 2024; Roundy et al., 2015).

Symbolic Realizations (SR)

If we assume V and W are two vector spaces, then we can define a transformation from V to W denoted as $L: V \rightarrow W$. Here, the transformation L is called linear if it satisfies the following two conditions for $\forall \mathbf{u}, \mathbf{w} \in V$ and $\forall c \in \mathbb{R}$: $L(\mathbf{u} + \mathbf{w}) = L(\mathbf{u}) + L(\mathbf{w})$ and $L(c\mathbf{u}) = cL(\mathbf{u})$ (Lay et al., 2022). Furthermore, there is a strong connection between matrices and LTs. For an LT defined between vector spaces V and W , if the bases of V and W are fixed, then we can associate a matrix to represent any LT (Lay et al., 2022). For example, if a mapping $M: \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfies two conditions above, then we can define a matrix transformation:

$$M: \mathbb{R}^n \rightarrow \mathbb{R}^m, M(x_1, x_2, \dots, x_n) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix},$$

where we map n -tuples to m -tuples through matrices. The *matrix-vector* equation above can be simplified to $A\mathbf{x} = \mathbf{b}$ where A shows the transformation matrix, $\mathbf{x}$ denotes the inputs, and $\mathbf{b}$ represents the outputs. Consequently, we can identify three interrelated SRs of LTs: (i) *algebraic* (such as the fundamental formulas above and or their combination into one condition (see Lay et al., 2022)), (ii) *function-related* (such as Dirichlet-Bourbaki notation and “mapping” notation for geometric linear transformations) (Turgut, 2019) and (iii) *matrix-related* (such as matrix-vector equations and/or the transformation matrix itself). Overall, we can also argue that symbolizing here is related to abstract mode of descriptions (Hillel, 2000).

Verbal Realizations (VR)

The term “Linear Transformation” focuses on two properties of *transformation* (correspondence), while the term *matrix transformation* describes how a transformation (e.g., *mapping* from $\mathbb{R}^n$ to $\mathbb{R}^m$) is implemented (Lay et al., 2022). However, the term *function* could be considered an embracing term for all LTs, while *mapping* could be used for geometric LTs. Another term *linearity* of a transformation refers to two specific conditions of LT, as discussed earlier, or the simplified equation.

Graphical Realizations (GR)

In general, to represent any LT from V to W (but also its inverse, compositions of LTs, range and kernel spaces), a *Venn diagram* can be used, as appears in various textbooks (e.g., Lay et al., 2022). There are also GRs about representing mappings, such as representing the transformation of one point, a vector, a geometric figure and/or a 3D object’s LT in $\mathbb{R}^2$ and $\mathbb{R}^3$ (e.g., specific grid plane applications), along with related descriptions that are associated with geometric mode (Hillel, 2000).

Numerical Realizations (NR)

The powerful tool of the matrix in LTs include numbers, as coefficients. These numbers generally refer to coordinates of geometric vectors as inputs. This could

involve algebraic mode of descriptions (Hillel, 2000), such as representing geometric vectors (such as $\mathbf{u} = (-3, \sqrt{2}, 2, -4)$ in $\mathbb{R}^4$) or through column matrices, row operations to explore characteristics of the transformation matrix (e.g., invertibility and finding the inverse LT), and carrying out algorithms.

Physical Realizations (PR)

The PR of the derivate is the measurement process before calculating the derivate (Roundy et al., 2015). Here, we adapt this conceptualization to LTs by considering—for example—a measurement process when recording data from experiments (related to real-world) to formulate an input matrix which later on could be transformed using an LT. An exemplary case could be the unit cost matrix, where the columns represent the number of products, and the rows could involve materials, labour and overhead.

METHODOLOGY

This study employs an intrinsic case study approach, a qualitative method that involves selecting a unique case requiring detailed description and analysis (Creswell & Poth, 2018). The selected case (https://youtu.be/kYB8IZa5AuE?si=xGB3XRhR_JFEmgZf) is from a highly viewed YouTube channel, *3Blue1Brown*, which, as of October 2024, has 6.57 million subscribers, and its content has been viewed over 564 million times. Regarding data analysis, we focused on how the five main realizations of mathematical objects (i.e., physical, numerical, verbal, symbolic, and graphical) were discussed in the video to aid in the learning of LTs.

RESULTS

The video starts with a summary of the content and the main goal, and later Grant, the content creator, focuses on several VRs: the terms LT (0:44), function (0:49) and the relationship/connection between these two. Grant further uses a different VR for outputs and input-output relation: “spit out” another point/vector (0:52) (to emphasize the function of transformation), with an example of NR and matrix-related SR, as vector inputs and outputs in $\mathbb{R}^2$ with algebraic SR (1:01) (Figure 1a).



Figure 1: left (a) Vector inputs (1:01), right (b) Transformation of all vectors (1:33). Then, the discussion goes back to VRs, particularly on why we use the word “transformation” (1:06) and “input-output relation”, stating “well, it’s to be suggestive of a certain way to visualize this input-output relation” (1:11). Grant refers to another VR, “movement” (1:16), where the use of the word transformation emphasizes. Then,

Grant explains input-output transformation (movement) on the grid plane (GR) (1:24). Then, by referring to GRs again, Grant explains all possible vectors' transformation to associated output vectors dynamically (1:33) (Figure 1b).

Then he refers to “points” on the grid plane (GR) (1:47) that corresponds to vectors and discusses the movement of points as transformations (VR) (1:59). Next, additional grid plane-related GRs follow, demonstrating the transformations of an infinite number of points while maintaining the original grid plane configuration (2:11). Later, Grant shows a number of GRs regarding “arbitrary transformations” (VR) (2:29) by speaking about the term “linear” in LTs (VR). Next, an endorsed narrative about LTs is highlighted by using GRs about the grid plane (Figure 2a): “[...] visually speaking, a transformation is linear if it has two properties: all lines remain lines, without getting curved, and the origin must remain fixed in place” (2:40–2:49). The video continues with examples of non-LTs (GR) by using a VR “get curved” (2:55) with grid plane transformations (GR) (Figure 2b). Later, Grant finalizes this part with another VR: “grid lines remain parallel and evenly spaced” (3:22) with GRs.

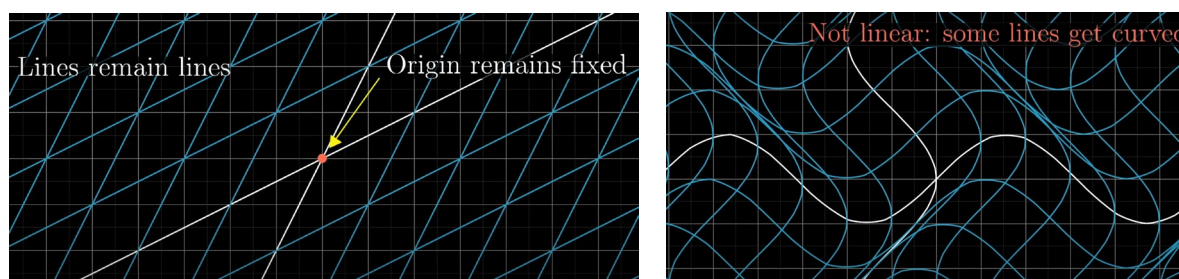


Figure 2: left (a) An LT on grid plane (2:49), right (b) A non-LT (2:55).

As a next step, the video gives examples of rotations about the origin (GR) (3:25) as a LT on the grid plane (GR) and emphasizes the “others” (VR) (referring to arbitrary LTs) which are not easy to describe with words (3:29). Then the discussion is about “representing transformations numerically” (VR) (3:32). Grant refers to computer graphics with matrix-related SR (3:42), how to formulate transformations by using input-output vectors. By using basis vectors' transformation on the plane (GR) (3:57), numerical examples are explained (NR) on grid plane (GR) with revisiting VRs regarding grid lines' positions (4:16). Grant explains the linear combination of transformed basis vectors (algebraic SR) referring to the coordinates on the grid plane (GR) but also with NRs (4:54–5:17) (Figure 3a).

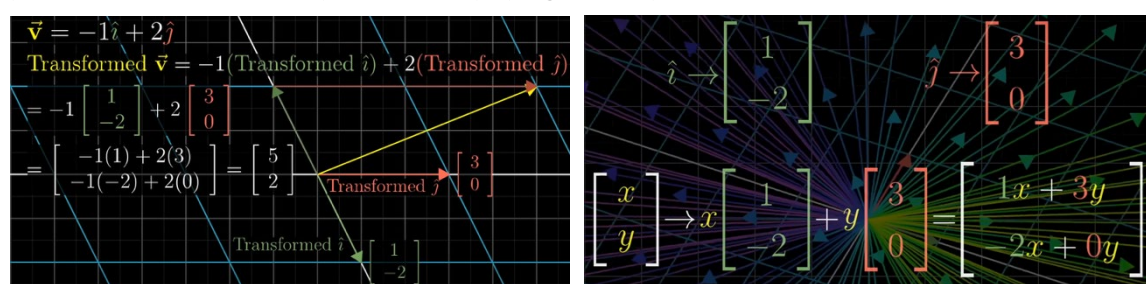


Figure 3: left (a) Linear combinations of outputs (5:17), right (b) Obtaining an arbitrary vector's outputs as a matrix (6:01).

Later, Grant revisits the grid plane transformations (GR) (5:27), with VRs: “[...] so long as we have a record of where $\hat{i}$ and $\hat{j}$ land without needing to watch the transformation itself”. Grant then shows a procedure for finding an arbitrary vector’s outputs by using column matrices, which is a combination of NRs, algebraic and matrix-related SRs (5:55) and later connects this to all vectors on the grid plane (GR) (6:01) (Figure 3b).

Going back to grid plane (GR), Grant characterizes this by (VR) “... a two-dimensional linear transformation is completely described by just four numbers: the two coordinates for where $\hat{i}$ lands and the two coordinates for where $\hat{j}$ lands” using NRs (6:16). Later, the video uses “ 2×2 grid of numbers” (VR) (6:22) to “package” the outputs and introduces 2×2 matrices (SR), as first column shows “where $\hat{i}$ lands” and the second column shows “where $\hat{j}$ lands” (6:28) (Figure 4a). Then Grant emphasizes how to operationalize this matrix (SR) as linear combinations (NR) with output matrices (SR) (6:50). Here, the video refers to a different VR, “adding the scaled versions of our new basis vectors” and shows this on the grid plane (GR) (6:52). Then, Grant explores a general case with a 2×2 matrix with four unknown coefficients (SR) (7:11) by applying this transformation to any vector, computes an arbitrary vector’s output (SR) (7:31).

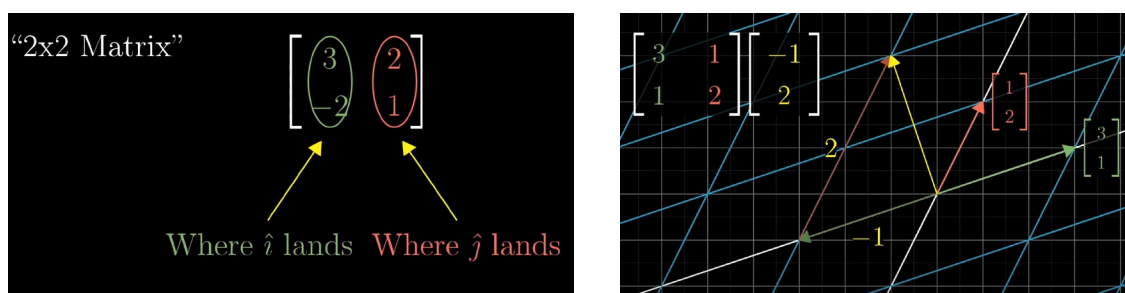


Figure 4: left (a) The “landing” characterization (6:28), right (b) The summary of the grid plane transformation (8:00).

Then, the discussion moves to represent this by matrix multiplication (SR) (7:39) and connects this to a “function” (function-related SR). Grant summarizes the transformation matrix by VRs stating, “these columns as the transformed versions of your basis vectors [...] results as the appropriate linear combination of those vectors” (7:55) with a GR (8:00) (Figure 4b). The video, later, continues with a few applications (GR) of counterclockwise rotation and explores the transformation matrix (SR) using landing coordinates of basis vectors (NR) and a matrix-vector multiplication (SR) (8:31). Grant repeats the same procedure for shear transformations (GR) using column/numerical matrices (NR) and 2×2 transformation matrix (SR) (8:46). As a next step, Grant discusses the other way around, starting with a matrix (NR), and what the transformation would be then. Then, Grant shows how column vectors (NR) position on the grid plane (GR) (9:16) and how the grid plane evenly spaced. Grant also explores the case of linearly dependent columns (NR) of the transformation matrix (SR), and how all the grid plane moves to the points on a line, where the column vectors (NR) lie on (GR). The video ends with a brief summary regarding specific VRs, NRs, GRs and

SRs of LTs.

CONCLUSIONS AND DISCUSSION

The analysed video offers a broad spectrum of different realizations of LTs, transitions between them, while explaining the related epistemological ideas, making it a valuable resource for learning LTs. The critical synergy among these different realizations mirrors transitions between various modes of description, which are key tools for teaching linear algebra (Hillel, 2000). For example, the video starts with several VRs to unpack the idea of using “transformation” instead of the term “function”. Second, the video establishes a strong connection between a number of different VRs and GRs, such as characterizing LTs and discussing them on the grid plane. We note that the synergy between these realizations might not be easily captured in a textbook, but YouTube resources could serve as valuable supplementary resources for teaching and learning (e.g., Esparza Puga & Aguilar, 2023; Pepin & Kock, 2021; Radmehr & Turgut, 2024). We note that the developer focuses exclusively on $\mathbb{R}^2$ in the analysed video; however, there are follow-up videos on the channel that address specific contexts of $\mathbb{R}^3$. Such a synergy between different realizations could facilitate objectification and could provide a smooth transition to (more) abstract mathematics objects, such as LT of abstract vector spaces. For example, regarding the abstract mode (Hillel, 2000), the analysed video highlights the transformation of grid planes as a whole. This may lead students to develop a narrative in which the transformation matrix applies only to given vectors, rather than to entire plane and/or space (Molina & Oktac, 2007). However, in our analysis, we have not observed any physical realizations regarding LTs. This could be because the use of function-related realizations which are not connected to measurement. Rather, these realizations are in relation to the set theory and functions, e.g., the abstract mode of representation (Hillel, 2000). In this paper, we focus on analysing the content of a YouTube learning resource; however, further research is needed to investigate how viewers interact with such resources. Additionally, it is important to explore how these resources can be effectively integrated into mathematics teaching, such as through a flipped classroom approach or by incorporating YouTube learning resources alongside traditional materials, such as textbooks, in classrooms or lectures.

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DEVELOPING PRESERVICE TEACHERS' DOCUMENTATION EXPERTISE FOR TEACHING MATHEMATICS IN A SWEDISH CONTEXT

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Documentation expertise, or a coherent use of resources is an essential part of a teacher's professionalism. Hence, it deserves a specific focus in mathematics teacher education. Yet, preservice teachers seldom learn to interact efficiently with instruction resources. In Sweden, it was not until 2024, that a related examination goal for teacher education came into force. Drawing on the documentational approach to didactics, research on teachers' use of digital resources, comparative textbook study, and the cultural specifics of using instruction resources in Sweden, I will propose a framework for developing preservice teachers' documentation expertise in teacher education in Sweden.

TEXTBOOKS, DIGITAL INSTRUCTION RESOURCES, AND TEACHER EDUCATION

Teachers use a multitude of resources for teaching mathematics: textbooks, digital resources, and manipulatives are just a few of the various set of resources used. Teachers use not only instruction resources during teaching, but also put a lot of effort in selecting and modifying resources, and plan for teaching with these resources outside of the classroom, often in collaboration with their colleagues in- and outside the school (Gueudet et al., 2013; Pepin et al., 2013; Sherin & Drake, 2009). Teachers' coherent use of a variety of instruction resources for teaching mathematics is thus an essential part of their professionalism (Gueudet & Trouche, 2009; Pepin & Gueudet, 2020; Trouche et al., 2018), and hence, it deserves a specific focus in mathematics teacher education. Yet, teachers are not often taught to interact with their instruction resources (Drake et al., 2014; Remillard, 2016a). In Sweden, it was not until July 1, 2024 that a specific examination goal about instruction resources for teacher education came into force. Preservice teachers are now expected to show competence in using and evaluating textbooks and other curriculum and instructional resources (Regeringskansliet, 2023). In working towards this goal, one might be tempted to draw on and apply existing frameworks to structure preservice teachers' interactions with instruction resources and their learning from it. But these frameworks are few and tend to focus primarily on the use of printed curriculum resources (e.g., Drake et al., 2014; Remillard, 2016a). Research also suggests that teachers' expertise in making use of instruction resources for teaching mathematics is culturally loaded (Wang, 2019),

meaning that one needs to be careful in applying frameworks developed in specific educational contexts.

The aim of this contribution is to propose a framework for developing preservice teachers' expertise in interacting with mathematics instruction resources in teacher education in Sweden. To develop the framework, I will draw on the documentary approach to didactics, research on teachers' use of digital instruction resources, comparative textbook study, and on the cultural specifics of using instruction resources in Sweden. I understand this contribution to fit within theme 4: Studies of the use of curriculum resources by students and/or teachers (including student teachers and teacher educators).

Inspired by Pepin and Gueudet (2020), I relate to instruction resources as all the resources developed, adapted, and/or used for teaching and learning mathematics. Many instruction resources are material, such as manipulatives and textbooks, but resources can also be cognitive (such as frameworks) and social (such as colleagues and students). Curriculum resources are a category of material instruction resources that contain a specific scope and are designed to cover a learning sequence (Pepin et al., 2017). Instruction resources and curriculum resources can be analogue and digital.

NEED FOR FRAMEWORKS TO SUPPORT PRESERVICE TEACHERS' USE OF A VARIETY OF RESOURCES

As already mentioned, frameworks to support preservice teachers' use of instruction resources for teaching mathematics tend to focus on printed curriculum resources (i.e., printed textbooks and teacher's guides). Drake et al. (2014) propose such a framework in a USA context. They propose that preservice teachers can learn about and from educative curriculum resources in teacher education. Thus, through their encounters with curriculum resources, preservice teachers can develop both their knowledge about working with the curriculum resource as well as their broader knowledge base. In order to do so, Drake et al. (2014, p. 159-160) propose five principles to guide preservice teachers' work with curriculum resources (Drake and colleagues use the term prospective teachers, abbreviated as PTs): 1) PTs need opportunities to learn that curriculum materials contain educative features to support teacher learning, as well as student learning, and to orient themselves to reading these materials in ways that attend to the educative features; 2) PTs require tools to serve as lenses through which the educative features of curriculum materials can be read, understood, evaluated, and adapted; 3) PTs require scaffolds to prompt the development of multiple knowledge bases through the use of curriculum materials, 4) PTs need to examine multiple lessons and units in order to identify and understand the development of content over time; 5) PTs need experiences reading across a variety of curriculum materials in order to compare and contrast the treatment of content.

Remillard (2016a) proposes that curriculum resources and teachers each come with complementary contributions to teaching mathematics and suggests to understand the teacher – curriculum resource relationship as a partnership. Remillard (2016a, p. 37-

38) proposes four strategies for teachers to partner with their curriculum resource: 1) Look for the big ideas; 2) Pay attention to the pathways; 3) Anticipate: What will ___ say?; 4) Collaborate with colleagues.

The two abovementioned frameworks focus primarily on teachers' use of printed curriculum resources. In many cases, textbooks are indeed a main instructional resource (Mullis et al., 2012) but it is not their sole resource. In practice, teachers use multiple resources (e.g., Gueudet & Trouche, 2009). Also, the textbook landscape has changed considerably, and it is nowadays almost impossible not to think of curriculum resources in digital formats. Furthermore, research suggesting that teachers' expertise in making use of instruction resources is culturally loaded (Wang, 2019) underscores the importance of considering the cultural specificity of being a mathematics teacher. I believe therefore that a framework for developing preservice teachers' expertise in interacting with mathematics instruction resources should target teachers' use of a variety of instruction resources, including digital resources, and be attuned to the cultural dimension of teaching mathematics. I try to do so below, grounding resource use in the theory of the documentational approach to didactics, research on teachers' use of digital resources, comparative textbook study, and some of the cultural specifics of using instruction resources in Sweden.

DOCUMENTATION WORK: (PRESERVICE) TEACHERS' INTERACTIONS WITH INSTRUCTION RESOURCES

Perhaps one of the most powerful conceptualizations of the teacher – instructional resource relationship is the documentational approach to didactics (DAD), developed by Ghislaine Gueudet, Luc Trouche, Birgit Pepin and colleagues (e.g., Gueudet & Trouche, 2009; Trouche et al., 2018). Within DAD, teachers' interactions with resources during processes such as searching, selecting, adapting, accumulating, and reflecting on resources are termed as documentation work (Wang, 2019). Documentation expertise is the expertise that teachers develop through their interactions with resources; it is highly contextualized and culture-specific (Wang, 2019). Through their documentation work and expertise, teachers develop coordinated and consistent ways of using sets of resources for specific teaching purposes, visible in observable regularities, termed “usages”, and building on their implicit knowledge bases, termed “operational invariants” (Gueudet & Trouche, 2009). Resources can be material, cognitive, and social such as collegial input or feedback from students (Pepin & Gueudet, 2020; Trouche et al., 2018). Thus, DAD explicitly acknowledges that teaching mathematics intersects with a coordinated use of a set of instruction resources – not just the textbook series – for specific purposes. This brings me to the first principle:

- Principle 1: Preservice teachers need opportunities to learn consistently using sets of instruction resources for specific purposes in teaching mathematics.

A considerable aspect of this first principle relates to the integration of digital instruction resources in the classroom, for which Ruthven (2009) proposes that teachers

develop specific craft knowledge concerning five structuring features. These key features are a) working environment or the physical location, classroom organization and procedures; b) resource system or the integration and coordination of digital resources in teachers' use of existing resources (e.g., textbook); c) activity format or patterns of interaction between students and teacher in specific lesson segments; d) curriculum script or the connection of ideas, representations, tasks, difficulties, and how digitalization may interfere with these scripts; and e) time economy.

The process of developing documents from sets of resources is termed a documentational genesis. It is a lengthy process, indicating that consistently using resources for specific purposes takes time. This long process suggests the necessity of including preservice teachers' use of instruction resources as a specific focus throughout their entire teacher education program rather than on a limited number of single occasions, as is often the case. This brings me to the second principle:

- Principle 2: Preservice teachers need opportunities to work with and learn from curriculum and instruction resources throughout their entire teacher education program.

TRANSFORMATIVE USE OF DIGITAL INSTRUCTION RESOURCES

Digital instruction resources are often named for their possibilities compared to printed curriculum resources, but these digital resources come with challenges as well. Characteristics named in literature related to digital curriculum resources are their non-linear (Remillard, 2016b), connection-rich (Gueudet et al., 2018), and opaque nature (Usiskin, 2013). The first two of these characteristics are often related to possibilities for personalization and coherence, whereas the latter brings in a challenge related to these resources' transparency.

Focusing on the possibilities of digital curriculum resources and reviewing related research, Pepin et al. (2017) have conceptualized how digital resources can transform the teaching and learning of mathematics. They distinguish between a presentation space, a problem space, work space, and navigation space. Although their conceptualization is related to digital curriculum resources, I think that the first three spaces relate well to digital instruction resources as well. The presentation space is about the mode through which mathematics is introduced to the students. In a printed book, mathematics can be presented through written text, symbols and images, which is also possible with digital resources. But digital resources additionally can also allow for presenting mathematics through video and sound, impossible in printed textbooks. The problem space has to do with the type of problem students are working on. In some ways, digital resources' problems can be more open with multiple ways toward the solution than in printed textbooks. The work space relates to the tools that students have at their disposal to work on mathematical tasks. Changing the shape of a graph by changing the values in an equation and in so doing helping the students making the connection between the equation and the graph is an example of such a digital tool. The navigation space relates explicitly to digital *curriculum* resources since it is about

how students navigate sequenced content. I understand this space to be related to the non-linear characteristic, meaning that there are different pathways to navigate the curriculum. Pepin et al.'s review found that when digital curriculum resources are used in a transformative manner, this is usually done so in relation to the presentation space, but relatively little in relation to the other spaces. Thus, digital resources are often used in non-transformative ways in spite of the possibilities that come with these resources (Pepin et al., 2017). I find Pepin et al.'s (2017) conceptualization interesting because it helps to think about how one can make use of digital resources to do things one cannot do with non-digital resources. But it also shows that digital resources by themselves will not change teaching. The teacher, ultimately, has a decisive role in how transformative digital resources are being used, and here teacher education can play a supportive role. This brings me to the third principle:

- Principle 3: Preservice teachers need opportunities to learn using digital resources in a transformative manner.

SWEDISH TEACHERS' INTERPRETATION SPACE AND TESTED EXPERIENCE

Comparative study can help not only to learn about alternative viewpoints, but also to put a magnifying glass on one's own taken-for-granted assumptions (Andrews, 2007; Stigler & Hiebert, 2009). For instance, from a comparative study of mathematics lesson guides, or the descriptions of mathematics lessons for teachers, from Sweden, USA, and Flanders, it appears that Swedish teachers are expected to make more individual judgements as to how to teach the lesson than their USA and Flemish counterparts (Van Steenbrugge & Remillard, 2023). Swedish lesson guides often describe what the lesson is about, but don't guide or strongly steer the teacher as to how to teach the lesson, leaving that up to the individual teacher. Teachers' autonomous interpretation space seems more strongly valued in lesson guides from Sweden compared to lesson guides from the USA and Flanders. This is also something I noticed in my encounters with both preservice and inservice teachers. When I let my students compare lesson guides from Sweden with lesson guides from the USA and Flanders, many react on several of the supportive features in the lesson guides from USA and Flanders. They typically note the specific supports in the American and Flemish lesson guides and the difference in length of a lesson guide. They also note that the descriptions in the American and Flemish lesson guides are much more directive than the descriptions in the Swedish lesson guides. They often describe the American and Flemish lesson guides as manuals, which comes with a negative connotation. Usually, a discussion then unfolds where some advocate for the adoption of these additional supports and others argue against because it doesn't value the profession of teaching. Almost always, at the end of the discussion, students come to point to a central feature of teaching mathematics in Sweden: the *tolkningsutrymme*, or Swedish teachers' considerable space for own interpretations. Comparing curriculum resources from various educational contexts helps to become aware of some of the, often unspoken,

assumptions that ground the teaching of mathematics, which brings me to the fourth principle:

- Principle 4: Preservice teachers need opportunities to compare instruction resources from various countries to shed light on their own cultural lens.

My students' reactions are in line with my experiences of introducing an American lesson guide to Swedish teachers. The group of teachers was interested in developing a new curriculum resource (Van Steenbrugge & Ryve, 2018) and I showed them American lesson guides. The teachers reacted very strongly on some of the supports, for instance, on the time indications for activities mentioning that curriculum authors cannot decide on the timing of activities and that the pacing of lessons instead belongs to the teachers' discretion. Swedish teacher guides usually don't contain much time pacing guidance; in fact, they often don't state more than suggesting that a chapter should take a specific number of weeks.

Teachers' specific interpretation space, as suggested in the analysis of Swedish lesson guides and in my encounters with preservice and inservice teachers, I believe, is part of the Swedish culture of teaching mathematics. There are traces of teachers' interpretation space in the 2010 school law, where the government's proposal for the 2010 school law mentions that teachers can decide themselves about the content and methods to reach the goals (Skolverket, 2013; Utbildningsdepartementet, 2010a). But there is also a collective dimension coupled to the interpretation space. The 2010 Swedish school law also stipulates that Swedish teachers' work should be based on research and tested experience ("beprovad erfarenhet") (Utbildningsdepartementet, 2010b: 800 1 kap. 5 §). Tested experience is tested during a lengthy period of collegial work, documented, and shared (Sahlin, 2017; Skolverket, 2013). Tested experience indicates that teachers should not just adopt textbooks that include other types of teacher support. Instead, teachers are supposed to try out these materials over lengthy periods of time, discuss related challenges and opinions with colleagues, and document and share their findings. Related to teacher education, preservice teachers should thus be provided with opportunities to collectively plan, implement, discuss, and document their work with curriculum and instruction resources. I believe that working toward this principle is imperative given the centrality of interpretation space and tested experience in the Swedish context. A fifth principle is thus:

- Principle 5: Preservice teachers need opportunities to collectively plan, implement, discuss, and document their work with instruction resources.

DEVELOPING PRESERVICE TEACHERS' DOCUMENTATION EXPERTISE: WHAT AND HOW

Within the proposed list of principles for working toward developing preservice teachers' documentation expertise, principles 1 (the coordinated use of a set of resources for a specific purpose) and 3 (transformative use of digital resources) talk to the substance of what to focus on in teacher education. Principles 2 (a continued focus throughout teacher education) and 5 (collectively plan, discuss, and document) talk

more toward how to do so. Principle 4 might help preservice teachers becoming aware of their own cultural lenses related to using instruction resources and teaching mathematics. That said, the principles are formulated in a general language. Indeed, the proposed principles are less concrete than the principles and strategies proposed by Drake et al. (2014) and Remillard (2016a). This limitation requires further work to operationalize the proposed principles.

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REVISION OF THE DUTCH INTENDED CURRICULUM FOR MATHEMATICS – IMPRESSIONS OF PROCESS AND RESULTS

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In the Netherlands, the legal intended curriculum is of great influence on the work of publishers and authors of textbooks and other curriculum resources. Recently, the government has commissioned the revision of the Dutch legal core objectives for primary and lower secondary education. For the development of the renewed core objectives for mathematics, an analysis was carried out regarding societal developments and educational trends. This paper reports upon the development process of the revision of the core objectives and the results thereof, with a focus on new and more emphasized learning content in the final draft core objectives.

INTRODUCTION

In the Netherlands, like in many other countries, the legal intended curriculum is a main influence on the work of publishers and authors of textbooks and other curriculum resources. Publishers aim and claim that their textbooks cover all the obligatory learning content. The legal intended curriculum for mathematics for primary and lower secondary education (age range from 4 to 15 years), consists of core objectives and a so-called Reference Framework for Dutch language and Mathematics. The current core objectives date from 2006 and are formulated in very general terms. For example, one of the eleven objectives for primary school mathematics is “students learn to use mathematical language”, and one of the nine objectives for lower secondary school mathematics is “students learn to recognize mathematics in practical situations and to use mathematics to solve problems, both individually and in collaboration with others.”

Recently, the Dutch Ministry of Education (2022a) commissioned a revision of the core objectives. The new core objectives to be developed should be future-proof and more concrete. The Netherlands institute for curriculum development SLO (hereafter: SLO) was appointed the assignment to develop the renewed core objectives.

CONTEXT AND LEADING QUESTION

The revision of the core objectives for mathematics is part of a larger operation to update all legal curriculum prescriptions in the Netherlands, including core objectives for other learning subjects and examination programs for upper secondary school. Due to ongoing societal concerns regarding students’ achievements in Dutch language and mathematics, the Dutch government decided to prioritize the revision of the core objective for these two learning subjects. Following specifications of the assignment

of the Ministry of Education, the leading question for the revision of the mathematics core objectives was: *What mathematical knowledge, skills, insights and attitudes do Dutch students need to be well prepared for their future in our changing society?*

METHOD

The core objectives to be developed had to describe desired learning outcomes. What desired learning outcomes entail is not an entirely neutral or scientific question, and thus cannot be answered by a scientific method alone. Nevertheless, a systematic approach was followed, including a problem analysis, a development stage and a trial stage (Figure 1). Three curriculum quality criteria (Thijs & Van den Akker, 2009) were considered. In the development stage these concerned *relevance* (the curriculum meets the need and is based on state-of-the art (scientific) knowledge) and *consistency* (the curriculum is designed logically), and in the trial stage the quality criterium was *usability* (the curriculum is useful in the settings for which it is intended).

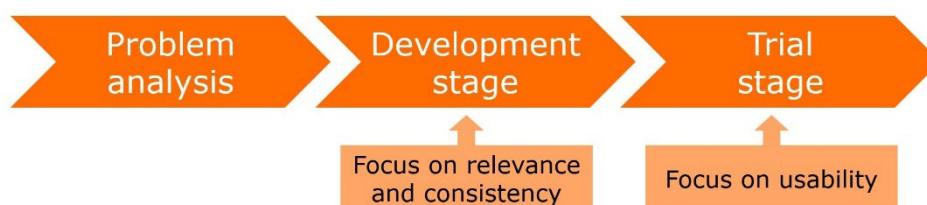


Figure 1. Stages in the revision process

For the development work, SLO put together teams and advisory boards. The mathematics team consisted of primary school teachers specialized in mathematics, secondary school mathematics teachers, mathematics experts and curriculum experts. The advisory board included representatives of all Dutch stakeholders regarding mathematics and mathematics education, primary school teachers, mathematics teachers, mathematics teacher educators, and scientists.

According to the assignment of the Ministry of Education, the mathematics team was expected to develop an introductory text characterizing mathematics education (the so-called ‘characterization’), a framework, and core objectives for primary and lower secondary education. The characterization should make clear, among other things, what the contribution of the mathematics curriculum should be to three main aims of education: qualification, socialization and subjectification (cf. Biesta, 2020).

Problem analysis

The problem analysis was meant to bring into view the educational and societal developments to be addressed in the revised core objectives. To that end, key publications of Dutch educational and mathematical institutions were studied, such as the Education Council, the Dutch Inspectorate of Education and the Dutch Mathematics Foundation. In addition, recent national and international (scientific) sources regarding societal developments and mathematics education were studied.

Development stage

The results of the problem analysis (Van Zanten & Schmidt, 2022), together with the 2006 core objectives formed the starting point of the development stage. Based upon this and other sources, such as mathematics curriculum documents of other countries, the mathematics team drew up a first draft of the characterization and framework to be developed and made first choices regarding learning content to be included in the objectives. After that, first drafts of a few core objectives to be developed were written. The advisory board provided feedback and advice regarding all of these drafts, based upon input they collected from the people and institutions they represented. The advisory board checked upon relevance and consistency of the drafts, and also on whether a good balance was reached across the various learning content to be included. In an iterative process the drafts were rewritten, refined and extended with other draft core objectives. During the process, feedback on the drafts was again collected from the advisory board, as well as from teachers, experts and scientists. Up to 17 versions of drafts were written during this stage.

Trial stage

In the trial stage, primary and secondary school teachers and head teachers judged the draft core objectives on usability. Mathematics coordinators and mathematics teachers of over 200 schools participated in regional conferences spread all over the country, in which they were informed about the draft core objectives, the changes compared to the 2006 versions, and the underlying ideas and intentions. Also, examples of possible educational activities regarding new elements in the draft objectives were discussed. These samples were developed by the mathematics team. After the conferences, the participants discussed the draft objectives with their colleagues. If possible, teachers at the schools involved tried out some of the sample activities with their students. The feedback of the participating schools was collected using a questionnaire with both closed-ended and open-ended questions. Furthermore, the drafts were again judged on relevance and consistency. To this end, a group of 29 experts and scientists discussed the drafts and provided feedback on all points they considered to be relevant. All information gathered in the trial stage was weighed and used to refine and finalize the draft core objectives.

ISSUES DEALT WITH AND RESULTS

Issues to be addressed in the revised core objectives

The problem analysis identified several societal developments and educational trends to be addressed in the revised core objectives. Among other things, the increasing role of IT affects the mathematical skills that people need for daily and professional life. Matters that cannot or hardly be taken over by IT, such as formulating problems in mathematical terms and assessing what a mathematical answer means for the issue or problem at hand, are therefore becoming more important and deserve more attention in education (Gravemeijer et al., 2017; cf. OECD, 2018). The increasing amount of

quantitative information calls for critical mathematical thinking and an understanding of mathematical language and mathematical communication (Cairo, 2019). More generally, handling data requires mathematical thinking methods such as modelling and algorithmic thinking (cf. Fry, 2018; Platform Wiskunde Nederland, 2012). An international trend in mathematics education in this context is to focus not only or mainly on procedural skills, but also on conceptual knowledge and more advanced mathematical skills and insights (Gravemeijer et al., 2017; Lesh et al., 2007; OECD, 2018). Examples include critical quantitative thinking and mathematical problem solving, a global understanding of the mathematics work that devices perform, and the ability to critically interpret its output (Wiskunde voor Morgen, 2019). At the same time, considering the earlier mentioned concerns about Dutch students' mathematics achievements, the renewed objectives to be developed also had to address basic mathematical knowledge, skills and insights, and should contribute to more students achieving an adequate level of numeracy (Ministry of Education, 2022b). In the renewed core objectives to be developed, a balance was to be reached between all these issues in order to adequately contribute to the mathematical knowledge, skills, insights and attitudes that nowadays' students need to be well prepared for their future.

A characterization of mathematics education

The characterization to be developed had to specify what should give direction to the mathematics curriculum. According to the assignment, as mentioned before, this includes the contribution of the mathematics curriculum to the qualification, socialization and subjectification of students. The problem analysis provided several points of interest for this. The mathematics team also looked into comparable curriculum documents of other countries, which were suggested by the advisory board. These documents included statements in line with points of interest of the problem analysis, such as the following.

People around the world have used and continue to use mathematical knowledge, skills, and attitudes to make sense of the world around them and develop new mathematical thinking and appreciation for mathematics (The Ontario Curriculum, 2020, p. 63).

Mathematics is the study of the relationships, connections and patterns that surround us and is thus intrinsic to our concept of the world. Mathematics greatly enhances our capacity to understand and engage fully with the world around us. (NCCA, 2022, p. 9).

Generally, the studied curriculum documents all mentioned the importance of mathematics for everyday life, other learning areas, further education and occupations. Guiding elements found in these documents included, for example, mathematics in every aspect of daily life, numeracy, mathematical thinking, appreciation for mathematics and mathematics to engage with the world around us. The mathematics team incorporated these and other points in the characterization. The following quote is the opening statement of the final draft, in which the contribution of the mathematics curriculum to qualification, socialization and subjectification is expressed.

Using and understanding mathematics is important to function in society, occupation, and for making choices in personal life. To promote equity, all students should gain a solid foundation in mathematics. All students need to learn to calculate fluently and flexibly and need to become numerate citizens. Numeracy enables people to understand reality and value information. In primary and lower secondary education, students use mathematics to understand information and phenomena in the world around them (SLO, 2024, p. 6).

Regarding qualification and socialization, the final draft characterization further mentions the importance of mathematical language, mathematics for critical thinking, and mathematical problem solving. With respect to qualification, socialization and subjectification, the characterization also emphasizes the importance of a mathematical attitude.

A framework for the revised core objectives

The development of the mathematics framework started with a comparison of existing Dutch and international frameworks. This concerned, among others, the mathematics frameworks of TIMSS (Mullis & Martin, 2017) and PISA (OECD, 2018), and the Dutch Mathematics Requirements for Vocational Education (Expertgroep Herijking Rekeneisen mbo, 2020). All frameworks included in this comparison distinguish mathematical concepts and mathematical thinking, the latter combined with applying mathematics. This distinguishment was translated into two domains in the framework for the revised core objectives: *Mathematical concepts* and *Mathematical thinking*. To emphasize the importance of mathematics for daily life, other subjects and occupation, in line with the characterization, applying mathematics was taken as a separate domain, with the label *Mathematics and the world*. To make clear that in daily education these domains cannot be separated, the mathematics team developed a visualization that shows just that (Figure 2).

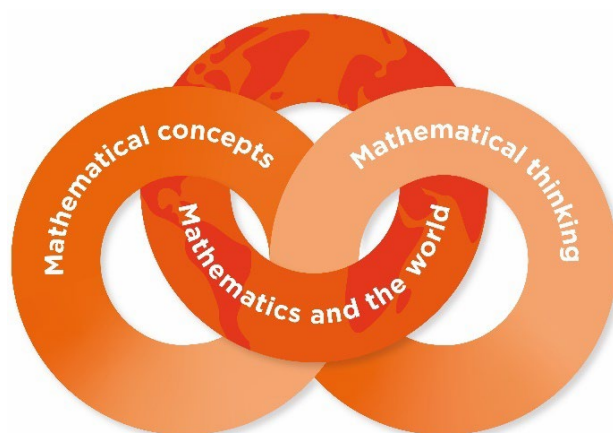


Figure 2. The visualization that emphasizes the cohesiveness of the domains *Mathematical concepts*, *Mathematical thinking*, and *Mathematics and the world*

Impression of the revised core objectives

In the final draft of the revised core objectives, a balance has been established between different kinds of mathematical knowledge, skills, insights and attitudes needed in light of the issues identified in the problem analysis. This means that both basic knowledge,

skills and insights, such as basic arithmetic and measurement, as well as more advanced mathematical ways of thinking, like mathematical problem solving, have been included in the renewed objectives. Some of the learning content included is new or more emphasized in comparison to the 2006 core objectives. This new learning content includes mathematical modelling and aspects of algorithmic thinking. Furthermore, the renewed core objectives put more emphasis on mathematical language, mathematical reasoning, critical mathematical thinking, and a mathematical attitude. Table 1 summarizes these new elements.

More emphasis	Partly new	New
Mathematical reasoning	Using heuristics in problem solving	Mathematical modelling
Critical mathematical thinking	Mathematics in different learning areas	Using and describing algorithms
Mathematical attitude		Probability (only secondary school)

Table 1: More emphasized and (partly) new elements in the revised core objectives

To provide a more detailed picture of how the draft core objectives aim to contribute to students' better preparation for a changing society, two examples are given below. The first example concerns the increasing need of critical thinking and understanding mathematical language, as identified in the problem analysis. This is addressed in the draft core objective *Use of mathematical language* (domain *Mathematical thinking*).

Use of Mathematical Language

The student uses mathematical language and mathematical representations. This involves:

- using mathematical symbols, notations and concepts;
- displaying calculations and mathematical problem approaches legibly;
- choosing and devising mathematical representations;
- assessing mathematical representations critically;
- establishing relations between different representations of mathematical concepts (SLO, 2024, p. 30).

Critical mathematical thinking also comes to the fore in other places, for instance in the specification “reasoning whether conclusions presented with diagrams and graphs are correct, incorrect or partially correct” of the draft core objective *Data* (domain *Mathematical concepts*). The second example concerns mathematical modelling, the importance of which was also highlighted in the problem analysis. Mathematical modelling is addressed in a separate draft core objective (domain *Mathematical thinking*) because of its importance for (mathematical) problem solving and, in a more general way, for the ability to relate reality to mathematics.

Mathematical Modelling

The student constructs and uses mathematical models. This involves:

- representing a situation schematically;
- representing a situation in mathematical language;
- selecting relevant features and leaving out not relevant features;
- using abstract models to demonstrate calculations, to interpret situations and to solve problems (SLO, 2024, p. 26).

TO CONCLUDE

The guiding question for the revision of the core objectives concerned what mathematical knowledge, skills, insights and attitudes nowadays' students need. In the characterization, an answer to this question has been formulated broadly as using and understanding mathematics to function in society, occupation and daily life. What mathematics this may entail more precisely, has been concretized further in the draft objectives as, among other things, critical mathematical thinking, understanding mathematical language and mathematical modelling. The inclusion of these and other new elements in the draft core objectives for mathematics, aims to contribute to students' adequate preparation for their future.

In the trial stage, it was found that the draft core objectives were generally well-received by both teachers and experts. A majority of teachers and experts endorsed the aims of the new and more emphasized elements of the draft objectives. In September 2024, the final drafts of the revised core objectives for mathematics, together with those for Dutch language, were delivered to the Ministry of Education. The Dutch State Secretary of Primary and Secondary Education has sent them to the Dutch parliament, which has the final say whether the draft core objectives will be put into law. It is expected that parliament will approve and that the draft core objectives will be legally established in 2026.

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VOCATIONAL ORIENTATION IN PRIMARY SCHOOL MATHEMATICS TEXTBOOKS

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Vocational orientation (VO) is a topic that can be addressed in the context of general education, including in primary school mathematics education. There is potential for vocational contexts to be combined with mathematical competences as well as for VO. However, this requires sufficient opportunities to learn (OTLs) within vocational contexts. This raises the question of the extent and quality of OTLs in a vocational context that can be identified in mathematics textbooks as representatives of the potentially implemented curriculum. We analysed 13,515 OTLs of 20 German mathematics textbooks (five different series, Grades 1 to 4) and identified 46 within a vocational context. Of these, only 14 showed a high level of potential for VO.

INTRODUCTION

One aim of general education is – among many others – to prepare students for working life. To achieve this, it is necessary to introduce vocational content and contexts at an early stage of primary education. Mathematics education is particularly well-suited to this type of orientation, given that mathematical work is to be carried out in a variety of application contexts, and mathematical modelling competences benefit from this. However, the extent to which such vocational application contexts become part of the implemented curriculum is largely dependent on the teacher, who often lacks professional experience outside the academic world. This underscores the vital role these materials play in the potential integration of vocational contexts into the curriculum, as exemplified by the case of mathematics textbooks. It is not only the quantity of opportunities to learn (OTLs) with a vocational context that matters, but also their quality, in the sense of a potential for vocational orientation (VO).

THEORETICAL BACKGROUND

Vocational orientation as a part of general education

The concept of VO can be viewed from two distinct perspectives. Firstly, VO can be understood as the process through which children and young adults develop an understanding of the world of work and identify a profession that aligns with their interests, beliefs, talents, and skills. Furthermore, interest in a particular occupation is regarded as the most significant predictor of job satisfaction (Hoff et al., 2020). In contrast, VO denotes the obligations imposed upon children and young adults by corporate entities and vocational training institutions. This offers a potential means of addressing social issues such as the shortage of skilled labourers, which is prevalent in

numerous countries, at an early stage. From the perspective of general education, VO is initially prioritised at the secondary school level (e.g., for Germany: Grades 5 to 12). A principal reason for this is the proximity in time to the transition from school to work. It is therefore to be expected that structures such as career fairs and aptitude tests will be more promising for both sides if they generate success (in the sense of satisfaction and employment contracts) in the foreseeable future. Consequently, VO is often included in the intended curriculum as one goal (among many others) of general education at the secondary level (e.g., for Germany: KMK, 2017). Nevertheless, numerous factors that influence students' career decisions exert an influence prior to the commencement of secondary education (Howard & Walsh, 2010). The prestige of professions, in addition to financial considerations, exerts an influence at the primary school level. Furthermore, as primary school students mature, their desired professions become increasingly aligned with their interests, beliefs, talents, and skills. However, the influence of the social environment and socially predetermined gender-specific role models also play a significant role in early VO. Primary school age is seen as the ideal starting point for a first broader VO (Porfeli & Lee, 2012). It is therefore beneficial to integrate VO processes at an early stage as part of general education, providing students with a diverse range of perspectives for their future. Vocational contexts and requirements should ideally be incorporated from primary school onwards, at an appropriate level for the students, enabling coherent VO and cumulative learning across all grades.

Vocational orientation as a part of mathematics education

The integration of vocational contexts into general education is a cross-cutting task for all subjects, not exclusive to mathematics education. Yet, integrating vocational contexts into mathematics education results in synergy effects that go beyond VO. For example, vocational competences can also be developed in the sense of vocational preparation, so that cumulative competence acquisition at the transition from general education to working life is facilitated. Furthermore - and this is of particular importance in primary school, in addition to the aspect of VO - general mathematical competence can also be strengthened by integrating vocational contexts. General mathematical competence is understood to be based on mathematical literacy, which is organised into mathematical processes, mathematical content, and contexts (cf. Turner & Adams, 2012). Competence is further defined by Weinert (2001, p. 46) as "learning dispositions that are available to individuals and/or the members of social groups and that must be used to solve demanding problems".

One area of general mathematical competence that is particularly associated with mathematics in various contexts is mathematical modelling. Here, real-world problems are translated into the mathematical world, worked on there, and then related back to the real-world problem (Blomhøj & Jensen, 2003). The *radius of action*, i.e., the extent to which different contexts can be integrated in mathematical modelling, can be understood as a dimension of modelling competence (Niss & Højgaard, 2011). This refers to the range of contexts to which modelling competences can be applied. They

assume that modelling competences can be acquired best when OTLs (stimuli such as tasks, problems, examples, information texts, and hints that get the students to engage with the mathematical content) occur in relevant contexts and situations, i.e., the targeted radius of action is already considered during the acquisition of competences. Approaches from cognitive psychology, such as *situated cognition theory*, support this thesis that mathematical competences are always acquired within the corresponding application context (Evans, 1999). There are also empirical results within research on mathematics education that confirm this thesis. For example, in an interview study with trainees as industrial clerks, differences in the processing of vocation-related and non-vocation-related problems with an equivalent mathematical structure were demonstrated (von Hering et al., 2018).

Curriculum resources (CR) for vocational OTLs

According to situated cognition theory, OTLs in diverse contexts are required to acquire mathematical competences in equally diverse contexts. Early mathematical interaction with different contexts, therefore, requires a variety of OTLs with different extra-mathematical contexts at primary level. Contexts are described as extra-mathematical if they are situated in a real-world situation. Such extra-mathematical contexts are frequently classified as *private*, *school*, *vocational*, and *other* (e.g., social contexts). For example, an arithmetic task is situated in a private context when it involves arithmetic with one's own toys. It is situated in a school context when arithmetic is performed with objects from the classroom or school building, for example, when it involves the number of chairs in the classroom. If a vocational perspective is adopted, for example, when calculating the number of products in a business (e.g., cakes in a bakery), this is a vocational context.

For mathematics education to achieve its objective of preparing students for working life, it is imperative that OTLs in vocational contexts be integrated into mathematics lessons. The mathematics textbook is a decisive factor in its role as a potentially implemented curriculum between the intended curriculum and the implemented curriculum (Valverde et al., 2002). Studies on textbook use have shown that the mathematics textbook is the main source for lesson preparation for 75% of mathematics teachers at primary schools in OECD countries (Mullis et al., 2012, 75% in Germany). Furthermore, studies have demonstrated a positive correlation between the scope of a topic in mathematics textbooks and corresponding characteristics of instruction or competence development (Schmidt et al., 2001). This legitimises the analysis of textbooks as an approximation of the implemented curriculum.

Given the nature of their biographies, the majority of teachers have accumulated only a limited amount of professional experience beyond the confines of the educational institution. In comparison to other contexts, such as school or private contexts, this is a distinctive feature of vocational contexts. Therefore, on the one hand, vocational contexts should accurately reflect the vocational requirements to foster genuine interest in the presented professions, thus facilitating VO. On the other hand, the mathematical

requirements should be age-appropriate, which occasionally presents a challenge in reconciling with the authenticity of requirements in primary school. Vocational OTLs must be incorporated into the curriculum to a sufficient extent for teachers to be able to fully utilise them. This is because teachers are largely guided by them when planning their lessons. Nevertheless, it remains unclear whether mere presence in CR can be considered sufficient. In an analysis of 18 widely used mathematics textbooks for German secondary schools, it was found that 6% of all OTLs were in a vocational context (von Hering et al., 2017). This proportion is described as sufficient. However, the majority of these OTLs only had a low potential for vocational preparation. Furthermore, 33% of all OTLs had an extra-mathematical context. These values cannot be used as a benchmark for primary schools, as different conditions prevail there. However, they can serve as a basis for comparing extra-mathematical contexts in CR.

RESEARCH QUESTIONS

It is imperative that mathematics education at the primary school level plays a role in VO. It is therefore necessary to ensure that sufficient OTLs with vocational contexts are provided. However, the extent to which such OTLs occur in lessons and the way they are designed is still unclear. Thus, the following two research questions arise:

- (1) To what extent do OTLs with an extra-mathematical context appear in primary school mathematics textbooks with a special focus on vocational contexts?
- (2) What is the potential of VO in the OTLs situated within a vocational context?

Given the well-documented effectiveness of the transition from the potentially implemented curriculum to the implemented curriculum, particularly in the context of primary education in Germany, this study takes on a unilateral perspective and focuses on the transition from the intended to the potentially implemented curriculum.

METHODS

Sampling

To gain an overview and classification of OTLs in extra-mathematical contexts, particularly with regard to VO, we consider printed mathematics textbooks as CR. Five German primary school textbook series, each covering Grades 1 to 4, were examined. The total of 20 textbooks represents the most widely used textbooks in Schleswig-Holstein, Germany. The unit of analysis is defined as a OTL, which encompasses not only tasks and problems but also examples, information texts, hints, and memorisation rules. In this manner, we received and subsequently analysed a total of 13,515 OTLs.

Identification and classification of contextualised OTLs

To determine the extent and nature of OTLs situated within an extra-mathematical context, we conducted a two-stage analysis. First, all OTLs with an extra-mathematical context were identified. The initial assessment did not consider the specific characteristics of the extra-mathematical context, such as its type, authenticity, or complexity. Second, the OTLs identified in this way were classified according to

various context categories during a qualitative content analysis. The deductive category system for the classification comprises four categories: *private*, *school*, *vocational*, and *other*. The coding rules for the vocational context include, for example, presented activities such as selling from a non-private perspective or delivering products. The interrater reliability based on double coding 10% of the OTLs is $\kappa = .99$ for the step of identifying the extra-mathematical contexts and $\kappa = .96$ for the classification step.

Content analysis for the potential for vocational orientation

In the following phase of the study, we sought to examine the potential for VO. To this end, we carried out a qualitative content analysis of all OTLs within a vocational context. For each task, we analysed vocation-related variables and categorized each of them into three levels, such as the perspective taken in the OTL (0 = private, 1 = unknown, 2 = vocational), the authenticity of the problem, and the degree of mathematisation from the perspective of the respective profession. To categorize these two variables, we examined whether the activities occur in these occupations in the described form and whether they are typically solved with the help of mathematics. Based on this analysis, we distinguished three levels for both of them: 0 = low (no occurrence / typically not solved with mathematics), 1 = medium (partly authentic / partly solved with mathematics), and 2 = high (authentic occurrence / typically solved with mathematics). Subsequently, we combined all three variables into a one-dimensional scale ranging from 0 to 6 points. This scale was then categorised into three levels to determine the potential for VO: 0–2 = low, 3–4 = medium, 5–6 = high. The analysis is based on a deductive category system comprising coding rules and anchor examples, and, where appropriate, involved the analysis of training curriculum documents for the targeted occupations.

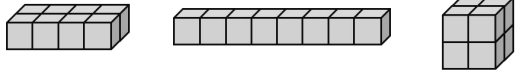
Mrs. Brause cleans the school. She usually fills her cleaning bucket with water eleven times per school day to mop. A bucket holds about 15 liters of water. How many liters of water does Mrs. Brause use for mopping?	Mrs. Grün brings a trailer with sugar beets to the sugar factory. The sugar factory scales show 15.825 tons. The empty trailer weighs 6.045 tons. How many tons of beets are in the trailer?
A packaging manufacturer wants to offer cuboid 8-litre bag containers. For cost reasons, he wants to use as little cardboard as possible. He starts his considerations with the help of decimetre cubes. a) Which option would you recommend? Give reasons. b) Research which shape is predominantly used. Explain. 	

Figure 1: Examples for OTLs for Grade 4 with low (top left), high (top right), and medium (bottom) potential for VO (translated from German).

Figure 1 shows OTLs with low, medium, and high potential for VO. The task involving a cleaner is presented from a vocational perspective (2); however, it is not an authentic problem (0), and mathematics is not typically used in such situations in practice (0; overall score of 2). The task involving a packaging operator is also presented from a vocational perspective (2); it shows a medium level of authenticity (1), and the requirements were partly solved with mathematics in practice (1; overall score of 4). The task involving a farmer is presented from an unknown perspective (1); it shows a high level of authenticity (2), and the requirements were typically solved with mathematics in practice (2; overall score of 5). The ratings of all OTLs were conducted by two independent researchers, with a subsequent consensus procedure.

RESULTS

In the initial phase, a total of 3,702 OTLs were identified across 20 textbooks that were situated within an extra-mathematical context. This represents 27% of the total number of OTLs. We observed no significant differences between grade levels ($\chi^2(df = 3) = 5.36, p = .15$), but between textbook series ($\chi^2(df = 4) = 33.3, p < .001$).

Grade	Private	School	Vocational	Other
1	11.3	76.9	0.7	11.1
2	22.5	61.1	0.3	16.1
3	17.4	61.3	1.3	20.0
4	15.9	60.3	2.8	21.0

Table 1: Percentages of OTLs divided into context categories by grade level.

Table 1 illustrates the distribution of the OTLs identified in this manner, situated within an extra-mathematical context, across the four context categories. With regard to the total number of 13,515 OTLs, the proportion of OTLs within a vocational context is 0.4%. In total, there are 49 OTLs within a vocational context across all textbooks. It is notable that nine of the 20 textbooks do not contain any vocational contexts. There were significant differences in the distribution of the context categories between grade levels ($\chi^2(df = 9) = 203.41, p < .001$) and between series ($\chi^2(df = 12) = 245.08, p < .001$). Vocational contexts occur less frequently than average in Grades 1 and 2 and more frequently than average in Grade 4. The subsequent content analysis examined the 49 OTLs in a vocational context for their potential to facilitate VO (see Table 2). Fourteen OTLs displayed a high level of potential for VO, with 11 situated in Grade 4.

Grade	Low Potential for VO	Medium Potential for VO	High Potential for VO
1	3	2	0
2	1	0	2
3	5	6	1
4	8	10	11

Table 2: Number of OTLs (low, medium, and high potential for VO) by grade level.

DISCUSSION

One of the aims of general education is to equip students with the requisite skills and knowledge to successfully transition to their working lives. Consequently, primary school mathematics education should also encompass sufficient OTLs with the potential for VO. In this study, the mathematics textbook serves as an indicator for this in terms of the potentially implemented curriculum. To answer the first research question, it can be observed that while approximately 27% of OTLs have an extra-

mathematical context, most of them are situated in private and school contexts. Vocational contexts are virtually absent, accounting for only 0.4% of all OTLs. In Grade 4, this value is slightly higher (0.9%), but still considerably below the 6% value used as a comparative value. Nevertheless, this proportion of OTLs situated in a vocational context does not seem to be adequate when evaluated against the aims of general education and against the results on early vocational behaviour. In addition to the number of OTLs in vocational contexts that each student receives, the quality of these opportunities is also an important factor in their VO. To answer the second research question, in 71% of the few vocational OTLs, the potential for VO can be classified as low or medium. Furthermore, an OTL with a high level of potential for VO is almost non-existent in the lower grades. Even in an average mathematics textbook for Grade 4, there are only two OTLs with a high level of potential for VO. The integration of vocational contexts seems to have been a rather subordinate criterion in the development of the CRs under consideration. This could be since the relevance of early VO is not seen by the authors of the textbooks as a task for mathematics education. This seems to change slightly in Grade 4, where there are more mathematical opportunities to use mathematics in an authentic vocational context. Beyond that, it is unclear whether the respective teachers implement these elements of the potentially implemented curriculum into their mathematics lessons. It is even conceivable that precisely these OTLs are selected less frequently, since teachers are less familiar with the contexts due to their own biography. A limitation of this study is that, when analysing the OTLs for their potential for VO, each OTL and its vocational context were considered individually, rather than the entire textbook or textbook series. Yet, the overall number of vocationally situated OTLs in the (potentially) implemented curriculum is decisive for the students' actual VO. Nevertheless, our findings indicate that individual textbooks contain too few OTLs to provide sufficient VO.

The present study examines the status of VO in primary school mathematics textbooks in Germany. However, it should be noted that the vocational education system differs greatly on an international scale in some cases. Other countries may have different aims regarding vocational education. Nevertheless, fundamental aspects of opening the students' social (and therefore also vocational) environment can always be integrated into general education using mathematics. Comparative international textbook studies could prove an invaluable opportunity to gain a deeper understanding of the role of VO on the one hand and the role of textbooks in mathematical learning on the other.

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ANALYSING THE IMPACT OF LINGUISTIC MODIFICATIONS ON HOW STUDENTS READ AND LEARN FROM MATHEMATICS TEXTBOOKS

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Research investigating how students read their mathematics textbooks is limited and unpromising. The existing literature finds that mathematical text is the least-used feature of undergraduate mathematics textbooks, and that undergraduates struggle to read this text – even if they are proficient general readers. In this study, we aimed to gain insights into which aspects of mathematical texts support students' mathematical reading ability. We used eye-tracking technology to investigate the impact of various linguistic modifications on how students read mathematical texts in the style of textbook expositions. Our findings suggest that problematic linguistic features did not impact reading as expected, and we propose necessary areas of research to explore these findings further. We also conclude by suggesting implications for future mathematics textbook exposition design.

FOCUS AND RATIONALE

Despite the ever-increasing field of research investigating mathematics textbooks, research into how undergraduate students use their textbooks remains under-represented (Fan et al., 2013; Rezat, 2024). Of the limited research, we find that undergraduate students report using the textual exposition the least (Weinberg et al., 2012), and that they struggle with reading mathematics proficiently, even if their general reading skills are good (Shepherd et al., 2012).

Such findings also fuel conceptions that mathematical texts are particularly difficult to read, despite the lack of empirical evidence for this claim. Most recently, Österholm and Bergqvist (2013) reviewed articles which made claims about “any type of linguistic property of mathematical texts” (p. 754), only finding two weakly empirically verified claims. Furthermore, both of these claims contradicted predominant assumptions, finding that mathematical texts were less dense and lower in complexity than those of other subjects. Similarly, Österholm (2006) suggested that mathematical texts share similar demands to other technical texts; however, he proposed that this was only true when controlling for symbols. He compared students' learning outcomes when reading two mathematics texts (one with symbols and one without) and a history text, finding that the presence of mathematical symbols led to the only meaningful differences in reading comprehension. Using this finding, Rezat et al. (2022) controlled for symbols when designing their mathematical reading task; however, they found that many

students still experienced problems whilst reading mathematics. This was also despite Rezat et al. designing tasks which scaffolded beneficial reading strategies; these strategies were designed based on previous research studying the reading behaviour of relative mathematical experts (e.g., Berger, 2019; Shepherd & van de Sande, 2014). Therefore, even when potentially problematic characteristics (e.g., symbols) are limited and successful reading strategies are encouraged, issues with mathematical reading persist.

In an effort to understand why, we turn to the wider STEM (Science, Technology and Mathematics) literature, where there has been a greater focus on the role of linguistic features in STEM texts, and how they might impact reading comprehension and learning from texts (see Cruz Neri & Retelsdorf, 2022). The majority of this literature is also inconsistent, with contradictions in the sizes and frequency of effects. However, Strohmaier et al. (2023) conducted a meta-analysis that identified the overall effects of linguistic modifications on reading comprehension and learning from STEM texts. They found that three of their eight categories of linguistic modifications had a positive impact on learning and comprehension: (i) global clarity and elaboration (reducing syntactic complexity, increasing global links within the text, and increasing global explicitness and elaboration with prior knowledge), (ii) local clarity and elaboration (reducing word-level complexity, increasing links within the text, and increasing local explicitness and elaboration with prior knowledge), and (iii) personalisation (increasing engagement and motivation using personal pronouns and active, rather than passive, voice).

As such, we investigated the impact of implementing these linguistic modifications in a mathematical domain. We measured impact via students' learning from the texts, and we also used eye-tracking technology to explore reading behaviour in more detail, aiming to establish how such modifications affect students' reading (if they do). Our two research questions are: (a) do these linguistic modifications support students' learning from mathematics textbook expositions, and (b) do these linguistic modifications affect how students read mathematics expositions?

THEORETICAL AND METHODOLOGICAL BACKGROUND

We focussed on the three linguistic modifications identified from Strohmaier et al.'s (2023) meta-analysis of STEM texts (as defined above): global clarity and elaboration, local clarity and elaboration, and personalisation. According to Strohmaier et al., these three modifications supported students' learning from STEM texts, especially for learners with lower prior knowledge. Therefore, we aimed to understand whether this was the case for mathematics (using post-tests of learning) and then aimed to examine their impact in more detail by measuring reading behaviour using participants' eye movements whilst they read.

Eye movements indicate various cognitive processes during reading, including visual processing as well as the integration of prior knowledge and contextual information; as such, eye-tracking technology has been used intensively as a tool to investigate

cognitive processes whilst reading (Rayner, 1998). Therefore, using eye-tracking technology allowed us to investigate whether any of the linguistic modifications affected how students read mathematical texts.

STUDY DESIGN

We pre-registered the study design and analyses, see <https://aspredicted.org/7pfk-699s.pdf>. For this conference paper, we present a selection of the pre-registered eye-tracking analyses due to space.

We selected eight expositions from English mathematics textbooks aimed at 17-18 year-old students. We re-formatted and adapted these expositions for readability, and then implemented each of the modifications in the following ways:

- for global clarity and elaboration, we gave each exposition a title and included a brief summary at the end;
- for local clarity and elaboration, we included additional explanations relating to prior knowledge (especially when new words were introduced);
- for personalisation, we used active voice (rather than passive) and personal pronouns where possible.

Using these manipulations, the study had a $2 \times 2 \times 2$ within-subject design, with global clarity and elaboration, local clarity and elaboration, and personalisation as factors. We used expositions on 8 different topics, resulting in 64 stimuli, as each exposition was written with every different combination of the 3 factors. Each participant was randomly assigned into 1 of 8 groups; each group contained all 8 exposition topics, with each topic written as a different version, so that all participants saw all possible combinations of manipulations across the expositions. Expositions were randomly presented within groups.

Based on a power analysis to detect a medium effect between within-subject factors, we aimed for a sample size of 57 participants. To account for data loss, we recruited data from 72 participants, excluding 11 participants due to data quality issues (e.g., bad calibration, missing data, recording issues etc.) which resulted in a final sample of 61 participants (slightly above the pre-registered sample of 57 due to technology issues during data collection). Participants were Loughborough University undergraduates from programs other than mathematics to reduce the likelihood that they were familiar with the materials.

We gained ethical approval from Loughborough University, and all participants gave full informed consent prior to testing. All participants were given an inconvenience allowance of a £10 voucher. Within the study design, we also included emotion measures, and participants were fitted with an electro-dermal activity (EDA) device; however, we do not report on these analyses or findings in this paper as they are not relevant to this paper's research questions.

Participants were first given a pre-test booklet, containing a questionnaire on mathematics-related trait emotions, a general reading comprehension task, a demographic questionnaire (including mathematical qualifications and degree subject), and a general mathematics calculus task. Participants were then asked to read 8 expositions, each of which was followed by questions about participants' familiarity with the exposition topic and a questionnaire on mathematics-related state emotions, and a three-question post-test on the exposition content. We used a Tobii Pro Spectrum eye-tracker with a sampling rate of 1200Hz, and we used Tobii Pro Lab v1.232 software for data collection.

ANALYSIS

For the analyses of the effect on learning, we used linear mixed models with participant and text as random effects, all three linguistic modifications as fixed effects, and post-test scores as the dependent variable.

For the analyses of eye movements, we used the parameters *total fixation duration* (as an indicator of overall processing speed), *ratio of regressions* (as an indicator of disruptions in the reading process), and *mean fixation duration* (as an indicator of processing effort and load). All three parameters were investigated for the whole text, and for the *core text* (the parts of the text that were the same across all manipulations). For all the eye tracking parameters, we conducted the same linear mixed model as described above.

RESULTS AND DISCUSSION

None of the linguistic modifications had a significant effect on learning outcomes ($p = .548$). Regarding eye movement parameters, total fixation duration was longer for texts with increased local and global clarity and elaboration ($p < .001$), but not when only the core text was considered ($p = .122$). This suggests that expositions which included headings, summaries, and additional relevant explanations increased participants' processing time whilst reading these expositions but that this did not hold when comparing how participants read the unchanged text between expositions.

Overall, the ratio of regressions was lower when local clarity and elaboration was increased ($p = .046$) and higher when global clarity and elaboration was increased ($p = .019$), but the ratio of regressions did not differ for the core text ($p = .173$). As such, additional explanations appeared to reduce disruptions in the reading process, whereas headings and summaries increased disruption; again, there was no change when considering the unchanged text between expositions.

Mean fixation duration did not differ significantly between any exposition version, either for the whole text or the core text. We might infer that none of the modifications affected depth of processing across participants (see Rayner, 1998).

Overall, we found that the linguistic modifications had no effect on learning outcomes or reading behaviour. The differences in eye-tracking measures due to global and local

elaboration and clarity were likely a consequence of both modifications lengthening the text; when we considered the core text only (the text that did not change across the exposition versions), we found no significant differences in any of the eye movement measures.

One explanation for our findings might be the redundancy principle from the wider instructional design literature: that presenting information in multiple ways or unnecessarily elaborating interferes with learning, rather than facilitating it (Kalyuga & Sweller, 2014). We found that although these students spent longer reading the additional explanations, there was a decrease in regressions (suggesting that the explanations required less cognitive effort, and therefore were potentially easier to read) and there was no effect of learning. As such, the additional explanations might have been unnecessary or redundant, and therefore triggered the redundancy principle and impeded learning rather than supporting it. There is limited research specifically investigating the redundancy principle in mathematics texts, and we propose that such research would be beneficial to understand whether this principle extends to mathematics. This is particularly interesting due to the common perception that mathematical texts tend to be ‘dense’ (Österholm & Bergqvist, 2013): if the redundancy principle does extend to mathematics, then adding more explanation might not necessarily be a straightforward or beneficial solution. This is particularly important to establish for mathematics textbook design, as mathematical expositions should be designed to support the learner as best possible.

In addition, we propose that our findings perhaps lend support to long-established narratives that mathematical texts are unique, in that manipulations, which appear to support the learning and comprehension of STEM texts in general, do not seem to have much impact here. Through our study design, we manipulated various elements of these expositions, reducing the complexity, density, and impersonal nature of the expositions but finding no effect on learning outcomes or overall reading behaviour. As such, we consider insights from Österholm (2006), who suggested that symbols in mathematical texts impacted mathematical reading comprehension. We did not control for symbols within our expositions, with all expositions containing symbols of some kind, so perhaps this influenced our findings. Given that our study was not designed to investigate this claim explicitly, we propose that it would be worthwhile to further investigate the impact of symbols on students’ mathematical reading comprehension. We suggest that eye-tracking technology might be beneficial to investigate how symbols are processed and how this compares to more familiar mathematical symbols or text. Again, understanding the impact of mathematical symbols on mathematical reading behaviour could help textbook designers to ensure that symbols are displayed and introduced in the most beneficial, appropriate way.

Of course, there were limitations to our study design, which should be taken into account when interpreting these results. We recognise that our power analysis was calculated under the assumption that we would detect a medium effect, based on the evidence from the STEM literature. As such, it is possible that increasing the sample

size might have increased the likelihood of finding an effect. Similarly, although excluding participants due to data quality is a common process in eye-tracking research, we recognise that this potentially affects the generalisability of our findings. We propose that more research is needed in this area of mathematics to understand the nuances of mathematical text design and do not recommend any practical implications based on the findings of our research.

We also note that how we implemented the modifications and students' general familiarity with the texts, might have influenced the likelihood of detecting an effect. We recognise that further controlling for students' prior knowledge or ensuring that students were unfamiliar with the exposition topics might have led to these modifications being more beneficial to students (as found by Strohmaier et al., 2023). As such, in forthcoming analyses, we will take into account several indicators of prior knowledge (mathematical abilities, reading abilities, content familiarity and expertise) to investigate possible interactions.

CONCLUSION

In sum, our linguistic modifications did not significantly affect how undergraduate students read or learnt from a set of mathematical textbook expositions. This includes the addition of title headings, summary textboxes, additional (relevant) explanations, active text, and personal pronouns. Such findings are surprising given the existing evidence base suggesting that these features impact students' learning from and reading of STEM texts. In our discussion, we explored whether there might be other features of mathematical texts that have a greater impact than those studied, and we suggested potential approaches. However, we also recognise the limitations of our study and emphasise that this research serves to motivate future empirical inquiry into understanding difficulties with mathematical reading and texts. We propose that a greater understanding of mathematical texts would help textbook designers ensure that mathematics textbooks' expositions are as beneficial for learning as possible.

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NEW DIRECTIONS IN THE DESIGN OF CURRICULUM RESOURCES FOR TEACHING GEOMETRY: THE USE OF VISUAL AND ANALYTIC GENERALISATIONS AS DIDACTICAL TOOLS IN TEACHER TRAINING

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The paper discusses how visual and analytic generalisations of theorems in geometry textbooks can serve as didactical tools to enhance mathematics content knowledge in teacher training and can contribute to the development of teachers' skills to promote learning with understanding. The novelty of this method is in combining the elements of content knowledge with specially designed geometry tasks where generalisation plays the leading role towards establishing different properties and their proofs.

THEORETICAL BACKGROUND AND METHOD OF TEACHER TRAINING

In the recent years, a significant amount of research in mathematics education is dedicated to the different aspects of the process of generalisation (Crompton, 2015; De Villiers, 2016; Komatsu, 2017). While a wide range of professional learning models having relation to teaching geometry are available in literature (Jacobs et al., 2020; Smith et al., 2005) and known from practical implications (Seago et al., 2010), the search for new resources and educational tools that can be developed and used for professional training and development of mathematics teachers remains one of the strategic goals. The proposed model of teachers' professional development is based on the use of visual and analytic generalisations of theorems in geometry textbooks affiliated with meaningful tasks that require the use of generalising while making reasoning and argumentation. It is considered from the perspective of activity theory, where mathematical ideas are regarded as historical artifacts of human culture (Vygotsky, 1987). The art of generalising requires special skills and, therefore, can be seen as more difficult to acquire. This method of training aims to address these issues so teachers can develop practical skills in using generalisation while making reasoning, which is expected to have a positive effect on their content knowledge and enrich their professional experience. Specifically, the paper addresses the following research question: How can curriculum resources for geometry integrate geometrical configurations, meaningful tasks, and the use of visual and analytic generalisations, and be incorporated in teachers' professional work through teacher training?

This study is part of a long-term project that focuses on the professional development of in-service mathematics teachers and their attitudes towards mathematics content

knowledge and the ways in which the latter can be improved through specially designed rich activities in geometry. The focus of this study is on using visual and analytic generalisations of theorems in geometry textbooks that can serve as didactical tools to enhance content knowledge of mathematics teachers and their skills in geometrical reasoning.

The concept of generalisation is most understood as a duality between going from particular to general and seeing the particular through the general. The generalisations discussed in the paper are considered in the broader sense of how an attempt to generalise a geometrical object or its properties can be developed. Under a visual generalisation in geometry, we understand a change in the visual representation of a geometrical object by making some characteristics or properties of that object more general. Under an analytic generalisation in geometry, we understand a change in the description of a geometrical object, its characteristics, or properties by formal means. For example, a visual generalisation for a square can be a regular polygon with n sides, while an analytic generalisation for a square, or more precisely, for its angle of 90° , can be the angle size of a regular polygon with n sides. Visual and analytic generalisations of any geometrical object are not unique and provide space for creativity. For example, by equally changing two opposite sides of a square, we obtain a rectangle as the result of visual generalisation. Then, by equally changing two opposite angles of a rectangle, we obtain a parallelogram, again as the result of visual generalisation. On the other hand, the length of a diagonal of a square or rectangle can be found by applying Pythagoras' theorem, and Cosine Law is the result of an analytic generalisation for the length of a diagonal of a parallelogram.

In some activities, generalisations represent open-ended questions with a broad direction to extend the given property further under new conditions or with different geometrical objects/shapes, while in other activities, they are formulated in the form of conjectures or through the meaningful discussions of proofs. A geometrical configuration representing some theorem or lemma of geometry is assigned with a task to complete using visual and analytic generalisations. In some cases, a proof (or justification) of the given configuration is provided together with its statement; in other cases, participants are encouraged to prove some statements on their own before moving on to work on the tasks. The solution to complete the task requires participants to use some specified or unspecified generalisations. It also requires searching for the ways of how a special configuration assigned to the task can be used as part of the solution for that task. While working on each task, teachers refresh their content knowledge represented through configurations on the diagrams and make acquaintance with other configurations. Whatever their case is, they need to demonstrate the use of visual and analytic generalisations to complete the task successfully. This gives participants the opportunity to work out solutions or at least some conjectures to the given tasks at the time when they are involved in studying geometrical content and applying it in their practical work. Modules containing configurations and their assigned tasks within a certain topic are called activities. This method of training

includes different activities that represent the theoretical content and show the practical implementation of acquired skills in different tasks. Both types of activities aim to stimulate teachers' thinking and enhance their content knowledge of the topic under consideration. Due to the limitations of the paper, we focus entirely on the theoretical reasoning of the tasks to demonstrate the novelty of the method in detail and with clarity.

This method of training was influenced by rich traditions of teaching mathematics and school curriculum in eastern European countries (Hungary, Poland, Romania and Ukraine) and conducted in Australia for high school teachers specialising in teaching mathematics for Grade 11 and 12 of high school (last two years of secondary education in Australia) and lecturers who teach geometry courses for teachers at tertiary level.

VISUAL AND ANALYTIC GENERALISATIONS AS DIDACTICAL TOOLS

We begin with the two famous theorems of Euclidean geometry: Morley's trisector theorem and Ceva's theorem. The proofs of both theorems can be found in Coxeter and Greitzer (1967). The proofs of the two lemmas that follow the theorems, and other statements can be found in Altshiller-Court (2007).

Morley's trisector theorem

The three intersections of the trisectors of the angles of a triangle, lying near the three sides respectively, form an equilateral triangle (Figure 1).

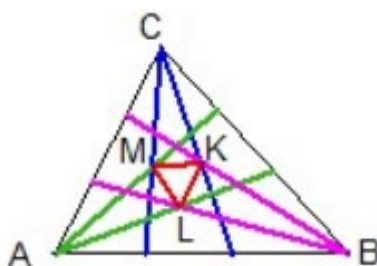


Figure 1: KLM is Morley's equilateral triangle.

Ceva's theorem

Three concurrent lines drawn from the vertices A , B and C of a triangle and called cevians (Figure 2) divide the opposite sides BC , CA and AB respectively in ratios the

product of which is 1, i.e. $\frac{AF}{FC} \cdot \frac{CE}{EB} \cdot \frac{BD}{DA} = 1$.

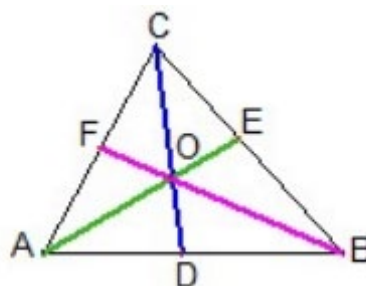


Figure 2: AE , BF , and CD are cevians of triangle ABC .

We begin with the following property:

Angle bisectors' Lemma 1

The three angle bisectors of a triangle are concurrent at a point which is the incentre of that triangle (Figure 3).

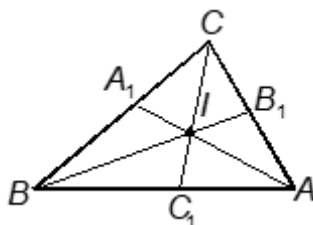


Figure 3: The incentre I is the intersection of the three angle bisectors.

This property is present in almost every geometry textbook, it represents a nice and simple fact. We can use it as one of the starting points for possible generalisations. Of course, it can be interpreted and seen as a stand-alone statement derived from the fact that any angle bisector is equidistant from its angle sides. However, we want to use a different interpretation here. We can see the aforementioned property as the first step (with $n=2$) of the generalisation process where each angle of a triangle is divided into n equal parts. Indeed, as discussed above, for $n=2$ the outcome of this process is a special point which is the centre of the circle inscribed into the given triangle, i.e. the incentre. We can continue this process and divide each angle of a triangle into three equal parts (with $n=3$ which is the next number of equal parts of each angle), i.e. replacing angle bisectors with angle trisectors. Then a construct of an equilateral triangle – Morley's triangle – appears to be quite striking and draws attention in any sketch of the diagram as shown in Figure 1. Looking back, we see that angle bisectors of a triangle have been the main source of generalising. We can say that we have proceeded through the generalisation process of dividing each angle of a triangle into some number of equal parts. We also note that Morley's triangle (as a conjecture) comes to the scene in the form of visual generalisation. It is important to mention that while visualisation cannot help to validate the correctness of any conjecture, it still helps significantly to see what some conjecture statements could consist of. We continue with angle bisectors of a triangle as the source for other generalisations. Looking for analytic generalisations, one can quickly realise that they cannot be identified as easily as visual generalisations. Indeed, they are generally more difficult to deal with (this comes as no surprise) and we have to figure out first what way could work in case of analytic generalisation. We refer to the following property of angle bisectors which can be seen as the first step of analytic generalisation.

Angle bisectors' Lemma 2

Angle bisectors of a triangle divide the opposite sides into the parts that are proportional to the corresponding adjoining sides of the triangle. For example, if c is

divided into parts with the lengths a_1 and b_1 (Figure 4), then $\frac{a_1}{a} = \frac{b_1}{b}$.

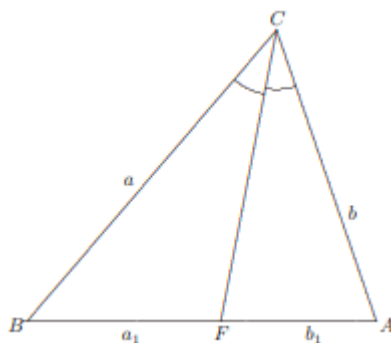


Figure 4: Angle bisectors' property.

Before we proceed to the outcome of analytic generalisation for this property, it is important to note that angle bisectors (like many other concepts and constructs in geometry) are an excellent example to demonstrate the benefits of visual and analytic generalisations used as didactical tools. The importance of this example is in its inclusiveness: it contains a visual component (the incentre of a triangle) and it also has an analytic component (the ratios between special elements of a triangle). Therefore, analytic generalisation of this property can follow as another type of proportionality between the parts of the opposite sides and corresponding adjoining sides of a triangle. Thus, we can generalise the result of **Angle bisectors' Lemma 2** to the form

$$\frac{c_1}{a^k} = \frac{d_1}{b^k}.$$

The next value of the power k for investigation is $k=2$, where c_1 and d_1 are now other parts of the side c . Are the parts of the opposite sides of a triangle's angles divided by the feet of their angle bisectors proportional to the squares of the corresponding adjoining sides of the triangle? Not surprisingly, the newly constructed parts satisfying the new proportionality have lost the angle bisectors' property (Figure 5).

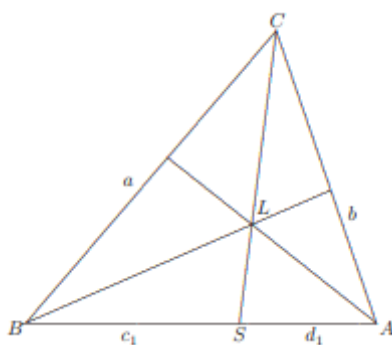


Figure 5: Three symmedians and the Lemoine point L.

They are no longer angle bisectors. However, instead of that property, another property based on visual generalisation can be suggested where a visual component in terms of concurrency of these special line segments comes to the scene:

The lines through the vertices of triangle ABC that divide the opposite sides BC , CA and AB into the parts proportional to the squares of the corresponding adjoining sides of the triangle are concurrent.

We note that the proof of the latter statement leads to establishing the Lemoine point L of a triangle. This special point can be defined in many different ways. In particular, it can be introduced as a point of concurrency of the three line segments connecting the vertices of a triangle with the opposite sides with each of them being symmetric to the corresponding median with respect to the angle bisector emanating from the same vertex (Figure 6).

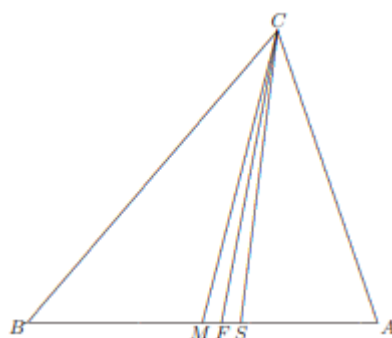


Figure 6: Symmedian CS is symmetric to median CM with respect to CF .

These line segments are called symmedians and the Lemoine point is the point where symmedians of a triangle intersect. Here we have come to an interesting observation. In fact, the Lemoine point L is the result of analytic generalisation of the angle bisectors' Lemma 2, where $k=2$. However, we can see the same point as the result of visual generalisation for a point of concurrency of the three angle bisectors. In other words, we construct three line segments, i.e. cevians, joining the vertices of a triangle with their opposite sides being concurrent while not requiring them to be angle bisectors. This example demonstrates the interplay between conjecture and proof in geometry textbooks in terms of possible generalisations. It shows that analytic and visual generalisations do not stand alone from each other. They are intertwined in many ways, and they influence and mutually enrich each other despite the absence of any resemblance at all between them. It also highlights that generalisation can be of a different nature. As shown above, in some cases it is more convenient to change some characteristics (parameters) involved in the description of the model which paves the way to analytic generalisation, while in other cases the model itself can be changed towards a more general model, mostly for visual generalisation.

In the case of Ceva's theorem, we encounter another example of how generalisation works. There should be no surprise now that angle bisectors serve as a source for generalisations again. Indeed, as it was already discussed, on one hand, we can focus on a point of concurrency of the three cevians without them being angle bisectors, using visual generalisation. On the other hand, we can focus on the ratios of the parts each side of a triangle is divided into by the three cevians. This will follow as the result of the analytic generalisation of the aforementioned ratios. Ceva's theorem may be seen

as a challenging task to be established. However, a logical path does exist, and we are just a few steps away from that. Indeed, we know that the three medians of a triangle are also concurrent, and each ratio of the corresponding parts is equal to 1 in that case.

Thus, we can come to the following conclusion: $\frac{AF}{FC} \cdot \frac{CE}{EB} \cdot \frac{BD}{DA} = 1$

in the case of AE , BF , and CD being medians (Figure 7).

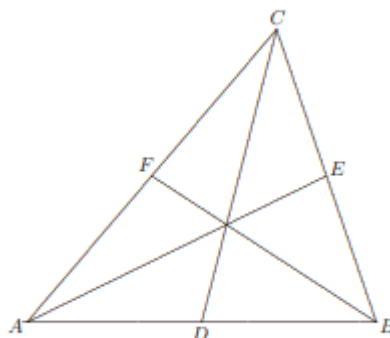


Figure 7: Triangle ABC and its medians.

Therefore, we can proceed to the statement of Ceva's theorem not requiring them to be medians: Is the above expression valid in general terms? Or under some special conditions only? The statement finally comes to the scene as the result of analytic generalisation. However, it has a visual component too since we want to keep the three cevians concurrent, but without being medians. In other words, we enquire which requirements should be put on those three concurrent cevians so that the equality holds. Thus, an example of cevians can also be seen as based on visual generalisation of the medians (or angle bisectors) of a triangle.

Returning to the concept of angle bisector in **Angle bisectors' Lemma 2** we can point out one more question to generalise the latter property through consideration of external angle bisectors.

Generalisation of Lemma 2

If CD is an external angle bisector of triangle ABC , what can change in the angle bisectors' property (Figure 8)?

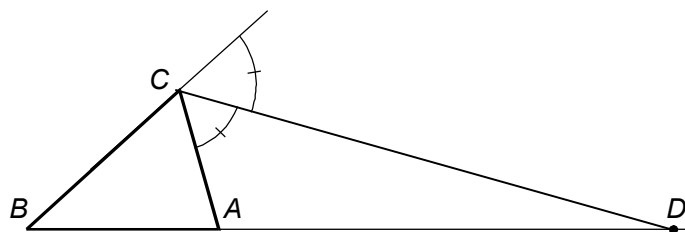


Figure 8: External angle bisector CD of triangle ABC .

CONCLUDING REMARKS

Visual and analytic generalisations in geometry represent an interesting and challeng-

ing direction in teacher training and professional development. These are the two processes whose cognitive functions help to identify different theorems and illustrate them, while proofs make a bridge to justify those theorems. It is important that much more attention should be paid to generalising activities in geometry textbooks along proof and proving activities. Hanna (1995) emphasised that “the most important challenge to mathematics educators in the context of proof is to enhance its role in the classroom by finding more effective ways of using it as a vehicle to promote mathematical understanding” (p.42). We suggest that introducing teachers to the process of generalising in teacher training could be the way to enhance mathematics content knowledge and promote mathematical understanding of how visual and analytic generalisations could arise and be validated in the classroom.

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HOW TO DESIGN TEXTBOOKS FOR PROMOTING STUDENTS' STRUCTURAL LEARNING: EXPLORATION OF DESIGNING BNUP JUNIOR HIGH SCHOOL TEXTBOOKS

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One of the ultimate goals of designing structured textbooks is to promote students' capacity to learning and thinking structurally. In this paper, it is reported when designing the junior high school mathematics textbook published by Beijing Normal University Press (BNUP Textbooks) in mainland China, how to help students build up their structure of learning mathematics. Students are expected to gain insights into four types of structures, that is, the logical structure between mathematics content, the growing structure of specific mathematical objects, the research structure of similar knowledge, and the learning structure within each unit. Several suggestions, together with examples in the textbook series, are illustrated.

RESEARCH BACKGROUND

In mainland China, the Compulsory Education Mathematics Curriculum Standards are revised approximately every ten years. The most recent version, issued in 2022, emphasizes the integration of curriculum content based on its underlying structure to foster students' core competencies. The curriculum encourages teachers to holistically analyse content to help students establish a structured mathematical knowledge system, one that reflects the essence of mathematics and supports future learning (MoE, 2022). Compared to previous versions, this version places a stronger focus on the structural organization of curriculum content and teaching practices (Shi & Cao, 2022).

Textbooks, as the cornerstone of Chinese mathematics education, are the most important resource for teachers' daily instruction (Li, Chen, & Kulm, 2009). Teachers in mainland China typically study textbooks thoroughly, as these materials serve as a vital bridge between national mathematics curriculum standards and classroom teaching. Consequently, textbook design plays a crucial role in translating curriculum objectives into structured content that promotes students' structural learning.

The significance of structured learning has long been recognized in educational psychology. Piaget (1970), in his theory of "genetic epistemology," emphasized that structure is a system governed by overarching laws or attributes, and cognitive development inherently involves structured adaptation. Similarly, Bruner (1960) proposed that the educational process should aim to expand and deepen knowledge through basic and general concepts. Without a proper structure to connect acquired

knowledge, such knowledge is prone to being forgotten. Scandura's (1973) theory of structural learning further highlights the importance of organizing information into meaningful patterns and structures, enabling students to develop critical thinking and problem-solving skills. Structured learning, therefore, is not just about acquiring knowledge but about understanding the underlying framework that makes knowledge transferable and applicable in future contexts.

Given this background, there are three main reasons why textbooks in mainland China must be designed to support students' structural learning. First, they must align with the goals outlined in the Mathematics Curriculum Standards (MoE, 2022). Second, they should incorporate insights from educational psychology to create effective learning experiences. Third, they must meet the practical learning needs of students. These considerations lead us to the central question: How can textbooks be systematically designed to promote students' structural learning? This paper explores how the junior high school mathematics textbooks published by Beijing Normal University Press (BNUP Textbooks) address this question. Specifically, the study defines structural learning as the process of understanding why to learn, what to learn, and how to learn (including the sequence and methods of learning), leading to a deep comprehension of knowledge structures and their construction (Zhang, 2008). Furthermore, structural learning involves the ability to transfer these structured concepts independently to new learning scenarios (Zhang, 2018). Therefore, this study aims to address the following two questions:

Q1: What types of structures should students build to support their learning of mathematics?

Q2: What strategies can textbook designers employ to systematically foster students' structural learning?

RESEARCH DESIGN AND METHODOLOGY

Referring to Design Research in Education by Bakker (2018), this study adopts a design research approach to systematically investigate and develop strategies for enhancing structural learning through textbook design. Guided by principles of instructional design and educational psychology, the research framework integrates theoretical foundations (e.g., logical and cognitive structures) with practical strategies, ensuring that students can build, recognize, and utilize mathematical structures effectively during learning.

A qualitative content analysis approach was employed to examine how these four structures are presented and operationalized within the BNUP textbooks (Ma, 2024). This study adopts a design-based research methodology, focusing on iterative textbook development and evaluation. First, textbook content was analysed for its alignment with the four structures and their coherence with curriculum standards. Next, teaching strategies and examples were incorporated to make these structures explicit to students.

Finally, feedback from teachers and small-scale classroom trials informed further revisions.

FINDINGS

Four structures for helping students learn mathematics

To answer Q1, this study concluded four types of structures that help students learn mathematics. There is a logical structure, a growing structure, a research structure, and an organizing structure.

Logical structure of core mathematics knowledge

The logical structure of core mathematics knowledge. In mathematics learning, different mathematical objects are related to each other in a logical way, which in turn determines in what order students are supposed to obtain them. However, since the content in textbooks is often organized in a spiral way, the structures may be hidden during the long process of mathematics learning. In some cases, what is learned successively is clearly related. It is clear in Figure 1, from “parallelogram” to “rectangle and rhombus” and to “square”, that it is from the more general concept to the specific ones. Also, it happens when the related mathematical concepts are arranged in different chapters or even different volumes. While learning, students need to be aware of such relationships. In the same mathematics domains, some knowledge has parallel relationships, although the extent of complexity may be different; some are progressively related. In addition, the logical structures are extended and cover mathematical concepts in different domains. For example, based on the Pythagorean Theorem, irrational numbers are generated, which lead to real numbers; real numbers correspond to the real axis, and are extended to two-dimensional space to produce a plane rectangular coordinate system; after having such a powerful tool, the image of functions can be studied. When designing textbooks, such logical structures need to be visible to students in some manner (Zhang et al., 2022).

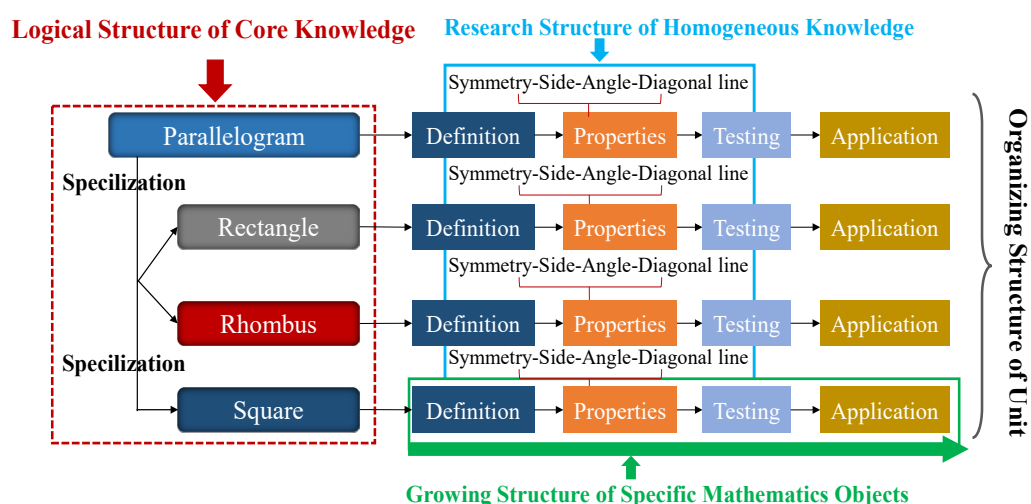


Figure 1: Four structures in the unit of Parallelogram in the BNUP Textbooks

Growing structure of specific mathematical objects

The second structure is the growing structure of specific mathematical objects. For certain mathematics knowledge, the route of learning or study can be found. Like geometric figures, normally it is examined from definition to properties, followed by testing and applications. And for equations, it is always studied from definition to solution and to application. In sum, students need to experience and summarize by themselves how to study particular mathematical concepts.

Research structure of homogeneous mathematical objects

According to the logical structure of core mathematics knowledge and the growing structure of specific mathematical objects, students can transfer what they have learned and realize the research structure of homogeneous mathematical objects. Take learning geometric figures' properties as an example, often it is studied from a holistic perspective and then zoomed into specific components. As shown in Figure 1, when studying parallelogram's properties, students are guided to examine the figure as a whole and find its symmetry and then focus on its edges and angles. Another example is learning mathematical concepts, the learning starts from making an abstraction from context and then generalizing the essence and formulating its definition. After that, students need to deepen their understanding and apply the newly acquired concepts.

Organizing structure of the unit

Last but not least important, students need to be familiar with the organizing structure of the unit. Each textbook has its own characteristics. By understanding basically how a unit is organized, students are more likely to guide and reflect on their learning. In the BNUP Textbooks, each unit will start with providing students with an overview. With such a cognitive map, students get into each sub-unit to learn and make summaries and revisions in the end. Such a "whole-part-whole" structure would guide students to form a higher point of view and a more comprehensive understanding.

Three strategies to make the structures visible

Students' ability of structural learning is not innate, and it requires the guidance of textbook and teachers. In order to promote students' structural learning, textbooks should first organize mathematics content according to a certain structure. In other words, textbooks indeed need and have structure. Secondly, the structure is supposed to be appropriately explicit so that students can perceive the structure. Thirdly, tasks need to be designed in order to offer opportunities to promote students' perception and construction of structure by themselves. Specifically, there are three suggestions and strategies that are explained in detail for answering Q2.

Keeping and following the structures consistently

In the first place, it makes clear that structured thinking should be reflected during the design of textbooks. The textbook developers arrange the learning sequences according to the logical structures of the core knowledge, trying to keep the homogeneous mathematical objects approached and studied in the same manner. In short, the logical structure of the core knowledge, the growing structure of the specific objects, and the

research structure of the homogenous knowledge were kept as much as possible. Besides the examples given above, when explaining the four structures, another one can be the parallel ways of learning linear and quadratic functions. Students are invited to explore the special and the simplest functions from the very beginning, and then a relatively more complex function is studied by gradually adding parameters.

Making and showing structures explicitly

How to make the structures visible or recognizable to students and teachers is also part of the design. Various ways are used to make the structures clear. For example, the logical structure of the core knowledge and the growing structure of the specific objects is often given through the table of contents at the very beginning of the units. However, the table of contents is relatively refined and may not be able to show more detailed structures. Back to the example of Figure 1, in the unit of parallelogram, rectangle, and rhombus, it is difficult to see the growing structure of these specific objects “definition-properties-testing-application” from the table of contents. Therefore, it is a crucial design task to make it visible to students. When learning parallelograms, students are guided by a “Thinking and Communication” activity to think about whether the parallelogram has symmetry and what characteristics can be obtained, and then observe its sides, angles, and diagram. Such growing structures were repeated when learning rectangle, rhombus, and square. As a result, it is not that difficult for students to perceive and furthermore grasp it.

When it comes to the research structure of specific knowledge, another challenge for textbook design is the short learning time. Take the linear function as an example, in the BNUP Textbooks, it is tried to provide the research structure from a specific to a general manner. In the beginning, students receive a probe question: “What is the graph of a linear function? Is it really a straight line? How to study it? Let’s start with the graph of a simple proportional function”. Then it costs a lesson to study the proportional function of $y=kx$, focusing on the effect of the parameter k on the graph of the function. In the next lesson, the topic is studying the graph of a general linear function. The research question will be clarified again. “What is the graph of a linear function? Compared with a proportional function, a linear function has another parameter b . What effect will b have on the graph of the function? With this conjecture, we continue to study the graph of a linear function!” In short, by introducing a series of probes, how to study function is explicit. The study map from the specific to the general, gradually adding parameters, and then the focus is on the effect of parameters on the graph of the function. With the foundation of studying the graph of a linear function, the study of a quadratic function guides students to transfer their previous learning experience, hoping to independently construct a similar sequence of studying.

Offering opportunities through tasks to think structurally

In the third place, students are invited to complete tasks for participating them to establishing their structures. Although the textbooks show the structure, many students may not have the awareness and habit of perceiving the structures. If the textbooks do

not have clear requirements for perceiving or constructing the structure, many students may not be able to know the structure, let alone construct it. Therefore, the textbook design cannot only stop at giving the explicit structure but also needs to guide students to build up the structure in an appropriate way on their own. To be more specific, there are four deliberative activity designs in order to deal with systematically in the BNUP Textbooks, which are (1) *Guidance in the preface*, (2) *“Review and Reflection” within a chapter*, (3) *“Revision and Thinking” by the end*, and (4) *Structured exercises*.

The preface to a chapter is the prelude, and it has multiple functions. Except for stimulating learning interest, it also reveals learning maps. In the BNUP Textbooks, it adopts the “A+B+C” structure. “A” means to begin with a familiar everyday life or mathematical phenomenon, which makes clear the necessity of learning this chapter. “B” is mainly to make a connection between what we have learned with the coming knowledge, briefly explaining what will be learned in this chapter, how to learn, and what will happen after learning. “C” provides several relatively “macro” questions, and it is encouraged to think continuously throughout the whole chapter. Some of the questions may point to the understanding of the core mathematical concepts; some may deal with the research methods with general significance; others are designed to trigger students’ structural thinking on the mathematical learning.

As shown in Figure 2, it is the preface to the chapter on Rational Number and Its Operations. The second paragraph briefly explains the content, methods and core competencies to be developed in this chapter. The third paragraph raises two questions to be thought throughout the chapter: why we should learn negative numbers and how to study a type of “new” number. These questions require students to reflect and comprehend many times during the learning process and finally form a general thinking path for learning numbers and operations, and further understanding the expansion of the number system.

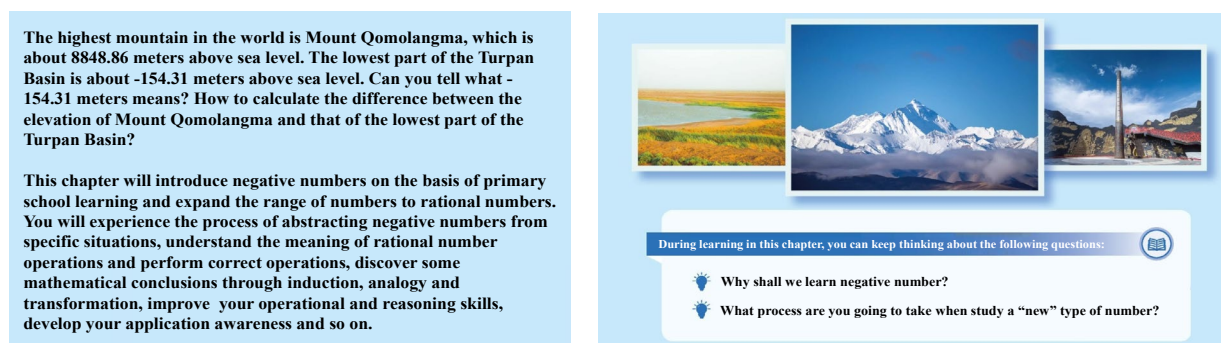


Figure 2: Preface to the chapter on Rational Number in the BNUP Textbooks

Within the chapter, there are designed activities called “Review and Reflection”, which offer opportunities that purposely guide students to focus on their thinking process and learning and experience. Such questions asked in the chapter regarding rational numbers include “After learning the addition of rational numbers, what smart strategy do you experience to make calculation easier and quicker?” “Looking back on the study of rational number operations, what kind of exploration process have you gone through?”

What research problems have you accumulated?” Such questions echo the ones at the beginning of the chapter.

By the end of the chapter, the “Revision and Thinking” section guides students to have an overview of the content, methods and process of this chapter through a series of questions once more, and finally organize and present the knowledge structure in the chapter in an appropriate manner. In this way, students’ ability to think and sort out knowledge structure is gradually developed. Of course, in order to give teachers and students sufficient help, especially students who have not had much similar experience before, the teacher’s book provides the knowledge map as a reference. In Figure 3, the “Revision and Thinking” of the chapter on Rational Number and Its Operations is provided to illustrate.

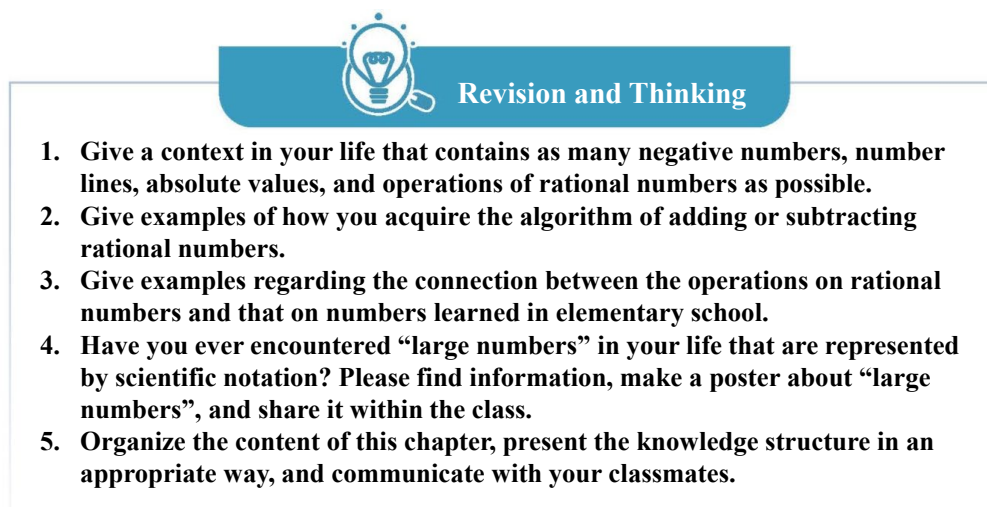


Figure 3: “Revision and Thinking” in the chapter on Rational Number

In addition, exercises in textbooks are not only a way to consolidate and deepen students’ learning but also an effective vehicle for students’ independent learning and reflection. Therefore, the textbook has designed some exercises to trigger students’ structural thinking; at the same time, it also promotes students to summarize and articulate their learning experience, improving their ability of self-regulation. There are three types of structured exercises included in the BNUP Textbooks, respectively pointing to self-learning, thinking backward, and thinking forwards. Such as after learning the chapter on “Linear Function”, students are encouraged to communicate. “Based on the learning experience of the current chapter, could you please tell what other functions we may learn in the future? And for a specific function, what do you think we may study?”

CONCLUSION AND DISCUSSION

This study demonstrates how textbooks can be systematically designed to promote students’ structural learning by emphasizing the integration of logical, growing, and research structures. By making these structures explicit and incorporating tasks that guide students’ active engagement, the BNUP junior high school mathematics

textbooks provide a robust framework for fostering structured thinking in mathematics education. This study has theoretical and practical contributions to mathematics education. Theoretically, it bridges the gap between theoretical perspectives on structural learning and practical textbook design. The integration of logical, growing, and research structures ensures that students can navigate mathematics not as isolated pieces of knowledge but as an interconnected system. Practically, the actionable strategies for fostering structural learning can be adapted to a variety of educational contexts, extending their relevance beyond the Chinese mathematics curriculum. With the BNUP textbooks newly introduced in September 2024, and this study providing a foundational framework for designing textbooks to promote structural learning, ongoing empirical research is needed to evaluate their effectiveness in real classroom settings and identify areas for improvement.

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CHANGES IN PROBLEM CONTEXTS IN MIDDLE SCHOOL MATHEMATICS TEXTBOOKS: A LONGITUDINAL COMPARATIVE STUDY OF EQUATION TOPICS IN CHINA

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Mathematics curriculum guidelines emphasize the critical role of problem contexts in developing students' mathematical literacy. Over the past two decades, Chinese middle school textbooks have increasingly integrated diverse problem contexts, especially in the "equations" unit. This study analyses two popular textbooks from the People's Education Press (2003–2024), utilizing a framework that examines both internal features and contexts types. The results reveal enhanced mathematical complexity, task difficulty, and authenticity, with higher cognitive demands. Additionally, there is a notable increase in real-life, occupational, and mathematical contexts, improving relevance. These shifts highlight the educational value of problem contexts and suggest ways for teachers to more effectively utilize textbook materials.

INTRODUCTION

Problem contexts are crucial in mathematics education, serving as a bridge between abstract concepts and real-world application (Reinke, 2020). Well-designed contexts can enhance students' mathematical reasoning and problem-solving skills. In the "equations" unit, they play a significant role, helping students grasp complex algebraic ideas through familiar scenarios. In China, mathematics textbooks are central to education, widely used by students and teachers alike. Over time, the development of problem contexts has evolved, aligning with shifts in curriculum standards (Li & Li, 2018). Earlier editions emphasized procedural knowledge, while recent ones prioritize contextualized problems to boost cognitive demand and relevance. This shift aims to develop core competencies through deeper mathematical engagement.

This study investigates changes in problem contexts within the "equations" unit of Chinese middle school mathematics textbooks from the People's Education Press (PEP), spanning 2003 to 2024. By examining both internal features and types of problem contexts, the research highlights how textbooks have adapted to educational standards, offering insights for effective mathematics instruction.

LITERATURE REVIEW

Problem contexts in mathematics textbooks are essential for developing students'

problem-solving skills and understanding. Research indicates that many textbooks rely heavily on traditional and routine problems, often lacking diverse real-world scenarios. For example, Swedish and Spanish textbooks tend to present standardized problems, limiting engagement with practical contexts (Brehmer et al., 2016; Tárraga-Mínguez et al., 2021). Comparisons between countries, such as the U.S. and China, highlight differences in the use of application and open-ended problems, which influence students' performance and problem-solving strategies (Zhu & Fan, 2006).

The scarcity of context-rich tasks can hinder students' ability to apply mathematical concepts in real-world scenarios. Studies on Indonesian textbooks, for instance, revealed a limited number of context-based problems, leading to challenges in practical application (Wijaya et al., 2015). In this study, context refers to the situation or setting that gives meaning to a mathematical problem (Wijaya et al., 2015). The degree to which a problem reflects real-world situations is described as its authenticity (Roach et al., 2018), indicating how well it connects abstract mathematics to students' lived or societal experiences. Overall, enhancing the variety and authenticity of problem contexts could better support the development of mathematical literacy. This study investigates the changes in problem contexts within Chinese textbooks over the past two decades, focusing on how these contexts have evolved in the "equations" unit.

METHODOLOGY

Textbook selection

This study examined two widely used sets of seventh- to ninth-grade mathematics textbooks in mainland China: PEP2003 (2003-2011) and PEP2012 (2012-2024). PEP2003 followed the Mathematics Curriculum Standards for Compulsory Education (Experimental Version), while PEP2012 adhered to the updated 2011 standards. These editions reflect key shifts in Chinese middle school mathematics education over the last two decades. According to the Chinese mathematics curriculum (Ministry of Education of the People's Republic of China 2022), linear equations in one variable are introduced in Grade 7, systems of equations in Grade 8, and quadratic equations in Grade 9. The learning objectives progress from building basic algebraic manipulation skills to solving contextualized problems and developing abstract reasoning. The study focused on the "equations" unit, covering linear equations, systems of equations, fractional equations, and quadratic equations. Consistent structural elements in both editions allowed for categorization: "Chapter Introduction," "Problem," "Exploration," and similar sections were grouped as "Knowledge Introduction," while "Exercises," "Review," and "Comprehensive Application" were classified under "Knowledge Consolidation."

Analytic frameworks

Internal Feature Analysis Framework

This study utilizes a framework (Table 1) analysing four dimensions: mathematical level, representational level, task level (Li, 2000; Brehmer et al., 2016), and

authenticity level. The notion of authenticity is informed by previous work on authentic learning and authentic mathematical tasks (Palm, 2008; Roach et al., 2018). Building on this literature, we propose an analytical typology of four context types to describe different degrees and forms of connection to the real world or cognitive experience: *Constructed contexts*-purely hypothetical (e.g., “solve $3x + 5 = 20$ ”); *Experiential contexts*-derived from students’ everyday life (e.g., calculating phone bills); *Rational contexts*-requiring reflection beyond daily life, such as considering carbon-emission quotas; *Speculative contexts*-entirely abstract, focusing on mathematical reasoning (e.g., properties of quadratic equations).

Internal Feature	Level	Description	Code	Value
Mathematical Level (ML)	Level 1	No calculations	ML1	1
	Level 2	One-step calculation	ML2	2
	Level 3	Two-step calculation	ML3	3
	Level 4	Three or more steps	ML4	4
Representational Level (RL)	Level 1	Pure numbers or phrases	RL1	1
	Level 2	Story/narrative	RL2	2
	Level 3	Story/narrative + diagram/table	RL3	3
	Level 4	Story/narrative + diagram+table	RL4	4
Task Level (TL)	Level 1	Memorization tasks	TL1	1
	Level 2	Conceptual understanding	TL2	2
	Level 3	Problem solving	TL3	3
	Level 4	Problem investigation	TL4	4
Authenticity Level (AL)	Level 1	Constructed context	AL1	1
	Level 2	Experiential context	AL2	2
	Level 3	Rational context	AL3	3
	Level 4	Speculative context	AL4	4

Table 1. Internal Feature Analysis Framework

The levels within each dimension are calculated using a weighted average method:

$$d_i = \frac{1}{n} \sum_j n_{ij} d_{ij} (i = 1, 2, 3, 4; j = 1, 2, 3, 4)$$

Where d_i represents the weighted average value for the four dimensions: "Mathematical Level," "Representational Level," "Task Level," and "Authenticity Level." The value d_{ij} represents the weight assigned to each level within each

dimension (ranging from 1 to 4); n is the total number of contexts, and n_{ij} is the number of contexts at level j for dimension i .

Context Types Analysis Framework

Building on the PISA 2022 classification of context types, which describe the situational domain in which a mathematical task is embedded – namely, personal, occupational, societal, and scientific contexts (OECD, 2022), this study adds a separate category for mathematical contexts to capture the unique nature of mathematical problem scenarios. As a result, problem contexts in this study are categorized into five types: Personal context (PC), occupational context (OC), societal context (SC), scientific context (SciC), and mathematical context (MC).

For example: A student bought 3 copies of the same mathematics book and 2 copies of the same science book, spending a total of 94 yuan. The price of one science book is 8 yuan more than that of one mathematics book. The task is to determine the price of each mathematics book. (from PEP2012, Grade 7 Mathematics)

To solve this problem, it is necessary to first translate the verbal description into an algebraic equation by introducing a variable for the price of the mathematics book. This involves two steps of calculation, with the mathematical characteristic denoted as ML3 (basic linear equations). The problem is presented in a narrative text format without diagrams or tables, and thus the representation characteristic is labelled RL2. The problem belongs to the problem-solving type, with the task characteristic denoted as TL3. The context of the problem reflects a familiar consumer experience—buying books—which most students have encountered in real life. Therefore, the problem context is categorized as an experiential context (AL2). According to the type of context, it belongs to a personal context (PC).

Textbook analysis procedure

The analysis involved three experts in mathematics education—a PhD student, a faculty member, and a practicing teacher—following the two frameworks. Tasks were systematically counted, with sub-items within a problem considered as a single task. Each task was coded for internal features (mathematical, representational, task, and authenticity levels) and external features (context types). Internal feature complexity was assessed using a weighted average, while external features were categorized qualitatively.

To ensure consistency, the coders initially discussed the framework. Two coders independently analysed the first two units, achieving a Kappa coefficient of 0.88. Any discrepancies were resolved through discussion, with a third coder reviewing unresolved cases. Tasks were then classified as low, medium, or high complexity based on internal feature scores.

RESULTS AND DISCUSSION

Changes in Internal Features of Problem Contexts

The analysis reveals that the PEP2012 edition shows notable improvements in mathematical complexity, task difficulty, and authenticity over PEP2003 (see Table 2). These changes align with the shift in curriculum standards, emphasizing problem-solving skills and critical thinking over procedural knowledge. The rise in "conceptual understanding" and "problem-solving" tasks in PEP2012 highlights this trend.

Internal Feature	Level	Value	Frequency		Average Value	
			PEP2003	PEP2012	PEP2003	PEP2012
ML	ML1	1	116	107	2.0782	2.2471
	ML2	2	35	36		
	ML3	3	49	54		
	ML4	4	43	58		
RL	RL1	1	102	103	1.7490	1.7176
	RL2	2	105	126		
	RL3	3	31	21		
	RL4	4	5	5		
TL	TL1	1	60	60	2.4897	2.5216
	TL2	2	24	26		
	TL3	3	139	145		
	TL4	4	20	24		
AL	AL1	1	98	98	2.2049	2.2353
	AL2	2	43	50		
	AL3	3	54	56		
	AL4	4	48	51		

Table 2. Internal Feature Levels for Problem Contexts in the "Equations" Section.

PEP2012's increased use of real-world examples, such as replacing a seed planting scenario with a phone billing situation, enhances task authenticity for students by making problems more relatable. This shift reflects China's rapid urbanization and economic development over the past two decades, which has made phone billing scenarios more aligned with the everyday experiences of most students. However, this does not imply that agricultural contexts, such as seed planting, are unimportant; they remain meaningful in connecting mathematics to rural and environmental realities. Both textbook editions show a predominance of basic mathematical tasks, with limited

examples requiring complex calculations or higher-order reasoning. Representational levels mainly consist of numeric descriptions or phrases, with minimal use of visual aids. While problem-solving tasks are prevalent, there are fewer tasks focused on conceptual understanding.

In this study, we focus specifically on the authenticity of mathematical contexts. Among the four categories, the constructed context appears most frequently, as it typically includes problems aimed at reinforcing students' computational fluency with equations. The next most prevalent type is the rational context. To further clarify one of the most prominent categories—rational contexts—this study identifies three major subtypes commonly found in Chinese mathematics textbooks:

1. Contexts drawn from current production and daily life but unfamiliar to some students. For instance, high-speed train scenarios are frequently presented in textbook problems and examination items. However, students who have never experienced high-speed rail travel may interpret such contexts differently from those who have, introducing unintended bias. Similar challenges have also been reported in large-scale educational assessments, where students from inland or underdeveloped regions misinterpreted unfamiliar contexts such as “wharf.” This example further illustrates the importance of precise and context-sensitive problem design to minimize potential bias and ensure fairness across diverse student backgrounds.

2. Contexts based on real-life production or occupational scenarios that are unfamiliar to most students. Some textbook problems are set in industrial or occupational contexts, such as comparing the properties of materials or the efficiency of production processes. While these contexts are grounded in authentic industrial practices, they typically lie outside the everyday experiences of secondary school students. This example reflects a broader issue in mathematics education—the challenge of balancing contextual authenticity with students' experiential familiarity.

3. Contexts derived from human civilization, history, or scientific and technological advancement. For example, when introducing the concept of the golden ratio, some instructional materials reference cultural artifacts such as the Mona Lisa or the Parthenon. For students with exposure to art appreciation or higher aesthetic awareness, these contexts feel more authentic and enhance the learning experience, as they can more easily perceive the embedded mathematical structures.

These examples demonstrate that rational contexts demand careful consideration not only in terms of accuracy and relevance, but also in terms of cognitive accessibility. While experiential contexts remain dominant in contemporary Chinese mathematics education, relying solely on them may obscure the deeper connections between authenticity and mathematical reasoning. It is therefore essential for educators to place greater emphasis on rational contexts. Doing so can help students engage with broader scientific, technological, and societal themes that extend beyond their immediate environments. This approach not only prepares students for future academic and

professional challenges but also enhances their ability to transfer knowledge across unfamiliar contexts, thereby deepening their mathematical understanding.

Changes in Problem Context Types

Table 3 indicates an increase in the total number of problem contexts in PEP2012 compared to PEP2003. The radar chart in Figure 1 illustrates that both versions prioritize mathematical contexts, highlighting a strong focus on mathematics throughout the textbooks.

For personal and occupational contexts, PEP2003 contained 18.1% and 18.9% respectively, while PEP2012 saw a slight increase to 19.2% and 21.2%. This reflects an enhanced effort in PEP2012 to connect mathematics with students' daily lives and potential career paths (Kolar & Hodnik, 2021). However, societal and scientific contexts remained low, with PEP2003 showing 5.8% and 9.9%, and PEP2012 indicating a slight decline to 3.9% and 7.5%, suggesting limited emphasis on real-world societal and scientific applications.

Edition	PC	OC	SC	SciC	MC	Total
PEP2003	44	46	14	24	115	243
PEP2012	49	54	10	19	123	255

Table 3. Comparison of Problem Context Types in PEP2003 and PEP2012.

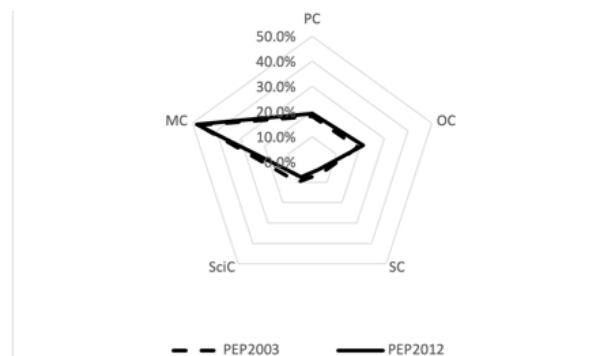


Figure 1: Problem Context Types for PEP2003 and PEP2012.

SUMMARY AND CONCLUSIONS

Since 2001, China's curriculum reforms have shifted from focusing on traditional basics to fostering core competencies, emphasizing problem contexts as essential tools for skill development. Over the past 20 years, the "equations" sections in textbooks have evolved to better align with students' real-life experiences and mathematical understanding, enhancing cognitive demands and fostering deeper mathematical thinking.

This study's analysis of PEP textbooks from 2003 and 2012 identified notable changes in mathematical complexity, task demands, and authenticity, with a broader inclusion of personal, occupational, and mathematical contexts, thereby enhancing their educational value. With the revised 2022 curriculum standards, new textbooks will be

introduced starting September 2024 to June 2027. Future research should examine these editions to understand how they adapt problem contexts to align with evolving educational goals and emphasize core competencies.

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GENERATIVE AI-DRIVEN DESIGN AND IMPLEMENTATION OF INTERDISCIPLINARY CURRICULUM RESOURCES: AN EXPLORATORY CASE STUDY ON THE “GOLDEN RATIO”

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Designing and implementing interdisciplinary curriculum resources remains a challenge for mathematics teachers. This study explores the potential of Generative AI (Gen-AI) to address these challenges, focusing on the design and implementation of an interdisciplinary module using Gen-AI. Data were collected from one instructor and two seventh-grade students in Shanghai through interviews and human-Gen-AI interaction data. The results reveal that Gen-AI plays a “supporting” rather than a “replacing” role and can perform seven roles at different stages. Participants gave generally positive evaluations on Gen-AI’s roles: positive (3), neutral (3), and negative (1), though greater attention must be given to the potential educational risks that accompany Gen-AI use. Limitations and future directions are discussed in the end.

INTRODUCTION

Existing literature (e.g., Kokazhaeva et al., 2024) and current mathematics curriculum reforms in many countries (e.g., Ministry of Education, 2022) both emphasize the crucial role of interdisciplinarity in the teaching and learning of mathematics. However, numerous challenges exist for mathematics teachers to teach interdisciplinarity in classrooms. Generative Artificial Intelligence (Gen-AI), powered by deep learning technologies, can respond to prompts in diverse formats—such as text, audio, images, simulations, code, and videos—within a remarkably short time frame (e.g., Mao et al., 2024), which may offer promising solutions to these challenges. However, few studies have investigated the use of Gen-AI for generating interdisciplinary curriculum resources, especially in mathematics-arts integration.

Therefore, this study seeks to address the research gap by conducting an exploratory case study on using Gen-AI to design and implement a mathematics-arts interdisciplinary curriculum module: “The Golden Ratio: A Harmonious Chapter Between Mathematics and Music”. Specifically, the study aims to address the following research question: What roles can Gen-AI play in the design and implementation of interdisciplinary curriculum resources?

THEORETICAL CONSIDERATION

In mathematics education, research on Gen-AI is burgeoning. Studies have begun to explore the role of Gen-AI in facilitating various tasks. For example, Chiu et al. (2023)

highlight Gen-AI's potential to support personalized learning by assigning tailored tasks, analysing student work, enabling human-AI dialogue, and enhancing interactivity. For teachers, Gen-AI aids in using tools like GeoGebra and SPSS (Manik, 2024), supports organizational tasks (Karaman & Göksu, 2024), and helps student-centred lesson designs (Wardat et al., 2023). It also helps explain complex concepts and assists in automated assessment and timely feedback (Pepin et al., 2025). Although debates remain over Gen-AI's mathematical expertise (Korkmaz Guler et al., 2024), Gen-AI has generally received positive feedback (Kaplan-Rakowski et al., 2023).

To date, existing research has predominantly focused on mathematics teachers' perceptions of Gen-AI in daily practice or its use by pre-service teachers for lesson planning (e.g., Gurl et al., 2025). However, research on the embedding of Gen-AI throughout the entire process of actual mathematics instruction remains scarce, particularly within interdisciplinary educational settings. Alan and Yurt's (2024) study on flipped mathematics classrooms integrates Gen-AI, outlining the roles and tasks of students and teachers across three phases: preparation, outside-classroom teaching, and inside-classroom teaching. This study offers practical guidelines for fostering effective interactions with Gen-AI in each phase.

Building on this, to investigate how Gen-AI facilitates the design and implementation of interdisciplinary curriculum resources across multiple stages of instruction, the present study examines three stages: (1) Before the class: Instructor conducts lesson preparation and designs resources, (2) During the class: Instructor and students jointly implement the resources, and (3) After the class: Instructor implements assessment for students. Also, researchers gather reflections about the roles and user evaluation of Gen-AI from both the instructor and students.

METHODS

This study adopts a case study approach, comprising one instructor (a doctoral student majoring in mathematics education) and two seventh-grade students from an international middle school in Shanghai, having obtained informed consent. Both students have prior experience with Gen-AI, forming a learning unit together.

Interviews and human-Gen-AI interaction data were both collected in August 2024. Both students were on summer break and participated in the study in an informal, non-school mathematics learning environment—a quiet, off-school small study room—with a class duration of 90 minutes. Before the lesson, the instructor used Gen-AI to search for appropriate topics, produce lesson plans, and design some classroom activities. During the lesson, two students used Gen-AI to collect musical scores, request modifications or enhancements to their project plans, and create PowerPoint slides based on their research reports. After the lesson, both students participated in a simultaneous, unstructured interview lasting approximately 10 minutes, during which they shared their reflections on the role and impact of Gen-AI in their learning process. Also, the instructor utilized Gen-AI to generate relevant tests for interdisciplinary knowledge and skill. Due to time and space constraints, students were required to

complete the test at home under parental supervision and submit it to the instructor via social media. Finally, the instructor offered reflections on the role of Gen-AI throughout the entire process of the interdisciplinary curriculum.

The interview with students and the instructor's self-reflections focused primarily on two questions: "What did you use Gen-AI for? Do you think it was helpful?" Using content analysis, the researchers recorded and coded the responses in conjunction with the human-Gen-AI interaction data and finally produced Gen-AI's roles before, during, and after class (see Table 1). For example, in one after-class task, the instructor prompted Gen-AI to "...design an interdisciplinary knowledge and skills test...". The system generated multiple questions. However, the instructor negatively evaluated the output as "...scientifically erroneous and overly simplistic...". This instance was thus coded as *Generating tests* and *Negative evaluation*.

Considering the accessibility and convenience of use, the hardware used in this study was iPads, and the software utilized was Kimi AI (<https://kimi.moonshot.cn/>), which included text dialogue and PPT generation features. Additionally, the iPad provided access to other mainstream search engines for cross-referencing when necessary.

RESULTS

Overall, the instructor serves as the primary Gen-AI user in the before-class and after-class phases, while in the during-class phase, students are the main Gen-AI users, with the instructor providing appropriate guidance. It is important to note that Gen-AI serves only an assisting role rather than a substitutive one. The effectiveness of Gen-AI requires human oversight. Gen-AI's various roles received an overall positive evaluation, categorized as positive (3), neutral (3), and negative (1), as detailed in Table 1 below.

Dimension	Role	Prompt (example) & Evaluation
Before class	Assist in selecting topics	Prompt: I would like to design an interdisciplinary curriculum that combines mathematics with the humanities and arts. Please recommend research themes. Instructor evaluation: Positive; Gen-AI can offer various possibilities, helping to overcome knowledge blind spots.
	Assist in establishing teaching objectives	Prompt: See Figure 1 Instructor evaluation: Neutral; Gen-AI can generally list the teaching objectives across various dimensions; however, the statements are rather vague and lack specificity.
	Assist in conducting	Prompt: Based on the aforementioned teaching themes, objectives, and fundamental inquiry approach, please

During class	activity pre-structuring	provide three different design proposals for interdisciplinary learning activities. Instructor evaluation: Neutral; The optimal localized design can be obtained by comparing the different plans generated by Gen-AI. However, the designs for mathematical inquiry typically lack distinctiveness. Prompt: Please provide the musical score for “March of the Volunteers”.
	Simplify the retrieval of procedural resources	Student evaluation: Positive; Gen-AI is very useful in the collection of musical scores. It aggregates available web links, significantly reducing the considerable amount of time I would spend searching for them individually.
	Evaluate the initial plan	Prompt: Please evaluate the rationality and feasibility of the following plan and provide suggestions for revision. Student evaluation: Positive; Gen-AI offers suggestions for modifications regarding both the overall layout and specific details of the plan.
	Intelligently generate presentation materials	Prompt: Please create a PowerPoint based on the following document/text (research report). Student evaluation: Neutral; Gen-AI provides a wide range of PowerPoint templates, significantly reducing production time. However, the content layout is relatively simple and lacks personalized elements.
		Prompt: See Figure 2
After class	Generating tests	Instructor evaluation: Negative; The test items autonomously generated by Gen-AI contain scientific errors and are overly simplistic.

Table 1: Overview of various roles of Gen-AI with participants' evaluation.

Before class

In the before-class phase, Gen-AI can primarily perform three different roles. Taking “establishing teaching objectives” as an example, one can approach this from five guiding dimensions: knowledge base, contextual understanding, problem-solving, communication interaction, and attitude orientation. By briefly describing the basic

interdisciplinary concepts in mathematics classrooms to Gen-AI, one can gradually refine these concepts to form initial goal settings. The instructor can then revise these goals in conjunction with subsequent designs (see Figure 1). However, these objectives may be overly vague and require further refinement in conjunction with the design of subsequent planning stages. Consequently, the instructor maintained neutral.

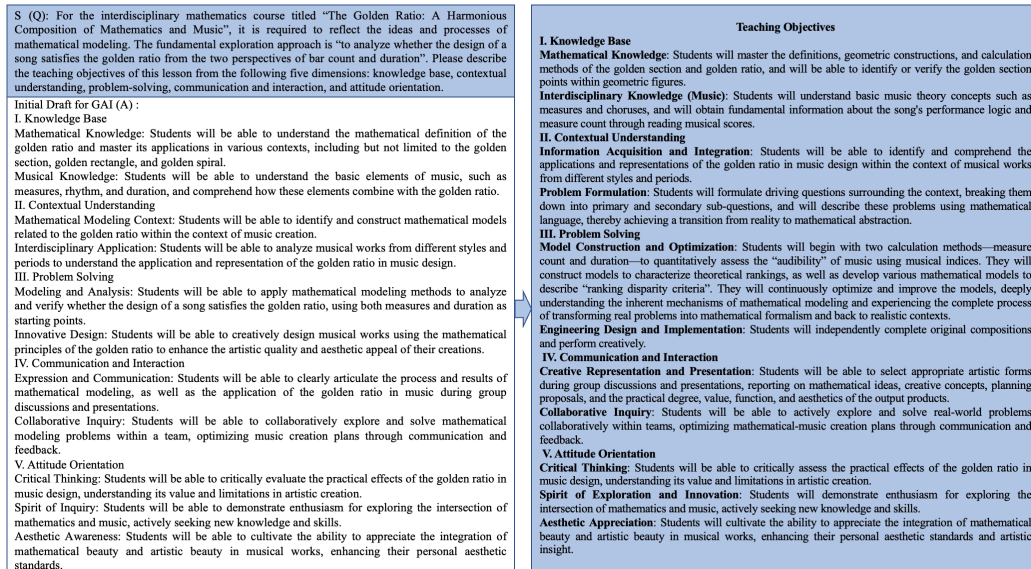


Figure 1: Gen-AI-assisted establishment of teaching objectives.

During class

Regarding the during-class phase, taking “evaluating the initial plan” as an example, two students took multiple rounds of human-Gen-AI discussions with the assistance of the instructor, leveraging Gen-AI’s unique context perception and memory capabilities. This collaborative effort allowed them to systematically address and rectify the shortcomings in the original plan, ultimately leading to the formulation of a final research proposal. Students were actively engaged in constructing their research plans, and this autonomous learning experience significantly enhanced their engagement and learning attitudes. As a result, Gen-AI received a positive evaluation.

After class

In the after-class phase, Gen-AI can assist in assessments by “generating tests”. Gen-AI follows specific instructions, not only aggregating links to existing subject-related questions but also autonomously creating new ones (see Figure 2). However, Gen-AI exhibits general deficiencies in mathematical professionalism when autonomously generating questions, leading to negative evaluation from the instructor. For instance, while assisting in the formulation of multiple-choice question 2 in Figure 2, Gen-AI’s

initial output included two duplicate options labeled $\frac{T}{\phi}$ and $T \times \frac{1}{\phi}$.

S (Q): Based on the aforementioned teaching objectives and research plan, please design an interdisciplinary knowledge and skills test for the interdisciplinary mathematics course, *The Golden Ratio: A Harmonious Composition of Mathematics and Music*.

GAI (A) :

Section I: Multiple Choice Questions

- Given that point C is on segment AB and satisfies $AC^2 = BC \cdot AB$, which of the following equations holds? ()
 A. $\frac{AC}{BC} = \frac{\sqrt{5}-1}{2}$ B. $\frac{AC}{AB} = \frac{\sqrt{5}-1}{2}$ C. $\frac{BC}{AB} = \frac{\sqrt{5}-1}{2}$ D. $\frac{BC}{AC} = \frac{\sqrt{5}+1}{2}$
- If the total duration of a musical piece is T seconds and it is divided according to the golden ratio, with $\phi = \frac{\sqrt{5}+1}{2}$, the duration of the longer part is ().
 A. $\frac{T}{\phi}$ B. $T \times \frac{2}{\phi}$ C. $T \times \frac{\phi-1}{\phi}$ D. $T \times \frac{1+\phi}{2}$

3.....

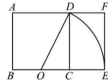
Section II: Fill-in-the-Blank Questions

- Given that the total duration of a piece of music is 120 seconds, the durations of the two parts divided according to the golden ratio are ____ and _____. If the first half contains 40 measures, then the number of measures in the second half would be ____ (assuming the average duration of each measure is the same).

2.....

Section III: Answer Questions

- For many painters and artists, the “golden ratio” is a guiding principle they must deeply comprehend in their real-world creations. Photographers are no exception; in photography, there is a technique called the golden ratio composition method. Its principle is as follows: as shown in the diagram, take the midpoint O of side BC of square $ABCD$, use O as the center of a circle, and the segment OD as the radius to draw a circle, which intersects the extension of side BC at point E . In this way, square $ABCD$ is extended to form a golden rectangle $ABEF$. If $CE = 4$, find the length of AB .



2.....

Section IV: Open-Ended Questions

- Some believe that the application of the golden ratio in music composition is too formulaic and may limit the creativity of artists. Do you agree with this view? Please provide your reasoning and discuss how to effectively utilize the golden ratio while maintaining innovation.

2.....

Figure 2: Gen-AI-assisted test generation

CONCLUDING REMARKS

Through the analysis of human-with-Gen-AI interaction data and reflections from participants involved in a mathematics-arts interdisciplinary curriculum module, the case study reveals that Gen-AI exhibited seven roles. Compared with prior research (Chiu et al., 2023), these roles appear markedly expanded in interdisciplinary contexts. Gen-AI not only streamlines access to procedural resources but also rapidly generates presentation materials tailored to interdisciplinary needs. Such extended roles surpass its relatively limited uses in mathematics classrooms, highlighting Gen-AI's unique value in supporting complex, interdisciplinary learning. However, it should be noted that these expanded roles remain primarily at the auxiliary support level and have not yet substantially transformed students' learning of mathematics. Meanwhile, participants gave positive evaluations on the role Gen-AI plays in the design and implementation of interdisciplinary curriculum resources. Overall, Gen-AI offers significant convenience to the instructor and students during information collection, with some shortcomings leading to negative experiences for some participants, such as scientific errors and a lack of individualized elements.

While the convenience of interdisciplinary curriculum resources provided by Gen-AI presents significant benefits, it is crucial to acknowledge that potential educational risks cannot be overlooked. Given the occurrence of mathematical errors in Gen-AI, refining tools to optimize output is essential. For example, integrating ChatGPT-4 with the Wolfram Alpha plugin can ensure correct answers, accurately detect calculation errors, and offer direct explanations (Davis, 2023). Additionally, establishing a framework for critically evaluating mathematical outputs may become indispensable in the era of Gen-AI (Gurl et al., 2025).

The weakening of instructors' identity—where students may no longer view instructors as their primary source of knowledge, leading to fewer teacher-student interactions—

requires further consideration. On the one hand, attention should be paid to fostering students' emotional engagement through strategies such as using encouraging language, eye contact, and motivational experience sharing to ensure that each student's emotional needs are addressed more effectively. On the other hand, the curriculum material suggested by Gen-AI is often teacher-centered and repetitive, with little understanding of students' needs (Gurl et al., 2025). This presents an opportunity for instructors to intervene. By supplementing and adapting the curriculum to better align with students' unique needs and knowledge, instructors can demonstrate their distinct advantage over machines, emphasizing the irreplaceable value of human educators.

This study has certain limitations. Firstly, only two students participated in this exploratory case study. The limited sample size may affect the generalizability of the findings, and a larger and more diverse group of students could provide a more comprehensive understanding of this issue. Future interventions could be conducted in authentic campus settings with diverse groups of students, along with varied Gen-AI usage. Secondly, the representativeness of the interdisciplinary module should be enhanced by incorporating more cases that integrate mathematics with other disciplines, which may yield different findings and warrant further exploration. Thirdly, incorporating diverse qualitative data—such as reflective journals and in-depth interviews (which were not collected in this study)—for example, by presenting excerpts of student/instructor reflections, could offer a more comprehensive understanding of both students' and teachers' experiences with Gen-AI, warranting further exploration in future research.

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INVESTIGATING CHINESE TEACHERS' CURRICULAR NOTICING THROUGH A PROJECT-BASED LEARNING LESSON STUDY

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This study uses a new type of noticing, termed curricular noticing, to address the question: How do teachers attend to, interpret, and respond to textbooks through the project-based learning (PjBL) lesson study? This study included multiple case studies of several teachers and their classes. Data consist of lesson plans, videotaped lessons, interviews, discussions, and reflection reports. Findings indicate that lesson study can enhance teachers' curricular noticing, strengthening their ability to interact with textbooks and redesign written materials for the effective implementation of PjBL.

INTRODUCTION

Project-based learning (PjBL), one of the fundamental teaching reforms in China, was first introduced in the Mathematics Curriculum Standards for Compulsory Schools from the Ministry of Education (MOE) of China (MOE of China, 2022a) to help students develop problem-solving skills and foster mathematics literacy. To promote this curriculum reform, a series of policy documents has been issued to embed PjBL in the school mathematics curriculum (MOE of China, 2022b; Shanghai Education Commission, 2020, 2023). The MOE of China emphasized that no less than 10% of class time should be devoted to the implementation of interdisciplinary thematic learning, which at the lower secondary level takes the form of PjBL (MOE of China, 2022a). In addition, Shanghai has also issued a series of policy documents, with the aim of covering all schools in compulsory education by 2024 and regularising the teaching of PjBL by 2026 (Shanghai Education Commission, 2020, 2023). However, being in the early stages of curriculum reform, textbooks have not yet adequately provided content about PjBL in China. Although written curriculum materials (like textbooks) are often seen as essential tools for teachers to accomplish their goals for a long time, recent research emphasizes the vital interaction between teachers and the resources to accomplish their goals (Cai & Hwang, 2021; Trouche et al., 2018). Thus, in this light, the lack of PjBL content in textbooks can be seen as an opportunity for teachers to reinterpret and redesign these written materials to include PjBL. Most available relevant studies have concentrated generally on how teachers collaborate in designing PjBL (e.g., Qi et al., 2023), while seldom focus on how curriculum materials and teachers interact to implement this reform-oriented teaching. Therefore, this study, which focused on secondary school, aims to investigate teachers' curricular noticing in

a PjBL-oriented lesson study. This study is also one of the findings of the Research and Development of the Beijing Normal University Edition Textbooks.

THEORETICAL FRAMEWORK

The concept of noticing is not unique to the study of teaching. The idea of professional vision was first developed by Goodwin (1994) to explain why experts and amateurs perceive the same events and phenomena extremely differently. Just as Jacobs et al. (2010) argue about teachers' interactions with students' thinking, teachers' interactions with curriculum materials require a special kind of noticing. This noticing is referred to as curriculum noticing (Dietiker et al., 2018). Drawing on the specialised noticing work of children's mathematical thinking, curriculum noticing has been described as a set of skills that make up the curriculum work of teaching and learning mathematics, such as curriculum attending, curriculum interpreting and curriculum responding.

The origin of the concept of PjBL begins with Dewey's idea of learning by doing (Dewey, 1997). The Buck Institute for Education (BIE) has a more authoritative definition of the concept of PjBL, which is a systematic form of teaching and learning organisation where students continue to explore and learn by experiencing designed projects and a series of tasks in a complex, authentic and problematic learning situation (Hallermann et al., 2011). BIE provides the principles of PjBL, namely authenticity, academic rigor, assessment practices, applied learning, active exploration, and adult connections, with the first three principles being even more important in the process of teacher-textbook interaction (Markham et al., 2006).

Table 1 shows the specific content of the three sub-skills of curricular noticing in the context of PjBL. Combined with authenticity, assessment practice, and academic rigor (Markham et al., 2006), *curricular attending* is further specified in attending authentic situations in textbooks that are relevant and meaningful to students' lives, opportunities to implement assessments that reflect processes and outcomes, and core knowledge that needs to be developed by students. The notion of *curricular interpreting* is connecting the ideas found in the textbooks with the teachers' mathematical knowledge for teaching, which means teachers interpret the materials based on their evidence. We adopted the interpret code (lack of, limited, robust evidence) of Jacobs et al. (2010) in this study. The key concept of *curricular responding* means making a curriculum decision, specifically the decisions teachers make based on the PjBL principles in this study. Therefore, we assigned a responding code (offloading, adapting, improvising) when teachers made a curriculum decision, using the types defined by Brown (2009) to characterise teachers' responding.

Skill	Dimension	Detail
Curricular attending	Authenticity	Teachers attended to elements from textbooks related to the context of the task, the authentic background of the PjBL, e.g., loans, instalments.

	Academic rigor	Teachers attended to elements from textbooks related to the content of knowledge from mathematics or other subjects, e.g., operations with rational numbers.
	Assessment practice	Teachers attended to elements from textbooks related to questions, assessment tools, assessment methods, PjBL assessment objectives.
Curricular interpreting	Lack of evidence	Teachers simply made further statements about the elements attended to, staying at the descriptive stage, and providing no evidence of interpretation.
	Limited evidence	Teachers provided evidence of interpretation of the elements attended to, but this interpretation came more from personal experience and was not linked to the rest of the curriculum material or to knowledge of mathematics teaching and learning.
	Robust evidence	Teachers provided strong evidence for the elements attended to in terms of interpretation, making links to curriculum materials or mathematical pedagogical knowledge.
Curricular responding	Offloading	Teachers used the curriculum materials from textbooks in a literal manner, following the materials as closely as possible.
	Adapting	Teachers adapted the curriculum resources from textbooks in ways that reflected contributions of both the materials and their personal resources.
	Improvising	Teachers improvised their own strategies for instruction with minimal reliance on the materials from textbooks.

Table 1: Curricular noticing in the context of PjBL.

Based on this theoretical framework, the research question was refined as follows: *How do teachers attend to, interpret, and respond to textbooks through PjBL lesson study?*

METHODOLOGY

Participants and setting

The participants were junior high school mathematics teachers from Shanghai, China, who were involved in a large-scale design research project on which this study was based. All teachers had more than ten years of teaching experience. The school principals and the teachers gave consent for participation in the sample. They took part in three PjBL lesson study (LS) in two years, which were all designed by the authors. In this study, we chose one PjBL-LS as a case for analysis. A total of 6 teachers were included, and the case study focused on one mathematics teaching topic. The PjBL-LS

consisted of 3 LS cycles in which the participants went through the process of studying, planning, implementing, and reflecting. For a more detailed description of the process, see Qi et al. (2023).

The focus on mathematical content was the rational numbers chapter of the Shanghai HUJIAO textbook for the first year of junior high school students. Furthermore, the textbooks in this study also include teacher books. In this case, the PjBL was called “My Financial Plan”, in which students were asked to make a financial book to record and organise their income and expenditure over a period.

The PjBL- LS for this study was with reference to Lewis et al.’s (2019) LS cycle. Throughout the LS, teachers developed a total of three versions of the lesson plan. After the completion of each lesson plan, the teachers answered the questions designed by the authors in the form of a reflective log. Each PjBL-LS is flexible in terms of frequency and duration of meetings, depending on the choice of topic. Meanwhile, we used a mixed online and offline format to conduct the LS, with a team that included university researchers and PhD students in addition to mathematics teachers.

Data collection and analysis

The data in this study focused on curricular noticing when teachers design the PjBL, therefore, we designed the following questions, adapted from Christopher et al. (2023) and Hallermann et al. (2011), for teachers to respond. Sample questions included: What elements related to academic rigor are present in the textbooks? Why do you consider these elements to represent academic rigor? How will you design your PjBL approach based on these elements? Similar questions were asked for authenticity and assessment practices. Responses to the questions on three key elements of the materials related to PjBL, which responded to the teachers’ lesson plan, as well as evidence of their curricular attending, curricular interpreting, and curricular responding, were collected. In addition, video and audio recordings were made during the teachers’ group meetings. Furthermore, semi-open-ended interviews with teachers were conducted at the end of this LS to find possible reasons for teachers’ changes in curricular noticing.

Our data analysis consisted of the following four steps. The first step focuses on teachers’ curricular attending to elements related to academic rigor, authenticity, and assessment practices in the textbooks during PjBL design. The unit of analysis was each teacher’s response to the first sub-question (What type questions) in each string of questions. We extracted the elements that teachers attended to, based on the textbook. The second step focuses on their curricular interpreting of these elements, and the data of analysis was the response to the second sub-question (Why type question). We read each teacher’s responses, and the units of analysis were the teachers’ interpretations of each element of their attention. The third step focuses on how they responded to these interpretations in their decision-making, and the data of analysis was the response to the third sub-question (How type question). We also read the responses, and each decision stated by the teachers was divided as a unit of analysis for this step. Finally, coding was completed using the framework in Table 1. All coding was done

independently by two PhD students in mathematics education. The inter-coder reliability coefficient of Cohen's Kappa between the two coders was 0.84, demonstrating good reliability.

RESULTS

Teachers' curricular noticing in PjBL-LS

Based on the textbook chapters selected for this study, teachers were asked to identify the elements related to academic rigor, authenticity, and assessment practice in the textbook when implementing their PjBL. Teachers had their own selective attention, which means that they filtered them after reading the textbook. In terms of elements related to academic rigor, the material covered the definition of rational numbers (Q), arithmetic (addition, subtraction, multiplication and division in Q), exponentiation to integer power, absolute value and scientific notation. Teachers selectively ignored exponentiation to integer power and scientific notation in LS cycle 1, but returned to the exponentiation to integer power in LS cycle 2 and LS cycle 3, as explained by one teacher in an interview:

If you look at my first draft, I didn't cover the exponentiation to integer power because it didn't feel as if students' calculation of money needed that knowledge. And then it was, and it was suggested by a lot of teachers, that there could be exponentiation to integer power for loans and things like that, for compound interest rates. So, in the second draft we added it, but we felt like we were getting away from the students' life. Just kept wrestling with the discussion there, in terms of opening the mind.

The results of teachers' curricular interpreting in textbooks during the PjBL-LS are shown in Table 2. There was an improvement in teachers' skill of curricular interpreting during the PjBL-LS, as evidenced by a decrease in interpretations that had a lack of evidence and an increase in interpretations that had robust evidence. Of these three elements related to PjBL, teachers made the highest number of interpretations in the element related to academic rigor, followed by authenticity, and finally assessment practice, which may be related to the amount of relevant content in the textbooks. Interpretations of elements related to assessment practice had similar results to authenticity, but at the same time, this type of element was the least frequently used for interpretation by teachers. They were able to provide robust evidence of the interpretation of this element at the beginning of the PjBL-LS. In addition, we found that more than 50 % of the teachers' interpretations of such elements at the end of LS cycle 3 had robust evidence. As one teacher mentioned in the interview:

In the early stages of lesson study, our knowledge of PjBL was incomplete, which led us to interpret elements of the textbook in a way that did not connect well with PjBL. For example, in addition and subtraction of rational numbers, we initially thought that this was a skill that students needed to learn, but we did not understand that students did need to master this skill in financial bookkeeping.

Curricular interpreting	LS cycle 1		LS cycle 2		LS cycle 3	
	Number	Percent	Number	Percent	Number	Percent
Lack of evidence	23	43.40%	14	21.88%	10	14.93%
Limited evidence	20	37.74%	30	46.88%	30	44.78%
Robust evidence	10	18.87%	20	31.25%	27	40.30%

Table 2: Teachers' curricular interpreting in textbooks during the PjBL-LS.

In analysing how teachers responded to their interpretations of the three aspects of design and planning the elements related to academic rigor, authenticity, and assessment practice, we divided all codes into three different specific response: offloading, adapting, improvising. Overall, teachers' offloading was gradually decreasing, while adapting and improvising were gradually increasing (see Table 3). This meant that teachers' changes to textbook elements became more frequent. As one teacher explained:

Because in the second LS cycle we discussed and decided to integrate instalment loan or time deposit into our overall PjBL real-world context, we made further modifications to the relevant content in the textbook, which involved several detailed contextual changes.

Curricular responding	LS cycle 1		LS cycle 2		LS cycle 3	
	Number	Percent	Number	Percent	Number	Percent
Offloading	21	51.22%	20	37.04%	18	30.00%
Adapting	12	29.27%	20	37.04%	24	40.00%
Improvising	8	19.51%	14	25.93%	18	30.00%

Table 3: Teachers' curricular responding in textbooks during PjBL-LS.

DISCUSSION

This study examined how teachers' curricular noticing developed during three cycles of PjBL-LS. First, teachers' selective attention shifted modestly over time. In the early stages, they tended to overlook mathematical ideas that they perceived as less relevant to students' real-life experiences, such as exponentiation to integer power. Later discussions encouraged some teachers to reconsider the potential value of these elements, especially when connected to financial contexts. While not all teachers adopted these ideas consistently, the shift indicates a growing awareness of how academic rigor may be integrated into PjBL tasks. Second, teachers' curricular interpreting showed signs of improvement, though progress varied across individuals and elements. The proportion of interpretations supported by stronger evidence

increased across cycles, and unsupported interpretations became less common. Academic rigor remained the most frequently interpreted element, likely reflecting the textbook's content structure. Interpretations related to authenticity and assessment were less frequent and developed more gradually. This suggests that while teachers became more capable of articulating their curricular decisions, some areas—particularly assessment—may require further support. Third, teachers' curricular responding moved slightly away from offloading toward more adapting and improvising. The increase in adapting and improvising indicates that teachers became somewhat more willing to adjust or modify textbook materials to fit the PjBL context. However, offloading still appeared in each cycle, showing that teachers continued to rely on textbook structures to some extent.

Taken together, these findings suggest that PjBL-LS can provide opportunities for teachers to reconsider and refine aspects of their curricular noticing. The development observed was gradual rather than dramatic, and some dimensions—particularly authenticity and assessment-related noticing—remained relatively challenging. Future work could examine how these noticing patterns influence classroom enactment and how professional learning structures might better support teachers in areas where change appeared limited.

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Part IV

Posters

A THREE-DIMENSIONAL FRAMEWORK FOR TEXTBOOK ANALYSIS

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There is a tradition of extensive use of textbooks in mathematics (e.g., Boesen et al., 2014), which is noteworthy because the choice of teaching material impacts the teaching methods (Fan et al., 2023) as well as students' learning (van den Ham & Heinze, 2018). The importance of high-quality curriculum resources (CR) has received attention, whereas frameworks for textbook analysis that combine several established theoretical perspectives are scarce. We present a three-dimensional theoretical framework intended for textbook analysis for professionals and to be used in research.

Three dimensions of didactic relevance

The theoretical foundation of the framework lies in curriculum theory, which views teaching as opportunities for meaning making (Englund, 1997). This perspective emphasises that the content is contingent rather than predetermined and assumes that some selection of teaching content always occurs. Three established theoretical dimensions constitute the basis for the framework and can contribute to the understanding of the kind of learning situation that is stressed with a particular teaching material. These dimensions are described below.

Through classical *learning theories (how)* – behaviourism, constructivism, and sociocultural theory – perspectives on learning can be distinguished. Each theory provides insights into different aspects of learning opportunities, shaping our understanding of how individuals develop knowledge and skills. Several *competency frameworks (what)* in mathematics education have been designed and used in the last decades (e.g., Niss & Højgaard, 2011). Analysing competences offers insight into what mathematics proficiency entails and, consequently, what needs to be developed. *Curriculum emphases (why)* capture different purposes of teaching (Roberts, 1982) and have been widely used to analyse intentions in science curricular materials, teachers' beliefs, and textbooks. Similar emphases have been developed to capture purposes within mathematics curricula (Dyrvold & Bråting, 2019; Ernest, 2004). Using our framework, with indicators of each of its dimensions, the relationship between learning theories, competences and curriculum emphases in textbooks can be explored.

Implementation of the framework

Reflections on opportunities for meaning making offered in teaching materials from a

pilot study of 13 pre-service teachers' (PSTs) shows that the framework has potential to be useful in teacher education and for teaching. While still needing refinement, the indicators of each dimension helped the PSTs to understand the framework. However, to apply the framework as intended, sufficient knowledge about its dimensions is required. The PSTs in our pilot only received a one-hour lecture before using the framework. Even though the PSTs' reflections, analysed regarding different qualities (cf. Hatton & Smith, 1995), to some extent were of a dialogical character, their focus rather concerned the meaning of the three dimensions themselves than the analysis of the teaching materials. None of the reflections were characterised as critical.

The framework also has the potential to be used in research, for example to systematically investigate which aspects of the three dimensions that are emphasised in various CR and whether and how they covary.

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AI COMPETENCIES FOR PRE-SERVICE MATHEMATICS TEACHERS: BELIEFS, SELF-EFFICACY, AND USAGE

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INTRODUCTION

As Artificial Intelligence (AI) as a resource transforms educational practices, its integration into mathematics education offers promising tools for individualized learning, adaptive feedback, and complex problem-solving (Pepin et al., 2025). Preparing future teachers to effectively mediate these technologies is essential. This study explores the AI-related competencies of $N=343$ mathematics pre-service teachers across six German universities, focusing on their beliefs, self-efficacy, and openness to using AI technologies, particularly for mathematics instructional purposes.

THEORETICAL BACKGROUND

Within the context of developing AI-related competencies, beliefs and self-efficacy play central roles. Teachers' beliefs, often defined as deeply held attitudes and perceptions about teaching and learning (Philipp, 2007), shape their openness to integrating new technologies in their instructional practices. Our study draws, amongst others, on the Theory of Planned Behavior (TPB), which posits that behavior is influenced by attitudes, subjective norms, and perceived behavioral control (Ajzen, 1991). Teachers' beliefs about AI's benefits and challenges shape their attitudes toward its use, while self-efficacy impacts their confidence in managing AI in the classroom.

METHODOLOGY

Developed to assess various perspectives on AI in mathematics teaching and learning, the survey encompassed five core sections with a total of 87 items. The item set was designed based on existing AI literacy frameworks (e.g., Laupichler et al., 2023; Wang et al., 2023) and tailored to the domain-specific needs of mathematics education. To ensure validity, an expert panel, including mathematics educators and AI researchers, reviewed and refined the items in iterative rounds. The final instrument captures both cognitive and affective dimensions of AI literacy, addressing beliefs, self-efficacy, and ethical concerns related to AI integration in teaching. Each section focuses on specific aspects of pre-service teachers' views, challenges, and readiness for incorporating AI into their math instruction. Data collection took place between February and July 2024. Participants responded on a five-point Likert scale ranging from "Strongly disagree" to "Strongly agree." The survey sections include:

- (S1) Benefits and Applications of AI for Students (“*Generative AI tools can help students understand mathematical content by generating tailored explanations*”).
- (S2) Challenges and Risks for Students (“*The use of AI technologies in math classes could lead to students’ dependence on technology*”).
- (T1) Benefits and Applications of AI for Teachers: (“*AI can assist teachers in designing personalized learning paths*”).
- (T2) Challenges and Risks for Teachers (“*The use of AI in classrooms could reduce the importance of the teacher’s role in student learning*”).
- (C1) Self-Reported AI Competencies and Openness (“*I am willing to accept the risks associated with generative AI tools to experiment with them in class*”).

PRELIMINARY RESULTS AND ONGOING WORK

Currently, data analysis is ongoing, with exploratory and confirmatory factor analyses being conducted to identify underlying structures in pre-service teachers’ beliefs regarding AI in mathematics education. A three-part structure was identified (S1/S2; T1/T2; C1), each of which is differentiated into sub-factors. Preliminary descriptive results offer a foundational understanding of teachers’ perspectives, with further analysis expected to reveal more nuanced patterns. While pre-service teachers overall exhibit openness to AI and recognize its potential in mathematics education in general, self-efficacy and actual usage are limited. A publication is currently being worked on that compares the results of the beliefs of pre-service teachers with the results of 99 in-service teachers.

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DISCERNMENTS NEEDED WHEN INTERPRETING REPRESENTATIONS IN PRINT

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One distinguishing feature of mathematics textbooks is the prevalent use of words from an everyday register, such as “price” and “answer” (Ribeck, 2015). Because mathematics proficiency includes the ability to interpret and use mathematics in a real-life context, many mathematics tasks are contextualised, and the reader of mathematics textbooks cannot simply give the technical sense precedence in the interpretation process; the reader must instead navigate between senses. An example of an *ambiguous representation* in mathematics is the word “find”, as in “find all solutions”, which must be interpreted in a technical sense to grasp the intended meaning.

Learners’ difficulties with ambiguous vocabulary in mathematics textbooks have long been recognized (cf. Durkin et al., 1985), but other representations than words can also be ambiguous. For example, parentheses and the slash sign have specific technical senses in mathematics (e.g., $f(p) = p / 5$), but these signs can also appear in everyday senses in mathematics textbooks. Spatial representations (diagrams, images) can also take specialised meanings in mathematics. The same figure can, for example, in an everyday sense represent a stroke, and in a technical sense, a line. Altogether, mathematics text is highly dependent on representations that utilise a variety of semiotic resources (Mainali, 2021), and readers must correctly interpret these, both those unique to the subject and those used with a unique meaning within the subject.

THE IRIS MODEL

The point of departure for our presentation is the model of *Interpreting Representations in their Intended Senses* (IRIS; Dyrvold & Ribeck Nyström, in press). IRIS provides a theoretical frame capturing the discernments a reader of subject text engages in when faced with the task of interpreting ambiguous representations, that is discerning within which register a particular representation should be interpreted. The interpretation process goes from seeing a representation, via repeated discernments based on either *knowledge* or *context*, to finally ending up in an everyday, academic, or technical sense. Consider the following textbook example.

When evaluating the expression $5 + 6 \div 3 - 6$, $6 \div 3$ is calculated first.

In this text, for example, the word “calculated” and the symbol ‘÷’ is unproblematic to interpret because they unambiguously belong to the technical register. The word “expression”, on the other hand, has one technical meaning and another meaning in everyday language. To discern which meaning is intended in this case, one must consider the textual context (which is technical). Another ambiguous representation is the comma, which is crucial for understanding the logical structure of the text. However, the immediate context of this symbol is mathematical notation, indicating that it might be a technical symbol (in Swedish, the comma and the decimal point look the same). In this example, one must use knowledge about mathematical and syntactic conventions to understand that the symbol, in fact, is not mathematical notation. Rather, it is a symbol that separates two clauses (and two expressions) from one another, and it should therefore be interpreted in an academic sense, as a comma. All these different processes of discerning the intended meaning are captured and visualised by IRIS.

DIDACTIC IMPLICATIONS

Our presentation contributes to research on curriculum resources regarding how the complexity of ambiguous mathematics representations can be addressed in didactical discussions on mathematics text. Due to the heavy reliance on non-verbal resources in mathematics text, the types of discernments required from a reader are quite varied. With IRIS, we pinpoint the complexity of discerning the intended sense while reading mathematics, which needs to be handled in teaching. For experienced readers, the interpretation process is fully automated, but for learners, interpreting representations may require conscious efforts.

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A BILINGUAL TEACHER'S INTERACTION WITH CURRICULUM RESOURCES TO SUPPORT MULTILINGUAL STUDENTS' LEARNING IN MATHEMATICS

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INTRODUCTION

Globalization and immigration have increased the number of multilingual students in Norwegian schools, yet research on bilingual teachers, who work with these students, is sparse (Daugaard & Dewilde, 2017). Bilingual teachers (BLTs) use both students' mother tongue and Norwegian to support learning in subjects like mathematics. Still, our understanding of what a BLT's work of teaching in mathematics entails is limited. Therefore, the research question is: *How does a bilingual teacher interact with and adapt curriculum resources in mathematics?*

THEORETICAL FRAMING

How a BLT interacts with curriculum resources (CR) can be understood in the context of a mathematics teacher's design capacity (Brown, 2009). According to the Knowledge for Mathematical Teaching framework (Ball et al., 2008), "Knowledge of Content and Students" (KCS), and "Knowledge of Content and Teaching" (KCT), are essential for selecting tasks and representations, as well as for anticipating the potential difficulties students may encounter in understanding them. Justnes and Gättschmann (2023) found that common teaching strategies in mathematics often reduce cognitive demands for multilingual students, but that a BLT employed strategies, such as designing alternative contexts, that help maintain cognitive demands. How teachers balance challenge and support for their students, which is crucial for managing cognitive demands, could be understood with Marianis' quadrant model (Mariani, 1997). The model categorises teaching strategies into 1) High challenge, High support, 2) High challenge, Low support, 3) Low challenge, High support, and 4) Low challenge, Low support. The frameworks support the understanding of how BLT interacts with CR, balancing between challenge and support.

METHODS

Data consisted of observations and interviews with a BLT in a Norwegian primary school. The initial interview and observation occurred a year before the second interview and were analysed via thematic analysis (Creswell, 2018).

This poster presents findings from a single case study. It is particularly relevant due to the BLT's 25 years of experience providing bilingual math support across various

primary schools and levels. To further explore BLTs' interactions with CR in mathematics, 20 additional BLTs are recruited for interviews in spring 2025.

PRELIMINARY RESULTS AND REFLECTIONS

The interactions of the BLT with the CR occur in two stages: "design *for* teaching" and "design *in* teaching." In the first stage, the BLT reviews the CR that the main teacher intends to use and anticipates potential difficulties that students may face, in line with KCS and KCT (Ball et al., 2008). The BLT then designs alternative contexts, wording and visual aids, tailored to meet the needs of multilingual students. In the second stage, the BLT makes real-time adjustments during the lesson, based on observations and student feedback, translating terms into the students' mother tongue, simplifying complex language, and providing visual support. Throughout both stages, the BLT aims to balance high challenge with high support (Mariani, 1997).

The BLT finds the CR often falls short in supporting multilingual learners and feels confident in his design choices in both stages. The "invisible work" of design *for* and *in* teaching for multilingual learners often falls to BLTs and is crucial for ensuring participation in mathematics, a key aspect of inclusion (Roos, 2019). Collaborative efforts between teachers and BLTs can enhance teachers' design capacity towards more inclusive practices in mathematics (Brown, 2009).

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INTRODUCTION AND EXPANSION OF MULTIDIGIT NUMBERS: A COMPARATIVE ANALYSIS OF KOREAN AND JAPANESE ELEMENTARY MATHEMATICS TEXTBOOKS

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RESEARCH BACKGROUND

The Hindu-Arabic numeral system, based on the base-10 system, operates on the principle that the value of a digit is determined by both its face value and its position within a number (Lozin & Pinhas, 2025). Ong and Kawata (2023) highlight that the systematic organization of East Asian number systems, grounded in the base-10 concept, enables children in East Asia to easily remember number words by grouping them into tens and units. Moreover, Korea and Japan, both part of the Chinese character cultural sphere, use a numeration system that reads and segments multidigit numbers into four-digit units. To facilitate reading large numbers in four-digit groupings, Korea and Japan teach numbers like 10^4 (만/万), 10^8 (억/億), and 10^{12} (조/兆) using unique numerical concepts and their own numeration systems. However, research examining curriculum resources related to this content is scarce. Thus, this study aims to conduct a comparative analysis of Korean and Japanese textbooks to deepen the comprehension of multidigit numbers. This research seeks to offer educational implications for teaching of multidigit numbers and aims to enhance the global understanding of instruction related to the Asian number system.

To deepen the understanding of multidigit numbers, our research question is as follows: Which numbers are introduced at what Grade levels, how are they introduced, and what are the similarities and differences between them?

METHODS

This study analysed the research question according to Grade levels in mathematics textbooks used for Grades 1 through 4 in Korea and Japan, focusing on textbook pages. Qualitative research methods were utilized, comparing one type of textbook per country and Grade level. Given that both Korea and Japan are top-ranked countries in TIMSS 2023 (von Davier et al., 2024), a comparison of their mathematics textbooks provides valuable insights. Furthermore, both textbooks have been revised recently.

RESULTS AND IMPLICATIONS

The textbooks of both countries exhibited similarities in content and Grade distribution but also notable differences. First, the numbers introduced regarding place value were identical in both countries: 10, 10^2 , 10^3 , 10^4 , 10^8 , 10^{12} . This reflects the incorporation

of commonly used multidigit numbers into the textbooks of both Korea and Japan. A notable shared feature is the use of specific four-digit place-value units— 10^4 (만/万), 10^8 (억/億), and 10^{12} (조/兆)—rather than pure numeric strings. This convention facilitates a concise representation of multidigit numbers and establishes a fundamental scale for subsequent mathematical learning. Second, although both countries introduce and expand multidigit numbers across Grades 1 through 4—suggesting a similar overarching timeline in instruction—there are differences in the specifics. For example, Korea takes an intensive approach in Grade 4 to deepen students’ understanding of multidigit relationships. This is achieved by comprehensively introducing key place-value units from 10^4 to 10^{12} and using a vertical model that extends to 10^{15} to reinforce the ten-times multiplicative structure (Pang et al., 2025). In contrast, Japan mitigates the learning load by segmenting the introduction of these numbers across three Grade levels. This focuses on the recursive structure of the 10^4 unit and promotes flexible transformations within a narrower numerical range (Shimizu et al., 2024). While based on a shared number system, Korean and Japanese mathematics textbooks have evolved distinct approaches reflecting their respective pedagogical aims. This study suggests a potential pedagogical synthesis: Integrating the Korean comprehensive approach—where numbers from 1 to a quadrillion are scaled by successive powers of 10—with the Japanese emphasis on four-digit increments may highlight the overall numerical relationships while revealing a recurring four-digit structural pattern. Such a hybrid approach may enhance students’ conceptual understanding of multidigit number expansion within a place-value framework.

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ENHANCING GEOMETRIC UNDERSTANDING IN STEM EDUCATION THROUGH INTEGRATED 3D MODULES FOR PRE-SERVICE MATHEMATICS TEACHERS

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INTRODUCTION AND OBJECTIVES

Advancements in educational technology are reshaping STEM education by fostering deeper engagement and understanding of complex concepts, particularly within mathematics (Wu et al., 2013). As an example, 3D printing has proven valuable in making abstract mathematical ideas more tangible, directly supporting pre-service mathematics teachers (PSETs) in visualizing and interacting with concepts traditionally difficult to grasp in two-dimensional formats (Segerman, 2016; Tejera et al., 2025). In this study, we explored how PSETs engaged with two innovative, integrated STEM modules—Crystal Lattices and Astronomical Distance. These modules were specifically designed to help PSETs better understand three-dimensional geometric concepts, an area that often poses challenges for students (Ferrara & Mammana, 2014).

The poster presents a study that addresses the research question: To what extent do integrated STEM modules affect pre-service elementary teachers' geometric learning? The study illustrates how technology, such as 3D printing, can improve pre-service teachers' learning experiences. Specifically, the study investigates how integrated STEM modules enhance understanding of complex geometric concepts and strengthen spatial reasoning in three dimensions. This includes how to represent 3D objects, how these objects are structured spatially, how their mathematical properties are conceptualized, and how they are measured (Pittalis & Christou, 2010).

METHODOLOGY

A qualitative method was utilized for this study. Thirty-nine PSETs enrolled in a mathematics content course at a large U.S. university were recruited to participate. Their written critical reflections were analysed to gain deeper insight into their learning through the integrated STEM modules: (i) *Crystal Lattices Module*: this module enables PSETs to explore the geometric structure and spatial relationships of crystal formations, which is supposed to deepen understanding of symmetry and three-dimensional space; (ii) *Astronomical Distance Module*: by using 3D models to illustrate astronomical scales and distances, this module is supposed to help PSETs develop spatial reasoning skills essential for understanding vast scales and relative

positioning in space (Liang & Wang, 2023). Kolb's learning cycle (Kolb, 1984) was adapted as the guiding framework to analyse and document PSETs' learning experiences as reflected in their writing.

FINDINGS

Our findings suggest that the integration of 3D modules in STEM education not only enriches PSETs' mathematical learning but also cultivates a deeper conceptual understanding of geometric relationships through visualization and hands-on exploration. By engaging with 3D modelling and printing tasks, PSETs develop spatial reasoning skills and connect abstract mathematical representations to tangible, real-world contexts. These experiences appear to enhance their confidence and pedagogical readiness to design and implement geometry lessons that promote inquiry, creativity, and problem-solving in their future classroom. (Liang & Wang, 2023).

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USING REDESIGNED TEXTBOOK ACTIVITIES TO TEACH THE PROPERTIES OF EVEN AND ODD NUMBERS

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OVERVIEW

Blanton et al. (2022) examined the development of young learners' reasoning about even and odd numbers, highlighting the importance of engaging students in valid mathematical arguments about general claims. According to Kim and Pang (2023), the timing of instruction on even and odd numbers varies internationally: Finnish, Singaporean, and Japanese textbooks introduce this concept in grades 1, 2, and 5, respectively, with Finnish textbooks explicitly addressing the properties of even and odd numbers. In contrast, Korean textbooks define even and odd numbers in Grade 1, based on pairing, but do not further discuss these properties in subsequent grades. Kim and Pang (2023) emphasize the need for explicit instruction on these properties. This study aims to enable students to explore and justify the properties of even and odd numbers by redesigning textbook activities in alignment with the research objectives. The research questions are as follows: (a) What core activities are included in the redesigned textbook? (b) How do students justify the properties?

METHODS

To teach the properties of even and odd numbers, we implemented redesigned activities in a fourth-grade Korean classroom. Since division with remainders had been introduced in Grades 3 and 4, students were expected to use definitions of even and odd beyond simple pairing. In addition, the "Finding Patterns" unit, which covers addition patterns, was adapted to emphasize these properties. For the redesign process, activities from Finnish textbooks and relevant studies (Isler et al., 2013; Lin & Tsai, 2016) were analysed, and three main activities were extracted. These activities were then developed for two 40-minute sessions. To examine how students justified their reasoning, we collected the activity sheets and recorded the class sessions. The activity sheets were categorized according to two criteria: (a) which definitions of even and odd were used, and (b) the justification levels (specific numbers or generalized expressions) based on Blanton et al. (2022).

RESULTS

Three Redesigned Textbook Activities

Three core activities were developed for the redesigned textbook. The first activity,

“Exploring Even and Odd Sums in an Addition Table” drew on Finnish textbooks and the work of Lin and Tsai (2016). Students used different colors to differentiate even and odd sums, which allowed them to notice patterns intuitively and infer properties inductively. The second and third activities, “Justifying Properties with a Partner” and “Whole-Class Discussion”, were adapted from Isler et al. (2013), who observed that students begin to use generalized expressions once prompted to consider the universal applicability of their findings. In these activities, students wrote justifications on activity sheets with their partners and then shared their reasoning in whole-class discussions.

Student Justifications Using the Redesigned Textbook Activities

Analysis of the students’ activity sheets revealed four main definitions: “Pairing”, “Multiples of Two”, “Division by Two”, and “Other.” Across both sessions, most students used the pairing definition, which they had learned in first grade. The number of generalizations rose from three in the first session to six in the second session. During discussions, students shared their justifications in various ways, building on one another’s ideas. As the discussions progressed, students who initially relied on specific numerical examples began to generalize the properties of even and odd numbers. Examples of these justifications will be included in the poster.

CONCLUSIONS

The redesigned textbook activities provided fourth-grade students with opportunities to justify the properties of even and odd numbers. By sharing and discussing their justifications, students progressed toward using more generalized expressions. Since students rely on definitions of even and odd for their justifications, it is crucial to offer them opportunities to explore multiple definitions.

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THE INTERPLAY OF EDUCATIVE FEATURES IN PRINTED AND DIGITAL CURRICULUM RESOURCES

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OVERVIEW

Curriculum resources that provide specific information about what and how to teach in lessons support student learning while also foster teacher learning (Stein et al., 2007). With the advent of the digital age, the form of *educative* curriculum resources is expanding from printed to digital formats. Digital curriculum resources are characterized by their flexibility in design and modification, allowing users to explore vast resources according to their own needs and preferences (Remillard, 2016). The interaction between teachers and digital curriculum resources supports teachers' professional learning (Gueudet & Pepin, 2020). Pang et al. (2023) explored the ways in which printed educative curriculum resources support teacher learning and found that teacher guidebooks contain a variety of educative features but also exhibit strengths and weaknesses. Building on this background, this poster explores how the educative features of printed resources have been expanded, supplemented, maintained, or augmented through interplay between printed and digital curriculum resources.

METHODS

This poster analyses the entire set of mathematics curriculum resources within digital curriculum platforms operated by eight publishers of elementary school mathematics textbooks in Korea. To analyse the data, we first categorized the elements into those that support teacher learning and those that support teaching convenience. The filtered elements related to educative features were categorized into seven dimensions and additional categories (Pang et al., 2023), a framework for analysing educative features of mathematics curriculum resources in Korea. To explore the interplay of educative features between printed and digital resources, we categorized the educative features of teacher guidebooks into strengths and weaknesses, and analysed whether these strengths and weaknesses were expanded, supplemented, maintained, or augmented when combining them with digital resources (see Table 1).

Types	Description
Expansion	The strengths of the educative features of the teacher guidebooks are enhanced by combining them with digital curriculum resources.
Supplement- ation	The weaknesses of the educative features of the teacher guidebooks are mitigated by combining them with digital curriculum resources.

Maintenance	The strengths and weaknesses of the educative features of the teacher guidebooks are maintained even when combining them with digital curriculum resources.
Augmentation	Educative features that were not found in teacher guidebooks are newly introduced in digital curriculum resources.

Table 1: Types of interplay of educative features in printed and digital curriculum resources.

RESULTS AND IMPLICATIONS

In this poster, we present typical examples of each type of interplay between educative features and explore the remaining weaknesses in terms of teacher learning and the issues that emerge from digital resources, even though the format of curriculum resources has expanded from printed to digital. Due to space limitations, we use the Supplementation Type as an example. Teacher guidebooks provide information on manipulative materials and technology use for only a few topics, which is insufficient. However, digital curriculum resources richly offer a variety of digital manipulatives across several content strands, supplementing teacher learning about the use of manipulative materials and technology. Nevertheless, some of the digital manipulatives have functional limitations, such as the inability to change pre-determined conditions, which prevent students from freely exploring, or automated operations and answers with simple clicks that replace important mathematical thinking and processes. Meanwhile, the teacher guidebooks richly provide assessment items but do not provide enough examples of student responses or feedback for each item. However, the digital curriculum resources supplement teacher learning about assessment in mathematics by providing analyses of individual achievement and individualized assessment items, based on an AI-based individualized assessment system. However, the way the AI system provides feedback is limited to providing similar, easier, or harder follow-up items. These specific examples of each type of interplay are expected to provoke discourse on what needs to be paid attention to when combining printed and digital resources to enhance teacher learning.

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USING TOOLS FROM TIME SERIES ANALYSIS FOR ANALYSING TASKS IN TEXTBOOKS

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INTRODUCTION: TEXTBOOK TASKS AS A TIME SERIES

Rezat and Strässer (2015) described three types of research on textbooks: influences on textbook design (curriculum perspective), impacts of textbooks (user perspective), and the textbook itself (content perspective). They concluded that the latter typically analyse the content of a textbook. Now, a large proportion of the content in several textbooks are tasks that reflect curricular learning goals (CLG). Petersson et al. (2020) used tools from time series analysis to graphically display how CLG are distributed within a textbook. Namely, moving averages give a momentary view of the proportion of CLG, and cumulative graphs give an accumulated (relative or absolute) frequency of CLG.

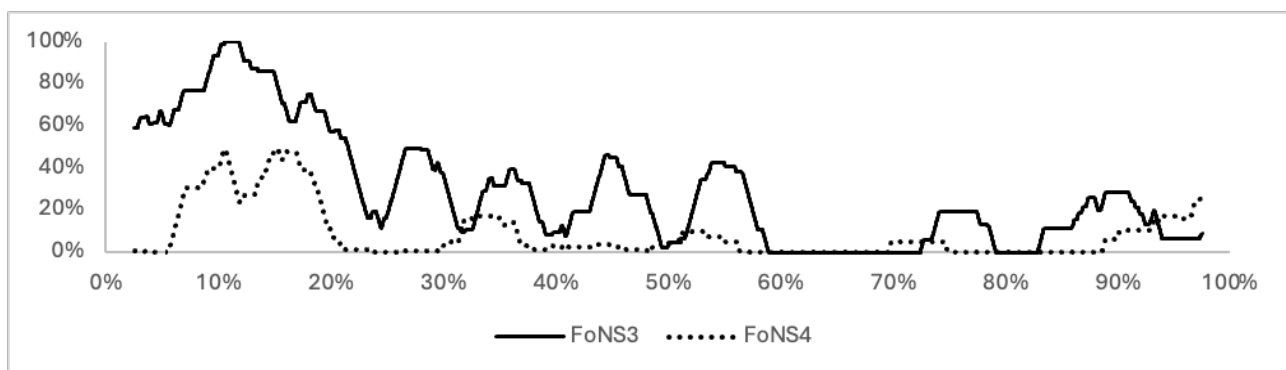


Figure 1. Moving averages of how two learning goals are distributed within a book.

As an example, Figure 1 displays how the two learning goals that Petersson et al. (2020) label FoNS3 (cardinality of objects; counting) and FoNS4 (discrimination between numbers; /in-/equality) in primary school year 1 occur as peaks in the beginning of the book and then are repeated a few times from start at 0% until the end at 100% of the tasks in the textbook. The height of a peak, say 25%, tells that about a quarter of the tasks around that peak contains that CLG. While there are methods for investigating the similarity of two textbooks visually (Petersson et al., 2020) and quantitatively (Petersson et al., 2022), the aim of this study is to develop a method for investigating

how two CLG occur in the proximity of each other within a textbook, measured through the conditional co-occurrence ($Y|X$) between two different CLG.

METHOD: CONDITIONAL OCCURRENCE OF LEARNING GOALS

A methodological challenge with exact co-occurrence is that two CLG might not occur in the same task but instead in neighbour tasks. To catch this neighbour co-occurrence,

the conditional occurrence $(\hat{Y}|\hat{X}) = \sqrt{\Sigma(\hat{Y}_k \cdot \hat{X}_k) / \Sigma(\hat{X}_k^2)}$ was calculated for short moving averages $\hat{X}_k = \frac{1}{3}(X_{k-1} + X_k + X_{k+1})$ instead of on the original data. While Figure 1 shows that FoNS3 occurs more often than FoNS4 and at similar positions in the time series of textbook tasks, Table 1 quantifies these conditional occurrences for the investigated CLG. For example, the value 0.45 in column FoNS3 in Table 1 responds quantitatively to the following question: For tasks containing FoNS3, to what extent does FoNS4 occur in their neighbourhood?

	FoNS3	FoNS4
FoNS3	1.00	0.84
FoNS4	0.45	1.00

Table 1: Conditional occurrence.

IMPLICATIONS AND REFLECTIONS

Conditional occurrence seems not to have been investigated previously. It quantifies how two different CLG occur in the proximity of each other. Further, it indicates how complex in composition of different CLG a task is. Hence, conditional occurrence can be an inspiration in the design of both textbook tasks and teacher-led activities with the aim to combine CLG suitably and in possibly innovative ways. This tool for analysing the conditional occurrence of CLG could further extend the analysis in Petersson et al. (2020) and Petersson et al. (2022) of the implemented and intended curriculum as seen through textbooks and teacher-led activities.

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LEARNING OPPORTUNITIES FOR LENGTH MEASUREMENT IN TAIWANESE AND GERMAN ELEMENTARY TEXTBOOKS

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Length measurement plays a prominent role in measurement education in most countries (Buys & de Moor, 2008; Clements, 2003). Despite this, poor learning outcomes have been reported, particularly in relation to the measurement concepts underlying measurement procedures (Smith et al., 2011). In line with international studies (Lee & Smith, 2011), our former textbook analysis (Ruwisch & Huang, 2018) showed that procedural requirements clearly exceed conceptual learning opportunities. In 2018, the integration of real-life scenarios and measurement skills was strengthened in the new Taiwanese curriculum, but not in Germany. Thus, the question arose as to whether differences in learning opportunities for understanding length could be identified in the textbooks of the two countries.

LENGTH LEARNING

Length understanding, length measurement, and length estimation are thought to be complex concepts with different conceptual underpinnings as well as concrete actions. There seems to be a great consistency in these aspects, even if some researchers combine them differently than others (Barrett et al., 2017; Clements, 2003). Since our research groups looked and still look for different length estimation abilities in both countries (e.g., Hoth et al., 2022), a new textbook analysis on the learning opportunities may give hints for interpreting differences in the results of the length estimation test.

RESEARCH QUESTIONS

Q1: How long is the intended instructional time on length learning in both countries?

Q2: Do procedural aspects still dominate the Taiwanese and the German curriculum?

Q3: Which concrete differences in the conceptual and procedural opportunities to learn length understanding, measurement, and estimation can be observed between the textbooks of the two countries? What changes can be observed in our first analysis?

METHOD

The same textbook series—three Taiwanese from the publishers with the highest market share; four German, two of which are widely used, one of which can be considered challenging, and one of which is designed for 'individual learning', and the same analysis method as in the previous analysis was used (Ruwisch & Huang, 2018). We included the pages that were signed as units of length measurement, looked up their

intended teaching time with respect to the distribution plans of the publishers, and used the coding scheme of Lee and Smith (2011, 689). Each sentence, question, or demand served as a coding unit and was assigned up to two codes each.

RESULTS

First, preliminary results show that the intended teaching time for length learning changed in different directions. Whereas all Taiwanese textbooks increased the time for length comprehension by a total of 240 to 480 minutes, the time for German length measurement units remained nearly the same. One German textbook series even reduced the learning opportunities.

Concerning the second question, procedural requirements dominate in both countries: the German textbooks present three times as many, whereas the Taiwanese textbooks show about twice as many procedural requirements as conceptual ones.

With respect to the third question, the poster will present examples of the main conceptual and procedural categories and their numbers in detail. In both countries, the most prominent procedural categories are *unit conversion* and *calculating lengths given in symbolic representation*. Whereas in all German textbooks the most dominant conceptual category is *benchmark learning*, *units can be converted* dominates as the most prominent conceptual category in all Taiwanese textbooks.

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FROM ARITHMETIC TO ALGEBRA: ANALYSING PRAXEOLOGICAL CHANGE VIA THE INTRODUCTION OF A JAPANESE TEXTBOOK IN DENMARK

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CURRICULUM AND TEXTBOOK MATERIAL AS A RESOURCE

Japan and Singapore consistently achieve top results in international mathematics assessments such as PISA (OECD, 2023). These high-performing nations have in common the fact that their mathematics textbooks are extensively based on empirical studies and are subject to a system of official quality screening (Yoshikawa, 2008). It is generally acknowledged that teachers rely on textbook material to guide their teaching (Lloyd et al., 2010). Therefore, it is interesting to investigate whether empirically tested Japanese textbooks could improve teaching in Denmark. However, what are the conditions and constraints of using such materials in a different national curriculum and school context?

ATD AND THE NOTION OF PRAXEOLOGY AS A THEORETICAL FRAMEWORK

In The Anthropological Theory of Didactics (ATD), we use the notion of praxeology to model school mathematics activities. A praxeology consists of task types, techniques, technology, and theory (Bosch, 2015). Praxeologies are constructed over time through activities and can be seen as dynamic entities that can be represented and remodelled. We use the concept of praxeological change (Putra, 2019) to describe these mathematical and didactic praxeological reconstructions. In light of this framework, the present study seeks to explore what kinds of praxeological change emerge when Danish mathematics teachers work with a Japanese textbook chapter on algebraic expressions in Danish lower secondary classrooms.

METHODOLOGY

The textbook material Junior High School Mathematics: 1 (Isoda & Tall, 2019) is based on systematic empirical research, and the chapter “Algebraic Expressions” was translated into Danish and given to the teachers without further instructions. To answer the research question, we used classroom observations of three Danish grade 8 classes (age 14) with photos of the whiteboards and students’ notebooks. After the lessons, teacher interviews were conducted based on the photo elicitation interview method (Harper, 2002).

RESULTS AND CONCLUSION

The explicit description of the use of algebraic notation and the distinction between laws and conventions are key elements of the textbook. This potential is not being fully realized, but there are signs in the interviews that a praxeological change is emerging for the teachers. The opening problem of the textbook chapter, a repeated arithmetic expression, becomes the introduction of a general algebraic expression, which is the first step toward a praxeological change from arithmetic to algebra. The problem provides an opportunity to work with key techniques for simplifying linear equations, technological elements such as the use of algebraic expressions with letters, and key theoretical elements such as the distributive property.

The Japanese textbook serves as a valuable resource for mathematics teachers, making algebraic notation conventions explicit and reducing their ad hoc nature for students. However, challenges arise in effectively distinguishing and teaching algebraic conventions, laws, and definitions in a theoretically coherent way as suggested in the textbook. Teacher interviews indicate that these challenges can be mitigated over time, allowing students to benefit from the explicit theoretical insights provided by the textbook. This suggests potential for realizing a praxeological change from algebra dominated by an ad hoc technology to algebra based on a few and rich theoretical connections.

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