Polynomial Approximation and Pluripotential Theory

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Dissertation submitted in partial fulfillment of a
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Abstract

We study approximation of holomorphic functions by polynomials of several complex variables with exponents restricted to the dilates of a fixed compact convex set $S \subset \mathbb{R}^n_+$ that contains zero. This restriction defines a graded subring $\mathcal{P}^S(\mathbb{C}^n)$ of the polynomials in $\mathbb{C}^n$. We study the properties of the weighted extremal functions $\Phi^S_{K,q}$ and $V^S_{K,q}$ associated to these polynomial rings. These functions are generalizations of the Siciak function and the pluricomplex Green function. We prove a generalization of the Siciak-Zakharyuta Theorem, that $V^S_{K,q} = \log \Phi^S_{K,q}$ on $\mathbb{C}^{*n}$. We also prove versions of the Runge-Oka-Weil Theorem on the approximation of holomorphic functions by polynomials from $\mathcal{P}^S(\mathbb{C}^n)$.

Ágrip

Við skoðum nálganir á fáguðum föllum með margliðum af mörgum tvinnbreytistærðum með vísa sem einskorðaðir eru við stríkkanir kúpts þjappaðs mengis $S \subset \mathbb{R}^n_+$ sem inniheldur núll. Þessi einskorðun skilgreinir stigaðan hlutbaug $\mathcal{P}^S(\mathbb{C}^n)$ í margliðunum á $\mathbb{C}^n$. Við skoðum eiginleika útgildisfallanna $\Phi^S_{K,q}$ og $V^S_{K,q}$ sem ákvarðast af þessum margliðubaugum. Þessi föll eru alhæfingar Siciak fallsins og Green fallsins í tvinnfallagreiningu af mörgum breytistærðum. Við sönnum alhæfingu Siciak-Zakharyuta setningarinnar, sem kveður á um að $V^S_{K,q} = \log \Phi^S_{K,q}$ á $\mathbb{C}^{*n}$. Auk þess sönnum við útgáfur af Runge-Oka-Weil setningunni um nálganir á fáguðum föllum með margliðum úr $\mathcal{P}^S(\mathbb{C}^n)$.

To the memory of
Hjalti Þór Ísleifsson.

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Introduction

The holomorphic polynomials of one complex variable can be characterized by their growth. One way to study the growth of functions is by considering $\log^+|f|$ where $\log^+ = \max\{0, \log\}$. If f is an entire function and $\log^+|f(z)| \leq m\log^+|z| + c$ for all $z \in \mathbb{C}$ and some constant c, then f is a polynomial of degree $\leq m$. If p_1 and p_2 are two polynomials of the same degree m, then $\log^+|p_1| - \log^+|p_2|$ is bounded, so $\log^+|p_1|$ and $\log^+|p_2|$ have the same growth. We are interested in understanding the relationship between the degree and growth of polynomials of several complex variables.

A polynomial $p(z) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha$ in the variable $z \in \mathbb{C}^n$ is of standard degree $\leq m$ if all $\alpha \in \mathbb{N}^n$ such that $a_\alpha \neq 0$ have $|\alpha_1| + \cdots + |\alpha_n| \leq m$. Let $\Sigma \subset \mathbb{R}^n_+$ denote the unit simplex $\Sigma = \mathrm{ch}\{0, e_1, \ldots, e_n\}$, where ch is the convex hull and e_j are the standard unit vectors. Then a polynomial p of degree $\leq m$ can also be presented as $p(z) = \sum_{\alpha\in m\Sigma\cap\mathbb{N}^n} a_\alpha z^\alpha$. For two distinct polynomials p_1, p_2 on $\mathbb{C}^n$, $n \geq 2$ of degree $m \geq 1$, then $\log^+|p_1| - \log^+|p_2|$ is not necessarily bounded. We could take for example $p_1(z) = z_1^m$ and $p_2(z) = z_2^m$. Instead we need some further tools to study the growth of polynomials of several complex variables. The set $\mathrm{ch}\{\alpha \in \mathbb{N}^n\,;\, a_\alpha \neq 0\}$ is called the *Newton polytope* of the polynomial p. If p_1 and p_2 have different Newton polytopes, then $\log^+|p_1| - \log^+|p_2|$ is unbounded.

The number of roots of a generic polynomial of one variable equals its degree. By generic we mean that its coefficients are chosen from some particular open dense set. There is no sense in counting the number of roots of a polynomial of several variables as it is generically infinite. But we can count the simultaneous roots of a system of n polynomials of n variables. The Bernstein–Kushnirenko Theorem [9] from 1975 states that if $p_1, \ldots, p_n$ are polynomials with Newton polytopes $S_1, \ldots, S_n$, and such that their coefficients are generic, then the system of equations $p_1 = \cdots = p_n = 0$ has precisely $n!V(S_1, \ldots, S_n)$ solutions, where $V(S_1, \ldots, S_n)$ is the mixed volume of the sets $S_1, \ldots, S_n$. In particular if $S = S_1 = \cdots = S_n$, the number of roots is simply $n!\,\mathrm{vol}(S_1, \ldots, S_n)$.

We would like to not only gauge the number of solutions to such a system of equations, but also where those solutions are expected to be located. Bayraktar [4] studies systems of random polynomials whose Newton polytopes are dilates mS_j of some fixed polytopes S_j, $j = 1, \ldots, n$ with non-empty interior and vertices in $\mathbb{Z}^n$. He provides a description of the expected distribution of the zeros of such a system as m tends to infinity. Again to simplify, we focus on the case when $S = S_1 = \cdots = S_n$.

In order for the problem to be well posed, the space $\mathcal{P}_m^S(\mathbb{C}^n)$ of polynomials whose Newton polytope is contained in mS must be endowed with a probability measure, which is chosen with respect to a non-pluripolar compact set $K \subset \mathbb{C}^n$ and an admissible weight $q\colon K \to \mathbb{R}\cup\{+\infty\}$. A weighted global extremal function $V_{K,q}^S$ is then defined emulating the pluricomplex Green function which is an important tool used in potential theory. Bayraktar's findings essentially describe how the normalized point masses at the solutions of a system $p_1 = \cdots = p_n = 0$ of random $p_j \in \mathcal{P}_m^S(\mathbb{C}^n)$, $j = 1, \ldots, n$ tend to the Monge-Ampère measure $(dd^c V_{K,q}^{S*})^n$ as the dilation m tends to infinity. Understanding the properties of the function $V_{K,q}^S$ is the main topic of this thesis. A fundamental building block in Bayraktar's result is a generalization of the Siciak-Zakharyuta Theorem for the extremal function $V_{K,q}^S$, which is of independent interest on its own. Generalizing that result is the main objective of Paper II.

Even though a Newton polytope of a single polynomial is a polytope with vertices in $\mathbb{Z}^n$, one may as well consider polynomial spaces $\mathcal{P}_m^S(\mathbb{C}^n)$ where the set S in question is more general. Trefethen [43] presents a case when it is practical to study the polynomials graded with their *euclidean degree*, which is the grading of polynomials obtained by studying $\mathcal{P}_m^T(\mathbb{C}^n)$ where T is the intersection of the euclidean unit ball $B(0,1)$ with $\mathbb{R}_+^n$. This case in question regards the uniform approximation of holomorphic functions f in certain neighborhoods of the hypercube $[-1,1]^n$ by polynomials. The best approximation $\inf\{\|f-p\|_{[-1,1]^n}\,;\, p \in \mathcal{P}_m^S(\mathbb{C}^n)\}$ in this case decays equally rapidly for $S = T$ as it does for $S = [0,1]^n$ which represents the maximal degree. The dimension of $\mathcal{P}_m^T(\mathbb{C}^n)$ is the number of points in $B(0,m) \cap \mathbb{N}^n$ which is smaller than the dimension of $\mathcal{P}^{[0,1]^m}(\mathbb{C}^n)$ which is $(m+1)^n$. Any algorithm to find the polynomial $p \in \mathcal{P}_m^S(\mathbb{C}^n)$ minimizing $\|f-p\|_{[-1,1]^n}$ should therefore be more efficient for $S = T$ than for $S = [0,1]^n$.

The authors Bayraktar, Bloom, Bos, Hussung, Levenberg, Lu and Perera published a series of papers titled *Pluripotential theory and convex bodies* [6, 7, 8, 14, 25, 30], where as the name suggests, S is assumed to be a convex body; a convex set with non-empty interior. Notably [14] presents a version of the Bernstein-Walsh Theorem which quantifies the uniform approximation of holomorphic functions by polynomials from $\mathcal{P}^S(\mathbb{C}^n)$, where $S \subset \mathbb{R}_+^n$ is assumed to be a convex body containing a neighborhood of the origin. Magnússon, Sigurðsson and Snorrason [28] generalize this result to allow S to not necessarily contain a neighborhood of the origin. If S does not contain a neighborhood of the origin, then $\mathcal{P}^S(\mathbb{C}^n)$ does not contain all the polynomials in $\mathbb{C}^n$. In order to quantify the approximation in this case, one must also verify the existence of an approximation. Paper III regards the existence of a uniform approximation of holomorphic functions by polynomials from $\mathcal{P}^S(\mathbb{C}^n)$. It contains a generalization of the Runge-Oka-Weil Theorem, which relies on our Siciak-Zakharyuta result in a subtle way.

The contents of this thesis and the forthcoming Ph.D. thesis of Bergur Snorrason is the product of a research project carried out by these two candidates and

their supervisors Benedikt Steinar Magnússon and Ragnar Sigurðsson. The goal of this studywas to provide a thorough analysis of the analogues of foundational results from pluripotential theory for the extremal functions $V^S_{K,q}$ and the corresponding Siciak functions $\Phi^S_{K,q}$, with the assumptions on the set $S \subset \mathbb{R}^n_+$ as general as is natural. Paper I is mostly self-contained for a reader with some background knowledge of pluripotential theory. The rest of this introductory section will provide the necessary background and introduce the classical results from pluripotential theory, using Klimek's excellent book [23] as the main reference.

1 The Dirichlet problem and Perron method

Laplace's equation $\Delta h = 0$ appears in many branches of physics and describes for example heat at equilibrium. The twice continuously differentiable solutions h to Laplace's equation on a domain $\Omega \subset \mathbb{R}^n$ satisfy

$$h(x) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x,r)} h \, d\sigma, \tag{1.1}$$

for any euclidean ball $B(x,r)$ centered at x of radius r that is contained in Ω, where σ is the $(n-1)$-dimensional surface measure and ω_n is the σ-volume of $B(0,1)$. We say that a function h on Ω is *harmonic* if it satisfies (1.1). The twice continuously differentiable harmonic functions are precisely the solutions to Laplace's equation.

The value of a harmonic function at a given point is completely determined by its value on the boundary of any sphere centered at the point and contained in its domain. Then arises the question whether the same is true for any bounded region containing the point. The classical Dirichlet problem is as follows: Given a function $f\colon \partial K \to \mathbb{R}$ where K is a region in $\mathbb{R}^n$, does there exist a unique continuous v on K that is harmonic in the interior of K and $v = f$ on ∂K? The answer is affirmative, if K has a sufficiently nice boundary. In that case, K is said to be *regular* with respect to the Dirichlet problem. If there exists a solution to the Dirichlet problem, it is always unique by the maximum principle [23], Theorem 2.2.5. Whether a solution exists or not, one can always write down a candidate for a solution by the Perron method, described here below. But first we need the notion of subharmonic functions.

The *upper regularization* of a function $u\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ defined on a region $\Omega \subset \mathbb{R}^n$ is a function $u^*\colon \overline{\Omega} \to \mathbb{R} \cup \{-\infty\}$ defined by

$$u^*(z) = \overline{\lim_{\Omega \ni w \to z}} u(w).$$

We say that u is *upper semicontinuous* if $u = u^*|_\Omega$. An upper semicontinuous u is *subharmonic* if for every closed $\overline{B}(x,r) \subset \Omega$ we have

$$u(x) \leq \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x,r)} u \, d\sigma.$$

A twice continuously differentiable $u \in \mathcal{C}^2(\Omega)$ is subharmonic if and only if $\Delta u \geq 0$ by [23], Theorem 2.5.1. Denote the set of subharmonic functions on Ω by $\mathcal{SH}(\Omega)$, excluding any function that is identically $-\infty$ on any component of Ω. This exclusion ensures that all members of $\mathcal{SH}(\Omega)$ are locally integrable [19], Corollary 3.2.8.

The *Perron function* of a bounded domain $\Omega \subset \mathbb{R}^n$ and $f \colon \partial\Omega \to \mathbb{R}$ is defined by

$$v(x) = \sup\{u(x)\,;\, u \in \mathcal{SH}(\Omega),\, u^*|_{\partial\Omega} \leq f\}.$$

When the Dirichlet problem has a solution, then the Perron function is that solution. The Perron function is always harmonic in the interior of Ω, but it may not reach the desired values f on the boundary.

The Laplace operator in the complex plane satisfies the identity

$$\Delta u = 4\frac{\partial^2 u}{\partial z \partial \bar{z}}.$$

When regarding functions of several complex variables, we want to regard an operator that generalizes the Laplace operator in one complex variable, and is not oblivious to the complex structure on $\mathbb{C}^n$. This inspires us to consider the following form-valued operator

$$dd^c u = 2i \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k.$$

We want to regard functions on domains Ω in $\mathbb{C}^n$ such that the form $dd^c u$ is positive in some sense. A way to interpret that is that the *Levi-form*

$$\langle \mathcal{L}u(a)b, c\rangle = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(a) b_j \bar{c}_k, \qquad b, c \in \mathbb{C}^n,$$

should by positive-definite at every point $a \in \Omega$. The functions $u \in \mathcal{C}^2(\Omega)$ that satisfy this have a clear description, they are precisely those which are plurisubharmonic [23], Theorem 2.9.11. An upper semicontinuous function u on a domain Ω in $\mathbb{C}^n$ is *plurisubharmonic* if the function $u(a + \zeta b)$ is a subharmonic function of $\zeta \in \mathbb{C}$ where it is defined, for every $a, b \in \mathbb{C}^n$. We denote by $\mathcal{PSH}(\Omega)$ the set of plurisubharmonic functions on Ω that are not identically $-\infty$ on any component of Ω. The plurisubharmonic functions on a domain in $\mathbb{C}$ are simply the subharmonic functions.

We can find an analogy to the Perron function using plurisubharmonic functions on a bounded domain $\Omega \subset \mathbb{C}^n$. We can extend $u \in \mathcal{PSH}(\Omega)$ to the boundary $\partial\Omega$, in an upper semicontinuous way by letting

$$u^*(z) = \overline{\lim_{\substack{w \to z \\ w \in \Omega}}} u(w), \qquad z \in \partial\Omega.$$

Given a continuous $f\colon \partial\Omega \to \mathbb{R}$ we can define the Perron-Bremermann function of Ω and f as

$$v(z) = \sup\{u(z)\,;\, u \in \mathcal{PSH}(\Omega),\, u^*|_{\partial\Omega} \leq f\}.$$

The Perron function is harmonic in the interior of its domain, but what is special about the Perron-Bremermann function inside Ω?

2 Maximal plurisubharmonic functions

A subharmonic function h on Ω is harmonic if and only if for every compact $G \subset \Omega$ and a $u \in \mathcal{SH}(G) \cap \mathcal{C}(\mathring{G})$ and then $u|_{\partial G} \leq h|_{\partial G}$ implies $u|_G \leq h|_G$. This can be seen by comparing h with the solution of the Dirichlet problem on G with boundary condition $u|_{\partial G}$. The plurisubharmonic functions that play a similar role to the harmonic functions are the maximal plurisubharmonic functions. We say that $v \in \mathcal{PSH}(\Omega)$ on a domain $\Omega \subset \mathbb{C}^n$ is *maximal* if for any $u \in \mathcal{PSH}(\Omega)$ and any compact set $G \subset \Omega$ then $u|_{\partial G} \leq v|_{\partial G}$ implies $u|_G \leq v|_G$.

As the Perron function of Ω is harmonic in the interior of $\Omega \subset \mathbb{R}^n$, the Perron-Bremermann function is maximal in the interior $\Omega \subset \mathbb{C}^n$. That fact requires much work to prove, as [23] devotes the entire Chapter 4 for that purpose. A $\mathcal{C}^2$ function h on a domain in $\mathbb{C}$ is harmonic if and only if $\Delta h = 0$. In a similar way can maximal $\mathcal{C}^2$ plurisubharmonic functions also be identified through a differential operator.

Proposition 2.1 ([23], Proposition 3.1.6 and Corollary 3.1.8.) *A function $u \in \mathcal{C}^2 \cap \mathcal{PSH}(\Omega)$ on an open $\Omega \subset \mathbb{C}^n$ is maximal if and only if the Monge-Ampère operator*

$$(dd^c u)^n = \underbrace{dd^c u \wedge \cdots \wedge dd^c u}_{n \text{ times}}$$

vanishes everywhere in Ω.

If u is not $\mathcal{C}^2$ we can extend the definition of $dd^c u$ in terms of currents. It is not as easy to generalize $(dd^c u)^n$, since a wedge product of currents is not possible to define in general. For a reference on currents we refer the reader to Demailly's online book [15].

For locally bounded plurisubharmonic functions u, there is a way to give $(dd^c u)^n$ a consistent meaning. For every decreasing sequence of smooth $u_j \in \mathcal{PSH}(\Omega)$ tending to $u \in \mathcal{PSH}(\Omega)$, the weak*-limit $\lim_{j\to\infty}(dd^c u_j)^n$ will be consistent. Plurisubharmonic functions can be approximated from above by smooth plurisubharmonic functions via a convolution with a radially symmetric smoothing kernel χ. If $\chi_\delta = (1/\delta)^{2n}\chi(\cdot/\delta)$, then $u * \chi_\delta$ decreases pointwise to the function u. Therefore $(dd^c u)^n$ can be given the meaning $\lim_{\delta\to 0}(dd^c(u * \chi_\delta))^n$. Locally bounded $u \in \mathcal{PSH}(\mathbb{C}^n)$ are maximal if and only if $(dd^c u)^n = 0$ with this generalized meaning, see [23], Theorem 4.4.2.

3 Pluricomplex Green functions

The Perron and Perron-Bremermann function are both examples of extremal functions, that is functions that are defined as the supremum over some family of (pluri-)subharmonic functions. Extremal functions can also be defined on an unbounded domain, say $\mathbb{C}^n$, but then one has to restrict the growth of the functions the supremum is taken over. For example, if K is any compact set in $\mathbb{C}^n$ then the expression $\sup\{v(z)\,;\, v \in \mathcal{PSH}(\mathbb{C}^n),\, v|_K \leq 0\}$ is $+\infty$ at all points far enough away from K.

The *Lelong class* $\mathcal{L}(\mathbb{C}^n)$ consists of all functions $u \in \mathcal{PSH}(\mathbb{C}^n)$ with logarithmic growth, that is they satisfy a global bound $u \leq \log^+ |\cdot| + c_u$ for some constant $c_u \in \mathbb{R}$ only depending on u. For any subset E of $\mathbb{C}^n$ we can define the *pluricomplex Green function of the set* E by

$$V_E(z) = \sup\{u(z)\,;\, u \in \mathcal{L}(\mathbb{C}^n), u \leq 0 \text{ on } E\}.$$

One can also call this function the *Siciak-Zakharyuta function* in their honor, and that is the name we will use for its generalization, which will be introduced in the next section. The function V_E is not necessarily plurisubharmonic. However, its upper regularization $V_E^*(z) = \overline{\lim}_{w\to z} V_E(w)$ is, unless if there exists some $u \in \mathcal{PSH}(\mathbb{C}^n)$ such that $u|_E = -\infty$. In the latter case we say that E is *pluripolar*, and then V_E^* is identically $+\infty$ by [23], Corollary 5.2.2. The same corollary also states that if E is non-pluripolar, then $V_E^* \in \mathcal{L}(\mathbb{C}^n)$. If E is compact, which we then denote by $E = K$, then V_K is a member of the class $\mathcal{L}_+(\mathbb{C}^n)$ of all $u \in \mathcal{L}(\mathbb{C}^n)$ such that $u \geq \log^+ |\cdot| - c_u$ for some constant c_u dependent on u. This can be seen by taking the constant as $\sup_K \log^+ |\cdot|$.

The study of plurisubharmonic functions is called pluripotential theory, a name which hints at its heritage from physics. For $n = 1$, the function V_K has the following physical interpretation: If a cloud of electrons lies in a connected compact conductor K in the plane, then the electric potential generated by this charge at its equilibrium is V_E. For an even more physically minded reader who prefers three-dimensional space over the plane, the set conductor could be taken as an infinitely long rod whose crosscut is K. If the conductor K is in a world where there is an external electric field, that will influence the equilibrium of our charge and the resulting potential too. This motivates the study of weighted pluricomplex Green functions with the presence of an external field. If w is an external field, then one regards the weight $q = -\log w$.

Admissible weights $q\colon E \to \mathbb{R} \cup \{+\infty\}$ are defined in Paper I, Definition 4.4. We may define for any admissible q on E a *weighted Siciak-Zakharyuta function*

$$V_{E,q}(z) = \sup\{u(z)\,;\, u \in \mathcal{L}(\mathbb{C}^n), u \leq q \text{ on } E\}. \tag{3.1}$$

Let $\mathcal{P}_m(\mathbb{C}^n)$ denote the space of polynomials on $\mathbb{C}^n$ of degree $\leq m$. If $p \in \mathcal{P}_m(\mathbb{C}^n)$, then $\log|p|^{1/m}$ is plurisubharmonic and has logarithmic growth, so it is a member of $\mathcal{L}(\mathbb{C}^n)$. Here we mean $\log(|p|^{1/m})$, but the parenthesis is a cumbersome notation. A fundamental theorem in pluripotential theory is the

Siciak-Zakharyuta theorem, which states that the supremum (3.1) is reached by taking only the supremum over the subclass of $\mathcal{L}(\mathbb{C}^n)$ of functions of the form $\log|p|^{1/m}$ where p is a polynomial. A proof of the classical, unweighted Siciak-Zakharyuta theorem can be found in [23], Theorem 5.1.7. The theorem has a weighted version which is stated here.

Theorem 3.1 ([33], Theorem 2.8 of Appendix B) *If E is closed and q is an admissible weight on E, then*

$$V_{E,q}(z) = \sup\{\log|p(z)|^{1/m}\,;\, p \in \mathcal{P}_m(\mathbb{C}^n),\, m \in \mathbb{N}^*,\, u \leq q \text{ on } E\}. \tag{3.2}$$

This theorem describes in some sense the density of functions of the form $\log|p|^{1/m}$ in the Lelong class. The right hand side of (4.3) is $\log\Phi_{E,q}$ where $\Phi_{E,q}$ is the *weighted Siciak function* for q on E.

4 Extremal functions associated to a convex set

The logarithmic growth of $\mathcal{L}(\mathbb{C}^n)$ is not the only kind of growth that can make an expression like (3.1) make sense. We have some freedom to choose a $\psi \in \mathcal{PSH}(\mathbb{C}^n)$ and regard expressions of the form

$$\sup\{u(z)\,;\, u \in \mathcal{PSH}(\mathbb{C}^n), u \leq \psi + c_u, u \leq q \text{ on } E\}, \tag{4.1}$$

where c_u is again a constant depending on u. We would like this expression to define a function with characteristics similar to the pluricomplex Green function. Then it is convenient if there is some $\sigma > 0$ such that $\psi \in \sigma\mathcal{L}(\mathbb{C}^n)$, since that would allow us to translate many properties of $V_{E,q}$ onto (4.1), in particular the finiteness of the expression at any $z \in \mathbb{C}^n$ if E is non-pluripolar. A version of the Siciak-Zakharyuta theorem would then concern the density of $\log|p|^{1/m}$ where p is any polynomial such that $\log|p|^{1/m} \leq \psi + c_p$ for some constant c_p.

If ψ was chosen as an *exhaustion* of $\mathbb{C}^n$, that is with all of its sublevel sets $\{z \in \mathbb{C}^n\,;\, \psi(z) < c\}$ bounded for any $c \in \mathbb{R}$, then the space $\mathcal{P}^\psi_m(\mathbb{C}^n)$ of entire functions p satisfying the bound $\log|p|^{1/m} \leq \psi + c_p$ is referred to in the literature as ψ-polynomials, [41]. In general this condition is not very descriptive. The basic example $\psi = \log^+|\cdot|$ gives $\mathcal{P}^\psi_m(\mathbb{C}^n)$ as the standard m-th degree polynomials, which is a very concrete criterion on the allowed exponents of the polynomial.

If we do not require ψ to be an exhaustion, then we can find a family of functions from $\mathcal{PSH}(\mathbb{C}^n)$ that have a similarly concrete description of their ψ-polynomials. For a compact and convex subset S of $\mathbb{R}^n_+$, we define its *supporting function* as $\varphi_S\colon \mathbb{R}^n \to \mathbb{R}$, $\varphi_S(\xi) = \sup_{s\in S}\langle s, \xi\rangle$ and its *logarithmic supporting function* by $H_S(z) = \varphi_S(\log|z_1|, \ldots, \log|z_n|)$ on $\mathbb{C}^{*n}$. The function H_S extends as a continuous plurisubharmonic function on $\mathbb{C}^n$ by Proposition 3.4 in Paper I. The ψ-polynomials for $\psi = H_S$ are easy to describe. Theorem 3.6 implies that $\log|p|^{1/m} \leq H_S + c_p$ precisely if the Newton polytope of p is contained in mS. In short, this theorem states that $\mathcal{P}^{H_S}_m(\mathbb{C}^n) = \mathcal{P}^S_m(\mathbb{C}^n)$

This neat description of the polynomials whose growth is governed by the functions H_S motivates the study of the class $\mathcal{L}^S(\mathbb{C}^n)$ of all $u \in \mathcal{PSH}(\mathbb{C}^n)$ such that $u \leq H_S + c_u$ for some constant c_u depending on u. Note that the unit simplex Σ in the role of S yields the Lelong class, $\mathcal{L}^\Sigma(\mathbb{C}^n) = \mathcal{L}(\mathbb{C}^n)$. The fundamental object of study in the papers that follow is the *Siciak-Zakharyuta function associated to* S

$$V^S_{E,q}(z) = \sup\{u(z)\,;\, u \in \mathcal{L}^S(\mathbb{C}^n), u \leq q \text{ on } E\}. \tag{4.2}$$

We generalize the Siciak-Zakharyuta theorem in Paper II, Theorem 1.1, and show that

$$V^S_{K,q}(z) = \sup\{u(z)\,;\, u = \log|p|^{1/m},\, p \in \mathcal{P}^S_m(\mathbb{C}^n),\, m \in \mathbb{N},\, u \leq q \text{ on } K\} \tag{4.3}$$

for all $z \in \mathbb{C}^{*n}$ if and only if the rational points $S \cap \mathbb{Q}^n$ are dense in S. The exponents of polynomials in $\mathcal{P}^S_m(\mathbb{C}^n)$ are the points of $(mS) \cap \mathbb{N}^n$. Those points only come from dilates of the rational points of S, that is $(mS) \cap \mathbb{N}^n = m(S \cap \mathbb{Q}^n) \cap \mathbb{N}^n$. If S does not have plenty of rational points, then $\mathcal{P}^S_m(\mathbb{C}^n)$ will not contain enough polynomials for the equality (4.3) to hold. This theorem was applied by Snorrason in [42] to show a generalized product property for V^S_K.

5 Conditions on the set S

Throughout Papers I, II and III, we will always work with S which is assumed to be compact, convex, containing 0 and contained in $\mathbb{R}^n_+$. Compactness implies that each of the spaces $\mathcal{P}^S_m(\mathbb{C}^n)$ is finite dimensional and that $H_S(z)$ is finite at any point $z \in \mathbb{C}^n$. Convexity implies that $\mathcal{P}^S(\mathbb{C}^n)$ is a graded ring and that the dilation of S acts as a grading of the ring: If $p_1 \in \mathcal{P}^S_{m_1}(\mathbb{C}^n)$ and $p_2 \in \mathcal{P}^S_{m_2}(\mathbb{C}^n)$, then $p_1 p_2 \in \mathcal{P}^S_{m_1+m_2}(\mathbb{C}^n)$. Also, S has the same logarithmic supporting function H_S as its convex hull, so it is better for each of those functions H_S to only have one representative S. Assuming that $0 \in S$ implies that the $\mathcal{P}^S(\mathbb{C}^n)$ contains the constants. It further implies, along with convexity on S, that the spaces are nested, that if $m_1 \leq m_2$ then $\mathcal{P}^S_{m_1}(\mathbb{C}^n) \subseteq \mathcal{P}^S_{m_2}(\mathbb{C}^n)$.

We choose to only regard $S \subset \mathbb{R}^n_+$, because of the philosophy that this should be a project about polynomials. Then and only then does the function H_S extend across $\mathbb{C}^n \setminus \mathbb{C}^{*n}$ as a locally bounded plurisubharmonic function. We study the classes $\mathcal{P}^S(\mathbb{C}^n)$ and $\mathcal{L}^S(\mathbb{C}^n)$, each containing functions that are well-defined along $\mathbb{C}^n \setminus \mathbb{C}^{*n}$. It turns out that there are many complications to our theory exactly on the coordinate hyperplanes $\mathbb{C}^n \setminus \mathbb{C}^{*n}$, not all of which yet resolved. In Paper I, Proposition 5.4, we are only able to show that $V^S_{K,q}$ is lower semicontinuous on $\mathbb{C}^{*n}$. That has later the effect that in Paper II we are only able to prove our Siciak-Zakharyuta Theorem at all points of $\mathbb{C}^{*n}$. When studying S not contained in $\mathbb{R}^n_+$, we only have classes $\mathcal{P}^S(\mathbb{C}^{*n})$ and $\mathcal{L}^S(\mathbb{C}^{*n})$ defined outside the axis. Then one avoids having to regard such complications. It should be mentioned that in the works of Bayraktar [4], the integral polytopes

S are assumed to be subsets of $\mathbb{R}^n$. The corresponding *Laurent polynomials* are of interest e.g. for the study of toric varieties.

In Theorem 5.8 of Paper I we show that $\mathcal{L}^S(\mathbb{C}^n)$ is translation invariant if and only if S is a *lower set*, defined in the same section. The lower condition appears in previous papers studying the extremal functions $V^S_{K,q}$, e.g. [14], assumed either explicitly or implicitly. The conditions we impose on the set S influence the properties of the polynomial space $\mathcal{P}^S(\mathbb{C}^n) = \bigcup_{m\in\mathbb{N}} \mathcal{P}^S_m(\mathbb{C}^n)$. In many previous papers, e.g. [8, 14], S is assumed to contain a neighborhood of zero, which in this context means that $r\Sigma \subset S$ for some $r > 0$. Then $\mathcal{P}^S(\mathbb{C}^n)$ is the entire polynomial space $\mathcal{P}(\mathbb{C}^n)$, just with a different grading. That condition would be uninteresting in the context of Paper III, which concerns approximation of holomorphic functions by $\mathcal{P}^S(\mathbb{C}^n)$, which would then just reduce onto the classical Runge-Oka-Weil theorem.

Most previous results assume that S has nonempty interior, that it is a *convex body*. This condition is convenient to have for the reason that it allows one to compare the function H_S to a logarithmic supporting function of some ball contained in S, which is something we can compute directly. Such an argument is used in our proof of our Siciak-Zakharyuta Theorem 1.1. However, convex bodies S are not the most general choice of a convex set S for which a Siciak-Zakharyuta Theorem holds. That is why we spend much effort in Paper II to describe $V^S_{K,q}$ for S with empty interior in terms of $V^T_{K',q'}$ for some lower-dimensional T that is a convex body.

6 The Runge-Oka-Weil Theorem

In spirit, the Siciak-Zakharyuta theorem concerns the density of polynomials. It is related to other questions on the density of polynomials, such as the Runge-Oka-Weil Theorem. In Paper III we study a generalization of the Runge-Oka-Weil Theorem and seek to answer the following question: Which holomorphic functions can be uniformly approximated by polynomials from $\mathcal{P}^S(\mathbb{C}^n)$?

The Runge theorem in the complex plane states that a holomorphic function defined in a neighborhood of a compact set K which is simply connected can be uniformly approximated by polynomials on K. This theorem has an analogue in higher dimensions, the Runge-Oka-Weil Theorem stated here below. There simple connectedness is replaced by polynomial convexity. If K is a compact subset of $\mathbb{C}^n$, then its *polynomial hull* is

$$\widehat{K} = \{z \in \mathbb{C}^n \,;\, |p(z)| \leq \|p\|_K \text{ for all polynomials } p\},$$

where $\|p\|_K = \sup_{z\in K} |p(z)|$. The set K is said to be *polynomially convex* if $K = \widehat{K}$. The polynomially convex compact sets in the complex plane are precisely the simply connected compact sets.

Theorem 6.1 (Runge-Oka-Weil) *Let $\Omega \subset \mathbb{C}^n$ be a neighborhood of a polynomially convex compact set K. Then every $f \in \mathcal{O}(\Omega)$ can be approximated uniformly on $\widehat{K}$ by polynomials.*

We provide two versions of this theorem on approximation by polynomials from a ring $\mathcal{P}^S(\mathbb{C}^n)$. In the specific case when the compact set K is Reinhardt, that is invariant under rotation in each variable, then we have a precise description of the holomorphic functions that can be uniformly approximated on K by Theorem 1.1 in Paper III. A holomorphic function in a neighborhood of a Reinhardt set K is approximable by polynomials from $\mathcal{P}^S(\mathbb{C}^n)$ if and only if its domain can be extended to contain $\widehat{K}^S = \{z \in \mathbb{C}^n \,;\, |p(z)| \leq \|p\|_K,\, p \in \mathcal{P}^S(\mathbb{C}^n)\}$, and f is bounded on $\widehat{K}^S$. Furthermore, if K contains some point from $\mathbb{C}^{*n}$, this is equivalent to f having a power series expression around the origin of the form $f(z) = \sum_{\alpha \in \mathbb{R}_+ S \cap \mathbb{N}^n} a_\alpha z^\alpha$. If K is not a Reinhardt set, but is a compact subset of $\mathbb{C}^{*n}$ with $K = \widehat{K}^S$ and $V_K^{S*} = 0$, then any holomorphic function in a neighborhood of K can be uniformly approximated on K by polynomials from $\mathcal{P}^S(\mathbb{C}^n)$. This is Theorem 1.2 in Paper III.

7 Convexity in $\mathbb{R}^n$

The definition of plurisubharmonic functions bears much resemblance to the definition of convex functions of several real variables, so one expects there to be some direct correspondence between them. We have such a correspondence when we regard plurisubharmonic functions on $\mathbb{C}^n$ that effectively only depend on real variables. We say that a function u on $\mathbb{C}^{*n}$ is *rotationally invariant* if it is rotationally invariant in each variable, that is if $u(z) = u(\zeta z)$ for all ζ from the unit torus $\mathbb{T}^n$, where ζz denotes $(\zeta_1 z_1, \ldots, \zeta_n z_n)$. The arguments are superfluous to rotationally invariant functions, making them effectively functions of n real variables. The plurisubharmonic functions on $\mathbb{C}^{*n}$ that are rotationally invariant are in a one-to-one correspondence with convex functions on $\mathbb{R}^n$, as will be illustrated here below.

If $u \in \mathcal{PSH}(\mathbb{C}^{*n})$ is rotationally invariant, we can find a sequence of smooth $u_\ell \in \mathcal{PSH}(\mathbb{C}^{*n})$ such that $u_\ell \searrow u$ as $\ell \to \infty$ such that u_ℓ are rotationally invariant. One way to obtain such a sequence is to regard the pullback of u through the holomorphic map $\mathbb{C}^n \ni \zeta \to e^\zeta = (\zeta_1, \ldots, \zeta_n)$. Then $v(\zeta) = u(e^\zeta)$ is plurisubharmonic and only depending on the real part of its inputs. Given any $\chi \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ and $\delta > 0$ and writing $\chi_\delta(\zeta) = (1/\delta^n)\chi(\delta\zeta)$, the convolutions $v_\delta = v * \chi_\delta$ are smooth plurisubharmonic functions, $v_\delta \searrow v$ and v_δ are only dependent on the real parts of ζ. Then $u_\delta(z) = v_\delta(\mathrm{Log}(z))$ are smooth functions on $\mathbb{C}^{*n}$ that are rotationally invariant and $u_\delta \searrow u$. We denote $\xi_j = \mathrm{Re}(\zeta_j)$ for $j = 1, \ldots, n$, and may regard v and v_δ as a functions of $\xi \in \mathbb{R}^n$. Observe that

$$\frac{\partial^2 u_\delta(z)}{\partial z_j \partial \overline{z}_k} = \frac{1}{4 z_j \overline{z}_k} \cdot \frac{\partial^2 v_\delta(\xi)}{\partial \xi_j \partial \xi_k}, \qquad z \in \mathbb{C}^{*n}. \tag{7.1}$$

The smooth function u_δ is plurisubharmonic if and only if $\langle \mathcal{L}(u_\delta)b, b\rangle \geq 0$ for all $b \in \mathbb{C}^n$ and a smooth function v_δ on $\mathbb{R}^n$ is convex if its Hesse matrix $\nabla^2 v$ has $\langle (\nabla^2 v)t, t\rangle \geq 0$ for all $t \in \mathbb{R}^n$. By (7.1), $u_\delta \in \mathcal{PSH}(\mathbb{C}^{*n})$ if and only if v_δ is convex.

Furthermore, u extends as a plurisubharmonic function to all of $\mathbb{C}^n$ if and only if it is locally bounded above on $\mathbb{C}^n \setminus \mathbb{C}^{*n}$. That implies that for all $j = 1, \ldots, n$ we have

$$\lim_{t\to\infty} v(\xi_1, \ldots, -t, \ldots, \xi_n) = \lim_{t\to\infty} u(z_1, \ldots, e^{-t}, \ldots, z_n) < +\infty.$$

Since $t \mapsto v(\xi_1, \ldots, -t, \ldots, \xi_n)$, $t \geq 0$ is a bounded convex function, it is increasing (decreasing in $-t$). We can infer that v is increasing in each variable if and only of u extends as $u \in \mathcal{PSH}(\mathbb{C}^n)$. Furthermore, given some compact convex $S \subset \mathbb{R}^n_+$ with $0 \in S$, then $u \in \mathcal{L}^S(\mathbb{C}^n)$ with $u \leq H_S + c$ if and only if $v \leq \varphi_S + c$ with the same constant.

The correspondence presented here is exploited in Paper I, Section 6 and Paper III, Sections 3 and 4. In both of these papers, we derive the following formula, relating the real Monge-Ampère mass of the convex function v at a Borel set $E \subset \mathbb{R}^n$ to the complex Monge-Ampère mass of u by the following relation

$$\begin{aligned}\int_{\mathrm{Log}^{-1}(E)} (dd^c u)^n &= n!\,\big(\mathcal{MA}_{\mathbb{R}}(\varphi_S) \otimes d\theta\big)(E \times [0, 2\pi]^n) \\ &= (2\pi)^n n!\,\mathcal{MA}_{\mathbb{R}}(v)(E).\end{aligned}$$

The real Monge-Ampère mass of a convex function v at E can be given a geometric interpretation by studying the *subdifferential* of v at each point point $x \in E$, which is the set

$$\mathrm{Sub}_v(x) = \{\xi \in \mathbb{R}^n\,;\, \langle y, \xi\rangle \leq v(x + y) - v(x),\ \forall y \in \mathbb{R}^n\}.$$

The *gradient image* of v over E is $\mathrm{Grad}_v(E) = \bigcup_{x\in E} \mathrm{Sub}_v(x)$. The volume of the gradient image is precisely the real Monge-Ampère mass.

Theorem 7.1 ([16], Theorem A.31) *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, and let $u\colon \Omega \to \mathbb{R}$ be a convex function of class $\mathcal{C}^{1,1}_{\mathrm{loc}}$. Then*

$$\mathrm{vol}(\mathrm{Grad}_v(E)) = \int_E \mathcal{MA}_{\mathbb{R}}(v).$$

We apply this identity in Paper I, Theorem 6.2 to obtain a neat formula for the Monge-Ampère mass of Siciak-Zakharyuta functions, $(dd^c V_K^S)^n = (2\pi)^n n!\,\mathrm{vol}(S)$. In Paper III, Theorem we use this theorem again to show that the function $x \mapsto \sup_{s\in S}(\langle s, x\rangle - \varphi_A(s))$ has no real Monge-Ampère mass outside a set $A \subset \mathbb{R}^n$. This will imply that the plurisubharmonic function $\sup_{s\in S}(\langle s, \mathrm{Log}\, z\rangle - \varphi_A(s))$ is maximal outside $K = \mathrm{Log}^{-1} A$, and is then precisely the function $V_K^S(z)$.

Paper I

Polynomials with exponents in compact convex sets and associated weighted extremal functions: Fundamental results

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Abstract

This paper is a collection of fundamental results for the study of polynomial rings $\mathcal{P}^S(\mathbb{C}^n)$ where the m-th degree polynomials have exponents restricted to mS, where $S \subseteq \mathbb{R}^n_+$ is compact, convex and $0 \in S$. We study the relationship between $\mathcal{P}^S(\mathbb{C}^n)$ and the class $\mathcal{L}^S(\mathbb{C}^n)$ of global plurisubharmonic functions where the growth is determined by the logarithmic supporting function of S. We present properties of their respective weighted extremal functions $\Phi^S_{K,q}$ and $V^S_{K,q}$ in connection with properties of S. Our ambition is to give detailed proofs with minimal assumptions of all results, thus creating a self contained exposition.

1 Introduction

Approximation theory deals with problems of determining whether a given function in some prescribed function space can be approximated by functions in a certain subspace. The theorems of Runge and Mergelyan are prototypes of results from approximation theory. The Runge theorem states that every function f holomorphic in some neighborhood of a simply connected compact subset K of $\mathbb{C}$ can be approximated in the uniform norm $\|\cdot\|_K$ on K by polynomials and the Mergelyan theorem states that it is enough to assume that f is continuous on K and holomorphic in the interior of K.

Quantitative approximation theory deals with problems of relating properties of the given function f to the error in approximation which is usually measured as the distance from f to a certain finite dimensional subspaces of the approximating subspace. The Bernstein-Walsh theorem is a prototype of a result from quantitative approximation theory. It states that a holomorphic function f defined in some neighborhood of a compact simply connected subset K of $\mathbb{C}$ extends as a holomorphic function to the sublevel set $\{z \in \mathbb{C}\,;\, g_{\mathbb{C}\setminus K}(z,\infty) < \log R\}$ if and only if $\overline{\lim}_{m\to+\infty} d_m(f,K)^{1/m} \le 1/R$, where $R > 1$, $g_{\mathbb{C}\setminus K}(\cdot,\infty)$ is the Green function of $\mathbb{C}\setminus K$ with logarithmic pole at ∞, $d_m(f,K)$ is the distance from f to the space $\mathcal{P}_m(\mathbb{C})$ of all polynomials of degree $\le m$ with respect to the uniform norm on K, and it is assumed that the domain $\mathbb{C}\setminus K$ is regular for the Dirichlet problem in the sense that $g_{\mathbb{C}\setminus K}(\cdot,\infty)$ vanishes at the boundary of K.

The Runge theorem is generalized to several variables where simple connectedness of K generalizes as polynomial convexity. This generalization is usually called the Oka-Weil theorem. There only exist fragmentary, but interesting, results generalizing the Mergelyan theorem to several complex variables. See Levenberg [24].

The origin of the subject of the present paper is the generalization by Siciak [35] of the Bernstein-Walsh theorem. There he introduced the extremal function $\Phi_K = \overline{\lim}_{m\to\infty} \Phi_{K,m}$, where $\Phi_{K,m} = \sup\{|p|^{1/m}\,;\, p \in \mathcal{P}_m(\mathbb{C}^n), \|p\|_K \le 1\}$ and $\mathcal{P}_m(\mathbb{C}^n)$ is the space of all polynomials of degree $\le m$. In his work $\log \Phi_K$ plays the role of $g_{\mathbb{C}\setminus K}(\cdot,\infty)$ and the regularity condition at ∂K is that $\Phi_K^*(z) = \overline{\lim}_{\zeta\to z} \Phi_K(\zeta) \le 1$ for every $z \in \partial K$. Siciak's paper is a seminal work on the understanding of the interrelation between quantitative approximation theory in several complex variables and pluripotential theory. He studied this subject further in many of his works, for example [36, 37, 38, 39]. See also Bloom's Appendix B in the monograph by Saff and Totik [33].

We will systematically work with subspaces of the polynomial space $\mathcal{P}(\mathbb{C}^n)$ in n variables. For every non-empty bounded subset S of $\mathbb{R}^n_+$ we let $\mathcal{P}^S_m(\mathbb{C}^n)$ denote the space of all polynomials p of the form $p(z) = \sum_{\alpha\in(mS)\cap\mathbb{N}^n} a_\alpha z^\alpha$, $z \in \mathbb{C}^n$, and let $\mathcal{P}^S(\mathbb{C}^n) = \bigcup_{m\in\mathbb{N}} \mathcal{P}^S_m(\mathbb{C}^n)$. The standard simplex $\Sigma = \operatorname{ch}\{0, e_1, \dots, e_n\}$, where $\operatorname{ch} A$ denotes the convex hull of a set A and $(e_1, \dots, e_n)$ is the standard basis in $\mathbb{R}^n$, yields the standard grading of polynomials of degree $\le m$, that is $\mathcal{P}^\Sigma_m(\mathbb{C}^n) = \mathcal{P}_m(\mathbb{C}^n)$. If S is a compact convex set in $\mathbb{R}^n_+$ with $0 \in S$ then the space $\mathcal{P}^S(\mathbb{C}^n)$ forms a graded polynomial ring where the degree of a polynomial p is the minimal $m \in \mathbb{N}$ such that $p \in \mathcal{P}^S_m(\mathbb{C}^n)$.

The polynomial spaces $\mathcal{P}^S_m(\mathbb{C}^n)$ appear in Shiffman-Zelditch [34] with S as an integral polytope and later in Bayraktar [4] as sparse polynomials. These polynomial classes and their relation to pluripotential theory have been studied by Bloom, Levenberg and their collaborators [1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 14, 25, 30]. Unfortunately, in some of these papers there are false results stated that we have not seen corrected. We point out and correct them as far as we can.

In Section 2 we define the Siciak function with respect to S, E, and q for every function $q\colon E \to \mathbb{R}\cup\{+\infty\}$ on a subset E of $\mathbb{C}^n$ by $\Phi^S_{E,q} = \overline{\lim}_{m\to\infty} \Phi^S_{E,q,m}$, where

$$\Phi^S_{E,q,m} = \sup\{|p|^{1/m}\,;\, p \in \mathcal{P}^S_m(\mathbb{C}^n), \|pe^{-mq}\|_E \le 1\}, \qquad m = 1, 2, 3, \dots.$$

We drop S in the superscript if $S = \Sigma$ and q in the subscript if $q = 0$. The grading of the polynomial classes $\mathcal{P}^S_m(\mathbb{C}^n)$ implies that for every $z \in \mathbb{C}^n$ the sequence $(-\log\big(\Phi^S_{E,q,m}(z)\big)^m)_{m\in\mathbb{N}^*}$ is subadditive and a lemma by Fekete implies that

$$\Phi^S_{E,q} = \lim_{m\to\infty} \Phi^S_{E,q,m} = \sup_{m\in\mathbb{N}^*} \Phi^S_{E,q,m}$$

without any restriction on S or q. This was first proved by Siciak [35], Theorem 6.1, for $S = \Sigma$. For the reader's convenience we prove the Fekete Lemma 2.3,

because ingredients from the proof are needed in the proof of Proposition 2.2, where it is shown that the convergence is uniform on a compact subset X, if q is bounded below on E and $\Phi^S_{E,q}$ is continuous on X. This statement on uniform convergence appears in many arguments in pluripotential theory. See for example Bloom and Shiffman [13], Lemma 3.2, Bayraktar [1, 4], Theorem 2.10, and Bayraktar, Hussung, Levenberg, and Perera [8], Theorem 1.1. The statements on uniform convergence in these papers all follow directly from Proposition 2.2 only with the conditions that S is compact and convex with $0 \in S$ and q is bounded below.

In Section 3 we introduce the *logarithmic supporting function* H_S of S and the *Lelong class with respect to* S. The function H_S is defined on $\mathbb{C}^{*n}$ as the supporting function φ_S of S, $\varphi_S(\xi) = \sup_{s\in S} \langle s, \xi\rangle$, $\xi \in \mathbb{R}^n$, in logarithmic coordinates extended to $\mathbb{C}^n \setminus \mathbb{C}^{*n}$ as an upper semicontinuous function and the Lelong class $\mathcal{L}^S(\mathbb{C}^n)$ with respect to S is defined as the set of all $u \in \mathcal{PSH}(\mathbb{C}^n)$ satisfying $u \leq c_u + H_S$, where c_u is a constant only depending on u and $\mathcal{PSH}(X)$ denotes the set of plurisubharmonic functions on an open set $X \subset \mathbb{C}^n$. We have $H_\Sigma(z) = \log^+ \|z\|_\infty$, which implies that $\mathcal{L}^\Sigma(\mathbb{C}^n)$ is equal to the Lelong class $\mathcal{L}(\mathbb{C}^n)$.

What values H_S takes at points on the coordinate hyperplanes $\mathbb{C}^n \setminus \mathbb{C}^{*n}$ is opaque from its definition itself, but Proposition 3.3 provides an explicit formula. Additionally, this formula shows that the zero set $\mathcal{N}(H_S)$ of H_S may very well be unbounded. We prove in Proposition 3.4 that H_S extends from $\mathbb{C}^{*n}$ as a continuous plurisubharmonic function on $\mathbb{C}^n$.

The usual grading of the polynomials can be characterized by growth, and so is also the case for $\mathcal{P}^S(\mathbb{C}^n)$. It is clear that if $p \in \mathcal{P}^S_m(\mathbb{C}^n)$ then $|p| \leq C_p e^{H_S}$ for some constant $C_p > 0$. In Theorem 3.6 we prove a Liouville type theorem which states that if $f \in \mathcal{O}(\mathbb{C}^n)$ satisfies a growth estimate $|f(z)| \leq C(1+|z|)^a e^{mH_S(z)}$ for some $C > 0$ and some $a > 0$ strictly less than the distance from mS to $\mathbb{N}^n \setminus mS$ in L^1-norm, then $f \in \mathcal{P}^S_m(\mathbb{C}^n)$.

In Section 4 we define the *Siciak-Zakharyuta function*

$$V^S_{E,q} = \sup\{u\,;\, u \in \mathcal{L}^S(\mathbb{C}^n),\ u|_E \leq q\}$$

with respect to S and any $q\colon E \to \mathbb{R} \cup \{+\infty\}$ on a subset E of $\mathbb{C}^n$. We drop S in the superscript if $S = \Sigma$ and q in the subscript if $q = 0$. By Klimek [23], Example 5.1.1, $V_K(z) = \log^+(\|z-a\|/r)$ for any norm $\|\cdot\|$ and $K = \{z \in \mathbb{C}^n\,;\, \|z-a\| \leq r\}$, $r > 0$, the closed ball in $\|\cdot\|$ with center a and radius r. These classical examples can not be generalized for V^S_E for the simple reason that the Lelong class $\mathcal{L}^S(\mathbb{C}^n)$ with respect to S does not need to be translation invariant. We prove in Proposition 4.3 that $V^S_E = H_S$ for any subset E of $\mathbb{C}^n$ such that $\mathbb{T}^n \subseteq E \subseteq \mathcal{N}(H_S)$, where $\mathbb{T}$ denotes the unit circle in $\mathbb{C}$ and $\mathcal{N}(H_S)$ the zero set of H_S. This is a generalization of Bayraktar [4], Example 2.3.

We introduce *admissible weights* in Definition 4.4, where we follow Bloom [33], Appendix B, with the natural generalization for the Lelong classes with respect to S given in Bayraktar, Bloom, and Levenberg [6]. In Proposition 4.5 we prove that for every compact convex $S \subset \mathbb{R}^n_+$ with $0 \in S$ and admissible weight q on a closed subset E the upper regularization $V^{S*}_{E,q}$ of $V^S_{E,q}$ is in $\mathcal{L}^S_+(\mathbb{C}^n)$, the set of $u \in \mathcal{L}^S(\mathbb{C}^n)$ such that $H_S - c_u \le u$ for some constant c_u.

It is a fundamental problem to characterize those S and admissible weights q for which $V^S_{E,q} = \log \Phi^S_{E,q}$. The classical Siciak-Zakharyuta theorem states that $V_K = \log \Phi_K$, for every compact subset K of $\mathbb{C}^n$, see [23], Theorem 5.1.7. It was first proved by Zakharyuta [45] and generalized to $V_{K,q} = \log \Phi_{K,q}$ for continuous q on a compact K by Siciak [36], Theorem 4.12. Magnússon, Sigurðardóttir, and Sigurðsson [26] prove that for every compact and convex $S \subset \mathbb{R}^n_+$ with $0 \in S$ and every admissible weight q on a closed $E \subset \mathbb{C}^n$, the equation $V^S_{E,q} = \log \Phi^S_{E,q}$ holds on $\mathbb{C}^{*n}$ if and only if $S \cap \mathbb{Q}^n$ is dense in S. In the case when $V^S_{E,q}$ is lower semicontinuous equality holds on $\mathbb{C}^n$. In Proposition 4.7 we show that if $S \cap \mathbb{Q}^n$ is not dense in S, then $V^S_{E,q} \ne \log \Phi^S_{E,q}$ for any admissible weight q on a closed set $E \subseteq \mathbb{C}^n$. Magnússon, Sigurðsson, and Snorrason [28] prove a generalization of the Bernstein-Walsh-Siciak theorem to the weighted setting with approximation by polynomials from the polynomial ring $\mathcal{P}^S(\mathbb{C}^n)$.

In Section 5 we study regularity of $V^S_{E,q}$. We introduce a regularization operator R_δ that preserves the class $\mathcal{L}^S(\mathbb{C}^n)$ and has the property that $R_\delta u \in \mathcal{C}^\infty(\mathbb{C}^{*n}) \cap \mathcal{L}^S(\mathbb{C}^n)$ for every $u \in \mathcal{L}^S(\mathbb{C}^n)$ and $R_\delta u \searrow u$ as $\delta \searrow 0$. This property enables us to prove in Proposition 5.4 that $V^S_{K,q}$ is lower semicontinuous on $\mathbb{C}^{*n}$ for every S and every admissible weight q on a compact set K. It is crucial to know if the upper regularization $V^{S*}_{E,q}$ of $V^S_{E,q}$ satisfies $V^{S*}_{E,q} \le q$ on K, for then $V^{S*}_{E,q} = V^S_{E,q}$ and $V^S_{E,q}$ is continuous at every point where it is lower semicontinuous.

It is a natural question to ask under which conditions on S the class $\mathcal{L}^S(\mathbb{C}^n)$ is preserved under the standard method for regularization of plurisubharmonic functions, that is convolution $u \mapsto u * \psi$, where $\psi \in \mathcal{C}^\infty_0(\mathbb{C}^n)$ with $\psi \ge 0$, and $\int_{\mathbb{C}^n} \psi \, d\lambda = 1$. In Theorem 5.8 we prove that $\mathcal{L}^S(\mathbb{C}^n)$ is preserved under convolution if and only if S is a lower set, which says that the cube $C_s = [0, s_1] \times \cdots \times [0, s_n]$ is contained in S for every $s \in S$. This is a complete answer to a question raised by Bayraktar, Hussung, Levenberg, Perera [8], Section 2.

In Section 6 we consider an equilibrium measure $\mu^S_{E,q} = \left(dd^c V^{S*}_{E,q}\right)^n$ for $V^{S*}_{E,q}$. We prove in Theorem 6.1 that its support is located where $V^{S*}_{K,q} \ge q$ and in Theorem 6.2 we prove that its total mass is $\mu^S_{E,q}(\mathbb{C}^n) = (2\pi)^n n! \operatorname{vol}(S)$ where vol denotes the euclidean volume. See also Bayraktar [4], Proposition 2.7, and Rashkovskii [31], Section 3.

In Section 7 we return to the problem of characterizing the classes $\mathcal{P}^S_m(\mathbb{C}^n)$ by growth properties we started in Theorem 3.6. Our main result Theorem 7.2 is that every entire function f in the weighted L^2-space $L^2(\mathbb{C}^n, \psi)$ consisting of all measurable functions which are square integrable with respect to the Lebesgue

measure λ on $\mathbb{C}^n$ with weight $e^{-\psi}$ and $\psi = 2mH_S + a\log(1+|\cdot|^2)$ is a polynomial in $\mathcal{P}^{\widehat{S}_\Gamma}_m(\mathbb{C}^n)$, where $\widehat{S}_\Gamma$ is the Γ-hull of S defined by $\widehat{S}_\Gamma = \{x \in \mathbb{R}^n_+ ; \langle x, \xi\rangle \leq \varphi_S(\xi), \forall \xi \in \Gamma\}$ for a certain cone $\Gamma \subset \mathbb{R}^n$. Bayraktar, Hussung, Levenberg, and Perera [8], Proposition 4.3 claim that $f \in \mathcal{P}^S_m(\mathbb{C}^n)$ for any polytope S and a sufficiently small. Example 7.4 shows that their claim is false even if S is a polytope containing a neighborhood of 0 in $\mathbb{R}^n_+$. The Siciak-Zakharyuta type theorem [8], Theorem 1.1, does not have a sound proof as it is based on their Proposition 4.3. As far as we can see the proof is only valid if S is a lower set.

In Section 8 we continue the discussion started in Perera [30] and show that for any polynomial map $f\colon \mathbb{C}^n \to \mathbb{C}^\ell$ and any compact convex $0 \in S \subseteq \mathbb{C}^n$, there is a canonical minimal choice of S' such that $f^*\colon \mathcal{P}^S_m(\mathbb{C}^n) \to \mathcal{P}^{S'}_m(\mathbb{C}^\ell)$ is well-defined for all $m \in \mathbb{N}$ and $f^*\colon \mathcal{L}^S(\mathbb{C}^n) \to \mathcal{L}^{S'}(\mathbb{C}^\ell)$ is well-defined. In the case when $\ell = n$ we show when such pullbacks are bijective.

Snorrason [42] generalized the Siciak product formula, Siciak [35, 36] and Klimek [23], Theorem 5.1.8. He shows in Corollary 1.3, that the generalization of Bos and Levenberg [14] of Siciak's product formula only holds for lower sets. This shows that both Levenberg and Perera [25], Proposition 1.3, and Nguyen Quang Dieu and Tang Van Long [29], Theorem 1.3, do not hold. The error in both the papers is the same, the authors implicitly assume that $\varphi_S(\xi) = \varphi_S(\xi^+)$ holds for every $\xi \in \mathbb{R}^n$, where $\xi^+ = \max\{0, \xi\}$, but in Theorem 5.8 we prove that this identity holds if and only if S is a lower set. In [42], Section 5, Snorrason shows that the sublevel sets of V^S_K are not convex in general. This contradicts Nguyen Quang Dieu and Tang Van Long [29], Theorem 1.2, where they claim that for every convex body S and every compact convex $K \subset \mathbb{C}^n$ the sublevel sets of V^S_K are convex. All these mistakes showed us the importance of a careful study of the values of H_S near points on the union of the coordinate hyperplanes as we have done in Propositions 3.3 and 3.4.

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2 Weighted polynomial classes and Siciak functions

Let S be a bounded subset of $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n; x_j \geq 0 \text{ for all } j = 1, \dots, n\}$. For every $m \in \mathbb{N}$ we associate to S the space $\mathcal{P}^S_m(\mathbb{C}^n)$ of all polynomials in n complex variables of the form

$$p(z) = \sum_{\alpha \in (mS) \cap \mathbb{N}^n} a_\alpha z^\alpha, \qquad z \in \mathbb{C}^n, \tag{2.1}$$

with the standard multi-index notation and let $\mathcal{P}^S(\mathbb{C}^n) = \bigcup_{m\in\mathbb{N}} \mathcal{P}^S_m(\mathbb{C}^n)$. If S is the standard simplex $\Sigma = \mathrm{ch}\{0, e_1, \dots, e_n\}$, then the space $\mathcal{P}^\Sigma_m(\mathbb{C}^n)$ consists of all polynomials of degree $\leq m$, which we denote by $\mathcal{P}_m(\mathbb{C}^n)$. We let $\mathcal{P}(\mathbb{C}^n) = \bigcup_{m\in\mathbb{N}} \mathcal{P}_m(\mathbb{C}^n)$ denote the space of all polynomials in n complex variables.

Assume now that S is a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$. If $\alpha \in jS$ and $\beta \in kS$ for some $j, k \in \mathbb{N}^*$, say $\alpha = ja$ and $\beta = kb$ with $a, b \in S$, then convexity of S gives $\alpha + \beta = (j+k)\big((1-\lambda)a + \lambda b\big) \in (j+k)S$, where $\lambda = k/(j+k) \in [0,1]$. Thus, $z^\alpha z^\beta \in \mathcal{P}^S_{j+k}(\mathbb{C}^n)$ and by taking linear combinations of products of monomials we get

$$\mathcal{P}^S_j(\mathbb{C}^n)\mathcal{P}^S_k(\mathbb{C}^n) \subseteq \mathcal{P}^S_{j+k}(\mathbb{C}^n). \tag{2.2}$$

This tells us that $\mathcal{P}^S(\mathbb{C}^n)$ is a graded ring. For every $p \in \mathcal{P}^S(\mathbb{C}^n)$ we define the *S-degree* $\deg^S(p)$ of p as the infimum over m for which $p \in \mathcal{P}^S_m(\mathbb{C}^n)$. We have

$$\deg^S(p_1 + p_2) \leq \max\{\deg^S(p_1), \deg^S(p_2)\}, \tag{2.3}$$

$$\deg^S(p_1 p_2) \leq \deg^S(p_1) + \deg^S(p_2). \tag{2.4}$$

Equality does not hold in general in either of these inequalities.

Definition 2.1 Let $S \subset \mathbb{R}^n_+$ be a compact convex set, $0 \in S$, and $q\colon E \to \mathbb{R} \cup \{+\infty\}$ be a function on $E \subset \mathbb{C}^n$. For $m \in \mathbb{N}^* = \{1, 2, 3, \dots\}$ the *m-th Siciak extremal function with respect to S, E, and q* is defined by

$$\Phi^S_{E,q,m}(z) = \sup\{|p(z)|^{1/m}\,;\, p \in \mathcal{P}^S_m(\mathbb{C}^n), \|pe^{-mq}\|_E \leq 1\}, \qquad z \in \mathbb{C}^n,$$

the *Siciak extremal function with respect to S, E, and q* is defined by

$$\Phi^S_{E,q}(z) = \varlimsup_{m\to\infty} \Phi^S_{E,q,m}(z), \qquad z \in \mathbb{C}^n.$$

We drop S in the superscripts if $S = \Sigma$ and q in the subscripts if $q = 0$. Note that the family $\{p \in \mathcal{P}^S_m(\mathbb{C}^n); \|pe^{-mq}\|_E \leq 1\}$ is never empty since it always contains the zero polynomial. Furthermore, we define $\Phi^S_{E,q,0} = 1$.

Observe that our definition of $\Phi^S_{E,q,m}$ deviates from the original definition of Siciak [35], which is $\big(\Phi^S_{E,q,m}\big)^m$ in our notation. We have that $\Phi^S_{E,q}, \Phi^S_{E,q,m}$ are lower semicontinuous on $\mathbb{C}^n$ for $m = 1, 2, 3, \dots$, for all these functions are suprema of continuous functions.

If q is bounded below, say by the real number q_0, then the constant polynomial $p(z) = e^{mq_0}$ is in $\mathcal{P}^S_m(\mathbb{C}^n)$ and $\|pe^{-mq}\|_E = \|e^{-m(q-q_0)}\|_E \leq 1$. Hence, it follows that $\Phi^S_{E,q,m}(z) \geq e^{q_0}$ for every $z \in \mathbb{C}^n$ and $m \in \mathbb{N}^*$.

Proposition 2.2 *Let $S \subset \mathbb{R}^n_+$ be a compact convex set with $0 \in S$, $E \subset \mathbb{C}^n$, and $q\colon E \to \mathbb{R} \cup \{+\infty\}$ be a function. Then for $j,k = 1,2,3,\ldots$*

$$\big(\Phi^S_{E,q,j}(z)\big)^j \big(\Phi^S_{E,q,k}(z)\big)^k \le \big(\Phi^S_{E,q,j+k}(z)\big)^{j+k}, \qquad z \in \mathbb{C}^n, \tag{2.5}$$

and

$$\Phi^S_{E,q}(z) = \lim_{m\to\infty} \Phi^S_{E,q,m}(z) = \sup_{m\ge 1} \Phi^S_{E,q,m}(z), \qquad z \in \mathbb{C}^n. \tag{2.6}$$

If q is bounded below and $\Phi^S_{E,q}$ is continuous on some compact subset X of $\mathbb{C}^n$, then the convergence is uniform on X.

We need ingredients from the proof from Tsuji [44], Lemma after Theorem III.25, p. 73:

Lemma 2.3 (Fekete lemma) *Let $(a_m)_{m\in\mathbb{N}^*}$ be a subadditive real sequence, that is $a_{j+k} \le a_j + a_k$ for $j,k = 1,2,3,\ldots$. Then $\lim_{m\to\infty} a_m/m = \inf_{m\ge 1} a_m/m$.*

Proof: Denote the infimum of a_m/m by α and take $\beta \in \mathbb{R}$ such that $-\infty \le \alpha < \beta$. Then there exists $j \in \mathbb{N}^*$ such that $a_j/j < \beta$. Every $m > j$ can be written as $m = sj + r$ for some $s, r \in \mathbb{N}$ with $s \ge 1$ and $0 \le r < j$. By assumption, we have $a_m \le a_{sj} + a_r \le sa_j + a_r$, so

$$\frac{a_m}{m} \le \frac{sa_j + a_r}{m} < \beta\big(1 - r/m\big) + \frac{\max_{\ell<j} a_\ell}{m}, \tag{2.7}$$

which implies $\overline{\lim}_{m\to\infty} a_m/m \le \beta$. Since β is arbitrary we have

$$\overline{\lim_{m\to\infty}}\, a_m/m \le \inf_{m\ge 1} a_m/m \le \underline{\lim_{m\to\infty}}\, a_m/m.$$

Hence, the limit (possibly $-\infty$) exists and the equality holds. □

Proof of Proposition 2.2: Take $p_j \in \mathcal{P}^S_j(\mathbb{C}^n)$ with $\|p_j e^{-jq}\|_E \le 1$ and $p_k \in \mathcal{P}^S_k(\mathbb{C}^n)$ with $\|p_k e^{-kq}\|_E \le 1$. Then $\|p_j p_k e^{-(j+k)q}\|_E \le 1$ and (2.2) implies $p_j p_k \in \mathcal{P}^S_{j+k}(\mathbb{C}^n)$. Hence, $|p_j(z)p_k(z)| \le \big(\Phi^S_{E,q,j+k}(z)\big)^{j+k}$. By taking supremum over p_j and p_k (2.5) follows. By (2.5) the sequence $a_m = -\log\big(\Phi^S_{E,q,m}(z)\big)^m$ is subadditive for every z, so (2.6) follows from Lemma 2.3.

Assume now that $q \ge q_0$ for some $q_0 \in \mathbb{R}$ and that $\Phi^S_{E,q}$ is continuous on X. By the discussion after Definition 2.1 we have $\Phi^S_{E,q,m} \ge e^{q_0}$. Since $\Phi^S_{E,q} = \sup_{m\ge1} \Phi^S_{E,q,m}$, it follows by a simple compactness argument that it is sufficient to show that for every $z_0 \in X$ and every $\varepsilon > 0$ there exists $\delta > 0$ and $k \in \mathbb{N}^*$ both depending on z_0 and ε such that

$$\Phi^S_{E,q}(z) - \Phi^S_{E,q,m}(z) < \varepsilon, \qquad z \in B(z_0,\delta) \cap X,\ m \ge k. \tag{2.8}$$

Let $c = \sup_X \Phi^S_{E,q}$ and choose $\gamma > 0$ such that $c(1 - e^{-\gamma}) < \frac{1}{4}\varepsilon$ and $j \in \mathbb{N}^*$ so large that $\Phi^S_{E,q}(z_0) - \Phi^S_{E,q,j}(z_0) < \frac{1}{4}\varepsilon$. Since $\Phi^S_{E,q}$ is continuous on X and $\Phi^S_{E,q,j}$ is lower semicontinuous on $\mathbb{C}^n$, there exists $\delta > 0$ such that

$$\Phi^S_{E,q}(z) - \Phi^S_{E,q}(z_0) < \tfrac{1}{4}\varepsilon \quad \text{and} \quad \Phi^S_{E,q,j}(z_0) - \Phi^S_{E,q,j}(z) < \tfrac{1}{4}\varepsilon, \quad z \in B(z_0,\delta) \cap X.$$

The three estimates imply

$$\Phi^S_{E,q}(z) - \Phi^S_{E,q,j}(z) < \tfrac{3}{4}\varepsilon, \qquad z \in B(z_0,\delta) \cap X, \tag{2.9}$$

and (2.8) follows from (2.9) if we can prove that there exists $k > j$ such that

$$\Phi^S_{E,q}(z) - \Phi^S_{E,q,m}(z) \le \Phi^S_{E,q}(z) - \Phi^S_{E,q,j}(z) + \tfrac{1}{4}\varepsilon, \qquad z \in X,\ m \ge k. \tag{2.10}$$

For every $z \in \mathbb{C}^n$ the sequence $a_m = -\log\big(\Phi^S_{E,q,m}(z)\big)^m$ is subadditive. By (2.7) we have for every $m > j$ written as $m = sj + r$ with $s \in \mathbb{N}^*$ and $r \in \mathbb{N}$ with $0 \le r < j$ that

$$-\log \Phi^S_{E,q,m}(z) \le \frac{-sj \log \Phi^S_{E,q,j}(z) - r \log \Phi^S_{E,q,r}(z)}{m}, \qquad z \in \mathbb{C}^n.$$

We have $sj/m = 1 - r/m$, $-\log \Phi^S_{E,q,j}(z) \ge -\log c$ for every $z \in X$, and $\log \Phi^S_{E,q,r}(z) \ge q_0$ for every $z \in \mathbb{C}^n$, so

$$\begin{aligned}\log \Phi^S_{E,q,m}(z) &\ge \log \Phi^S_{E,q,j}(z) + \frac{-r \log \Phi^S_{E,q,j}(z) + r \log \Phi^S_{E,q,r}(z)}{m} \\ &\ge \log \Phi^S_{E,q,j}(z) - \frac{j \log(c/e^{q_0})}{m}, \qquad z \in X.\end{aligned}$$

We choose $k > j$ so large that $j \log(c/e^{q_0})/k < \gamma$. Then $\Phi^S_{E,q,m}(z) \ge \Phi^S_{E,q,j}(z)e^{-\gamma}$ and

$$\Phi^S_{E,q}(z) - \Phi^S_{E,q,m}(z) \le \Phi^S_{E,q}(z) - \Phi^S_{E,q,j}(z) + \Phi^S_{E,q,j}(z)\big(1 - e^{-\gamma}\big)$$

for every $z \in X$ and $m \ge k$. Since $\Phi^S_{E,q,j}(z) \le c$ for every $z \in X$ and $c(1 - e^{-\gamma}) < \frac{1}{4}\varepsilon$ the estimate (2.10) holds. □

3 The Lelong class with respect to a convex set

Let us begin by setting some notation for the sequel. We denote by $\mathcal{H}(X)$, $\mathcal{SH}(X)$, $\mathcal{PSH}(X)$, $\mathcal{O}(X)$, $\mathcal{LSC}(X)$, and $\mathcal{USC}(X)$ the classes of harmonic and subharmonic functions on a domain X in $\mathbb{C}$, plurisubharmonic and holomorphic functions on a complex manifold X, and lower and upper semicontinuous functions on a topological space X, respectively. We define the coordinate-wise logarithm of the modulus, exponential function, and positive part, by

$$\begin{aligned}\mathrm{Log}\colon \mathbb{C}^{*n} \to \mathbb{R}^n, \quad \mathrm{Log}(z) &= (\log|z_1|, \dots, \log|z_n|), \qquad z \in \mathbb{C}^{*n}, \\ \mathrm{Exp}\colon \mathbb{R}^n \to \mathbb{R}^n_+, \quad \mathrm{Exp}(\xi) &= e^\xi = (e^{\xi_1}, \dots, e^{\xi_n}), \qquad \xi \in \mathbb{R}^n, \\ {}^+\colon \mathbb{R}^n \to \mathbb{R}^n_+, \quad \xi^+ &= (\xi_1^+, \dots, \xi_n^+), \quad \xi_j^+ = \max\{\xi_j, 0\} \quad \xi \in \mathbb{R}^n.\end{aligned}$$

We let $\mathbb{D}$ denote the unit disc and $\mathbb{T}$ the unit circle in $\mathbb{C}$. The Lelong class $\mathcal{L}(\mathbb{C}^n)$ is the set of all $u \in \mathcal{PSH}(\mathbb{C}^n)$ such that for some constant c_u depending on u we have

$$u(z) \le c_u + \log^+ \|z\|_\infty, \qquad z \in \mathbb{C}^n.$$

It is clear that $\log^+ \|\cdot\|_\infty$ can be replaced by $\log^+ \|\cdot\|$ or $\log(1+\|\cdot\|)$ for any norm $\|\cdot\|$ on $\mathbb{C}^n$.

Definition 3.1 For every compact subset of $\mathbb{R}^n_+$ with $0 \in S$ we define the *supporting function of* S as

$$\varphi(x) = \sup_{s\in S}\langle s, x\rangle, \qquad x \in \mathbb{R}^n,$$

and the *logarithmic supporting function of* S as the function $H_S\colon \mathbb{C}^n \to \mathbb{R}_+$ defined on $\mathbb{C}^{*n}$ by

$$H_S(z) = (\varphi_S \circ \operatorname{Log})(z) = \max_{s\in S}\big(s_1 \log|z_1| + \cdots + s_n \log|z_n|\big), \qquad z \in \mathbb{C}^{*n},$$

and extended to $\mathbb{C}^n \setminus \mathbb{C}^{*n}$ by

$$H_S(z) = \varlimsup_{\mathbb{C}^{*n}\ni w\to z} H_S(w), \qquad z \in \mathbb{C}^n \setminus \mathbb{C}^{*n}.$$

The real number $\sigma_S = \varphi_S(\mathbf{1})$, where $\mathbf{1} = (1,\dots,1) \in \mathbb{R}^n_+$, is called the *logarithmic type* of H_S.

Since φ_S is positively homogeneous of degree 1 and convex, that is $\varphi_S(t\xi) = t\varphi_S(\xi)$ and $\varphi_S(\xi+\eta) \le \varphi_S(\xi) + \varphi_S(\eta)$ for every $t \in \mathbb{R}_+$ and $\xi, \eta \in \mathbb{R}^n$, we have

$$H_S(z) = \tfrac{1}{\lambda} H_S(|z_1|^\lambda, \dots, |z_n|^\lambda), \qquad \lambda \in \mathbb{R}^*_+,\ z \in \mathbb{C}^{*n}, \tag{3.1}$$

$$H_S(z_1w_1, \dots, z_nw_n) \le H_S(z) + H_S(w), \qquad z, w \in \mathbb{C}^{*n}. \tag{3.2}$$

Observe that $\varphi_S(-\mathbf{1}) = 0$ and

$$H_S(\lambda z) \le H_S(z) + H_S(|\lambda|\mathbf{1}) = H_S(z) + \sigma_S \log^+|\lambda|, \qquad z \in \mathbb{C}^{*n},\ \lambda \in \mathbb{C}^*. \tag{3.3}$$

If we write $z = \|z\|_\infty w$ with $\|w\|_\infty = 1$, then this formula implies $H_S/\sigma_S \in \mathcal{L}(\mathbb{C}^n)$,

$$H_S(z) \le \sigma_S \log^+ \|z\|_\infty, \qquad z \in \mathbb{C}^n. \tag{3.4}$$

Directly from the definition we see that $H_S \in \mathcal{PSH}(\mathbb{C}^{*n}) \cap \mathcal{C}(\mathbb{C}^{*n})$. Since $\mathbb{C}^n \setminus \mathbb{C}^{*n}$ is pluripolar and H_S is locally bounded above we have $H_S \in \mathcal{PSH}(\mathbb{C}^n)$.

Proposition 3.2 *Let $S \subset \mathbb{R}^n_+$ be compact convex and with $0 \in S$. Then for every $z \in \mathbb{C}^{*n}$ and $w \in \mathbb{C}^n$ we have*

$$H_S(z+w) \le H_S(z) + \varphi_S(|w_1|/|z_1|, \dots, |w_n|/|z_n|),$$

and in particular, for every $w \in \overline{\mathbb{D}}^n$ and $\delta \in]0,1[$ we have

$$H_S(\mathbf{1} + \delta w) \le \delta\sigma_S.$$

Furthermore,

$$H_S(z+w) \le H_S(z) + \varphi_S(\log^+(|w_1|/|z_1|), \dots, \log^+(|w_n|/|z_n|)) + (\log 2)\sigma_S.$$

Proof: By plurisubharmonicity of H_S and convexity of

$$w \mapsto \varphi_S(|w_1/|z_1|, \dots, |w_n|/|z_n|)$$

we may assume that $w, z+w \in \mathbb{C}^{*n}$. For some $t \in S$ we have

$$H_S(z+w) = \log(|z_1+w_1|^{t_1} \cdots |z_n+w_n|^{t_n}) \le \langle t, \operatorname{Log} z\rangle + \sum_{j=1}^{n} t_j \log(1+|w_j|/|z_j|).$$

Since $\log(1+x) \le x$, for $x \ge 0$ the first estimate follows. Since $\mathbf{1} + \delta w \in \mathbb{C}^{*n}$ for $w \in \overline{\mathbb{D}}^n$, $\delta \in]0,1[$, and $H_S(\mathbf{1}) = 0$, the second estimate follows. For the third estimate we use the fact that $\log(1+x) \le \log 2 + \log^+ x$ for $x \ge 0$. □

The zero set $\mathcal{N}(H_S)$ of H_S can be understood in terms of the zero set of φ_S which is a cone. Since $\mathcal{N}(H_S) \cap \mathbb{C}^{*n} = \operatorname{Log}^{-1}(\mathcal{N}(\varphi_S))$ and $\mathbb{R}^n_- \subseteq \mathcal{N}(\varphi_S)$, the closed unit polydisc $\overline{\mathbb{D}}^n$ is contained in $\mathcal{N}(H_S)$. Furthermore, $\mathcal{N}(H_S)$ is equal to $\overline{\mathbb{D}}^n$ if and only if $\mathbb{R}_+ S = \mathbb{R}^n_+$. We have a complete description of the values of H_S at every point in $\mathbb{C}^n \setminus \mathbb{C}^{*n}$, the union of the coordinate hyperplanes.

Proposition 3.3 *Let S be a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$. For every $a \ne 0$ in some coordinate hyperplane we have*

$$H_S(a) = H_{S_J}(a_{j_1}, \dots, a_{j_\ell}),$$

where $J \subset \{1, \dots, n\}$ consists of the indices $j_1 < \cdots < j_\ell$ of the non-zero coordinates $a_{j_1}, \dots, a_{j_\ell}$ of a and $S_J \subseteq \mathbb{R}^\ell$ consists of all $t \in \mathbb{R}^\ell$ such that if $s \in \mathbb{R}^n_+$ is defined by $s_{j_k} = t_k$ for $j_k \in J$ and $s_j = 0$ for $j \notin J$, then $s \in S$.

Proof: After renumbering the variables we may assume that $J = \{1, \dots, \ell\}$ and $a_{\ell+1} = \cdots = a_n = 0$. We write $z = (z', z'') \in \mathbb{C}^n$, where $z' \in \mathbb{C}^\ell$ and $z'' \in \mathbb{C}^{n-\ell}$. Then $a' \in \mathbb{C}^{*\ell}$, so H_{S_J} is continuous at a' and we have

$$\begin{aligned} H_{S_J}(a') &= \lim_{\mathbb{C}^{*\ell} \ni w' \to a'} H_{S_J}(w') = \lim_{\mathbb{C}^{*m} \ni w' \to a'} \sup_{t \in S_J} \langle t, \operatorname{Log} w' \rangle \\ &= \lim_{\mathbb{C}^{*m} \ni w' \to a'} \sup_{s=(t,0) \in S} \langle s, (\operatorname{Log} w', 0, \dots, 0) \rangle \\ &\le \varlimsup_{\mathbb{C}^{*n} \ni w \to a} \sup_{s \in S} \langle s, \operatorname{Log} w \rangle = H_S(a). \end{aligned}$$

In order to prove the converse inequality we take $\mathbb{C}^{*n} \ni w_j = (w_{j,1}, \dots, w_{j,n}) \to a$ such that $\lim_{j\to\infty} H_S(w_j) = H_S(a)$. There exists $s_j \in S$ such that $H_S(w_j) = \langle s_j, \operatorname{Log} w_j \rangle$ for every j. Since $H_S \ge 0$ and $\log|w_{j,\kappa}| \to -\infty$ as $j \to \infty$ for $\kappa = \ell+1, \dots, n$, it follows that $s_{j,\kappa} \to 0$ as $j \to \infty$ for $\kappa = \ell+1, \dots, n$. By compactness of S there exists a subsequence s_{j_k} converging to $(t, 0) \in S$. We have $t \in S_J$ and conclude that

$$H_S(a) = \lim_{k\to\infty} H_S(w_{j_k}) = \lim_{k\to\infty} \langle s_{j_k}, \operatorname{Log} w_{j_k} \rangle = \langle t, \operatorname{Log} a' \rangle \le H_{S_J}(a').$$

□

With a proof similar to Rashkovskii [32], Proposition 2.2, we are able to show that $H_S \in \mathcal{C}(\mathbb{C}^n)$:

Proposition 3.4 *Let S be a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$. Then H_S is plurisubharmonic and continuous on $\mathbb{C}^n$.*

Proof: Let $0 \neq a \in \mathbb{C}^n \setminus \mathbb{C}^{*n}$. Since $H_S \in \mathcal{PSH}(\mathbb{C}^n) \cap \mathcal{C}(\mathbb{C}^{*n})$, it suffices to prove that H_S is lower semicontinuous at a. After renumbering the variables we may assume $a_{\ell+1} = \cdots = a_n = 0$ and $a_j \neq 0$ for $j \leq \ell$. We also write $z = (z', z'') \in \mathbb{C}^n$, where $z' \in \mathbb{C}^\ell$ and $z'' \in \mathbb{C}^{n-\ell}$. Let $z_j \in \mathbb{C}^n$ be such that $z_j \to a$. Since H_S is rotationally invariant in each variable it takes the constant value $H_S(z_j)$ on the distinguished boundary of the $n-\ell$ dimensional polydisc $\{(z', \zeta'')\,;\, |\zeta''_k| \leq |z''_{j,k}|, k = 1, \ldots, n-\ell\}$, so by the maximum principle $H_S(z'_j, 0) \leq H_S(z_j)$. By Proposition 3.3 we have that $H_S(z'_j, 0) = H_{S_J}(z'_j)$, for $J = \{1, \ldots, \ell\}$. By the continuity of H_{S_J} at a' we have

$$\varliminf_{j\to\infty} H_S(z_j) \geq \varliminf_{j\to\infty} H_S(z'_j, 0) = \varliminf_{j\to\infty} H_{S_J}(z'_j) = H_{S_J}(a') = H_S(a).$$

This proves the lower semicontinuity of H_S at a. □

Definition 3.5 For every compact convex subset S of $\mathbb{R}^n_+$ with $0 \in S$ we define the *S-Lelong class* $\mathcal{L}^S(\mathbb{C}^n)$ as the set of all $u \in \mathcal{PSH}(\mathbb{C}^n)$ such that

$$u(z) \leq c_u + H_S(z), \qquad z \in \mathbb{C}^n,$$

for some constant c_u depending on u, and define $\mathcal{L}^S_+(\mathbb{C}^n)$ as the subclass of functions u, that have the same asymptotic behavior at infinity as the function H_S, that is

$$-c_u + H_S(z) \leq u(z) \leq c_u + H_S(z), \qquad z \in \mathbb{C}^n. \tag{3.5}$$

The Liouville theorem tells us that an entire function $f \in \mathcal{O}(\mathbb{C}^n)$ which satisfies a growth estimate $|f(z)| \leq C(1+|z|)^{a+m}$, $z \in \mathbb{C}^n$, for some $m \in \mathbb{N}$ and $a \in [0, 1[$, is a polynomial of degree $\leq m$, that is $f \in \mathcal{P}_m(\mathbb{C}^n)$. The following is a Liouville type theorem for the polynomial classes $\mathcal{P}^S_m(\mathbb{C}^n)$:

Theorem 3.6 *Let d_m denote the distance between mS and $\mathbb{N}^n \setminus mS$ in the L^1-norm. Then for every $f \in \mathcal{O}(\mathbb{C}^n)$ the following are equivalent:*

(i) $f \in \mathcal{P}^S_m(\mathbb{C}^n)$.

(ii) $\log|f|^{1/m} \in \mathcal{L}^S(\mathbb{C}^n)$.

(iii) *there exists $a \in [0, d_m[$ such that $|f|e^{-mH_S - a\log^+\|\cdot\|_\infty} \in L^\infty(\mathbb{C}^n)$.*

(iv) *there exists $a \in [0, d_m[$ and a constant $C > 0$ such that*

$$|f(z)| \leq C(1+|z|)^a e^{mH_S(z)}, \qquad z \in \mathbb{C}^n.$$

Proof: **(i)⇒(ii):** If $\alpha \in mS$, then $|z^\alpha| \le e^{mH_S(z)}$, so for $f(z) = \sum_{\alpha\in(mS)\cap\mathbb{N}^n} a_\alpha z^\alpha$, we have $\log|f(z)|^{1/m} \le c_p/m + H_S(z)$ with $c_p = \log\sum_{\alpha\in(mS)\cap\mathbb{N}^n}|a_\alpha|$.

(ii)⇒(iii): We have $|f|e^{-mH_S - a\log^+\|\cdot\|_\infty} \le |f|e^{-mH_S} \in L^\infty(\mathbb{C}^n)$.

(iii)⇒(i): Observe first that $m\varphi_S(\xi) + a\|\xi^+\|_\infty = \varphi_{mS+a\Sigma}(\xi)$ for every $\xi \in \mathbb{R}^n$. Let $f(z) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha$ be the power series expansion of f at 0. We need to show that $a_\alpha = 0$ for all $\alpha \in \mathbb{N}^n \setminus mS$. Since $a < d_m$ we have $(mS + a\Sigma)\cap\mathbb{N}^n = (mS)\cap\mathbb{N}^n$, so $\alpha \in \mathbb{N}^n \setminus (mS + a\Sigma)$. Hence, there exists $\xi \in \mathbb{R}^n$ such that $\langle\alpha,\xi\rangle > \varphi_{mS+a\Sigma}(\xi)$. We let C_t denote the polycircle with center 0 and polyradius $(e^{t\xi_1},\dots,e^{t\xi_n})$ and observe that by the Cauchy formula for derivatives we have

$$a_\alpha = \frac{1}{(2\pi i)^n}\int_{C_t}\frac{f(\zeta)}{\zeta^\alpha}\frac{d\zeta_1\cdots d\zeta_n}{\zeta_1\cdots\zeta_n}.$$

For $\zeta = (e^{t\xi_1+i\theta_1},\dots,e^{t\xi_n+i\theta_n})$ on C_t we have $|f(\zeta)|/|\zeta^\alpha| \le Ce^{-t(\langle\alpha,\xi\rangle-\varphi_{mS+a\Sigma}(\xi))}$, so the right-hand side tends to 0 as $t\to+\infty$, and we conclude that $a_\alpha = 0$.

(iii)⇔(iv): Follows from the equivalence of the euclidean norm $|\cdot|$ and $\|\cdot\|_\infty$. □

Recall from Klimek [23], p. 87, that a real valued $u \in \mathcal{PSH}(X)$ on an open subset X of $\mathbb{C}^n$ is said to be *maximal*, if for every relatively compact open subset G of X and every $v \in \mathcal{USC}(\overline{G})\cap\mathcal{PSH}(G)$ satisfying $v \le u$ on ∂G we have $v \le u$ on G. For the reader's convenience we prove the following well known result.

Lemma 3.7 *Let X be an open subset of $\mathbb{C}^n$ and $u \in \mathcal{PSH}(X)$ be real valued. Assume that for every relatively compact open subset G of X there exists a family $(g_z)_{z\in G}$ of holomorphic maps $g_z\colon D_z \to \mathbb{C}^n$ defined on open subsets D_z of $\mathbb{C}$, such that $K_z = g_z^{-1}(\overline{G})$ is compact, $z = g_z(\tau_z)$ for some $\tau_z \in K_z$, and $u\circ g_z$ is harmonic on D_z. Then u is maximal on X.*

Proof: By Klimek [23], Proposition 3.1.1, we may take $v \in \mathcal{PSH}(X)$ in the definition of maximality. We assume that $v \le u$ on ∂G and need to prove that $v(z) \le u(z)$ for every $z \in G$. The set $K_z = g_z^{-1}(\overline{G})$ is compact, the function $s_z = v\circ g_z \in \mathcal{SH}(D_z)$ is less than or equal to $h_z = u\circ g_z \in \mathcal{H}(D_z)$ on the boundary of K_z. By the maximum principle for harmonic functions $v(z) = s_z(\tau_z) \le h_z(\tau_z) = u(z)$. □

For every $z \in \mathbb{C}^{n*}$ we define the parametric curve $f_z = (f_{z,1},\dots,f_{z,n})\colon \mathbb{C}\to\mathbb{C}^n$ by

$$f_{z,j}(\tau) = \begin{cases} e^{-i\tau\log|z_j|}(z_j/|z_j|), & z_j \ne 0,\\ 0, & z_j = 0,\end{cases} \qquad \tau\in\mathbb{C}. \tag{3.6}$$

We have $f_z(i) = z$ and $\|f_z(\tau)\|_\infty = 1$ for every $\tau\in\mathbb{R}$. If $\|z\|_\infty > 1$, then for j with $|z_j| = \|z\|_\infty$ we have

$$f'_{z,j}(\tau) = -ie^{-i\tau\log|z_j|}(\log|z_j|)(z_j/|z_j|) \ne 0. \qquad \tau\in\mathbb{C}, \tag{3.7}$$

Hence f_z parametrizes an open Riemann surface in $\mathbb{C}^n$ through the point z. Furthermore, we have $H_S(f_z(\tau)) = \mathrm{Im}\tau\, H_S(z)$ for $\mathrm{Im}\tau \geq 0$. The function $\mathbb{C} \ni \tau \mapsto H_S(f_z(\tau))$ is subharmonic, harmonic in the upper half plane, equal to 0 on the real axis, and takes the value $H_S(z)$ at i. By Lemma 3.7 with f_z in the role of g_z and H_S in the role of u we get:

Proposition 3.8 *Let S be a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$. Then H_S is maximal on $\mathbb{C}^n \setminus \partial\mathcal{N}(H_S)$, where $\mathcal{N}(H_S)$ is the zero set of H_S.*

4 Weighted Siciak-Zakharyuta functions

Definition 4.1 Let $S \subset \mathbb{R}^n_+$ be a compact convex set, $0 \in S$, and $q\colon E \to \mathbb{R} \cup \{+\infty\}$ be a function on $E \subset \mathbb{C}^n$. The *Siciak-Zakharyuta function with respect to S, E, and q* is defined by

$$V^S_{E,q}(z) = \sup\{u(z)\,;\, u \in \mathcal{L}^S(\mathbb{C}^n),\ u|_E \leq q\}, \qquad z \in \mathbb{C}^n.$$

We drop S in superscripts if $S = \Sigma$ and q in subscripts if $q = 0$.

From Theorem 3.6 it follows that $\log \Phi^S_{E,q} \leq V^S_{E,q}$ for every compact convex $S \subset \mathbb{R}^n_+$ and every $q\colon E \to \mathbb{R} \cup \{+\infty\}$ on $E \subset \mathbb{C}^n$. We will need a variant of the Phragmén-Lindelöf principle, see [21], Lemma 2.1.

Lemma 4.2 *Let v be subharmonic in the upper half plane $\mathbb{C}_+ = \{z \in \mathbb{C}\,;\, \mathrm{Im}z > 0\}$ such that for some real constants C and A we have $v(z) \leq C + A|z|$ for all $z \in \mathbb{C}_+$, and $\overline{\lim}_{\mathbb{C}_+ \ni z \to x} v(z) \leq 0$ for all $x \in \mathbb{R}$. Then $v(z) \leq A\,\mathrm{Im}z$ for all $z \in \mathbb{C}_+$.*

By Klimek [23], Example 5.1.1, $V_K(z) = \log^+(\|z-a\|/r)$ if $\|\cdot\|$ is any norm and $K = \{z \in \mathbb{C}^n\,;\, \|z-a\| \leq r\}$, $r > 0$, is the closed ball in this norm with center a and radius r. The polynomial classes $\mathcal{P}^S_m(\mathbb{C}^n)$ are in general not translation invariant, so we can not expect to have a generalization of this example. The following is proved in special cases by Bos and Levenberg [14]:

Proposition 4.3 *Let S be a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$ and let E be a subset of $\mathbb{C}^n$ such that $\mathbb{T}^n \subseteq E \subseteq \mathcal{N}(H_S)$. Then $V^S_E = H_S$.*

Proof: Since $H_S \in \mathcal{L}^S(\mathbb{C}^n)$ and $H_S|_E = 0$ we have $H_S \leq V^S_E$, so it is sufficient to prove that if $u \in \mathcal{L}^S(\mathbb{C}^n)$ with $u|_E \leq 0$ we have $u(z) \leq H_S(z)$ for every $z \in \mathbb{C}^{*n}$ such that $H_S(z) > 0$. Define f_z by (3.6) and $v \in \mathcal{SH}(\mathbb{C})$ by $v(\tau) = u(f_z(\tau))$, $\tau \in \mathbb{C}$. Since $u \in \mathcal{L}^S(\mathbb{C}^n)$ we have $v(\tau) = u(f_z(\tau)) \leq c_u + H_S(f_z(\tau)) = c_u + \mathrm{Im}\tau\, H_S(z)$, $\mathrm{Im}\tau \geq 0$, and since $f_z(\mathbb{R}) \subseteq \overline{\mathbb{D}}^n$ we have $v \leq 0$ on $\mathbb{R}$. Lemma 4.2 gives $u(z) = v(i) \leq H_S(z)$. □

Since $\mathbb{T}^n \subset \mathcal{N}(H_S)$ for every S the maximum principle implies that $\overline{\mathbb{D}}^n \subset \mathcal{N}(H_S)$, where $\mathbb{D}$ denotes the unit disc in $\mathbb{C}$.

Definition 4.4 Let $0 \in S \subset \mathbb{R}^n_+$ be compact and convex and $q\colon E \to \mathbb{R} \cup \{+\infty\}$ be a function on $E \subset \mathbb{C}^n$. We say that q is an *admissible weight with respect to* S *on* E if

(i) q is lower semicontinuous,

(ii) the set $\{z \in E\,;\, q(z) < +\infty\}$ is non-pluripolar, and

(iii) if E is unbounded, then $\lim\limits_{E \ni z, |z| \to \infty} (H_S(z) - q(z)) = -\infty$.

This definition is taken from Bloom [33], Appendix B, Definition 2.1. Some authors use the term *admissible external field* for q rather than *weight* in this situation and then they refer to e^{-q} as a weight. Observe that if q is an admissible weight, then E is non-pluripolar and that if E is unbounded then $q = 0$ does not need to be an admissible weight. In the case when E is unbounded and S is a neighborhood of 0 in $\mathbb{R}^n_+$, then $q = 0$ can not be an admissible weight.

Proposition 4.5 *Let S be a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$ and q be an admissible weight with respect to S on a compact subset K of $\mathbb{C}^n$. Then $V^{S*}_{K,q} \in \mathcal{L}^S_+(\mathbb{C}^n)$.*

Proof: The upper regularization $V^{S*}_{E,q}$ of $V^S_{E,q}$ is plurisubharmonic in $\mathbb{C}^n$ if q is an admissible weight with respect to S on E. Since $V^S_{K,q} \le q$ on K, q is an admissible weight on K and $\{z \in K\,;\, q(z) < +\infty\} \subseteq \{z \in \mathbb{C}^n\,;\, V^S_{K,q}(z) < +\infty\}$, the set on the right is non-pluripolar. By Klimek [23], Proposition 5.2.1, it follows that the family $\mathcal{U} = \{u \in \mathcal{L}^S(\mathbb{C}^n)\,;\, u|_K \le q\} \subset \sigma_S \mathcal{L}(\mathbb{C}^n)$ is locally uniformly bounded above, where $\sigma_S = \varphi_S(\mathbf{1})$ is the logarithmic type of H_S. Let $c > 0$ be such that $u|_{\overline{\mathbb{D}}^n} \le c$ for all $u \in \mathcal{U}$. Then by Proposition 4.3, we have $V^S_{K,q} \le V^S_{\overline{\mathbb{D}}^n} + c = H_S + c$. Hence, $V^{S*}_{K,q} \in \mathcal{L}^S(\mathbb{C}^n)$. If $c = \max_{w \in K} H_S(w) - \min_{w \in K} q(w)$ then $H_S - c \le V^S_{K,q}$, so $V^{S*}_{K,q} \in \mathcal{L}^S_+(\mathbb{C}^n)$. □

Admissible weights on unbounded closed sets yield the same Siciak and Siciak-Zakharyuta functions as some compact subsets. Admissible weights do not obstruct the growth of Siciak-Zakharyuta functions. See Bloom [33], Appendix B.

Proposition 4.6 *Let S be a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$ and q an admissible weight on a closed subset E of $\mathbb{C}^n$, and set $E_R = E \cap \overline{B}(0, R)$ for every $R > 0$. Then $V^S_{E,q} = V^S_{E_R,q}$ and $\Phi^S_{E,q} = \Phi^S_{E_R,q}$ for R sufficiently large. Furthermore, $V^{S*}_{E,q} \in \mathcal{L}^S_+(\mathbb{C}^n)$.*

Proof: Since $E_R \subseteq E$ for every $R > 0$ we have $\Phi^S_{E,q} \le \Phi^S_{E_R,q}$ and $V^S_{E,q} \le V^S_{E_R,q}$, so we need to prove the reverse inequalities. By Definition 4.4, E is non-pluripolar and by [23], Corollary 4.7.7, a countable union of pluripolar sets is pluripolar. It follows that E_{R_0} is non-pluripolar for some $R_0 > 0$ and consequently E_R is pluripolar for every $R \ge R_0$. By Proposition 4.5, $V^{S*}_{E_R,q} \in \mathcal{L}^S(\mathbb{C}^n)$ and it follows that for some constant $c > 0$

$$V^{S*}_{E_R,q}(z) \le c + H_S(z), \qquad z \in \mathbb{C}^n. \tag{4.1}$$

Condition (iii) in Definition 4.4 implies that we can choose $R_1 \geq R_0$ such that

$$q(z) - H_S(z) \geq c, \qquad z \in E \setminus E_{R_1}. \tag{4.2}$$

Now we take $u \in \mathcal{L}^S(\mathbb{C}^n)$ and assume that $u \leq q$ on E_{R_1}. By (4.1) we have $u \leq c + H_S(z)$ for all $z \in \mathbb{C}^n$ and by (4.2) we have $u \leq q$ on $E \setminus E_{R_1}$. Hence, $u \leq V^S_{E,q}$.

Let $p \in \mathcal{P}^S_m(\mathbb{C}^n)$ for some $m \in \mathbb{N}$ be such that $\|pe^{-mq}\|_{E_{R_1}} \leq 1$. By Theorem 3.6 we have $u = \log|p|^{1/m} \in \mathcal{L}^S(\mathbb{C}^n)$ and $u \leq q$ on E_{R_1}. Again by (4.1) and (4.2) we have $u \leq q$ on $E \setminus E_{R_1}$ as well. Then $\|pe^{-mq}\|_E \leq 1$ and $|p|^{1/m} \leq \Phi^S_{E,q}$. The last statement follows directly from Proposition 4.5. □

For a general compact convex S with $0 \in S$, let $S' = \overline{S \bigcap \mathbb{Q}^n}$ be the closure of the set of rational points in S. Then it is clear that $\mathcal{P}^S_m(\mathbb{C}^n) = \mathcal{P}^{S'}_m(\mathbb{C}^n)$ for every $m \in \mathbb{N}^*$ and consequently $\log\Phi^S_{E,q} = \log\Phi^{S'}_{E,q} \leq V^{S'}_{E,q} \leq V^S_{E,q}$. Observe that $S = S'$ if S is a convex body but $S \neq S'$ for example if S is a line segment in $\mathbb{R}^2$ with irrational slope.

Proposition 4.7 *Let $S \subset \mathbb{R}^n_+$ be compact and convex with $0 \in S$. If $S \cap \mathbb{Q}^n$ is not dense in S, then for every admissible weight q on a closed $E \subset \mathbb{C}^n$ we have $\log\Phi^S_{E,q} \neq V^S_{E,q}$.*

Proof: Since $S' = \overline{S \cap \mathbb{Q}^n} \subsetneq S$, there exists a $\xi \in \mathbb{R}^n$ with $\varphi_{S'}(\xi) < \varphi_S(\xi)$. By Proposition 4.5 there exist constants c and c' such that

$$H_S(z) - c \leq V^S_{E,q} \quad \text{and} \quad V^{S'}_{E,q}(z) \leq H_{S'}(z) + c'.$$

For $r > 0$ sufficiently large we have $\varphi_{S'}(r\xi) + c' < \varphi_S(r\xi) - c$, so for $z = (e^{r\xi_1}, \ldots, e^{r\xi_n})$

$$\log\Phi^S_{E,q}(z) = \log\Phi^{S'}_{E,q}(z) \leq V^{S'}_{E,q}(z) \leq H_{S'}(z) + c' < H_S(z) - c \leq V^S_{E,q}(z).$$

□

The next result regards the Siciak-Zakharyuta functions with respect to S, E and q when we have decreasing sequences of sets $S_j \searrow S$ or increasing sequences of weights $q_j \nearrow q$.

Proposition 4.8 *Let S_j, $j \in \mathbb{N}$ and S be compact convex subsets of $\mathbb{R}^n_+$ with $0 \in S$ and $S_j \searrow S$, q be an admissible weight on a compact subset K of $\mathbb{C}^n$, $(K_j)_{j\in\mathbb{N}}$ be a decreasing sequence of compact sets with $\bigcap_j K_j = K$, and $(q_j)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{LSC}(K_j)$ such that $q_j \nearrow q$. Then:*

- **(i)** $\mathcal{L}^S(\mathbb{C}^n) = \bigcap_{j\in\mathbb{N}} \mathcal{L}^{S_j}(\mathbb{C}^n)$.
- **(ii)** *If $V^{S_j*}_{K,q} \leq q$ on K for some j, then $V^{S_j}_{K,q} \searrow V^S_{K,q}$ as $j \to \infty$.*
- **(iii)** *Every q_j is an admissible weight on K_j, $V^S_{K_j,q_j} \nearrow V^S_{K,q}$ and $\Phi^S_{K_j,q_j} \nearrow \Phi^S_{K,q}$ as $j \to \infty$.*

Proof: **(i)** Obviously $\mathcal{L}^S(\mathbb{C}^n) \subseteq \bigcap_j \mathcal{L}^{S_j}(\mathbb{C}^n)$. Let $u \in \bigcap_j \mathcal{L}^{S_j}(\mathbb{C}^n)$ and set $c_u = \sup_{\overline{\mathbb{D}}^n} u$. Then by Proposition 4.3 we have $u - c_u \leq V^{S_j}_{\overline{\mathbb{D}}^n} = H_{S_j}$ for every j. We have $H_{S_j} \searrow H_S$, so $u \leq c_u + H_S$ and $u \in \mathcal{L}^S(\mathbb{C}^n)$.

(ii) We have $V^S_{K,q} \leq V^{S_j}_{K,q}$ and since $V^{S_j *}_{K,q} \leq q$ the equation $V^{S_j *}_{K,q} = V^{S_j}_{K,q}$ holds. By Proposition 4.5 we have $V^{S_j}_{K,q} \in \mathcal{L}^{S_j}(\mathbb{C}^n)$. Since the sequence is decreasing $V = \lim_{j\to\infty} V^{S_j *}_{K,q} \in \bigcap_j \mathcal{L}^{S_j}(\mathbb{C}^n) = \mathcal{L}^S(\mathbb{C}^n)$ and $V \leq q$ on K. Hence,

$$V^S_{K,q} \leq \lim_{j\to\infty} V^{S_j}_{K,q} \leq V \leq V^S_{K,q}.$$

(iii) Since $F = \{z \in K\,;\, q(z) < +\infty\} \subseteq F_j = \{z \in K_j\,;\, q_j(z) < +\infty\}$ for every j and F is non-pluripolar, the set F_j in non-pluripolar and every q_j is an admissible weight on K_j. Since the sequence $(V^S_{K_j,q_j})_{j\in\mathbb{N}}$ is increasing and bounded above by $V^S_{K,q}$ we need to show that $V^S_{K,q} \leq V = \lim_{j\to\infty} V^S_{K_j,q_j}$. For that purpose we take $u \in \mathcal{L}^S(\mathbb{C}^n)$ with $u \leq q$ on K and $\varepsilon > 0$. For every $z_0 \in K$ there exists j_ε such that $u(z_0) - \varepsilon < q_{j_\varepsilon}(z_0)$. Since $u \in \mathcal{USC}(\mathbb{C}^n)$ and $q_{j_\varepsilon} \in \mathcal{LSC}(K_j)$ it follows that we have $u(z) - \varepsilon < q_{j_\varepsilon}(z)$ for all $z \in K_j \cap U_0$ for some neighborhood U_0 of z_0 in $\mathbb{C}^n$. A simple compactness argument gives that there exists an open neighborhood U of K such that $u - \varepsilon < q_j$ on $K_j \subset U$ for $j \geq j_\varepsilon$, possibly with j_ε replaced by a larger number. Hence, $u - \varepsilon \leq V$, and since ε is arbitrary we conclude that $u \leq V$. By taking supremum over u we get $V^S_{K,q} \leq V$. The same argument for $\log|p|^{1/m}$, $p \in \mathcal{P}^S_m(\mathbb{C}^n)$, in the role of u implies that $\Phi^S_{K,q} \leq \lim_{j\to\infty} \Phi^S_{K_j,q_j}$ by Proposition 2.2. □

5 Regularity of Siciak-Zakharyuta functions

In this section we study regularity of the functions $\Phi^S_{E,q}$ and $V^S_{E,q}$, where we assume that S is a convex subset of $\mathbb{R}^n_+$ with $0 \in S$ and that $q\colon E \to \mathbb{R}\cup\{+\infty\}$ is a function on a subset E of $\mathbb{C}^n$. Since the Siciak function $\Phi^S_{E,q}$ is the supremum of a subclass of $\mathcal{C}(\mathbb{C}^n)$ we have $\Phi^S_{E,q} \in \mathcal{LSC}(\mathbb{C}^n)$. If q is an admissible weight on a closed set E, then by Proposition 4.6 we have $V^{S*}_{E,q} \in \mathcal{L}^S_+(\mathbb{C}^n)$.

If $V^{S*}_{E,q} \leq q$ on E then $V^S_{E,q} = V^{S*}_{E,q}$ and we conclude that $V^S_{E,q}$ is continuous at every point where it is lower semicontinuous. In this section we prove that $V^S_{E,q} \in \mathcal{LSC}(\mathbb{C}^{*n})$ for every admissible weight on a compact set K. For that purpose we need to discuss regularization of plurisubharmonic functions, but we begin with local $\mathcal{L}$-regularity of sets.

Definition 5.1 A subset E of $\mathbb{C}^n$ is said to be *$\mathcal{L}$-regular* at a point $a \in \overline{E}$ if V_E is continuous at a and E is said to be *locally $\mathcal{L}$-regular* at a if $E \cap U$ is $\mathcal{L}$-regular at a for every open neighborhood U of a. We say that E is (*locally*) *$\mathcal{L}$-regular* if E is (locally) $\mathcal{L}$-regular at every $a \in \overline{E}$.

The function V_E is continuous at every interior point of E. Moreover, V_E is

continuous at $a \in \overline{E}$ if and only if $V_E^*(a) = 0$. Thus, it is sufficient to check the condition for local $\mathcal{L}$-regularity at boundary points only.

Lemma 5.2 *Let E be a closed subset of $\mathbb{C}^n$, $a \in E$ and assume that there exists a norm with closed unit ball $\mathbb{B}$ such that for some $\delta > 0$ and $b \in E$ we have $a \in B = b + \delta\mathbb{B} \subset E$. Then E is locally $\mathcal{L}$-regular at a.*

Proof: Let $\|\cdot\|$ denote the norm and U be open with $a \in U$. Choose $\tau > 0$ so small that $C = c + \tau\delta\mathbb{B} \subset U$, where $c = (1-\tau)a + \tau b$. We have $C \subset B \subset E$, so

$$0 \le V_E^*(a) \le V_C^*(a) = \log^+(\|a-c\|/\tau\delta) = \log^+(\|a-b\|/\delta) = 0.$$

□

Observe that the lemma implies that every set of the form $E = A + \delta\mathbb{B}$ is locally $\mathcal{L}$-regular, where $A \subset \mathbb{C}^n$ is closed and $\mathbb{B}$ is the closed unit ball with respect to some norm. The following result is a generalization of Siciak [36], Proposition 2.16.

Proposition 5.3 *For every continuous function q on a locally $\mathcal{L}$-regular closed subset E of $\mathbb{C}^n$ we have $V_{E,q}^{S*} \le q$ and consequently $V_{E,q}^S = V_{E,q}^{S*}$.*

Proof: Let $a \in E$ and take $\varepsilon > 0$. Since q is continuous on E there exists an open neighborhood U of a such that $q(z) \le q(a) + \varepsilon$ for every $z \in E \cap U$. Since E is locally $\mathcal{L}$-regular we have $V_{E\cap U}^*(a) = 0$ and

$$V_{E,q}^{S*}(a) \le V_{E\cap U, q(a)+\varepsilon}^{S*}(a) \le \sigma_S V_{E\cap U}^*(a) + q(a) + \varepsilon = q(a) + \varepsilon.$$

Since a and ε are arbitrary the inequality holds. □

Convolution is a standard tool for approximating functions $u \in \mathcal{PSH}(\mathbb{C}^n)$. We define a convolution operator $L^1_{\mathrm{loc}}(\mathbb{C}^n) \to L^1_{\mathrm{loc}}(\mathbb{C}^n)$, $u \mapsto u * \mu$, for a given positive Borel measure μ with compact support,

$$u * \mu(z) = \int_{\mathbb{C}^n} u(z-w)\, d\mu(w), \qquad z \in \mathbb{C}^n. \tag{5.1}$$

If $u \in \mathcal{PSH}(\mathbb{C}^n)$ then $u * \mu \in \mathcal{PSH}(\mathbb{C}^n)$ and if μ is a probability measure then $u \mapsto u * \mu$ preserves $\mathcal{L}(\mathbb{C}^n)$. In the case that μ is presented by a $\mathcal{C}^\infty$ function, $\mu = \psi\, d\lambda$, then $u * \mu = u * \psi$ is a $\mathcal{C}^\infty$ function. If we define $\psi_\delta \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ by $\psi_\delta(z) = \delta^{-2n}\psi(z/\delta)$, $\delta > 0$, with $\psi \in \mathcal{C}_0^\infty(\mathbb{C}^n)$ radially symmetric, $\psi \ge 0$, and $\int_{\mathbb{C}^n} \psi\, d\lambda = 1$, then $u * \psi_\delta \searrow u$ as $\delta \searrow 0$. See Klimek [23] or Hörmander [17, 18]. Siciak [36], Proposition 2.12, used convolution to prove that $V_{E,q} \in \mathcal{LSC}(\mathbb{C}^n)$ for compact E and $q \in \mathcal{LSC}(E)$. In general, the class $\mathcal{L}^S(\mathbb{C}^n)$ is not preserved under convolution (cf. Theorem 5.8).

In order to preserve a particular subclass of $\mathcal{PSH}(\mathbb{C}^n)$ under regularization, special methods are sometimes needed. For example homogeneity is preserved under

$$R_\delta u(z) = \int_G u(Az)\psi_\delta(A)\, d\mu(A), \qquad z \in \mathbb{C}^n, \tag{5.2}$$

where G is some group of $n \times n$ matrices with real or complex entries and μ is a positive measure on the matrix space $\mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$. The smoothing kernel ψ_δ is chosen so that it converges to the Dirac measure δ_I at the identity I as $\delta \to 0$. This method only gives a $\mathcal{C}^\infty$ function on $\mathbb{C}^n \setminus \{0\}$ when we integrate over the group of *complex* invertible matrices, and it gives a $\mathcal{C}^\infty$ function on $\mathbb{C}^n \setminus \mathbb{CR}^n$, where $\mathbb{CR}^n = \{\lambda x; \lambda \in \mathbb{C}, x \in \mathbb{R}^n\}$, when we integrate over the group of *real* invertible matrices. See Sigurdsson [40] and Hörmander and Sigurdsson [21].

In order to preserve the classes $\mathcal{L}^S(\mathbb{C}^n)$ for a compact convex $S \subset \mathbb{R}^n_+$ with $0 \in S$ it is natural to choose G as the group of invertible diagonal matrices. This group can be identified with $\mathbb{C}^{*n}$ with coordinate wise multiplication, so it is natural to choose μ as the Lebesgue measure λ on $\mathbb{C}^n$.

In the following text we allow us a slight abuse of notation by identifying a vector denoted by a lower case letter with a diagonal matrix denoted by the corresponding upper case letter. Thus, we identify the vector $a \in \mathbb{C}^n$ with the diagonal matrix A with diagonal a and in particular the vector $\mathbf{1}$ with the identity matrix I. We define

$$\begin{aligned} R_\delta u(z) &= \int_{\mathbb{C}^n} u(Az)\psi_\delta(A)\,d\lambda(A) = \int_{\mathbb{C}^n} u((I+\delta B)z)\psi(B)\,d\lambda(B) \qquad (5.3)\\ &= \int_{\mathbb{C}^n} u((1+\delta w_1)z_1,\dots,(1+\delta w_n)z_n)\psi(w)\,d\lambda(w), \qquad z \in \mathbb{C}^n, \end{aligned}$$

where we choose a function $0 \le \psi \in \mathcal{C}_0^\infty(\mathbb{C}^n)$, rotationally symmetric in each variable with $\int_{\mathbb{C}^n} \psi\,d\lambda = 1$, and set $\psi_\delta(z) = \delta^{-2n}\psi((z-\mathbf{1})/\delta)$. By the Fubini-Tonelli theorem $R_\delta \colon L^1_{\mathrm{loc}}(\mathbb{C}^n) \to L^1_{\mathrm{loc}}(\mathbb{C}^n)$ and $R_\delta u \to u$ in the L^1_{loc} topology as $\delta \to 0$ and with local uniform convergence for $u \in \mathcal{C}(\mathbb{C}^n)$. Furthermore, $R_\delta \colon \mathcal{PSH}(\mathbb{C}^n) \to \mathcal{PSH}(\mathbb{C}^n)$. This implies that if $u \in \mathcal{L}^S(\mathbb{C}^n)$ then $R_\delta u \in \mathcal{L}^S(\mathbb{C}^n)$, more precisely, if $u \le c_u + H_S$, Proposition 3.2 implies

$$\begin{aligned} R_\delta u(z) &\le \int_{\mathbb{C}^n} \big(c_u + H_S(\mathbf{1}+\delta w) + H_S(z)\big)\psi(w)\,d\lambda(w)\\ &\le c_u + C\sigma_S\delta + H_S(z), \qquad z \in \mathbb{C}^n, \end{aligned}$$

where $C = \sup_{w\in\operatorname{supp}\psi} \|w\|_\infty$. The linear map $\mathbb{C}^n \to \mathbb{C}^n$, $A \mapsto Az$, has the Jacobi determinant $|z_1\cdots z_n|^2$, so for every $z \in \mathbb{C}^{*n}$ and corresponding matrix Z with z on the diagonal we have

$$\begin{aligned} R_\delta u(z) &= \int_{\mathbb{C}^n} u(w)\psi_\delta(Z^{-1}w)\,|z_1\cdots z_n|^{-2}\,d\lambda(w)\\ &= \int_{\mathbb{C}^n} u(w)\psi_\delta(w_1/z_1,\dots,w_n/z_n)\,|z_1\cdots z_n|^{-2}\,d\lambda(w). \end{aligned}$$

By applying the Lebesgue dominated convergence theorem we may differentiate with respect to z_j under the integral sign infinitely often, so this shows that for every $u \in L^1_{\mathrm{loc}}(\mathbb{C}^n)$ we have $R_\delta u \in \mathcal{C}^\infty(\mathbb{C}^{*n})$.

Proposition 5.4 *Let S be a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$ and q be an admissible weight on a compact subset K of $\mathbb{C}^n$. Then*

(i) *$V^S_{K,q} \in \mathcal{LSC}(\mathbb{C}^{*n})$, and if $\mathcal{L}^S(\mathbb{C}^n)$ is preserved under convolution (see Theorem 5.8) then $V^S_{K,q} \in \mathcal{LSC}(\mathbb{C}^n)$.*

(ii) *If $V^{S*}_{K,q} \le q$ on K then $V^S_{K,q} \in \mathcal{L}^S(\mathbb{C}^n) \cap \mathcal{C}(\mathbb{C}^{*n})$ and if $\log \Phi^S_{K,q} = V^S_{K,q}$ then $V^S_{K,q} \in \mathcal{C}(\mathbb{C}^n)$.*

Proof: (i) It is sufficient to show that there exists an increasing sequence $(u_j)_{j\in\mathbb{N}}$ in $\mathcal{L}^S(\mathbb{C}^n)\cap\mathcal{C}(\mathbb{C}^{*n})$ such that $u_j \le q$ on K for every j and $\lim_{j\to\infty} u_j(z) = V^S_{K,q}(z)$ for every $z \in \mathbb{C}^{*n}$. (See Klimek [23], Section 2.3.) We take $u \in \mathcal{L}^S(\mathbb{C}^n)$, $\varepsilon > 0$, and let R_δ be the regularization operator (5.3). Since $R_\delta u \searrow u$ and $q \in \mathcal{LSC}(K)$ there exists δ_ε such that $u - \varepsilon \le R_\delta u - \varepsilon < q$ on K for every $\delta \le \delta_\varepsilon$. These estimates tell us that there exists a family $\mathcal{F} \subseteq \mathcal{L}^S(\mathbb{C}^n) \cap \mathcal{C}^\infty(\mathbb{C}^{*n})$ such that $V^S_{K,q} = \sup \mathcal{F}$. By the Choquet lemma $V^S_{K,q} = \sup \mathcal{G}$ for some countable subfamily $\mathcal{G}$. By arranging the elements of $\mathcal{G}$ into a sequence $(v_j)_{j\in\mathbb{N}}$ and then setting $u_j = \max_{k\le j} v_k$, we have $u_j \nearrow V^S_{K,q}$ as $j \to \infty$. Hence, $V^S_{K,q} \in \mathcal{LSC}(\mathbb{C}^{*n})$. This is a modification the proof of [36], Proposition 2.12, where the regularization operator is given by (5.1), and that is the second statement.

(ii) We have $V^{S*}_{K,q} \ge V^S_{K,q}$ and by the definition of $V^S_{K,q}$ we have $V^{S*}_{K,q} \le V^S_{K,q}$. Hence, $V^S_{K,q} = V^{S*}_{K,q} \in \mathcal{USC}(\mathbb{C}^n)$ and from **(i)** it follows that $V^S_{K,q} \in \mathcal{C}(\mathbb{C}^{*n})$. The last statement follows from the fact that $\Phi^S_{E,q} \in \mathcal{LSC}(\mathbb{C}^n)$. □

Our next task is to characterize the $\mathcal{L}^S(\mathbb{C}^n)$ classes that are invariant under the convolution operator $u \mapsto u * \psi$ given by (5.1). For that purpose we need to review a few facts from convexity theory and define the hull of a convex set in $\mathbb{R}^n_+$ with respect to a cone. Recall that a point x in a convex set S is said to be an *extreme point of S* if the only representation of x as a convex combination $x = (1-t)a + tb$ with $a, b \in S$ and $t \in]0,1[$ is the case $x = a = b$. We let $\operatorname{ext} S$ denote the set of all extreme points in the convex set S. Note that by the Minkowski theorem [19], Theorem 2.1.9, every non-empty compact convex set S is the closed convex hull of its extreme points and

$$\varphi_S(\xi) = \max_{x\in \operatorname{ext} S} \langle x, \xi\rangle, \qquad \xi \in \mathbb{R}^n. \tag{5.4}$$

Every affine hyperplane $\{x \in \mathbb{R}^n\,;\, \langle x,\xi\rangle = \varphi_S(\xi)\}$ with $\xi \in \mathbb{R}^{n*}$ is called a *supporting hyperplane of the set S*. For every $s \in \partial S$ the set $N^S_s = \{\xi \in \mathbb{R}^n\,;\, \langle s,\xi\rangle = \varphi_S(\xi)\}$ is a convex cone which is called the *normal cone of S at the point s*. For every ξ the upper bound in (5.4) is attained at some $s \in \operatorname{ext} S$, so $\mathbb{R}^n = \bigcup_{s\in\operatorname{ext} S} N^S_s$.

Every normal cone N^S_s is an intersection of closed half spaces in $\mathbb{R}^n$ with the origin in the boundary hyperplanes, that is $N^S_s = \bigcap_{t\in\partial S}\{\xi \in \mathbb{R}^n\,;\, \langle s-t,\xi\rangle \ge 0\}$. Observe that ∂S can be replaced by the set of extreme points $\operatorname{ext} S$. In the case

S is a convex polytope and s is an extreme point, then N_s^S is an intersection of finitely many half spaces in $\mathbb{R}^n$ with boundary hyperplanes containing the origin. See Figure 2(b).

If $\xi \in \mathbb{R}^{n*}$ and $\xi_0 \in \mathbb{R}$, then $|\langle x, \xi\rangle + \xi_0|/|\xi|$ is the distance from the point $x \in \mathbb{R}^n$ to the hyperplane $\{y \in \mathbb{R}^n\,;\, \langle y, \xi\rangle + \xi_0 = 0\}$. For supporting hyperplanes with normal ξ we have $\xi_0 = -\varphi_S(\xi)$. Hence, $|\varphi_S(\xi)|/|\xi|$ is the euclidean distance from the origin in $\mathbb{R}^n$ to the supporting hyperplane $\{x \in \mathbb{R}^n\,;\, \langle x, \xi\rangle = \varphi_S(\xi)\}$.

Recall that a subset Γ of $\mathbb{R}^n$ is said to be a cone if $t\xi \in \Gamma$ for every $\xi \in \Gamma$ and $t \in \mathbb{R}_+$. The *dual cone* $\Gamma^\circ = \{x \in \mathbb{R}^n\,;\, \langle x, \xi\rangle \geq 0\ \forall \xi \in \Gamma\}$ of Γ is a closed convex cone and if $\Gamma \neq \mathbb{R}^n$ is closed and convex, then $\Gamma^{\circ\circ} = \Gamma$.

Definition 5.5 For every cone $\Gamma \subset (\mathbb{R}^n \setminus \mathbb{R}^n_-) \cup \{0\}$ with $\Gamma \neq \{0\}$ and every subset S of $\mathbb{R}^n_+$ we define the Γ-*hull* of S by

$$\widehat{S}_\Gamma = \{x \in \mathbb{R}^n_+\,;\, \langle x, \xi\rangle \leq \varphi_S(\xi)\ \forall \xi \in \Gamma\},$$

and we say that S is Γ-*convex* if $S = \widehat{S}_\Gamma$.

Note that if $\Gamma_1 \subseteq \Gamma_2$ then $\widehat{S}_{\Gamma_2} \subseteq \widehat{S}_{\Gamma_1}$. For every compact and convex S we have $S = \{x \in \mathbb{R}^n\,;\, \langle x, \xi\rangle \leq \varphi_S(\xi)\ \forall \xi \in \mathbb{R}^n\} = \widehat{S}_{\mathbb{R}^n}$, which implies that $S \subseteq \widehat{S}_\Gamma$ for every cone $\Gamma \subseteq \mathbb{R}^n$.

Proposition 5.6 *Let S be a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$ and Γ be a proper closed convex cone containing at least one point in $\mathbb{R}^{*n}_+$. Then*

$$\widehat{S}_\Gamma = (S - \Gamma^\circ) \cap \mathbb{R}^n_+,$$

and $\varphi_{\widehat{S}_\Gamma}(\xi) = \varphi_S(\xi)$ holds for every $\xi \in \Gamma$. If for every $\xi \in \mathbb{R}^n$ and every extreme point x of $\widehat{S}_\Gamma$ there exists $\eta \in \Gamma$ such that $\langle x, \xi\rangle \leq \langle x, \eta\rangle$ and $\varphi_S(\eta) = \varphi_S(\xi)$, then S is Γ-convex.

Proof: Take $a = s - t \in S' = (S - \Gamma^\circ) \cap \mathbb{R}^n_+$ with $s \in S$ and $t \in \Gamma^\circ$. For every $\xi \in \Gamma$ we have $\langle t, \xi\rangle \geq 0$ which implies $\langle a, \xi\rangle = \langle s, \xi\rangle - \langle t, \xi\rangle \leq \varphi_S(\xi)$ and $a \in \widehat{S}_\Gamma$.

Conversely, we take $a \notin S'$ and prove that $a \notin \widehat{S}_\Gamma$. Without restriction, we may assume that $a \in \mathbb{R}^n_+$. Since $S - \Gamma^\circ$ is convex the Hahn-Banach theorem implies that $\{a\}$ and $S - \Gamma^\circ$ can be separated by an affine hyperplane. Hence, there exists $\xi \in \mathbb{R}^n \setminus \{0\}$ and $c \in \mathbb{R}$ such that $\langle a, \xi\rangle > c$ and $\langle x, \xi\rangle \leq c$ for every $x \in S - \Gamma^\circ$. By replacing c by $\sup_{x \in S - \Gamma^\circ} \langle x, \xi\rangle$ we may assume that there exists $s \in S$ and $t \in \Gamma^\circ$ such that $\langle s - t, \xi\rangle = c$. Now we need to prove that $\xi \in \Gamma = \Gamma^{\circ\circ}$ by showing that $\langle y, \xi\rangle \geq 0$ for every $y \in \Gamma^\circ$. Since Γ° is a convex cone, we have $t + y \in \Gamma^\circ$ and $c - \langle y, \xi\rangle = \langle s - t - y, \xi\rangle \leq c$. Hence, $\langle y, \xi\rangle \geq 0$. This implies that $\langle a, \xi\rangle > c \geq \varphi_S(\xi)$ and we conclude that $a \notin \widehat{S}_\Gamma$.

Let $\xi \in \Gamma$ and take $a = s - t \in \widehat{S}_\Gamma$, such that $s \in S$, $t \in \Gamma^\circ$, and $\varphi_{\widehat{S}_\Gamma}(\xi) = \langle s - t, \xi\rangle$. Since $\langle s, \xi\rangle \leq \varphi_S(\xi) \leq \varphi_{\widehat{S}_\Gamma}(\xi) = \langle s, \xi\rangle - \langle t, \xi\rangle$ and $\langle t, \xi\rangle \geq 0$, we conclude that

$t = 0$ and $\varphi_S(\xi) = \varphi_{\widehat{S}_\Gamma}(\xi)$. If for every $\xi \in \mathbb{R}^n$ and every extreme point x of $\widehat{S}_\Gamma$ there exists $\eta \in \Gamma$ such that $\langle x, \xi\rangle \le \langle x, \eta\rangle$ and $\varphi_S(\eta) = \varphi_S(\xi)$, then $\widehat{S}_\Gamma = \widehat{S}_{\mathbb{R}^n} = S$. □

Definition 5.7 We say that a subset S of $\mathbb{R}^n_+$ is a *lower set* if for every $s \in S$ the cube $C_s = [0, s_1] \times \cdots \times [0, s_n]$ is a subset of S and we call

$$\widehat{S}_{\mathbb{R}^n_+} = \bigcup_{s\in S} C_s = \big(S - \mathbb{R}^n_+\big) \cap \mathbb{R}^n_+$$

the *lower hull of* S.

We have a characterization of lower sets:

Theorem 5.8 *Let $S \subset \mathbb{R}^n_+$ be compact and convex with $0 \in S$ and $\sigma_S = \varphi_S(\mathbf{1})$. Then the following are equivalent:*

- **(i)** *$S \subset \mathbb{R}^n_+$ is a lower set,*
- **(ii)** *$S = \widehat{S}_{\mathbb{R}^n_+}$,*
- **(iii)** *$\varphi_S(\xi) = \varphi_S(\xi^+)$ for every $\xi \in \mathbb{R}^n$,*
- **(iv)** *$H_S(z - w) \le \sigma_S\|w\|_\infty + H_S(z)$ for every $z, w \in \mathbb{C}^n$,*
- **(v)** *$\mathcal{L}^S(\mathbb{C}^n)$ is translation invariant, and*
- **(vi)** *if $u \in \mathcal{L}^S(\mathbb{C}^n)$ then $u * \psi \in \mathcal{L}^S(\mathbb{C}^n)$ for every $\psi \in \mathcal{C}^\infty_0(\mathbb{C}^n)$ with $\psi \ge 0$ and $\int_{\mathbb{C}^n} \psi\, d\lambda = 1$.*

Proof: (i)⇔(ii)⇔(iii): Observe that the extreme points of C_s are $t = (t_1, \ldots, t_n)$ with $t_j = 0$ or $t_j = s_j$, so $\varphi_{C_s}(\xi) = \sup_{t\in \operatorname{ext} C_s} \langle t, \xi\rangle = \langle s, \xi^+\rangle$ for every $\xi \in \mathbb{R}^n$. Hence, for every lower set S we have $\varphi_S(\xi) = \sup_{s\in S} \langle s, \xi^+\rangle = \varphi_S(\xi^+)$ for $\xi \in \mathbb{R}^n$. This equivalence follows from Proposition 5.6 with $\eta = \xi^+$.

(iii)⇒(iv): Let $w \in \mathbb{C}^n$, $z \in \mathbb{C}^{*n}$ and assume that $z - w \in \mathbb{C}^{*n}$. Since $\varphi_S(\xi) - \varphi_S(\eta) \le \varphi_S(\xi - \eta)$ and $\varphi(\xi) \le \sigma_S\|\xi\|_\infty$ for every $\xi, \eta \in \mathbb{R}^n$ and $|\log^+ x - \log^+ y| \le |x - y|$ for every $x, y \in \mathbb{R}_+$, **(iii)** implies that

$$\begin{aligned} &H_S(z - w) - H_S(z) = \varphi_S(\operatorname{Log}^+(z - w)) - \varphi_S(\operatorname{Log}^+(z)) \\ &\le \sigma_S \max_j |\log^+ |z_j - w_j| - \log^+ |z_j|| \le \sigma_S \max_j ||z_j - w_j| - |z_j|| \le \sigma_S\|w\|_\infty. \end{aligned}$$

By continuity of the function H_S the inequality follows.

(iv)⇒(v): If $u \le c_u + H_S$, then $u(\cdot - w) \le c_u + \sigma_S\|w\|_\infty + H_S$ for every $w \in \mathbb{C}^n$.

(v)⇒(vi): It is sufficient to show that $H_S * \psi \in \mathcal{L}^S(\mathbb{C}^n)$. We observe that for every $\gamma \in]0, 1[$ the Riemann sum $A_\gamma = \sum_{\alpha\in\mathbb{Z}^{2n}} \psi(\gamma\alpha)\gamma^{-2n}$ tends to $1 = \int_{\mathbb{C}^n} \psi\, d\lambda$ as $\gamma \to 0$. This implies that the function $u_\gamma \colon \mathbb{C}^n \to \mathbb{R}$ defined by the Riemann sum

$$u_\gamma(z) = \sum_{\alpha\in\mathbb{Z}^{2n}} H_S(z - \gamma\alpha)\psi(\gamma\alpha)\gamma^{-2n}/A_\gamma$$

tends to $H_S * \psi$ as $\gamma \to 0$. By **(v)** the function u_γ is in $\mathcal{L}^S(\mathbb{C}^n)$ for it is a convex combination of functions in $\mathcal{L}^S(\mathbb{C}^n)$ and we have $u_\gamma - c_\gamma \le V^S_{\overline{\mathbb{D}}^n} = H_S$, where $c_\gamma = \sup_{\overline{\mathbb{D}}^n} u_\gamma$. Furthermore, since $H_S \in \mathcal{C}(\mathbb{C}^n)$ the convergence is locally uniform, so by Proposition 4.3 we have $H_S * \psi(z) \le \sup_{\overline{\mathbb{D}}^n} H_S * \psi + H_S(z)$ for $z \in \mathbb{C}^n$ and conclude that $H_S * \psi \in \mathcal{L}^S(\mathbb{C}^n)$.

(vi)⇒(iii): We begin by taking $\psi(\zeta) = \chi(\zeta_1) \cdots \chi(\zeta_n)$, where $0 \le \chi \in \mathcal{C}^\infty_0(\mathbb{C})$ is rotationally invariant, $\operatorname{supp}\chi \subset \overline{\mathbb{D}}$, and $\int_{\mathbb{C}} \chi \, d\lambda = 1$ and observe that with this choice of ψ we have $(f * \psi)(z) = \sum_{j=1}^n (f_j * \chi)(z_j)$ for any locally integrable function of the form $f(z) = \sum_{j=1}^n f_j(z_j)$.

Let $\eta \in \mathbb{R}^{*n}$ have at least one strictly negative coordinate, enumerate the coordinates so that $\eta_j < 0$ for $j = 1, \dots, \ell$ and $\eta_j > 0$ for $j = \ell + 1, \dots, n$, and take $s \in S$ such that $\varphi_S(\eta) = \langle s, \eta \rangle$. Then for every $t > 0$ we have

$$\begin{aligned}
H_S * \psi(e^{t\eta}) - H_S(e^{t\eta}) &= \int_{\mathbb{C}^n} \varphi_S(\operatorname{Log}(e^{t\eta} - \zeta)) \, \psi(\zeta) \, d\lambda(\zeta) - \langle s, t\eta \rangle \\
&\ge \int_{\mathbb{C}^n} \langle s, \operatorname{Log}(e^{t\eta} - \zeta) - t\eta \rangle \, \psi(\zeta) \, d\lambda(\zeta) \\
&= \sum_{j=1}^n s_j \int_{\mathbb{D}} \left(\log |e^{t\eta_j} - \zeta_j| - t\eta_j \right) \chi(\zeta_j) \, d\lambda(\zeta_j) \\
&= -t\langle s', \eta' \rangle + \sum_{j=1}^{\ell} s_j \int_{\mathbb{D}} \log |e^{t\eta_j} - \zeta_j| \chi(\zeta_j) \, d\lambda(\zeta_j) \\
&+ \sum_{j=\ell+1}^{n} s_j \int_{\mathbb{D}} \log |1 - e^{-t\eta_j} \zeta_j| \chi(\zeta_j) \, d\lambda(\zeta_j).
\end{aligned}$$

We let $v = \log|\cdot| \in \mathcal{SH}(\mathbb{C}) \cap \mathcal{H}(\mathbb{C}^*)$. Then for $j = 1, \dots, \ell$

$$\int_{\mathbb{D}} \log |e^{t\eta_j} - \zeta_j| \, \chi(\zeta_j) \, d\lambda(\zeta_j) = (v * \chi)(e^{t\eta_j}) \to v * \chi(0), \qquad t \to +\infty.$$

Since $v \in \mathcal{H}(\mathbb{C}^*)$, χ is rotationally invariant, and $\operatorname{supp}\chi \subset \overline{\mathbb{D}}$, the mean value property gives for $j = \ell + 1, \dots, n$

$$\int_{\mathbb{D}} \log |1 - e^{-t\eta_j} \zeta_j| \, \chi(\zeta_j) \, d\lambda(\zeta_j) = \log 1 = 0.$$

Hence,

$$H_S * \psi(e^{t\eta}) - H_S(e^{t\eta}) \ge -t\langle s', \eta' \rangle + \sum_{j=1}^{\ell} s_j (v * \chi)(e^{t\eta_j}). \tag{5.5}$$

Assume that **(iii)** does not hold and take $\xi_0 \in \mathbb{R}^n$ such that $\varphi_S(\xi_0) < \varphi_S(\xi_0^+)$. Then at least one coordinate of ξ_0 is strictly negative. By continuity we can choose $\xi_0 \in \mathbb{R}^{*n}$, and we can renumber the coordinates so that $\xi_{0,j} < 0$ for $j = 1, \dots, \ell$ and $\xi_{0,j} > 0$ for $j = \ell+1, \dots, n$. By continuity there exists an open neighborhood

U of ξ_0^+ such that $\varphi_S(\xi_0) < \varphi_S(\eta)$ for every $\eta \in U$. We fix $\eta = (\eta', \xi_0'') \in U$ with $\eta_j < 0$ for $j = 1, \dots, \ell$. There exists a point $s = (s', s'') \in \partial S$, such that $\langle s, \eta\rangle = \langle s', \eta'\rangle + \langle s'', \eta''\rangle = \varphi_S(\eta)$. We have $\langle s', \eta'\rangle \le 0$. Equality is excluded for it would imply $s' = 0'$ and $\varphi_S(\eta) = \langle s'', \xi_0''\rangle = \langle (0', s''), \xi_0\rangle = \langle p, \xi_0\rangle \le \varphi_S(\xi_0)$, contradicting the choice of η. Hence, $\langle s', \eta'\rangle < 0$ and the estimate (5.5) implies that $H_S * \psi - H_S$ is unbounded, contradicting **(vi)**. □

6 Monge-Ampère masses

The equilibrium measure for a bounded non-pluripolar set $E \subset \mathbb{C}^n$ is the Monge-Ampère operator of V_E, defined as $\mu_E = (dd^c V_E^*)^n$ where $d^c = i(\overline{\partial} - \partial)$ and $(dd^c V_E^*)^n = dd^c V_E^* \wedge \cdots \wedge dd^c V_E^*$ is defined in terms of currents. Similarly denote the Monge-Ampère measure of $V_{E,q}^{S*}$ by

$$\mu_{E,q}^S = \left(dd^c V_{E,q}^{S*}\right)^n.$$

Theorem 6.1 *Let $S \subset \mathbb{R}_+^n$ be compact and convex with $0 \in S$, $E \subset \mathbb{C}^n$ be closed, and q be an admissible weight on E. Then*

(i) $\operatorname{supp} \mu_{E,q}^S \subseteq \{z \in E\,;\, V_{E,q}^{S*}(z) \ge q(z)\}$, *and*

(ii) $\{z \in E\,;\, V_{E,q}^{S*}(z) > q(z)\}$ *is pluripolar.*

Proof: (i) We need to prove that $V_{E,q}^{S*}$ is maximal in

$$U = (\mathbb{C}^n \setminus E) \cup \{z \in E\,;\, V_{E,q}^{S*}(z) < q(z)\},$$

which is open. Take $a \in U$. If $a \in \mathbb{C}^n \setminus E$ then we take $r > 0$ such that $B(a, r) \in \mathbb{C}^n \setminus E$. If on the other hand $a \in E$ and $V_{E,q}^{S*}(a) < q(a)$ then by upper semicontinuity of $V_{E,q}^{S*}$ and lower semicontinuity of q there exists $r > 0$ such that

$$\sup_{\zeta \in B(a,r)} V_{E,q}^{S*}(\zeta) < \inf_{\zeta \in E \cap B(a,r)} q(\zeta).$$

We need to prove that the restriction of $\mu_{E,q}^S$ to some $B(a, r)$ is the zero measure. We let $V \in \mathcal{PSH}(\mathbb{C}^n)$ be the function given by

$$V(z) = \begin{cases} V_{E,q}^{S*}(z), & z \in \mathbb{C}^n \setminus B(a, r), \\ u(z), & z \in B(a, r), \end{cases}$$

where u is the Perron-Bremermann function on $B(a, r)$ with boundary values $V_{E,q}^{S*}$, i.e.,

$$u(z) = \sup\{v(z)\,;\, v \in \mathcal{PSH}(B(a, r)),\, v^* \le V_{E,q}^{S*} \text{ on } \partial B(a, r)\}.$$

Then $V_{E,q}^{S*} \le V$ and since V only deviates from $V_{E,q}^{S*}$ on a compact set we have $V \in \mathcal{L}^S(\mathbb{C}^n)$ by Proposition 4.6. Furthermore, in the case $V_{E,q}^{S*}(a) < q(a)$ the maximum principle implies

$$V(z) \le \sup_{\zeta \in B(a,r)} V_{E,q}^{S*}(\zeta) < \inf_{\zeta \in B(a,r)} q(\zeta) \le q(z), \qquad z \in B(a,r),$$

and it implies that $V \le q$ on E. Hence, $V = V_{E,q}^{S*}$, and by Klimek [23], Theorem 4.4.1, $\mu_{E,q}^S = \big(dd^c V_{E,q}^{S*}\big)^n = 0$ on $B(a,r)$.

(ii) Since $V_{E,q}^S \le q$ on E we have $\{z \in E\,;\, q(z) < +\infty\} \subseteq \{z \in \mathbb{C}^n\,;\, V_{E,q}^S < +\infty\}$ Since q is admissible, the left-hand side is non-pluripolar, and then so is the right-hand side. Since $\mathcal{L}^S(\mathbb{C}^n)/\varphi_S(\mathbf{1}) \subset \mathcal{L}(\mathbb{C}^n)$ it follows from Klimek [23], Proposition 5.2.1 and Theorems 5.2.4 and 4.7.6 the set $\{z \in E\,;\, q < V_{E,q}^{S*}(z)\}$ $\subseteq \{z \in \mathbb{C}^n\,;\, V_{E,q}^S(z) < V_{E,q}^{S*}(z)\}$ is pluripolar. □

The complex Monge-Ampère mass of H_S can be described in terms of the real Monge-Ampère mass of φ_S. Let $U_\ell = H_S * \psi_{\delta_\ell} \searrow H_S$, where $0 < \delta_\ell \searrow 0$ and $\psi_{\delta_\ell}(\zeta) = \delta_\ell^{-2n}\psi(\zeta/\delta_\ell)$, $\psi(\zeta) = \chi(\zeta_1)\cdots\chi(\zeta_n)$, where $0 \le \chi \in \mathcal{C}_0^\infty(\mathbb{C})$ is rotationally invariant, $\operatorname{supp}\chi \subset \overline{\mathbb{D}}$, and $\int_{\mathbb{C}} \chi\, d\lambda = 1$. Then $U_\ell \in \mathcal{C}^\infty(\mathbb{C}^n) \cap \mathcal{PSH}(\mathbb{C}^n)$ and $U_\ell(z) = U_\ell(|z_1|,\dots,|z_n|)$ holds. We set $v_\ell(\xi) = U_\ell(e^{\xi_1},\dots,e^{\xi_n})$. Since

$$\frac{\partial^2 U_\ell(z)}{\partial z_j \partial \overline{z}_k} = \frac{1}{4 z_j \overline{z}_k} \cdot \frac{\partial^2 v_\ell(\xi)}{\partial \xi_j \partial \xi_k}, \qquad z \in \mathbb{C}^{*n}$$

it follows that

$$\det\left(\frac{\partial^2 U_\ell(z)}{\partial z_j \partial \overline{z}_k}\right) = \frac{1}{4^n |z_1 \cdots z_n|^2} \det\left(\frac{\partial^2 v_\ell(\xi)}{\partial \xi_j \partial \xi_k}\right)\Big|_{\xi = \operatorname{Log} z}.$$

With $z_j = e^{\xi_j + i\theta_j}$, $\theta_j \in [0, 2\pi]$ we have

$$\big(\tfrac{i}{2}\big)^n\, dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n = |z_1 \cdots z_n|^2 d\xi d\theta$$

and on $\mathbb{C}^{*n}$ we have

$$(dd^c U_\ell)^n = n! \det\left(\frac{\partial^2 v_\ell(\xi)}{\partial \xi_j \partial \xi_k}\right) d\xi d\theta.$$

The real Monge-Ampère measure of v, denoted $\mathcal{MA}_{\mathbb{R}}(v)$, is defined for $\mathcal{C}^2$ function by

$$\mathcal{MA}_{\mathbb{R}}(v) = \det\left(\frac{\partial^2 v(\xi)}{\partial \xi_j \partial \xi_k}\right) d\xi,$$

and extended to convex functions by locally uniform limits. This is done in detail in Proposition 2.6 in Figalli's book [16].

Letting $\ell \to \infty$, we have for every Borel set $E \subset \mathbb{R}^n$

$$\int\limits_{\operatorname{Log}^{-1}(E)} (dd^c H_S)^n = n!\,\big(\mathcal{MA}_{\mathbb{R}}(\varphi_S) \otimes d\theta\big)(E \times [0,2\pi]^n) = (2\pi)^n n!\, \mathcal{MA}_{\mathbb{R}}(\varphi_S)(E).$$

In particular

$$\int_{\mathbb{T}^n} (dd^c H_S)^n = (2\pi)^n n!\, \mathcal{MA}_{\mathbb{R}}(\varphi_S)(\{0\}). \tag{6.1}$$

This will be useful for our next result. Its proof is borrowed from Rashkovskii [32], Theorem 3.4.

Theorem 6.2 *Let $S \subset \mathbb{R}^n_+$ be compact and convex with $0 \in S$, $E \subset \mathbb{C}^n$ be closed, and q be an admissible weight on E. Then*

$$\mu^S_{E,q}(\mathbb{C}^n) = (2\pi)^n n!\, \mathrm{vol}(S),$$

where vol *denotes the euclidean volume.*

Proof: By Proposition 4.5, $V^S_{K,q} - H_S$ is bounded. The comparison principle, Klimek [23], Theorem 3.7.1, then implies that

$$\mu^S_{K,q}(\mathbb{C}^n) = \int_{\mathbb{C}^n} (dd^c H_S)^n = \mu^S_{\mathbb{T}^n,0}(\mathbb{C}^n).$$

By Theorem 6.1 **(i)**, $\mu^S_{\mathbb{T}^n,0}(\mathbb{C}^n) = \int_{\mathbb{T}^n} (dd^c H_S)^n$. We already established in (6.1) that

$$\int_{\mathbb{T}^n} (dd^c H_S)^n = (2\pi)^n n! \mathcal{MA}_{\mathbb{R}}(\varphi_S)(\{0\})$$

By Blocki [10], see also Figalli [16], we have

$$\mathcal{MA}_{\mathbb{R}}(\varphi_S)(\{0\}) = \mathrm{vol}(\{s \in \mathbb{R}^n\,;\, \langle s, \xi\rangle \le \varphi_S(\xi),\ \forall \xi \in \mathbb{R}^n\}) = \mathrm{vol}(S).$$

□

7 Characterization of polynomials by L^2-estimates

In this section we study characterization of the polynomial spaces $\mathcal{P}^S_m(\mathbb{C}^n)$ with weighted L^2-norms of entire functions. Recall that the Liouville type Theorem 3.6 states that if $f \in \mathcal{O}(\mathbb{C}^n)$ and $|f|e^{-mH_S - a\log^+\|\cdot\|_\infty} \in L^\infty(\mathbb{C}^n)$ for some $a \in [0, d_m[$, where d_m is the distance between mS and $\mathbb{N}^n \setminus mS$ in the L^1-norm, then $f \in \mathcal{P}^S_m(\mathbb{C}^n)$. We take a measurable function $\psi\colon \mathbb{C}^n \to \overline{\mathbb{R}}$ and let $L^2(\mathbb{C}^n, \psi)$ denote the space of all measurable $f\colon \mathbb{C}^n \to \mathbb{C}$ such that

$$\|f\|^2_\psi = \int_{\mathbb{C}^n} |f|^2 e^{-\psi}\, d\lambda < +\infty. \tag{7.1}$$

Proposition 7.1 *Let $f \in L^2(\mathbb{C}^n, \psi) \cap \mathcal{O}(\mathbb{C}^n)$ for some measurable $\psi\colon \mathbb{C}^n \to \overline{\mathbb{R}}$ with power series expansion $f(z) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha$ at the origin. Then for every polyannulus $A_{\sigma,\tau} = \{\zeta \in \mathbb{C}^n\,;\, e^{\sigma_j} \le |\zeta_j| < e^{\tau_j}\}$ in $\mathbb{C}^n$, where $\sigma, \tau \in \mathbb{R}^n$, $\sigma_j < \tau_j$ for $j = 1, \dots, n$, with volume $v(A_{\sigma,\tau}) = \pi^n \prod_{j=1}^n (e^{2\tau_j} - e^{2\sigma_j})$, we have*

$$|a_\alpha| \le \frac{\|f\|_\psi}{v(A_{\sigma,\tau})} \Big(\int_{A_{\sigma,\tau}} \frac{e^{\psi(\zeta)}}{|\zeta^\alpha|^2}\, d\lambda(\zeta) \Big)^{1/2}. \tag{7.2}$$

If ψ is rotationally invariant in each variable ζ_j, then for $\chi(\xi) = \frac{1}{2}\psi(e^{\xi_1}, \dots, e^{\xi_n})$ and $K_{\sigma,\tau} = \prod_{j=1}^n [\sigma_j, \tau_j] \subset \mathbb{R}^n$ we have

$$|a_\alpha| \le \frac{\|f\|_\psi}{\prod_{j=1}^n (1 - e^{-2(\tau_j - \sigma_j)})} e^{-\langle \mathbf{1}, \tau\rangle} \left(\int_{K_{\sigma,\tau}} e^{2(\chi(\xi) - \langle \alpha, \xi\rangle)}\, d\lambda(\xi) \right)^{1/2}. \tag{7.3}$$

Proof: By the Cauchy formula for derivatives

$$\begin{aligned} a_\alpha &= \frac{1}{(2\pi i)^n} \int_{C_r} \frac{f(\zeta)}{\zeta^\alpha} \cdot \frac{d\zeta_1 \cdots d\zeta_n}{\zeta_1 \cdots \zeta_n} \\ &= \frac{1}{(2\pi)^n} \int_{[-\pi,\pi]^n} \frac{f(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})}{r^\alpha e^{i\langle \alpha, \theta\rangle}}\, d\theta_1 \cdots d\theta_n, \end{aligned}$$

where $C_r = \{z \in \mathbb{C}^n\,;\, |z_j| = r_j\}$, is any polycircle with center 0 and polyradius $r \in \mathbb{R}_+^{*n}$. We parametrize C_r by $[-\pi, \pi]^n \ni \theta \mapsto (r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})$, multiply the integral by $r_1 \cdots r_n\, dr_1 \cdots dr_n$, integrate with respect to r_j over $[e^{\sigma_j}, e^{\tau_j}]$, observe that $\int_{[e^{\sigma_j}, e^{\tau_j}]} r_j\, dr_j = \frac{1}{2}(e^{2\tau_j} - e^{2\sigma_j})$, set $L_{\sigma,\tau} = \prod_{j=1}^n \big([e^{\sigma_j}, e^{\tau_j}] \times [-\pi, \pi]\big)$, and get

$$\begin{aligned} a_\alpha &= \frac{1}{v(A_{\sigma,\tau})} \int_{L_{\sigma,\tau}} \frac{f(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n})}{r^\alpha e^{i\langle \alpha, \theta\rangle}} (r_1\, dr_1 d\theta_1) \cdots (r_n\, dr_n d\theta_n) \\ &= \frac{1}{v(A_{\sigma,\tau})} \int_{A_{\sigma,\tau}} \frac{f(\zeta)}{\zeta^\alpha}\, d\lambda(\zeta) = \frac{1}{v(A_{\sigma,\tau})} \int_{A_{\sigma,\tau}} f(\zeta) e^{-\psi(\zeta)/2} \cdot \frac{e^{\psi(\zeta)/2}}{\zeta^\alpha}\, d\lambda(\zeta). \end{aligned}$$

Now (7.2) follows from the Cauchy-Schwarz inequality and (7.3) by integrating over the angular variables and using the fact that $v(A_{\sigma,\tau}) = \pi^n \prod_{j=1}^n (e^{2\tau_j} - e^{2\sigma_j})$. □

Theorem 7.2 *Let S be a compact convex subset of $\mathbb{R}_+^n$ with $0 \in S$, $m \in \mathbb{N}^*$, and $d_m = d(mS, \mathbb{N}^n \setminus mS)$ denote the euclidean distance between mS and $\mathbb{N}^n \setminus mS$. If $f \in \mathcal{O}(\mathbb{C}^n)$ and for some $a \in [0, d_m[$*

$$\int_{\mathbb{C}^n} |f|^2 (1 + |\zeta|^2)^{-a} e^{-2mH_S}\, d\lambda < +\infty, \tag{7.4}$$

then $f \in \mathcal{P}_m^{\widehat{S}_\Gamma}(\mathbb{C}^n)$, where $\widehat{S}_\Gamma$ is the Γ-hull of S and $\Gamma = \Gamma_a$ consists of all ξ such that the angle between the vectors $\mathbf{1} = (1, \dots, 1)$ and ξ is $\le \arccos(-(d_m - a)/\sqrt{n})$.

Observe that the largest possible d_m is 1 and smallest possible a is 0, which implies that the largest possible opening angle of the cone Γ is $\arccos(-1/\sqrt{n})$ in the case $d_m = 1$ and $\Gamma = \mathbb{R}^n \setminus \mathbb{R}_-^{*n}$. If a_0 is the infimum of a such that (7.4) holds, then $\Gamma_{a_0} = \bigcup_{a > a_0} \Gamma_a$. Therefore f is a polynomial with exponents in $m\widehat{S}_{\Gamma(a_0)} = \bigcap_{a > a_0} m\widehat{S}_{\Gamma(a)}$.

We are interested in conditions on cones $\Lambda \subseteq \Gamma$ which ensure that $\widehat{S}_\Lambda = S$:

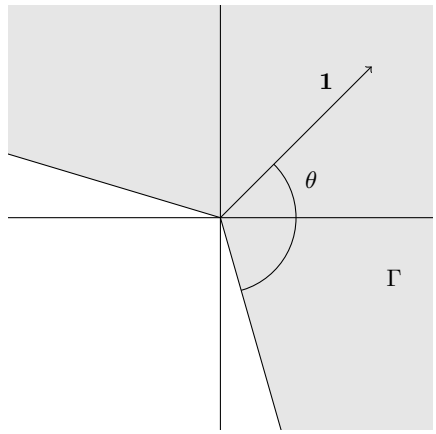

Figure 1: The cone Γ has opening angle $\theta = \arccos(-(d_m - a)/\sqrt{n})$

Corollary 7.3 *The function f in Theorem 7.2 is in $\mathcal{P}^S_m(\mathbb{C}^n)$ in the cases:*

(i) *$S = \widehat{S}_\Lambda$ for some cone Λ contained in $\{\xi \in \mathbb{R}^n\,;\, \langle \mathbf{1}, \xi\rangle \geq 0\}$.*

(ii) *S is a lower set, that is $S = \widehat{S}_{\mathbb{R}^n_+}$.*

(iii) *$(mS) \cap \mathbb{N}^n = (m\widehat{S}_\Gamma) \cap \mathbb{N}^n$.*

Proof of Theorem 7.2: Let $f(z) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha$ be the Taylor expansion of f at the origin. We need to show that $a_\alpha = 0$ for every $\alpha \in \mathbb{N}^n \setminus m\widehat{S}_\Gamma$. Since $\alpha \notin m\widehat{S}_\Gamma$, there exists $\tau \in \Gamma$ such that $|\tau| = 1$ and $\langle \alpha, \tau\rangle > m\varphi_S(\tau)$. By rotating τ we may assume that τ is an interior point of Γ. Since the angle between τ and $\mathbf{1}$ is $\leq \arccos(-(d_m - a)/\sqrt{n})$ we have $-\langle \mathbf{1}, \tau\rangle < d_m - a$. We choose $\varepsilon > 0$ such that $d_m - a - \varepsilon > 0$, and $-\langle \mathbf{1}, \tau\rangle < d_m - a - \varepsilon$. Recall that $\langle \alpha, \tau\rangle - m\varphi_S(\tau)$ is the euclidean distance from α to the supporting hyperplane $\{x\,;\, \langle x, \tau\rangle = m\varphi_S(\tau)\}$, so by assumption $m\varphi_S(\tau) - \langle \alpha, \tau\rangle \leq -d_m$. Hence

$$-\langle \mathbf{1}, \tau\rangle + m\varphi_S(\tau) - \langle \alpha, \tau\rangle < -a - \varepsilon.$$

We take $\sigma \in \mathbb{R}^n \setminus \{0\}$ with $\sigma_j < \tau_j$ for $j = 1, \ldots, n$ such that

$$-\langle \mathbf{1}, \tau\rangle + m\varphi_S(\xi) - \langle \alpha, \xi\rangle < -(a + \varepsilon)|\xi|, \qquad \xi \in K_{\sigma,\tau} = \prod_{j=1}^n [\sigma_j, \tau_j].$$

By homogeneity we get

$$-t\langle \mathbf{1}, \tau\rangle + m\varphi_S(\xi) - \langle \alpha, \xi\rangle < -(a + \varepsilon)|\xi|, \qquad t > 0,\ \xi \in tK_{\sigma,\tau}.$$

Let $\xi_j = \log|\zeta_j|$ and observe that $(1 + |\zeta|^2)^a \leq (n + 1)^a \max\{1, \|\zeta\|^{2a}_\infty\}$. From this inequality and (7.4) it follows that $f \in L^2(\mathbb{C}^n, \psi)$, where

$$\tfrac{1}{2}\psi(\zeta) = a \log\|\zeta\|_\infty + mH_S(\zeta) = a\|\xi\|_\infty + m\varphi_S(\xi).$$

We set $\chi(\xi) = \frac{1}{2}\psi(e^{\xi_1}, \dots, e^{\xi_n})$. Then

$$-t\langle \mathbf{1}, \tau\rangle + \chi(\xi) - \langle \alpha, \xi\rangle < -\varepsilon|\xi|, \qquad t > 0,\ \xi \in tK_{\sigma,\tau}.$$

The estimate (7.3) with $tK_{\sigma,\tau}$ in the role of $K_{\sigma,\tau}$ gives

$$\begin{aligned} |a_\alpha| &\le \frac{\|f\|_\psi}{\prod_{j=1}^n (1 - e^{-2(\tau_j - \sigma_j)t})} e^{-t\langle \mathbf{1}, \tau\rangle} \left(\int_{tK_{\sigma,\tau}} e^{2(\chi(\xi) - \langle \alpha, \xi\rangle)}\, d\lambda(\xi) \right)^{1/2} \\ &\le \frac{\|f\|_\psi}{\prod_{j=1}^n (1 - e^{-2(\tau_j - \sigma_j)t})} e^{-\varepsilon|\sigma|t} t^{n/2} v(K_{\sigma,\tau})^{1/2} \to 0, \qquad t \to +\infty, \end{aligned}$$

and we conclude that $a_\alpha = 0$. □

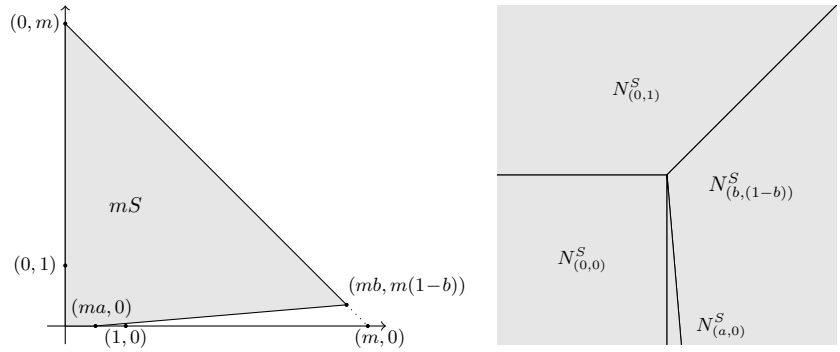

Figure 2: (a) The set S. (b) The normal cones N_s^S of the extreme points of S.

The Γ-hull $\widehat{S}_\Gamma$ can not be replaced by S in Theorem 7.2:

Example 7.4 Let $m \ge 4$ and $S \subseteq \mathbb{R}^2_+$ be the quadrilateral

$$S = \mathrm{ch}\{(0,0), (a,0), (b, 1-b), (0,1)\}.$$

where $0 < a < 1/m$, $0 < a < b < 1$, $m(1-b) < 1$, and $(b-a)/(1-b) > m-2-am$. Then $(1,0), (2,0), \dots, (m-3, 0) \notin mS$, but the calculations below show that $\|p_k\|_{2mH_S} < +\infty$ for $p_k(z) = z^{(k,0)} = z_1^k$ with $k = 1, \dots, m-3$.

Since the map $(\mathbb{R}\times]-\pi, \pi[)^2 \to \mathbb{C}^2$, $(\xi_1, \theta_1, \xi_2, \theta_2) \mapsto (e^{\xi_1 + i\theta_1}, e^{\xi_2 + i\theta_2})$ has the Jacobi determinant $e^{2\xi_1 + 2\xi_2}$, we have

$$\|p_k\|^2_{2mH_S} = \int_{\mathbb{C}^2} |z_1|^{2k} e^{-2mH_S(z)}\, d\lambda(z) = 4\pi^2 \int_{\mathbb{R}^2} e^{2(k+1)\xi_1 + 2\xi_2 - 2m\varphi_S(\xi)}\, d\xi_1 d\xi_2,$$

Now

$$\varphi_S(\xi) = \max_{x\in\operatorname{ext}(S)} \langle x,\xi\rangle = \begin{cases} 0, & \xi \in N^S_{(0,0)},\\ a\xi_1, & \xi \in N^S_{(a,0)},\\ b\xi_1 + (1-b)\xi_2, & \xi \in N^S_{(b,(1-b))},\\ \xi_2, & \xi \in N^S_{(0,1)}. \end{cases}$$

We split the integral over $\mathbb{R}^2$ into the sum of the integrals over the normal cones at the extreme points of S, which we calculate as

$$\int_{N^S_{(0,0)}} e^{2(k+1)\xi_1+2\xi_2}\,d\xi = \int_{-\infty}^0 e^{2(k+1)\xi_1}\,d\xi_1 \int_{-\infty}^0 e^{2\xi_2}\,d\xi_2 = \frac{1}{4(k+1)},$$

$$\begin{aligned}\int_{N^S_{(a,0)}} e^{2(k+1)\xi_1+2\xi_2-2ma\xi_1}\,d\xi &= \int_0^\infty e^{2(k+1-ma)\xi_1} \int_{-\infty}^{-\xi_1(b-a)/(1-b)} e^{2\xi_2}\,d\xi_2\,d\xi_1\\ &= \frac{1}{4((b-a)/(1-b)+ma-1-k)},\end{aligned}$$

$$\begin{aligned}&\int_{N^S_{(b,1-b)}} e^{2(k+1)\xi_1+2\xi_2-2m(b\xi_1+(1-b)\xi_2)}\,d\xi\\ &\quad= \int_0^\infty e^{2(k+1-mb)\xi_1} \int_{-\xi_1(b-a)/(1-b)}^{\xi_1} e^{2(1-m(1-b))\xi_2}\,d\xi_2\,d\xi_1\\ &\quad= \frac{1}{4(1-m(1-b))}\left(\frac{1}{m-2-k} + \frac{1}{(b-a)/(1-b)+ma-1-k}\right),\end{aligned}$$

$$\begin{aligned}\int_{N^S_{(0,1)}} e^{2(k+1)\xi_1+2\xi_2-2m\xi_2}\,d\xi &= \int_0^\infty e^{2(1-m)\xi_2} \int_{-\infty}^{\xi_2} e^{2(k+1)\xi_1}\,d\xi_1\,d\xi_2\\ &= \frac{1}{4(k+1)(m-2-k)}.\end{aligned}$$

This shows that (7.4) is satisfied with p_k in the role of f, but $p_k \notin \mathcal{P}^S_m(\mathbb{C}^n)$.

8 Pullbacks of polynomial classes by polynomial maps

Let $S \subset \mathbb{R}^n_+$ be compact and convex with $0 \in S$ and $f = (f_1,\dots,f_n)\colon \mathbb{C}^\ell \to \mathbb{C}^n$ be a polynomial map. If $f_j(z) = \sum_{\alpha\in\mathbb{N}^\ell} a_{j,\alpha} z^\alpha$, $I_j = \{\alpha \in \mathbb{N}^\ell;\, a_{j,\alpha} \neq 0\}$, and $S_j \subset \mathbb{R}^\ell_+$ be the convex hull of I_j. Then $f_j \in \mathcal{P}^{S_j}_1(\mathbb{C}^\ell)$ and for every $p(w) = \sum_{\beta\in(mS)\cap\mathbb{N}^n} b_\beta w^\beta$ in $\mathcal{P}^S_m(\mathbb{C}^n)$ we have

$$(f^*p)(z) = \sum_{\beta\in(mS)\cap\mathbb{N}^n} b_\beta f_1(z)^{\beta_1}\cdots f_n(z)^{\beta_n}.$$

By Theorem 3.6 we have $|f_j(z)| \le e^{c_j + H_{S_j}(z)}$ which implies that for every $z \in \mathbb{C}^{*\ell}$

$$|(f^*p)(z)| \le \sum_{\beta\in(mS)\cap\mathbb{N}^n} |b_\beta| e^{\langle c,\beta\rangle+\beta_1 H_{S_1}(z)+\cdots+\beta_n H_{S_n}(z)}$$
$$\le \Big(\sum_{\beta\in(mS)\cap\mathbb{N}^n} |b_\beta| e^{\langle c,\beta\rangle}\Big) e^{m\varphi_S(\varphi_{S_1}(\operatorname{Log} z),\ldots,\varphi_{S_n}(\operatorname{Log} z))}.$$

We have for every $\xi\in\mathbb{R}^\ell$ that

$$\varphi_S(\varphi_{S_1}(\xi),\ldots,\varphi_{S_n}(\xi))$$
$$= \sup\{\langle x_1 s_1+\cdots+x_n s_n,\xi\rangle;\, x\in S,\, s_j\in S_j,\, j=1,\ldots,n\}$$
$$= \sup_{x\in S}\varphi_{x_1S_1+\cdots+x_nS_n}(\xi) = \varphi_{S'}(\xi),$$

where $S'=\bigcup_{x\in S} x_1S_1+\cdots+x_nS_n$. The set $S'\subset\mathbb{R}^\ell_+$ is compact and convex.

Proposition 8.1 *Assume that $\overline{S\cap\mathbb{Q}^n}=S$. Then with the notation above S' is the smallest compact convex subset T of $\mathbb{R}^\ell_+$ with $0\in T$ for which $f^*\big(\mathcal{P}^S_m(\mathbb{C}^n)\big)\subseteq\mathcal{P}^T_m(\mathbb{C}^\ell)$ for all $m\in\mathbb{N}$.*

Proof: Assume that $T\subsetneq S'$ such that $f^*\big(\mathcal{P}^S_m(\mathbb{C}^n)\big)\subseteq\mathcal{P}^T_m(\mathbb{C}^\ell)$ for every $m\in\mathbb{N}$. Then there exists $\xi\in\mathbb{R}^\ell$ such that $\varphi_T(\xi)<\varphi_{S'}(\xi)$ and since $S\cap\mathbb{Q}^n$ is dense in S we can choose $r\in S\cap\mathbb{Q}^n$ such that

$$\varphi_T(\xi) < r_1\langle\alpha_1,\xi\rangle+\cdots+r_n\langle\alpha_n,\xi\rangle\le\varphi_{S'}(\xi). \tag{8.1}$$

Now we fix $m\in\mathbb{N}$ such that $\gamma=mr\in\mathbb{N}^n$, define $p\in\mathcal{P}^S_\mu(\mathbb{C}^n)$ by $p(w)=w^\gamma$. By Theorem 3.6, there exists a constant c_p such that

$$|f^*p(z)|\le e^{c_p+mH_T(z)},\qquad z\in\mathbb{C}^\ell. \tag{8.2}$$

Since S_j is a convex polytope in $\mathbb{R}^\ell_+$, the set $\bigcap_{j=1}^n\bigcup_{\alpha\in\operatorname{ext}S_j}\mathring{N}^{S_j}_\alpha$ is dense in $\mathbb{R}^\ell$, where $\mathring{N}^{S_j}_\alpha$ is the interior of $N^{S_j}_\alpha$. Hence ξ may be chosen from $\mathring{N}^{S_j}_{\alpha_j}$ where $\alpha_j\in\operatorname{ext}S_j$ and $\varphi_{S_j}(\xi)=\langle\alpha_j,\xi\rangle>\langle\alpha,\xi\rangle$ for every $\alpha\in I_j\setminus\{\alpha_j\}$. All the α_j are from $\mathbb{N}^\ell$ by the definition of S_j. We define the sequence $(\zeta_k)_{k\in\mathbb{N}}$ in $\mathbb{C}^\ell$ by $\zeta_k=(e^{k\xi_1},\ldots,e^{k\xi_\ell})$ and we will show that (8.1) implies that the sequence $\big(f^*p(\zeta_k)e^{-mH_T(\zeta_k)}\big)_{k\in\mathbb{N}}$ is unbounded, contradicting the estimate (8.2). First we observe that $\zeta_k^{\alpha_j}=e^{k\langle\alpha_j,\xi\rangle}$ and then that (8.1) implies

$$(\zeta_k^{\alpha_1},\ldots,\zeta_k^{\alpha_n})^\gamma e^{-mH_T(\zeta_k)} = e^{km(r_1\langle\alpha_1,\xi\rangle+\cdots+r_n\langle\alpha_n,\xi\rangle-\varphi_T(\xi))}\to+\infty,\quad k\to+\infty. \tag{8.3}$$

Next we observe that

$$f_j(\zeta_k)/\zeta_k^{\alpha_j} = a_{j,\alpha_j}+\sum_{\alpha\in I_j\setminus\{\alpha_j\}} a_{j,\alpha}e^{-k(\langle\alpha_j,\xi\rangle-\langle\alpha,\xi\rangle)}\to a_{j,\alpha_j}\ne0,\qquad k\to+\infty, \tag{8.4}$$

$$f^*p(\zeta_k)/(\zeta_k^{\alpha_1},\ldots,\zeta_k^{\alpha_n})^\gamma = (f_1(\zeta_k)/\zeta_k^{\alpha_1})^{\gamma_1}\cdots(f_n(\zeta_k)/\zeta_k^{\alpha_n})^{\gamma_n}. \tag{8.5}$$

By combining (8.3), (8.4), and (8.5), we see that $\big(f^*p(\zeta_k)e^{-mH_T(\zeta_k)}\big)_{k\in\mathbb{N}}$ is unbounded. □

Assume now that f is a proper map and that q is a given admissible weight function on a compact set $K \subseteq \mathbb{C}^n$. Then f^*q is lower semicontinuous and

$$\{z \in f^{-1}(K)\,;\, f^*q(z) < +\infty\} = f^{-1}\big(\{w \in K\,;\, q(w) < +\infty\}\big).$$

Since inverse images of non-pluripolar sets by proper maps are non-pluripolar it follows that f^*q is an admissible weight on $f^{-1}(K)$. Furthermore, we have

$$\|f^*p e^{-mf^*q}\|_{f^{-1}(K)} = \|p e^{-mq}\|_K.$$

From Proposition 8.1 we conclude that $f^*\big(\Phi^S_{K,q,m}\big) \le \Phi^{S'}_{f^{-1}(K),f^*q,m}$, for every $m \in \mathbb{N}^*$, consequently $f^*\big(\Phi^S_{K,q}\big) \le \Phi^{S'}_{f^{-1}(K),f^*q}$ and equality holds if $f^*\colon \mathcal{P}^S_m(\mathbb{C}^n) \to \mathcal{P}^{S'}_m(\mathbb{C}^\ell)$ is surjective.

Next we look at the pullback of Lelong classes. Let $u \in \mathcal{L}^S(\mathbb{C}^n)$, say $u \le c_u + H_S$. Then for every $z \in \mathbb{C}^\ell$ with $\mathrm{Log} f(z) \in \mathbb{C}^{*n}$ we have

$$\begin{aligned}(f^*u)(z) &\le c_u + H_S(f(z)) = c_u + \varphi_S(\log|f_1(z)|, \dots, \log|f_1(z)|)\\ &\le c_u + \varphi_S(c_1 + \varphi_{S_1}(\mathrm{Log} z), \dots, c_n + \varphi_{S_n}(\mathrm{Log} z))\\ &\le c_u + \varphi_S(c) + \varphi_S(\varphi_{S_1}(\mathrm{Log} z), \dots, \varphi_{S_n}(\mathrm{Log} z))\\ &= c_u + \varphi_S(c) + H_{S'}(z)\end{aligned}$$

and conclude that $f^*u \in \mathcal{L}^{S'}(\mathbb{C}^\ell)$. If $u \le q$ on K, then $f^*u \le f^*q$ on $f^{-1}(K)$, we have $f^*\big(V^S_{K,q}\big) \le V^{S'}_{f^{-1}(K),f^*q}$ and that equality holds if $f^*\colon \mathcal{L}^S(\mathbb{C}^n) \to \mathcal{L}^{S'}(\mathbb{C}^\ell)$ is surjective.

When $\ell = n$ we have the following weighted transformation rule, which is a generalization of Klimek [23], Theorem 5.3.1, and Perera [30], Theorem 1:

Proposition 8.2 *If $f\colon \mathbb{C}^n \to \mathbb{C}^n$ is proper, then the following are equivalent:*

- **(i)** *The difference $f^*H_S - H_{S'}$ is bounded, that is $f^*H_S \in \mathcal{L}^{S'}_+(\mathbb{C}^n)$.*
- **(ii)** *For every compact set $K \subset \mathbb{C}^n$ and every admissible weight q on K we have $f^*V^S_{K,q} = V^{S'}_{f^{-1}(K),\,f^*q}$.*

Proof: (i)⇒(ii): Let $u \in \mathcal{L}^{S'}(\mathbb{C}^n)$ with $u|_{f^{-1}(K)} \le f^*q$. By Klimek [23], Theorem 2.9.26, the function $v(z) = \max\{u(w)\colon w \in f^{-1}(z)\}$ is plurisubharmonic on $\mathbb{C}^n$. Let c be a constant such that $f^*H_S - H_{S'} \ge -c$. Then $v|_K \le q$ and

$$v(z) \le \max_{w\in f^{-1}(z)} H_{S'}(w) + c_u \le \max_{w\in f^{-1}(z)} f^*H_S(w) + c + c_u = H_S(z) + c + c_u.$$

It follows that $u \le f^*v \le f^*V^S_{K,q}$.

(ii)⇒(i): Let $c = \max_{w\in f^{-1}(\overline{\mathbb{D}}^n)} H_{S'}(w)$ and $c' = \max_{w\in f(\overline{\mathbb{D}}^n)} H_S(w)$. Then

$$H_{S'} - c \le V^{S'}_{f^{-1}(\overline{\mathbb{D}}^n)} = f^*V^S_{\overline{\mathbb{D}}^n} = f^*H_S$$

and

$$H_{S'} = V^{S'}_{\overline{\mathbb{D}}^n} \ge V^{S'}_{f^{-1}(f(\overline{\mathbb{D}}^n))} = f^*V^S_{f(\overline{\mathbb{D}}^n)} \ge f^*H_S - c'. \tag{8.6}$$

□

Paper II

Polynomials with exponents in compact convex sets and associated weighted extremal functions: The Siciak-Zakharyuta theorem

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Abstract

The classical Siciak-Zakharyuta theorem states that the Siciak-Zakharyuta function V_E of a subset E of $\mathbb{C}^n$, also called a pluricomplex Green function or global extremal function of E, equals the logarithm of the Siciak function Φ_E if E is compact. The Siciak-Zakharyuta function is defined as the upper envelope of functions in the Lelong class that are negative on E, and the Siciak function is the upper envelope of m-th roots of the modulus of polynomials p in $\mathcal{P}_m(\mathbb{C}^n)$ of degree $\leq m$ such that $|p| \leq 1$ on E. We generalize the Siciak-Zakharyuta theorem to the case where the polynomial space $\mathcal{P}_m(\mathbb{C}^n)$ is replaced by $\mathcal{P}_m^S(\mathbb{C}^n)$ consisting of all polynomials with exponents restricted to sets mS, where S is a compact convex subset of $\mathbb{R}_+^n$ with $0 \in S$. It states that if q is an admissible weight on a closed set E in $\mathbb{C}^n$ then $V_{E,q}^S = \log \Phi_{E,q}^S$ on $\mathbb{C}^{*n}$ if and only if the rational points in S form a dense subset of S.

1 Introduction

A polynomial in n complex variables $p(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$, $z \in \mathbb{C}^n$ is supported in a set $S \subset \mathbb{R}_+^n$ if $a_\alpha \neq 0$ implies that $\alpha \in S$. Given a compact $S \subset \mathbb{R}_+^n$ and $m \in \mathbb{N}$, let the space $\mathcal{P}_m^S(\mathbb{C}^n)$ consist of all polynomials p supported in mS, of the form

$$p(z) = \sum_{\alpha \in (mS) \cap \mathbb{N}^n} a_\alpha z^\alpha, \qquad z \in \mathbb{C}^n.$$

If S is convex and $0 \in S$ then $\mathcal{P}^S(\mathbb{C}^n) = \cup_{m \in \mathbb{N}} \mathcal{P}_m^S(\mathbb{C}^n)$ forms a graded polynomial ring where the degree of a polynomial p is the minimal m such that p is supported in mS. The standard grading is obtained when S is the unit simplex Σ, the convex hull of the union of 0 and the unit basis $\{e_1, \ldots, e_n\}$. Polynomials with prescribed supports are often called sparse polynomials.

Bayraktar [4] introduces methods from pluripotential theory to find the zero distribution of systems of random sparse polynomials as their degree tends to infinty. That paper is the naissance of pluripotential theory with respect to convex bodies, further developed by Bayraktar, Bloom, Levenberg and their collaborators in [6, 7, 8]. A self-contained exposition of the background to this paper is found in Magnússon, Sigurðardóttir, Sigurðsson and Snorrason [27].

A fundamental result from classical pluripotential theory is the Siciak-Zakharyuta theorem which states an equivalence between two extremal functions, $V_{E,q} = \log \Phi_{E,q}$, for every admissible weight $q : E \to \mathbb{R} \cup \{+\infty\}$ on a compact subset E of $\mathbb{C}^n$, see Siciak [36]. Here $V_{E,q}(z)$ is the pluricomplex Green function, defined as the supremum of functions $u \in \mathcal{PSH}(\mathbb{C}^n)$ satisfying a growth estimate $u \leq \log^+ |\cdot| + c_u$ for some constant c_u and such that $u|_E \leq q$. The Siciak function $\Phi_{E,q}$ is the supremum of $|p|^{1/m}$ where p is an m-th degree polynomial satisfying $\|pe^{-mq}\|_E \leq 1$, with the sup-norm on E is denoted by $\|\cdot\|_E$. For a proof of the original Siciak-Zakharyuta theorem with $q = 0$ and a reference on plurpotential theory, see Klimek [23] and for a reference on the weighted theory see Bloom's Appendix B in [33].

The definition of $\Phi_{E,q}$ can be modified using the grading of the ring $\mathcal{P}^S(\mathbb{C}^n)$. Instead of regarding $|p|^{1/m}$ where m is the standard degree of p, we take m such that p is supported in mS.

For every function $q\colon E \to \mathbb{R} \cup \{+\infty\}$ defined on $E \subseteq \mathbb{C}^n$ and $m \in \mathbb{N}^* = \{1, 2, 3, \dots\}$ we define Siciak functions with respect to S, E, q and m for $z \in \mathbb{C}^n$ by

$$\Phi^S_{E,q,m}(z) = \sup\{|p(z)|^{1/m}\,;\, p \in \mathcal{P}^S_m(\mathbb{C}^n), \|pe^{-mq}\|_E \leq 1\},$$

and the Siciak function with respect to S, E and q by

$$\Phi^S_{E,q} = \varlimsup_{m\to\infty} \Phi^S_{E,q,m}.$$

To generalize the Siciak-Zakharyuta theorem we study an extremal function $V^S_{E,q}$ to match $\log \Phi^S_{E,q}$. To do so, the logarithmic growth of $V_{E,q}$ is replaced by growth that reflects the grading of $\mathcal{P}^S(\mathbb{C}^n)$. The supporting function φ_S of S, $\varphi_S(\xi) = \sup_{s\in S} \langle s, \xi\rangle$ with $\xi \in \mathbb{R}^n$, is positively homogeneous and convex and determines the set S uniquely. The logarithmic supporting function $H_S \in \mathcal{PSH}(\mathbb{C}^n)$ of S is defined by

$$H_S(z) = \varphi_S(\log|z_1|, \dots, \log|z_n|)$$

on $\mathbb{C}^{*n} = (\mathbb{C}^*)^n$ and the definition is extended to the coordinate hyperplanes by

$$H_S(z) = \varlimsup_{\mathbb{C}^{*n}\ni w\to z} H_S(w), \qquad z \in \mathbb{C}^n \setminus \mathbb{C}^{*n}.$$

The class $\mathcal{L}^S(\mathbb{C}^n)$ is defined as the set of all $u \in \mathcal{PSH}(\mathbb{C}^n)$ satisfying a growth estimate of the form $u \leq H_S + c_u$ for some constant c_u. The Siciak-Zakharyuta function with respect to S, E and q is defined by

$$V^S_{E,q}(z) = \sup\{u(z)\,;\, u \in \mathcal{L}^S(\mathbb{C}^n),\, u|_E \leq q\}, \qquad z \in \mathbb{C}^n.$$

A function $p \in \mathcal{O}(\mathbb{C}^n)$ is in $\mathcal{P}^S_m(\mathbb{C}^n)$ if and only if $\log|p|^{1/m}$ is in $\mathcal{L}^S(\mathbb{C}^n)$, by [27], Theorem 3.6. By [27], Proposition 2.2, $\log \Phi^S_{K,q}$ is the supremum of the subclass of all $u \in \mathcal{L}^S(\mathbb{C}^n)$ with $u|_K \leq q$ that are of the form $u = \log|p|^{1/m}$ for some $p \in \mathcal{P}^S_m(\mathbb{C}^n)$. This implies that $\log \Phi^S_{E,q} \leq V^S_{E,q}$.

The standard simplex Σ, which yields the standard grading of polynomials, has the supporting function $\varphi_\Sigma(\xi) = \max\{\xi_1^+ \dots, \xi_n^+\}$ where $\xi_j^+ = \max\{\xi_j, 0\}$. Its logarithmic supporting function is $H_\Sigma(z) = \log^+ \|z\|_\infty$, which implies that the class $\mathcal{L}^\Sigma(\mathbb{C}^n)$ is the standard Lelong class $\mathcal{L}(\mathbb{C}^n)$. We drop the superscript S in the case $S = \Sigma$ and the subscript q in the case $q = 0$.

A function $q\colon E \to \mathbb{R} \cup \{+\infty\}$ is an *admissible weight with respect to S on E* if

(i) q is lower semicontinuous,

(ii) the set $\{z \in E\,;\, q(z) < +\infty\}$ is non-pluripolar, and

(iii) if E is unbounded, $\lim_{z\in E, |z|\to\infty}(H_S(z) - q(z)) = -\infty$.

Condition **(i)** excludes any function with a negative pole, which otherwise might force $V^S_{E,q}$ to be the supremum over an empty family. Without **(ii)** there would exist an $u \in \mathcal{PSH}(\mathbb{C}^n)$ such that $u = -\infty$ on $\{z \in E\,;\, q(z) < +\infty\}$. Then $u + c \le q$ on E for every constant c so if $u \in \mathcal{L}^S(\mathbb{C}^n)$ then $V^S_{E,q}(z) = +\infty$ for all $z \in \mathbb{C}^n$ such that $u(z) > -\infty$. Admissible weights ensure that the upper regularization $V^{S*}_{E,q}$ is a member of $\mathcal{L}^S(\mathbb{C}^n)$, see [27], Propositions 4.5 and 4.6. The condition **(iii)** additionally implies that the Siciak-Zakharyuta function satisfies a growth estimate $V^S_{E,q} \ge H_S - c$ for some constant c, which is used in [27], Proposition 4.6, to show that $V^S_{E,q} = V^S_{K,q}$ and $\Phi^S_{E,q} = \Phi^S_{K,q}$ for some compact $K \subseteq E$. Therefore admissible weights allow us to study mostly $E = K$ compact.

Exponents in $mS \cap \mathbb{N}^n$ can only appear as dilates of rational points from S so $\mathcal{P}^S_m(\mathbb{C}^n) = \mathcal{P}^{S'}_m(\mathbb{C}^n)$ for $S' = \overline{S \cap \mathbb{Q}^n}$. If $S \cap \mathbb{Q}^n$ does not form a dense subset of S, then there exists a point $\xi \in \mathbb{R}^n$ for which $\varphi_{S'}(\xi) < \varphi_S(\xi)$. For every admissible weight q on a compact set K the function $z \mapsto H_S(z) - c$, where $c = \max_{w\in K} H_S(w) - \min_{w\in K} q(w)$, is in $\mathcal{L}^S(\mathbb{C}^n)$ and $\le q$ on K. Hence $H_S - c \le V^S_{K,q}$. We also have $V^{S'}_{K,q} \le V^{S'*}_{K,q} \le H_{S'} + c'$ for some constant c'. By homogeneity of the supporting functions we have for $r > 0$ sufficiently large that $\varphi_{S'}(r\xi) + c' < \varphi_S(r\xi) - c$, so if we set $z = (e^{r\xi_1}, \dots, e^{r\xi_n})$, then

$$\log \Phi^S_{K,q}(z) = \log \Phi^{S'}_{K,q}(z) \le V^{S'}_{K,q}(z) \le H_{S'}(z) + c' < H_S(z) - c \le V^S_{K,q}(z).$$

To summarize, we have observed that if $S \cap \mathbb{Q}^n$ is not dense in S, then for every admissible weight q on a compact set K we have $V^S_{K,q} \ne \log \Phi^S_{K,q}$.

This necessary condition on S is almost sufficient, as seen by our main result.

Theorem 1.1 *Let $S \subset \mathbb{R}^n_+$ be compact and convex with $0 \in S$ and let q be an admissible weight on a closed $E \subset \mathbb{C}^n$. Then $V^S_{E,q}(z) = \log \Phi^S_{E,q}(z)$ for every $z \in \mathbb{C}^{*n}$ if and only if $S \cap \mathbb{Q}^n$ is dense in S. Furthermore, if $V^S_{E,q}$ is lower semicontinuous then the equality extends to all $z \in \mathbb{C}^n$.*

The last statement follows from the fact that $\Phi^S_{K,q}$ is lower semicontinuous, as it is the supremum over a family of continuous functions. The only remaining problem is to characterize when $V^S_{E,q}$ is lower-semicontinuous. The supremum

over a family of continuous plurisubharmonic functions is a lower-semicontinuous function, see [23], Proposition 2.3.3. Therefore $V^S_{K,q}$ is lower semicontinuous whenever any $u \in \mathcal{L}^S(\mathbb{C}^n)$ with $u \le q$ on K can be approximated by a continuous member of the same family. Therefore identifying lower-semicontinuous $V^S_{K,q}$ can be done by finding appropriate regularizing methods that preserve the class $\mathcal{L}^S(\mathbb{C}^n)$, see the discussion in Section 5 of [27]. When S is a lower set, and only then, can we use the traditional regularizing method of convolution with a smoothing kernel, by [27], Theorem 5.8. When S is assumed to contain a neighborhood of zero, the method of Ferrier approximation can be employed, see Section 3 of [8], so in that case $V^S_{K,q}$ is lower-semicontinuous.

Bayraktar, Hussung, Levenberg and Perera state in Theorem 1.1 of [8] that for every convex set $S \subset \mathbb{R}^n_+$ that contains a neighborhood of 0 and q an admissible weight on a closed E, then $\log \Phi^S_{E,q} = V^S_{E,q}$. Their proof is based on Proposition 4.3 in [8] which, as far as we can see, only holds if S is a lower set, that is if $[0, s_1] \times \cdots \times [0, s_n] \subset S$ for every $s \in S$. A counterexample to its claim is found in [27], Example 7.4. The statement of Theorem 1.1 in [8] is however valid as it follows from our theorem.

Previous works [6, 7, 8] refer to this subject as *Pluripotential theory and convex bodies*, as the support S is throughout assumed to be a convex body, that is with non-empty interior. With the goal of relaxing the conditions on S as much as possible, we study separately the case when S has empty interior. If S has empty interior but is convex and contains zero, then S lies in some subspace of dimension $\ell < n$. Therefore S is the image of some linear L map of some convex set T in $\mathbb{R}^\ell$. The next theorem asserts that this linear map L can be chosen so that T is a convex body in $\mathbb{R}^\ell_+$ and how the extremal functions with respect to S and T are related. This result gives us a reduction of the proof of Theorem 1.1 to the case when S is a convex body.

Theorem 1.2 *Let $S \subset \mathbb{R}^n_+$ be compact and convex with $0 \in S$. Assume that S is of dimension $\ell \le n$ and that $S \cap \mathbb{Q}^n$ is dense in S. Then there exists a linear map $L\colon \mathbb{R}^\ell \to \mathbb{R}^n$ and a compact convex subset T of $\mathbb{R}^\ell_+$ with $0 \in T$ such that $L(T) = S$, $L(e_k) \in \mathbb{Z}^n$ for $k = 1, \dots, \ell$ and $L^{-1}(\mathbb{Z}^n) = \mathbb{Z}^\ell$. Let $F_L\colon \mathbb{C}^{*n} \to \mathbb{C}^{*\ell}$ denote the holomorphic map given by*

$$F_L(z) = (z^{L(e_1)}, \dots, z^{L(e_\ell)}), \qquad z \in \mathbb{C}^{*n}. \tag{1.1}$$

Then for all $m \in \mathbb{N}$ there are well-defined bijective pullbacks

$$F_L^*\colon \mathcal{P}^T_m(\mathbb{C}^\ell) \to \mathcal{P}^S_m(\mathbb{C}^n), \qquad F_L^*\colon \mathcal{L}^T(\mathbb{C}^\ell) \to \mathcal{L}^S(\mathbb{C}^n) \tag{1.2}$$

and for every admissible weight function q on a compact $K \subset \mathbb{C}^n$ and $m \in \mathbb{N}^*$

$$\Phi^S_{K,q,m} = F_L^*\big(\Phi^T_{K',q',m}\big) \qquad \text{on } \mathbb{C}^{*n}, \quad \text{and} \tag{1.3}$$

$$V^S_{K,q} = F_L^*\big(V^T_{K',q'}\big) \qquad \text{on } \mathbb{C}^{*n}, \tag{1.4}$$

where $K' = F_L(K)$ and $q'(w) = F_{L*}q(w) = \inf\{q(z)\,;\, z \in F_L^{-1}(w) \cap K\}$.

Magnússon, Sigurðsson and Snorrason continue the discussion in [28], where a Bernstein-Walsh-Siciak theorem on approximation by polynomials in $\mathcal{P}^S(\mathbb{C}^n)$ is proved.

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2 Convex sets S of lower affine dimension

If $S \subset \mathbb{R}^n$ is convex with $0 \in S$ and has empty interior then the smallest subspace W of $\mathbb{R}^n$ that contains S is a proper subspace of $\mathbb{R}^n$. For any linear map $L\colon \mathbb{R}^\ell \to \mathbb{R}^n$ whose image is W the set $T = L^{-1}(S)$ is convex with non-empty interior. Before we make an explicit choice of the linear map L, we show that in general L induces a map between the polynomial rings $\mathcal{P}^T(\mathbb{C}^\ell)$ and $\mathcal{P}^S(\mathbb{C}^n)$ and the Lelong classes $\mathcal{L}^T(\mathbb{C}^\ell)$ and $\mathcal{L}^S(\mathbb{C}^n)$.

Let S be a compact convex subset of $\mathbb{R}^n_+$ with $0 \in S$. Let $L\colon \mathbb{R}^\ell \to \mathbb{R}^n$ be a linear map and assume that $S = L(T)$, where T is a compact convex subset of $\mathbb{R}^\ell_+$ with $0 \in T$. The relation between φ_S and φ_T is given by

$$\varphi_S(\xi) = \sup_{t\in T} \langle L(t), \xi\rangle = \sup_{t\in T} \langle t, L^*(\xi)\rangle = \varphi_T(L^*(\xi)), \tag{2.1}$$

where $L^*\colon \mathbb{R}^n \to \mathbb{R}^\ell$ denotes the adjoint of L. If $\eta_k = L(e_k) \in \mathbb{Z}^n$ for $k = 1, \dots, \ell$ then we have a well defined rational map

$$F_L\colon \mathbb{C}^{*n} \to \mathbb{C}^{*\ell}, \qquad F_L(z) = (z^{\eta_1}, \dots, z^{\eta_\ell}). \tag{2.2}$$

Observe that for every $t \in \mathbb{R}^\ell$ and $z \in \mathbb{C}^{*n}$ we have

$$\langle t, L^*(\mathrm{Log} z)\rangle = \sum_{j=1}^{n}\sum_{k=1}^{\ell} t_k \eta_{k,j} \log|z_j| = \langle t, \mathrm{Log}\, F_L(z)\rangle, \tag{2.3}$$

so by (2.1) we have

$$H_S(z) = \varphi_T(\mathrm{Log}(F_L(z))) = H_T(F_L(z)), \qquad z \in \mathbb{C}^{*n}. \tag{2.4}$$

By continuty this gives a relation between H_S and H_T,

$$H_S = F_L^* H_T \tag{2.5}$$

and a pullback

$$F_L^* \colon \mathcal{L}^T(\mathbb{C}^\ell) \to \mathcal{L}^S(\mathbb{C}^n). \tag{2.6}$$

If $p \in \mathcal{P}_m^T(\mathbb{C}^\ell)$ with $p(w) = \sum_{\beta \in mT} b_\beta w^\beta$, then

$$\begin{aligned} F_L^* p(z) &= \sum_{\beta \in mT} b_\beta \big(F_L(z)\big)^\beta \\ &= \sum_{\beta \in mT} b_\beta \big(z^{\beta_1 L(e_1)} \cdots z^{\beta_\ell L(e_\ell)}\big) = \sum_{\beta \in mT} b_\beta z^{L(\beta)}. \end{aligned} \tag{2.7}$$

Since $L(mT) = mS$ this equation shows that we have a pullback

$$F_L^* \colon \mathcal{P}_m^T(\mathbb{C}^\ell) \to \mathcal{P}_m^S(\mathbb{C}^n) \tag{2.8}$$

which is surjective if $L(mT \cap \mathbb{Z}^\ell) = mS \cap \mathbb{Z}^n$.

We need an appropriate weight on $F_L(K) \subset \mathbb{C}^\ell$. We have natural definitions of push-forwards F_* of the classes $\mathcal{USC}(K)$ and $\mathcal{LSC}(K)$ of upper and lower semicontinuous functions by a continuous map $F \colon K \to \mathbb{C}^\ell$ such that

$$F_* \colon \mathcal{USC}(K) \to \mathcal{USC}(F(K)) \quad \text{and} \quad F_* \colon \mathcal{LSC}(K) \to \mathcal{LSC}(F(K)),$$

but their definitions are different. The function

$$w \mapsto \sup\{u(z)\,;\, z \in F^{-1}(w)\}$$

is in $\mathcal{USC}(F(K))$ for every $u \in \mathcal{USC}(K)$ and

$$w \mapsto \inf\{u(z)\,;\, z \in F^{-1}(w)\}$$

is in $\mathcal{LSC}(F(K))$ for every $u \in \mathcal{LSC}(K)$. In this context it is natural to use the second definition of F_*:

Proposition 2.1 *Let $q \colon K \to \mathbb{R} \cup \{+\infty\}$ be an admissible weight on a compact subset K of $\mathbb{C}^n$ and $F \colon U \to \mathbb{C}^\ell$ be an open holomorphic map defined on some neighborhood of K. Then the push-forward $F_* q \colon F(K) \to \mathbb{R} \cup \{+\infty\}$ of q by F, given as*

$$F_* q(w) = \inf\{q(z)\,;\, z \in F^{-1}(w) \cap K\}, \qquad w \in F(K),$$

is an admissible weight on $F(K)$.

Proof: Take $w \in F(K)$ and $w_j \to w$ such that

$$\lim_{j \to \infty} F_* q(w_j) = \varliminf_{\zeta \to w} F_* q(\zeta)$$

and $\varepsilon_j \searrow 0$. Then there exists $z_j \in F^{-1}(w_j) \cap K$ such that $q(z_j) \le F_*q(w_j) + \varepsilon_j$. Let (z_{j_k}) be a subsequence of (z_j) converging to $z \in K$. Since q is lower semicontinuous we have

$$F_*q(w) \le q(z) \le \varliminf_{k\to\infty} q(z_{j_k}) \le \varliminf_{k\to\infty} F_*q(w_{j_k}) + \varepsilon_{j_k} = \varliminf_{\zeta\to w} F_*q(\zeta).$$

Hence F_*q is lower semicontinuous on $F(K)$.

Let $A = \{z \in K\,;\, q(z) < +\infty\}$. Since A is non-pluripolar and F is open, it follows that $F(A) \subseteq \{w \in F(K)\,;\, F_*q(w) < +\infty\}$ are non-pluripolar. Hence F_*q is admissible. □

Let $K \subseteq \mathbb{C}^n$ be compact and q an admissible weight on K. For ease of notation, write $K' = F_L(K)$ and $q' = F_{L*}q$. We have $F_L^*q' \le q$, so if $v \in \mathcal{PSH}(\mathbb{C}^\ell)$ and $v \le q'$ on K', then $u = F_L^*v$ satisfies $u \le q$ on K'. Hence

$$F_L^*\big(V^T_{K',q'}\big) \le V^S_{K,q} \tag{2.9}$$

and equality holds if (2.6) is surjective.

If $p \in \mathcal{P}^T_m(\mathbb{C}^\ell)$ and $\log|p|^{1/m} \le q'$ then $\log|F_L^*p|^{1/m} \le q$ on K. Hence

$$F_L^*\big(\Phi^T_{K',q',m}\big) \le \Phi^S_{K,q,m}, \qquad m \in \mathbb{N}, \tag{2.10}$$

and equality holds if the pullback (2.8) is surjective.

3 Preliminary algebra

In order to prove Theorem 1.2 we show that there exists a linear map $L\colon \mathbb{R}^\ell \to \mathbb{R}^n$ which maps a compact convex subset T of $\mathbb{R}^\ell_+$ with $0 \in T$ onto S such that the two pullbacks (2.6) and (2.8) are surjective. For that purpose we need some preliminary results.

Lemma 3.1 *Let W be a subspace of $\mathbb{R}^n$ of dimension ℓ spanned by vectors in $\mathbb{Q}^n$. Then there exists a linear map $L\colon \mathbb{R}^\ell \to \mathbb{R}^n$ such that $L(\mathbb{R}^\ell) = W$, $L(\mathbb{Z}^\ell) \subseteq \mathbb{Z}^n$, $L^{-1}(\mathbb{Z}^n) = \mathbb{Z}^\ell$, and $L^{-1}(\mathbb{R}^n_+) \subseteq \mathbb{R}^\ell_+$.*

Proof: The group $W \cap \mathbb{Z}^n$ is a subgroup of $\mathbb{Z}^n$ and is therefore isomorphic to $\mathbb{Z}^\nu$ for some $\nu \le n$. It is generated by ν elements $g_1, \dots, g_\nu$ which are independent, so $\nu \le \ell$. In order to prove that $\nu = \ell$ we observe first that since W is spanned by vectors in $\mathbb{Q}^n$ it is sufficient to show that that $g_1, \dots, g_\nu$ span $W \cap \mathbb{Q}^n$ over $\mathbb{Q}$. Take $x \in W \cap \mathbb{Q}^n$ and $\lambda \in \mathbb{N}^*$ such that $\lambda x \in \mathbb{Z}^n$. Then $\lambda x \in W \cap \mathbb{Z}^n$ which gives $\lambda x = x_1 g_1 + \cdots + x_\nu g_\nu$, with $x_k \in \mathbb{Z}$. Hence $x = (x_1/\lambda)g_1 + \cdots + (x_\nu/\lambda)g_\nu$ and $\nu = \ell$.

Let $M\colon \mathbb{R}^\ell \to \mathbb{R}^n$ be defined by $M(e_k) = g_k$, $k = 1, \dots, \ell$ and let $\Gamma = M^{-1}(\mathbb{R}^n_+)$. Since the generators are linearly independent M is injective and consequently the adjoint M^* is surjective. Hence we can find indices $j_1, \dots, j_\ell$

such that $a_k = M^*(e_{j_k})$, $k = 1, \dots, \ell$ span $\mathbb{R}^\ell$. The formula for M^* is $M^*(y) = (\langle g_1, y\rangle, \dots, \langle g_\ell, y\rangle)$ so $a_k \in \mathbb{Z}^\ell$ for $k = 1, \dots, \ell$. The dual cone $\Gamma^\circ = \{y \in \mathbb{R}^\ell\,;\, \langle y, x\rangle \geq 0 \text{ for all } x \in \Gamma\}$ contains a_k because if $x \in \Gamma$ then $M(x) \in \mathbb{R}^n_+$ and $\langle x, a_k\rangle = \langle M(x), e_{j_k}\rangle \geq 0$. Since the vectors a_k are linearly independent, Γ° is the closed cone spanned by $a_1, \dots, a_\ell$.

Define $P(\xi_1, \dots, \xi_\ell) = \big\{ \sum_{k=1}^\ell \lambda_k \xi_k\,;\, \lambda_k \in [0,1[, k = 1, \dots, \ell\big\}$ for $\xi_1, \dots, \xi_\ell \in \mathbb{R}^\ell$. We will now show that there exist linearly independent $\xi_1, \dots, \xi_\ell \in \mathbb{Z}^\ell \cap \Gamma^\circ$ such that $P(\xi_1, \dots, \xi_\ell) \cap \mathbb{Z}^\ell = \{0\}$. If $P(a_1, \dots, a_\ell) \cap \mathbb{Z}^\ell = \{0\}$ we are done. Assume that

$$0 \neq x \in P(a_1, \dots, a_\ell) \cap \mathbb{Z}^\ell,$$

$x = \lambda_1 a_1 + \cdots + \lambda_\ell a_\ell$ with $\lambda_k \in [0,1[$ and $\lambda_m \neq 0$. We form a new sequence $a_1^{(1)}, \dots, a_\ell^{(1)}$ in $\mathbb{Z}^\ell$ by setting $a_k^{(1)} = a_k$ for $k \neq m$ and $a_m^{(1)} = x$. After renumbering $a_1, \dots, a_\ell$ we may assume that $m = 1$.

The vectors $a_1^{(1)}, \dots, a_\ell^{(1)}$ are linearly independent. In fact, if $\mu_1 a_1^{(1)} + \cdots + \mu_\ell a_\ell^{(1)} = 0$ for $\mu_k \in \mathbb{R}$, then $\mu_1\lambda_1 a_1 + (\mu_1\lambda_2 + \mu_2)a_2 + \cdots + (\mu_1\lambda_\ell + \mu_\ell)a_\ell = 0$ holds, and since $a_1, \dots, a_\ell$ are linearly independent and $\lambda_1 \neq 0$ we have $\mu_1 = \cdots = \mu_\ell = 0$.

Next we construct an injective map G from the set $P(a_1^{(1)}, \dots, a_\ell^{(1)}) \cap \mathbb{Z}^\ell$ into $P(a_1, \dots, a_\ell) \cap \mathbb{Z}^\ell$ which does not take the value x. If $y \in P(a_1^{(1)}, \dots, a_\ell^{(1)}) \cap \mathbb{Z}^\ell$, then $y = \sum_{k=1}^\ell y_\ell a_k^{(1)} = y_1\lambda_1 a_1 + \sum_{k=2}^\ell (y_1\lambda_k + y_\ell)a_k$ with $y_k \in [0,1[$. Since $a_k \in \mathbb{Z}^\ell$ the map G is well defined by

$$G(y) = y_1\lambda_1 a_1 + \sum_{k=2}^{\ell} \big(y_1\lambda_k + y_k - \lfloor y_1\lambda_k + y_k\rfloor\big)a_k.$$

If $G(y) = G(z)$, then $y_1\lambda_1 = z_1\lambda_1$ and for $k = 2, \dots, \ell$,

$$y_1\lambda_k + y_k - \lfloor y_1\lambda_k + y_k\rfloor = z_1\lambda_k + z_k - \lfloor z_1\lambda_k + z_k\rfloor.$$

Since $\lambda_1 \neq 0$ we have $y_1 = z_1$ and consequently

$$y_k - \lfloor y_1\lambda_k + y_k\rfloor = z_k - \lfloor z_1\lambda_k + z_k\rfloor, \quad k = 2, \dots, \ell.$$

The two integer parts in this equation are 0 or 1. They must be the same number, for otherwise we would have a strictly negative value on one side and a positive value on the other side. Hence $y_k = z_k$ for $k = 2, \dots, \ell$ and G is injective. The point $x = a_1^{(1)} \in P(a_1, \dots, a_\ell) \cap \mathbb{Z}^\ell$ is not in the image of G, for $x = G(y)$ would imply that $y_1\lambda_1 = \lambda_1$ for some $y_1 \in [0,1[$. Hence $\#\big(P(a_1^{(1)}, \dots, a_\ell^{(1)}) \cap \mathbb{Z}^\ell\big) < \#\big(P(a_1, \dots, a_\ell) \cap \mathbb{Z}^\ell\big)$. Observe that $a_1^{(1)}, \dots, a_\ell^{(1)}$ are in Γ° which is the closed convex cone spanned by $a_1, \dots, a_\ell$.

By repeating this argument with $a_1^{(1)}, \dots, a_\ell^{(1)}$ in the role of $a_1, \dots, a_\ell$ finitely many times we conclude that there exist linearly independent vectors $\xi_1, \dots, \xi_\ell$ in $\mathbb{Z}^\ell \cap \Gamma^\circ$ such that $P(\xi_1, \dots, \xi_\ell) \cap \mathbb{Z}^\ell = \{0\}$.

We define $B\colon \mathbb{R}^\ell \to \mathbb{R}^\ell$ by $B(x) = (\langle \xi_1, x\rangle, \dots, \langle \xi_\ell, x\rangle)$ and observe that $B^*(x) = x_1\xi_1 + \cdots + x_\ell\xi_\ell$. This gives that $B^*(\mathbb{Z}^\ell) \subseteq \mathbb{Z}^\ell$ and since

$$P(\xi_1, \dots, \xi_\ell) \cap \mathbb{Z}^\ell = \{0\}$$

it follows that $B^*(\mathbb{Z}^\ell) = \mathbb{Z}^\ell$. We have

$$Bx \in \mathbb{Z}^\ell \;\Leftrightarrow\; \langle Bx, y\rangle \in \mathbb{Z}^\ell \;\forall y \in \mathbb{Z}^\ell \;\Leftrightarrow\; \langle x, B^*y\rangle \in \mathbb{Z}^\ell \;\forall y \in \mathbb{Z}^\ell \;\Leftrightarrow\; x \in \mathbb{Z}^\ell.$$

Finally, we define the linear map $L = M \circ B^{-1}$. Since M and B are injective and W is the image of M we have $L(\mathbb{R}^\ell) = W$,

$$L(\mathbb{Z}^\ell) = M(B^{-1}(\mathbb{Z}^\ell)) = M(\mathbb{Z}^\ell) = W \cap \mathbb{Z}^n.$$

Any $y \in W \cap \mathbb{R}^n_+$ has $x = M^{-1}(y) \in \Gamma$ and since $\xi_k \in \Gamma^\circ$, we have $\langle x, \xi_k\rangle \geq 0$ for $k = 1, \dots, \ell$, which shows that $B(x) \in \mathbb{R}^\ell_+$. Thus $L^{-1}(\mathbb{R}^n_+) = B(M^{-1}(\mathbb{R}^n_+)) \subseteq \mathbb{R}^\ell_+$. □

Lemma 3.2 *If $L\colon \mathbb{R}^\ell \to \mathbb{R}^n$ is an injective linear map such that $L(\mathbb{Z}^\ell) \subseteq \mathbb{Z}^n$ and $L^{-1}(\mathbb{Z}^n) = \mathbb{Z}^\ell$, then $L^*(\mathbb{Z}^n) = \mathbb{Z}^\ell$, where L^* denotes the adjoint of L.*

Proof: Let A be the matrix for L with respect to the standard bases on $\mathbb{R}^\ell$ and $\mathbb{R}^n$ and let $D = SAT$ be the Smith normal form of A, see Hungerford [22], Propostion 2.11, p. 339. This means that we factor A into $A = S^{-1}DT^{-1}$, with $S \in \mathbb{Z}^{n\times n}$ such that $x \mapsto Sx$ is invertible on $\mathbb{Z}^n$, $T \in \mathbb{Z}^{\ell\times\ell}$ such that $x \mapsto Tx$ is invertible on $\mathbb{Z}^\ell$ and $D = SAT$ is an $n \times \ell$ matrix with all entries zero exccept the diagonal entries $a_{j,j}$ which are of the form $a_{j,j} = \pm d_j/d_{j-1} \neq 0$ where d_j is the greatest common divisor of all determinants of $j \times j$ minors of L.

Since $D(e_j/a_{j,j}) = e_j \in \mathbb{Z}^n$ and D only maps integer vectors into $\mathbb{Z}^n$ it follows that $a_{j,j} = \pm 1$. Hence $D(\mathbb{Z}^\ell) \subseteq \mathbb{Z}^n$ and $D^{-1}(\mathbb{Z}^n) = \mathbb{Z}^\ell$. The matrix $D^* = T^*A^*S^*$ is the normal form of A^* and since the first k columns of D^* are $\pm e_j$ we see that $L^*(\mathbb{Z}^n) = \mathbb{Z}^\ell$. □

4 Surjectivity of the pullback F_L^*

Let us continue the study of the map F_L now assuming that $L\colon \mathbb{R}^\ell \to \mathbb{R}^n$ satisfies the conditions in Lemma 3.1 with $W = L(\mathbb{R}^\ell) = \mathrm{Span}_{\mathbb{R}}(S)$ and $T = L^{-1}(S)$. Since $L^{-1}(\mathbb{R}^n_+) \subseteq \mathbb{R}^\ell_+$ and $L(\mathbb{Z}^\ell) \subseteq \mathbb{Z}^n$ we have $L(mT \cap \mathbb{Z}^\ell) = mS \cap \mathbb{Z}^n$ and it follows that the pullback (2.8) is surjective. Recall that $\eta_k = L(e_k)$ for $k = 1, \dots, \ell$. Our goal is then to prove that the pullback in (2.6) is surjective.

We observe that if A is an $m \times n$ matrix with entries in $\mathbb{Z}$, $a_1, \dots, a_m$ are its line vectors, $A_1, \dots, A_n$ are its column vectors and $b \in \mathbb{Z}^n$ is a column vector, then for every $t \in \mathbb{C}^{*m}$ we have $(t^{A_1}, \dots, t^{A_n})^b = t^{Ab}$. In order to show that $F_L\colon \mathbb{C}^{*n} \to \mathbb{C}^{*\ell}$ is surjective we take $w \in \mathbb{C}^{*\ell}$, write $w_j = e^{b_j}$ for some $b_j \in \mathbb{C}$, and let A be the $\ell \times n$ matrix with line vectors $\eta_1, \dots, \eta_\ell$, i.e. A is the matrix

for L^* with respect to the standard bases on $\mathbb{R}^n$ and $\mathbb{R}^\ell$. By Lemma 3.2 L^* is surjective, so there is a solution $c \in \mathbb{C}^n$ of the linear system $Ac = b$. If we set $z_j = e^{c_j}$, then

$$F_L(z) = (z^{\eta_1}, \dots, z^{\eta_\ell}) = (e^{\langle \eta_1, c\rangle}, \dots, e^{\langle \eta_\ell, c\rangle}) = (e^{b_1}, \dots, e^{b_\ell}) = w. \tag{4.1}$$

which shows that F_L is surjective.

For every $z \in \mathbb{C}^{*n}$ we have

$$\frac{\partial F_{L,k}}{\partial z_j}(z) = \frac{\partial}{\partial z_j}\big(z_1^{\eta_{k,1}} \cdots z_n^{\eta_{k,n}}\big) = \frac{\eta_{k,j} F_{L,k}(z)}{z_j}. \tag{4.2}$$

Since L is injective, the vectors $\eta_1, \dots, \eta_\ell$ are linearly independent, so the matrix

$$\begin{bmatrix} \eta_{1,1} & \cdots & \eta_{1,n} \\ \vdots & \ddots & \vdots \\ \eta_{\ell,1} & \cdots & \eta_{\ell,n} \end{bmatrix}$$

has rank ℓ. We choose $1 \le j_1 < \cdots < j_\ell \le n$ such that the columns number $j_1, \dots, j_\ell$ are linearly independent. Then the determinant of the corresponding $\ell \times \ell$ submatrix of the Jacobi matrix of F_L is

$$\begin{vmatrix} \eta_{1,j_1} F_{L,1}(z)/z_{j_1} & \dots & \eta_{1,j_\ell} F_{L,1}(z)/z_{j_\ell} \\ \vdots & \ddots & \vdots \\ \eta_{\ell,j_1} F_{L,\ell}(z)/z_{j_1} & \dots & \eta_{\ell,j_\ell} F_{L,\ell}(z)/z_{j_\ell} \end{vmatrix}$$
$$= \frac{F_{L,1}(z) \cdots F_{L,\ell}(z)}{z_{j_1} \cdots z_{j_\ell}} \begin{vmatrix} \eta_{1,j_1} & \cdots & \eta_{1,j_\ell} \\ \vdots & \ddots & \vdots \\ \eta_{\ell,j_1} & \cdots & \eta_{\ell,j_\ell} \end{vmatrix} \neq 0.$$

Let $w_0 = F_L(z_0)$. By the implicit function theorem the fiber $F_L^{-1}(w_0)$ through $z_0 \in \mathbb{C}^{*n}$ is a complex manifold of dimension $n - \ell$. We can parametrize the fibers of F_L. In order to do so we extend $\eta_1, \dots, \eta_\ell$ to a basis in $\mathbb{R}^n$ by choosing $\eta_{\ell+1}, \dots, \eta_n$ as a basis for $W^\perp = \operatorname{Ker} L^*$. Since W is spanned by vectors in $\mathbb{Q}^n$ the same holds for $W^\perp$, so it is always possible to choose η_j from $\mathbb{Z}^n$. We let $M\colon \mathbb{R}^{n-\ell} \to \mathbb{R}^n$ be the linear map with $M(e_j) = \eta_j$, for $j = \ell+1, \dots, n$. Here we identify $\mathbb{R}^{n-\ell}$ with the subspace $\{0\} \times \mathbb{R}^{n-\ell}$ in $\mathbb{R}^n$ and view the vectors $e_{\ell+1}, \dots, e_n$ in the standard basis in $\mathbb{R}^n$ as the standard basis for $\mathbb{R}^{n-\ell}$. Furthermore, we write $t = (t', t'') \in \mathbb{C}^n$ with $t' \in \mathbb{C}^\ell$ and $t'' \in \mathbb{C}^{n-\ell}$. We let $N\colon \mathbb{R}^n \to \mathbb{R}^n$ be defined by $N(t) = L(t') + M(t'')$. Then

$$N^*(x) = (\langle \eta_1, x\rangle, \cdots, \langle \eta_n, x\rangle) = (L^*(x), M^*(x)). \tag{4.3}$$

If B is the matrix for N^* with respect to the standard basis, then $\eta_1, \dots, \eta_n$ are the line vectors of B, so if we let $B_1, \dots, B_n$ be the column vectors in B, then

for every $b \in \mathbb{Z}^n$ we have $(z_1 t^{B_1}, \dots, z_n t^{B_n})^b = z^b t^{Bb}$. If we apply this formula with $b = \eta_j^*$ for $j = 1, \dots, \ell$, then

$$B\eta_j^* = (\langle \eta_1, \eta_j \rangle, \dots, \langle \eta_\ell, \eta_j \rangle, 0, \dots, 0) = (L^*(e_j), 0) = (A_j^*, 0)$$

and consequently

$$F_L(z_1 t^{B_1}, \dots, z_n t^{B_n}) = (z^{\eta_1} t^{B\eta_1^*}, \dots, z^{\eta_\ell} t^{B\eta_\ell^*}) = (z^{\eta_1} t'^{A_1}, \dots, z^{\eta_\ell} t'^{A_\ell}).$$

This formula tells us that every point of the form $(z_1 t^{B_1}, \dots, z_n t^{B_n})$ with $(t'^{A_1}, \dots, t'^{A_\ell}) = (1, \dots, 1)$ is in the fiber of F_L through z. Our observations lead to:

Lemma 4.1 *Let $B \in \mathbb{Z}^{n\times n}$ be the matrix with line vectors $\eta_1, \dots, \eta_n$ and $B_1, \dots, B_n$ denote its columns. Let $z \in \mathbb{C}^{*n}$ and*

$$\Xi_z \colon \mathbb{C}^{*n} \to \mathbb{C}^n, \qquad \Xi_z(t) = (z_1 t^{B_1}, \dots, z_n t^{B_n}).$$

Then,

$$\Upsilon_z \colon \mathbb{C}^{*(n-\ell)} \to \mathbb{C}^n, \qquad \Upsilon_z(t) = \Xi_z(1, \dots, 1, t'')$$

is a parametrization of the fiber of F_L through z.

Proof: From our calculations before the statement of the lemma it follows that we only need to prove that every $\zeta \in \mathbb{C}^{*n}$ which satisfies $F_L(\zeta) = F_L(z)$ is of the form $\zeta = (z_1 t^{B_1}, \dots, z_n t^{B_n})$ for some $t = (1, \dots, 1, t'')$ with $t'' \in \mathbb{C}^{*(n-\ell)}$.

We choose $s = (s_1, \dots, s_n) \in \mathbb{C}^n$ such that $\zeta_j = z_j e^{s_j}$ for $j = 1, \dots, n$ and observe that $\zeta^{\eta_k} = z^{\eta_k}$ implies that $e^{\langle \eta_k, s \rangle} = 1$ for $k = 1, \dots, \ell$. Hence there exists $\mu = (\mu_1, \dots, \mu_\ell) \in \mathbb{Z}^\ell$ such that $\langle \eta_k, s \rangle = 2\pi i \mu_k$ for $k = 1, \dots, \ell$. By Lemma 3.2 there exists $\nu \in \mathbb{Z}^n$ such that $L^*(\nu) = \mu$. We have $L^*(s) = (\langle \eta_1, s \rangle, \dots, \langle \eta_\ell, s \rangle)$, so

$$s - 2\pi i \nu \in \operatorname{Ker}_{\mathbb{C}} L^* = \operatorname{Span}_{\mathbb{C}} \{\eta_{\ell+1}, \dots, \eta_n\},$$

where $\operatorname{Ker}_{\mathbb{C}} L^*$ is the kernel of the natural extension of L^* to a $\mathbb{C}$-linear map $\mathbb{C}^n \to \mathbb{C}^\ell$. We take $u'' = (u_{\ell+1}, \dots, u_n) \in \mathbb{C}^{n-\ell}$ such that

$$s - 2\pi i \nu = u_{\ell+1}\eta_{\ell+1} + \cdots + u_n \eta_n,$$

set $u = (0, \dots, 0, u'') \in \mathbb{C}^n$ and $t = (t_1, \dots, t_n) \in \mathbb{C}^n$ with $t_j = e^{u_j}$ for $j = 1, \dots, n$. Then

$$z_j t^{B_j} = z_j e^{u_{\ell+1}\eta_{\ell+1,j} + \cdots + u_n \eta_{n,j}} = z_j e^{s_j - 2\pi i \nu_j} = \zeta_j.$$

Hence $\zeta = \Upsilon_z(t'')$ and Υ_z parametrizes of the fiber of F_L through z.

□

End of proof of Theorem 1.2: Take $u \in \mathcal{L}^S(\mathbb{C}^n)$, fix $z \in \mathbb{C}^n$ and define $u_z \in \mathcal{PSH}(\mathbb{C}^{*(n-\ell)})$ by $u_z(t) = u(\Upsilon_z(t))$. Then we have the estimate

$$\begin{aligned} u_z(t) \le c_u + H_S(\Upsilon_z(t)) &= c_u + H_T(F_L(\Upsilon_z(t))) \\ &= c_u + H_T(F_L(z)) = c_u + H_S(z). \end{aligned}$$

This gives us extension of u_z to a plurisubharmonic function on $\mathbb{C}^{n-\ell}$ by the formula

$$u_z(t) = \overline{\lim_{\mathbb{C}^{*(n-\ell)} \ni \tau \to t}} u_z(\tau), \qquad t \in \mathbb{C}^{n-\ell}.$$

We have $u_z(t) \le c_u + H_S(z)$ for every t, so by the Liouville theorem for plurisubharmonic functions u_z is the constant function taking the value $u(z)$.

Define $v\colon \mathbb{C}^{*\ell} \to \mathbb{R} \cup \{-\infty\}$ by $v(w) = u(z)$ for any $z \in F_L^{-1}(w) \subset \mathbb{C}^{*n}$. This is well-defined since F_L is surjective and u is constant on $F_L^{-1}(w)$. Since F_L is a surjective submersion of constant rank there exists for every $w_0 \in \mathbb{C}^{*\ell}$ an open neighborhood U_0 of w_0 and a holomorphic map $s\colon U_0 \to \mathbb{C}^n$ such that $s(w_0) = z_0$ and $F_L(s(w)) = w$ for every $w \in U_0$. Then $v(w) = u(s(w))$ on U_0, which shows that $v \in \mathcal{PSH}(\mathbb{C}^{*\ell})$.

We have $v(w) = u(z) \le c_u + H_S(z) = c_u + H_T(w)$ for every $w \in \mathbb{C}^{*\ell}$ which implies that v extends to a function in $\mathcal{L}^T(\mathbb{C}^\ell)$ satisfying $u = F_L^* v$. Hence the pullback (2.6) is surjective. □

Corollary 4.2 *Let $S \subset \mathbb{R}^n_+$ be compact and convex with $0 \in S$ and assume that S has empty interior and that $S \cap \mathbb{Q}^n$ is dense in S. Then $V^{S*}_{K,q}$ is maximal.*

Proof: By Theorem 1.2 we have $V^{S*}_{K,q} = V^{T*}_{K',q'} \circ F_L$ on $\mathbb{C}^{*n}$. From Lemma 4.1 follows that for every $z \in \mathbb{C}^{*n}$ we have $V^{S*}_{K,q} \circ \Upsilon_z = V^{T*}_{K',q'} \circ F_L \circ \Upsilon_z$. Since $F_L \circ \Upsilon_z$ is constant it follows that $V^{S*}_{K,q}$ is maximal on $\mathbb{C}^{*n}$. Since $V^{S*}_{K,q}$ is locally bounded and $\mathbb{C}^n \setminus \mathbb{C}^{*n}$ is pluripolar, $V^{S*}_{K,q}$ is maximal on $\mathbb{C}^n$. □

5 Solutions of Cauchy-Riemann equations

To construct functions in $\mathcal{P}^S_m(\mathbb{C}^n)$ we are going to apply a special case of Hörmander's theorem on the existence of solutions of the Cauchy-Riemann equation with L^2 estimates. Before we can state it we need to introduce some notation. Let $\varphi\colon X \to \overline{\mathbb{R}}$ be a measurable function on an open subset X of $\mathbb{C}^n$ and let $L^2(X,\varphi)$ denote the set of all functions $u \in L^2_{\mathrm{loc}}(X)$ such that

$$\int_X |u|^2 e^{-\varphi}\, d\lambda < +\infty, \tag{5.1}$$

where λ denotes the Lebesgue measure. Then $L^2(X,\varphi)$ is a Hilbert space with inner product

$$\langle u, v\rangle_\varphi = \int_X u\bar{v}\, e^{-\varphi} d\lambda, \qquad u, v \in L^2(X,\varphi), \tag{5.2}$$

and corresponding norm $\|u\|_\varphi = \langle u, u\rangle_\varphi^{1/2}$ for $u \in L^2(X,\varphi)$.

Similarly, we let $L^2_{(0,1)}(X,\varphi)$ denote the set of all $(0,1)$-forms $f = f_1\,d\bar z_1 + \cdots + f_n\,d\bar z_n$, with coefficients $f_j \in L^2(X,\varphi)$. Then $L^2_{(0,1)}(X,\varphi)$ is a Hilbert space with inner product

$$\langle f, g\rangle_\varphi = \int_X \langle f, g\rangle e^{-\varphi} d\lambda, \tag{5.3}$$

where $\langle f, g\rangle = f_1\bar g_1 + \cdots + f_n\bar g_n$ for $f, g \in L^2_{(0,1)}(X,\varphi)$ and norm $\|f\|_\varphi = \langle f, f\rangle_\varphi^{1/2}$ for $f \in L^2_{(0,1)}(X,\varphi)$.

The function $u \in L^2_{\mathrm{loc}}(X)$ is a solution of the Cauchy-Riemann equation $\bar\partial u = f$, i.e. $\bar\partial_j u = f_j$ for $j = 1,\dots,n$, in the sense of distributions if

$$-\int_X u\bar\partial_j v\,d\lambda = \int_X f_j v\,d\lambda, \qquad v \in \mathcal{C}^\infty_0(X). \tag{5.4}$$

A necessary condition for existence of solution is that

$$\bar\partial f = \sum_{k=1}^n \bar\partial f_k \wedge d\bar z_k = \sum_{j<k} \big(\bar\partial_k f_j - \bar\partial_j f_k\big) d\bar z_j \wedge d\bar z_k = 0.$$

Theorem 5.1 (Hörmander) *Let X be a pseudoconvex domain of $\mathbb{C}^n$, $\varphi \in \mathcal{PSH}(X)$ and define for $a \in \mathbb{R}$,*

$$\varphi_a(z) = \varphi(z) + a\log(1+|z|^2), \qquad z \in X. \tag{5.5}$$

Then for every $a > 0$ and $f \in L^2_{(0,1)}(X,\varphi_{a-2})$ satisfying $\bar\partial f = 0$ there exists a solution $u \in L^2(X,\varphi_a)$ of $\bar\partial u = f$ satisfying the estimate

$$\begin{aligned}\|u\|^2_{\varphi_a} &= \int_X |u|^2(1+|z|^2)^{-a} e^{-\varphi}\,d\lambda \\ &\le \frac{1}{a}\int_X |f|^2(1+|z|^2)^{-a+2} e^{-\varphi}\,d\lambda = \frac{1}{a}\|f\|^2_{\varphi_{a-2}}.\end{aligned} \tag{5.6}$$

If $f_j \in \mathcal{C}^\infty(X)$ for $j = 1,\dots,n$, then $u \in \mathcal{C}^\infty(X)$.

For a proof see Hörmander [19], Theorem 4.2.6. The smoothness statement follows from the fact that the Laplace operator $\Delta = 4\sum_{j=1}^n \partial_j\bar\partial_j$ is is hypoelliptic, i.e. every distribution solution $v \in \mathcal{D}'(X)$ of $\Delta v = g$ with $g \in \mathcal{C}^\infty(X)$ is a $\mathcal{C}^\infty$ function on X. If u satisfies $\bar\partial u = f$ in the sense of distributions, then

$$\Delta u = 4\sum_{j=1}^n \partial_j\bar\partial_j u = 4\sum_{j=1}^n \partial_j f_j \in \mathcal{C}^\infty(X), \tag{5.7}$$

so $u \in \mathcal{C}^\infty(X)$. Observe that if $e^{-\varphi}$ is not integrable in any neighborhood of $z \in X$, then a solution $u \in L^2(X,\varphi_a)$ satisfying (5.6) vanishes at z.

Let us now review a simple method for estimating values $u(z)$ of a solution u of $\bar\partial u = f$ satisfying (5.6) for some measurable $\varphi\colon X \to \overline{\mathbb{R}}$. If $\bar B(z,\delta) \subseteq X \setminus \operatorname{supp} f$,

then u is holomorphic in a neighborhood of $\bar{B}(z,\delta)$ and the mean value theorem gives

$$u(z) = \mathcal{M}_\delta u(z) = \frac{1}{\Omega_{2n}\delta^{2n}} \int_{B(z,\delta)} u \, d\lambda. \tag{5.8}$$

By the Cauchy-Schwarz inequality where Ω_{2n} denotes the volume of the unit ball

$$\begin{aligned} |u(z)| &\le \frac{1}{\Omega_{2n}\delta^{2n}} \int_{B(z,\delta)} |u| e^{-\varphi_a/2} \cdot e^{\varphi_a/2} \, d\lambda \\ &\le \frac{\|u\|_{\varphi_a}}{\Omega_{2n}\delta^{2n}} \Big(\int_{B(z,\delta)} e^{\varphi_a} \, d\lambda \Big)^{1/2} \\ &\le a^{-1/2} \Omega_{2n}^{-1/2} \delta^{-n} \|f\|_{\varphi_{a-2}} \cdot \sup_{|w|\le\delta} e^{\varphi_a(z+w)/2}. \end{aligned} \tag{5.9}$$

In our proof of the Siciak-Zakharyuta theorem we need more sophisticated methods for deriving uniform estimates from L^2 estimates. (See Hörmander [17], Chapter 15, [20], p. 328 and Sigurdsson [40], Lemma 1.3.5.) These methods are based on a formula for the fundamental solution $E \in L^1_{\mathrm{loc}}(\mathbb{C}^n) \cap \mathcal{C}^\infty(\mathbb{C}^n \setminus \{0\})$ of the Laplace operator, which for $n \ge 2$ is given by

$$\begin{aligned} E(z) &= -\tfrac{1}{2} |z|^{-2n+2} / (\omega_{2n}(n-1)) \\ &= -\tfrac{1}{2} |z_1\bar{z}_1 + \cdots + z_n\bar{z}_n|^{-n+1} / (\omega_{2n}(n-1)) \end{aligned} \tag{5.10}$$

where ω_{2n} denotes the volume of the unit sphere in $\mathbb{R}^{2n}$, and has partial derivatives given by

$$\partial_j E(z) = \tfrac{1}{2} |z_1\bar{z}_1 + \cdots + z_n\bar{z}_n|^{-n} \bar{z}_j / \omega_{2n} = \tfrac{1}{2} |z|^{-2n} \bar{z}_j / \omega_{2n}. \tag{5.11}$$

Recall that for every distribution $v \in \mathcal{E}'(\mathbb{C}^n)$ with compact support we have

$$v = E * (\Delta v) = 4 \sum_{j=1}^n (\partial_j E) * (\bar{\partial}_j v). \tag{5.12}$$

Assume now that $u \in C^1(X)$, $\varphi \colon X \to \overline{\mathbb{R}}$ is a measurable function on X, $\bar{\partial} u = f$, and that $\|u\|_{\varphi_a} \le a^{-1/2} \|f\|_{\varphi_{a-2}}$ for some $a > 0$. Let $\chi \in \mathcal{C}^\infty_0(\mathbb{C}^n)$ with $\chi = 1$ in some neighborhood of 0 and $\chi_z(w) = \chi(z-w)$. By (5.11) we have for every $z \in X$ such that $z - K \subset X$ with $K = \operatorname{supp} \chi$,

$$\begin{aligned} u(z) = \chi_z(z) u(z) &= 4 \sum_{j=1}^n \Big(\big(\partial_j E\big) * \big(\chi_z \bar{\partial}_j u\big)(z) + \big(\partial_j E\big) * \big(u \bar{\partial}_j \chi_z\big)(z) \Big) \\ &= \frac{2}{\omega_{2n}} \int_K \Big(|w|^{-2n} \chi(w) \sum_{j=1}^n \bar{w}_j \bar{\partial}_j u(z-w) \\ &\qquad - |w|^{-2n} \sum_{j=1}^n \bar{w}_j \bar{\partial}_j \chi(w) u(z-w) \Big) \, d\lambda(w). \end{aligned} \tag{5.13}$$

We choose $\chi(w) = \chi_0(|w|^2/\delta^2)$ for $\delta > 0$, where $\chi_0 \in \mathcal{C}^\infty(\mathbb{R})$, $\chi_0(x) = 1$ for $x \leq \frac{1}{4}$, χ_0 is decreasing and $\chi_0 = 0$ for $x \geq 1$. Then $\chi(w) = 1$ for $|w| \leq \frac{1}{2}\delta$, $0 \leq \chi \leq 1$, and $\chi(w) = 0$ for $|w| \geq \delta$. We have $\bar\partial_j \chi(w) = \chi_0'(|w|^2/\delta^2) w_j/\delta^2$ and that χ_0 can be chosen such that $|\chi_0'| \leq 2$. We use this information in the second sum in the integrand in (5.13) and apply the Cauchy-Schwarz inequality to the first sum to get the estimate

$$
\begin{aligned}
|u(z)| &\leq \frac{2}{\omega_{2n}} \int\limits_{|w|\leq\delta} |w|^{-2n+1}\, d\lambda(w) \cdot \sup_{|w|\leq\delta} |f(z+w)| \\
&\quad + \frac{4}{\omega_{2n}\delta^2} \int\limits_{\frac{1}{2}\delta\leq|w|\leq\delta} |w|^{-2n+2} |u(z-w)|\, d\lambda(w). \qquad (5.14)
\end{aligned}
$$

Now we apply the Cauchy-Schwarz inequality as in (5.9)

$$
\begin{aligned}
|u(z)| &\leq 2\delta \cdot \sup_{|w|\leq\delta} |f(z+w)| \\
&\quad + \frac{4}{\omega_{2n}\delta^2} \Big(\int\limits_{\frac{1}{2}\delta\leq|w|\leq\delta} |w|^{-4n+4} e^{\varphi_a(z-w)}\, d\lambda(w) \Big)^{1/2} \|u\|_{\varphi_a} \\
&\leq 2\delta \cdot \sup_{|w|\leq\delta} |f(z+w)| + a^{-1/2} c_n \delta^{-n} \|f\|_{\varphi_{a-2}} \cdot \sup_{|w|\leq\delta} e^{\varphi_a(z+w)/2}, \qquad (5.15)
\end{aligned}
$$

and

$$
\begin{aligned}
c_n &= \frac{4}{\omega_{2n}} \Big(\int\limits_{\frac{1}{2}\leq|w|\leq 1} |w|^{-4n+4}\, d\lambda(w) \Big)^{1/2} \\
&= \begin{cases} 4\big(\log 2/\omega_{2n}\big)^{1/2}, & n = 2, \\ 4\big((2^{2n-4}-1)/(\omega_{2n}(2n-4))\big)^{1/2}, & n > 2. \end{cases} \qquad (5.16)
\end{aligned}
$$

We summarize our calculations in the next result.

Theorem 5.2 *Let $u \in \mathcal{C}^1(X)$ denote a solution of the Cauchy-Riemann equations given in Theorem 5.1. Then for every $z \in X$ and $0 < \delta < d(z, \partial X)$ we have*

$$
|u(z)| \leq 2\delta \cdot \sup_{|w|\leq\delta} |f(z+w)| + a^{-1/2} c_n \delta^{-n} \|f\|_{\varphi_{a-2}} \sup_{|w|\leq\delta} e^{\varphi_a(z+w)/2}, \qquad (5.17)
$$

where the constant c_n is given by (5.16).

In the proof of the Siciak-Zakharyuta theorem we will apply Theorem 5.2 and the following variant of Lemma 1.3.5 in Sigurdsson [40]:

Lemma 5.3 *Let $v \in \mathcal{C}^1(\mathbb{C}^n)$ and assume that there exist positive constants a, C_1, C_2 and a measurable function ψ such that for every $z \in \mathbb{C}^n$ we have*

$$\int_{B(z,1)} |v(w)|^2 (1+|e^w|^2)^{-a} e^{-2\psi(w)} d\lambda(w) < C_1 \tag{5.18}$$

and

$$|\overline{\partial} v(z)| \le C_2 e^{\psi(z)}. \tag{5.19}$$

Then there exists a positive constant C_3 such that

$$|v(z)| \le C_3 (1+|e^z|)^a \sup_{w\in\mathbb{B}} e^{\psi(z+w)}, \qquad z \in \mathbb{C}^n. \tag{5.20}$$

Proof: Let $I(z)$ denote the square root of the integral in (5.18). Then, with the same notation as above, (5.13) holds with $u = v$, $\delta = 1$, and $K = \mathbb{B}$, so by (5.14)

$$|v(z)| \le \frac{2}{\omega_{2n}} \int_{\mathbb{B}} |w|^{-2n+1}\, d\lambda(w) \sup_{w\in\mathbb{B}} |\bar\partial v(z+w)| + \frac{4}{\omega_{2n}} \int_{\frac12 \le |w| \le 1} |w|^{-2n+2} |v(z-w)|\, d\lambda(w).$$

By (5.19) and the Cauchy-Schwarz inequality

$$|v(z)| \le 2C_2 \sup_{w\in\mathbb{B}} e^{\psi(z+w)} + \frac{4I(z)}{\omega_{2n}} \Big(\int_{\frac12 \le |w| \le 1} |w|^{-4n+4} (1+|e^{z+w}|^2)^a e^{2\psi(z+w)}\, d\lambda(w) \Big)^{1/2}.$$

Finally, by (5.16) and (5.18) we get for every $z \in \mathbb{C}^n$

$$|v(z)| \le \big(2C_2 + \big(e^a C_1\big)^{1/2} c_n\big)(1+|e^z|)^a \sup_{w\in\mathbb{B}} e^{\psi(z+w)},$$

and have proved (5.20). □

6 Proof of the Siciak-Zakharyuta theorem

Before we go into details of the proof of Theorem 1.1 we describe its main ideas. The inequality $V^S_{E,q} \ge \log \Phi^S_{E,q}$ is immediate. By [27], Proposition 4.6, $V^S_{E,q} = V^S_{K,q}$ and $\Phi^S_{E,q} = \Phi^S_{K,q}$ for some compact $K \subseteq E$, so it suffices to prove that $V^S_{K,q} = \log \Phi^S_{K,q}$ for a compact set $K \subseteq \mathbb{C}^n$. Since the admissible weight q is lower-semicontinuous, there is a sequence of continuous functions $q_j \nearrow q$ pointwise on K. By [27], Proposition 4.8 (iii) we have $V^S_{K,q_j} \nearrow V^S_{K,q}$ and $\Phi^S_{K,q_j} \nearrow \Phi^S_{K,q}$, so it suffices to prove that $V^S_{K,q} \le \log \Phi^S_{K,q}$ on $\mathbb{C}^{*n}$ for continuous weights q.

Before we apply Theorem 1.2, we reduce onto the case when K avoids the union of the coordinate hyperplanes $H = \mathbb{C}^n \setminus \mathbb{C}^{*n}$.

Lemma 6.1 *Let $S \subset \mathbb{R}^n_+$ be compact and convex with $0 \in S$. Let q be a continuous function on a compact subset K of $\mathbb{C}^n$. Then there exists a sequence of compact subsets K_j of $\mathbb{C}^{*n}$ and continuous functions q_j on $\mathbb{C}^{*n}$ such that $V^{S*}_{K_j,q_j} \le q_j$ on K_j and*

$$\lim_{j\to\infty} V^S_{K_j,q_j} = V^S_{K,q} \qquad \text{and} \qquad \lim_{j\to\infty} \Phi^S_{K_j,q_j} = \Phi^S_{K,q}.$$

Proof: Let $\varepsilon > 0$ and $\delta_j \searrow 0$, $j \in \mathbb{N}$. Write $E_j = K + \delta_j \overline{\mathbb{D}}^n$, where $\mathbb{D}$ is the open unit disc. By Tietze's theorem, we can extend q continuously to E_1. Then there is a j_ε such that for all $j > j_\varepsilon$ we have $\|q(x) - q(y)\|_\infty < \varepsilon$ whenever $x, y \in E_1$ and $\|x - y\|_\infty < 2\delta_j$. Let $q_j = V^S_{E_j,q}$ and define the set $K_j = (E_j \setminus (H + \frac{1}{2}\delta_j \mathbb{D}^n)) + \frac{1}{4}\delta_j \overline{\mathbb{D}}^n$.

Let $v \in \mathcal{L}^S(\mathbb{C}^n)$ with $v|_{E_j} \le q$. Then $v \le V^S_{E_j,q} = q_j$ and $V^S_{E_j,q} \le V^S_{K_j,q_j}$. Let $u \in \mathcal{L}^S(\mathbb{C}^n)$ with $u|_{K_j} \le q_j$. If $z \in K \setminus K_j$ we can arrange the coordinates so $|z_k| < \frac{1}{2}\delta_j$ if $k \le \ell$ and $|z_k| \ge \frac{1}{2}\delta_j$ if $k > \ell$. If $w \in \mathbb{T}^\ell \times \{0\} \subset \mathbb{C}^n$ then $z + \delta_j w \in K_j$ and

$$u(z) \le \max_{w\in\mathbb{T}^\ell\times\{0\}} q_j(z + \delta_j w) \le \max_{w\in\mathbb{T}^\ell\times\{0\}} q(z + \delta_j w) \le q(z) + \varepsilon$$

if $j > j_\varepsilon$. Then $u - \varepsilon \le q$ on K and $V^S_{K_j,q_j} - \varepsilon \le V^S_{K,q}$ for all $j > j_\varepsilon$.

Fix $z_0 \in \mathbb{C}^n$. By [27], Proposition 4.8 (iii), there is a $j_0 \ge j_\varepsilon$ such that for all $j \ge j_0$ we have $V^S_{E_j,q}(z_0) + \varepsilon \ge V^S_{K,q}(z_0)$. Then $|V^S_{K,q}(z_0) - V^S_{K_j,q_j}(z_0)| < \varepsilon$ as desired. The argument for $\Phi^S_{K,q}$ is analoguous, by taking $u = \log|p|^{1/m}$ with $p \in \mathcal{P}^S_m(\mathbb{C}^n)$.

By [27], Lemma 5.2, Propositions 5.3 and 5.4 (ii), E_j and K_j are locally $\mathcal{L}$-regular, $q_j^* = V^{S*}_{E_j,q} \le q$ on E_j and $q_j = q_j^* \in \mathcal{C}(\mathbb{C}^{*n})$, therefore $V^{S*}_{K_j,q_j} \le q_j$ on K_j. □

By Lemma 6.1 we may assume $K \subset \mathbb{C}^{*n}$ is compact and $V^{S*}_{K,q} \le q$ on K. By Theorem 1.2 there is a convex body $T \subset \mathbb{R}^\ell_+$ and a rational map $F_L \colon \mathbb{C}^{*n} \to \mathbb{C}^{*\ell}$ such that $V^S_{K,q} = F_L^* V^T_{K',q'}$ and $\Phi^S_{K,q} = F_L^* \Phi^T_{K',q'}$ on $\mathbb{C}^{*n}$, where $q' = F_{L*}q$ is an admissible weight on the compact set $K' = F_L(K) \subset \mathbb{C}^{*\ell}$ by Proposition 2.1. Since F_L is a surjective submersion, $V^{T*}_{K',q'} \le q'$ on K', thus by [27], Proposition 5.4, $V^T_{K',q'}$ is continuous on $\mathbb{C}^{*\ell}$.

It follows that it is sufficient to prove that $\log \Phi^S_{K,q} \ge V^S_{K,q}$ on $\mathbb{C}^{*n}$ if $V^S_{K,q}$ is continuous on $\mathbb{C}^{*n}$, S is a convex body and $K \subset \mathbb{C}^{*n}$ is compact. Write $V = V^{S*}_{K,q}$ and note that $V = V^S_{K,q}$ on $\mathbb{C}^{*n}$.

We will prove that for every $z_0 \in \mathbb{C}^{*n}$ and $\varepsilon > 0$ there exists a $p \in \mathcal{O}(\mathbb{C}^n)$ such that $0 < a_m < d(mS, \mathbb{N}^n \setminus mS)$ for some $m \in \mathbb{N}^*$ and a constant $C > 0$ we have

$$|p(z)| \le C(1+|z|)^{a_m} e^{mH_S(z)}, \qquad z \in \mathbb{C}^n, \tag{6.1}$$

$$|p(z)| \le e^{mV(z)}, \qquad z \in K, \quad \text{and} \tag{6.2}$$

$$|p(z_0)| = e^{m(V(z_0)-3\varepsilon)}. \tag{6.3}$$

Then, by (6.1) and Theorem 3.6 in [27], we have $p \in \mathcal{P}^S_m(\mathbb{C}^n)$. Since $V \le q$ on K, (6.2) implies that $\|pe^{-mq}\|_K \le 1$ and (6.3) implies that

$$\log \Phi^S_{K,q}(z_0) \ge \log \Phi^S_{K,q,m}(z_0) \ge \log |p(z_0)|^{1/m} = V(z_0) - 3\varepsilon.$$

Since z_0 and $\varepsilon > 0$ are arbitrary we conclude that $\log \Phi^S_{K,q} \ge V$ on $\mathbb{C}^{*n}$.

We will apply Theorem 5.1 to show that p can be chosen of the form

$$p(z) = e^{m(V(z_0)-3\varepsilon)}\big(\chi(z) - u(z)\big), \qquad z \in \mathbb{C}^n, \tag{6.4}$$

where $\chi(z) = \chi_0(|z-z_0|^2/\gamma^2)$, with $\chi_0 \in \mathcal{C}^\infty(\mathbb{R})$, $\chi_0(x) = 1$ for $x \le \frac{1}{4}$, χ_0 decreasing, $\chi_0 = 0$ for $x \ge 1$, and 2γ is chosen smaller than the distance from z_0 to the coordinate hyperplanes. Then $\bar\partial u = \bar\partial \chi$ and we will choose a certain weight function $\varphi \in \mathcal{PSH}(\mathbb{C}^n)$ such that

$$\|u\|_{\varphi_{a_m}} \le a_m^{-1/2} \|\bar\partial \chi\|_{\varphi_{a_m-2}}. \tag{6.5}$$

Observe that $\chi(z) = 1$ for $|z-z_0| \le \frac{1}{2}\gamma$, $0 \le \chi \le 1$, $\chi(z) = 0$ for $|z-z_0| \ge \gamma$, $\bar\partial_j \chi(z) = \chi_0'(|z-z_0|^2/\gamma^2)(z_j - z_{0,j})/\gamma^2$ and that χ_0 can be chosen such that $|\chi_0'| \le 2$.

Proof of Theorem 1.1: As we have already noted, we may assume that S is a convex body and K is a compact subset of $\mathbb{C}^{*n}$ and it is sufficient to prove that for every $z_0 \in \mathbb{C}^{*n}$ and $\varepsilon > 0$ there exist $m \in \mathbb{N}^*$ and $p \in \mathcal{O}(\mathbb{C}^n)$, such that (6.1), (6.2) and (6.3) hold.

Recall that the function $z \mapsto H_S(z) - \max_{w\in K} H(w) + \min_{w\in K} q(w)$ is in $\mathcal{L}^S(\mathbb{C}^n)$ and is $\le q$ on K. This tells us that $V \in \mathcal{L}^S_+(\mathbb{C}^n)$, i.e. for some $c_V \in \mathbb{R}$ we have

$$-c_V + H_S(z) \le V(z) \le c_V + H_S(z), \qquad z \in \mathbb{C}^n. \tag{6.6}$$

Since V is continuous on $\mathbb{C}^{*n}$ there exists $\gamma > 0$ such that

$$|V(z+w) - V(z)| < \varepsilon, \qquad z \in K \cup \{z_0\}, \quad |w| \le 2\gamma. \tag{6.7}$$

Since S is a convex body, there exists a closed ball $\bar B(s_0, 2\varrho_0) \subseteq S$. Then for every $t \in \bar B(s_0, \varrho_0)$ the closed ball $\bar B(t, \varrho_0)$ is contained in S. Its supporting function is $\mathbb{R}^n \ni \xi \mapsto \varrho_0|\xi| + \langle t, \xi\rangle \le \varphi_S(\xi)$, so we have

$$\varrho_0 |\operatorname{Log} z| + \langle t, \operatorname{Log} z\rangle \le H_S(z), \qquad t \in B(s_0, \varrho_0), \quad z \in \mathbb{C}^{*n}.$$

Observe that if $|z| > |z_0|$ then

$$\log|z-z_0| \le \log(|z|+|z_0|) \le \log|z| + \log 2 \le \log\|z\|_\infty + \log(2\sqrt{n}).$$

We have $\log\|z\|_\infty \le |\operatorname{Log} z|$ for every $z \in \mathbb{C}^n$, so if we choose $c_0 \le -\varrho_0 \log(2\sqrt{n})$ such that $\varrho_0 \log|z-z_0| + c_0 \le 0$ for $|z| \le |z_0|$ then it follows that

$$\varrho_0 \log|z-z_0| + \langle t, \operatorname{Log} z\rangle + c_0 \le H_S(z), \tag{6.8}$$

for every $t \in B(s_0,\varrho_0)$ and $z \in \mathbb{C}^{*n}$. Now we fix $t_0 = s_0 + \varrho_0 \mathbf{1}/n \in \bar{B}(s_0,\varrho_0)$ and observe that for every $m > n/\varrho_0$ we have $\varphi \in \mathcal{PSH}(\mathbb{C}^n)$, where

$$\begin{aligned}\varphi(z) = 2(m-n/\varrho_0)(V(z)-\varepsilon)\\ + (2n/\varrho_0)\big(\varrho_0\log|z-z_0| + \langle t_0, \operatorname{Log} z\rangle + c_0\big).\end{aligned} \tag{6.9}$$

We choose χ as described after (6.4) and let u be a solution of $\bar\partial u = \bar\partial\chi$ satisfying (6.5). Before we make our choice of m let us show that (6.1) and (6.3) hold for every $m > n/\varrho_0$.

Since $\varphi(z) = 2n\log|z-z_0| + \psi(z)$, where the function ψ is bounded near z_0, the function $e^{-\varphi}$ is not locally integrable in any neighborhood of z_0 and we conclude that $u(z_0) = 0$. Hence (6.3) holds. By (6.6) and (6.8) with $t = s_0$ we have for every $z \in \mathbb{C}^n$ that

$$\begin{aligned}-\varphi(z) + 2(m-n/\varrho_0)c_V &\ge -2mH_S(z) - 2\langle \mathbf{1}, \operatorname{Log} z\rangle\\ &\quad + (2n/\varrho_0)\Big(H_S(z) - \varrho_0\log|z-z_0| - \langle s_0, \operatorname{Log} z\rangle - c_0\Big)\\ &\ge -2mH_S(z) - 2\langle \mathbf{1}, \operatorname{Log} z\rangle,\end{aligned}$$

so (6.5) implies

$$\begin{aligned}\int_{\mathbb{C}^n} |u(z)|^2(1+|z|^2)^{-a_m} e^{-2mH_S(z) - 2\langle \mathbf{1}, \operatorname{Log} z\rangle} d\lambda(z)\\ \le e^{2(m-n/\varrho_0)c_V}\|u\|^2_{\varphi_{a_m}} < +\infty.\end{aligned} \tag{6.10}$$

The Jacobi determinant of $\zeta \mapsto e^\zeta = (e^{\zeta_1},\dots,e^{\zeta_n})$ viewed as a mapping $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is equal to $|e^{\zeta_1}\cdots e^{\zeta_n}|^2 = e^{2\langle \mathbf{1},\xi\rangle}$. We define $v\colon \mathbb{C}^n \to \mathbb{C}$ by $v(\zeta) = u(e^\zeta)$, write $\zeta = \xi + i\eta$, for $\xi,\eta \in \mathbb{R}^n$, fix $\zeta_0 = \xi_0 + i\eta_0 \in \mathbb{C}^n$ such that $z_0 = e^{\zeta_0}$, and set

$$A = \prod_{j=1}^n \mathbb{R} \times [\eta_{0,j} - \pi, \eta_{0,j} + \pi].$$

By (6.10) we have

$$\begin{aligned}\int_A |v(\zeta)|^2(1+|e^\zeta|^2)^{-a_m} e^{-2m\varphi_S(\xi)} d\lambda(\zeta)\\ = \int_A |v(\zeta)|^2(1+|e^\xi|^2)^{-a_m} e^{-2m\varphi_S(\xi) - 2\langle \mathbf{1},\xi\rangle} e^{2\langle \mathbf{1},\xi\rangle} d\lambda(\zeta)\\ = \int_{\mathbb{C}^n} |u(z)|^2(1+|z|^2)^{-a_m} e^{-2mH_S(z) - 2\langle \mathbf{1}, \operatorname{Log} z\rangle} d\lambda(z) < +\infty.\end{aligned}$$

Since $\bar\partial u$ has compact support it follows that there exists a constant $C_2 > 0$ such that

$$|\bar\partial v(\zeta)| \le C_2 \le C_2 e^{m\varphi_S(\xi)}, \qquad \zeta = \xi + i\eta \in \mathbb{C}^n.$$

By Lemma 5.3 there exists a constant $C_3 > 0$ such that for every $\zeta = \xi + i\eta \in \mathbb{C}^n$

$$|v(\zeta)| \le C_3\big(1 + |e^\zeta|\big)^{a_m} \sup_{w\in\mathbb{B}} e^{m\varphi_S(\xi + \operatorname{Re} w)} \le C_4\big(1 + |e^\zeta|\big)^{a_m} e^{m\varphi_S(\xi)},$$

where $C_4 = C_3 \sup_{w\in\mathbb{B}} e^{m\varphi_S(\operatorname{Re} w)}$. We change the coordinates back to $z = e^\zeta$, use the fact that $\operatorname{Log} z = \xi$ and conclude that (6.1) holds for p as in (6.4).

It remains to show that if m is large enough then (6.2) holds. By Theorem 5.2 we have for every $z \in \mathbb{C}^n$ and $\delta > 0$ that

$$|p(z)| \le e^{m(V(z_0)-3\varepsilon)}\bigg(\chi(z) + 2\delta \cdot \sup_{|w|\le\delta} |\bar\partial\chi(z+w)| + a_m^{-1/2} c_n \delta^{-n} \|\bar\partial\chi\|_{\varphi_{a_m-2}} \sup_{|w|\le\delta} e^{\varphi_{a_m}(z+w)/2}\bigg), \quad (6.11)$$

where the constant c_n is given by (5.16). We take $\delta = \gamma$ and $z \in K$. If $z \in \operatorname{supp}\chi$, then (6.7) implies that $V(z_0) - \varepsilon < V(z)$ and we get

$$e^{m(V(z_0)-3\varepsilon)}\chi(z) \le e^{-m\varepsilon} e^{mV(z)}. \quad (6.12)$$

This estimate trivially holds for $z \notin \operatorname{supp}\chi$. If $z + w \in \operatorname{supp}\bar\partial\chi$ for some w with $|w| \le \gamma$, then $|z - z_0| < 2\gamma$, so by (6.7) we have $V(z_0) - \varepsilon < V(z)$ and since $|\bar\partial\chi| \le 2/\gamma$ we get

$$e^{m(V(z_0)-3\varepsilon)} 2\gamma \sup_{|w|\le\gamma} |\bar\partial\chi(z+w)| \le 4e^{-m\varepsilon} e^{mV(z)}. \quad (6.13)$$

If $z + w \notin \operatorname{supp}\bar\partial\chi$ then this estimate holds for every $|w| \le \gamma$. Next we observe that by (6.6), (6.7) and (6.8) with $t = t_0$ we have for $z \in K$ and $K_\gamma = \{z \in \mathbb{C}^n\,;\, d(z,K) \le \gamma\}$

$$\begin{aligned} e^{\varphi_{a_m}(z+w)/2} &\le \sup_{\zeta\in K_\gamma} (1 + |\zeta|^2)^{a_m/2} e^{(m - n/\varrho_0)(V(z+w)-\varepsilon) + (n/\varrho_0)H_S(z+w)} \\ &\le \sup_{\zeta\in K_\gamma} (1 + |\zeta|)^{a_m} e^{mV(z) + (n/\varrho_0)(\varepsilon + c_V)}. \end{aligned} \quad (6.14)$$

If we combine (6.11), (6.12), (6.13) and (6.14), we get

$$|p(z)| \le e^{-m\varepsilon} e^{mV(z)} \cdot \bigg(5 + a_m^{-1/2} c_n \delta^{-n} e^{(n/\varrho_0)(\varepsilon + c_V)} \sup_{\zeta\in K_\gamma} (1 + |\zeta|)^{a_m} e^{m(V(z_0)-2\varepsilon)} \|\bar\partial\chi\|_{\varphi_{a_m-2}}\bigg).$$

If we can prove that $e^{m(V(z_0)-2\varepsilon)}\|\bar\partial\chi\|_{\varphi_{a_m-2}}$ is a bounded function of m then it follows that we can choose m sufficiently large for (6.2) to hold. For every $\zeta\in\operatorname{supp}\bar\partial\chi$ we have $V(z_0)-2\varepsilon\le V(\zeta)-\varepsilon\le V(z_0)$, so

$$\begin{aligned}e^{2m(V(z_0)-2\varepsilon)}\|\bar\partial\chi\|^2_{\varphi_{a_m-2}}&\\ \le\int\limits_{\frac12\gamma\le|\zeta-z_0|\le\gamma}&|\bar\partial\chi(\zeta)|^2(1+|\zeta|^2)^2e^{(2n/\varrho_0)V(z_0)}|\zeta-z_0|^{-2n}\\ &\cdot\big(|\zeta_1|^{t_{0,1}}\cdots|\zeta_n|^{t_{0,n}}\big)^{-2n/\varrho_0}e^{-2nc_0/\varrho_0}\,d\lambda(\zeta).\end{aligned}$$

Since 2γ is smaller than the distance from z_0 to $\mathbb{C}^n\setminus\mathbb{C}^{*n}$, we have $|\zeta_j|\ge\gamma$ for $|\zeta-z_0|\le\gamma$, so the integral is convergent and independent of m. □

Paper III

Holomorphic approximation by polynomials with exponents restricted to a convex cone

Álfheiður Edda Sigurðardóttir

Abstract

We study approximations of holomorphic functions of several complex variables by proper subrings of the polynomials. The subrings in question consist of polynomials of several complex variables whose exponents are restricted to a prescribed convex cone $\mathbb{R}_+S$ for some compact convex $S \in \mathbb{R}^n_+$. Analogous to the polynomial hull of a set, we denote the hull of K with respect to the given ring by $\widehat{K}^S$. By studying an extremal function $V^S_K(z)$, we show a version of the Runge-Oka-Weil Theorem on approximation by these subrings on compact subsets of $\mathbb{C}^{*n}$ that satisfy $K = \widehat{K}^S$ and $V^{S*}_K|_K = 0$. We show a sharper result for compact Reinhardt sets K, that a holomorphic function is uniformly approximable on $\widehat{K}^S$ by members of the ring if and only if it is bounded on $\widehat{K}^S$. We also show that if K is a compact Reinhardt subsets of $\mathbb{C}^{*n}$, then we have $V^S_K(z) = \sup_{s\in S}(\langle s, \operatorname{Log} z\rangle - \varphi_A(s))$, where φ_A is the supporting function of $A = \operatorname{Log} K = \{(\log|z_1|, \ldots, \log|z_n|)\,;\, z \in K\}$.

1 Introduction

Polynomials in $\mathbb{C}^n$ whose exponents are restricted to a convex cone $\Gamma \subset \mathbb{R}^n_+$ form a subring of the polynomials. We regard Γ as the scaling of some compact convex $S \subset \mathbb{R}^n_+$ with $0 \in S$. Throughout this article, S will denote such a set. Let the space $\mathcal{P}^S_m(\mathbb{C}^n)$ consist of all polynomials p of the form

$$p(z) = \sum_{\alpha\in(mS)\cap\mathbb{N}^n} a_\alpha z^\alpha, \qquad z \in \mathbb{C}^n.$$

Then $\mathcal{P}^S(\mathbb{C}^n) = \bigcup_{m\in\mathbb{N}} \mathcal{P}^S_m(\mathbb{C}^n)$ is the ring of the polynomials whose exponents are restricted to $\Gamma = \mathbb{R}_+S$. The ring is graded in terms of the dilates of S, if $p_1 \in \mathcal{P}^S_{m_1}(\mathbb{C}^n)$ and $p_2 \in \mathcal{P}^S_{m_2}(\mathbb{C}^n)$, then $p_1p_2 \in \mathcal{P}^S_{m_1+m_2}(\mathbb{C}^n)$. The standard grading of polynomials is obtained by letting S be the standard unit simplex Σ, which is the convex hull of 0 and the unit basis $\{e_1, \ldots, e_n\}$.

By the Runge-Oka-Weil theorem, a holomorphic function defined in a neighborhood of the polynomial hull $\widehat{K}$ of compact $K \subset \mathbb{C}^n$ can be uniformly approximated by polynomials on K. If $\mathcal{F}$ is any family of complex valued functions on a set X, we define the $\mathcal{F}$-hull of a compact subset K of X by

$$\widehat{K}^{\mathcal{F}} = \{z \in X\,;\, |p(z)| \le \|p\|_K \text{ for all } p \in \mathcal{F}\}, \tag{1.1}$$

where $\|p\|_K = \sup\{|p(z)|\,;\, z \in K\}$. The polynomial hull is obtained by taking $\mathcal{F}$ as the family of polynomials on $\mathbb{C}^n$.

Regard a locally bounded complex valued function f defined on a neighborhood of Ω of $\widehat{K}^{\mathcal{F}}$ for some compact subset K of X. Assume that f can be uniformly approximated by members of $\mathcal{F}$ on compact subsets of $\widehat{K}^{\mathcal{F}}$. Unlike the polynomial hull, an $\mathcal{F}$-hull of a compact set is not necessarily compact, yet we can show that f has to be bounded on $\widehat{K}^{\mathcal{F}}$, and in fact $\|f\|_{\widehat{K}^{\mathcal{F}}} = \|f\|_K$. Indeed, for any $\varepsilon > 0$ and $z_0 \in \widehat{K}^{\mathcal{F}}$ there is a $p \in \mathcal{F}$ such that $\|p - f\|_{K\cup\{z_0\}} < \varepsilon$. Denote $C = \|f\|_K$. Then $\|p\|_K < C + \varepsilon$. Since $z_0 \in \widehat{K}^{\mathcal{F}}$ we have $|p(z_0)| \leq \|p\|_K < C + \varepsilon$. Therefore $|f(z_0)| < C + 2\varepsilon$. As this holds for any $\varepsilon > 0$ and any $z_0 \in \widehat{K}^{\mathcal{F}}$, we can conclude that $\|f\|_{\widehat{K}^{\mathcal{F}}} = C$.

In the setting of the Runge-Oka-Weil theorem, a continuous function defined on a neighborhood of $\widehat{K}$ for a compact K is automatically bounded on the compact set $\widehat{K}$. We have just presented a necessary condition for any generalization of the Runge-Oka-Weil Theorem for approximation by some family $\mathcal{F}$ that an approximable f must satisfy $\|f\|_{\widehat{K}^{\mathcal{F}}} = \|f\|_K$. The hull with respect to the family $\mathcal{F} = \mathcal{P}^S(\mathbb{C}^n)$ as defined in (1.1) will be referred to as the S-hull for short. We say that a set $K \subset \mathbb{C}^n$ is S-convex if $\widehat{K}^S = K$.

We first present a result in the specific case when the compact set K is a *Reinhardt set*, that is rotationally symmetric in each variable, $(\zeta_1 z_1, \ldots, \zeta_n z_n) \in K$ for all $z \in K$ and ζ in the unit torus $\mathbb{T}^n$. We can approximate a holomorphic function uniformly on the S-hull of compact Reinhardt sets, whether the hull itself is compact or not, if and only if the function is bounded on the S-hull.

Theorem 1.1 *Let X be a neighborhood of $\widehat{K}^S$ where K is a compact Reinhardt set and let $f \in \mathcal{O}(X)$. Then the following are equivalent:*

(i) f can be approximated uniformly on $\widehat{K}^S$ by polynomials from $\mathcal{P}^S(\mathbb{C}^n)$.

(ii) f can be approximated uniformly on all compact subsets of $\widehat{K}^S$ by polynomials from $\mathcal{P}^S(\mathbb{C}^n)$.

(iii) $\|f\|_{\widehat{K}^S} = \|f\|_K$.

(iv) f is bounded on $\widehat{K}^S$.

(v) There exists a $h \in \mathcal{O}(X)$ defined by a convergent power series of the form $h(z) = \sum_{\alpha\in\mathbb{R}_+ S} a_\alpha z^\alpha$ such that $f = h$ on $\widehat{K}^S$.

When condition (i) is assumed, it is quite clear that f is bounded on $\widehat{K}^S$. Condition (i) could then be phrased as so: f is in the closure of $\mathcal{P}^S(\mathbb{C}^n)$ in the uniform topology on $\mathcal{C}_b(\widehat{K}^S)$, where $\mathcal{C}_b(\widehat{K}^S)$ is the space of continuous bounded functions on $\widehat{K}^S$. Condition (ii) could be phrased as: f is in the closure of $\mathcal{P}^S(\mathbb{C}^n)$ in the topology on $\mathcal{C}(\widehat{K}^S)$ generated by the seminorms $\|\cdot\|_A$ for all compact $A \subset \widehat{K}^S$.

Any neighborhood X of $\widehat{K}^S$ for a Reinhardt set K contains a Reinhardt neighborhood of $\widehat{K}^S$, which contains the origin. Any function $f \in \mathcal{O}(X)$ can therefore be expressed uniquely as a power series centered at the origin which is normally

convergent in a neighborhood of $\widehat{K}^S$. If K also contains a point $z \in \mathbb{C}^{*n}$, it contains the polycircle $\{(|z_1|e^{i\theta_1}, \dots, |z_n|e^{i\theta_n})\,;\, \theta \in \mathbb{R}^n\}$. In that case a holomorphic function f is approximable by $\mathcal{P}^S(\mathbb{C}^n)$ precisely if its power series is of the form

$$f(z) = \sum_{\alpha \in \mathbb{R}_+ S} a_\alpha z^\alpha. \tag{1.2}$$

If K contains no point of $\mathbb{C}^{+n}$, then K is a subset of $\mathbb{C}^n \setminus \mathbb{C}^{*n}$ which is pluripolar so we do not expect to determine the series of f only by its values on K. Instead we can say that f coincides on K with some function whose series is of the form (1.2). Later, in Proposition 4.4 we see that if f has a series expansion of the form (1.2) only known to be convergent in a neighborhood of a compact $K \subset \mathbb{C}^{*n}$, then f extends as a holomorphic function in a neighborhood of $\widehat{K}^S$. The implication (v)⇒(i) is then true for the extension of f. Our next result is an approximating result without needing to assume that the set K is Reinhardt. But first we need to introduce certain extremal functions that are studied in a series of papers [26, 27, 42].

The polynomials from $\mathcal{P}^S(\mathbb{C}^n)$ can be characterized by comparing their growth to the *logarithmic supporting function* H_S of S defined by

$$\begin{aligned} H_S(z) &= \sup_{s \in S} \langle s, \operatorname{Log} z\rangle, && z \in \mathbb{C}^{*n} \\ H_S(z) &= \overline{\lim_{\mathbb{C}^{*n} \ni w \to z}} H_S(w), && z \in \mathbb{C}^n \setminus \mathbb{C}^{*n}. \end{aligned}$$

where $\operatorname{Log} z = (\log|z_1|, \dots, \log|z_n|)$. Let $\mathcal{L}^S(\mathbb{C}^n)$ be the class of all $u \in \mathcal{PSH}(\mathbb{C}^n)$ such that $u \le H_S + c_u$ where c_u is a constant only depending on u. Theorem 3.6 of [27] implies that a $p \in \mathcal{O}(\mathbb{C}^n)$ is a member of $\mathcal{P}^S_m(\mathbb{C}^n)$ if and only if $\log(|p|^{1/m})$ belongs to the class $\mathcal{L}^S(\mathbb{C}^n)$.

We associate to S and a compact $K \subset \mathbb{C}^n$ the m-th *Siciak functions*

$$\Phi^S_{K,m}(z) = \sup\{|p(z)|^{1/m}\,;\, p \in \mathcal{P}^S_m(\mathbb{C}^n),\, \|p\|_K \le 1\},$$

the *Siciak function*, which can by [27], Proposition 2.2, be equivalently defined as

$$\Phi^S_K = \sup_{m \in \mathbb{N}} \Phi^S_{K,m} = \lim_{m \to \infty} \Phi^S_{K,m},$$

and the *Siciak-Zakharyuta function*

$$V^S_K(z) = \sup\{u(z)\,;\, u \in \mathcal{L}^S(\mathbb{C}^n),\, u|_K \le 0\}.$$

From the definition follows that $\widehat{K}^S = \{z \in \mathbb{C}^n\,;\, \Phi^S_K(z) = 1\}$. Replacing S with $\overline{S \cap \mathbb{Q}^n}$ does not change the polynomial ring $\mathcal{P}^S(\mathbb{C}^n)$, so we may assume throughout the article that S satisfies $\overline{S \cap \mathbb{Q}^n} = S$. Then [26], Theorem 1.1 states that $V^S_K(z) = \log \Phi^S_K(z)$ for all $z \in \mathbb{C}^{*n}$. It is a regularity condition on K whether the upper regularization $V^{S*}_K(z) = \limsup_{w \to z} V^S_K(w)$ satisfies $V^{S*}_K|_K = 0$ or not.

Theorem 1.2 *Let $K \subset \mathbb{C}^{*n}$ be compact, S-convex and such that $V_K^{S*}|_K = 0$. If f is holomorphic in a neighborhood of K, then f can be uniformly approximated on K by polynomials from $\mathcal{P}^S(\mathbb{C}^n)$.*

We need to know that Theorem 1.2 is not a statement on an empty family of sets. Section 2 explores the properties of S-hulls in terms of the properties of the set S itself. We show in Proposition 2.4 that if S has non-empty interior, then compact subsets K of $\mathbb{R}_+^{*n}$ have $\widehat{K}^S \subset \mathbb{R}_+^{*n}$. Locally $\mathcal{L}$-regular sets, see [27], Definition 5.1, have the property that $V_K^{S*}|_K = 0$. For example, sets with a $\mathcal{C}^2$ boundary are locally $\mathcal{L}$-regular. We see that polynomially convex subsets of $\mathbb{R}_+^{*n}$ with a $\mathcal{C}^2$ boundary are S-convex if S is a convex body.

Theorem 1.2 then implies that $\widehat{K}^S = \widehat{K}$ whenever $\widehat{K}^S$ is a bounded locally $\mathcal{L}$-regular subset of $\mathbb{C}^{*n}$. Indeed, if $z \in \widehat{K}^S \setminus \widehat{K}$, there exists a polynomial f such that $|f(z)| > \|f\|_K$. By Theorem 1.2, there exists a $p \in \mathcal{P}^S(\mathbb{C}^n)$ such that $\|f - p\|_{\widehat{K}^S} < (|f(z)| - \|f\|_K)/2$. Then $|p(z)| > \|p\|_K$, which contradicts the assumption that $z \in \widehat{K}^S$. Therefore $\widehat{K}^S \setminus \widehat{K}$ is empty and $\widehat{K}^S = \widehat{K}$.

We also see in Section 2 that if S has an empty interior, then there are no non-empty compact S-convex subsets of $\mathbb{C}^{*n}$. In Section 3 we describe the S-hulls of Reinhardt sets. Since $\widehat{K}^S \cap \mathbb{C}^{*n} = \{z \in \mathbb{C}^{*n}\,;\, V_K^S(z) = 0\}$, it is useful to derive a formula for $V_K^S(z)$.

Theorem 1.3 *Let $K \subset \mathbb{C}^n$ be a compact Reinhardt set and $A = \operatorname{Log}(K \cap \mathbb{C}^{*n})$. Assume $S \cap \mathbb{R}_+^{*n} \neq \emptyset$ or $K = \overline{K \cap \mathbb{C}^{*n}}$. Then for all $z \in \mathbb{C}^{*n}$ we have*

$$V_K^S(z) = \sup_{s \in S}(\langle s, \operatorname{Log} z\rangle - \varphi_A(s)).$$

Note that the right hand side is the composite of Log with the Legendre transform of $x \mapsto \varphi_A(x)$ if $x \in S$ and $x \mapsto +\infty$ otherwise. This theorem implies that $\operatorname{Log} z$ maps $\widehat{K}^S \cap \mathbb{C}^{*n}$ onto the hull of $\operatorname{Log} K$ with respect to the cone $\Gamma = \mathbb{R}_+ S$ in $\mathbb{R}^n$. Furthermore, Proposition 3.5 provides a description of $\widehat{K}^S$ on $\mathbb{C}^n \setminus \mathbb{C}^{*n}$, giving us a complete description of S-hulls of Reinhardt sets in $\mathbb{C}^{*n}$.

Section 4 is devoted to the proof of Theorem 1.1 and Section 5 to the proof of Theorem 1.2. Section 6 is an exploration of specific cases of S and K such that we have a stronger approximating result than is obtained from Theorem 1.2, where the assumption that K avoids $\mathbb{C}^n \setminus \mathbb{C}^{*n}$ is not needed.

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2 S-convex sets

In order to appreciate Theorem 1.2 it is necessary to identify some compact S-convex subsets of $\mathbb{C}^{*n}$. The set $S \subset \mathbb{R}^n_+$ is a neighborhood of zero in $\mathbb{R}^n_+$ if there exists an $r > 0$ such that $r\Sigma \subset S$, where Σ is the standard unit simplex. Then $\mathcal{P}^S(\mathbb{C}^n)$ contains all the polynomials in $\mathbb{C}^n$, perhaps with a different grading, so that case gives no new approximating result. If S is not a neighborhood of zero, there is some unit vector e_k for $k \in [n] = \{1, \dots, n\}$ such that $e_k \notin \mathbb{R}_+ S$. Every $p \in \mathcal{P}^S(\mathbb{C}^n)$ can then be written of the form $p(z) = c + \sum_{j\neq k} z_j p'_j(z)$ where c is a constant and p'_j is a polynomial for $j \in [n] \setminus \{k\}$. Then p is constant on the axis $\mathbb{C}e_k$ in $\mathbb{C}^n$. So if a set K contains a point from $\mathbb{C}e_k$, then $\widehat{K}^S$ contains $\mathbb{C}e_k$.

In the same vein we can identify more subspaces where polynomials from $\mathcal{P}^S(\mathbb{C}^n)$ are fixed depending on S, but for that we need some notation. For an ordered subset $J = (j_1, \dots, j_\ell)$ of $[n]$ we denote $\mathbb{C}^J = \{z \in \mathbb{C}^n \,;\, z_j = 0 \text{ if } j \notin J\}$ and $\mathbb{C}^{*J} = \{z \in \mathbb{C}^n \,;\, z_j \neq 0 \text{ if and only if } j \in J\}$ and define $\mathbb{R}^J = \mathbb{C}^J \cap \mathbb{R}^n$. Let $\pi_J \colon \mathbb{C}^n \to \mathbb{C}^\ell$, $\pi_J(z) = (z_{j_1}, \dots, z_{j_\ell})$ and let $S_J = \pi_J(S \cap \mathbb{R}^J)$. Proposition 3.3 of [27], states that

$$H_S(z) = H_{S_J}(\pi_J(z)), \qquad z \in \mathbb{C}^J. \tag{2.1}$$

Every $p \in \mathcal{P}^S_m(\mathbb{C}^n)$ satisfies $\log|p| \leq mH_S + c$ for all $z \in \mathbb{C}^n$ and some constant c by [27], Theorem 3.6. If $J \subset [n]$ is such that $S_J = \{0\}$, the right hand side of (2.1) is merely zero. Any $p \in \mathcal{P}^S(\mathbb{C}^n)$ is then bounded, hence constant, on $\mathbb{C}^J$, which is an unbounded set if $J \neq \emptyset$.

While singletons on $\mathbb{C}^n \setminus \mathbb{C}^{*n}$ may have unbounded S-hulls, we show in Corollary 2.2 that the singletons in $\mathbb{C}^{*n}$ are S-convex if S has non-empty interior, referred to as a *convex body*. This follows from the fact that the family $\mathcal{P}^S(\mathbb{C}^n)$ separates the points of $\mathbb{C}^{*n}$, meaning that for every pair $x, y \in \mathbb{C}^{*n}$, $x \neq y$ there exists $p \in \mathcal{P}^S(\mathbb{C}^n)$ such that $p(x) \neq p(y)$. Furthermore, this can be done only using monomials.

Proposition 2.1 *Let S be a convex body. Then the monomials in $\mathcal{P}^S(\mathbb{C}^n)$ separate the points of $\mathbb{C}^{*n}$.*

Proof: Let $z, w \in \mathbb{C}^{*n}$, $z \neq w$ and let $\xi, \eta, \theta, \phi \in \mathbb{R}^n$ be such that $e^{\xi_j + i\theta_j} = z_j$ and $e^{\eta_j + i\phi_j} = w_j$ for $j = 1, \dots, n$. Assume first that $\xi \neq \eta$. Let the points $\alpha_1, \dots, \alpha_n \in \mathbb{R}_+ S \cap \mathbb{N}^n$ be linearly independent. Then there is some $k \in [n]$ such that $\langle \alpha_k, \xi\rangle \neq \langle \alpha_k, \eta\rangle$ which implies that $|z^{\alpha_k}| \neq |w^{\alpha_k}|$.

Otherwise, there exists some $j \in [n]$ such that $\theta_j - \phi_j \notin 2\pi\mathbb{Z}$. Since the cone $\mathbb{R}_+ S$ has nonempty interior, there exists some $\alpha \in \mathbb{R}_+ S \cap \mathbb{N}^n$ such that $\alpha + \Sigma \subset \mathbb{R}_+ S$. Then $\frac{1}{2\pi}\left(\langle \alpha + e_j, \theta - \phi\rangle - \langle \alpha, \theta - \phi\rangle\right) = \frac{1}{2\pi}\langle e_j, \theta - \phi\rangle$ is not an integer, so either

$\beta = \alpha + e_j$ or $\beta = \alpha$ is such that $\frac{1}{2\pi}\langle \beta, \theta - \phi\rangle$ is not an integer. Then z^β and w^β have different arguments. □

Corollary 2.2 *If S is a convex body and $x \in \mathbb{C}^{*n}$, then $\{x\}$ is S-convex.*

We will show in Proposition 2.4 that if S is a convex body, then S-hulls of compact subsets of $\mathbb{R}^{*n}_+$ satisfy the hypothesis of Theorem 1.2, that is are S-convex subsets of $\mathbb{C}^{*n}$. But first we need a Lemma.

Lemma 2.3 *Let $\alpha_1, \ldots, \alpha_n \in \mathbb{N}^n$ be linearly independent. Then the map $F\colon \mathbb{C}^{*n} \to \mathbb{C}^{*n}$ defined by $F(z) = (z^{\alpha_1}, \ldots, z^{\alpha_n})$ is proper.*

Proof: Let $L\colon \mathbb{R}^n \to \mathbb{R}^n$ be the linear map that maps e_j to α_j. Denote the adjoint of L by L^* and denote $\operatorname{Log} z = (\log|z_1|, \ldots, \log|z_n|)$. Regard that $\operatorname{Log} F(z) = L^* \operatorname{Log} z$. If K is a compact set in $\mathbb{C}^{*n}$, it is contained in some polyannulus,

$$K \subset \{z \in \mathbb{C}^n\,;\, e^{a_j} \le |z_j| \le e^{b_j},\ j = 1, \ldots, n\} = \operatorname{Log}^{-1}\Big(\prod_{j=1}^n [a_j, b_j]\Big).$$

Then $F^{-1}(K) \subset (\operatorname{Log} F)^{-1}\big(\prod_{j=1}^n [a_j, b_j]\big) = (L^* \operatorname{Log})^{-1}\big(\prod_{j=1}^n [a_j, b_j]\big)$.

Now L^* is linear and bijective since L is, so $(L^*)^{-1}\prod_{j=1}^n [a_j, b_j]$ is compact, and is therefore contained in some box $\prod_{j=1}^n [c_j, d_j]$. Then finally

$$F^{-1}(K) \subset \operatorname{Log}^{-1}\Big(\prod_{j=1}^n [c_j, d_j]\Big) = \{z \in \mathbb{C}^n\,;\, e^{c_j} \le |z_j| \le e^{d_j},\ j = 1, \ldots, n\}$$

so it is compact. We have shown that F is proper as a map $\mathbb{C}^{*n} \to \mathbb{C}^{*n}$. □

Proposition 2.4 *Assume S is a convex body and let K be a compact subset of $\mathbb{R}^{*n}_+$. Then $\widehat{K}^S$ is a compact subset of $\mathbb{R}^{*n}_+$.*

Proof: Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}_+ S \cap \mathbb{N}^n$ be linearly independent. By Lemma 2.3, $F\colon \mathbb{C}^{*n} \to \mathbb{C}^{*n}$ defined by $F(z) = (z^{\alpha_1}, \ldots, z^{\alpha_n})$ is proper. Now $F(K)$ is bounded and it is contained in some polyannulus $R\mathbb{D}^n \setminus (\mathbb{H} + r\mathbb{D}^n)$ where $\mathbb{H} = \mathbb{C}^n \setminus \mathbb{C}^{*n}$. Then $\widehat{K}^S$ is contained in $F^{-1}(R\mathbb{D}^n)$. For each j regard the polynomial $p_j(z) = R - z^{\alpha_j}$. Then $\|p_j\|_K = R - |z^{\alpha_j}| \le R$ and $|p_j(z)| > R - r$ whenever $|z^{\alpha_j}| < r$. The set

$$\{z \in \mathbb{C}^n\,;\, |z^{\alpha_j}| < r, \text{ for some } j = 1, \ldots, n\} = F^{-1}(\mathbb{H} + r\mathbb{D}^n)$$

does therefore not intersect $\widehat{K}^S$. Therefore $\widehat{K}^S \subset F^{-1}(R\mathbb{D}^n \setminus (\mathbb{H} + r\mathbb{D}^n)) \subset \mathbb{C}^{*n}$ which is compact since F is proper.

Let $z \in \mathbb{C}^{*n} \setminus \mathbb{R}^n$. By Proposition 2.1 there is an $\alpha \in \mathbb{R}_+ S \cap \mathbb{N}^n$ such that $z^\alpha \ne r^\alpha$ for $r = (|z_1|, \ldots, |z_n|)$. Since $|z^\alpha| = |r^\alpha|$, the arguments of z^α and r^α must be different. Now $r^\alpha \in \mathbb{R}_+$, so $z^\alpha \notin \mathbb{R}_+$. Let $p'(w) = w^\alpha$, and $K' = p'(K)$.

Then K' is a compact subset of $\mathbb{R}$, so it is polynomially convex. Then there exists a polynomial $p \in \mathcal{P}(\mathbb{C})$ such that $|p(z^\alpha)| > \|p\|_{K'}$, which implies that $p \circ p' \in \mathcal{P}^S(\mathbb{C}^n)$ is such $|p \circ p'(z)| > \|p \circ p'\|_K$. Therefore $z \notin \widehat{K}^S$, and we conclude that $\widehat{K}^S \subset \mathbb{R}^n_+$. □

The case when S has empty interior is thoroughly studied in [26], and we will review some of the results from there. Then the convex set S is of some lower dimension $\ell < n$ and there exists a linear map $L\colon \mathbb{R}^\ell \to \mathbb{R}^n$ whose image contains S. We may assume that $\overline{S \cap \mathbb{Q}^n} = S$ throughout this article, since replacing S with $\overline{S \cap \mathbb{Q}^n}$ does not affect the polynomial ring $\mathcal{P}^S(\mathbb{C}^n)$. This condition is also the sufficient and necessary for [26], Theorem 1.2. It states that L can be chosen so it maps the lattice points $\mathbb{Z}^\ell \subset \mathbb{R}^\ell$ onto the lattice points of its image, that is $L(\mathbb{Z}^\ell) = (\operatorname{span}_{\mathbb{R}} S) \cap \mathbb{Z}^n$. Furthermore, L can be chosen so the compact convex set $T = L^{-1}(S)$ is a subset of $\mathbb{R}^\ell_+$.

By Theorem 1.2 of [26], the map $F_L\colon \mathbb{C}^{*n} \to \mathbb{C}^{*\ell}$, $F_L(z) = (z^{L(e_1)}, \dots, z^{L(e_\ell)})$ is such that every polynomial $p \in \mathcal{P}^S_m(\mathbb{C}^n)$ can be factored as $p = p' \circ F_L$ for some $p' \in \mathcal{P}^T_m(\mathbb{C}^\ell)$. The fiber of p through a point $z \in \mathbb{C}^{*n}$ therefore contains the fiber of F_L through z, which is an unbounded $(n-\ell)$-dimensional submanifold. Therefore the S-hull of any K with $K \cap \mathbb{C}^{*n} \neq \emptyset$ is unbounded.

We can even parameterize $\widehat{\{z\}}^S$ for any point $z \in \mathbb{C}^{*n}$. Let $\beta'_1, \dots, \beta'_{n-\ell} \in \mathbb{N}^n$ be generators of the lattice orthogonal to S, that is $({}^\perp\operatorname{span}_{\mathbb{R}} S) \cap \mathbb{N}^n$, and regard the dual set of vectors $\beta_{jk} = \beta'_{kj}$. By [26], Lemma 4.1, for every $z \in \mathbb{C}^{*n}$, the image of the map $\Upsilon_z\colon \mathbb{C}^{(n-\ell)*} \to \mathbb{C}^{*n}$, $\Upsilon_z(t) = (z_1 t^{\beta_1}, \dots, z_n t^{\beta_n})$ is precisely the fiber of F_L that contains z, all of which is contained in $\widehat{\{z\}}^S$. For any point $w \in \mathbb{C}^{*n}$ outside this fiber, that is satisfying $F_L(w) \neq F_L(z)$, there exists by Proposition 2.1 a monomial $p' \in \mathcal{P}^T(\mathbb{C}^\ell)$ such that $p'(F_L(w)) \neq p'(F_L(z))$, which implies that $w \notin \widehat{\{z\}}^S$. We summarize:

Proposition 2.5 *If S has empty interior, then for any point $z \in \mathbb{C}^{*n}$ we have*

$$\widehat{\{z\}}^S \cap \mathbb{C}^{*n} = F_L^{-1}(F_L(z)) = \Upsilon_z[\mathbb{C}^{(n-\ell)*}].$$

Furthermore, the S-hull of any $K \subset \mathbb{C}^n$ that intersects $\mathbb{C}^{*n}$ is unbounded.

3 S-hulls of Reinhardt sets

Consider the simplest polynomials, the monomials. The modulus of monomials can be expressed solely in terms of an inner product with their exponent, $|z^\alpha| = e^{\langle \alpha, \operatorname{Log} z\rangle}$, $z \in \mathbb{C}^{*n}$. In logarithmic coordinates $x = \operatorname{Log} z$, the sublevel sets of monomials are halfspaces. The polynomial hull of a set $K \subset \mathbb{C}^{*n}$ is contained in the intersection of all sublevel sets of monomials that contain K, that is $\widehat{K} \subset \{z \in \mathbb{C}^n\,;\, |z^\alpha| \leq \max_{w\in K} |w^\alpha|,\ \alpha \in \mathbb{N}^n\}$. In logarithmic coordinates, the right hand side is the intersection of all halfspaces that contain $A = \operatorname{Log} K$,

which form its convex hull $\mathrm{ch}(A)$. The intersection of all halfspaces containing A whose normal is from some cone Γ is

$$\widehat{A}_\Gamma = \{x \in \mathbb{R}^n\,;\, \langle x, \xi\rangle \le \varphi_A(\xi) \text{ for all } \xi \in \Gamma\}. \tag{3.1}$$

where $\varphi_A(\xi) = \sup_{a\in A}\langle a, \xi\rangle$. Note that this definition of $\widehat{A}_\Gamma$ differs from [27], Definition 5.5 which is $\widehat{A}_\Gamma \cap \mathbb{R}^n_+$. The hull $\widehat{K}^S$ is contained in the intersection of all sublevel sets of monomials from $\mathcal{P}^S(\mathbb{C}^n)$ that contain K,

$$\widehat{K}^S \subset \{z \in \mathbb{C}^n\,;\, |z^\alpha| \le \sup_{w\in K} |w^\alpha| \text{ for all } \alpha \in \mathbb{R}_+S \cap \mathbb{N}^n\}.$$

By switching into logarithmic coordinates, $x = \mathrm{Log}\, z$, and letting $\Gamma = \mathbb{R}_+S$,

$$\mathrm{Log}[\widehat{K}^S \cap \mathbb{C}^{*n}] \subset \{x \in \mathbb{R}^n\,;\, \langle x, \alpha\rangle \le \varphi_A(\alpha) \text{ for all } \alpha \in \mathbb{R}_+S \cap \mathbb{N}^n\}. \tag{3.2}$$

The inner product is homogeneous, so the right hand side can be taken with α from $\mathbb{R}_+S \cap \mathbb{Q}^n$, which is dense in $\mathbb{R}_+S$ if $\overline{S \cap \mathbb{Q}^n} = S$. Then the right hand side is $\widehat{A}_\Gamma$. This inclusion is an equality for a certain class of sets.

Definition 3.1 We say that $K \subset \mathbb{C}^n$ is *Reinhardt* if for all $z \in K$ and $\zeta \in \mathbb{T}^n$ then $\zeta z = (\zeta_1 z_1, \ldots, \zeta_n z_n) \in K$.

As mentioned in the introduction, $\widehat{K}^S \cap \mathbb{C}^{*n} = \{z \in \mathbb{C}^{*n}\,;\, V_K^S(z) = 0\}$, so identifying V_K^S is helpful to find the S-hull of a set K. We will obtain a complete characterization of $\widehat{K}^S$ for Reinhardt sets from Propositions 3.3 and 3.5.

Reinhardt sets K have the property that if $u \in \mathcal{PSH}(\mathbb{C}^n)$ has $u|_K \le 0$, then

$$u'(z) = \sup_{\zeta\in\mathbb{T}^n} u(\zeta_1 z_1, \ldots, \zeta_n z_n)$$

is plurisubharmonic, rotationally symmetric in each coordinate, satisfies $u'|_K \le 0$ and $u' \ge u$. That means that V_K^S equals the supremum over the rotationally symmetric $u \in \mathcal{L}^S(\mathbb{C}^n)$ with $u|_K \le 0$.

If $u \in \mathcal{PSH}(\mathbb{C}^n)$ is rotationally symmetric, then $v\colon \mathbb{R}^n \to \mathbb{R}$, $v(\xi) = u(e^\xi)$ is a convex function that is increasing in each variable. The convexity of v can best be seen by taking a sequence $u_\ell \searrow u$ of $u_\ell \in \mathcal{C}^2 \cap \mathcal{PSH}(\mathbb{C}^n)$. By replacing u_ℓ by $\sup_{\zeta\in\mathbb{T}^n} u_\ell(\zeta z)$, we may assume u_ℓ are also rotationally symmetric. Taking $v_\ell(\xi) = u_\ell(e^\xi)$ we observe that

$$\frac{\partial^2 u_\ell(z)}{\partial z_j \partial \overline{z}_k} = \frac{1}{4 z_j \overline{z}_k} \cdot \frac{\partial^2 v_\ell(\xi)}{\partial \xi_j \partial \xi_k}, \qquad z \in \mathbb{C}^{*n},$$

which implies that

$$\det\left(\frac{\partial^2 u_\ell(z)}{\partial z_j \partial \overline{z}_k}\right) = \frac{1}{4^n |z_1 \cdots z_n|^2} \det\left(\frac{\partial^2 v_\ell(\xi)}{\partial \xi_j \partial \xi_k}\right)\Bigg|_{\xi = \mathrm{Log}\, z\,,} \tag{3.3}$$

and the positivity of one side of this equation implies the positivity of the other. Therefore $u_\ell \in \mathcal{PSH}(\mathbb{C}^n)$ implies that v_ℓ is convex. The limit v is

therefore also convex. In fact this is a bijection between rotationally symmetric $\mathcal{PSH}(\mathbb{C}^n)$ functions and convex functions increasing in each variable, since u can be recovered by $u(z) = v(\operatorname{Log} z)$.

The formula (3.3) further implies that the complex Monge-Ampère operator $(dd^c u)^n$ can be identified through the real Monge-Ampère operator

$$\mathcal{MA}_{\mathbb{R}}(v) = \det\left(\frac{\partial^2 v(\xi)}{\partial \xi_j \partial \xi_k}\right) d\xi.$$

The mass of $\mathcal{MA}_{\mathbb{R}}(v)$ at E has a geometric interpretation. The *subdifferential* of a convex v on an open $\Omega \subset \mathbb{R}^n$ at a point $x \in \Omega$ is the set

$$\operatorname{Sub}_v(x) = \{\xi \in \mathbb{R}^n\,;\, \langle y, \xi\rangle \leq v(x+y) - v(x),\, \forall y \in \mathbb{R}^n\}.$$

The *gradient image* of v over E is $\operatorname{Grad}_v(E) = \bigcup_{x\in E} \operatorname{Sub}_v(x)$. In [16], Theorem A.31 it is proven that the volume of the gradient image is precisely the real Monge-Ampère mass over E,

$$\int_E \mathcal{MA}_{\mathbb{R}}(v) = \operatorname{vol}(\operatorname{Grad}_v(E)). \tag{3.4}$$

A locally bounded plurisubharmonic function is maximal on open sets where the complex Monge-Ampère operator has no mass [23], Theorem 4.4.2. The plurisubharmonic function V_K^{S*} is maximal outside K by [27], Theorem 6.1.

Let $\mathcal{L}_+^S(\mathbb{C}^n)$ denote the class of all $u \in \mathcal{L}^S(\mathbb{C}^n)$ that also satisfy a bound $u \geq H_S - c$ for some constant c. Snorrason showed in [42], Proposition 2.2 that if S is a neighborhood of 0 and $V_K^{S*}|_K = 0$, then V_K^{S*} is the unique function in $\mathcal{L}_+^S(\mathbb{C}^n)$ that is 0 on K and maximal outside K.

If K is a Reinhardt set, then V_K^S coincides on $\mathbb{C}^{*n}$ with $V_{K'}^S$ for $K' = \overline{K \cap \mathbb{C}^{*n}}$, if we make the minor assumption that $S \cap \mathbb{R}_+^{*n} \neq \emptyset$. The case when that assumption is not satisfied will be sorted in Proposition 3.5.

Lemma 3.2 *Assume that $S \cap \mathbb{R}_+^{*n} \neq \emptyset$. Let $K \subset \mathbb{C}^n$ be a compact Reinhardt set, and let $K' = \overline{K \cap \mathbb{C}^{*n}}$. Then $V_K^S|_{\mathbb{C}^{*n}} = V_{K'}^S|_{\mathbb{C}^{*n}}$ and $V_{K'}^{S*}|_{K'} = 0$.*

Proof: Clearly $V_K^S \leq V_{K'}^S$. Let $s \in S \cap \mathbb{R}_+^{*n}$. The function $v(z) = \langle s, \operatorname{Log} z\rangle \in \mathcal{L}^S(\mathbb{C}^n)$ takes the value $-\infty$ on $\mathbb{C}^n \setminus \mathbb{C}^{*n}$ and is finite on $\mathbb{C}^{*n}$.

Let $u \in \mathcal{L}^S(\mathbb{C}^n)$ be such that $u|_{K'} \leq 0$ and let $\varepsilon > 0$. Then $(1-\eta)u + \eta v \in \mathcal{L}^S(\mathbb{C}^n)$ is $\leq \varepsilon$ on K for all $\eta > 0$ small enough. Therefore $u \leq V_K^S$ on $\mathbb{C}^{*n}$. We conclude that $V_{K'}^S = V_K^S$ on $\mathbb{C}^{*n}$.

If $u \in \mathcal{PSH}(\mathbb{C}^n)$ is rotationally symmetric and $u(z) \leq 0$ for some point $z \in \mathbb{C}^{*n}$, then $u \leq 0$ on the polydisc $D_z = \{w \in \mathbb{C}^n\,;\, |w_j| \leq |z_j|,\, j = 1, \ldots, n\}$. This implies that for $\widetilde{K} = \bigcup_{z\in K\cap\mathbb{C}^{*n}} D_z$ we have $V_{K'}^S \leq V_{\widetilde{K}}^S$ on $\mathbb{C}^n$. Furthermore, $V_K^S|_{\mathbb{C}^{*n}} = V_{K'}^S|_{\mathbb{C}^{*n}} \geq V_{\widetilde{K}}^S|_{\mathbb{C}^{*n}}$, concluding the proof of the first statement.

Each $w \in \widetilde{K}$ is a member of D_z for some $z \in \mathbb{C}^{*n}$, and D_z is the unit ball in some norm. Lemma 5.2 in [27] therefore implies that $\widetilde{K}$ is locally $\mathcal{L}$-regular. By [27], Proposition 5.3 and 5.4 $V^{S*}_{\widetilde{K}}|_{\widetilde{K}} = 0$, from which follows that $V^{S*}_{K'}|_{K'} = 0$. □

We can now derive a formula for V^S_K for all compact Reinhardt sets $K \subset \mathbb{C}^n$ that satisfy $\overline{K \cap \mathbb{C}^{*n}} = K$.

Proof of Theorem 1.3: Let $A = \mathrm{Log}(K \cap \mathbb{C}^{*n})$. In light of Lemma 3.2, we may assume that $K = \overline{K \cap \mathbb{C}^{*n}}$ which implies that $V^{S*}_K|_K = 0$. Let

$$U_S(z) = \sup_{s\in S}(\langle s, \mathrm{Log}\, z\rangle - \varphi_A(s)), \qquad z \in \mathbb{C}^{*n}$$

and observe that $U_S|_K = 0$. To show that U_S is upper semicontinuous on $\mathbb{C}^{*n}$ it suffices to show for every point $z \in \mathbb{C}^{*n}$ and every sequence $\{z_j\}_{j\in\mathbb{N}}$ in $\mathbb{C}^{*n}$ tending to z, that $\limsup_{j\to\infty} U_S(z_j) \le U_S(z)$. First let $\{z_{j_\kappa}\}_{\kappa\in\mathbb{N}}$ be a subsequence of $\{z_j\}_{j\in\mathbb{N}}$ such that $\limsup_{j\to\infty} U_S(z_j) = \lim_{\kappa\to\infty} U_S(z_{j_\kappa})$. Let $s_j \in S$ be such that $U_S(z_j) = \langle s_j, \mathrm{Log}\, z_j\rangle - \varphi_A(s_j)$. Since S is compact, there exists a convergent subsequence $\{s_{j_\kappa}\}_{\kappa\in\mathbb{N}}$ whose limit $s = (\lim_{\kappa\to\infty} s_{j_\kappa})$ is in S. Then

$$\begin{aligned}\limsup_{j\to\infty} U_S(z_j) &= \lim_{\kappa\to\infty} U_S(z_{j_\kappa}) = \lim_{\nu\to\infty} U_S(z_{j_{\kappa_\nu}})\\ &= \lim_{\nu\to\infty} \langle s_{j_{\kappa_\nu}}, \mathrm{Log}\, z_{j_{\kappa_\nu}}\rangle - \varphi_A(s_{j_{\kappa_\nu}}) = \langle s, \mathrm{Log}\, z\rangle - \varphi_A(s) \le U_S(z)\end{aligned}$$

which shows that U_S is upper semicontinuous on $\mathbb{C}^{*n}$. If we extend the definition of U_S to $\mathbb{C}^n$ by the formula $U_S(z) = \limsup_{w\to z} U_S(w)$, then $U_S \in \mathcal{PSH}(\mathbb{C}^n)$. Since φ_A is continuous and S is compact, there exists some constant c such that $\varphi_A(s) \in [-c, c]$ for all $s \in S$. Then $H_S - c \le U_S \le H_S + c$, which implies that $U_S \in \mathcal{L}^S_+(\mathbb{C}^n)$.

In order to apply [42], Proposition 2.2, we need to show that U_S is maximal outside K. The convex function $v\colon \mathbb{R}^n \to \mathbb{R}$, $v(x) = \sup_{s\in S}(\langle s, x\rangle - \varphi_A(s))$ has the subdifferential

$$\mathrm{Sub}_v(x) = \{\xi \in \mathbb{R}^n\,;\, \langle y, \xi\rangle \le v(x+y) - v(x) \quad \forall y \in \mathbb{R}^n\}.$$

Denote $v'(x; y) = \lim_{h\to 0+}(v(x + hy) - v(x))/h$ and observe that

$$\mathrm{Sub}_v(x) = \{\xi \in \mathbb{R}^n\,;\, \langle y, \xi\rangle \le v'(x; y) \quad \forall y \in \mathbb{R}^n\}.$$

By [19], Theorem 2.2.11, the function $y \mapsto v'(x; y)$ is the supporting function of the set $E = \{\xi \in \mathbb{R}^n\,;\, \langle x, \xi\rangle = v(x) + \tilde{v}(\xi)\}$ where $\tilde{v}$ is the Legendre transform of v, which implies that $\mathrm{Sub}_v(x) = E$. The convex function f defined by $f(\xi) = \varphi_A(\xi)$ if $\xi \in S$ and $f(\xi) = +\infty$ otherwise, has the Legendre transform $\tilde{f} = v$. By [19], Theorem 2.2.4, this implies that $\tilde{v} = f$. We can conclude that

$$\mathrm{Sub}_v(x) = \{\xi \in S\,;\, \langle x, \xi\rangle - \varphi_A(\xi) = \sup_{s\in S}(\langle s, x\rangle - \varphi_A(s))\}.$$

If ξ is an interior point of S, then there is an $r > 1$ such that $r\xi \in S$ and $\langle x, r\xi\rangle - \varphi_A(r\xi) > \langle x, \xi\rangle - \varphi_A(\xi)$. Therefore $\mathrm{Grad}_v(x) \subset \partial S$ for all $x \notin A$, which implies that $\mathrm{vol}(\mathrm{Grad}_v(x)) = 0$. Then U_S is maximal on $\mathbb{C}^{*n} \setminus K$, and therefore on $\mathbb{C}^n \setminus K$ since $\mathbb{C}^n \setminus \mathbb{C}^{*n}$ is pluripolar.

If S is a neighborhood of zero, it now follows from [42], Proposition 2.2, that V_K^S is the unique function from $\mathcal{L}_+^S(\mathbb{C}^n)$ that is 0 on K and is maximal outside K. Therefore $V_K^S = U_S$ in this case, and our next goal is to extend this identity to S that are not neighborhoods of zero.

Let $\delta_j \searrow 0$, $j \in \mathbb{N}$, and let $S_j = S + \delta_j \Sigma$. By [27], Proposition 4.8 (ii), $V_K^{S_j} \searrow V_K^S$. What is left to show is that $V_K^{S_j}(z) = \sup_{s \in S_j}(\langle s, \mathrm{Log}\, z\rangle - \varphi_A(s))$ tends to $U_S(z)$ as $j \to \infty$ for any fixed $z \in \mathbb{C}^{*n}$. For all j there is a particular $s_j \in S_j$ such that $V_K^{S_j}(z) = \langle s_j, \mathrm{Log}\, z\rangle - \varphi_A(s_j)$.

As S_1 is compact, there is a convergent subsequence $\{s_{j_\kappa}\}_{\kappa\in\mathbb{N}}$ of $\{s_j\}_{j\in\mathbb{N}}$, and the limit $s = \lim_{\kappa\to\infty} s_{j_\kappa}$ is in S. The convergent sequence $\{V_K^{S_j}(z)\}_{j\in\mathbb{N}}$ has the same limit as its subsequence $\{V_K^{S_{j_\kappa}}(z)\}_{\kappa\in\mathbb{N}}$, which is $\langle s, \mathrm{Log}\, z\rangle - \varphi_A(s)$. This shows that $V_K^S(z) = \langle s, \mathrm{Log}\, z\rangle - \varphi_A(s) \le U_S(z)$. □

The identity $\widehat{K}^S \cap \mathbb{C}^{*n} = \{z \in \mathbb{C}^{*n}\,;\, V_K^S(z) = 0\}$ then provides a corollary:

Corollary 3.3 *Assume that $S \cap \mathbb{Q}_+^{*n} \neq \emptyset$ or $K = \overline{K \cap \mathbb{C}^{*n}}$. Let K be a compact Reinhardt subset of $\mathbb{C}^n$, $A = \mathrm{Log}(K \cap \mathbb{C}^{*n})$ and $\Gamma = \mathbb{R}_+ S$. Then*

$$\widehat{K}^S \cap \mathbb{C}^{*n} = \mathrm{Log}^{-1}\widehat{A}_\Gamma.$$

Now $\widehat{A}_\Gamma$ can be described in terms of the dual cone

$$\Gamma^\circ = \{x \in \mathbb{R}^n\,;\, \langle x, \xi\rangle \ge 0 \quad \forall \xi \in \Gamma\}.$$

The dual cone of $\Gamma = \mathbb{R}_+ S$ is $\Gamma^\circ = -\mathcal{N}(\varphi_S) = \{\xi \in \mathbb{R}^n\,;\, \varphi_S(-\xi) = 0\}$. The following is a slight modification of [27], Proposition 5.6.

Proposition 3.4 *Let A be a subset of $\mathbb{R}^n$ with $0 \in S$ and Γ be a proper closed convex cone. Then*

$$\widehat{A}_\Gamma = \mathrm{ch}\, A - \Gamma^\circ.$$

Proof: Take $a \in \mathrm{ch}\, A$ and $t \in \Gamma^\circ$ and let $x = a - t$. For every $\xi \in \Gamma$ we have $\langle t, \xi\rangle \ge 0$ which implies $\langle x, \xi\rangle = \langle a, \xi\rangle - \langle t, \xi\rangle \le \varphi_A(\xi)$ and $a \in \widehat{A}_\Gamma$.

Conversely, we take $x \notin \mathrm{ch}\, A - \Gamma^\circ$ and prove that $x \notin \widehat{A}_\Gamma$. Since $\mathrm{ch}\, A - \Gamma^\circ$ is convex the Hahn-Banach theorem implies that $\{x\}$ and $\mathrm{ch}\, A - \Gamma^\circ$ can be separated by an affine hyperplane. Hence there exist $\xi \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that $\langle x, \xi\rangle > c$ and $\langle a, \xi\rangle \le c$ for every $a \in \mathrm{ch}\, A - \Gamma^\circ$. By replacing c with $\sup_{a\in \mathrm{ch}\, A - \Gamma^\circ}\langle a, \xi\rangle$ we may assume there exists $a \in \mathrm{ch}\, A$ and $t \in \Gamma^\circ$ with $\langle a - t, \xi\rangle = c$. Now we need to prove that $\xi \in \Gamma = \Gamma^{\circ\circ}$ by showing that $\langle y, \xi\rangle \ge 0$ for every $y \in \Gamma^\circ$. Since Γ° is a convex cone, we have $t + y \in \Gamma^\circ$ and $c - \langle y, \xi\rangle = \langle a - t - y, \xi\rangle \le c$.

Hence $\langle y,\xi\rangle \geq 0$. This implies that $\langle x,\xi\rangle > c \geq \varphi_S(\xi)$ and we conclude that $a \notin \widehat{A}_\Gamma$. $\square$

Several times in the next section will we use the fact that if K is a polycircle with polyradius $e^\varrho \in \mathbb{R}_+^{*n}$, then $\widehat{K}^S \cap \mathbb{C}^{*n} = \mathrm{Log}^{-1}\widehat{\{\varrho\}}_\Gamma = \mathrm{Log}^{-1}(\varrho + \mathcal{N}(\varphi_S))$, by Proposition 3.4. We can also describe $\widehat{K}^S$ for a Reinhardt K on $\mathbb{C}^n \setminus \mathbb{C}^{*n}$, though the description will not be as crisp as in Corollary 3.3. The description will be in terms of the non-zero coordinates $J \subset [n]$ of the points. For that we use notation laid out in Section 2, and let $K_J = \pi_J(K)$.

Proposition 3.5 *Let $K \subset \mathbb{C}^n$ be a compact Reinhardt set. For every ordered subset $J \subset \{1,\ldots,n\}$ and every $z \in \mathbb{C}^J$ we have $V_K^S(z) = V_{K_J}^{S_J}(\pi_J(z))$ and $\Phi_K^S(z) = \Phi_{K_J}^{S_J}(\pi_J(z))$. Furthermore, $\pi_J(\widehat{K}^S \cap \mathbb{C}^J) = \widehat{K_J}^{S_J}$.*

Proof: By rearranging the coordinates we may assume that $J = \{1,\ldots,\ell\}$. Regard a point $z \in \mathbb{C}^J$, so $z = (z',0)$ with $z' \in \mathbb{C}^\ell$. If $u' \in \mathcal{L}^{S_J}(\mathbb{C}^\ell)$ has $u'|_{K_J} \leq 0$, then $u(z',z'') = u'(z')$ defines member of $\mathcal{L}^S(\mathbb{C}^n)$ with $u|_K \leq 0$. Then $u'(z') = u(z',0) \leq V_K^S(z',0)$, implying that $V_{K_J}^{S_J}(z') \leq V_K^S(z',0)$.

For $u \in \mathcal{L}^S(\mathbb{C}^n)$ with $u|_K \leq 0$, we regard $u'(z') = u(z',0)$. Proposition 3.3 from [27] then implies that $u'(z') \leq H_S(z',0) + c_u = H_{S_J}(z') + c_u$, so $u' \in \mathcal{L}^{S_J}(\mathbb{C}^\ell)$. Since K is Reinhardt, $K_J \times \{0\}^{n-\ell} \subset \widehat{K}$, which implies that $u|_{K_J\times\{0\}^{n-\ell}} \leq 0$. Then $u'|_{K_J} \leq 0$ and $u(z',0) = u'(z') \leq V_{K_J}^{S_J}(z')$. Therefore $V_K^S(z',0) \leq V_{K_J}^{S_J}(z')$.

Repeat this argument except let $u = \log|p|^{1/m}$ with $p \in \mathcal{P}_m^S(\mathbb{C}^n)$ and let $u' = \log|p'|^{1/m}$ with $p' \in \mathcal{P}_m^{S_J}(\mathbb{C}^\ell)$ for some $m \in \mathbb{N}$. That yields $\log\Phi_K^S(z',0) = \log\Phi_{K_J}^{S_J}(z')$. The last assertion then follows from the fact that $\widehat{K}^S = \{z \in \mathbb{C}^n\,;\,\Phi_K^S(z) = 1\}$. $\square$

If $S \cap \mathbb{R}_+^{*n} = \emptyset$, then there is some largest $J \subset [n]$ of $\#J = \ell < n$ such that $S_J \cap \mathbb{R}_+^{*\ell} \neq \emptyset$. Then V_K^S is independent of its variables from $[n] \setminus J$ and can be described by $V_K^S(z) = V_{K_J}^{S_J}(\pi_J(z))$, and the right hand side can be understood by Theorem 1.3.

We can now determine the S-hull of the unit polydisc $\mathbb{T}^n$. Corollary 3.3 provides the fact $(\widehat{\mathbb{T}}^n)^S \cap \mathbb{C}^{*n} = \{z \in \mathbb{C}^{*n}\,;\,H_S(z) = 0\}$. Proposition 3.3 from [27] shows that for $a \in \mathbb{C}^J$ we have $H_S(a) = H_{S_J}(\pi_J(a))$.

By Proposition 3.5, then $(\widehat{\mathbb{T}}^n)^S = \{z \in \mathbb{C}^n\,;\,H_S(z) = 0\}$. Furthermore, this is an unbounded set if and only if S is not a neighborhood of zero. Another implication of Proposition 3.5 is that if K is Reinhardt, then $V_K^S = \log\Phi_K^S$ is true everywhere in $\mathbb{C}^n$.

4 Approximation on Reinhardt sets

If we regard the holomorphic function defined by a convergent power series of the form $f(z) = \sum_{\alpha\in\Gamma} c_\alpha z^\alpha$ where $\Gamma = \mathbb{R}_+ S$, then the partial sums are in $\mathcal{P}^S(\mathbb{C}^n)$. The partial sums provide a uniform approximation of the holomorphic function on compact subsets of its domain of convergence. Theorem 1.1 will also provide approximation on some unbounded sets, namely the S-hulls of compact sets.

Theorem 1.1 will be proven over the course of a few propositions. The step (ii)⇒(iii) was motivated in the introduction, and the steps (iii)⇒(iv) and (i)⇒(ii) are clear. Next we prove (iv)⇒(v) with the simplification that K contains a point from $\mathbb{C}^{*n}$. Then K contains some polycircle, which implies that the values of f on K determine the function completely. The condition (v) simplifies then onto the condition on the series expansion for f itself. This extra condition will be removed in Proposition 4.2.

The S-hull of a Reinhardt set is always connected and contains zero. For any $z \in \mathbb{C}^n$ with $r_j = |z_j|$, $j = 1, \ldots, n$, the polynomial hull of the (possibly lower dimensional) polycircle $\rho_r = \{w \in \mathbb{C}^n\,;\, |w_j| = r_j, j = 1, \ldots, n\}$ is the polydisc $D_r = \{w \in \mathbb{C}^n\,;\, |w_j| \le r_j, j = 1, \ldots, n\}$. Now D_r is a connected set that contains both the point z and the origin. If K is Reinhardt, then $\widehat{K}^S$ is Reinhardt as well, so for any $z \in \widehat{K}^S$ we have $\rho_r \subset \widehat{K}^S$. We then have $D_r = \widehat{\rho}_r \subset \widehat{\widehat{K}^S} = \widehat{K}^S$. Since all points in $\widehat{K}^S$ are connected to the origin, then set itself is connected.

The reason for this remark is that if X is a neighborhood of $\widehat{K}^S$, we now know that X has a component that contains the origin and all of $\widehat{K}^S$. That particular component contains a Reinhardt domain that contains the origin, and any holomorphic function on such a domain is uniquely expressed as a convergent power series.

Lemma 4.1 *Let X be a neighborhood of $\widehat{K}^S$ where K is a compact Reinhardt set with $K \cap \mathbb{C}^{*n} \neq \emptyset$. Let $f \in \mathcal{O}(X)$ have series expansion $f(z) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha$ around zero. If f is bounded on $\widehat{K}^S$, then $a_\alpha = 0$ for all $\alpha \notin \mathbb{R}_+ S$.*

Proof: The dual of $\Gamma = \mathbb{R}_+ S$ is $\Gamma^\circ = -\mathcal{N}(\varphi_S) = \{\xi \in \mathbb{R}^n\,;\, \varphi_S(-\xi) = 0\}$. Since Γ is a closed convex cone then $\Gamma^{\circ\circ} = \Gamma$. So if $\alpha \notin \mathbb{R}_+ S$, there exists a $\xi' \in \Gamma^\circ$ such that $\langle \alpha, \xi' \rangle < 0$. Then $\xi = -\xi'$ has $\varphi_S(\xi) = 0$ and $\langle \alpha, \xi \rangle > 0$.

Let $z \in K \cap \mathbb{C}^{*n}$ and $r = (|z_1|, \ldots, |z_n|)$. Let C_t denote the polycircle with center 0 and polyradius $(r_1 e^{t\xi_1}, \ldots, r_n e^{t\xi_n})$. Corollary 3.3 implies that $C_t \subset E$ for all $t \ge 0$. The component Ω' of $\cap_{\zeta\in\mathbb{T}^n} \zeta\Omega$ that contains $\widehat{K}^S$ is a Reinhardt domain that contains zero.

By [18], Theorem 2.4.5, f is expressible by a normally convergent power series $f(z) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha$ in Ω'. By the Cauchy formula for derivatives we have

$$a_\alpha = \frac{1}{(2\pi i)^n} \int_{C_t} \frac{f(\zeta)}{\zeta^\alpha} \frac{d\zeta_1 \cdots d\zeta_n}{\zeta_1 \cdots \zeta_n}.$$

For $\zeta = (r_1 e^{t\xi_1 + i\theta_1}, \ldots, r_n e^{t\xi_n + i\theta_n})$ on C_t we have $|f(\zeta)|/|\zeta^\alpha| \leq \|f\|_E \cdot r^{-\alpha} e^{-t\langle \alpha, \xi\rangle}$. The right hand side tends to 0 as $t \to +\infty$ and we conclude that $a_\alpha = 0$. $\square$

The assumption $K \cap \mathbb{C}^{*n} \neq \emptyset$ cannot be removed from the previous result. Take for example $S = \mathrm{ch}\{(0,0),(1,0),(1,1)\} \subset \mathbb{R}^2_+$ and $K \subset \{0\} \times \mathbb{C}$. Then $\widehat{K}^S = \{0\} \times \mathbb{C}$ and $z^{(1,2)}$ is bounded on $\widehat{K}^S$, even though $(1,2) \notin S$. What we can say instead is that f coincides on $\widehat{K}^S$ with a function that satisfies the conclusion of Proposition 4.1.

Proposition 4.2 *Let X be a neighborhood of $\widehat{K}^S$ where K is a compact Reinhardt set. Let $f \in \mathcal{O}(X)$ be bounded on $\widehat{K}^S$. Then there exists a $h \in \mathcal{O}(X)$ with a convergent series expansion $h(z) = \sum_{\alpha \in \mathbb{R}_+ S} a_\alpha z^\alpha$ around zero such that $f = h$ on $\widehat{K}^S$.*

Proof: In this proof we will use the index $k \in [n] = \{1, \ldots, n\}$ to denote the ordered subset $J = [n] \setminus \{k\}$ in the notation from Section 2. To be precise, denote $Z_k = \{z \in \mathbb{C}^n \,;\, z_k = 0\}$ and $X_k = \{x \in \mathbb{R}^n \,;\, x_k = 0\}$. Let $\pi_k \colon \mathbb{C}^n \to \mathbb{C}^{n-1}$ denote $\pi_k(z) = (z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n)$ and use the same notation for its restriction to $\mathbb{R}^{n-1}$. Let $K_k = K \cap Z_k$ and $S_k = \pi_k(S \cap X_k) \subset \mathbb{R}^{n-1}_+$. Additionally, we define $\mu_k \colon \mathbb{C}^{n-1} \to \mathbb{C}^n$ by $\mu_k(w) = (w_1, \ldots, w_{k-1}, 0, w_k, \ldots, w_{n-1})$. Observe that for all $k \in [n]$ we have $\pi_k \circ \mu_k = \mathrm{id}_{\mathbb{C}^{n-1}}$ and on Z_k we have $\mu_k \circ \pi_k = \mathrm{id}_{Z_k}$.

We induct over the dimension n. The base case $n = 1$ is easy: We covered the case $K \cap \mathbb{C}^* \neq \emptyset$ in Lemma 4.1 and if $K \cap \mathbb{C}^* = \emptyset$ then $K = \{0\}$. If $S = \{0\}$ then $\widehat{K}^S = \mathbb{C}$ and f is constant, and $h = f \in \mathcal{P}^S(\mathbb{C})$. If $S = [0, s]$, $s > 0$ then $\widehat{K}^S = \{0\}$. The constant function $h = f(0)$ is in $\mathcal{P}^S(\mathbb{C})$ and $f = h$ on $\widehat{K}^S$.

Assume the result is true in the dimension $n-1$. If $\widehat{K}^S$ contains a point $z \in \mathbb{C}^{*n}$ the result follows by Lemma 4.1 for the set $K \cup \{z\}$. So we may assume that $\widehat{K}^S \subset \mathbb{C}^n \setminus \mathbb{C}^{*n}$. The holomorphic function $f \circ \mu_k$ is bounded in a neighborhood of the $\mathcal{P}^{S_k}(\mathbb{C}^{n-1})$-hull of $\pi_k(K_k)$ by Proposition 3.5.

By the inductive hypothesis, there exists a holomorphic h_k with a series expansion $h_k(w) = \sum_{\beta \in S_k} a^k_\beta w^\beta$ defined in a neighborhood Ω_k of $\pi_k(K_k) \subset \mathbb{C}^{n-1}$ such that $f \circ \mu_k = h_k$ on $\pi_k(K_k)$. Then f coincides with $h_k \circ \pi_k$ on K_k.

If $z \in K_j \cap K_k = K \cap Z_j \cap Z_k$, then $f(z) = h_j(\pi_j(z)) = h_k(\pi_k(z))$ which implies that $\sum_{\alpha \in S \cap X_j \cap X_k} c^j_{\pi_j(\alpha)} z^\alpha = \sum_{\alpha \in S \cap X_j \cap X_k} c^k_{\pi_k(\alpha)} z^\alpha$. This implies that $c^k_{\pi_k(\alpha)}$ is the same number for all k such that $\alpha_k = 0$. We put $a_\alpha = c^k_{\pi_k(\alpha)}$ for some k such that $\alpha_k = 0$ and $a_\alpha = 0$ if $\alpha \in \mathbb{R}^{*n}$ and regard the series $h(z) = \sum_{\alpha \in \mathbb{R}_+ S} a_\alpha z^\alpha$. Then $h = h_k \circ \pi_k$ on K_k, therefore $f = h$ on $\widehat{K}^S$.

We also need to show that h is convergent. Since each of the series $h_k(w)$ is convergent in the neighborhood Ω_k then by [18], Theorem 2.4.2, there exists a constant $C_k > 0$ such that $|c^k_\beta w^\beta| \leq C_k$ for all $\beta \in \mathbb{N}^{n-1}$. Take $C = \max_{k \in [n]} C_k$, and observe that $|a_\alpha z^\alpha| \leq C_k$ for all $\alpha \in \mathbb{N}^n$ on the neighborhood $\bigcap_{k=1}^n \pi_k^{-1} \Omega_k$ of K. $\square$

What remains to prove in Theorem 1.1 is the implication (v)⇒(i).

Proposition 4.3 *Let X be a neighborhood $\widehat{K}^S$ for a compact Reinhardt set $K \subset \mathbb{C}^n$. Assume that $f \in \mathcal{O}(X)$ has a series expansion centered at zero of the form $f(z) = \sum_{\alpha \in \mathbb{R}_+ S} a_\alpha z^\alpha$. Then f can be approximated uniformly on $\widehat{K}^S$ by polynomials from $\mathcal{P}^S(\mathbb{C}^n)$.*

Proof: The polynomials $f_N(z) = \sum_{|\alpha| \le N} a_\alpha z^\alpha$ converge to f uniformly on K. For each N and each of the finitely many points $\alpha \in N\Sigma \cap \mathbb{R}^+ S$ there is an m such that $\alpha \in mS$. Therefore there is an $m_N \in \mathbb{N}$ such that $f_N \in \mathcal{P}^S_{m_N}(\mathbb{C}^n)$. By rearranging the indices, we have a sequence $f_m \in \mathcal{P}^S_m(\mathbb{C}^n)$ tending to f uniformly on K. So let $\varepsilon > 0$ and $m \in \mathbb{N}$ be such that $\tilde{f} = f - f_m$ has $\|\tilde{f}\|_K \le \varepsilon$.

For $z_0 \in \widehat{K}^S$ we want to show that $|\tilde{f}(z_0)| \le \varepsilon$ as that would imply that $f_m \to f$ uniformly on $\widehat{K}^S$. Recall that we may assume that $\overline{S \cap \mathbb{Q}^n} = S$.

Rearrange the coordinates if needed so

$$\begin{aligned}\{1, \dots, \ell\} &= \{j \in [n]\,;\, z_{0,j} \neq 0 \text{ and } \exists s \in S, s_j \neq 0\}\\ \{\ell+1, \dots, \ell+k\} &= \{j \in [n]\,;\, z_{0,j} \neq 0 \text{ and } \forall s \in S, s_j = 0\}\\ \{\ell+k+1, \dots, n\} &= \{j \in [n]\,;\, z_{0,j} = 0\}\end{aligned}$$

Write $z_0 = (z_0', z_0'', 0)$ with $z_0' \in \mathbb{C}^{*\ell}$, $z_0'' \in \mathbb{C}^{*k}$. Let $J = \{1, \dots, \ell\}$ and use the notation laid out in Section 2. Then $S_J \cap \mathbb{R}_+^{*\ell} \neq 0$. The function

$$u(z') = \max_{\zeta \in \mathbb{T}^\ell} \log |\tilde{f}(\zeta z', 0, 0)|$$

is plurisubharmonic on the open Reinhardt set $\Omega = \cap_{\zeta \in \mathbb{T}^\ell} \zeta \cdot \pi_J(X \cap \mathbb{C}^J) \subset \mathbb{C}^\ell$ which is a neighborhood of K_J and $u(\zeta z) = u(z)$ for all $\zeta \in \mathbb{T}^\ell$.

Then $\mathbb{R}^\ell \ni x \mapsto u(e^x)$ is a convex function of ℓ real variables. Since $u(e^x) \le \log \varepsilon$ holds for $x \in \operatorname{Log} K_J$, it is also true for $x \in \operatorname{ch}(\operatorname{Log} K_J)$. By Proposition 3.5, $z' \in \widehat{K}_J^{S_J} \subset \Omega$. By Proposition 3.3, we have

$$\widehat{K}_J^{S_J} \cap \mathbb{C}^{*\ell} = \operatorname{Log}^{-1}\big(\operatorname{ch}(\operatorname{Log} K_J) + \mathcal{N}(\varphi_{S_J})\big),$$

so $\operatorname{Log} z_0 = \varrho + \xi$ for some $\varrho \in \operatorname{ch}(\operatorname{Log} K_J)$ and $\xi \in \mathcal{N}(\varphi_S)$. Furthermore, $e^{\varrho + t\xi} \in \widehat{K}_J^{S_J}$ and therefore $(e^{\varrho + t\xi}, 0) \in \widehat{K}^S$ for all $t \ge 0$.

The function $v\colon \mathbb{R}_+ \to \mathbb{R}$, $v(t) = u(e^{\varrho + t\xi})$ is convex and $v(0) \le \log \varepsilon$. Since and both f and f_m are bounded on $\widehat{K}^S$, v is bounded above on $\mathbb{R}_+$. Bounded convex functions on $\mathbb{R}_+$ are decreasing, so we can conclude that $v \le \log \varepsilon$. In particular $v(1) \le \log \varepsilon$ so $|\tilde{f}(z_0', 0, 0)| \le \varepsilon$. Since $\tilde{f}$ is independent of the $(\ell+1), \dots, (\ell+k)$-th variables, we can conclude that $|\tilde{f}(z_0)| \le \varepsilon$. □

If f is expressed by a power series of the form

$$f(z) = \sum_{\alpha \in \mathbb{R}_+ S} a_\alpha z^\alpha \tag{4.1}$$

with uniform convergence on some compact set K, then the domain of convergence of (4.1) is some Reinhardt domain that contains K. The domains of convergence of power series are well characterized in Section 2.4 of Hörmander's book [18]. His Theorem 2.4.3 asserts that the domain of convergence Ω of any power series is such that its image in logarithmic coordinates $\operatorname{Log}\Omega$ is an open convex set and that for every $\varrho \in \operatorname{Log}\Omega$, then $\varrho - \mathbb{R}^n_+ \subset \operatorname{Log}\Omega$. By Proposition 3.4, this can be phrased so that the closure of $\operatorname{Log}\Omega$ is convex with respect to the cone $\mathbb{R}^n_+ = \mathbb{R}_+\Sigma$. With the added information that the series in question is of the form (4.1), we can show that the closure of its domain of convergence in logarithmic coordinates is convex with respect to the cone $\mathbb{R}_+S$.

Proposition 4.4 *Let Ω be a Reinhardt domain containing the origin and let $f \in \mathcal{O}(\Omega)$ have series expansion* (4.1)*. Denote $D = \operatorname{Log}(\Omega \cap \mathbb{C}^{*n})$ and let $\Gamma = \mathbb{R}_+S$. Let $\widetilde{\Omega}$ be the interior of $\operatorname{Log}^{-1}(\widehat{D}_\Gamma)$. Then the convergence of* (4.1) *is normal in $\widetilde{\Omega}$ and f extends to a holomorphic function on $\widetilde{\Omega}$.*

Proof: By [18], Theorem 2.4.2, the series (4.1) is normally convergent in the interior of the set B of all $z \in \mathbb{C}^n$ such that $|c_\alpha z^\alpha| \leq C$ for all $\alpha \in \mathbb{N}^n$. By [18], Theorem 2.4.6, $\operatorname{Log}(B \cap \mathbb{C}^{*n})$ contains $\operatorname{ch} D$. If $z \in \operatorname{Log}^{-1}(\widehat{D}_\Gamma)$, then $\operatorname{Log} z = b + \xi$ where $b \in \operatorname{ch} D$ and $\varphi_S(\xi) = 0$ by Proposition 3.4. For any $\alpha \in \mathbb{R}_+S \cap \mathbb{N}^n$ we then have $|c_\alpha z^\alpha| = |c_\alpha| e^{\langle \alpha, b+\xi\rangle} \leq |c_\alpha| e^{\langle \alpha, b\rangle} \leq C$. This shows that $z \in B$ and that the open set $\widetilde{\Omega}$ lies in the interior of B so (4.1) is normally convergent on $\widetilde{\Omega}$. □

We saw in (3.2) that the S-hull of compact set K restricted to $\mathbb{C}^{*n}$ is contained in $\operatorname{Log}^{-1}\widehat{A}_\Gamma$ where $A = \operatorname{Log} K$. Whenever Ω is a Reinhardt domain, then $\widetilde{\Omega}$ will contain the S-hull of every compact subset of Ω.

Corollary 4.5 *Let Ω be a Reinhardt domain containing the origin and let K be a compact subset of Ω. If $f \in \mathcal{O}(\Omega)$ has a series expansion* (4.1)*, then f extends as a holomorphic function in a neighborhood of $\widehat{K}^S$.*

Furthermore if X is any neighborhood of a compact Reinhardt set K, then the connected component of $\Omega = \cap_{\zeta \in \mathbb{T}^n} \zeta X$ that contains K is a Reinhardt domain. Note that Proposition 4.4 implies that if f is only assumed to be defined on any connected Reinhardt domain containing K and the origin, then f can be extended to a neighborhood of $\widehat{K}^S$. By Proposition 4.3, this extension can be approximated uniformly by $\mathcal{P}^S(\mathbb{C}^n)$ polynomials on $\widehat{K}^S$.

5 Proof of Theorem 1.2

The main tool in the proof of Theorem 1.2 is Hörmander's L^2-methods, so for the reader's convenience we present his Theorem 4.2.6 from [19].

Theorem 5.1 (Hörmander) *Let X be a pseudoconvex domain in $\mathbb{C}^n$ and $\varphi \in \mathcal{PSH}(X)$. Let $\varphi_a(z) = \varphi(z) + a \log(1 + |z|^2)$ for some $a > 0$. For every*

$f \in L^2_{(0,1)}(X, \varphi_{a-2})$ satisfying $\bar\partial f = 0$ there exists a solution $u \in L^2(X, \varphi_a)$ of $\bar\partial u = f$ satisfying the estimate

$$\begin{aligned} \|u\|^2_{\varphi_a} &= \int_X |u|^2 (1+|z|^2)^{-a} e^{-\varphi}\, d\lambda \\ &\le \frac{1}{a} \int_X |f|^2 (1+|z|^2)^{-a+2} e^{-\varphi}\, d\lambda = \frac{1}{a} \|f\|^2_{\varphi_{a-2}}. \end{aligned} \tag{5.1}$$

If $f_j \in \mathcal{C}^\infty(X)$ for $j = 1, \dots, n$, then $u \in \mathcal{C}^\infty(X)$.

Our task is to find an appropriate weight φ in order to be assured that the corresponding L^2-estimate implies that a holomorphic function is from $\mathcal{P}^S(\mathbb{C}^n)$.

The prototype for such a result is Theorem 3.6 from [27], which states that an entire function p on $\mathbb{C}^n$ is a member of $\mathcal{P}^S_m(\mathbb{C}^n)$, $m \in \mathbb{N}$ if and only if for some constants $C > 0$ and a smaller than the distance between mS and $\mathbb{N}^n \setminus mS$ in the L^1-norm, we have

$$|p(z)| \le C(1+|z|)^a e^{mH_S(z)}, \qquad z \in \mathbb{C}^n. \tag{5.2}$$

Therefore we need a result where a finite L^2-estimate can imply (5.2). Our trick is that the weight should accommodate the real Jacobian of the change of coordinates into logarithmic coordinates.

Proposition 5.2 *Let $u \in \mathcal{C}^1(\mathbb{C}^n)$ be such that $\overline\partial u$ has compact support. Denote $\nu(z) = \langle \mathbf{1}, \operatorname{Log} z\rangle$. If*

$$\int_{\mathbb{C}^n} |u(z)|^2 (1+|z|^2)^{-a} e^{-2mH_S(z) - 2\nu(z)} d\lambda(z) < +\infty, \tag{5.3}$$

then there exists a constant $C > 0$ such that

$$|u(z)| \le C(1+|z|)^a e^{mH_S(z)}, \qquad z \in \mathbb{C}^n. \tag{5.4}$$

Proof: The Jacobi determinant of $\zeta \mapsto e^\zeta = (e^{\zeta_1}, \dots, e^{\zeta_n})$ viewed as a mapping $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is equal to $|e^{\zeta_1} \cdots e^{\zeta_n}|^2 = e^{2\langle \mathbf{1}, \xi\rangle}$. We define $v\colon \mathbb{C}^n \to \mathbb{C}$ by $v(\zeta) = u(e^\zeta)$, write $\zeta = \xi + i\eta$, for $\xi, \eta \in \mathbb{R}^n$. Let

$$A = \{\zeta = \xi + i\eta \in \mathbb{C}^n\,;\, \xi \in \mathbb{R}^n \text{ and } \eta_j \in [-\pi, \pi],\, j = 1, \dots, n\}$$

By (5.3) we get the estimate

$$\begin{aligned} &\int_A |v(\zeta)|^2 (1+|e^\zeta|^2)^{-a} e^{-2m\varphi_S(\xi)} d\lambda(\zeta) \\ &\qquad = \int_{\mathbb{C}^n} |u(z)|^2 (1+|z|^2)^{-a} e^{-2mH_S(z) - 2\nu(z)} d\lambda(z) < +\infty. \end{aligned}$$

Since $\bar\partial u$ has compact support it follows that there exists a constant $C_2 > 0$ such that

$$|\bar\partial v(\zeta)| \le C_2 \le C_2 e^{m\varphi_S(\xi)}, \qquad \zeta = \xi + i\eta \in \mathbb{C}^n. \tag{5.5}$$

By [27], Lemma 5.3, there exists a constant $C_3 > 0$ such that for every $\zeta \in \mathbb{C}^n$, written as $\zeta = \xi + i\eta$ for $\xi, \eta \in \mathbb{R}^n$, we have

$$|v(\zeta)| \le C_3\big(1 + |e^\zeta|\big)^a \sup_{w\in\mathbb{B}} e^{m\varphi_S(\xi+\operatorname{Re} w)} \le C\big(1 + |e^\zeta|\big)^a e^{m\varphi_S(\xi)},$$

where $C = C_3 \sup_{w\in\mathbb{B}} e^{m\varphi_S(\operatorname{Re} w)}$ and $\mathbb{B}$ is the open euclidean unit ball in $\mathbb{C}^n$. We change the coordinates back to $z = e^\zeta$, use the fact that $\operatorname{Log} z = \xi$ and conclude (5.4). □

We now see that an entire function with a finite estimate of the form (5.1) with a weight φ that grows like $2mH_S + 2\nu + a\log(1+|z|^2)$ is a member of $\mathcal{P}^S(\mathbb{C}^n)$, if a is smaller than the distance d_m between mS and $\mathbb{N}^n \setminus mS$ in the L^1-norm. It suffices then also if a is smaller than the euclidean distance $\operatorname{dist}(mS, \mathbb{N}^n \setminus mS)$. One difficulty is that d_m is dependent on m so it is not clear that one can choose a number a that is smaller than this distance for all m large enough.

The subset $S_m = \operatorname{ch}(S \cap (1/m)\mathbb{N}^n)$ of S is such that the polynomial space $\mathcal{P}^{S_m}_m(\mathbb{C}^n)$ contains the same polynomials as $\mathcal{P}^S_m(\mathbb{C}^n)$. The set mS_m has a larger distance than mS to the next lattice point. Also mS_m is an *integral polytope*, meaning that its vertices are lattice points. We can estimate the euclidean distance from a convex integral polytope to the next lattice point using methods from linear algebra.

Lemma 5.3 *Let P be a convex integral polytope with nonempty interior contained in some box $[0, M]^n$ where $M > 0$. Then*

$$\operatorname{dist}(P, \mathbb{Z}^n \setminus P) \ge 1/(\sqrt{n}(n-1)!M^{n-1}).$$

Proof: Let $P = \operatorname{ch}\{a_1, \dots, a_N\}$ with vertices $a_j \in \mathbb{N}^n$ for $j = 1, \dots, N$. The boundary of P lies in the union of its boundary hyperplanes, each passing through some $(n+1)$-tuple of the vertices of P in a general position. Since P has nonempty interior such an n-tuple will exist.

Let $a_{\ell_1}, \dots, a_{\ell_n}$ be any such tuple and A be the hyperplane through those points. Let $v_j = a_{\ell_j} - a_{\ell_1}$ for $j = 2, \dots, n$. A normal $\eta = (\eta_1, \dots, \eta_n)$ to A can be found with the coordinates $\eta_j = \det(e_j, v_2, \dots, v_n)$, where e_j is the j-th unit vector.

For each vector v_j, $j = 2, \dots, n$, each coordinate $v_{j,k}$, $k \in [n]$ is an integer with $|v_{j,k}| \le M$. That implies that η_j is an integer with $|\eta_j| \le (n-1)!M^{n-1}$. Hence $|\eta| \le \sqrt{n}(n-1)!M^{n-1}$.

The distance from $x \in \mathbb{Z}^n \setminus P$ to A is $|\langle \eta, x\rangle - \langle a_1, x\rangle|/|\eta|$. If $x \notin A$ then this distance is at least $1/|\eta|$. If $x \in A$ then the point of P closest to x is on some lower dimensional face of P. That face lies on the intersection of A with some other boundary hyperplane A' of P with $x \notin A'$. The distance of x from P is therefore greater or equal to its distance to the boundary hyperplanes of P that x does not lie on, which is at least $1/(\sqrt{n}(n-1)!M^{n-1})$. □

This lemma implies that the distance of mS_m to the next lattice point grows slowly enough for our purposes.

Corollary 5.4 *Let $S_m = \mathrm{ch}(S \cap (1/m)\mathbb{Z}^n)$. Then*

$$\mathrm{dist}(mS_m, \mathbb{Z}^n \setminus (mS_m))^{1/m} \to 1 \quad \text{as } m \to \infty.$$

Proof: Clearly $\mathrm{dist}(mS_m, \mathbb{Z}^n \setminus (mS_m)) \leq 1$ for all $m \in \mathbb{N}$. Each mS_m is an integral polytope contained in $[0, m\varphi_S(\mathbf{1})]^n$, so by Lemma 5.3, we have

$$\mathrm{dist}(mS_m, \mathbb{Z}^n \setminus (mS_m))^{1/m} \geq 1/(\sqrt{n}(n-1)!m\varphi_S(\mathbf{1}))^{1/m} \to 1.$$

□

A good candidate as a weight function φ in (5.1) is some plurisubharmonic function with the same growth as $2mH_S + 2\nu$, in light of Proposition 5.2, which is preferably large outside the set K. The function V_K^S is therefore an ideal candidate as a weight.

We will need to establish some lower bound on V_K^S outside K, in other words that we should be able to escape one of its sublevel sets. We therefore show that a neighborhood of $\widehat{K}^S$ of V_K^S contains some sublevel set of V_K^S. Furthermore, if $V_K^{S*}|_K = 0$, then the set X_R produced in the following lemma is a neighborhood of K.

Lemma 5.5 *If Ω is a neighborhood of a S-convex compact set $K \subset \mathbb{C}^n$, then there is an $R > 1$ such that $X_R = \{z \in \mathbb{C}^n\,;\, V_K^{S*} < \log R\}$ is a relatively compact subset of Ω.*

Proof: We may assume that Ω is bounded. Let $B \subset \mathbb{C}^n$ be any compact set that contains Ω. For every $z \in B \setminus \Omega$ there exists a $p_z \in \mathcal{P}^S_{m_z}(\mathbb{C}^n)$ for some $m_z \in \mathbb{N}$ with $\|p_z\|_K = 1$ and with $|p_z(z)| > 1$. Let r_z be any number with $1 < r_z < |p_z(z)|^{1/m_z}$.

Since $B \setminus \Omega$ is compact there exist finitely many $z_1, \ldots, z_N$ such that $p_j = p_{z_j}$, $m_j = m_{z_j}$ and $r_j = r_{z_j}$ satisfy

$$B \setminus \Omega \subset \{w \in \mathbb{C}^n\,;\, |p_j(w)|^{1/m_j} > r_j,\ j = 1, \ldots, N\}.$$

Let $v = \max_{j=1,\ldots,N} \log |p_j|^{1/m_j}$ and $R = \min\{r_1, \ldots, r_N\}$. Then $v|_{B\setminus\Omega} > R$ and since $B \setminus \Omega$ is compact and v is continuous it takes some lowest value $\min_{B\setminus\Omega} v > R$. Let $\min_{B\setminus\Omega} v > R' > R$. The function

$$u(z) = \begin{cases} \max\{\log R', v(z)\}, & z \in \mathbb{C}^n \setminus B \\ v(z), & z \in B. \end{cases}$$

is plurisubharmonic by the gluing theorem. Since $u \in \mathcal{L}^S(\mathbb{C}^n)$ and $u|_K \leq 0$ we can conclude that $X_R \subset \{z \in \mathbb{C}^n\,;\, u(z) \leq \log R\}$, which is a closed bounded subset of Ω.

□

Lemma 5.6 *If $V_K^{S*}|_K = 0$, then $V_K^{S_m} \nearrow V_K^S$ as $m \to \infty$ uniformly on compact subsets of $\mathbb{C}^{*n}$, where $S_m = \mathrm{ch}(S \cap (1/m)\mathbb{N}^n)$.*

Proof: By [26], Theorem 1.1, $V_K^S = \log \Phi_K^S$ on $\mathbb{C}^{*n}$ and these function are continuous on $\mathbb{C}^{*n}$ by [27], Proposition 5.4. By [27], Proposition 2.2, $\Phi_{K,m}^S \nearrow \Phi_K^S$ as $m \to \infty$ uniformly on compact subsets of $\mathbb{C}^{*n}$. Now $\mathcal{P}_m^S(\mathbb{C}^n) = \mathcal{P}_m^{S_m}(\mathbb{C}^n)$, which implies that $\Phi_{K,m}^S = \Phi_{K,m}^{S_m}$. Then

$$\Phi_{K,m}^S = \Phi_{K,m}^{S_m} \leq \Phi_K^{S_m} \leq \Phi_K^S$$

which implies that $\Phi_K^{S_m} \nearrow \Phi_K^S$ as $m \to \infty$ uniformly on compact subsets of $\mathbb{C}^{*n}$. Since $\log\colon [1, +\infty) \to \mathbb{R}_+$ is Lipschitz, we have a uniform limit $\log \Phi_K^{S_m} \nearrow \log \Phi_K^S = V_K^S$. Finally, since $\log \Phi_{K,m}^{S_m} \leq V_K^{S_m} \leq V_K^S$, we have a uniform limit $V_K^{S_m} \nearrow V_K^S$ on compact subsets of $\mathbb{C}^{*n}$ as $m \to \infty$. □

We are now ready to start the proof of our main result.

Proof of Theorem 1.2: Let Ω be the domain of f. By Lemma 5.5 there is an $R > 1$ such that the open set $X_R = \{z \in \mathbb{C}^n\,;\, V_K^{S*} < \log R\}$ is a relatively compact subset of $\Omega \cap \mathbb{C}^{*n}$. Let $1 < r < R$ and $0 < \gamma < 1$ be such that $\gamma^2 > 1/r$. Now $V_K^{S_m} \nearrow V_K^S$ uniformly on the bounded set X_R as $m \to \infty$ by Lemma 5.6, so for some $m_0 \in \mathbb{N}$ we have $V_K^{S_m} > V_K^S - \log(1/\gamma)$ on X_R for all $m \geq m_0$.

By Corollary 5.4, there is an $m_1 \geq m_0$ such that $\operatorname{dist}(mS_m, \mathbb{N}^n \setminus mS_m)^{1/m} > 1/2$ for all $m \geq m_1$. By [27], Proposition 5.4 (ii), V_K^S is continuous on X_R, hence X_r is relatively compact in X_R. We may therefore take $\chi \in \mathcal{C}_0^\infty(X_R)$ with $0 \leq \chi \leq 1$, and $\chi = 1$ in X_r. Denote $V_m = V_K^{S_m *}$ and $\nu(z) = \langle \mathbf{1}, \operatorname{Log} z\rangle$. Define for every $z \in \mathbb{C}^n$

$$\begin{aligned}\psi_m(z) &= 2mV_m(z) + 2\nu(z) + 2^{-m}\log(1 + |z|^2),\\ \eta_m(z) &= \psi_m(z) - 2\log(1 + |z|^2).\end{aligned}$$

Since f is holomorphic in a neighborhood of $\operatorname{supp}\chi$, we have $\|f\bar\partial\chi\|_{\eta_m} < +\infty$ and $\bar\partial(f\bar\partial\chi) = 0$.

By Theorem 5.1, there exists a solution $u_m \in \mathcal{C}^\infty(\mathbb{C}^n)$ of $\bar\partial u_m = f\bar\partial\chi$ satisfying

$$\|u_m\|_{\psi_m}^2 = \int_{\mathbb{C}^n} |u_m|^2 (1 + |z|^2)^{-1/2^m} e^{-2mV_m(z) - 2\nu(z)}\, d\lambda \leq 2\|f\bar\partial\chi\|_{\eta_m}^2. \quad (5.6)$$

Now $V_m \leq V_K^{S*} \leq H_S + c_V$ for some constant $c_V \in \mathbb{R}$ by [27], Proposition 4.5. Therefore

$$\int_{\mathbb{C}^n} |u(z)|^2 (1 + |z|^2)^{-1/2^m} e^{-2mH_S(z) - 2\nu(z)} d\lambda(z) \leq e^{2mc_V}\|u\|_{\psi_m}^2 < +\infty. \quad (5.7)$$

We let $p_m = f\chi - u_m \in \mathcal{O}(\mathbb{C}^n)$. By Corollary 5.2, $|p_m(z)| \leq C(1+|z|)^{1/2^m} e^{mH_S(z)}$ for all $z \in \mathbb{C}^n$ for some constant $C > 0$. Theorem 3.6 from [27] then implies that $p_m \in \mathcal{P}_m^S(\mathbb{C}^n)$ for all $m \geq m_1$. Our goal is to show that $\|u_m\|_K = \|f - p_m\|_K$ can be made arbitrarily small by taking $m \geq m_1$ large enough.

Denote the mean of $v \in L^1_{\mathrm{loc}}(\mathbb{C}^n)$ over a ball $B(z,\delta)$ with center z and radius $\delta > 0$ by

$$\mathcal{M}_r v(z) = \frac{1}{\Omega_{2n} r^{2n}} \int_{B(z,r)} v \, d\lambda$$

where Ω_{2n} is the volume of the unit ball in $\mathbb{R}^{2n}$.

Since $u_m \in \mathcal{O}(\mathbb{C}^n \setminus \operatorname{supp} \bar\partial\chi)$, the mean value theorem implies that for every $z \in \mathbb{C}^n$ we have $u_m(z) = \mathcal{M}_\delta u_m(z)$ and consequently, by the Cauchy-Schwarz inequality

$$\begin{aligned}
|u_m(z)| &\le \Omega_{2n}^{-1}\delta^{-2n} \int_{B(z,\delta)} |u_m| \, d\lambda \\
&= \Omega_{2n}^{-1}\delta^{-2n} \int_{B(z,\delta)} |u_m| e^{-\psi_m/2} \cdot e^{\psi_m/2} \, d\lambda \\
&\le \Omega_{2n}^{-1}\delta^{-2n} \|u_m\|_{\psi_m} \Big(\int_{B(z,\delta)} e^{\psi_m} d\lambda \Big)^{1/2} \\
&\le \Omega_{2n}^{-1/2}\delta^{-n}\sqrt{2} \|f\bar\partial\chi\|_{\eta_m} \Big(\mathcal{M}_\delta\big(e^{\psi_m}\big)(z) \Big)^{1/2}. \qquad (5.8)
\end{aligned}$$

Let $0 < \varepsilon < \log(\gamma r)$ and denote $Y = \operatorname{supp} \bar\partial\chi$. Let $0 < \delta < \operatorname{dist}(K, Y)$ be such that $V_K^{S*}(z+w) \le V_K^{S*}(z) + \varepsilon = \varepsilon$ for all $z \in K$ and all $w \in B(0,\delta)$. Furthermore

$$V_m(\zeta) \le V_K^{S*}(\zeta) \le \varepsilon, \qquad z \in K,\, \zeta \in B(z,\delta).$$

Since the set $\overline{X}_r \subset \mathbb{C}^{*n}$ is compact, we have $B = \max_{z\in\overline{X}_r} |z| < +\infty$ and $b = \min_{z\in\overline{X}_r} \nu(z) > -\infty$. We have

$$\begin{aligned}
\mathcal{M}_\delta\big(e^{\psi_m}\big)(z) &= \frac{1}{\Omega_{2n}\delta^{2n}} \int_{B(z,\delta)} (1+|\zeta|^2)^{1/2^m} e^{2\nu(z)} e^{2mV_m(\zeta)} \, d\lambda(\zeta) \\
&\le (1+B^2)^{1/2^m} e^{2n\log B} e^{2m\varepsilon}. \qquad (5.9)
\end{aligned}$$

Since $V_m \ge \log(\gamma r)$ holds on $\mathbb{C}^n \setminus X_r$ for all $m \ge m_1$, we observe next that

$$\begin{aligned}
\|f\bar\partial\chi\|_{\eta_m} &= \Big(\int_Y |f\bar\partial\chi|^2 (1+|\zeta|^2)^{2-1/2^m} e^{-2mV_m(z)-2\nu(z)} \, d\lambda \Big)^{1/2} \\
&\le \Big(\int_Y |\bar\partial\chi|^2 \, d\lambda \Big)^{1/2} \cdot (1+B^2)^{2-1/2^m} e^{-2b} \cdot \frac{\|f\|_{X_r}}{(\gamma r)^m}. \qquad (5.10)
\end{aligned}$$

Denoting $c_\chi = \int_Y |\bar\partial\chi|^2 \, d\lambda < +\infty$, we see that (5.8), (5.9) and (5.10) together imply that

$$|u_m(z)| \le \Omega_{2n}^{-1/2}\delta^{-n}\sqrt{2} c_\chi (1+B^2)^2 e^{n\log B - 2b} \|f\|_{X_r} \left(\frac{e^\varepsilon}{\gamma r}\right)^m.$$

The constants δ, B, b and c_χ were chosen independent of m, and $|e^\varepsilon/\gamma r| < 1$. Therefore $\|u_m\|_K$ can be made arbitrarily small by choosing m large enough. We conclude that p_m tend to f uniformly on K as m tends to infinity. $\square$

6 Further discussions on Theorem 1.2

Here we present a condition on the set S where Theorem 1.2 holds without the assumption that K is a subset of $\mathbb{C}^{*n}$. In this article we are not concerned with the precise degree of our approximating polynomials in the ring $\mathcal{P}^S(\mathbb{C}^n)$, so it suffices to consider some larger set $\widehat{S}$ that scales into the same cone, $\mathbb{R}_+\widehat{S} = \mathbb{R}_+S$. For some cases of S, the following procedure will produce a hull of S with this property. Let

$$J = \{j \in [n]\,;\, e_j \in \mathbb{R}_+S\}, \tag{6.1}$$

$$\Xi = \{\xi \in \mathbb{R}^n\,;\, \xi_j \geq 0 \text{ for all } j \notin J\}. \tag{6.2}$$

The dual cone of Ξ is

$$\Xi^\circ = \{\xi \in \mathbb{R}^n\,;\, \xi_j \geq 0 \text{ if } j \notin J \text{ and } \xi_j = 0 \text{ if } j \in J\}$$

so Proposition 3.4 implies that

$$\widehat{S}_\Xi \cap \mathbb{R}^n_+ = \bigcup_{s\in S}\{x \in \mathbb{R}^n\,;\, 0 \leq x_j \leq s_j \text{ if } j \notin J \text{ and } x_j = s_j \text{ if } j \in J\}.$$

For some cases of S, the set $\widehat{S} = \widehat{S}_\Xi \cap \mathbb{R}^n_+$ has $\mathbb{R}_+\widehat{S} = \mathbb{R}_+S$. This is true for example in the case $n = 2$, when $J = \{1\}$ or $J = \{2\}$. In the cases when $\mathbb{R}_+\widehat{S} = \mathbb{R}_+S$ we are able to prove a version of Theorem 1.2, without needing to assume that $K \subset \mathbb{C}^{*n}$. First we have the following modification of [27], Theorem 3.6.

Proposition 6.1 *Let d_m denote the distance between $m\widehat{S}$ and $\mathbb{N}^n \setminus m\widehat{S}$ in the L^1-norm. Then $p \in \mathcal{O}(\mathbb{C}^n)$ belongs to $\mathcal{P}^{\widehat{S}}_m(\mathbb{C}^n)$ if and only if there exists an $a \in [0, d_m[$ and a constant $C > 0$ such that*

$$|p(z)| \leq C(1 + |z|)^a e^{mH_S(z)}, \qquad z \in \mathrm{Log}^{-1}\Xi. \tag{6.3}$$

Proof: If $p \in \mathcal{P}^{\widehat{S}}_m(\mathbb{C}^n)$ then [27], Theorem 3.6 provides the existence of $a \in [0, d_m[$ and $C > 0$ such that $|p(z)| \leq C(1 + |z|)^a e^{mH_{\widehat{S}}(z)}$ for all $z \in \mathbb{C}^n$. Since $\varphi_S(\xi) = \varphi_{\widehat{S}}(\xi)$ for all $\xi \in \Xi$ the result follows.

Assume the latter. Let $p(z) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha z^\alpha$ be the power series expansion of p at 0. We need to show that $a_\alpha = 0$ for all $\alpha \in \mathbb{N}^n \setminus m\widehat{S}$. Since $a < d_m$ we have $(m\widehat{S} + a\Sigma) \cap \mathbb{N}^n = (m\widehat{S}) \cap \mathbb{N}^n$, so $\alpha \in \mathbb{N}^n \setminus (m\widehat{S} + a\Sigma)$. Note that $m\widehat{S} + a\Sigma$ is the restriction of the Ξ-hull of $mS + a\Sigma$ to $\mathbb{R}^n_+$. Therefore we may find a $\xi \in \Xi$ such that $\langle\alpha, \xi\rangle > \varphi_{mS+a\Sigma}(\xi)$. We let C_t denote the polycircle with center 0 and polyradius $(e^{t\xi_1}, \ldots, e^{t\xi_n})$.

Observe that $\log(1 + |z|) \leq \log 2 + \log^+|z| \leq \log(2\sqrt{n}) + \log^+\|z\|_\infty$, $z \in \mathbb{C}^n$. For every $\zeta \in C_t$ we therefore have

$$|p(\zeta)| \leq 2C\sqrt{n}\exp(m\varphi_S(\xi) + a\|\xi^+\|_\infty) = 2C\sqrt{n}\exp(\varphi_{mS+a\Sigma}(\xi))$$

where ξ^+ is defined by $\xi_j^+ = \max\{0, \xi_j\}$.

By the Cauchy formula for derivatives we have

$$a_\alpha = \frac{1}{(2\pi i)^n} \int_{C_t} \frac{p(\zeta)}{\zeta^\alpha} \frac{d\zeta_1 \cdots d\zeta_n}{\zeta_1 \cdots \zeta_n}.$$

For $\zeta \in C_t$ we have

$$|p(\zeta)|/|\zeta^\alpha| \leq 2C\sqrt{n} \exp(t(\varphi_{mS+a\Sigma}(\xi) - \langle \alpha, \xi \rangle)).$$

The right hand side tends to 0 as $t \to +\infty$ and we conclude that $a_\alpha = 0$. $\square$

Any polynomial $p \in \mathcal{P}^S(\mathbb{C}^n)$ is of the form $p(z) = c + \sum_{j \in J} z_j p_j(z)$ where c is a constant and p_j are polynomials for $j \in J$. Then $p = c$ on the set $Z_J = \mathbb{C}^{[n]\setminus J} = \{z \in \mathbb{C}^n\,;\, z_j = 0, j \in J\}$. Now Z_J is unbounded unless $J = [n]$, that is if S is a neighborhood of zero. If the set K therefore contains a point from Z_J, then $Z_J \subset \widehat{K}^S$. If $\widehat{K}^S$ is compact and S is not a neighborhood of zero, it therefore cannot intersect the set Z_J. The function

$$\nu(z) = \max_{j \in J} \log |z_j| + \sum_{j \notin J} \max\{-1, \log |z_j|\} \tag{6.4}$$

is therefore locally bounded on a compact S-convex set.

Now we present a variant of [26], Lemma 5.3, on how to maneuver a bounded L^2-estimate into a bounded uniform estimate. It can be proven the same way, except for z restricted to $\mathrm{Log}^{-1}\Xi$. Here we let $\mathbb{B}$ denote the open euclidean unit ball both in $\mathbb{C}^n$ and $\mathbb{R}^n$ depending on the context.

Lemma 6.2 *Let $v \in \mathcal{C}^1(\mathbb{C}^n)$ and assume that there exist positive constants a, C_1, C_2 and a measurable function ψ such that for every $z = \xi + i\eta$, $\xi \in \Xi$ we have*

$$\int_{B(z,1)} |v(w)|^2 (1 + |e^w|^2)^{-a} e^{-2\psi(w)} d\lambda(w) < C_2.$$

and for every $z = \xi + i\eta$ with $\xi \in \Xi + \mathbb{B}$ we have

$$|\overline{\partial} v(z)| \leq C_1 e^{\psi(z)}.$$

Then there exists a positive constant C_3 such that

$$|v(z)| \leq C_3 (1 + |e^z|)^a \sup_{w \in \mathbb{B}} e^{\psi(z+w)}, \qquad z = \xi + i\eta,\ \xi \in \Xi. \tag{6.5}$$

This has the following corollary:

Corollary 6.3 *Let $u \in \mathcal{C}^1(\mathbb{C}^n)$ be such that $\overline{\partial} u$ has compact support. Let J be as defined in* (6.1)*. Denote $W = \{z \in \mathbb{C}^n\,;\, |z_j| \geq e^{-1} \text{ if } j \notin J\}$. If*

$$\int_W |u(z)|^2 (1 + |z|^2)^{-a} e^{-2mH_S(z) - 2\nu(z)} d\lambda(z) < +\infty, \tag{6.6}$$

then there exists a constant $C > 0$ such that

$$|u(z)| \leq C(1 + |z|)^a e^{mH_S(z)}, \qquad z \in \mathrm{Log}^{-1}\Xi. \tag{6.7}$$

Proof: This proof is similar to that of Corollary 5.2, except we let

$$A = \{\zeta = \xi + i\eta \in \mathbb{C}^n\,;\, \xi \in \Xi + \mathbb{B} \text{ and } \eta_j \in [\eta_{0,j} - \pi, \eta_{0,j} + \pi],\, j = 1, \dots, n\}$$

For every $z \in W$ we have $\nu(z) = \max_{j\in J} \log|z_j| + \sum_{j\notin J} \log|z_j| \le \langle \mathbf{1}, \operatorname{Log} z\rangle$. By (6.6) we get the estimate

$$\begin{aligned}\int_A |v(\zeta)|^2 (1+|e^\zeta|^2)^{-a} e^{-2m\varphi_S(\xi)} d\lambda(\zeta) \\ &= \int_W |u(z)|^2 (1+|z|^2)^{-a} e^{-2mH_S(z) - 2\langle \mathbf{1}, \operatorname{Log} z\rangle} d\lambda(z) \\ &= \int_W |u(z)|^2 (1+|z|^2)^{-a} e^{-2mH_S(z) - 2\nu(z)} d\lambda(z) < +\infty.\end{aligned}$$

Since $\bar\partial u$ has compact support there exists a constant $C_2 > 0$ such that

$$|\bar\partial v(\zeta)| \le C_2 \le C_2 e^{m\varphi_S(\xi)}, \qquad \zeta = \xi + i\eta \in \mathbb{C}^n. \tag{6.8}$$

By Lemma 6.2 there exists a constant $C_3 > 0$ such that for every $\zeta = \xi + i\eta$ with $\xi \in \Xi$ we have.

$$|v(\zeta)| \le C_3 \big(1 + |e^\zeta|\big)^a \sup_{w\in\mathbb{B}} e^{m\varphi_S(\xi + \operatorname{Re} w)} \le C\big(1 + |e^\zeta|\big)^a e^{m\varphi_S(\xi)},$$

where $C = C_3 \sup_{w\in\mathbb{B}} e^{m\varphi_S(\operatorname{Re} w)}$. We change the coordinates back to $z = e^\zeta$, use the fact that $\operatorname{Log} z = \xi$ and conclude (5.4). □

Using ν defined in (6.4) in the proof of Theorem 1.2 instead of $\nu(z) = \langle \mathbf{1}, \operatorname{Log} z\rangle$, and taking X_R as a bounded subset of $\Omega \setminus Z_J$, yields the following variant of Theorem 1.2.

Theorem 6.4 *Assume that $\widehat{S} = \widehat{S}_\Xi \cap \mathbb{R}^n_+$ satisfies $\mathbb{R}_+\widehat{S} = \mathbb{R}_+ S$, where the cone Ξ is defined in* (6.2)*. Let $K \subset \mathbb{C}^n$ be compact, S-convex and such that $V_K^{\widehat{S}*}|_K = 0$. If f is holomorphic in a neighborhood of K, then f can be uniformly approximated on K by polynomials from $\mathcal{P}^S(\mathbb{C}^n)$.*

We conclude by verifying that this is a generalization of the Runge-Oka-Weil Theorem. The standard unit simplex Σ has $J = [n]$, $\Xi = \mathbb{R}^n$ and $\widehat{\Sigma}_{\mathbb{R}^n} \cap \mathbb{R}^n_+ = \Sigma$. Regard a set K that is polynomially convex and let $\Omega \supset K$ be a bounded open set. By [23], Corollary 5.1.5, $V_{K+\delta\mathbb{B}} \nearrow V_K$ as $\delta \to 0$.

Then $\bigcap_{\delta>0}\{z \in \mathbb{C}^n\,;\, V_{K+\delta\mathbb{B}}(z) = 0\} = \{z \in \mathbb{C}^n\,;\, V_K(z) = 0\}$. Let B be some compact set that contains Ω. The open sets $\{z \in \mathbb{C}^n\,;\, V_{K+\delta\mathbb{B}} > 0\}$ for $\delta > 0$ cover the compact set $B \setminus \Omega$, so there is a $\delta_0 > 0$ such that $\{z \in \mathbb{C}^n\,;\, V_{K+\delta_0\mathbb{B}} = 0\} \subset \Omega$. By [27], Lemma 5.2 and Proposition 5.3, $V^*_{K+\delta_0\mathbb{B}}|_{K+\delta_0\mathbb{B}} = 0$. By Theorem 6.4 we can approximate f uniformly on $K + \delta_0\mathbb{B}$, hence also on K. Theorem 6.4 therefore implies the Runge-Oka-Weil Theorem.

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