UNIVERSITY OF ICELAND

Ph.D. Dissertation
in Theoretical Physics

Quantum fluctuations of non-Lorentzian black holes

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August 2026

FACULTY OF PHYSICAL SCIENCES

Quantum fluctuations of non-Lorentzian black holes

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Dissertation submitted in partial fulfillment of a
Philosophiae Doctor degree in Theoretical Physics

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Reykjavik, August 2026

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Bibliographic information:
Matthias Harksen, 2026, *Quantum fluctuations of non-Lorentzian black holes*, Ph.D. Dissertation, Faculty of Physical Sciences, University of Iceland, 138 pp.

Author ORCID: 0000-0001-7908-1103
ISBN: 978-9935-585-00-4

Printing: Háskólaprent
Reykjavik, Iceland, August 2026

Abstract

This thesis explores quantum gravity through the lens of holography, with particular emphasis on non-Lorentzian gravity and black hole thermodynamics. Its central aim is to understand how ideas from relativistic holography and lower-dimensional gravity are modified when one considers non-Lorentzian symmetry structures such as Carrollian, Galilean, and Lifshitz symmetries. Rather than extending holographic models by adding new fields or extra dimensions, the thesis explores how far one can go by relaxing the underlying relativistic symmetry structure itself.

The core of the thesis consists of three research papers, each addressing a different aspect of non-Lorentzian gravity. The first studies Carrollian string theory, examining how strings may be described in the ultra-relativistic limit in which the speed of light tends to zero, a regime naturally relevant near black hole horizons and at null infinity. The second investigates quantum corrections to the thermodynamics of a two-dimensional dilaton gravity model obtained by dimensional reduction of a Lifshitz black hole, with the aim of exploring how ideas familiar from JT gravity and Schwarzian boundary dynamics extend beyond the relativistic setting when one reduces, rather than enlarges, the symmetry algebra. The third studies non-relativistic de Sitter space, focusing on the formulation of an action principle for a non-relativistic analogue of Jackiw-Teitelboim gravity in terms of Newton-Cartan geometry and on the associated implications for lower-dimensional gravity and cosmology.

In addition to the appended papers, the thesis contains an introductory chapter written for the general reader, including reflections on why fundamental research matters, what practical value it may have, and a short introduction to high energy physics as a subject. It also includes a pedagogical chapter based on lectures attended during my PhD, which develops the relativistic perspective on which the appended papers build.

Ágrip

Ritgerðin fjallar um þyngdarskammtafræði. Hún kannar djúpstæð tengsl sem eru til staðar milli þyngdarfræði eins og henni er lýst í almennu afstæðiskenningunni og hornrækinnar skammtasviðsfræði í einni lægri rúmvídd. Þessi samsvörun nefnist þyngdarfræðileg heilmyndun og hefur ýmsar birtingarmyndir, en ein sú einfaldasta felur í sér að þyngdarfræði í tvívíðu tímarúmi með fastan neikvæðan krappa sé jafngild hornrækinni skammtasviðskenningu sem býr á jaðri þess. Í ritgerðinni er síðan rannsakað hvernig þessi samsvörun og skyldar hugmyndir taka á sig nýja mynd þegar horfið er frá hefðbundnum afstæðilegum samhverfum yfir í óafstæðilegar samhverfur, svo sem Galíleó-, Carroll- og Lifshitz-samhverfur.

Meginuppistaða ritgerðarinnar eru þrjár rannsóknagreinar sem fjalla um ólíka þætti óafstæðilegrar þyngdarfræði. Fyrsta greinin fjallar um Carroll-strengjafræði, þar sem athugað er hvernig lýsa megi strengjum í markgildi þar sem hraði ljóss stefnir á núll, sem er viðeigandi lýsing í grennd við sjóndeild svarthola og á jaðri tímarúmsins. Önnur greinin fjallar um skammtafræðilega leiðréttingarliði við varmafræði tvívíðs þinilsviðslíkans sem fæst með víddarlækkun á Lifshitz-svartholi. Þar er markmiðið að varpa ljósi á hvernig varmafræðilegir skammtaeiginleikar svarthola hegða sér þegar horfið er frá hefðbundnum afstæðilegum samhverfum. Þriðja greinin fjallar um óafstæðilegt Galíleó-de Sitter rúm. Þar er leitast við að skilja betur óafstæðilega Newton-Cartan rúmfræði og afleiðingar hennar fyrir heimsfræði þar sem afstæðilegar forsendur eru ekki lengur til staðar í sinni hefðbundnu mynd.

Auk greinanna þriggja inniheldur meginmál ritgerðarinnar alþýðlegan inngang. Þar má finna hugleiðingar um það hvers vegna grunnrannsóknir eru mikilvægar fyrir mannkynið og jafnframt stutta kynningu fyrir hinn almenna lesanda á því fræðasviði í háorkueðlisfræði sem greinarnar leggja sitt af mörkum til. Þar að auki fylgir fræðilegur kafli sem byggður er á röð fyrirlestra sem ég sótti í doktorsnáminu mínu. Sá kafli skýrir hið afstæðilega sjónarmið sem greinarnar sjálfar byggja á.

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Paper I: Matthias Harksen, Diego Hidalgo, Watse Sybesma and Lárus Thorlacius, 2024, *Carroll strings with an extended symmetry algebra*. *Journal of High Energy Physics* **05** (2024) 206. Contribution of the thesis author: The thesis author contributed to the development of the main idea of the paper, the construction and analysis of the Carroll string models, the derivation of their symmetry structure, and the writing of the manuscript.

Paper II: Matthias Harksen and Watse Sybesma, 2025, *The spectrum of a quantum Lifshitz black hole in two dimensions*. *Journal of High Energy Physics* 02 (2025) 214. Contribution of the thesis author: The thesis author contributed to the derivation of the effective boundary description, the computation of the logarithmic entropy correction and writing of the manuscript.

Paper III: Matthias Harksen, Diego Hidalgo and Watse Sybesma, 2026, *Quantum Fluctuations and Newton-Cartan Geometry for Non-Relativistic de Sitter space*. Preprint, accepted for publication in *Journal of High Energy Physics*. Contribution of the thesis author: The thesis author contributed to the analysis of the non-relativistic de Sitter boundary action, the one loop partition function, the associated Newton-Cartan bulk geometry, and the writing of the manuscript.

Acknowledgements

Inspired by the first book of Marcus Aurelius' *Meditations*, where he reflects on the people who shaped his character and what he learned from them, I would like to acknowledge those who have influenced my life during these years and what I have learned from each of them.

- Af **Eddu** minni, að þegar að vandamál virðast óyfirstíganleg þá þarf stundum bara annað sjónarhorn; að leita í náttúruna, hreyfingu og útivist þegar lífið verður erfitt og að umfram allt er það miklu auðveldara að ganga í gegnum erfiða tíma þegar maður gengur þá ekki einn.
- From **Watse Sybesma**, that good social skills and the ability to encourage the people around you are among the most important qualities a scientist can have; that when faced with a difficult decision one should ask: "What is the better story?" and that this simple question often leads to the right choice, both in research and in life; and that one should engage with people with genuine interest, notice their efforts, take an interest in their stories, and invite them generously into one's life.
- Af **Lárusi Thorlacius**, að gefa öðrum óskipta athygli þegar þeir óska eftir henni; að segjast aldrei vera of upptekinn þessa stundina, því sú stund kemur aldrei þar sem maður er minna upptekinn; að hafa þolinmæði til að fara vandlega í gegnum röksemdarfærslur annarra með opnum huga; að hugsa um rannsóknir í víðara samhengi samfélagsins; og fyrir að ryðja brautina fyrir mig og gefa mér tækifæri til að stunda doktorsnám í eðlisfræði, án þess að reyna að móta mig í eigin mynd heldur leyfa mér að finna mína eigin leið sem fræðimaður.
- Af **Valentinu G. M. Puletti**, að mæta öllum í mannlegum samskiptum með hlýju, hlátri og brosi; að hvetja fólk til að vera besta útgáfan af sjálfu sér og láta sig dreyma stórt, jafnvel þegar hugmyndir virðast á pappír óraunhæfar; að muna ávallt áminninguna hennar: "Don't be sloppy now, or you will be sloppy for the rest of your life"; fyrir að gefa sér ávallt tíma fyrir gesti, fara með þeim út að borða og verja með þeim kvöldum í góðri nærveru; fyrir útsýnið úr glugganum heima hjá henni, sem er að mínu mati hið fegursta í heimi, yfir Faxaflóann með útsýni til Snæfellsjökuls.
- Af **Friðriki Frey Gautasyni**, að maður þarf ekki alltaf að vera að toppa sjálfan sig; að treysta mínum eigin útreikningum; að stundum sé besta tækið í samningaviðræðum að láta hinn aðilann leitast við að lágmarka sitt eigið tap fremur en að hámarka sinn eigin ávinning; og að eltast ekki stöðugt við það sem aðrir segja að sé vinsælt heldur leyfa mínum eigin (sérviskulegu) hugmyndum að ráða för.
- From **Niels Anne Jacob Obers**, that one should not correct people in public, but instead give them the chance to save face by raising such matters gently in person and behind closed doors; for teaching me the phrase "It is very funny you say that!", which has the remarkable effect of throwing people slightly off balance and making them listen much more carefully to what comes next.

- From **Matthew McGeagh Roberts**, that perhaps the best way to introduce yourself to everyone is by sharing your own traditions with them; such as by preparing a Thanksgiving turkey for everyone in Iceland at the Christmas party.
- From **Diego Hidalgo**, that one should stay in the trenches of difficult calculations and never give up before every detail has been understood; that the artistic side of life deserves to shine just as brightly as the scientific one; and for explaining to me the method by which he manages to bring his guitar for free across international flights, although it still remains a mystery to me.
- From **Davide Astesiano**, that even the most ordinary events can be turned into a legendary story; that somewhere on the top of Esja there lives a dragon which we must defeat in order to free the town of Kópavogur; and that sometimes the only difference between dreams and nightmares is that nightmares come true.
- From **Victoria Martin**, that placing trust in younger students and taking them under one's wing can help them grow into far more than they initially believe themselves capable of.
- From **Rahul Poddar**, that everything that happens in physics can be fascinating if one looks at it with the right sense of wonder; that once one has learned something complicated (such as the Monster group) one suddenly begins to see it everywhere; and that one should never forget to constantly retie one's shoes.
- From **Vyshnav Mohan**, to be mindful of returning gratitude when one receives it; to run for the train when the moment calls for it, even when it feels beyond one's limits; and not to underestimate how often the most beautiful physics emerges from tinkering with creating diagrams.
- From **Alexia Nix**, to always be ready to say yes to new plans, even the slightly unreasonable ones; that building a wide circle of friends comes from showing up again and again for the people around you; and that beneath this energy there should always be a genuine care for how others are doing.
- From **Evangelos Tsolakidis**, that one can be close friends with someone whose views are very different from one's own, and still recognise that in the end we all want the same thing — for the world to be a good place to live; that in every place one visits, one should find something that becomes one's own — like skiing in Iceland; and that sometimes a single swim, in a pool in Akureyri, can change the course of an entire day.
- From **Phoebe Richman-Taylor**, that on long walks across Iceland, nothing more is needed than good company; and that she never quite fits the ideas people form about her, and never seems to feel the need to; and for all the stories about her cat Steve, whom I was convinced was entirely fictional until her last week in Iceland.
- From **Sixten Arthur Henrik Nordegren**, how to turn properly in the swimming pool, and that physical skills have a way of staying with you — to the point that I still think of him every time I turn; and that this alone is reason enough to teach such skills to others; and that when someone visits, it is worth clearing the day to see familiar places through their eyes — even something like the Vasa ship can feel new again when seen with different company.

- From **Luis Alberto León Andonayre**, that sometimes making a place one's own can be as simple as bringing a flag to the top of Esja; and that even the most ordinary paperwork can turn into something far more complicated than expected when one is far from home.
- Af **Sigurði Jens Albertssyni**, að segja hlutina eins og þeir eru og treysta því að skýr hugsun sé alltaf betri en flókin framsetning; að halda áfram án þess að festast í eigin hugsunum; og að það er ómetanlegt að eiga vin sem gefur af sér og er alltaf til staðar þegar á þarf að halda.
- From **Moira Aileen** that in doing research, whether on owls or anything else, one must not be too prudish to do the dirty field work, whether that means owl poop, vomit, or whatever else the work demands; that good research does not keep regular working hours, but is present when the subject is present; that one takes this upon oneself even when it means discomfort in the form of carsickness; and that through such work one may come to know Iceland better than many Icelanders do.
- From **Robert Andrew Scully**, that one should say yes to invitations because otherwise they may stop coming, whether to parties or in life; that small acts of generosity (such as lending someone money for a Rottweiler T-shirt) can become lasting memories of friendship; that the framework of physics can also be used to analyse different aspects of the world around us; and that he reminded me why I love teaching: to meet students where they are at, and to believe that even the smallest interaction can help them blossom later in life.
- Af **Breka Pálssyni**, að segja já að fyrra bragði við tillögum annarra því að þannig glæðir maður lífinu mestum lit; að það er alltaf tími fyrir smá vitleysu (eins og þríþraut) því lífinu fleytir stöðugt áfram og maður verður að grípa tækifærin á meðan að þau gefast.
- Af **Ásgeiri Tryggvasyni**, að kenna mér góða hestamennsku; og að vita nákvæmlega hvenær sólarvörn með brúnkukremi er nauðsynleg fyrir góða myndatöku.
- Af **Hjalta Þór Ísleifssyni**, að lífið er mikilvægara en akademían; að hugsa um lífið á lengri tímaskala og fyrir að veita mér innri styrk í erfiðum verkefnum; að það er ekki nóg að vera klár og gáfaður heldur þarf einnig að eyða botnlausu hafi af tíma ásamt þrautseigju og elju í að skilja raungreinar því enginn fæðist með skilning á Schrödinger-jöfnunni.
- Af **Stefaníu Katrínu Jónínu Finnsdóttur**, að minn stærsti styrkur geti einnig verið minn stærsti galli; að aldur sé aðeins tala og að fimmtugum sé allt fært; að muna að elska sumarsólstöður á Vestfjörðum; og að minna mig á að vinátta sé ævistarf sem spannar marga áratugi, þar sem báðir aðilar breytast með tímanum en halda þó sama kjarna inn að beini.
- Af **Nönnu Óttarsdóttur**, að sýna mér að Íslendingasögurnar eru ekki uppspuni; að það eru til Íslendingar sem geta kastað vel klyfjuðum lóðastöngum beint yfir höfuðið á sér — og þar með að það eru engin líkamleg takmörk sett á það hvaða afrek fólk getur unnið; og að styrkur er ekki bara stærð heldur einnig þrautseigja, vilji og sjálfsagi.
- Af **Jökli Sindra Gunnarssyni Breiðfjörð**, að það er alltaf tími fyrir smá FIFA þó að væf sé væf; og fyrir að sýna mér þann aðdáunarverða hæfileika að geta sem tenórsöngvari samt skemmt sér í karaoke með almúganum sem syngur falskt.

- Af **Aðalbjörgu Egilsdóttur**, að vera næmur fyrir því sem aðrir eru að ganga í gegnum; að gefa sér tíma til að hugsa um hvernig hlutirnir birtast frá sjónarhorni annarra; fyrir að vera mér félagsskapur í þverfimi; og að vera tilbúinn að endurhugsa hvernig ég vil að lífið mitt líti út.
- Af **Ragnheiði Birgisdóttur**, að það að hlúa að menningu, listum og bókmenntum skipti miklu máli og geri lífið bæði dýpra og fallegra; og að stundum búi ástin á Kaupmannahöfn, og hönnunarljósunum sem þaðan koma, ekki bara í borginni sjálfri heldur líka í æskuminningunum af fólkinu sem kynnti mann fyrir borginni fyrst.
- Af **Stefáni Óla Jónssyni**, að orð hafi sögu og vægi sem birtist stundum einna skýrast í rapptextum, en að það sé engu að síður mikilvægt að staldra við og minna fólk á að notkun þeirra er aldrei alveg saklaus eða sjálfsögð.
- Af **Þorgeiri Björnssyni**, að lífið er eins og Seinfeld þáttur og að það eitt að vita af því gefur manni styrk til að mæta öllum áskorunum með brosi á vör; að leyfa öðrum að tala frjálslega um það sem þeim þykir áhugavert — jafnvel þegar umræðuefnið er fiskihlaðborðið á Tjöruhúsinu.
- Af **Smára Yngvasyni**, að kenna mér að gera handgert vínarbrauð; að þegar fólk leitar ráða hjá mér að muna að spyrja fyrst áður en ég svara um hvers konar bakstur sé verið að ræða, svo ég skilji hvernig hveiti á að fara í baksturinn; að hver einasta manneskja á sitt eigið sérsvið þar sem hún býr yfir ótrúlegum hæfileikum; og að vera næmur fyrir líðan vina minna og hugsa um hvað gerir þá ánægða í hverjum aðstæðum.
- Af **Kristni Péturssyni**, að ríghalda í það sem veitir mér ánægju og lífshamingju; að það er ekki sjálfsagt að feta menntaveginn þó maður hafi gáfurnar til þess — það er einnig háð óstjórnanlegum samfélagslegum þáttum og heppni; og að mín eigin takmörk felast oft í því sem ég segi við sjálfan mig í huga mér.
- Af **Sölku Kristinsdóttur**, að búa yfir þeim sjaldgæfa hæfileika að leiða saman ólíkt fólk og láta því líða eins og það eigi náttúrulega samleið; að láta ekki skipulagskvíða eða ótta við hið óvænta stöðva sig þótt slíkt sé alltaf einhversstaðar til staðar; og að hafa kjark til að sækjast eftir því sem maður trúir að geti gefið lífinu meiri fyllingu og hamingju til lengdar, jafnvel þegar leiðin þangað felur í sér fórnir.
- Af **Bjarna Inga Sigurgíslasyni**, að leyfa ekki skoðunum mínum að litast af því sem aðrir segja heldur treysta eigin dómgreind og rökhugsun; að vera tilbúinn að ganga í burtu ef hlutir samrýmast ekki minni eigin siðferðislegu sannfæringu; að forvitni í leik og starfi sé besta leiðin til að lifa lífinu; og að tína myntu og hvönn í almenningsgörðum sem og kúmen í Garði.
- Af **Ægi Má Jónssyni**, að eltast stöðugt við ný áhugamál og demba mér af öllum krafti í þau þegar ég finn í þeim uppsprettu af hamingju; að vera ekki feiminn við að kynna mér hluti sem ég er ekki sérfræðingur í og ræða þá á mínum eigin forsendum við sérfræðinga; og að vera tilbúinn að færa fórnarkostnað til að mæta á viðburði, jafnvel þó það krefst þess að ferðast langt.
- Af **Guðrúnu Grétarsdóttur**, hvernig á að losa stein úr bremsuklossum á bíl með því að bakka eins hratt og maður getur; og að maður eigi alltaf að bjóða nýjum kunningjum með hlýju hjarta heim til sín.

- Af **Björgu Jessicu Harrendence**, að hafa hugrekkið til að stíga út fyrir þægindarammann, eins og með því að flytja á Bíldudal og hefja störf í kalkþörungaverksmiðju.
- Af **Jóni Emil Guðmundssyni**, að gefa mér tækifæri til að sinna vísindamiðlun og hvetja mig áfram í þeirri vegferð; að það að virkja yngri kynslóðina sé fjárfesting til framtíðar; og að sýna með eigin fordæmi að ástríða fyrir vísindum getur verið svo sterk að jafnvel eftir langa svefnlausa flugnótt má finna kraft til að mæta beint frá flugvellinum á vísindavöku Rannís.
- Af **Viðari Ágústssyni**, að mikilvægt er að sinna samfélaginu af ábyrgð; að óeigingjarnt starf unnið til að lyfta öðrum upp er ævistarf sem maður getur verið stoltur af; og að með því að halda líkamanum í hreyfingu — hvort sem er við hjólreiðar eða íssjóböð — endurheimtir maður orku og lífsgleði.
- Af **Ingibjörgu Haraldsdóttur**, fyrir þá útsjónarsemi að sjá möguleikana í mér áður en ég sá þá sjálfur og fyrir að hafa hvatt mig áfram í starfi mínu við eðlisfræðikeppnina, ekki aðeins með orðum sínum og verkum og með því að finna henni fjárhagslegan stuðning, heldur einnig með því að hvetja mig til að hugsa af alvöru hvert markmið keppninnar sé í samfélagslegu samhengi.
- Af **Unnari Bjarna Arnalds**, fyrir að hafa af örlæti veitt mér aðstoð við undirbúning fyrir atvinnuviðtalið mitt hjá DTU með því að kynna mig fyrir nanóvíratilrauninni sinni; fyrir samstarfið í gegnum árin í tengslum við eðlisfræðikeppnina, þar sem ég hef fengið að kynnast þeim einlæga áhuga sem hann hefur á verklegum tilraunum og þeim líkönum sem þeim fylgja.
- From **Judit Prat Martí**, that one can see the world in a different light; that preconceived ideas do not have to determine how one experiences it; for teaching me how to cross-country ski; for encouraging me to place greater emphasis on science communication; for giving me the courage to pursue interests beyond academia; and for reminding me to bring others along on adventures and to maintain a social rhythm — that Fridays are always reserved for friends, wherever they may be.
- Af **Sævari Helga Bragasyni**, fyrir að minna mig á að jafnvel bestu og færustu umsækjendur, með verkefni sem koma einungis einu sinni á öld, fá hátt í hundrað hafnanir á umsóknum sínum, en að þrátt fyrir það, verði maður bara að halda áfram að sækja um; að sýna mér að vinstri umhverfisstefna er samrýmanleg hægri efnahagsstefnu, því það er í hag okkar allra að varðveita þessa fallegu jörð sem við búum á; að minna mig á að manneskjan finnur oft mesta gleði í því að skapa og byggja eitthvað raunverulegt; og að stundum sé betra að láta verkin tala frekar en orðin, því alltof margir festist í fögrum hugsunum sem aldrei verða að veruleika.
- From **Paolo Di Vecchia**, that one can remain deeply engaged with science at any age; that one should share meals with the younger generation, bring one's own lunch, and outlast even the most ambitious of them; that the danger of creating something powerful (such as weapons) is that it will eventually be used; and that a true presence in physics is felt everywhere — to the point that I sometimes wondered whether he could be in several places at once.
- Af **Birgi Guðjónssyni**, að góð kennsla er lífsverk; að setja kröfur og standa við þær; og að vönduð menntun hefur áhrif langt út fyrir kennslustofuna.

- Af **Ragnheiði Guðmundsdóttur**, að með því að standa fast á þeim gildum sem maður stendur fyrir þá geti maður ýtt mörkum samfélagsins áfram.
- Af **Martin Jónasi Birni Swift** fyrir að hvetja mig til að sinna vísindamiðlunarstarfi og að minna mig á að þrátt fyrir það að það sé erfitt þá er það samt sem áður gefandi og verðugt verkefni. Tilgangur vísindamiðlunar er tvíþættur, almenningur lærir eitthvað um vísindi, en að áhrifin á vísindamiðlaran sjálfan séu ekkert síðri.
- Af **Degi Ara Kristjánssyni** og **Ými Braga Gíslasyni**, að það er mikilvægt að eiga sinn eigin þriðja stað í veröldinni; að maður eigi alltaf að fylgja lögum og reglum – sérstaklega þegar að maður heldur að enginn sjái til.
- Af **Sigurði Ólafssyni**, að það sé aðdáunarvert að vera þúsundþjalasmiður, bæði á sviði verkgreina og raungreina; að það sé dyggð að taka áhugamálin sín alvarlega, kafa djúpt í þau og mynda sér ákveðnar skoðanir á því hvernig best sé að gera hlutina; og að vera ekki feiminn við það að læra á ný verkfæri (eins og þrívíddarprentara) ef það þýðir að maður geti mótað heiminn í kringum sig eins og maður vill hafa hann.
- Af **Margréti Þorvaldsdóttur**, að það sé ekki til sá réttur á jarðríki sem væri ekki betri með smá fjallagrösum; að meta betur þau íslensku hráefni sem náttúran og Móadalurinn hafa upp á að bjóða; og að sumt af því sem maður bakar fyrir heiminn sé í raun alltaf ætlað kórnum, því enginn maður er eyland og maður lifir ekki aðeins fyrir sína nánustu heldur einnig fyrir það samfélag sem maður tilheyrir.
- Af **Líf Magneudóttur**, að ég get verið vinur fólks þó að stjórnmálaskoðanir okkar samræmast ekki; að vera alltaf spenntur fyrir að læra nýja hluti og taka að mér krefjandi verkefni sérstaklega ef það stangast á við þær forhugmyndir sem að aðrir hafa um mig.
- Af **Bjarnheiði Kristinsdóttur** að kynna mig fyrir bókum um þekkingarfræði og þeim menntaviðhorfum sem þeim fylgja; að aðstoða mig við að koma kennslufræðilegum hugmyndum mínum í betri farveg.
- Af **Líneyju Höllu Kristinsdóttur**, að besta ráðið við andvökunóttum sé einfaldlega að lesa þar til maður sofnar. Annaðhvort sofnar maður eða að maður klárar bókina – í báðum tilvikinu hefur nóttin því verið nytsamleg.
- Af **Mjöll Snæsdóttur**, að manneskjan lærir best með því að segja og heyra sögur; að sá fróðleikur sem situr lengst í okkur mótist af þeim dulræna sagnafróðleik sem við sönkum að okkur yfir ævina; fyrir að hafa verið íkveikjan að því að við Edda fórum til Miklagarðs og sáum rúnaáletrunina *Hálfdan var hér* í Ægissif.
- Af **Magneu J. Matthíasdóttur**, ömmu minni, að það er erfiðara að skila doktorsritgerð heldur en að fá fálkaorðuna; og þó, á sama tíma í annarri merkingu, að það sé erfiðara að fá fálkaorðuna en að skila doktorsritgerð; og fyrir það að hvetja mig til að sinna hugsjónarstarfi, ekki af því að það borgar vel, heldur af því að það veitir manni lífsánægju; að elska hið íslenska tungumál og þær sögur sem að við getum sagt á okkar eigin máli; að doktorsnám sé langhlaup en ekki spretthlaup.
- Af **Kristni Einarssyni**, afa mínum, að reyna eftir fremsta megni að forðast að taka lán; að mikilvægasti kostur í starfi sé ávallt sveigjanlegur vinnutími; og að besti tíminn til að byrja í líkamsrækt sé á eftirlaunaaldrinum, eða með öðrum orðum að það sé aldrei of seint að byrja á neinu.

- Af **Margréti Hallsdóttur**, að það besta sem fólk getur sagt um hana á afmælisdaginn hennar sé að hún sé að heiman, því þá er hún sjálf úti að upplifa eitthvað og lenda í ævintýrum; að taka þennan lærdóm til mín inn í eigið líf, ekki aðeins sem leið til að sneiða hjá gjöfum og húllumhæi heldur sem áminningu um hvað það er í raun sem maður lifir fyrir; og fyrir að kenna mér á skíði fyrir löngu síðan.
- Fra **Bodil Andersen** og **Josefine Birgit Andersen**, at det at være kusiner også er en aftale om ikke at sige noget til Birte om de farlige “das ist rot” situationer vi oplever sammen; for Bodils energi og erfaring, som førte os ud på en rigtig vandretur og et ægte eventyr på Fimmvörðuháls som vi kommer til med at huske for evigt; og for at Josefine turde vade over Krossá sammen med mig, hvilket dengang føltes nødvendigt, da tiden var knap, men i bakspejlet måske var en smule farligt.
- Af **Söru Dögg Lárudóttur**, fyrir að hafa haldið fast í draum sinn um að verða læknir þrátt fyrir mótlæti og daglega baráttu við veikindi sem krefst bæði styrks og þrautseigju.
- Af **Bjarka Baldurssyni Harksen**, bróður mínum, að kæra sig ekki um það sem samfélaginu finnst merkilegt bara af því að því finnst það merkilegt; eins og þegar hann vann ferð, gistingu og miða á Liverpool leikinn gegn Tottenham (5-2) frá Dominos, ekki vegna þess að hann sóttist sérstaklega eftir vinningnum heldur einfaldlega af því að tilboðið var ódýrara og 2L Coke fylgdi frítt með, og var síðan lengi vel ekki alveg meðvitaður um hversu eftirsóttur slíkur vinningur var, heldur vanmat hann um heila stærðargráðu.
- Fra **Edith Harksen Madsen**, min mormor, at samle svampe i skovene, fange rejer ved stranden og dyrke have, både blomster og nytteplanter, for der er noget særligt ved det, man selv frembringer med sine egne hænder, som man ikke kan købe i butikkerne; at det nogle gange er nemmere at skjule kaffen, når Wilfried er på besøg, end at fortælle ham at man synes den tyske kaffe smager dårligt; og for at have lært mig at sige *Gesundheit*, når andre nyser, således lever min mormors hellige ånd videre i mig for evigt.
- Fra **Birte Harksen**, min mor, at man skal have mod til at ændre sine omstændigheder og gribe chancerne, når de byder sig, selv om det måske ikke altid går på den måde, man havde håbet, for ellers sidder man bare tilbage med tanken: hvad nu hvis? Og at der altid kommer momentum i benene, når man først har fået gjort det første på listen færdigt, for så får man pludselig energi til alle de andre ting på listen; at man skal være lidt skør og pip-gok-påskeplim for at folk lægger mærke til én og hvordan man har tænkt sig at ændre verden; og for at have lært mig, hvordan man laver surdejsbrød.
- Af **Baldri Arnviði Kristinssyni**, föður mínum, fyrir það göfuga hugarfar að forgangsraða hlutunum þannig að maður geti alltaf verið til staðar fyrir sína nánustu svo að draumar þeirra fái að skína sem skærast, þó það kosti mann sjálfan stundum ákveðinn fórnarkostnað; fyrir að hafa kennt mér að tefla, en samt með þeim hætti að neyða mig aldrei til að stunda neitt af einskærri forræðishyggju; að hafa kennt mér að reyna ávallt að sjá hlutina frá sjónarhorni annarra, því með því að hugleiða hvers vegna fólk bregst við eins og það gerir skilur maður best hvernig maður á sjálfur að bregðast við á móti; að sækja ekki í einfaldan skáldskap sér til dægrastyttingar, heldur leita í bækur sem veita manni dýpri skilning á náttúrunni, mankyninu og því hvað það merkir yfir höfuð að vera til; en mest af öllu fyrir að vera mér fordæmi í fróðleiksfýsn og þekkingarleit.

1 Introduction

This introductory chapter is intended for the general reader and thus follows the well-known rule of thumb, often attributed to Stephen Hawking's anonymous editor, that each equation in the text halves the audience. It is therefore deliberately written in a popular style and kept as non-technical as possible. Calculations and technical details can be found in subsequent chapters; in this one, we will instead be concerned with stories, folklore, and the broader ideas that motivate the subject, although I will of course also be adding a few words about my own research in general outline.

A large part of my motivation for writing this introduction comes from a panel discussion during Eurostrings 2025, titled *The Role of Science in Times of Geopolitical Uncertainties*, which featured *Mikael Fogelström*, director of Nordita, *Ronald de Bruin*, director of COST and *Robbert Dijkgraaf*, former director of the Institute for Advanced Study in Princeton and former Minister of Education in the Netherlands. During the discussion, Dijkgraaf described the cartoon below, which he said captured his own transition from academia to public service.

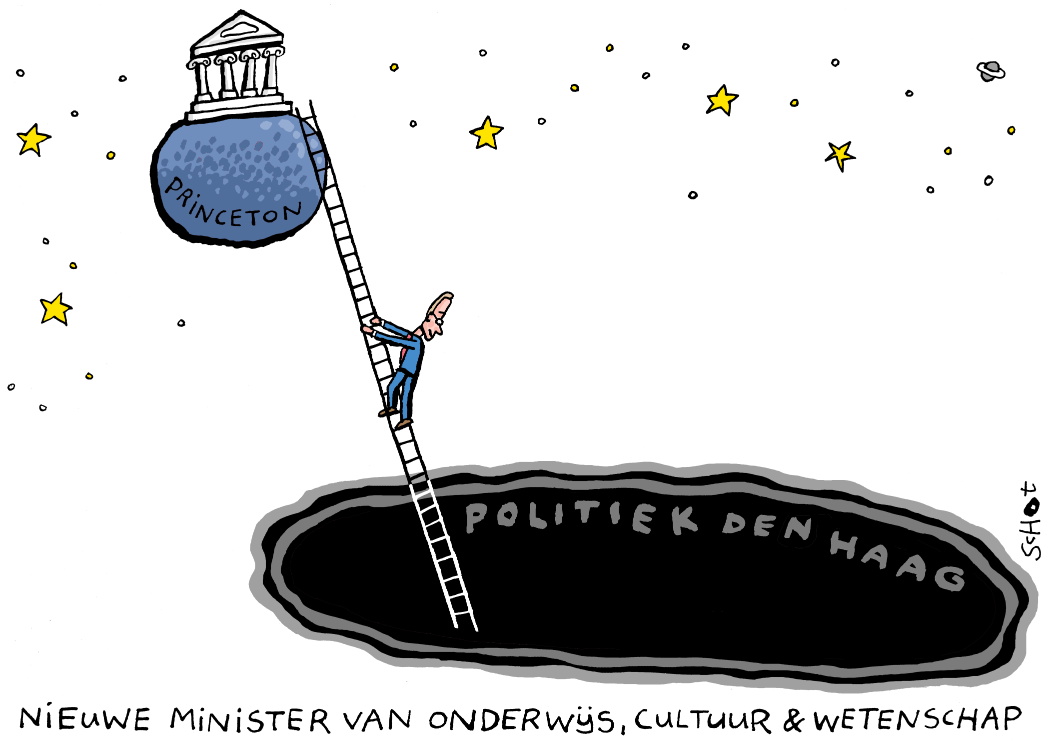

Figure 1.1. Robbert Dijkgraaf, former director of Princeton's Institute for Advanced Study, descends into the black hole of Dutch politics upon becoming Minister of Education in the Netherlands. We thank the artist of the cartoon, Bas van der Schot, for kindly granting permission to reproduce it in this thesis.

To me, this cartoon of Dijkgraaf has a symbolic significance. It represents the growing divide between academia and society. We physicists often like to think that we are simply misunderstood by society, but I believe that we, too, may have lost touch with it.

In his talk, Dijkgraaf emphasized that science has, in many ways, been one of the most successful collective projects in human history. Time and again, it has delivered far more than anyone could reasonably have expected. And yet, he argued, we may now be entering a different phase. For the first time in decades, there are visible signs of growing skepticism toward science. The trust it once enjoyed almost by default can no longer be taken for granted. Its value, though embedded everywhere in modern society, is no longer always perceived as self evident.

For decades, science benefited from a kind of quiet legitimacy. Its usefulness was so apparent that it rarely had to justify itself explicitly. But if that assumption is now being eroded the scientific community must learn not only to explain what it does, but also to defend why it matters. This raises a deeper question: how should one advocate for science in a world where its value is no longer taken for granted as a shared interest?

As a response to these concerns, I will, in the following sections, tell four short stories. I do so partly because I believe that large ideas are often easier to grasp when seen through small and concrete human episodes. Each of these stories, in its own way, points toward what I believe to be a central nugget of truth about the role of science in society.

The first is the story of Abraham Flexner, and what it reveals about the unpredictable power of curiosity-driven research. The most transformative ideas are rarely pursued for their immediate usefulness, but they often prove valuable in ways no one could have anticipated.

The second is the story of Erwin Finlay Freundlich, and what it reveals about the real strength of the scientific method. Science does not advance because everyone agrees from the outset, but because ideas are exposed to criticism, measurement, and revision until the evidence begins to point more clearly in one direction.

The third is a story told by Kip Thorne about the Teller-Ulam thermonuclear design. It is a story about what happens when knowledge is prevented from circulating freely. Scientific progress depends not only on brilliant ideas, but on the ability of those ideas to move and to be developed freely. When knowledge is enclosed, whether by military secrecy or by private control, it ceases to function fully as a public good.

The final story was told to me by Lárus Thorlacius on the first day of my PhD, and it addresses the question why we choose to do research at all. At its heart, it is also a story about why we choose to guide others along the same path. Science is sustained not only by discovery, but by mentorship and the decision to guide others into a life of curiosity.

Through these stories, I hope to bring the scientific and the human a little closer together, and to say something about why science matters in a time of rising skepticism towards science.

1.1 Which scientist has done the most for humanity?

In his 1939 essay *The Usefulness of Useless Knowledge* [1], Abraham Flexner, founding director of the Institute for Advanced Study in Princeton, recalls a conversation he had with the industrialist and major philanthropist George Eastman (founder of Kodak). Eastman had approached Flexner with the intention of donating a vast fortune but with a clear condition. The money, Eastman said, should be used for the promotion of research in useful subjects.

Flexner then proceeded to ask whom Eastman considered to be the most useful scientist in the world. Eastman replied instantaneously that this would be Marconi, to whom we owe the existence of radio and wireless communication. Eastman was then taken aback when Flexner responded:

"Mr. Eastman, Marconi was inevitable. Whatever pleasure we derive from the radio, or however wireless, Marconi's share was practically negligible. The real credit for everything that has been done in the field of wireless belongs, as far as such fundamental credit can be definitely assigned to anyone, to Professor Clerk Maxwell, who in 1865 carried out certain abstruse calculations in the field of magnetism and electricity. Maxwell reproduced his abstract equations in a treatise published in 1873. Fifteen years later, in 1888, Heinrich Hertz detected for the first time the electromagnetic waves which are the carriers of wireless signals. Neither Maxwell nor Hertz had any concern about the utility of their work; they had no practical objective. In fact, Hertz is supposed to have remarked, 'I do not think that the wireless waves I have discovered will have any practical application.' The inventor in the legal sense was, of course, Marconi — but what did Marconi invent? Merely the last technical detail, mainly the now obsolete receiving device called the coherer."

The most transformative discoveries in science have rarely been driven by the desire to be useful. They have emerged instead from a more fragile and more powerful impulse: *curiosity*. The search for understanding carried out with curiosity repeatedly proves to be the deepest source of practical progress. In an age increasingly preoccupied with measurable outcomes and immediate returns, this lesson seems to be forgotten. Funding tends to favor short term utility, identifiable deliverables and urgent problems. Yet history suggests that the opposite strategy is often wiser. The chain of utility begins with ideas that once appeared entirely useless and whose usefulness only emerges much later.

Another such example is Einstein's formulation of the general theory of relativity in 1915. He was driven by his curiosity to understand gravity at a fundamental level. Within the successful Newtonian framework he saw conceptual tensions which he wanted to resolve. Guided by a series of gedanken experiments, he was led to the equivalence principle and the conclusion that time itself must depend on the gravitational field: clocks placed higher in a weaker gravitational field run faster than clocks on Earth.

Remarkably, this effect is now part of our everyday technological infrastructure. The GPS system relies on atomic clocks aboard satellites. Due to the difference in the gravitational field these clocks tick faster than identical clocks on the surface of the Earth. Over the course of a year, the difference between these clocks grows to roughly 14 milliseconds. Without correcting for this relativistic effect, positional errors would accumulate to several kilometers within a single day. Perhaps engineers would eventually have discovered this once satellites were already in orbit. What is remarkable, however, is that the effect was calculated decades before the technology existed to make practical use of it, and it is difficult to imagine that Einstein could have anticipated such an application when he wrote down his theory of general relativity.

Sometimes the significance of an idea is not recognized within the lifetime of the person who proposes it. Vincent van Gogh sold almost no paintings while he lived and descended into mental illness in his later years. Something similar can be said of Ludwig Boltzmann. He proposed an atomic interpretation of thermodynamics at a time when many physicists of the late nineteenth century regarded atoms as mathematical conveniences rather than measurable physical entities (not unlike how dark matter or string theory are sometimes viewed today). Boltzmann insisted that entropy and thermodynamic behavior could only be understood statistically, in terms of microscopic quantities. His position met serious resistance, and the atomic hypothesis was not universally accepted during his lifetime.

In one of his annus mirabilis papers of 1905, Albert Einstein showed through his study of Brownian motion that there existed a concrete experimental method for verifying the existence of atoms. Yet Boltzmann had taken his own life only a few years before decisive experimental confirmation was established. Decades later, Richard Feynman would open his famous Lectures on Physics by claiming that if only one sentence of scientific knowledge could be preserved, it should be the atomic hypothesis: that all things are made of atoms. What had once been dismissed as metaphysical speculation now forms the most important conceptual foundation of physics, chemistry, molecular biology, and modern medicine.

In order to abide by Hinchliffe's rule, which states that if a research paper's title is posed as a yes–no question, the answer is almost certainly "no", we must give an inconclusive answer to the question posed at the start. The scientist who has done the most for humanity may well be someone who had no intention of doing so — and we may not yet know who that person is. But, in one thing I am certain: they were not motivated by thoughts of immediate utility.

1.2 An ode to Finlay-Freundlich

On August 21, 1914, a total solar eclipse was to provide one of the earliest real opportunities to test Einstein's prediction that gravity bends light. Erwin Finlay-Freundlich had organized an expedition for precisely this purpose. He was among the first astronomers to take Einstein's ideas seriously as an observational challenge, and Einstein himself followed the preparations closely, even contributing money from his own savings to secure the necessary instruments and photographic plates.

But history intervened. Before the eclipse could be observed, the First World War broke out. Freundlich and his collaborators were detained in Odessa for many weeks as foreigners under suspicion of espionage, their instruments were confiscated, and the expedition collapsed before any measurements could be carried out.

What followed is one of the most revealing stories about how science actually works. When Freundlich was finally allowed to return to Berlin in September 1914, his enthusiasm for testing relativity had not diminished. Nor did he give up after Eddington's celebrated 1919 expedition. He continued, joining further eclipse expeditions in 1922, 1925–26, and 1929, repeatedly trying to measure the deflection of starlight near the Sun. Twice the weather defeated him. Cloudy skies made the observations impossible in 1922 on Christmas Island and again during a later attempt in Sumatra. Only in 1929, more than fifteen years after his first attempt, did he finally obtain photographic plates of sufficient quality. Even then the story did not end cleanly, since his first calculations appeared to suggest a value larger than Einstein's prediction, before later reanalyses brought the result much closer to it [2].

As Subrahmanyan Chandrasekhar later mused, one cannot help wonder how the course of physics might have unfolded had the 1914 eclipse actually been observed. At that time Einstein had not yet arrived at the full prediction of general relativity. Because of an error in his earlier calculations, the value he expected for the deflection was only half the value he would obtain after completing the theory in 1915, and in fact coincided with the Newtonian result. Had Freundlich succeeded in 1914 and found the larger value, Einstein might well have seemed merely to be revising his theory after the fact. The deflection of light would then have lost much of its force as a genuine prediction [3].

One of the great failures of popular narratives about science is the tendency to portray theories either as triumphant truths or as embarrassing mistakes. The real process is slower, messier, and more interesting. To an outside observer, science can sometimes appear oddly unsettled. Specialists debate small differences between competing models, rule out alternatives one by one, and often seem reluctant to speak with a single voice too early. Yet this is not a weakness of the method but one of its strengths. Theories are formulated, their consequences are worked out, and they are confronted with the world. Some fail, others survive refined through scrutiny, and over time a clearer picture emerges. Consensus is produced by the repeated success of certain ideas in describing Nature better than their rivals.

This is the real strength of the scientific method. From an epistemological perspective, this should also be reflected in education. The world is not a collection of immutable facts handed down from above. It is an ongoing attempt to uncover the rules of a game whose pieces we did not design. Theories are provisional structures, beautiful and powerful within their domains, yet always subject to revision in the face of new insight. The strength of physics lies not in claiming certainty, but in its willingness to refine, replace, and sometimes even overturn its own constructions.

1.3 The Teller–Ulam design (So Zeldovich really knew?)

During the conference dinner of Eurostrings in 2025, a story was retold from the book Black Holes and Time Warps [4]. The story is a narrative told by nobel laureate Kip Thorne, who received the Nobel prize in 2017 for decisive contributions to the LIGO detector and the observation of gravitational waves. The episode takes place during the Cold War. Thorne was worrying about whether gas falling onto a neutron star, held back by intense X-ray radiation, might develop Rayleigh-Taylor instabilities. On Earth, when a dense fluid is supported by a lighter one, it tends to form unstable tongues. Thorne wondered whether something similar would happen for black holes.

Thorne first posed the question to the Soviet physicist Yakov Zeldovich. The answer was immediate and confident: the flow is stable, the gas does not develop tongues into the X-rays. When Thorne asked for details, however, Zeldovich became evasive and simply stated that this had been shown, but he could not provide any sources for this derivation when pressed by Thorne. Nothing had been published in any journals on these results. A few months later, hiking in the Sierra Nevada with the American physicist Stirling Colgate, Thorne asked the same question. Colgate gave precisely the same answer. The flow is stable. It has been shown. But again, he could give no references. When Thorne mentioned that Zeldovich had said the same thing, Colgate paused and reflected:

“Oh, that is fascinating. So Zeldovich really knew?”

Although Kip Thorne was asking and thinking about black hole physics, incidentally, these are also the same equations that govern the creation of the hydrogen bomb. The fact that Colgate learned that Zeldovich knew the answer to the question simply meant that he also in turn knew that the Russians had succeeded in creating the hydrogen bomb since these came up in classified nuclear weapons research. The astrophysics community was therefore forced to rediscover, more slowly and independently, something that was already known behind closed doors. When knowledge cannot circulate openly, it cannot be criticized, developed, or built upon in the usual way.

Today, we face a structurally similar challenge, though in a different form and it can be argued not in this line of research. A handful of technology companies now command research budgets that rival or exceed those of entire public national research programs. The issue is not that companies invest in research. On the contrary, such investment reflects a recognition that knowledge is the main driving force for prosperity. The deeper question is whether knowledge should primarily serve private accumulation or the shared advancement of humanity.

There is also a broader civilizational point here. Support for fundamental research does not merely produce knowledge. It helps determine the kinds of activities to which intellectual effort is directed. A society that sustains open inquiry creates space for its most talented minds to work on understanding Nature rather than the technologies of war. In that sense, the defense of basic research is not only a defense of knowledge, but also of the conditions under which knowledge may serve human flourishing rather than organized destruction.

1.4 Just around the next corner!

The following story was told to me by Lárus Thorlacius on my first day as a PhD student. I might get some of the exact details wrong, but I hope, nonetheless, that I manage to capture the message of the story.

During his first postdoc, Lárus was working closely with Leonard Susskind, more than twenty years his senior and already a famous name by then. They would frequently discuss physics while running. Susskind had been a track runner in his youth, though by then he was far removed from his days of chasing the four minute mile. Lárus was in the prime of his youth, although recurring ankle injuries had recently forced him to give up basketball.

One day, Lárus asked whether he could join Susskind and Robert Wagoner for a lunch run in the blistering California sun. What he had in mind was presumably an ordinary lunch run. But while tying their shoes, Susskind and Wagoner briefly discussed how far they should run, and settled, rather casually, on ten miles: five miles out, and then five miles back.

Once they had turned around at the five mile mark, Lárus felt that he was already close to giving up. The heat was overwhelming, and he had already begun running from shade to shade, trying to make use of every patch of cover he could find. More importantly, however, he did not know the way back on his own, and so he felt that he had to keep up with them for as long as possible. Only once he knew that he could make it home again did he allow himself to stop. He stopped, while Susskind and Wagoner kept going. I imagine that Lárus thought to himself: How can I ever look Susskind in the eye again? I would have to give up my entire physics career. Perhaps it was this very thought that gave him the strength to get up, intending to chase after Susskind to catch up with him. However, once he made it past the next corner, he saw that Susskind was himself there, having also slowed to a walk. He had not made it any further than the next corner.

For me, this story captures the essence of why we do research at all and why we mentor other people. The only way we know to help others achieve more than they thought possible is to set them an example, to raise the bar of what they believe they can live up to, and in doing so to go a little further than they themselves feel they can. Susskind himself was probably also close to giving up, but in setting this example I think Lárus was set on a trajectory toward achieving great things, both in his career as a physicist and beyond. Within the decade, Lárus would become a national marathon champion of Iceland, and perhaps he thought back to this run several times.

1.5 Where does this thesis fit into the framework?

I hope that the preceding subsections have provided a philosophical motivation for why we do fundamental research in physics, and that the stories I have told have, to some extent, helped narrow the widening gap between academia and society. I now turn to the more specifically academic motivation behind the line of research pursued in this thesis. I will begin by describing the field of high energy physics and its broader motivation for seeking a theory of quantum gravity, how the subject has developed over the last decades, and what a lay reader should know about this field in order to appreciate the context. Within this broader framework, I will then focus on the non-Lorentzian aspects and explain why my own research is primarily situated in that sector of high energy physics. This will form the basis of the discussion that follows.

1.5.1 The terrestrial meets the celestial

Some of the most important advances in our understanding of Nature have come when two principles, each deeply convincing in its own domain, begin to clash. For the Greeks, the heavens and the Earth belonged to two different worlds. The celestial realm was associated with perfection: stars and planets moved in regular patterns across the sky, as if governed by a more pristine and harmonious set of laws. The terrestrial realm, by contrast, was messy. Stones fell to the ground, carts slowed down.

Then there was a rock. Galileo realized that this distinction between the celestial and the terrestrial could not be fundamental. If one throws a stone high enough, it does not somehow cease to be earthly matter and become a celestial object. Rather, the same object participates in both descriptions. The implication was profound: there must be a single framework capable of describing both the falling stone and the motion of the planets. In this sense, Galileo's rock became the bridge between two seemingly separate realms of Nature.

Modern physics finds itself in a strikingly similar situation. On the one hand, we have quantum mechanics, which governs the microscopic world with extraordinary precision. It describes atoms, electrons, quarks and the probabilistic structure of matter at the smallest scales. On the other hand, we have general relativity, Einstein's theory of gravity, which describes the large scale structure of spacetime and the dynamics of stars, galaxies, and the Universe. Each theory is enormously successful within its own regime.

Black holes are Galileo's rock for modern physics. Black holes are not merely classical gravitational objects they also emit thermal radiation due to quantum fluctuations. In them, gravity, quantum theory, and thermodynamics coincide, and for this reason they demand a unified description.

1.5.2 From invisible stars to black holes

Long before general relativity, Michell and Laplace had already entertained the possibility of "dark stars", objects so massive that even light could not escape from them. It is worth stressing, however, that these Newtonian dark stars differ in an important conceptual way from the modern black hole. In the Newtonian picture, a light ray emitted from the surface does not necessarily fail to leave it altogether. Rather, it may travel outward for some finite distance before being pulled back. In that sense, the light does not reach asymptotic infinity, but it can still escape part of the way. Thus an observer located sufficiently close to the object could, at least in principle, still receive such light signals and then escape again. A Newtonian dark star is therefore not hidden behind an absolute horizon in the relativistic sense.

The formulation of black hole physics, and our understanding of its implications, changed profoundly with the advent of general relativity. The story begins in 1915, only months after Einstein published the theory, when Schwarzschild found the first exact solution to the Einstein equations. This solution seemed to contain singularities, and Einstein himself was deeply uncomfortable with such singularities, suspecting that they reflected a failure either of the theory or of its physical interpretation.

As it turned out, the Schwarzschild solution contains two very different kinds of singular behaviour. The first kind is an artefact of the coordinate system being used, much like the way a Mercator projection distorts the globe and fails to describe its poles. The second, however, is a genuine coordinate-independent singularity. It is this former which appears at the Schwarzschild radius and is now understood as the event horizon of a black hole. Once this horizon is crossed, no signal can escape back out, and every future directed path leads inward. In finite proper time, the infalling observer reaches the true singularity. It is this remarkable structure that opened the door to the modern physical and philosophical questions surrounding black holes, and to the question of what it really means to fall into one.

1.5.3 The golden age of black hole physics

Many pursuits in gravitational physics were halted during the wartime efforts and also in the years that followed during the Cold War, both because theoretical physicists found work connected to wartime research and because scientific priorities were redirected elsewhere. There was, however, a certain rejuvenation of these questions starting in the 1960s with the clarifications laid out by Roger Penrose, who, using ideas from differential topology, formulated a much more mathematically rigorous understanding of the singularities appearing in general relativity [5].

This was a remarkable development, because the ingredients of the argument went beyond standard differential geometry alone by adding something simple but essential, namely the implications of the Einstein equations themselves: that matter tells geometry how to curve and geometry tells matter how to move. There had been many arguments that such singularities might be evaded if the collapsing matter were not perfectly symmetric to begin with. One could hope that the singular behaviour seen in the highly idealized solutions was only a consequence of too much symmetry, and that more realistic asymmetries might avoid it. Penrose's work changed that discussion dramatically by demonstrating that singularities arise even in the presence of such asymmetries.

This era of physics is usually referred to as the golden age of black hole physics, and it culminated in a series of important developments, many of them associated with Hawking and others working in the years that followed. Hawking himself had risen to prominence in part by being an early adopter of Penrose's ideas, and by formulating similar arguments in a cosmological setting [6]. In this way, the same kind of reasoning that had been applied to black hole singularities could also be used to argue that spacetime itself might be traced back to a point in the past beyond which it could not be extended further. This became the mathematical formulation of what has since then come to be known as the Big Bang singularity.

There have by now been many classical tests of general relativity, and many tests of black holes themselves. Gravitational waves have perhaps provided the most striking examples, with Kip Thorne, later awarded the 2017 Nobel Prize in Physics, offering some of the most illuminating descriptions of mergers of objects that are at least compact enough to satisfy the expected conditions of black holes. Likewise, measurements of the supermassive compact object at the center of our galaxy by Reinhard Genzel and Andrea Ghez, recognized together with Roger Penrose in the 2020 Nobel Prize in Physics, have given very strong support to the black hole picture.

Yet it is worth keeping in mind what these observations do, and do not, establish. They provide strong evidence for extremely compact objects with horizons, or at least with external properties very close to those expected of black holes, but they do not amount to a direct observation of a singularity at the center. For this reason, many discussions in high energy physics still treat the singularity not as something that should necessarily be taken literally, but rather as a sign that classical general relativity has reached the limits of its domain of validity. In that sense, even when certain ideas have entered the lore of physics, people still return to the earlier reasoning, rethink the assumptions that led to that lore, and ask whether those assumptions still hold once more data has been explored, or whether they might be modified in ways that reveal other interesting possibilities. Along these lines, one may consider horizonless ultra compact objects that mimic the external behaviour of black holes and can, in certain regimes, be thermodynamically favoured over more traditional singular solutions. Such proposals are often referred to as black hole mimickers.

1.5.4 Black hole thermodynamics and the world as a hologram

Next to the scene was Jacob Bekenstein, who suggested that the black hole horizon area should be understood as playing the role of entropy, noting the striking similarity that both tend to increase irreversibly over time [7]. In this analogy, the surface gravity plays the role of temperature, the mass plays the role of energy, and, most intriguingly of all, the area of the horizon plays the role of entropy.

At first, this may have looked like merely a suggestive formal analogy. But Hawking then took the idea seriously and showed that black holes are not merely analogous to thermodynamic systems but that they truly have a temperature and that they radiate as thermal objects [8, 9]. In his original calculation, one considers a quantum field in the classical spacetime background of a black hole formed in gravitational collapse. Treating the matter field quantum mechanically, Hawking found that the black hole emits particles with a thermal spectrum. Black holes, in other words, have a temperature.

In order to illustrate this, Hawking presented a somewhat naive argument, which he himself admitted was nothing more than a crude analogy but which nevertheless has become something of a standard explanation of the phenomenon. The argument goes something like this: Due to quantum fluctuations governed by Heisenberg's uncertainty principle, there can be brief moments where a burst of energy comes into the world as a pair production of a particle and its corresponding antiparticle. This can happen at the boundary of the black hole such that one particle is created just inside the black hole horizon while the other is created just outside it, and subsequently in some cases manages to escape. Over a long time, this means that the black hole loses a bit of its mass and slowly "evaporates". On closer inspection, this analogy is a bit strange, for even in two-dimensional settings, it is not how Hawking radiation actually works. Nevertheless, it can still serve as a useful intuitive first step toward the actual phenomenon.

This was a remarkable moment, because it suggested that black holes provide a unique window into quantum phenomena in gravitational physics. Classical general relativity alone would never have led one to expect that a black hole could radiate. On the contrary, Hawking's area theorem suggests that, under classical evolution and suitable energy conditions, the area of a black hole horizon can never decrease [10]. And yet once quantum field theory is brought into the story, the picture changes completely. The result is what is now collectively referred to as black hole thermodynamics.

A closely related idea appears in the Unruh effect. There, a uniformly accelerated observer in flat spacetime is predicted to detect thermal radiation, even though an inertial observer would describe the same state as empty space. Again, temperature appears in a setting where one might not have expected it, and again it seems to be tied very deeply to horizons, observers, and the structure of spacetime itself.

The most mysterious part of Hawking's result was perhaps not that black holes have a temperature, but that their entropy is proportional to the area of the horizon. This is a very strange scaling. In most ordinary systems, one expects the number of degrees of freedom to scale roughly with volume. Here, instead, the relevant quantity seemed to live on a surface. This observation became one of the central clues in later attempts to understand quantum gravity. A major step in this direction came from string theory, where Strominger and Vafa famously showed that for certain supersymmetric black holes one can account microscopically for the Bekenstein-Hawking entropy by counting underlying quantum states [11].

It was precisely this feature that inspired a radically new way of thinking about the structure of spacetime. In independent work by 't Hooft [12] and Susskind [13], an idea emerged that perhaps a gravitational system can in some sense be fully described by degrees of freedom living on a lower dimensional boundary. This became known as the holographic principle.

This idea was soon followed by Maldacena's 1997 conjecture in *"The Large N Limit of Superconformal Field Theories and Supergravity"*, now usually referred to as AdS/CFT [14]. In its original form, it states that type IIB string theory on $\mathrm{AdS}_5 \times S^5$ is dual to $\mathcal{N} = 4$ super Yang–Mills theory living on the boundary. In other words, a gravitational theory in the bulk may be exactly equivalent to a quantum theory without gravity in one lower dimension. This marked a profound shift in the way high energy physicists think about quantum gravity. Rather than trying to force gravity into quantum mechanics in the most direct possible way, one was led to the possibility that gravity and quantum theory may already be two complementary descriptions of the same system.

1.5.5 The Kitaev lectures: A simple model of quantum holography

In 2015, Alexei Kitaev held a series of blackboard lectures in which he explained to an enthusiastic audience a striking new development in the study of quantum gravity [15]. The central realization, later developed more systematically in the influential work of Maldacena and Stanford [16], was that the Sachdev-Ye-Kitaev model [17, 15], a quantum mechanical model with random couplings, admits a simple universal description at low energies in terms of the Schwarzian theory. Remarkably, the same Schwarzian action also arises in the boundary description of nearly AdS_2 gravity [18], most elegantly formulated in terms of Jackiw-Teitelboim gravity [19, 20].

Why did this generate so much excitement? Because it provided perhaps the simplest tractable setting in which ideas of holography could be studied in detail. One had analytical control over both sides of the duality: on the one hand a quantum mechanical many body system, and on the other a two dimensional theory of gravity. In this sense, the SYK/JT duality offered a concrete way to answer questions about black holes, entropy, quantum chaos, and holography which could now be explored with an unusual degree of precision.

The excitement surrounding this development did not come only from the fact that one had found a particularly simple model of holography. It also came from the realization that a concrete bridge had emerged between condensed matter physics and black hole physics. Questions that appear difficult on the quantum mechanical side may become more tractable when translated into gravitational language, while gravitational questions about entropy, chaos and low-energy dynamics may be illuminated in the boundary description. In this sense, the duality is not merely a formal correspondence but also a practical tool, allowing difficult problems to be viewed from the angle in which they become simplest. It also sharpened an old conceptual question that often appears in discussions of holography: whether the bulk gravitational description or the lower-dimensional quantum theory should be regarded as more fundamental, or whether the duality instead teaches us that this distinction is itself misleading.

This lower-dimensional perspective forms much of the conceptual background for the work in this thesis. My own research has focused primarily on lower-dimensional gravitational models and on theories with non-Lorentzian symmetry. In the next two subsections, I will therefore explain why these two themes, lower-dimensional gravity and non-Lorentzian physics, have become central to the line of research pursued here.

1.5.6 Why lower dimensional holography?

One may ask why so much attention is devoted to lower-dimensional gravity, given that the physical world is four-dimensional. There are at least two reasons. First, lower-dimensional models are more tractable, and this simplicity often makes conceptual structures visible that would otherwise remain hidden in the full higher-dimensional problem. In particular, they allow one to study backreaction, quantum effects, and effective boundary dynamics in a controlled setting [21]. Second, they are not merely toy models. In many important cases, they arise as effective descriptions of higher-dimensional systems. A well-known example is the appearance of Jackiw–Teitelboim gravity in the near-horizon throat of near-extremal Reissner–Nordström black holes, which we review in section (2.7). In this sense, two-dimensional dilaton gravity captures universal infrared features that are already present in more realistic gravitational settings.

A common feature of many two-dimensional gravity models is the appearance of an additional dilaton field. This field does not arise as an independent component of the two-dimensional metric, but instead encodes information that remains from the higher-dimensional geometry after dimensional reduction [22]. Another well-studied example is Callan–Giddings–Harvey–Strominger gravity (CGHS) [23], which has likewise been used for exploring conceptual issues in black hole physics and quantum gravity. In the second paper of this thesis [24], we explore the implications of a two-dimensional dilaton gravity model that interpolates between JT and CGHS. The interpolation is governed by a parameter that can be traced back to the dynamical scaling of the higher-dimensional Lifshitz black hole from which the model is obtained by dimensional reduction.

1.5.7 Why non-Lorentzian holography?

The success of relativistic holography naturally raises the question of whether similar ideas can be extended beyond the Lorentzian setting. There are at least two reasons to expect that they should. The first comes from condensed matter physics. While the JT/SYK correspondence provides a particularly elegant bridge between black hole physics and quantum many body systems, many systems realized in the laboratory do not exhibit relativistic conformal symmetry near criticality. Instead, they often display anisotropic scaling between time and space, of the kind associated with Lifshitz symmetry. In such systems, time and space no longer enter on equal footing, and one is led away from Lorentz invariance toward a broader non-Lorentzian framework.

A second motivation comes from gravity itself. Non-Lorentzian structures are not merely artificial deformations of relativistic physics, but arise naturally within it. Galilean geometry appears in non-relativistic limits of general relativity and provides the geometric framework underlying Newton-Cartan theory. The Carrollian limit corresponds to the ultra-relativistic limit $c \to 0$. This structure appears naturally on null hypersurfaces, in particular at null infinity and on black hole horizons. The term “Carrollian” was inspired by Lewis Carroll, more specifically by the peculiar causal intuition associated with the world of *Through the Looking-Glass*. See the following figure for a playful visual reference.

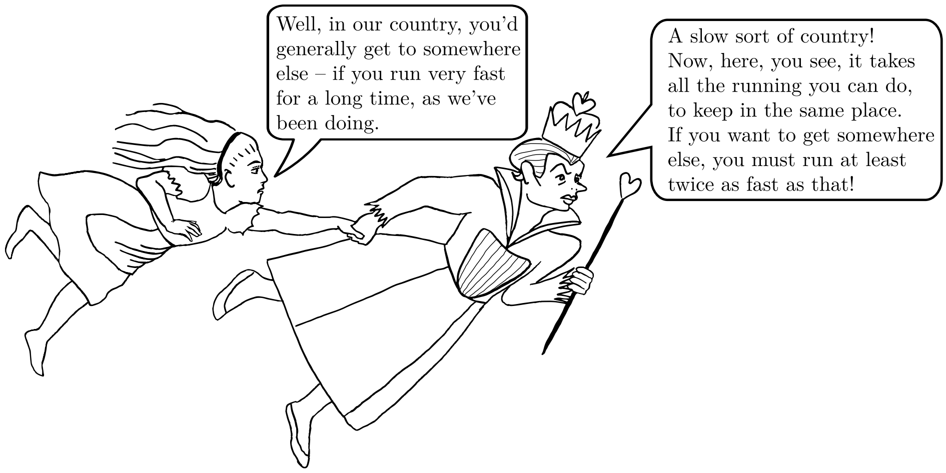

Figure 1.2. From Lewis Carroll's Red Queen's Race. We thank the artist, Álfheiður Edda Sigurðardóttir, for granting us permission to reproduce the figure in the thesis.

Lifshitz symmetry occupies a somewhat different role. Rather than being defined as a contraction of the relativistic symmetry group, it characterizes theories with anisotropic scaling between time and space, and for that reason is especially relevant in holographic descriptions of condensed matter theories. More broadly, non-Lorentzian physics may be viewed as an umbrella term that includes Galilean, Lifshitz and Carrollian structures. Galilean symmetry is the natural symmetry of non-relativistic spacetime, Lifshitz symmetry characterizes theories with anisotropic scaling between time and space, and Carrollian symmetry governs the opposite ultra-relativistic limit in which the speed of light is effectively taken to zero. These structures can be understood systematically through contractions and deformations of relativistic symmetry algebras [25, 26, 27].

The present thesis is motivated by the idea that some of the conceptual lessons of relativistic holography should persist in these non-Lorentzian regimes. In particular, the work explores how lower-dimensional gravitational models with non-Lorentzian symmetry capture aspects of black hole thermodynamics and boundary dynamics. In the first paper [24], we study Carrollian string theory and its extended symmetry structure. In the second paper [28], we investigate a two-dimensional dilaton gravity model related to Lifshitz holography, which in certain respects interpolates between JT and CGHS. The third paper then turns to non-relativistic de Sitter space and its associated Newton-Cartan geometry. Taken together, these works explore several corners of the non-Lorentzian landscape in which holography and gravitational thermodynamics can be extended beyond the purely relativistic setting.

1.5.8 Future directions

Let us conclude with a few remarks on possible future directions suggested by the work in this thesis. Each of the three papers points toward natural extensions, and in several cases the questions that remain open are at least as interesting as those that have already been addressed.

For the Carroll string model studied in the first paper [24], a natural next step would be to develop its quantization in more detail. Since the appearance of our work, a number of closely related developments have appeared in the literature [29, 30, 31, 32], suggesting that this line of research is both timely and structurally rich. In particular, it would be interesting to understand more precisely how our magnetic model is related to the actions analysed in these later works, and whether this relationship can be made exact through a suitable rewriting of the theory. From my perspective, the most interesting open question is whether the chiral version of the model can also be quantized by similar methods.

The Lifshitz dilaton model explored in the second paper [28] also opens several promising directions. One of the main motivations behind that work was to understand how the familiar lessons of JT gravity are modified once one moves into a non-Lorentzian setting. Our derivation of a Schwarzian-like boundary theory relied on the wiggling boundary construction, which proved to be a useful method in this context. It would be very interesting to compare this approach with the more recent program in which logarithmic corrections are extracted directly from bulk fluctuation operators, through the Lichnerowicz operator. A key question is how the quantum corrections can be recovered from the bulk point of view. More broadly, this model seems to provide a particularly clean arena in which to study quantum corrections to black hole thermodynamics away from the relativistic JT limit.

Finally, the most recent paper [33] suggests what may be a broader geometric reformulation of low-dimensional gravity itself. There we found a Newton-Cartan formulation for Jackiw-Teitelboim gravity derived directly from an action principle. If this perspective can be developed further, it may provide a new bridge between relativistic dilaton gravity and non-Lorentzian geometry. It would then be natural to ask how far this correspondence extends. In that sense, the result may open the door not only to a new way of viewing JT gravity, but also to a broader non-Lorentzian reformulation of familiar holographic models.

2 Quantum Corrections to Charged Black Hole Thermodynamics

There is a saying in mathematics that proofs found in research rarely resemble the polished versions presented in textbooks. Over time they are simplified and refined, much like stones in a river which are gradually smoothened by the current of the river over many generations. The ideas presented in this chapter are not new, rather, they represent concepts that have been shaped and refined through decades of work. The purpose of this chapter is therefore not to present new results, but to revisit and clarify a set of ideas that have become central to our modern understanding of black hole thermodynamics. Whether my writing can reflect the current of the river is then another matter but I hope that in paving this way, that at least I will be wiser about these connections!

Parts of this chapter are based on the lecture series *Black Hole Thermodynamics* given by Lárus Thorlacius at the GGI LACES school in Florence in 2025, as well as the lectures *Aspects of Low-Dimensional Quantum Gravity* given by Alejandra Castro at NBI in Copenhagen in the summer of 2024 in the summer school *Frontiers in gravity* organized by Troels Harmark and Niels Obers. Additional material is drawn from Grumiller's and Sheikh-Jabbari's excellent book *Black Hole Physics: From Collapse to Evaporation* and from the review article by Mertens and Turiaci, *Solvable Models of Quantum Black Holes: A Review on Jackiw–Teitelboim Gravity*. We also draw on several recent research papers, which are cited where relevant throughout the discussion. Parts of the presentation is also based on informal notes and calculations carried out with Diego Hidalgo, in particular concerning the first order formulation of gravity and its BF theory description. Some of the material furthermore grew out of a series of lectures given by my advisor, Watse Sybesma, to Davide Astesiano and myself in the late summer of 2023. More broadly, the perspective presented here was shaped by informal back-and-forth notes and many discussions with Watse over the course of my PhD, concerning various aspects of Jackiw–Teitelboim gravity and related structures.

We begin by reviewing the classical thermodynamic description of charged Reissner–Nordström black holes. We deviate slightly from the intended subject, and show some classical subtleties around the charged de Sitter solutions, summarizing them in the *shark fin diagram*. We then show that, at low temperatures and in the near-horizon limit of near-extremal solutions, the geometry factorizes into $\mathrm{AdS}_2 \times S^2$. The throat geometry admits a particularly elegant description since when integrating out the spherical part of the geometry one may dimensionally reduce the four-dimensional gravitational system to the two-dimensional Jackiw–Teitelboim model, where the additional degrees of freedom of the spherical geometry are captured by a so-called dilaton field. Finally, using the Schwarzian description of Jackiw–Teitelboim gravity, we review recent results showing how quantum gravitational effects modify the Hawking emission spectrum of near-extremal Reissner–Nordström black holes, leading to important corrections to the semiclassical prediction.

2.1 Classical Reissner–Nordström black holes

We begin with the four-dimensional Einstein–Maxwell action including a cosmological constant which will determine the asymptotic behaviour of the spacetime

$$S_{\rm EM} = \frac{1}{16\pi G_{\rm N}}\int_{\mathscr{M}} d^4x\sqrt{-g}(R-2\Lambda) - \frac{1}{4e^2}\int_{\mathscr{M}} d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu} \tag{1}$$

$$+\frac{1}{8\pi G_{\rm N}}\int_{\partial\mathscr{M}} d^3y\sqrt{-h}(K-K_0)\,. \tag{2}$$

Here g is the determinant of the spacetime metric, R denotes the Ricci scalar, $G_{\rm N}$ is Newton's constant. The field strength is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where $A_\mu = (\phi, \vec{A})$ is the gauge potential, and e is the associated gauge coupling of the $U(1)$ Maxwell field. The final boundary term is the Gibbons–Hawking–York boundary term which renders the variational problem well defined and it is supplemented with the counterterm K_0 which ensures that the on shell action remains finite. The quantity h denotes the determinant of the induced metric on the boundary $\partial\mathscr{M}$, typically evaluated at a large radial cutoff surface $r = r_0$ with $r_0 \to \infty$.

In four dimensions it is convenient to parameterize the cosmological constant as

$$\Lambda = \begin{cases} 0 & \text{asymptotically flat (Minkowski)}\,, \\ -\dfrac{3}{L^2} & \text{asymptotically Anti-de Sitter (AdS)}\,, \\ \dfrac{3}{\ell^2} & \text{asymptotically de Sitter (dS)}\,. \end{cases} \tag{3}$$

We denote the AdS length scale by L and the de Sitter length scale by ℓ. Although these parameters play analogous roles, they are related by the analytic continuation $L = i\ell$. Taking $L \to \infty$ reproduces the asymptotically flat limit.

Varying the Einstein-Maxwell action with respect to the metric and the gauge field, $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$ and $A_\mu \to A_\mu + \delta A_\mu$, one obtains the Einstein–Maxwell equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G_{\rm N}}{e^2}\left(F_{\mu\lambda}F_\nu{}^\lambda - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{4}$$

$$\nabla^\mu F_{\mu\nu} = 0. \tag{5}$$

The Einstein–Maxwell equations admit the Reissner–Nordström family of black hole solutions. For a purely electric configuration the solution takes the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2\,, \qquad f(r) = 1 - \frac{2G_{\rm N}M}{r} + \frac{G_{\rm N}Q^2}{r^2} - \frac{\Lambda}{3}r^2\,, \tag{6}$$

$$A = \frac{e}{2\sqrt{\pi}}\left(\mu - \frac{Q}{r}\right)dt. \tag{7}$$

Here $d\Omega_2^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ denotes the metric of the unit two sphere. The constant μ will later be identified with the chemical potential of the black hole. For now, let us simply note that regularity of the gauge potential at the horizon requires $A_t(r_+) = 0$, which fixes the chemical potential to be

$$\mu = \frac{Q}{r_+}\,. \tag{8}$$

The symmetries of the spacetime are generated by four Killing vectors. These are the same as for the Schwarzschild solution, since both spacetimes are static and spherically symmetric. The Killing vectors are

$$\xi = \partial_t\,, \quad L_z = \partial_\phi\,, \quad L_x = \cos\phi\,\partial_\theta - \cot\theta\sin\phi\,\partial_\phi\,, \quad L_y = \sin\phi\,\partial_\theta + \cos\phi\cot\theta\,\partial_\phi\,. \tag{9}$$

These satisfy the algebra

$$[L_x, L_y] = L_z\,, \quad [L_y, L_z] = L_x\,, \quad [L_z, L_x] = L_y\,, \tag{10}$$

while the time translation Killing vector ξ commutes with all the rotational generators. We recognize this as the Lie algebra of $SO(3)\times U(1)$ where the $U(1)$ factor corresponds to time translations.

Finally, let us mention that a second solution to the Einstein–Maxwell equations exists corresponding to a purely magnetic charge. In this case the Maxwell field is constrained on the two sphere and the solution takes the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2\,, \quad f(r) = 1 - \frac{2G_{\mathrm{N}}M}{r} + \frac{G_{\mathrm{N}}P^2}{r^2} - \frac{\Lambda}{3}r^2\,, \tag{11}$$

$$A = -\frac{eP}{2\sqrt{\pi}}\cos\theta\, d\phi\,. \tag{12}$$

The electric and magnetic solutions may be combined to form the general dyonic Reissner–Nordström solution with charge parameters Q and P, in which case the metric depends only on the total electromagnetic charge Q^2+P^2.

We highlight this since when deriving Jackiw–Teitelboim gravity it will be convenient to consider the purely magnetic solution. The reason for doing so is that in this case the magnetic flux lies entirely on the two sphere and does not depend on the radial coordinate, which greatly simplifies the dimensional reduction.

2.2 The horizon structure of the blackening factor

Since the theory contains a large parameter space spanned by (M, Q, Λ), it is useful to first examine several representative limits. We will briefly discuss a few characteristic corners of this parameter space and later summarize their relations in the so called "shark fin" diagram for the Reissner–Nordström–de Sitter solution.

It is instructive to begin with the case where the charge vanishes, $Q = 0$. In this limit the solutions reduce to the Schwarzschild family, whose qualitative behaviour depends on the value of the cosmological constant. The different possibilities are illustrated schematically in the diagram below.

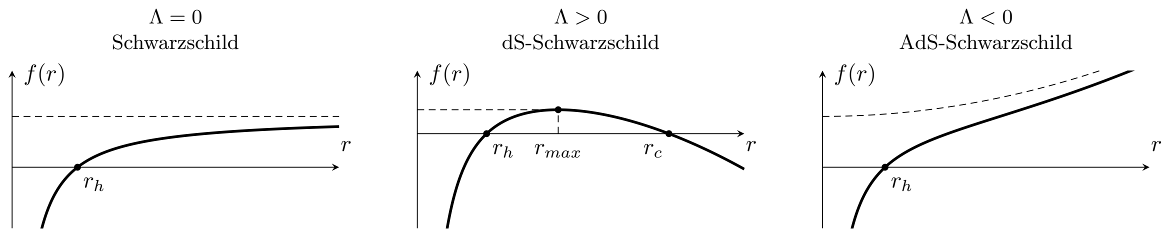

Figure 2.1. Schematic form of the blackening factor $f(r)$ for vanishing charge.

Note that in the absence of charge the asymptotically flat solution admits a single horizon located at $r_h = 2G_\mathrm{N}M$. For $\Lambda < 0$ (the asymptotically AdS case) the function $f(r)$ is monotonically increasing for $r > 0$, and therefore admits a single real root corresponding to the black hole horizon. For $\Lambda > 0$ (the de Sitter case) the function $f(r)$ develops a maximum, which allows the cubic equation to admit three real roots. One of these is always negative and therefore unphysical, while the remaining two correspond to the black hole and cosmological horizons respectively.

For the dS–Schwarzschild solution the maximum of $f(r)$ occurs at $r_\mathrm{max} = \left(\frac{3G_\mathrm{N}M}{\Lambda}\right)^{1/3}$. A necessary condition for the existence of horizons is therefore

$$\Lambda \leq \frac{1}{(3G_\mathrm{N}M)^2}\,. \tag{13}$$

The point where the two horizons coincide is known as the *Nariai* solution and it is simple to derive by demanding that $\Lambda = \frac{1}{(3G_\mathrm{N}M)^2}$ and $r_h = r_\mathrm{max}$ in which case one finds

$$f_\mathrm{Nariai}(r) = -\frac{\Lambda}{3r}(r-r_h)^2(r+2r_h)\,, \qquad r_h = \frac{1}{\sqrt{\Lambda}}\,, \quad G_\mathrm{N}M = \frac{1}{3\sqrt{\Lambda}}\,. \tag{14}$$

We now turn to the charged case. The corresponding possibilities are illustrated schematically in the diagram below.

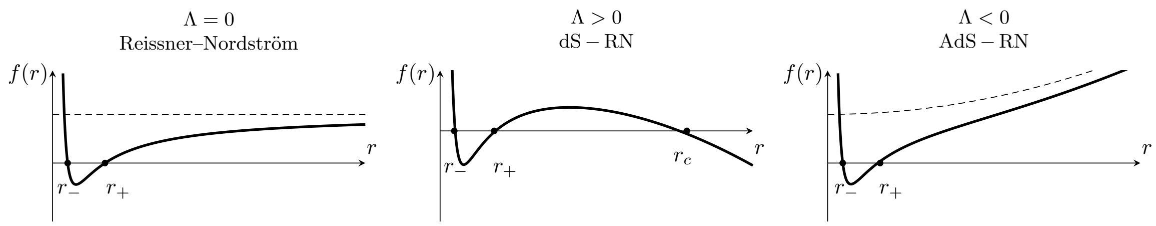

Figure 2.2. Schematic form of the blackening factor $f(r)$ for non-vanishing charge.

Note that when the charge is nonzero the qualitative behaviour of the blackening factor depends on the sign of the cosmological constant.

For $\Lambda = 0$ (Reissner–Nordström) the function $f(r)$ admits two positive roots corresponding to the inner Cauchy horizon, r_-, and the outer event horizon, r_+, given by

$$r_\pm = G_\mathrm{N}M \pm \sqrt{G_\mathrm{N}^2M^2 - G_\mathrm{N}Q^2}. \tag{15}$$

For $\Lambda < 0$ (AdS-RN) the function again admits two positive roots corresponding to the inner Cauchy horizon r_- and the outer black hole horizon r_+. For $\Lambda > 0$ (dS-RN) the function can admit three positive roots, corresponding to the Cauchy horizon r_-, the black hole horizon r_+ and the cosmological horizon r_c.

Let us finally derive the so-called ultracold solution for dS–RN black holes[1], corresponding to the case where the three horizons coincide, $r_- = r_+ = r_c$. Equivalently, the blackening factor must develop a triple root at $r = r_+$, i.e. $f(r_+) = f'(r_+) = f''(r_+) = 0$.

[1]We thank Lárus Thorlacius for explaining an elegant form of this derivation to us.

Hence starting from the blackening factor $f(r) = 1 - \frac{2G_N M}{r} + \frac{G_N Q^2}{r^2} - \frac{\Lambda}{3} r^2$, we note that using the horizon condition $f(r_+) = 0$, we may eliminate the mass parameter:

$$2G_N M = r_+ \left(1 + \frac{G_N Q^2}{r_+^2} - \frac{\Lambda}{3} r_+^2\right). \tag{16}$$

Substituting this back into $f(r)$ gives

$$f(r) = -\frac{(r - r_+)}{3r^2} \left(\Lambda r^3 + \Lambda r_+ r^2 + (\Lambda r_+^2 - 3) r + \frac{3G_N Q^2}{r_+}\right). \tag{17}$$

A double root at $r = r_+$ requires the cubic factor to vanish at $r = r_+$, which gives

$$G_N Q^2 = r_+^2 - \Lambda r_+^4. \tag{18}$$

The blackening factor then becomes $f(r) = -\frac{(r-r_+)^2}{3r^2} \left(\Lambda r^2 + 2\Lambda r_+ r + 3\Lambda r_+^2 - 3\right)$. To obtain a triple root, the quadratic factor must also vanish at $r = r_+$, which requires $6\Lambda r_+^2 - 3 = 0$ and hence $r_+ = \frac{1}{\sqrt{2\Lambda}}$ Substituting this back in, one concludes that the ultracold solution is

$$f_{\mathrm{uc}}(r) = -\frac{\Lambda (r - r_+)^3}{3r^2} (r + 3r_+), \quad r_+ = \frac{1}{\sqrt{2\Lambda}}, \quad G_N Q^2 = \frac{1}{4\Lambda}, \quad G_N M = \frac{2}{3\sqrt{2\Lambda}}. \tag{19}$$

We summarize the resulting parameter space in the following figure

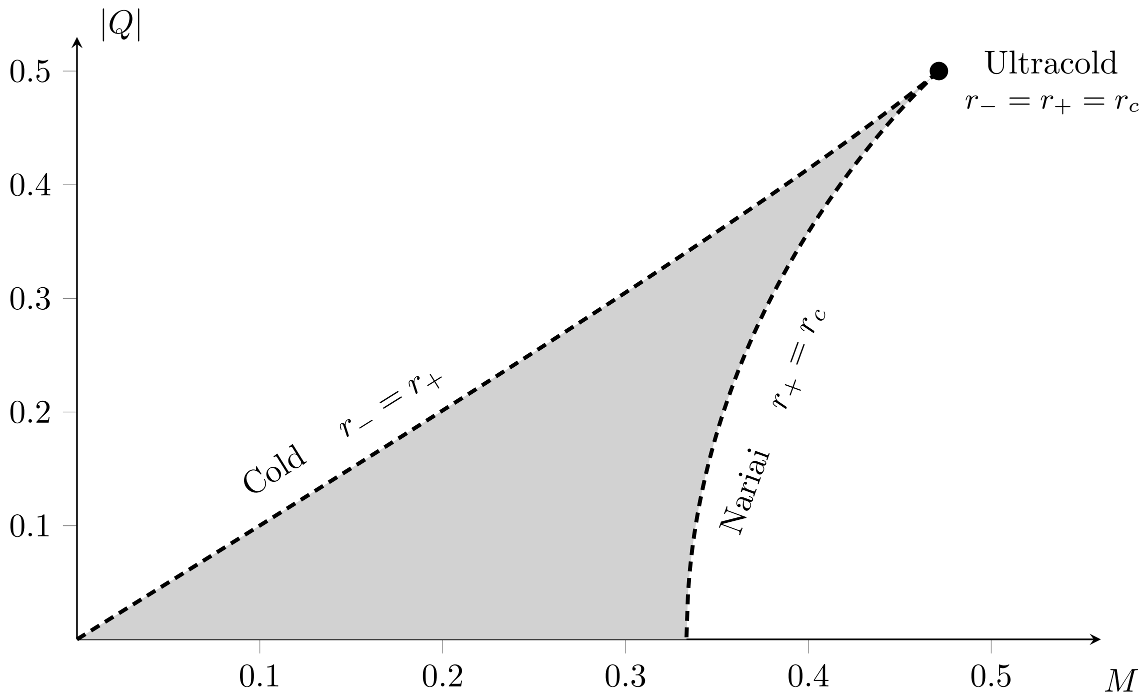

Figure 2.3. The shark fin diagram for dS–*RN black holes for* $G_N = \Lambda = 1$.

The boundaries of the shaded region can be obtained parametrically for $(|Q|, M)$ using equations (16) and (18). Requiring the charge to be real implies $0 \le r_+ \le \frac{1}{\sqrt{\Lambda}}$, which yields

$$(M; |Q|) = \left(\frac{r_+}{2G_N}\left(1 + \frac{\Lambda r_+^2}{3}\right); \sqrt{\frac{r_+^2}{G_N}\left(1 - \Lambda r_+^2\right)}\right), \qquad 0 \le r_+ \le \frac{1}{\sqrt{\Lambda}}. \tag{20}$$

2.3 Classical black hole thermodynamics

We now briefly summarize the basic laws of classical black hole thermodynamics. A central quantity is the surface gravity κ, which is associated with the Killing horizon generated by a Killing vector ξ^μ. On the horizon this vector becomes null, $\xi^\mu \xi_\mu = 0$, and the surface gravity may be defined invariantly by

$$\xi^\nu \nabla_\nu \xi^\mu = \kappa \xi^\mu \,. \tag{21}$$

There is also a useful equivalent expression for the surface gravity,

$$\kappa^2 = -\frac{1}{2} \left(\nabla_\mu \xi_\nu\right) \left(\nabla^\mu \xi^\nu\right) \Big|_{r=r_+} \,. \tag{22}$$

For static black holes the Killing vector generating the horizon coincides with the time translation symmetry, $\xi = \partial_t$, which is null at the horizon. For static, spherically symmetric metrics, $ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2$ this general expression for the surface gravity simplifies to

$$\kappa = \frac{1}{2} f'(r_+) \,. \tag{23}$$

To determine the temperature associated with the horizon we Wick rotate the time coordinate $t \to -i\tau$ and study the geometry near the horizon. Introducing a new near-horizon coordinate $r = r_+ + \frac{\kappa}{2}\rho^2$, the metric takes the form

$$ds^2 = d\rho^2 + \kappa^2 \rho^2 d\tau^2 + r_+^2 d\Omega_2^2 \,. \tag{24}$$

The (ρ, τ) part of the metric is therefore locally flat space written in polar coordinates with angular variable $\theta = \kappa\tau$. Regularity of the geometry at $\rho = 0$ requires the absence of a conical singularity, which implies that θ must have period 2π. Consequently the Euclidean time coordinate must be periodic, $\tau \sim \tau + \beta$, with $\beta = \frac{2\pi}{\kappa}$. Identifying the period β with the inverse temperature yields the Hawking temperature

$$T_{\mathrm{H}} = \frac{\kappa}{2\pi} \,. \tag{25}$$

Now consider a process in which the black hole absorbs a charged particle with mass $\delta M \ll M$ and charge $\delta Q \ll Q$. This means that the event horizon will now move to $r_+ \to r_+ + \delta r_+$ and we may find that to first order in the variations

$$0 = f_{M+\delta M, Q+\delta Q}(r_+ + \delta r_+) = f_{M,Q}(r_+) + \frac{\partial f(r_+)}{\partial Q}\delta Q + \frac{\partial f(r_+)}{\partial M}\delta M + \frac{\partial f(r_+)}{\partial r_+}\delta r_+ \,. \tag{26}$$

Since by assumption $f_{M,Q}(r_+) = 0$, solving for δM gives

$$\delta M = \frac{Q}{r_+}\delta Q + \frac{f'(r_+)}{2G_{\mathrm{N}}} r_+ \delta r_+ = \mu \delta Q + \frac{\kappa}{8\pi G_{\mathrm{N}}} \delta A_{\mathrm{H}} \,. \tag{27}$$

Here we recognised the chemical potential from equation (8) and the surface gravity from equation (23), using that $\delta A_{\mathrm{H}} = 8\pi r_+ \delta r_+$. This is the first law of black hole thermodynamics for a static charged black hole.

The argument presented above is intentionally kept quite general and applies to static, spherically symmetric black holes in a wide class of theories. The final expression resembles the first law of ordinary thermodynamics,

$$\delta E = \mu\, \delta Q + T\, \delta S\,. \tag{28}$$

Since we have already identified the temperature, this comparison allows us to define the Bekenstein–Hawking entropy of the black hole. Comparing the two expressions $\frac{\kappa}{8\pi G_{\mathrm{N}}}\,\delta A_{\mathrm{H}}$ and $T_{\mathrm{H}}\,\delta S_{\mathrm{BH}}$, we see that the Bekenstein–Hawking entropy of the black hole must be

$$S_{\mathrm{BH}} = \frac{A_{\mathrm{H}}}{4G_{\mathrm{N}}}\,. \tag{29}$$

Here A_{H} denotes the area of the black hole horizon. In the next section we will derive the entropy in a more formal way by evaluating the on-shell gravitational action.

The entropy also admits a natural thermodynamic interpretation. In the canonical ensemble, where the charge is held fixed, the free energy is defined such that

$$F = M - T_{\mathrm{H}}\, S_{\mathrm{BH}}\,. \tag{30}$$

One may also consider the grand canonical ensemble, where the chemical potential μ is held fixed. In the following we will mainly work in the canonical ensemble. Thermodynamic stability is then determined by the heat capacity

$$C = \frac{dM}{dT_{\mathrm{H}}}\,, \tag{31}$$

which measures the response of the black hole mass to changes in the temperature. A positive heat capacity indicates thermodynamic stability, while a negative heat capacity signals an instability.

In what remains of this section, we explore some of the thermodynamic behaviour of the Reissner–Nordström family in different regimes of the cosmological constant. As a warm up, it is useful to first consider the uncharged case, namely the cosmological Schwarzschild solutions. These already display several important qualitative features that will reappear in the charged case.

For $Q = 0$, the Hawking temperature is given in terms of r_h as

$$T_{\mathrm{H}}(r_h) = \frac{1}{4\pi r_h} - \frac{\Lambda}{4\pi} r_h\,. \tag{32}$$

Here the mass parameter has been eliminated using the horizon condition $f(r_h) = 0$. Depending on the sign of Λ, the Hawking temperature exhibits qualitatively different behaviours

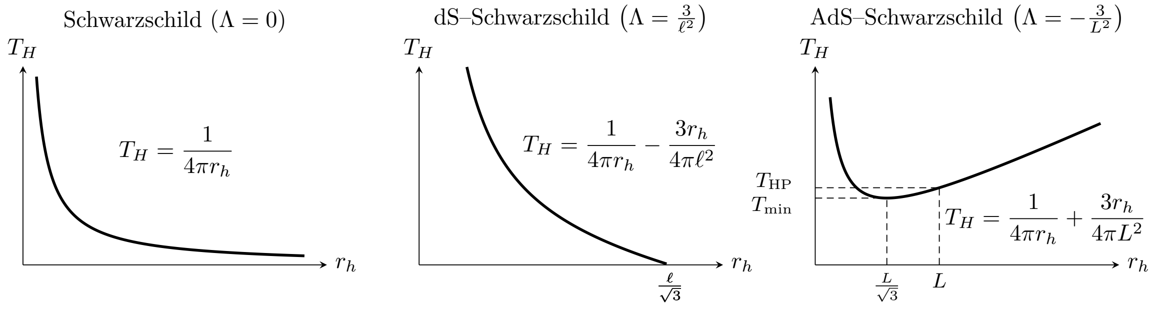

Figure 2.4. Hawking temperature as a function of the horizon radius for $Q = 0$.

For asymptotically flat Schwarzschild, the temperature decreases monotonically with increasing horizon radius. For dS Schwarzschild, the temperature also decreases, now vanishing at the maximal black hole radius. For AdS Schwarzschild, however, the temperature develops a minimum, so that for sufficiently large temperatures there are two black hole branches. The minimum occurs at $r_h = \frac{L}{\sqrt{3}}$, with $T_{\min} = \frac{\sqrt{3}}{2\pi L}$. Hence, for $T < T_{\min}$ there exists no AdS Schwarzschild black hole, while for $T > T_{\min}$ there are two branches: a small black hole branch and a large black hole branch. To determine which configuration is thermodynamically preferred, one studies the free energy,

$$F(r_h) = M - T_{\mathrm{H}} S_{\mathrm{BH}} = \frac{r_h}{4G_{\mathrm{N}}}\left(1 - \frac{r_h^2}{L^2}\right). \tag{33}$$

We note that the free energy changes sign at $r_h = L$ which corresponds to the Hawking–Page temperature

$$T_{\mathrm{HP}} = \frac{1}{\pi L}. \tag{34}$$

For $T < T_{\mathrm{HP}}$, thermal AdS is thermodynamically preferred, whereas for $T > T_{\mathrm{HP}}$, the large AdS black hole dominates. This is the Hawking–Page phase transition [34].

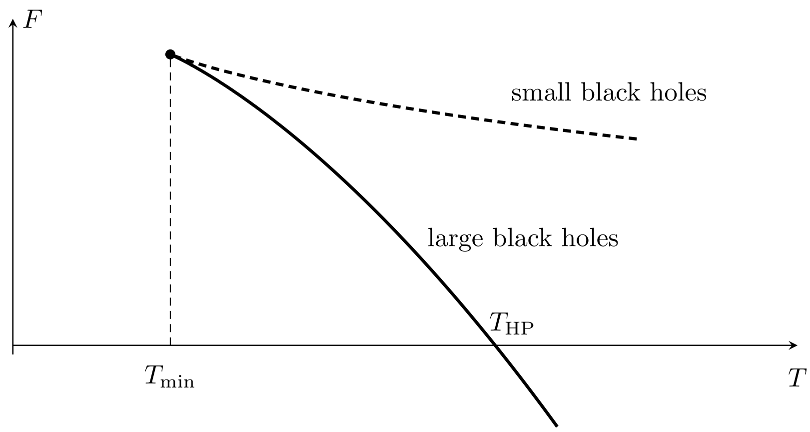

Figure 2.5. Hawking–Page transition for AdS-Schwarzschild black holes.

Where in order to draw the diagram we have used the parametric form $(T;F)$

$$(T;F) = \left(\frac{1}{4\pi r_h} + \frac{3r_h}{4\pi L^2};\ \frac{r_h}{4G_{\mathrm{N}}}\left(1 - \frac{r_h^2}{L^2}\right)\right), \qquad r_h > 0. \tag{35}$$

The thermodynamic stability can be analyzed from the heat capacity $C = dM/dT$. Since both M and T are functions of r_h, it is convenient to write

$$C(r_h) = \frac{dM}{dT} = \frac{dM/dr_h}{dT/dr_h} = -\frac{2\pi r_h^2}{G_{\mathrm{N}}}\frac{L^2 + 3r_h^2}{L^2 - 3r_h^2}. \tag{36}$$

The heat capacity diverges at $r_h = L/\sqrt{3}$, corresponding to the minimum temperature $T_{\min}$. Thus the small black hole branch ($r_h < L/\sqrt{3}$) has negative heat capacity and is thermodynamically unstable, whereas the large black hole branch ($r_h > L/\sqrt{3}$) has positive heat capacity.

Next we turn to the charged case, namely the Reissner–Nordström family. The Hawking temperature, free energy and heat capacity are expressed in terms of the horizon radius as

$$T_{\mathrm{H}}(r_+) = \frac{1}{4\pi r_+} - \frac{\Lambda r_+}{4\pi} - \frac{GQ^2}{4\pi r_+^3}\,, \tag{37}$$

$$F(r_+) = \frac{1}{4Gr_+}\left(r_+^2 - \frac{\Lambda}{3}r_+^4 + 3GQ^2\right)\,, \tag{38}$$

$$C(r_+) = \frac{2\pi r_+^2}{G}\frac{GQ^2 - r_+^2 + \Lambda r_+^4}{-3GQ^2 + r_+^2 + \Lambda r_+^4}\,. \tag{39}$$

In contrast to the uncharged case, the temperature now vanishes at an extremal radius r_{ext}, where the inner Cauchy horizon and the outer horizon coincide, $r_{\mathrm{ext}} = r_- = r_+$. For $r_+ > r_{\mathrm{ext}}$ the black hole has nonzero temperature and several thermodynamic branches may appear depending on the value of the cosmological constant. In the de Sitter case there exists an additional radius r_{N} where the temperature vanishes, corresponding to the Nariai limit where the black hole horizon and cosmological horizon coincide. The qualitative behaviour of the Hawking temperature as a function of the horizon radius is shown in the following figures

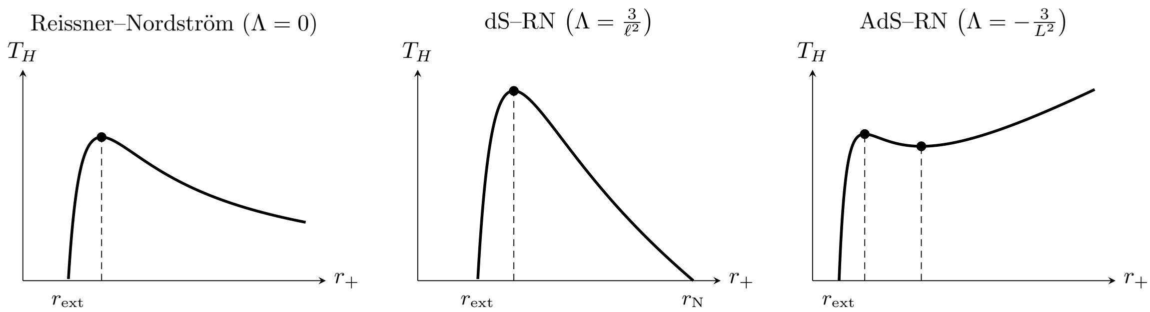

Figure 2.6. Hawking temperature as a function of horizon radius for charged black holes.

To determine which configuration is thermodynamically preferred one studies the free energy as a function of the temperature. For Reissner–Nordström and dS RN black holes the free energy remains positive and we find the following diagrams:

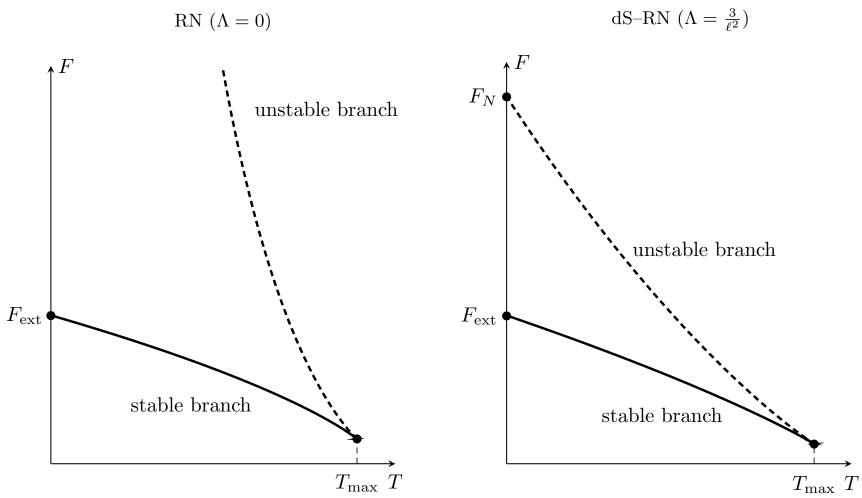

Figure 2.7. Free energy as a function of temperature for RN and dS-RN black holes.

The situation is qualitatively different in anti–de Sitter space. For AdS–RN black holes the free energy can become negative for sufficiently large horizon radius, leading to a phase transition between small and large black holes. This general feature holds irrespective of the charge. For sufficiently small charge the free energy develops a characteristic *swallowtail* structure (for further discussion see [35]).

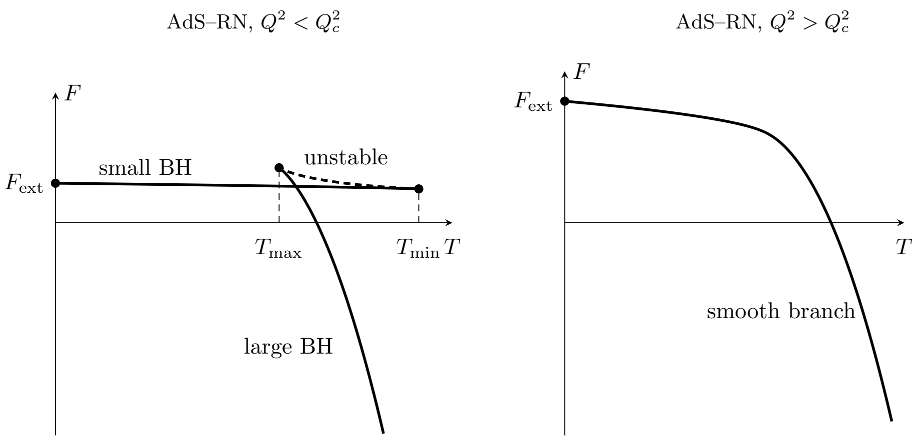

Figure 2.8. Free energy phase diagram for AdS–RN black holes.

The structure of the phase diagram depends on the charge. As the charge increases the swallowtail shrinks and eventually disappears at a critical value Q_c. This occurs when the local maximum and minimum of the temperature merge, which happens at [36]

$$r_c = \frac{L}{\sqrt{6}}, \qquad Q_c = \frac{L}{6\sqrt{G_\mathrm{N}}}, \qquad T_c = \frac{\sqrt{6}}{3\pi L}. \tag{40}$$

For a more detailed analysis see [36], where the cosmological constant is interpreted as a thermodynamic pressure and the resulting phase structure is shown to be analogous to that of the Van der Waals system.

2.4 The Euclidean on-shell action

In this section we formalize the derivation of the black hole entropy from the perspective of semiclassical gravity. The central object is the Euclidean gravitational path integral, which defines the thermal partition function of the theory. Following the approach of Gibbons and Hawking [37], we derive the black hole entropy by evaluating the Euclidean action on shell. The full gravitational path integral consists of doing the functional integral over Euclidean metrics with periodic Euclidean time of period β,

$$Z[\beta] = \int [\mathscr{D}g]\, e^{-S_E[g]}. \tag{41}$$

Strictly speaking the Euclidean gravitational path integral is not well defined since conformal transformations can make the action arbitrarily negative [38]. Nevertheless, in the semiclassical limit one may formally evaluate it by expanding around the classical saddle points. The partition function then takes the schematic form

$$Z[\beta] \sim \sum_{\text{saddles}} \exp\left(-\frac{1}{\hbar} I^{(0)} + I^{(1)} + \hbar I^{(2)} + \cdots\right), \tag{42}$$

where $I^{(0)}$ denotes the classical (tree level) contribution, which in the present case is simply the Euclidean action evaluated on the classical saddle,

$$I^{(0)} = S_E[g_\star]. \tag{43}$$

The quantities $I^{(1)}$ and $I^{(2)}$ correspond to the one loop and two loop quantum corrections respectively. In the classical limit $\hbar \to 0$ the dominant contribution comes from the classical saddle, $g_\star$, and the partition function reduces to

$$Z[\beta] \approx e^{-S_E[g_\star]}. \tag{44}$$

Our task, in this section, is therefore to evaluate the Euclidean action

$$S_{\rm E} = S_{\rm bulk} + S_{\rm F} + S_{\rm bdry} = -\frac{1}{16\pi G_{\rm N}} \int_{\mathscr{M}} d^4x\sqrt{g}(R-2\Lambda) + \frac{1}{4e^2}\int_{\mathscr{M}} d^4x\sqrt{g}F_{\mu\nu}F^{\mu\nu} \tag{45}$$

$$- \frac{1}{8\pi G_{\rm N}} \int_{\partial\mathscr{M}} d^3y\sqrt{h}(K-K_0)\,. \tag{46}$$

which we have decomposed into a bulk contribution $S_{\rm bulk}$, the Maxwell term $S_{\rm F}$, and the boundary term $S_{\rm bdry}$, which contains both the Gibbons–Hawking–York term and the counterterm K_0. The latter will be chosen so that the on-shell action is finite.

For concreteness we carry out the following derivation in the AdS case, with $\Lambda = -\frac{3}{L^2}$. The corresponding Reissner–Nordström solution (6) is therefore characterized by the blackening factor and gauge field

$$f(r) = 1 - \frac{2G_{\rm N}M}{r} + \frac{G_{\rm N}Q^2}{r^2} + \frac{r^2}{L^2}\,, \qquad A = \frac{e}{2\sqrt{\pi}}\left(\mu - \frac{Q}{r}\right)dt\,, \tag{47}$$

so that the only non-vanishing component of the Maxwell field strength is $F_{rt} = \frac{eQ}{2\sqrt{\pi}r^2}$. In this case the Euclidean geometry extends from the outer horizon r_+ to a large cutoff r_0, which is eventually sent to infinity. The asymptotically flat result is recovered in the limit $L \to \infty$. Evaluating the boundary contribution on a constant radial slice $r = r_0$, one finds[2]

$$R = \frac{2}{r^2} - \frac{2f(r)}{r^2} - \frac{4f'(r)}{r} - f''(r)\,, \quad \sqrt{g} = r^2\sin\theta\,, \quad F^2 = \frac{e^2Q^2}{2\pi r^4}\,, \tag{48}$$

$$K = \frac{2\sqrt{f(r_0)}}{r_0} + \frac{f'(r_0)}{2\sqrt{f(r_0)}}\,, \quad \sqrt{h} = r_0^2\sin\theta\sqrt{f(r_0)}\,. \tag{49}$$

Inserting all of these expressions into the action and evaluating the spherical integrals we find

$$S_{\rm bulk} = -\frac{1}{16\pi G_{\rm N}}\int_{r_+}^{\infty} dr \int_0^{\beta} d\tau \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\, \sin\theta\left(2 - 2f - 4rf' - r^2f'' - 2\Lambda r^2\right) \tag{50}$$

$$= \lim_{r_0\to\infty}\left\{\frac{3}{8\pi G_{\rm N}L^2}\int_{r_+}^{r_0} dr \int_0^{\beta} d\tau \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\, r^2\sin\theta\right\} = \lim_{r_0\to\infty}\left\{\frac{\beta}{2GL^2}(r_0^3 - r_+^3)\right\}. \tag{51}$$

[2]For the metric $ds^2 = f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2d\Omega_2^2$, the induced metric on the hypersurface $r = r_0$ is given by $h_{ab}dy^a dy^b = f(r_0)d\tau^2 + r_0^2 d\Omega_2^2$. The outward-pointing unit normal is $n = \sqrt{f(r)}\,\partial_r$. The trace of the extrinsic curvature is defined by $K = \nabla_\mu n^\mu$, where ∇_μ is the covariant derivative with respect to $g_{\mu\nu}$.

For the Maxwell field we find

$$S_{\mathrm{F}}=\frac{1}{4e^2}\int_{r_+}^{\infty}dr\int_0^{\beta}d\tau\int_0^{\pi}d\theta\int_0^{2\pi}d\phi\, r^2\sin\theta\frac{e^2Q^2}{2\pi r^4}=\lim_{r_0\to\infty}\left\{\frac{Q^2\beta}{2}\left(\frac{1}{r_+}-\frac{1}{r_0}\right)\right\}=\frac{Q^2\beta}{2r_+}. \tag{52}$$

For the boundary GHY and counter term we find

$$S_{\mathrm{bdry}}=\lim_{r_0\to\infty}\left\{-\frac{1}{8\pi G_{\mathrm{N}}}\int_0^{\beta}d\tau\int_0^{\pi}d\theta\int_0^{2\pi}d\phi\, r_0^2\sin\theta\left(\frac{2f(r_0)}{r_0}+\frac{1}{2}f'(r_0)-K_0\sqrt{f(r_0)}\right)\right\} \tag{53}$$

$$=\lim_{r_0\to\infty}\frac{\beta L}{2G}\left\{\left(K_0-\frac{3}{L}\right)\left(\frac{r_0^3}{L^2}-GM\right)+\left(K_0-\frac{4}{L}\right)\frac{r_0}{2}+\mathscr{O}\left(\frac{1}{r_0}\right)\right\}. \tag{54}$$

Where in the second line we have expanded the boundary term for large r_0. To cancel the divergences we must make a suitable choice for the counter term,

$$K_0=\frac{2}{L}+\frac{L}{2}\mathscr{R}[h], \tag{55}$$

where we defined $\mathscr{R}[h]=\frac{2}{r_0^2}$ which is the Ricci scalar of the induced boundary metric $h_{ab}dy^a dy^b=f(r_0)d\tau^2+r_0^2d\Omega_2^2$. With this choice the linear divergence vanishes, and the cubic divergence in S_{bdry} cancels exactly against the corresponding term in S_{bulk}. The remaining finite contribution yields the on-shell action

$$S_E[g_\star]=S_{\mathrm{bulk}}+S_{\mathrm{F}}+S_{\mathrm{bdry}}=\frac{\beta r_+}{2G}\left(\frac{GM}{r_+}+\frac{GQ^2}{r_+^2}-\frac{r_+^2}{L^2}\right)=-\frac{\beta r_+^2}{4G}f'(r_+). \tag{56}$$

Using the relations $\beta=\frac{2\pi}{\kappa}$, $\kappa=\frac{1}{2}f'(r_+)$ and $A_{\mathrm{H}}=4\pi r_+^2$, this expression may be written equivalently as $S_E[g_\star]=-\frac{A_{\mathrm{H}}}{4G}$. The semiclassical partition function therefore takes the form

$$Z(\beta)\approx e^{-S_E[g_\star]}. \tag{57}$$

The partition function is related to the free energy through $Z[\beta]=e^{-\beta F}$. Comparing this with the expression in (57) we therefore identify

$$F=\frac{S_E[g_\star]}{\beta}=-\frac{r_+^2}{4G}f'(r_+)=\frac{1}{4Gr_+}\left(r_+^2-\frac{\Lambda}{3}r_+^4+3GQ^2\right), \tag{58}$$

which agrees with the free energy obtained previously in (38). Finally the entropy follows from the thermodynamic identity $F=M-TS$. Solving for the entropy yields

$$S_{\mathrm{BH}}=\beta(M-F)=\frac{A_{\mathrm{H}}}{4G}, \tag{59}$$

which is precisely the Bekenstein–Hawking entropy that we used in the previous section. Equivalently, using $F=-\beta^{-1}\log Z$, and $S_{\mathrm{BH}}=-\frac{\partial F}{\partial T_{\mathrm{H}}}=\beta^2\frac{\partial F}{\partial\beta}$ the entropy may be written as

$$S_{\mathrm{BH}}=(1-\beta\partial_\beta)\log Z(\beta). \tag{60}$$

2.5 Near-Extremal Reissner-Nordström Black Holes

We now focus on the magnetically charged Reissner–Nordström solution with vanishing cosmological constant, $\Lambda = 0$, and work in units where $G_{\mathrm{N}} = 1$. As seen in Figure 2.6, the Hawking temperature vanishes at a particular value of the horizon radius where the inner and outer horizons coincide, $r_{\mathrm{ext}} = r_- = r_+$. This corresponds to an extremal black hole with Hawking temperature $T_{\mathrm{H}} = 0$. For later convenience, we consider the purely magnetic solution. In this case the electromagnetic flux is supported entirely on the two-sphere and does not depend on the radial coordinate, which simplifies the dimensional reduction leading to Jackiw–Teitelboim gravity.

To remind the reader, the corresponding spherically symmetric solution with magnetic charge $P > 0$, mass M, and $\Lambda = 0$ in Schwarzschild gauge takes the form from equation (11), namely

$$f(r) = 1 - \frac{2M}{r} + \frac{P^2}{r^2}, \qquad A = -\frac{eP}{2\sqrt{\pi}} \cos\theta \, d\phi. \tag{61}$$

It is often convenient to express the blackening factor as

$$f(r) = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right), \tag{62}$$

where r_+ and r_- denote the outer and inner horizons,

$$r_\pm = M \pm \sqrt{M^2 - P^2}, \qquad \text{i.e.} \qquad M = \frac{1}{2}(r_+ + r_-), \quad \text{and} \quad P = \sqrt{r_+ r_-}\,. \tag{63}$$

The extremal limit corresponds to the coincidence of the two horizons, $r_+ = r_- = r_0$ which occurs when $M^2 = P^2$. At extremality, the Hawking temperature

$$T_{\mathrm{H}} = \frac{r_+ - r_-}{4\pi r_+^2}, \tag{64}$$

vanishes, while the Bekenstein–Hawking entropy remains finite,

$$S_0 = \pi r_0^2\,. \tag{65}$$

Note that a non-zero entropy at zero temperature normally indicates a degenerate ground state. Indeed, if entropy is interpreted as a measure of the number of available states, a finite extremal entropy means that the black hole possesses a large number of microstates already at zero temperature. This is closely related to the fact that near-extremal excitations are separated from the extremal ground state by an effective mass gap, as will become more precise from the low-temperature expansion below.

For an extremal black hole one may further consider the near-horizon decoupling limit

$$r = r_0\left(1 + \frac{\lambda}{z}\right), \qquad t = r_0 \frac{\tau}{\lambda}, \tag{66}$$

where λ is a scaling parameter. In the limit $\lambda \to 0$, the metric reduces to

$$ds^2 = r_0^2\left(\frac{-d\tau^2 + dz^2}{z^2}\right) + r_0^2 d\Omega_2^2\,. \tag{67}$$

Thus the geometry becomes $\mathrm{AdS}_2 \times \mathrm{S}^2$ in the decoupling limit. This is the Robinson–Bertotti solution [39, 40], which is an exact solution of the Einstein–Maxwell equations of motion. In particular, the AdS_2 factor and the two-sphere have the same radius r_0. This near-horizon region is often depicted schematically as a long throat connecting the asymptotically flat region to the deep $\mathrm{AdS}_2 \times \mathrm{S}^2$ region.

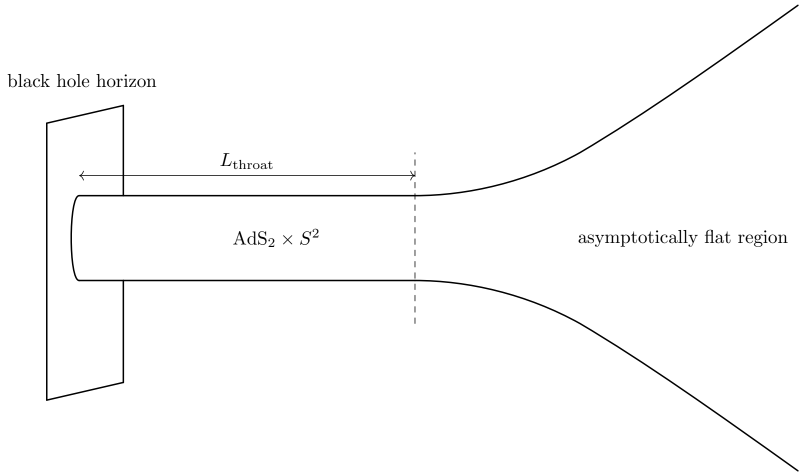

Figure 2.9. Schematic throat geometry of a near-extremal Reissner–Nordström black hole.

Far from the horizon the spacetime is asymptotically flat, while near the horizon it develops a long throat that is increasingly well approximated by $\mathrm{AdS}_2 \times \mathrm{S}^2$. In the extremal limit this throat becomes infinitely long. In the following, Jackiw–Teitelboim gravity will arise as an effective description of this throat region when one considers small deviations away from the extremal geometry.

There is also a natural temperature dependent version of the near horizon limit for a near extremal Reissner–Nordström black hole. Instead of imposing extremality from the outset, one keeps the Hawking temperature fixed but small and expands the geometry around the extremal throat [41, 42, 43]. For a Reissner–Nordström black hole close to extremality we therefore take $T_{\mathrm{H}} = T \ll \frac{1}{2\pi r_0}$, while keeping the charge fixed at its extremal value, $P = r_0$ (canonical ensemble). We then expand the outer and inner horizon radii as a power series in the temperature,

$$r_+(T) = a_0 + a_1 T + a_2 T^2 + a_3 T^3 + \dots, \tag{68}$$

$$r_-(T) = b_0 + b_1 T + b_2 T^2 + b_3 T^3 + \dots. \tag{69}$$

These coefficients are determined by expanding the relations

$$P = r_0 = \sqrt{r_+(T) r_-(T)}, \qquad T = \frac{r_+(T) - r_-(T)}{4\pi r_+(T)^2}, \tag{70}$$

for small temperature. One then finds

$$r_+(T) = r_0 + 2\pi r_0^2 T + 10\pi^2 r_0^3 T^2 + 64\pi^3 r_0^4 T^3 + \dots, \tag{71}$$

$$r_-(T) = r_0 - 2\pi r_0^2 T - 6\pi^2 r_0^3 T^2 - 32\pi^3 r_0^4 T^3 + \dots. \tag{72}$$

The corresponding mass is therefore

$$M(T) = \frac{1}{2}\big(r_+(T) + r_-(T)\big) = r_0 + 2\pi^2 r_0^3 T^2 + \mathcal{O}(T^3)\,. \tag{73}$$

Subtracting the extremal mass $M_{\text{ext}} = r_0$, we obtain the energy above extremality,

$$E_{\text{BH}} = M(T) - M_{\text{ext}} = 2\pi^2 r_0^3 T^2 + \mathcal{O}(T^3)\,. \tag{74}$$

This quadratic scaling has an important consequence. Since Hawking radiation is thermal, the characteristic energy of an emitted quantum is set by the peak of the blackbody spectrum. By Wien's displacement law this is proportional to the temperature, so

$$E_{\text{Hawking}} \sim T \tag{75}$$

These two scales are compared schematically in the following figure.

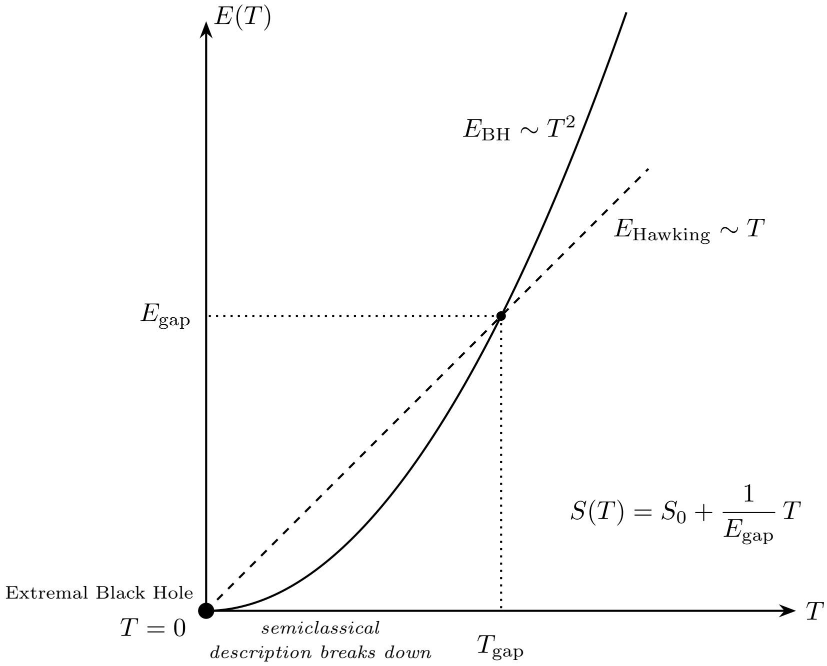

Figure 2.10. Comparison between the energy above extremality, $E_{\text{BH}} \sim T^2$, and the characteristic energy of a Hawking quantum, $E_{\text{Hawking}} \sim T$. Their intersection defines the scale T_{gap} below which the semiclassical description breaks down.

The semiclassical description therefore breaks down when the energy of a typical Hawking quantum becomes comparable to the total energy above extremality [44, 45, 46]. This defines the scale T_{gap}, given by

$$T_{\text{gap}} = \frac{1}{2\pi^2 r_0^3}\,. \tag{76}$$

At this temperature the corresponding energy scale is

$$E_{\text{gap}} = E_{\text{BH}}(T_{\text{gap}}) = \frac{1}{2\pi^2 r_0^3}\,. \tag{77}$$

Below this scale the semiclassical approximation breaks down, and the discrete structure of the spectrum becomes important. In the following sections we will see that logarithmic corrections to the entropy modify the thermodynamics precisely in this regime.

Let us also highlight that in this description there is a large ground state degeneracy and an apparent mass gap. Using $E_{\mathrm{BH}}(T) = T^2/E_{\mathrm{gap}}$, the first law $dE = T\,dS$ immediately gives

$$S(T) = S_0 + \frac{2}{E_{\mathrm{gap}}} T + \mathscr{O}(T^2). \tag{78}$$

To characterize the spectrum, it is convenient to introduce the density of states $\rho(E)$, defined through the Euclidean partition function

$$Z(\beta) = \int_0^\infty dE\, \rho(E)\, e^{-\beta E}. \tag{79}$$

Equivalently, the density of states may be recovered from the inverse Laplace transform

$$\rho(E) = \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{d\beta}{2\pi i} Z(\beta) e^{\beta E}. \tag{80}$$

In the previous section, we found in equation (57) that

$$Z_{\mathrm{cl}}(\beta) = e^{\pi r_+^2} = e^{S_0} e^{\frac{2}{E_{\mathrm{gap}}\beta}}. \tag{81}$$

Therefore, the classical density of states is

$$\rho_{\mathrm{cl}}(E) = e^{S_0} \left[\delta(E) + \sqrt{\tfrac{2}{E_{\mathrm{gap}} E}}\, I_1\!\left(2\sqrt{\tfrac{2E}{E_{\mathrm{gap}}}} \right) \right]. \tag{82}$$

Here $\delta(E)$ is the Dirac delta function, reflecting the large ground state degeneracy at $E = 0$, while I_1 is the modified Bessel function of the first kind. It is also useful to introduce the integrated density of states

$$N(E) = \int_0^E dE'\, \rho(E'), \tag{83}$$

which counts the number of states with energy less than E. This gives

$$N_{\mathrm{cl}}(E) = e^{S_0} \left[-1 + \Theta(E) + I_0\!\left(2\sqrt{\tfrac{2E}{E_{\mathrm{gap}}}} \right) \right]. \tag{84}$$

Here $\Theta(E)$ is the Heaviside step function. We will return later to the physical interpretation of these expressions once we have obtained the quantum corrections and can compare the two.

Let us conclude this section with a brief remark on the decoupling limit of the near-extremal geometry. Even away from exact extremality, one may still zoom into the near-horizon region while keeping a finite excitation above extremality. This is achieved by taking a temperature-dependent scaling limit in which the four-dimensional Hawking temperature is sent to zero while the near-horizon black hole patch remains finite.

$$t \to \frac{\mu\, t}{2\pi T}, \qquad r \to r_+(T) + \frac{2\pi r_0^2 T}{\mu}(r - \mu), \tag{85}$$

and take the limit $T \to 0$ with μ held fixed. In this limit the metric reduces to

$$ds^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = r_0^2\left(-(r^2-\mu^2)dt^2 + \frac{dr^2}{r^2-\mu^2}\right) + r_0^2 d\Omega_2^2\,. \tag{86}$$

The resulting geometry corresponds to the black hole patch of AdS_2. The corresponding background gauge field is given by $\bar{A} = -\frac{er_0}{2\sqrt{\pi}}\cos\theta\, d\phi$. Strictly speaking, the term decoupling limit is sometimes reserved for situations in which the near-horizon region also becomes infinitely long in a proper-distance sense, so that it genuinely separates from the asymptotically flat region. This is indeed what happens for near-extremal Reissner–Nordström black holes. Note that the parameter μ should therefore not be identified with the original four-dimensional Hawking temperature T, but rather with the temperature of the decoupled AdS_2 black hole, or equivalently with the finite excitation above extremality that remains in the throat after the limit is taken. It is in this sense that Jackiw–Teitelboim gravity should be understood as an effective description of the near-horizon throat. One may furthermore retain the first subleading correction in the four-dimensional temperature, so that the metric takes the form

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + T\,\delta g_{\mu\nu} + \mathscr{O}(T^2)\,, \tag{87}$$

with

$$T\,\delta g_{\mu\nu}dx^\mu dx^\nu = \frac{T}{\mu T_{\mathrm{gap}}}\left[(r+2\mu)\left((r-\mu)^2 dt^2 + \frac{dr^2}{(r+\mu)^2}\right) + r\,d\Omega_2^2\right]\,. \tag{88}$$

Retaining this subleading correction in temperature is essential for a well-defined semiclassical analysis. In the strict extremal limit, the AdS_2 throat exhibits an infinite set of zero modes associated with its enhanced symmetry, leading to infrared divergences in the one-loop partition function. Including the leading temperature correction lifts this degeneracy, rendering the path integral finite and giving rise to logarithmic corrections to the entropy [43].

2.6 The generalized Lichnerowicz operator

Before carrying out the dimensional reduction to Jackiw–Teitelboim gravity, let us return to the near-horizon geometry obtained in the previous section in the near extremal regime $T \ll T_{\mathrm{gap}}$. In the strict extremal limit, the metric $\bar{g}_{\mu\nu}$ in (86) develops an enhanced AdS_2 symmetry, and with it an infinite tower of metric zero modes. These are precisely the modes responsible for the infrared divergence of the extremal one-loop path integral, and they are lifted once the leading temperature correction $T\,\delta g_{\mu\nu}$ in (87) is included.

From the four-dimensional point of view, these modes arise as large diffeomorphisms of the extremal throat geometry. Once the theory is reduced to the Jackiw–Teitelboim description, much of this higher dimensional geometric structure is no longer manifest. It is therefore useful at this stage to make precise the notion of the generalized Lichnerowicz operator and the associated metric zero modes in the parent theory.

The discussion in this section is based on the distinction between small and large diffeomorphisms. Its main purpose is to clarify the origin of the tensor zero modes of the generalized Lichnerowicz operator that appear repeatedly in the recent literature.[3]

[3]Representative formulas include eq. (2.27) of [47], eq. (3.7) of [43], eq. (2.38) of [48], eq. (4.11) of [49], eq. (3.8) of [41], eq. (A.6) of [50], eq. (4.1) of [51], and in a mathematical setting, eq. (2.117) of [52].

Following the elegant formulation of [49], the near-horizon geometry takes the form of a direct product $\mathscr{M}_2 \times S^2$ where $\mathscr{M}_2$ denotes in our case the AdS_2 black hole patch in (86). This factorization allows one to organize fluctuations according to their transformation properties on S^2, writing

$$ds^2 = \bar{g}_{ab}dx^a dx^b + \bar{g}_{ij}dx^i dx^j\,, \qquad \bar{A} = \bar{A}_i\, dx^i\,, \tag{89}$$

where the indices a, b label coordinates on $\mathscr{M}_2$, while i, j label coordinates on S^2. Accordingly, one may distinguish three classes of fluctuations:

- **Tensor modes**, corresponding to diffeomorphisms along $\mathscr{M}_2$ acting on $\bar{g}_{ab}$,
- **Vector modes**, arising from mixed components $\bar{g}_{ai}$,
- **Gauge modes**, associated with $U(1)$ transformations of $\bar{A}_i$.

In this subsection we focus exclusively on the tensor sector. The relevant fluctuations are pure diffeomorphisms of the extremal background,

$$h_{ab} = \mathscr{L}_\xi \bar{g}_{ab}\,, \tag{90}$$

where $\mathscr{L}_\xi$ denotes the Lie-derivative with respect to the vector field ξ^a. Our first restriction on ξ is that the resulting metric perturbation h_{ab} satisfies the transverse traceless gauge conditions

$$\bar{g}^{ab}h_{ab} = 0\,, \qquad \bar{\nabla}^a h_{ab} = 0\,. \tag{91}$$

In two dimensions it is convenient to parametrize the generating vector in terms of a scalar potential Φ,

$$\xi^a = \varepsilon^{ab}\bar{\nabla}_b \Phi\,, \tag{92}$$

where ε_{ab} is the Levi-Civita tensor on $\mathscr{M}_2$. With this parametrization, the traceless condition is automatically satisfied, while the transverse condition reduces to

$$\left(\Box_{\mathscr{M}_2} + R^{(2)}\right)\Phi = 0\,. \tag{93}$$

In practice, it is often more convenient to solve the transverse traceless gauge conditions directly at the level of components, rather than working with the scalar equation above.[4] The corresponding metric perturbations,

$$h_{ab} = \mathscr{L}_\xi \bar{g}_{ab}\,, \tag{94}$$

therefore define the tensor zero modes of the extremal throat geometry. To make this explicit, we now solve these differential equations for the Euclidean geometry obtained in (86), namely for

$$\bar{g}_{ab}dx^a dx^b = (r^2-1)d\tau^2 + \frac{dr^2}{r^2-1}\,. \tag{95}$$

Assuming $\xi = e^{in\tau}\left[f_n(r)\,\partial_\tau + g_n(r)\,\partial_r\right]$ the traceless condition $\bar{g}^{ab}h_{ab} = 0$ implies

$$in\, f_n(r) + g_n'(r) = 0\,. \tag{96}$$

[4] We thank Watse Sybesma and Matthew J. Blacker for helpful discussions regarding this approach.

Hence $f_n(r) = \frac{i}{n} g_n'(r)$. Substituting this into the transverse condition $\bar{\nabla}^a h_{ab} = 0$, one finds that $g_n(r)$ obeys

$$g_n''' + \frac{4r}{r^2-1} g_n'' - \frac{n^2}{(r^2-1)^2} g_n' + \frac{2n^2 r}{(r^2-1)^3} g_n = 0\,, \tag{97}$$

$$g_n'' + \frac{2r}{r^2-1} g_n' + \frac{2-n^2-2r^2}{(r^2-1)^2} g_n = 0\,. \tag{98}$$

The former equation can be obtained by differentiating the latter, and it is therefore sufficient to solve the second-order equation. Requiring the generating vector field to be regular at the Euclidean horizon and to produce a normalizable metric perturbation at the asymptotic boundary selects the solution

$$g_n(r) = A_n\,(r+|n|) \left(\frac{r-1}{r+1}\right)^{\frac{|n|}{2}}\,, \tag{99}$$

For $|n| \geq 2$, the corresponding vector field takes the form

$$\xi = A_n e^{in\tau} \left(\frac{r-1}{r+1}\right)^{\frac{|n|}{2}} \left[\frac{i}{n}\frac{r^2+|n|r+n^2-1}{r^2-1}\partial_\tau + (r+|n|)\partial_r\right]\,, \tag{100}$$

and thus we conclude that $h_{ab} = \mathscr{L}_\xi \bar{g}_{ab}$ is given by

$$h_{ab}dx^a dx^b = 2A_n \frac{(n^2-1)}{(r^2-1)} \left(\frac{r-1}{r+1}\right)^{\frac{|n|}{2}} e^{in\tau} \left[-(r^2-1)d\tau^2 + \frac{2i|n|}{n} d\tau dr + \frac{dr^2}{r^2-1}\right]\,. \tag{101}$$

The tensor zero modes of the extremal throat are therefore labeled by integers satisfying $|n| \geq 2$. The modes with $n = 0, \pm 1$ are excluded, since they generate the $SL(2,\mathbb{R})$ isometries of the extremal AdS_2 throat and hence correspond to trivial metric perturbations, or equivalently to local rather than large diffeomorphisms. The zero modes $h_{ab}^{(n)}$ are orthonormal with respect to the inner product

$$4\pi^2 \int_0^{2\pi} d\tau \int_1^\infty dr\,\sqrt{\bar{g}}\,\big(h_{ab}^{(n)}\big)^\dagger \bar{g}^{ac}\bar{g}^{bd} h_{cd}^{(m)} = \delta^{mn}. \tag{102}$$

where the factor of $4\pi^2$ arises from integrating over the transverse two-sphere, the Euclidean time has period 2π, and the radial coordinate ranges over the exterior region $r \in [1,\infty)$. Evaluating this inner product fixes the normalization constant to

$$A_n = \frac{\sqrt{|n|}}{4\sqrt{2}\,\pi^{3/2}\sqrt{n^2-1}}\,. \tag{103}$$

We now turn to the crucial point. The infinite degeneracy of the extremal throat is lifted once one includes the leading near-extremal correction $T\,\delta g_{\mu\nu}$ in (88). To make this precise, one studies the quadratic fluctuation operator governing metric perturbations about the Reissner–Nordström background. Expanding the Euclidean action to second order, the metric sector takes the schematic form

$$I_h^{(2)} = \int d^4x \sqrt{\bar{g}}\, h_{\alpha\beta}^\dagger\, \Delta_L^{\alpha\beta,\mu\nu}\, h_{\mu\nu}\,, \tag{104}$$

where $\Delta_L^{\alpha\beta,\mu\nu}$ is the generalized Lichnerowicz operator [53, 48, 51, 49]. For Einstein–Maxwell theory without scalar fields, this operator reduces to (up to the normalization conventions of the Maxwell sector)

$$\Delta_L^{\alpha\beta,\mu\nu} = -\frac{1}{16\pi G}\left(\Delta_{\text{EH}}^{\alpha\beta,\mu\nu} - 2\Delta_F^{\alpha\beta,\mu\nu}\right). \tag{105}$$

where the Einstein–Hilbert and gauge-field contributions are respectively given by [53, 48, 51, 49]

$$\Delta_{\text{EH}}^{\alpha\beta,\mu\nu} = \frac{1}{2}\bar{g}^{\alpha\mu}\bar{g}^{\beta\nu}\bar{\Box} - \frac{1}{4}\bar{g}^{\alpha\beta}\bar{g}^{\mu\nu}\bar{\Box} + \bar{R}^{\alpha\mu\beta\nu} + \bar{R}^{\alpha\mu}\bar{g}^{\beta\nu} - \bar{R}^{\alpha\beta}\bar{g}^{\mu\nu} - \tfrac{\bar{R}}{2}\bar{g}^{\alpha\mu}\bar{g}^{\beta\nu} + \tfrac{\bar{R}}{4}\bar{g}^{\alpha\beta}\bar{g}^{\mu\nu}, \tag{106}$$

$$\Delta_F^{\alpha\beta,\mu\nu} = -\frac{1}{8}\bar{F}_{\rho\sigma}\bar{F}^{\rho\sigma}\left(2\bar{g}^{\alpha\mu}\bar{g}^{\beta\nu} - \bar{g}^{\alpha\beta}\bar{g}^{\mu\nu}\right) + \bar{F}^{\alpha\mu}\bar{F}^{\beta\nu} + 2\bar{F}^{\alpha\gamma}\bar{F}^{\mu}{}_{\gamma}\bar{g}^{\beta\nu} - \bar{F}^{\alpha\gamma}\bar{F}^{\beta}{}_{\gamma}\bar{g}^{\mu\nu}, \tag{107}$$

We will not evaluate this generalized Lichnerowicz operator explicitly here. Instead, we quote the results of the recent analyses of near-extremal black holes in [48, 51, 49], where the generalized Lichnerowicz operator is evaluated on the near-extremal background (87)-(88). The main point is that the near-extremal correction lifts the tensor zero modes of the extremal throat, rendering the one-loop determinant finite and giving rise to the corresponding logarithmic correction to the entropy. We will return to this issue later in the simpler setting of Jackiw–Teitelboim gravity. There, we will carry out the analogous analysis explicitly and compute the corresponding Lichnerowicz operator directly in the JT background. To our knowledge, this explicit evaluation has not appeared in the literature.

2.7 Dimensional reduction to Jackiw-Teitelboim gravity

In this section we dimensionally reduce four-dimensional Einstein–Maxwell theory to an effective two-dimensional dilaton gravity theory. Although our eventual interest is the asymptotically flat Reissner–Nordström black hole, the derivation below establishes a more general point: the near-extremal throat of a spherically symmetric Einstein–Maxwell black hole is described by Jackiw–Teitelboim gravity [19, 20] irrespective of the value of the four-dimensional cosmological constant. In this sense, the Jackiw–Teitelboim description is universal within this class of solutions.

To demonstrate this, we keep the cosmological constant throughout the reduction and first derive the effective two-dimensional theory for the magnetically charged AdS–Reissner–Nordström black hole. The asymptotically flat Reissner–Nordström case relevant for the subsequent discussion is then obtained by taking the flat space limit $L \to \infty$ at the end.

We take the standard spherically symmetric ansatz [54]

$$ds^2 = \frac{1}{\sqrt{2\Phi}}g_{ab}^{(2d)}\,dx^a dx^b + 2G_{\text{N}}\Phi\, d\Omega_2^2, \tag{108}$$

where $x^a = (t, r)$ and $\Phi = \Phi(x^a)$ is the dilaton field controlling the area of the transverse two-sphere. At this stage it becomes clear why it is convenient to work with the magnetically charged black hole solution. In the magnetic frame the only non-vanishing component of the gauge potential is $A_\phi = -\frac{eP}{2\sqrt{\pi}}\cos\theta$ so that the corresponding field strength has support only on the transverse sphere. After integrating out the angular coordinates $0<\theta<\pi$ and $0<\phi<2\pi$, and focusing only on the bulk contribution, we find that the reduced action takes the form

$$
\begin{aligned}
I_{\rm 2d} = &-\int dt\,dr\sqrt{-g^{(2d)}}\left[\frac{1}{2}R^{(2d)}\Phi + \frac{3\sqrt{\Phi}}{\sqrt{2}L^2} - \frac{P^2}{4\sqrt{2}G_{\rm N}\Phi^{3/2}} + \frac{1}{2\sqrt{2}G_{\rm N}\Phi^{1/2}}\right] \\
&-\frac{3}{4}\int dt\,dr\,\partial_a\left(\sqrt{-g^{(2d)}}g^{ab}\partial_b\Phi\right) .
\end{aligned}
\tag{109}
$$

The second line is a total derivative term which we discard in the bulk analysis. We therefore conclude that the effective two-dimensional description is captured by

$$
I_{\rm 2d} = -\frac{1}{2}\int d^2x\left[\sqrt{-g}\,\Phi R + V(\Phi)\right] ,
\tag{110}
$$

with dilaton potential

$$
V(\Phi) = \frac{3\sqrt{\Phi}}{\sqrt{2}L^2} - \frac{P^2}{4\sqrt{2}G_{\rm N}\Phi^{3/2}} + \frac{1}{2\sqrt{2}G_{\rm N}\Phi^{1/2}} .
\tag{111}
$$

This is the effective two-dimensional description of magnetically charged AdS–Reissner–Nordström black holes. To derive Jackiw–Teitelboim gravity, we look for a constant dilaton value Φ_0 such that

$$
V(\Phi_0) = 0 .
\tag{112}
$$

Solving this equation we find

$$
\Phi_0 = \frac{L^2}{12G_{\rm N}}\left(-1+\sqrt{1+\frac{12G_{\rm N}P^2}{L^2}}\right) = \frac{P^2}{2} + \mathscr{O}\left(\frac{1}{L^2}\right) .
\tag{113}
$$

We now consider small fluctuations around the extremal background by writing

$$
\Phi = \Phi_0 + \phi .
\tag{114}
$$

Expanding the potential to linear order around Φ_0, we have

$$
V(\Phi_0+\phi) = V(\Phi_0) + V'(\Phi_0)\phi = V'(\Phi_0)\phi ,
\tag{115}
$$

where we used that $V(\Phi_0) = 0$. Evaluating the derivative at Φ_0, we find

$$
V'(\Phi_0) = \frac{6\sqrt{6}\sqrt{G_{\rm N}}\left(L^2+12G_{\rm N}P^2 - L\sqrt{L^2+12G_{\rm N}P^2}\right)}{\left[L\left(-L+\sqrt{L^2+12G_{\rm N}P^2}\right)\right]^{5/2}} = \frac{1}{G_{\rm N}P^3} + \mathscr{O}\left(\frac{1}{L^2}\right) .
\tag{116}
$$

Substituting this into the reduced action and keeping only terms linear in ϕ, we arrive at Jackiw–Teitelboim gravity, namely

$$
I_{\rm JT} = -\frac{1}{2}\Phi_0\int d^2x\sqrt{-g}R - \frac{1}{2}\int d^2x\sqrt{-g}\phi\left(R+V'(\Phi_0)\right) .
\tag{117}
$$

The first term is topological and retains the classical entropy of the extremal higher-dimensional black hole, while the second term describes the fluctuations of the near-extremal throat geometry. We have therefore shown that the near-extremal throat of a spherically symmetric Einstein–Maxwell black hole is universally described by Jackiw–Teitelboim gravity, independently of the value of the four-dimensional cosmological constant.

For the remainder of this discussion, we specialize to the asymptotically flat case $L \to \infty$, which is the one relevant for the near-extremal Reissner–Nordström black hole discussed above. In this limit,

$$\Phi_0 = \frac{P^2}{2} + \mathcal{O}\left(\frac{1}{L^2}\right), \qquad V'(\Phi_0) = \frac{1}{G_\mathrm{N} P^3} + \mathcal{O}\left(\frac{1}{L^2}\right). \tag{118}$$

The reduced action therefore becomes

$$I_\mathrm{JT} = -\frac{P^2}{4}\int d^2x\sqrt{-g}\,R - \frac{1}{2}\int d^2x\sqrt{-g}\,\phi\left(R + \frac{1}{G_\mathrm{N} P^3}\right). \tag{119}$$

Moreover, with our ansatz the radius of the transverse two-sphere is determined by

$$r_0^2 = 2G_\mathrm{N}\Phi_0\,, \tag{120}$$

so that in the flat-space limit we recover

$$r_0^2 = G_\mathrm{N} P^2, \qquad r_0 = \sqrt{G_\mathrm{N}}\,P\,. \tag{121}$$

Thus the constant dilaton background selected by $V(\Phi_0) = 0$ is precisely the extremal Reissner–Nordström throat, around which the Jackiw–Teitelboim description emerges.

2.8 Classical aspects of Jackiw–Teitelboim gravity

Having shown how Jackiw–Teitelboim gravity arises as an effective description of the throat geometry of near-extremal Reissner–Nordström black holes, we now turn to the classical Jackiw–Teitelboim model itself. For general reviews and further background on Jackiw–Teitelboim gravity and its role in AdS_2 holography, see [55, 56]. As we saw in the previous subsection, the dimensional reduction also produces a topological term, which accounts for the classical entropy of the extremal higher-dimensional black hole. In the present subsection, however, we focus only on the dynamical Jackiw–Teitelboim sector. Denoting the AdS_2 radius of the effective two-dimensional theory by L_JT, we write the action as

$$S_\mathrm{JT}[g,\phi] = -\frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\,\phi\left(R + \frac{2}{L_\mathrm{JT}^2}\right) - \int_{\partial\mathcal{M}} dy\sqrt{h}\,\phi\left(K - \frac{1}{L_\mathrm{JT}}\right). \tag{122}$$

Here G_N denotes Newton's constant, ϕ is the dilaton field, R is the Ricci scalar of the metric $g_{\mu\nu}$, and g denotes its determinant. The cosmological constant is therefore $\Lambda = -1/L_\mathrm{JT}^2$. Furthermore, $\mathcal{M}$ denotes the two-dimensional spacetime manifold, y is a coordinate along the boundary $\partial\mathcal{M}$, K is the trace of the extrinsic curvature, and h is the determinant of the induced metric on the boundary. In the asymptotically flat Reissner–Nordström case discussed above, comparison with the reduced action shows that $L_\mathrm{JT}^2 = G_\mathrm{N} P^3/2$. For notational simplicity, we drop the subscript in the remainder of this subsection and write $L = L_\mathrm{JT}$.

Varying the action with respect to ϕ and $g_{\mu\nu}$, one obtains the equations of motion

$$R+\frac{2}{L^2}=0\,, \tag{123}$$

$$\nabla_\mu\nabla_\nu\phi-\left(\nabla^2\phi-\frac{1}{L^2}\phi\right)g_{\mu\nu}=0\,. \tag{124}$$

To begin with we will start by considering the static solution, given by

$$ds^2=L^2\left[(r^2-r_h^2)d\tau^2+\frac{dr^2}{r^2-r_h^2}\right]\,,\qquad \phi=\phi_b\,r\,. \tag{125}$$

The boundary is approached as $r\to\infty$. The isometries of the AdS_2 metric are generated by the Killing vectors

$$H=\frac{1}{L\mu}\,\partial_\tau\,, \tag{126}$$

$$P=\frac{r\cos(\mu\tau)}{L\mu\sqrt{r^2-\mu^2}}\,\partial_\tau+\frac{\sqrt{r^2-\mu^2}}{L}\sin(\mu\tau)\,\partial_r\,, \tag{127}$$

$$B=-\frac{r\sin(\mu\tau)}{\mu\sqrt{r^2-\mu^2}}\,\partial_\tau+\sqrt{r^2-\mu^2}\cos(\mu\tau)\,\partial_r\,. \tag{128}$$

These generators satisfy the $\mathfrak{so}(2,1)$ commutation relations

$$[B,H]=P,\qquad [B,P]=H,\qquad [H,P]=\frac{1}{L^2}B\,. \tag{129}$$

This algebra will later provide the foundation for the BF formulation of Jackiw–Teitelboim gravity, where the theory is constructed directly from the underlying symmetry algebra. Let us also note that, although these vector fields are isometries of the AdS_2 metric, the full background configuration is not invariant under all of them, since $\mathscr{L}_H\phi=0$ but $\mathscr{L}_B\phi\neq 0$ and $\mathscr{L}_P\phi\neq 0$. Thus the non-trivial dilaton profile breaks the full $\mathrm{SL}(2,\mathbb{R})$ isometry of the AdS_2 metric down to a $U(1)$ subgroup [57, 18]. It is in this sense that the resulting background is referred to as nearly-AdS_2, or NAdS_2.

As in subsection 2.3, we may again study the classical thermodynamic behaviour of the model by computing the Hawking temperature from the regularity of the Euclidean cigar geometry. Requiring smoothness at the horizon $r=\mu$ fixes the Euclidean periodicity to be $\beta=\frac{2\pi}{\mu}$ and hence the associated Hawking temperature is

$$T_{\mathrm{H}}=\frac{r_h}{2\pi}. \tag{130}$$

It is worth emphasizing that in AdS_2 this temperature is associated with the chosen black hole patch and its Euclidean periodicity. Evaluating the on-shell action with the same methods as in section 2.4 one finds

$$I_{\mathrm{cl}}=-\frac{2\pi^2\phi_b}{\beta}, \tag{131}$$

so that the classical partition function is

$$Z_{\rm cl} = e^{-I_{\rm cl}} = \exp\left(\frac{2\pi^2\phi_b}{\beta}\right). \tag{132}$$

The corresponding entropy is therefore

$$S = \left(1-\beta\partial_\beta\right)\log Z_{\rm cl} = \frac{4\pi^2\phi_b}{\beta}. \tag{133}$$

Notably, one may also compute the free energy, energy above extremality and heat capacity, which are given by

$$F = -\frac{1}{\beta}\log Z_{\rm cl} = -\frac{2\pi^2\phi_b}{\beta^2}, \quad E = F + T_{\rm H}S = \frac{2\pi^2\phi_b}{\beta^2}, \quad C = \frac{dE}{dT_{\rm H}} = \frac{4\pi^2\phi_b}{\beta}. \tag{134}$$

Here we are considering only the dynamical Jackiw–Teitelboim sector. Including the topological term discussed in the previous subsection would simply shift the logarithm of the partition function by the extremal entropy S_0, so that the full entropy becomes e.g.

$$S_{\rm JT} = S_0 + \frac{4\pi^2\phi_b}{\beta}. \tag{135}$$

Having discussed the classical thermodynamics of Jackiw–Teitelboim gravity, let us now turn to a coordinate system in which the underlying $\mathrm{SL}(2,\mathbb{R})$ symmetry is more manifest. For this purpose it is convenient to introduce Poincaré coordinates. One may pass between the black hole coordinates in (125) and the Poincaré patch by the transformation

$$Z(r,\tau) = \frac{r_h}{r-\sqrt{r^2-r_h^2}\cos\left(\frac{2\pi\tau}{\beta}\right)}, \quad T(r,\tau) = \frac{\sqrt{r^2-r_h^2}\sin\left(\frac{2\pi\tau}{\beta}\right)}{r-\sqrt{r^2-r_h^2}\cos\left(\frac{2\pi t}{\beta}\right)}. \tag{136}$$

In these coordinates the metric take the form

$$ds^2 = L^2\frac{dT^2+dZ^2}{Z^2}, \tag{137}$$

with $T\in(-\infty,\infty)$, and where the asymptotic boundary is approached as $Z\to 0$. The isometries of this metric are generated by

$$H = \partial_T, \quad D = T\,\partial_T + Z\,\partial_Z, \quad K = (T^2-Z^2)\,\partial_T + 2TZ\,\partial_Z. \tag{138}$$

These generators satisfy the $\mathfrak{sl}(2,\mathbb{R})$ algebra

$$[H,D] = H, \qquad [K,D] = -K, \qquad [H,K] = 2D. \tag{139}$$

They correspond respectively to translations, dilations, and special conformal transformations. This relativistic seed algebra will later provide the foundation for the first order BF formulation of Jackiw–Teitelboim gravity, where the theory is constructed directly from the underlying symmetry algebra.

2.9 Three ways to skin a cat: The Schwarzian action

In this section we present three complementary derivations of the Schwarzian action, which governs the boundary dynamics of Jackiw–Teitelboim gravity. Each derivation highlights a different aspect of the same low energy degree of freedom. First, following [58], we show how the Schwarzian arises from asymptotic boundary reparametrizations in a holographic renormalization approach. We then turn to the geometric *wiggling boundary* construction [18]. Finally, we discuss the BF theoretic perspective, in which the Schwarzian appears as the action of the would-be gauge mode [59, 60].

2.9.1 The holographic renormalization method

Following the logic of [58], we seek residual bulk diffeomorphisms that implement a boundary reparametrization while preserving the asymptotic form of the metric. We begin with Euclidean AdS_2 in Poincaré coordinates,

$$ds^2 = \frac{d\tau^2 + dz^2}{z^2}, \tag{140}$$

with Euclidean time identified as $\tau \sim \tau + \beta$. We would like to implement a boundary reparametrization $\tau \mapsto f(\tau)$ by means of a bulk diffeomorphism that preserves the gauge conditions

$$g_{\tau z} = 0, \qquad g_{zz} = \frac{1}{z^2}, \tag{141}$$

so that the metric remains in of the form,

$$ds^2 = \frac{dz^2}{z^2} + g_{\tau\tau}(\tau, z)\, d\tau^2. \tag{142}$$

Since the Euclidean time circle is periodic, the reparametrization must satisfy

$$f(\tau + \beta) = f(\tau) + \beta, \tag{143}$$

and we further require $f'(\tau) > 0$ so that the map is smooth and orientation preserving. The problem is therefore to determine the finite bulk diffeomorphism that realizes such a boundary reparametrization while maintaining the chosen gauge. An explicit finite transformation with these properties is

$$\tau \to f(\tau) - \frac{2f'(\tau)^2}{f''(\tau)} \frac{\left(\frac{f''(\tau)z}{2f'(\tau)}\right)^2}{1 + \left(\frac{f''(\tau)z}{2f'(\tau)}\right)^2}, \qquad z \to \frac{f'(\tau)z}{1 + \left(\frac{f''(\tau)z}{2f'(\tau)}\right)^2}. \tag{144}$$

Acting with this diffeomorphism on the AdS_2 metric yields

$$ds^2 = \frac{d\tau^2 + dz^2}{z^2} \mapsto \frac{dz^2}{z^2} + \left[1 - \frac{1}{2} z^2\, \mathrm{Sch}(f, \tau)\right]^2 \frac{d\tau^2}{z^2}, \tag{145}$$

where

$$\mathrm{Sch}(f, \tau) = \frac{f'''(\tau)}{f'(\tau)} - \frac{3}{2}\left(\frac{f''(\tau)}{f'(\tau)}\right)^2 \tag{146}$$

is the Schwarzian derivative. Thus the Schwarzian appears as the universal subleading term generated by boundary reparametrizations that preserve the asymptotic AdS_2 gauge. To obtain the corresponding boundary action in Jackiw–Teitelboim gravity, we must also transform the dilaton. The same bulk diffeomorphism gives

$$\phi = \frac{\phi_b}{z} \mapsto \frac{\phi_b}{f'(\tau)\, z} \left[1 + \left(\frac{f''(\tau)\, z}{2f'(\tau)} \right)^2 \right]. \tag{147}$$

One may verify that the transformed pair $(g_{\mu\nu}, \phi)$ continues to satisfy the JT equations of motion (123). We now evaluate the on-shell action. Since the bulk term in (122) vanishes on shell, only the boundary contribution remains. For a constant-z hypersurface, the induced metric and extrinsic curvature transform as

$$\sqrt{h} \mapsto \frac{1}{z} - \frac{1}{2} \mathrm{Sch}(f, \tau)\, z, \qquad K \mapsto \frac{1 + \frac{1}{2} \mathrm{Sch}(f, \tau)\, z^2}{1 - \frac{1}{2} \mathrm{Sch}(f, \tau)\, z^2}. \tag{148}$$

Moreover, the boundary measure transforms as

$$d\tau \mapsto f'(\tau)\, d\tau \, \frac{1 - \left(\frac{f''(\tau) z}{2f'(\tau)} \right)^2}{\left[1 + \left(\frac{f''(\tau) z}{2f'(\tau)} \right)^2 \right]^2} \left(1 - \frac{1}{2} \mathrm{Sch}(f, \tau)\, z^2 \right). \tag{149}$$

Substituting the transformed metric, dilaton, and measure into the Euclidean JT boundary term, we obtain

$$I_{\mathrm{JT}} = - \int_0^\beta d\tau \sqrt{h} \phi \left(K - \frac{1}{L} \right) \mapsto - \int_0^\beta d\tau \, \mathrm{Sch}(f, \tau) \left[\frac{1 - \left(\frac{f''(\tau) z}{2f'(\tau)} \right)^2}{1 + \left(\frac{f''(\tau) z}{2f'(\tau)} \right)^2} \left(1 - \frac{1}{2} \mathrm{Sch}(f, \tau)\, z^2 \right) \right]. \tag{150}$$

Taking the boundary limit $z \to 0$, we find the Schwarzian action

$$I_{\mathrm{Sch}} = -\phi_b \int_0^\beta d\tau \, \mathrm{Sch}(f, \tau). \tag{151}$$

2.9.2 The wiggling boundary method

We now present a second derivation of the Schwarzian action, following the geometric *wiggling boundary* approach [56, 18].

Consider a cutoff surface embedded in Euclidean AdS_2, parametrized by a curve $(t(u), z(u))$ in Poincaré coordinates,

$$ds^2 = \frac{dt^2 + dz^2}{z^2}. \tag{152}$$

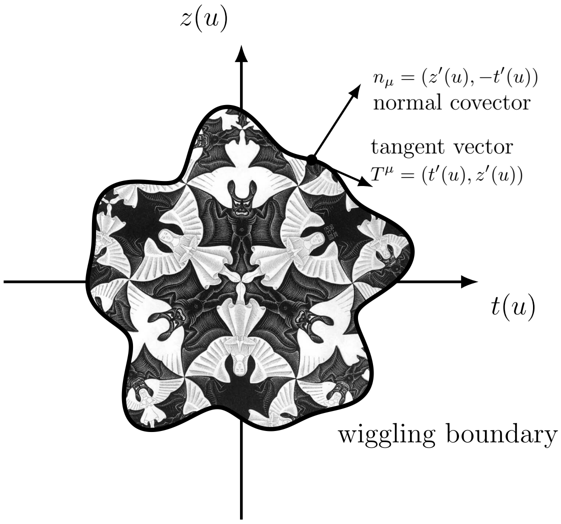

Figure 2.11. A wiggling boundary diagram with a reference to the Dutch artist Escher.

Here u is an arbitrary parameter along the boundary curve. Define the vielbein $e^\mu_u = \partial_u x^\mu$ so that $e^t_u = t'(u)$ and $e^z_u = z'(u)$ which coincides with the tangent vector $T^\mu = e^\mu_u$. Note that this means that the pullback of the metric becomes on the wiggling boundary

$$h_{uu} = e^\mu_u e^\nu_u g_{\mu\nu} = t'(u)^2 g_{tt} + z'(u)^2 g_{zz} = \frac{t'(u)^2 + z'(u)^2}{z(u)^2}\,, \tag{153}$$

Since the boundary is one-dimensional, the inverse metric is simply $h^{uu} = (h_{uu})^{-1}$. The extrinsic scalar curvature can be computed in the following manner

$$K = g^{\mu\nu} K_{\mu\nu} = h^{uu} e^\mu_u e^\nu_u \nabla_\mu n_\nu = h^{uu}\left[e^\nu_u \partial_u n_\nu - e^\mu_u e^\nu_u \Gamma^\rho_{\mu\nu} n_\rho\right]\,, \tag{154}$$

where the outwards pointing unit normal vector is given by

$$n_t = \frac{z'}{z\sqrt{t'^2 + z'^2}}, \quad n_z = -\frac{t'}{z\sqrt{t'^2 + z'^2}}\,, \tag{155}$$

and non-vanishing Christoffel-symbols are in this case given by:

$$\Gamma^z_{tt} = -\Gamma^t_{tz} = -\Gamma^z_{zz} = \frac{1}{z}\,. \tag{156}$$

Therefore we conclude that

$$K = h^{uu}\left[t'\partial_u n_t + z'\partial_u n_z + \frac{2t'z'}{z} n_t + \frac{z'^2 - t'^2}{z} n_z\right] = \frac{t'\left(t'^2 + z'^2 + zz''\right) - zz't''}{(t'^2 + z'^2)^{3/2}}\,. \tag{157}$$

We now impose the *wiggling boundary condition*, which fixes the induced metric to

$$h_{uu} = \frac{1}{\varepsilon^2}. \tag{158}$$

This constraint is solved by

$$z(u) = \varepsilon\, t'(u). \tag{159}$$

Substituting this into the expression for K and expanding for small ε, we obtain

$$K = 1 + \varepsilon^2 \mathrm{Sch}(t,u) + \mathscr{O}(\varepsilon^4). \tag{160}$$

From this we conclude that

$$I_{\mathrm{JT}} = \int du \sqrt{h_{uu}} \frac{\phi_b}{z}(K-1) = \int du \frac{1}{\varepsilon} \frac{\phi_b}{\varepsilon t'(u)} \left(-1 + 1 + \varepsilon^2 \mathrm{Sch}(t,u) + \mathscr{O}(\varepsilon^4)\right) \tag{161}$$

$$= \phi_b \int du \frac{1}{t'(u)} \mathrm{Sch}(t,u). \tag{162}$$

To express the action in a more standard form , one uses the transformation property of the Schwarzian derivative under inversion [60],

$$\mathrm{Sch}(t,u) = -t'(u)^2 \, \mathrm{Sch}(u,t), \tag{163}$$

we find after changing variables in the integral via $dt = t'(u)\, du$, that the action becomes

$$I_{\mathrm{JT}} = -\phi_b \int_0^\beta dt \, \mathrm{Sch}(u,t). \tag{164}$$

2.9.3 BF-theory: The would-be gauge mode method

We now present an alternative derivation of the Schwarzian action based on the dynamics of would-be gauge modes in a BF formulation of two-dimensional JT gravity. This approach, together with closely related particle-on-a-group and edge-mode formulations, has been studied extensively in the literature [61, 62, 63, 64, 65]. As this framework requires somewhat more background than the preceding formulations, we keep this subsection brief in order to emphasise the different mechanisms by which the Schwarzian action emerges. For the derivation presented here, we follow closely [66, 67]. Our starting point is the $SL(2,\mathbb{R})$ algebra, previously obtained from the isometries of the JT metric solution (139),

$$[H,D] = H\,, \qquad [K,D] = -K\,, \qquad [H,K] = 2D\,. \tag{165}$$

A convenient matrix representation is given by

$$H = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \qquad D = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{166}$$

We assume that $\mathfrak{g}$ admits a non-degenerate invariant bilinear form $\langle \cdot\,,\cdot \rangle$ which satisfies the following relation $\langle [X,Y],Z\rangle = \langle X,[Y,Z]\rangle$. This can be viewed as a natural notion of an inner product on the space of generators that is compatible with the algebraic structure. For semi-simple Lie algebras, such a form is unique up to normalization and is given by the Killing–Cartan form. With respect to the present representation, we may represent the bilinear form with the simple trace formula for the generators, namely

$$\langle X,Y\rangle = \mathrm{Tr}(XY)\,, \tag{167}$$

A general group element can then be written as

$$g = e^{\rho(t)H} e^{y(t)K} e^{u(t)D}. \tag{168}$$

A central ingredient in the BF formulation is the left-invariant Maurer–Cartan form $\theta = g^{-1}dg$, which evaluates to

$$\theta = \Big[e^{-u}\rho' H + \big(u' + 2y\rho'\big)D + e^{-u}\big(y' + y^2\rho'\big)K\Big]dt. \tag{169}$$

the boundary action therefore is given by

$$I_{\text{JT}} = \phi_b \int dt\, \langle \theta_t, \theta_t \rangle = \phi_b \int dt\, \left[\frac{1}{2}u'^2 + 2y\left(u'\rho' + \rho''\right)\right]. \tag{170}$$

where $\theta = \theta_t dt$ and we have integrated by parts to make explicit that y appears linearly and hence acts as a Lagrange multiplier. The action depends only on the pullback along the boundary, and thus describes the dynamics of the would-be gauge modes, which become physical in the presence of the boundary. Eliminating the auxiliary field y via its equation of motion,

$$u' = -\frac{\rho''}{\rho'}, \tag{171}$$

and substituting back into the action, we obtain the Schwarzian action (up to a total derivative) for the boundary reparametrization mode,

$$I_{\text{JT}} = -\phi_b \int dt\, \text{Sch}(\rho, t). \tag{172}$$

The BF formulation, while technically more involved, provides the most direct route to the Schwarzian action, as it is determined entirely by the underlying Lie algebra of the isometry group. This is particularly powerful in the non-relativistic setting, where the symmetry structure is well understood while an action principle is often less clear. Indeed, recent works have shown that non-relativistic versions of the Schwarzian action can be constructed precisely from this algebraic perspective [66, 68]. In this sense, the BF formulation offers the most natural route to a non-relativistic Schwarzian.

2.10 Ostrogradsky momenta and conserved charges

At this point, the Schwarzian action has been derived in several complementary ways, but one ingredient is still missing. If this boundary theory is to serve as the effective low energy description of Jackiw–Teitelboim gravity, one would like to understand not only its form as an action, but also its canonical structure and the charges associated with its symmetries. For higher derivative actions such as the Schwarzian, this is most naturally done using the Ostrogradsky formalism [69]. The purpose of the present subsection is therefore to introduce the generalized canonical momenta and to show how the conserved charges arise from them.

Consider a higher derivative action of the form

$$S = \int d\tau L(f', f'', f'''). \tag{173}$$

Consider an infinitesimal variation $f \to f + \varepsilon\, \delta f$ from which we find that

$$\delta S = \int d\tau \left[\frac{\partial L}{\partial f'}\delta f' + \frac{\partial L}{\partial f''}\delta f'' + \frac{\partial L}{\partial f'''}\delta f'''\right]. \tag{174}$$

Integrating by parts, this may be written in the standard form

$$\delta S = \int d\tau E(f)\delta f + [\Theta(\delta f)]_{t_i}^{t_f} \,, \tag{175}$$

where the equations of motion, are given by

$$E(f) = -\frac{d}{d\tau}\frac{\partial L}{\partial f'} + \frac{d^2}{d\tau^2}\frac{\partial L}{\partial f'} - \frac{d^3}{d\tau^3}\frac{\partial L}{\partial f'''} = 0\,. \tag{176}$$

Wheras the boundary term is given by

$$\Theta(\delta f) = \left(\frac{\partial L}{\partial f'} - \frac{d}{d\tau}\frac{\partial L}{\partial f''} + \frac{d^2}{d\tau^2}\frac{\partial L}{\partial f'''}\right)\delta f + \left(\frac{\partial L}{\partial f''} - \frac{d}{d\tau}\frac{\partial L}{\partial f'''}\right)\delta f' + \left(\frac{\partial L}{\partial f'''}\right)\delta f''\,. \tag{177}$$

Now define the generalized Ostragradsky momenta such that

$$P_1 = \frac{\partial L}{\partial f'} - \frac{d}{d\tau}\frac{\partial L}{\partial f''} + \frac{d^2}{d\tau^2}\frac{\partial L}{\partial f'''}\,, \qquad P_2 = \frac{\partial L}{\partial f''} - \frac{d}{d\tau}\frac{\partial L}{\partial f'''}\,, \qquad P_3 = \frac{\partial L}{\partial f'''}\,. \tag{178}$$

Then

$$\Theta(\delta f) = P_1\,\delta f + P_2\,\delta f' + P_3\,\delta f''\,. \tag{179}$$

If the variation is a total derivative such that $\delta L = \frac{d}{d\tau}K$ then the conserved charge is given by

$$Q[\delta f] = P_1\,\delta f + P_2\,\delta f' + P_3\,\delta f'' - K\,. \tag{180}$$

Let us now apply this general formalism to the Schwarzian action

$$I_{\text{Sch}} = C\int d\tau \text{Sch}(f,\tau)\,, \tag{181}$$

where $\text{Sch}(f,t) = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$. The Ostrogradsky canonical momenta are defined as

$$P_1 = \frac{\partial L}{\partial f'} - \frac{d}{d\tau}\frac{\partial L}{\partial f''} + \frac{d^2}{d\tau^2}\frac{\partial L}{\partial f'''} = \frac{C}{f'}\left(\frac{f'''}{f'} - \left(\frac{f''}{f'}\right)^2\right)\,, \tag{182}$$

$$P_2 = \frac{\partial L}{\partial f''} - \frac{d}{d\tau}\frac{\partial L}{\partial f'''} = -\frac{2Cf''}{f'^2}\,, \tag{183}$$

$$P_3 = \frac{\partial L}{\partial f'''} = \frac{C}{f'}\,. \tag{184}$$

From the previous section it follows that the conserved charges are given by

$$Q[f] = P_1\delta f + P_2\delta f' + P_3\delta f'' - K \tag{185}$$

To identify the conserved charges explicitly, we must now determine the infinitesimal transformations under which the Schwarzian action is invariant. In the present context, the relevant symmetry is the residual global $\mathrm{SL}(2,\mathbb{R})$ symmetry acting on f by Möbius transformations. Infinitesimally, these are generated by transformations of the form

$$f \to g(f) = f + \varepsilon\eta(f) \tag{186}$$

Under which we find that the Schwarzian action changes by

$$\delta L = f'^2 \eta'''(f) \tag{187}$$

which in this case implies that $\eta(f) = a + bf + cf^2$ and it is for these three values (a,b,c) which we shall find the conserved charges to be given by

$$Q_- = Q[1] = P_1 = \frac{C}{f'}\left[\frac{f'''}{f'} - \left(\frac{f''}{f'}\right)^2\right], \tag{188}$$

$$Q_0 = Q[f] = P_1 f + P_2 f' + P_3 f'' = \frac{C}{f'}\left[-f'' + f\left[\frac{f'''}{f'} - \left(\frac{f''}{f'}\right)^2\right]\right], \tag{189}$$

$$Q_+ = Q[f^2] = P_1 f^2 + P_2 2ff' + P_3\left(2f'^2 + 2ff''\right) \tag{190}$$

$$= \frac{C}{f'}\left[f^2\left[\frac{f'''}{f'} - \left(\frac{f''}{f'}\right)^2\right] - 2ff'' + 2f'^2\right]. \tag{191}$$

These three charges generate the residual $\mathrm{SL}(2,\mathbb{R})$ symmetry of the Schwarzian theory. Equivalently, they are the Noether charges associated with the infinitesimal Möbius transformations generated by $\eta(f) = 1$, $\eta(f) = f$, and $\eta(f) = f^2$. Computing their Poisson brackets, one finds that they close into the expected $\mathfrak{sl}(2,\mathbb{R})$ algebra, up to conventional choices of sign and normalization.

2.11 Ostrogradsky symplectic form

Having identified the conserved charges of the Schwarzian theory, the next natural step is to understand the corresponding phase space structure. In the present context, this means determining the symplectic form associated with the Schwarzian mode. This is important for two related reasons. First, it clarifies the geometry of the space of boundary reparametrizations around the classical saddle. Second, in the quantum theory, the symplectic form determines the local measure on field space and therefore enters directly into the one loop path integral.

Consider therefore the Schwarzian action:

$$S = \frac{1}{2}\int_0^\beta dt\left(\frac{\phi''^2}{\phi'^2} - \phi'^2\right). \tag{192}$$

The field $\phi(t)$ is taken to satisfy the quasi periodic boundary condition

$$\phi(t+\beta) = \phi(t) + 2\pi, \tag{193}$$

which ensures that ϕ defines a reparametrization of the thermal circle of period 2π. We furthermore require the fluctuation $\delta\phi$ to be periodic with period β, so that

$$\delta\phi(t+\beta) = \delta\phi(t). \tag{194}$$

Let us now expand the action around a general configuration ϕ by writing $\phi \to \phi + \delta\phi$. To quadratic order, one finds

$$S^{(0)} = \frac{1}{2}\int_0^\beta dt\left(\frac{\phi''^2}{\phi'^2} - \phi'^2\right), \tag{195}$$

$$S^{(1)} = \int_0^\beta dt\,\delta\phi\,\frac{d}{dt}\left(\frac{1}{\phi'}\left(\frac{\phi'''}{\phi'} - \frac{\phi''^2}{\phi'^2} + \phi'^2\right)\right), \tag{196}$$

$$S^{(2)} = \int_0^\beta dt\left[\left(\mathrm{Sch}(\phi,t) - \frac{1}{2}\phi'^2\right)\frac{(\delta\phi')^2}{\phi'^2} + \frac{(\delta\phi'')^2}{2\phi'^2}\right]. \tag{197}$$

From $S^{(1)}$ we may read off the equations of motion for ϕ which give that

$$\frac{d}{dt}\left[\frac{1}{\phi'}\left(\frac{\phi'''}{\phi'} - \frac{\phi''^2}{\phi'^2} + \phi'^2\right)\right] = 0. \tag{198}$$

This can be written more suggestively as

$$\frac{d}{dt}\mathrm{Sch}\left(\tan\frac{\phi}{2},t\right) = 0. \tag{199}$$

so that the classical solutions are precisely those quasi periodic functions ϕ for which

$$\mathrm{Sch}\left(\tan\frac{\phi}{2},t\right) = a, \tag{200}$$

with a constant. A simple particular solution of $\mathrm{Sch}(f,t) = a$ is

$$f_0(t) = \tan\left(\sqrt{\tfrac{a}{2}}\,t\right). \tag{201}$$

Since the Schwarzian derivative is invariant under Möbius transformations $f(t) \mapsto \dfrac{Af(t)+B}{Cf(t)+D}$, the most general solution can be written as

$$f(t) = \frac{A\tan\left(\sqrt{\frac{a}{2}}t\right)+B}{C\tan\left(\sqrt{\frac{a}{2}}t\right)+D}, \qquad AD - BC = 1. \tag{202}$$

This freedom corresponds to the global $\mathrm{SL}(2,\mathbb{R})$ symmetry of the Schwarzian action. When evaluating the path integral, the three zero modes associated with this symmetry are removed by fixing the $\mathrm{SL}(2,\mathbb{R})$ gauge, for instance by an equivalent choice of parameters such as $A = D = 1$, $B = C = 0$.

Imposing the quasi-periodic boundary condition $\phi(t+\beta) = \phi(t) + 2\pi$ then fixes the remaining constant a. Since $\tan(x+\pi) = \tan x$, this requires

$$\sqrt{\tfrac{a}{2}}\,\beta = n\pi, \qquad n \in \mathbb{Z}. \tag{203}$$

Choosing the minimal winding $n = 1$ gives $a = \frac{2\pi^2}{\beta^2}$. With this choice and the above gauge fixing, we find the classical saddle

$$\phi_{\mathrm{cl}}(t) = \frac{2\pi}{\beta}t. \tag{204}$$

In the subsequent one-loop computation we expand $\phi = \phi_{\text{cl}} + \delta\phi$ and remove the three $\mathrm{SL}(2,\mathbb{R})$ zero modes. We can now evaluate the on-shell action $\mathscr{I}^{(0)} = S^{(0)}[\phi_{\text{cl}}]$. With $\phi_{\text{cl}}(t) = \frac{2\pi}{\beta}t$ we have $\phi'_{\text{cl}}(t) = \frac{2\pi}{\beta}$ and $\phi''_{\text{cl}}(t) = 0$, so

$$S^{(0)}[\phi_{\text{cl}}] = \frac{1}{2}\int_0^\beta dt \left(\frac{\phi''^2_{\text{cl}}}{\phi'^2_{\text{cl}}} - \phi'^2_{\text{cl}}\right) = -\frac{1}{2}\int_0^\beta dt \left(\frac{2\pi}{\beta}\right)^2 = -\frac{2\pi^2}{\beta}. \tag{205}$$

Let us note that including the constants A,B,C,D would lead to a more complicated expression, but the value of the on-shell action is unchanged since all such configurations are related by the global $\mathrm{SL}(2,\mathbb{R})$ symmetry. By choice we find that $S^{(1)}[\phi_{\text{cl}}] = 0$. Now let us turn to the quadratic fluctuation term, $S^{(2)}$, here we find that

$$S^{(2)}[\phi_{\text{cl}} + \delta\phi] = \int_0^\beta dt \left[\left(\mathrm{Sch}(\phi_{\text{cl}},t) - \frac{1}{2}\phi'^2_{\text{cl}}\right)\frac{(\delta\phi')^2}{\phi'^2_{\text{cl}}} + \frac{(\delta\phi'')^2}{2\phi'^2_{\text{cl}}}\right] \tag{206}$$

$$= -\frac{1}{2}\int_0^\beta dt \left[(\delta\phi')^2 - \left(\frac{\beta}{2\pi}\right)^2 (\delta\phi'')^2\right]. \tag{207}$$

The corresponding symplectic form can be computed using the Ostrogradsky prescription for higher-derivative Lagrangians, which amounts to first finding the conjugate momenta. From the quadratic Lagrangian density

$$L^{(2)} = -\frac{1}{2}\left[(\delta\phi')^2 - \left(\frac{\beta}{2\pi}\right)^2 (\delta\phi'')^2\right], \tag{208}$$

we obtain

$$p_{\delta\phi} = \frac{\partial L^{(2)}}{\partial(\delta\phi')} - \frac{d}{dt}\left(\frac{\partial L^{(2)}}{\partial(\delta\phi'')}\right) = -\delta\phi' - \left(\frac{\beta}{2\pi}\right)^2 \delta\phi''', \tag{209}$$

$$p_{\delta\phi'} = \frac{\partial L^{(2)}}{\partial(\delta\phi'')} = \left(\frac{\beta}{2\pi}\right)^2 \delta\phi''. \tag{210}$$

The symplectic form is then obtained by

$$\Omega^{(2)} = \int_0^\beta dt \left[dp_{\delta\phi} \wedge d(\delta\phi)\right] \tag{211}$$

$$= -\int_0^\beta dt\, d(\delta\phi') \wedge d(\delta\phi) - \left(\frac{\beta}{2\pi}\right)^2 \int_0^\beta dt\, d(\delta\phi''') \wedge d(\delta\phi). \tag{212}$$

Integrating the last term by parts we find

$$\Omega^{(2)} = \int_0^\beta dt \left[\left(\frac{\beta}{2\pi}\right)^2 d(\delta\phi'') \wedge d(\delta\phi') - d(\delta\phi') \wedge d(\delta\phi)\right]. \tag{213}$$

Since the function $\delta\phi$ is β-periodic, we go to Fourier space and expand the fluctuation as

$$\delta\phi(t) = \sum_{n\in\mathbb{Z}} \delta\phi_n e^{i\frac{2\pi n}{\beta}t}. \tag{214}$$

Let us note at this stage that later we will make use of the following identity which relates the real and imaginary parts. Writing

$$\delta\phi_n = \delta\phi_n^{(R)} + i\delta\phi_n^{(I)}\,, \tag{215}$$

the reality condition on $\delta\phi$ means that $\delta\phi^* = \delta\phi$ and thus

$$\delta\phi^* = \sum_n \left(\delta\phi_n^{(R)} - i\delta\phi_n^{(I)}\right) e^{-i\frac{2\pi n}{\beta}t} = \sum_n \left(\delta\phi_{-n}^{(R)} - i\delta\phi_{-n}^{(I)}\right) e^{i\frac{2\pi n}{\beta}t}\,, \tag{216}$$

from which we read off the constraint

$$\delta\phi_{-n}^{(R)} = \delta\phi_n^{(R)}\,, \qquad \delta\phi_{-n}^{(I)} = -\delta\phi_n^{(I)}. \tag{217}$$

The implication of this is the following two relationships

$$\delta\phi_n\delta\phi_{-n} = \left(\delta\phi_n^{(R)}\right)^2 + \left(\delta\phi_n^{(I)}\right)^2, \qquad d\delta\phi_n \wedge d\delta\phi_{-n} = -2i\, d\delta\phi_n^{(R)} \wedge d\delta\phi_n^{(I)}\,. \tag{218}$$

The quadratic action becomes

$$\begin{aligned}
S^{(2)} &= -\frac{1}{2}\int_0^\beta dt\left[(\delta\phi')^2 - \left(\frac{\beta}{2\pi}\right)^2(\delta\phi'')^2\right] && (219)\\
&= \frac{2\pi^2}{\beta^2}\int_0^\beta dt \sum_{n,m}\left[nm + n^2m^2\right]\delta\phi_n\delta\phi_m e^{i\frac{2\pi(n+m)}{\beta}t} && (220)\\
&= \frac{2\pi^2}{\beta}\sum_n\left[n(-n) + n^2(-n)^2\right]\delta\phi_n\delta\phi_{-n} && (221)\\
&= \frac{2\pi^2}{\beta}\sum_n n^2(n^2-1)\left[\left(\delta\phi_n^{(R)}\right)^2 + \left(\delta\phi_n^{(I)}\right)^2\right] && (222)\\
&= \frac{4\pi^2}{\beta}\sum_{n\geq 2} n^2(n^2-1)\left[\left(\delta\phi_n^{(R)}\right)^2 + \left(\delta\phi_n^{(I)}\right)^2\right], && (223)
\end{aligned}$$

where in the third step we used $\int_0^\beta dt\, e^{i\frac{2\pi(n+m)}{\beta}t} = \beta\,\delta_{n+m,0}$, in the second to last step we used (218) and in the last step we used (217). From this we see that when $n = 0, \pm 1$ the contribution of $S^{(2)}$ vanishes. This is a direct manifestation of the underlying $\mathrm{SL}(2,\mathbb{R})$ symmetry of the Schwarzian theory. In the path integral, one removes these three modes to obtain a finite one-loop contribution. In the gravitational picture, these three boundary zero modes correspond to the bulk Killing vectors of Euclidean AdS_2.

Evaluating the symplectic form (213) with the Fourier modes (214), we find

$$\begin{aligned}
\Omega^{(2)} &= \int_0^\beta dt\left[\left(\frac{\beta}{2\pi}\right)^2 d(\delta\phi'') \wedge d(\delta\phi') - d(\delta\phi') \wedge d(\delta\phi)\right] && (224)\\
&= -i\frac{2\pi}{\beta}\int_0^\beta dt\sum_{n,m}\left(n^2m + n\right)e^{i\frac{2\pi(n+m)}{\beta}t} d\delta\phi_n \wedge d\delta\phi_m && (225)\\
&= i2\pi\sum_n\left(n^3 - n\right)d\delta\phi_n \wedge d\delta\phi_{-n} && (226)\\
&= 4\pi\sum_n\left(n^3 - n\right)d\delta\phi_n^{(R)} \wedge d\delta\phi_n^{(I)} && (227)\\
&= 8\pi\sum_{n\geq 2}\left(n^3 - n\right)d\delta\phi_n^{(R)} \wedge d\delta\phi_n^{(I)}\,, && (228)
\end{aligned}$$

where in the third step we used $\int_0^\beta dt\, e^{i\frac{2\pi(n+m)}{\beta}t} = \beta\,\delta_{n+m,0}$, in the second to last step we used (218) and in the last step we used (217). Where notably we know from the action that $n \neq 0, \pm 1$. Having collected these results, i.e.

$$S^{(0)}[\phi_{\rm cl}+\delta\phi] = -\frac{2\pi^2}{\beta}\,, \qquad S^{(2)}[\phi_{\rm cl}+\delta\phi] = \frac{4\pi^2}{\beta}\sum_{n\geq 2} n^2(n^2-1)\left[(\delta\phi_n^{(R)})^2 + (\delta\phi_n^{(I)})^2\right], \tag{229}$$

$$\Omega^{(2)} = 8\pi \sum_{n\geq 2} (n^3-n)\, d\delta\phi_n^{(R)} \wedge d\delta\phi_n^{(I)}\,, \tag{230}$$

the quadratic path integral thus becomes a Gaussian integral in each Fourier mode. The symplectic form determines the local measure on field space as

$$\mathscr{D}\delta\phi = \prod_{n\geq 2} \mathrm{Pf}\big[\Omega_n^{(2)}\big]\, d\delta\phi_n^{(R)} d\delta\phi_n^{(I)} = \prod_{n\geq 2} 8\pi n(n^2-1)\, d\delta\phi_n^{(R)} d\delta\phi_n^{(I)}. \tag{231}$$

We may now evaluate the one-loop partition function,

$$\begin{aligned}
Z(\beta) &= \int \mathscr{D}\phi\, e^{-S} && (232)\\
&\approx e^{-S^{(0)}} \int \mathscr{D}\delta\phi\, e^{-S^{(2)}} && (233)\\
&= e^{\frac{2\pi^2}{\beta}} \prod_{n\geq 2}\left(8\pi n(n^2-1)\int d\delta\phi_n^{(R)} d\delta\phi_n^{(I)} e^{-\frac{4\pi^2}{\beta}n^2(n^2-1)[(\delta\phi_n^{(R)})^2+(\delta\phi_n^{(I)})^2]}\right) && (234)\\
&= e^{\frac{2\pi^2}{\beta}} \prod_{n\geq 2} 8\pi n(n^2-1)\frac{\pi}{\frac{4\pi^2}{\beta}n^2(n^2-1)} && (235)\\
&= e^{\frac{2\pi^2}{\beta}} \prod_{n\geq 2} \frac{2\beta}{n} = \frac{1}{4\sqrt{\pi}\beta^{3/2}} e^{\frac{2\pi^2}{\beta}}\,. && (236)
\end{aligned}$$

Where in the second step we approximated the full path-integral with the 1-loop contribution, in the third step we performed two Gaussian integrals and finally we used zeta-function regularization, which relies on the analytic continuation of the Riemann zeta function defined by $\zeta(s) = \sum_{n\geq 1} n^{-s}$, with derivative $\zeta'(s) = -\sum_{n\geq 1} n^{-s}\log n$. Applying this we find

$$\begin{aligned}
\prod_{n\geq 2}\frac{2\beta}{n} &= \exp\left[\sum_{n\geq 2}\big(\log(2\beta) - \log n\big)\right] = \exp\big[\log(2\beta)(\zeta(0)-1) + \zeta'(0)\big] && (237)\\
&= (2\beta)^{\zeta(0)-1} e^{\zeta'(0)} = \frac{1}{2\sqrt{\pi}\,\beta^{3/2}}\,, && (238)
\end{aligned}$$

where we used the regularized values $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$.

It is worth emphasizing that the Schwarzian path integral is in fact one loop exact. Although we have organized the discussion here as an expansion around the classical saddle up to quadratic order, the exact result is already captured by this Gaussian analysis. Indeed, the fluctuation exhibits a simple pattern: all terms $S^{(n)}$ with $n \geq 3$ can be written as total derivatives, and therefore do not contribute for periodic fluctuations satisfying the boundary conditions imposed above.

2.12 A Lichnerowicz operator for JT-gravity

So far, the one loop structure has been derived from the boundary Schwarzian description. In this section we check whether one can obtain it in terms of quadratic Lichnerowicz operators.

Following the prescription in [38] we will expand the Jackiw-Teitelboim action for $g = g_0 + h$ and $\Phi = \phi_0 + \varphi$ to second order such that

$$I_{\mathrm{JT}}[g,\Phi] = I_0[g_0,\phi_0] + I_1[h,\phi_0] + I_1[g_0,\varphi] + I_2[h,\varphi] + I_2[g_0,\varphi] + I_2[h,\phi_0]\,. \tag{239}$$

Where we will choose to expand around

$$(g_0)_{ab}dx^a dx^b = d\rho^2 + \sinh^2(\rho)d\tau^2\,, \quad \phi_0 = \cosh(\rho). \tag{240}$$

Which solve the equations of motion of JT gravity. We start with

$$I_{\mathrm{JT}}[g,\Phi] = \int d^2x\sqrt{g}\Phi(R+2)\,. \tag{241}$$

Then we expand around the dilaton solution ϕ_0 and the AdS_2 background g_0 to find that I_0 vanishes identically since $R = -2$ for g_0. The next to leading order is given by

$$I_1 = \int d^2x\sqrt{g_0}\,\varphi(R+2) + \int d^2x\sqrt{g_0}\,\phi_0\left[h - R_{ab}h^{ab} + \frac{1}{2}Rh - \nabla^2 h + \nabla^a\nabla_b h^b_a\right]. \tag{242}$$

Using the fact that for 2d metrics we have $R_{ab} = \frac{1}{2}Rg_{ab}$ we find that the two corresponding equations of motion are

$$R + 2 = 0\,, \tag{243}$$

$$\left[g^{ab} + \nabla^a\nabla^b - g^{ab}\nabla^2\right]\phi_0 = 0\,. \tag{244}$$

where the first corresponds to the φ variation and the second to the h_{ab} variation. These are of course the JT equations of motion. At the next order we find however that

$$I_2 = 2\int d^2x\sqrt{g}\,\varphi\left[g^{ab} + \nabla^a\nabla^b - g^{ab}\nabla^2\right]h_{ab} \tag{245}$$

$$+ \int d^2x\sqrt{g}\,h_{cd}A^{abcd}h_{ab} \tag{246}$$

where

$$A^{abcd} = \phi_0\left(g^{ab}g^{cd} - 2g^{ac}g^{bd} - g^{ac}\nabla^b\nabla^d - \frac{1}{2}g^{ab}g^{cd}\nabla^2 + g^{cd}\nabla^a\nabla^b + \frac{1}{2}g^{ac}g^{bd}\nabla^2\right) \tag{247}$$

$$+ g^{ac}\nabla^b\phi_0\nabla^d - \frac{3}{2}g^{ac}g^{bd}\nabla_e\phi_0\nabla^e - g^{ab}\nabla^d\phi_0\nabla^c + 2g^{bc}\nabla^d\phi_0\nabla^a - \frac{1}{2}g^{ab}g^{cd}\nabla_e\phi_0\nabla^e - g^{ab}\nabla^c\phi_0\nabla^d\,.$$

We then find that

$$A^{abcd}g_{ce}g_{df}h^{(n)}_{ab} = -\frac{|n|}{2}h^{(n)}_{ef}\,, \tag{248}$$

where

$$h^{(n)}_{ab}dx^a dx^b = 2iA_n e^{in\tau}\tanh^n\left(\frac{\rho}{2}\right)|n|(n^2-1)\left[d\tau^2 - \frac{2i\,d\tau d\rho}{\sinh(\rho)} - \frac{d\rho^2}{\sinh^2\rho}\right]. \tag{249}$$

whereas

$$\left(g^{ab} + \nabla^a \nabla^b - g^{ab} \nabla^2\right) h^{(n)}_{ab} = 0\,. \tag{250}$$

Thus we take a linear combination of these metrics

$$h_{ab} = \sum_n c_n h^{(n)}_{ab} \tag{251}$$

with the measure given by $[\mathscr{D}g] = \prod_n dc_n$ since we have already apropriately normalized $h^{(n)}_{ab}$ so that they are orthogonal in the sense that

$$\int d^2x \sqrt{g}\, h^{(n)\dagger}_{ab} g^{ac} g^{bd} h^{(m)}_{cd} = \delta^{mn}\,. \tag{252}$$

This means that we find

$$\int [\mathscr{D}h][\mathscr{D}\varphi]\, e^{-I_{\rm JT}[g+h,\phi_0+\varphi]} \tag{253}$$

$$= \prod_n \int \mathrm{d}c_n\, e^{-I_{\rm JT}[g,\phi_0] + \int \mathrm{d}^2x \sqrt{g}\, h_{ab} A^{abcd} h_{cd}} \tag{254}$$

$$= e^{-S_0} \prod_n \int \mathrm{d}c_n\, e^{\int \mathrm{d}^2x \sqrt{g} \sum_n c_n h^{(n)}_{ab} A^{abcd} \sum_m c_m h^{(m)}_{cd}} \tag{255}$$

$$= e^{-S_0} \prod_n \int \mathrm{d}c_n\, e^{\sum_{n,m} c_n c_m \int \mathrm{d}^2x \sqrt{g}\, h^{(n)}_{ab} \left(-\frac{n}{2} g^{ac} g^{bd}\right) h^{(m)}_{cd}} \tag{256}$$

$$= e^{-S_0} \prod_n \int \mathrm{d}c_n\, e^{\sum_{n,m} c_n c_m \left(-\frac{|n|}{2} \delta^{mn}\right)} \tag{257}$$

$$= e^{-S_0} \prod_{n=-\infty}^{+\infty} \int \mathrm{d}c_n\, e^{-\sum_n \frac{|n|}{2} c_n^2} = e^{-S_0} \prod_{n\neq 0,\pm 1} \sqrt{\frac{2\pi}{|n|}} = e^{-S_0} \prod_{n\geq 2} \frac{2\pi}{n}\,. \tag{258}$$

The bulk Gaussian integral organized by the generalized Lichnerowicz operator reproduces the same one loop structure that was previously obtained from the boundary reparametrization mode.

2.13 Quantum corrected black hole thermodynamics

We are now in a position to return to the thermodynamic interpretation of the one loop results derived above. The Schwarzian partition function captures the temperature dependent contribution above extremality, while the full black hole partition function also includes the extremal entropy factor e^{S_0}. It is therefore convenient to write

$$Z_Q(\beta) = \frac{1}{2\sqrt{\pi}\,\beta^{3/2}} e^{S_0 + \frac{2\pi^2}{\beta}}\,, \qquad Z_{\rm cl}(\beta) = e^{S_0 + \frac{2\pi^2}{\beta}}\,. \tag{259}$$

Here S_0 denotes the extremal entropy, while the remaining β dependent factor describes the excitations above extremality. The corresponding density of states is obtained by inverse Laplace transform. One finds

$$\rho_Q(E) = \frac{e^{S_0}}{2\pi^2\sqrt{2}} \sinh\!\left(2\pi\sqrt{2E}\right), \qquad \rho_{\rm cl}(E) = e^{S_0}\left[\delta(E) + \frac{\sqrt{2}\,\pi}{\sqrt{E}} I_1\!\left(2\pi\sqrt{2E}\right)\right]. \tag{260}$$

The crucial difference between these two expressions is that the one loop prefactor removes the degenerate classical ground state structure. In this sense, the logarithmic correction is not merely a small quantitative modification, but changes the qualitative near extremal behavior of the theory. This is illustrated in the following figure:

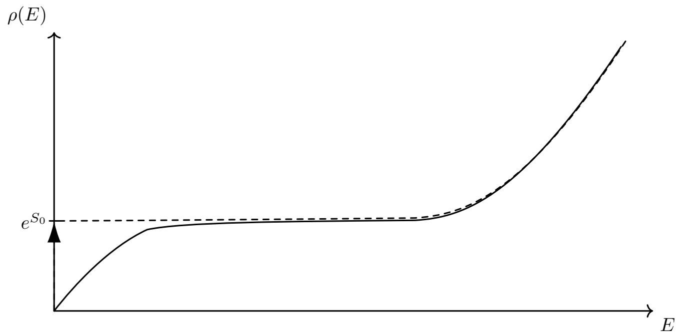

Figure 2.12. Comparison between classical (dashed) and quantum (full) density of states.

The thermodynamics are correspondingly modified. Using the relation $E(\beta) = -\partial_\beta \log Z(\beta)$, one finds for the energy above extremality

$$E_Q(T) = 2\pi^2 T^2 + \frac{3}{2}T, \qquad E_{\rm cl}(T) = 2\pi^2 T^2. \tag{261}$$

Likewise, the entropy follows from $S = (1-\beta\partial_\beta)\log Z$. One therefore finds

$$S_Q(T) = S_0 + 4\pi^2 T + \frac{3}{2}\log T - \log\left(2\sqrt{\pi}\right) + \frac{3}{2}, \qquad S_{\rm cl}(T) = S_0 + 4\pi^2 T. \tag{262}$$

Thus the leading quantum correction to the entropy is logarithmic, exactly as expected from the one loop determinant. This may be understood from a counting argument in which each generator of the underlying asymptotic symmetry algebra contributes a factor of $\frac{1}{2}$ to the one loop measure. The effect on the near extremal energy is summarized in the following figure.

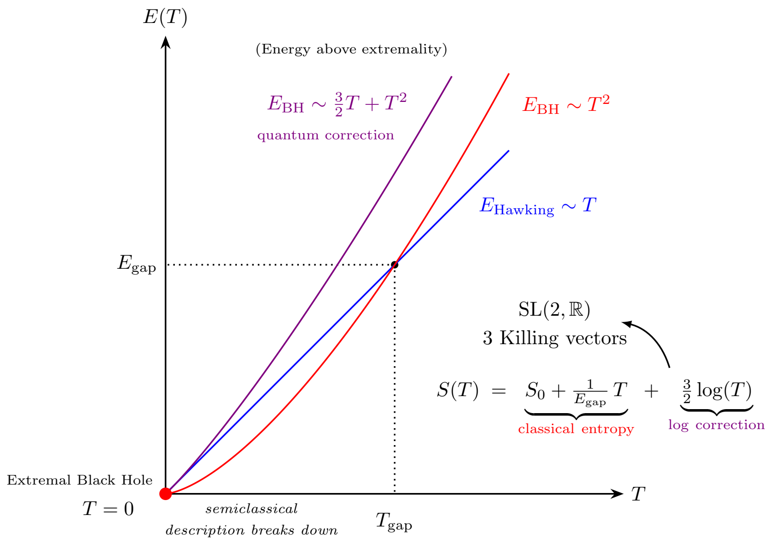

Figure 2.13. Comparison between classical and quantum energy above extremality.

Note that classically the energy above extremality scales as $E_{\mathrm{BH}} \sim T^2$, while the characteristic Hawking quantum scales as $E_{\mathrm{Hawking}} \sim T$. Their intersection defines the gap scale below which the semiclassical description breaks down, as previously illustrated in figure 2.10. After including the one loop correction, however, the black hole energy is shifted to $E_{\mathrm{BH}} \sim \frac{3}{2}T + T^2$, so that in the near extremal regime it remains above the Hawking scale, thereby resolving the puzzle discussed in [44].

Finally, let us mention that these results also have direct implications for black hole evaporation. In particular, [70] used this quantum corrected near extremal thermodynamics to derive a prediction for the corrected Hawking emission rate of near extremal charged black holes.

Appendix A: Publications

Paper I

Carroll strings with an extended symmetry algebra

Matthias Harksen, Diego Hidalgo, Watse Sybesma and Lárus Thorlacius

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The thesis author contributed to the development of the main idea of the paper, the construction and analysis of the Carroll string models, and the derivation of their symmetry structure. The thesis author also contributed to the writing of the manuscript.

Published for SISSA by Springer

Received: March 28, 2024
Accepted: April 22, 2024
Published: May 16, 2024

Carroll strings with an extended symmetry algebra

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Abstract: Starting from the Polyakov action we consider two distinct Carroll limits in target space, keeping the string worldsheet relativistic. The resulting magnetic and chiral Carroll string models exhibit different symmetries and dynamics. Both models have an infinite dimensional symmetry algebra with Carroll symmetry included in a finite dimensional subalgebra. For the magnetic model, this is the so-called string Carroll algebra. The chiral model realises an extended version of the string Carroll algebra. The magnetic model does not have any transverse string excitations. The chiral model is less restrictive and includes arbitrary left-moving modes that carry transverse momentum but do not contribute to the energy in target space.

Keywords: Bosonic Strings, Sigma Models, Global Symmetries, Space-Time Symmetries

ArXiv ePrint: 2403.01984

https://doi.org/10.1007/JHEP05(2024)206

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1 Introduction

String theory provides a robust framework for investigating quantum gravity in various decoupling limits. In this regard, non-relativistic limits of string theories [1–3] form a self-contained sector within string theory governed by a relativistic worldsheet metric with Galilean global symmetries on the target spacetime. We refer the interested readers to [4] for a recent review of non-relativistic string theory. In this note, we consider bosonic string theory in another non-Lorentzian target space limit called the Carroll limit [5–7]. A pictorial way to highlight the contrasting extremes of these limits is that rather than opening up the light cone, as happens in the Galilean limit, the lightcone collapses to a line in the Carroll limit. As a result, timelike particles are confined to the temporal axis. Tachyons still move in the Carroll limit but have no rest frame. Furthermore, Carroll boost symmetry requires energy flux to vanish. Particles that move are thus forced to have vanishing energy while particles at rest can have non-vanishing energy [8, 9]. Particle properties in the Carroll limit were further studied in [10–14], where it was demonstrated that non-trivial dynamics can arise for coupled Carroll particles. Our primary motivation for developing a Carroll limit of string theory is to see if Carroll string dynamics is less constraining than Carroll particle dynamics. Carroll strings with worldsheet Carroll symmetry were studied in [15–19], whereas target space Carroll symmetry was studied in [18, 20–22]. As we proceed, we will comment on similarities and differences between our work and these earlier papers.

We study two inequivalent Carroll limits in the target space of bosonic string theory by introducing different sets of auxiliary worldsheet fields into the Polyakov action. One of these limits corresponds to what is known throughout the literature as a magnetic limit [9, 23–25]. Our other Carroll limit breaks worldsheet chirality via a novel auxiliary field configuration.

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In parallel with the non-relativistic string [1], the Carroll symmetry appears as a global symmetry of the target spacetime, while the worldsheet remains relativistic. The magnetic limit reproduces an action that is invariant under the string Carroll algebra previously derived by Cardona, Gomis, and Pons (CGP) by considering a Carroll limit of the canonical form of the Nambu-Goto action [20].

The worldsheet dynamics of our magnetic Carroll string model is similar to the CGP model in that no motion is allowed in the transverse directions. The transverse target space momentum is unconstrained, but does not enter into the magnetic Carroll string dispersion relation. This is reminiscent of the behaviour of Carroll particles [9, 10, 13]. The chiral Carroll model also has unconstrained transverse momentum and a simple dispersion relation, where energy does not depend on transverse momentum. Unlike the magnetic model, transverse motion is allowed in the chiral model via arbitrary left-moving string excitations, thus departing from the Carroll particle picture. The symmetry algebra of the chiral model is an extension of the string Carroll algebra of the magnetic model. Furthermore, an infinite dimensional symmetry extension is realised in both our models in a similar manner as for strings with Galilean symmetry [26, 27].

The remainder of the paper is organised as follows. In section 2, we establish notation and construct the magnetic Carroll limit by introducing suitable auxiliary fields in the Polyakov action. The symmetry algebra of the magnetic Carroll model and its string dispersion relation are derived. In section 3 we present the chiral Carroll model and its extended Carroll string algebra. We then show how to extend the symmetry to an infinite dimensional algebra for both the chiral and magnetic models. In section 4 we summarise our results and discuss some future directions.

2 Magnetic Carroll string model

This section introduces the first of two Carroll limits for strings we consider. The Carroll structure of the model is observed on the target space, while the worldsheet metric remains relativistic. As we will see in section 2.1, the magnetic Carroll string arises from relativistic string theory via an asymmetric scaling of the longitudinal and transverse target space coordinate fields in the Polyakov action that renders the transverse sector 'dominant' over the longitudinal sector. The naive divergence that appears when taking this Carroll limit can be controlled by introducing auxiliary fields along the lines of the corresponding construction for non-relativistic strings in [1]. The presence of the auxiliary fields severely constrains the dynamics of the magnetic Carroll string model.

2.1 Relativistic origin of the magnetic Carroll model

Our starting point is the Polyakov action, given by

$$S_{\mathrm{P}} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\gamma}\gamma^{ab}\partial_a\hat{X}^\mu\partial_b\hat{X}^\nu\eta_{\mu\nu}\,. \tag{2.1}$$

Here lowercase Latin indices run over the worldsheet coordinates $a, b \in \{\sigma, \tau\}$ and γ^{ab} denotes the inverse of the worldsheet metric. The flat D-dimensional target-space metric is denoted by $\eta_{\mu\nu}$ with Lorentzian signature. The coupling constant α' is related to the string tension

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T via $T = 1/(2\pi\alpha')$. The worldsheet fields in (2.1) are labelled by hats in anticipation of a rescaling to be carried out immediately below. We reserve a hat-free notation for the rescaled fields that we work with in the Carroll limit.

We want to take a Carroll limit on the target space while allowing the worldsheet theory to retain its relativistic and Weyl symmetries. In order to achieve this, we introduce an effective coupling constant α'_{eff} and perform an asymmetric scaling of the worldsheet fields by the dimensionless factor of $\alpha'/\alpha'_{\text{eff}}$, where longitudinal coordinates in target space, denoted by uppercase Latin indices $A, B = 0, 1$, are scaled differently from the transverse coordinates, denoted by uppercase primed indices $A', B' = 2, \ldots, D-1$,

$$X^A = \sqrt{\frac{\alpha'_{\text{eff}}}{\alpha'}}\,\hat{X}^A\,, \qquad X^{A'} = \frac{\alpha'_{\text{eff}}}{\alpha'}\,\hat{X}^{A'}. \tag{2.2}$$

With this scaling, the Polyakov action (2.1) can be rewritten as

$$\begin{aligned} S_{\text{P}} &= -\frac{1}{4\pi\alpha'_{\text{eff}}}\int d^2\sigma\sqrt{-\gamma}\gamma^{ab}\left(\partial_a X^A\partial_b X_A + \frac{\alpha'}{\alpha'_{\text{eff}}}\partial_a X^{A'}\partial_b X_{A'}\right) \\ &= -\frac{1}{4\pi\alpha'_{\text{eff}}}\int d^2\sigma\sqrt{-\gamma}\gamma^{ab}\left(\partial_a X^A\partial_b X_A + 2\lambda_a^{A'}\partial_b X_{A'} - \frac{\alpha'_{\text{eff}}}{\alpha'}\lambda_a^{A'}\lambda_{bA'}\right)\,. \end{aligned} \tag{2.3}$$

In the second line we have introduced auxiliary fields $\lambda_a^{A'}$, which can be integrated out to recover the first line. Taking the $\alpha' \to \infty$ limit, we obtain

$$S_{\text{M}} = -\frac{1}{4\pi\alpha'_{\text{eff}}}\int d^2\sigma\sqrt{-\gamma}\gamma^{ab}\left(\partial_a X^A\partial_b X_A + 2\lambda_a^{A'}\partial_b X_{A'}\right)\,, \tag{2.4}$$

which we refer to as the *magnetic* Carroll model.[1] Upon adopting a conformal gauge for the worldsheet metric,

$$ds^2 = -d\sigma^+ d\sigma^-\,, \qquad \sigma^\pm = \tau \pm \sigma\,, \qquad \partial_\pm = \frac{1}{2}\left(\partial_\tau \pm \partial_\sigma\right)\,, \tag{2.5}$$

we can write the magnetic action (2.4) as

$$S_{\text{M}} = 2\,T_{\text{eff}}\int d^2\sigma\left(\partial_+ X^A\partial_- X^B\eta_{AB} + \lambda_+^{A'}\partial_- X_{A'} + \lambda_-^{A'}\partial_+ X_{A'}\right)\,, \tag{2.6}$$

where we have introduced the effective string tension $T_{\text{eff}} = 1/(2\pi\alpha'_{\text{eff}})$, and defined

$$\lambda_\pm^{A'} = \frac{1}{2}\left(\lambda_\tau^{A'} \pm \lambda_\sigma^{A'}\right)\,. \tag{2.7}$$

In conformal gauge, the field equations of the magnetic Carroll model are

$$\begin{aligned} \partial_+\partial_- X^A &= 0\,, & \partial_+ X^{A'} &= 0\,, \\ \partial_- X^{A'} &= 0\,, & \partial_+\lambda_-^{A'} + \partial_-\lambda_+^{A'} &= 0\,, \end{aligned} \tag{2.8}$$

[1] A corresponding *electric* Carroll model is obtained by a different scaling, $\hat{X}^A = X^A$, $\hat{X}^{A'} = \sqrt{\frac{\alpha'}{\alpha'_{\text{eff}}}}X^{A'}$. In this case, taking the $\alpha' \to \infty$ limit leads to $S_{\text{E}} = -\frac{1}{4\pi\alpha'_{\text{eff}}}\int d^2\sigma\sqrt{-\gamma}\gamma^{ab}\partial_a X^{A'}\partial_b X_{A'}$, which corresponds to the Polyakov action for the transverse coordinate fields alone.

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supplemented by constraints of the form

$$\partial_\pm X^A \partial_\pm X_A + 2\lambda_\pm^{A'} \partial_\pm X_{A'} = 0 \,. \tag{2.9}$$

It is immediately apparent that $X^{A'}$ is constant, i.e. there is no transverse motion in the magnetic Carroll model, and the constraints reduce to

$$\partial_\pm X^A \partial_\pm X_A = 0 \,. \tag{2.10}$$

The dispersion relation of magnetic Carroll strings,

$$(P^0)^2 = (P^1)^2 \,, \tag{2.11}$$

follows from the worldsheet field equations and constraints in a straightforward way. To see this we consider null combinations of the longitudinal coordinate fields, $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1)$, and fix one of them by the usual lightcone gauge condition,

$$X^+ = x^+ + \alpha'_{\text{eff}} P^+ \tau \,. \tag{2.12}$$

This is manifestly a solution of the two-dimensional wave equation as required by (2.8). The other longitudinal field is a solution of the wave equation and a periodic function of σ, which results in a general mode expansion of the form

$$X^- = x^- + \alpha'_{\text{eff}} P^- \tau + i\sqrt{\frac{\alpha'_{\text{eff}}}{2}} \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n^- e^{-in\sigma^-} + \tilde{\alpha}_n^- e^{-in\sigma^+} \right) . \tag{2.13}$$

The longitudinal fields must also satisfy the constraints (2.10), which reduce to

$$P^+ \partial_\pm X^- = 0 \,, \tag{2.14}$$

when the lightcone gauge condition (2.12) is satisfied. There are two cases to consider. If $P^+ \neq 0$ then the constraints require X^- to be a constant, which implies $P^- = 0$. On the other hand, if $P^+ = 0$ then X^- is unrestricted. In both cases the product of lightcone momenta vanishes, $P^+ P^- = 0$, which is equivalent to (2.11).

We note that the energy of the magnetic Carroll string is independent of the transverse momentum. The transverse momentum itself, however, remains unconstrained as it only appears in

$$P^{A'} = T_{\text{eff}} \int d\sigma \lambda_\tau^{A'} \,, \tag{2.15}$$

where the zero mode of $\lambda_\tau^{A'}$ remains undetermined by the constraints.

Finally, if the spatial longitudinal coordinate is compactified on a circle of radius R,

$$X^1 \sim X^1 + 2\pi R \,, \tag{2.16}$$

then the energy takes discrete values $P^0 = n/R$.

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2.2 Canonical formulation

In this subsection, we present the first-order formulation of the magnetic model to facilitate comparison with the Carroll string model found by Cardona, Gomis, and Pons (CGP) in [20]. The canonical momentum densities conjugate to the dynamical fields X^A and $X^{A'}$ of the magnetic Carroll action (2.4) are given by

$$\Pi_A = \frac{\delta \mathcal{L}_{\rm M}}{\delta \dot{X}^A} = -T_{\rm eff}\sqrt{-\gamma}\gamma^{a\tau}\partial_a X_A\,, \qquad \Pi_{A'} = \frac{\delta \mathcal{L}_{\rm M}}{\delta \dot{X}^{A'}} = -T_{\rm eff}\sqrt{-\gamma}\gamma^{a\tau}\lambda_{aA'}\,, \tag{2.17}$$

where the dot denotes a partial derivative with respect to τ. The canonical conjugate momenta of the worldsheet metric γ_{ab} and the auxiliary field $\lambda_{aA'}$ are identically zero and thus correspond to the primary constraints of the model. The canonical Hamilton density $\mathcal{H}_{\rm M} = \Pi_\mu \dot{X}^\mu - \mathcal{L}_{\rm M}$ can be written as a sum of a longitudinal and transverse part,

$$\begin{aligned}
\mathcal{H}^{\parallel}_{\rm M} &= \Pi_A\partial_\tau X^A + \frac{1}{2}T_{\rm eff}\sqrt{-\gamma}\gamma^{ab}\partial_a X^A\partial_b X_A \\
&= \frac{\sqrt{-\gamma}}{2\,T_{\rm eff}\,\gamma_{\sigma\sigma}}\left(\Pi_A\Pi^A + T_{\rm eff}^2\,\partial_\sigma X_A\partial_\sigma X^A\right) + \frac{\gamma_{\sigma\tau}}{\gamma_{\sigma\sigma}}\,\Pi_A\partial_\sigma X^A\,, \qquad (2.18)\\
\mathcal{H}^{\perp}_{\rm M} &= \Pi_{A'}\partial_\tau X^{A'} + T_{\rm eff}\sqrt{-\gamma}\gamma^{ab}\lambda_a^{A'}\partial_b X_{A'} \\
&= \frac{T_{\rm eff}\sqrt{-\gamma}}{\gamma_{\sigma\sigma}}\lambda_{\sigma A'}\partial_\sigma X^{A'} + \frac{\gamma_{\sigma\tau}}{\gamma_{\sigma\sigma}}\,\Pi_{A'}\partial_\sigma X^{A'}\,. \qquad (2.19)
\end{aligned}$$

We note that the longitudinal part has the standard form obtained from the Polyakov action. It follows that the Hamilton equations for the magnetic Carroll model are given by

$$\dot{\Pi}_A - \partial_\sigma\left(\frac{\gamma_{\sigma\tau}}{\gamma_{\sigma\sigma}}\Pi_A + \frac{T_{\rm eff}\sqrt{-\gamma}}{\gamma_{\sigma\sigma}}\,\partial_\sigma X_A\right) = 0\,, \tag{2.20}$$

$$\dot{X}^A - \frac{\gamma_{\sigma\tau}}{\gamma_{\sigma\sigma}}\partial_\sigma X^A - \frac{\sqrt{-\gamma}}{T_{\rm eff}\,\gamma_{\sigma\sigma}}\,\Pi_B\,\eta^{AB} = 0\,, \tag{2.21}$$

$$\dot{\Pi}_{A'} - \partial_\sigma\left(\frac{\gamma_{\sigma\tau}}{\gamma_{\sigma\sigma}}\,\dot{\Pi}_{A'} + \frac{T_{\rm eff}\sqrt{-\gamma}}{\gamma_{\sigma\sigma}}\,\lambda_{\sigma A'}\right) = 0\,, \tag{2.22}$$

$$\dot{X}^{A'} - \frac{\gamma_{\sigma\tau}}{\gamma_{\sigma\sigma}}\,\partial_\sigma X^{A'} = 0\,. \tag{2.23}$$

As expected, these equations reduce to (2.8) in the conformal gauge (2.5).

The action of the CGP Carroll string action in [20] is given by

$$S_{\rm CGP} = \int d^2\sigma\left(\Pi\cdot\dot{X} - \mu\,\Pi\cdot X' - \frac{e}{2}\left(\Pi_A\Pi^A + T_{\rm eff}^2\,\partial_\sigma X^{A'}\partial_\sigma X_{A'}\right)\right)\,, \tag{2.24}$$

where $\Pi\cdot\dot{X} \equiv \Pi_A\partial_\tau X^A + \Pi_{A'}\partial_\tau X^{A'}$ and $\Pi\cdot X' \equiv \Pi^A\partial_\sigma X_A + \Pi^{A'}\partial_\sigma X_{A'}$. Here μ and e are Lagrange multipliers enforcing the primary constraints. The action of the magnetic Carroll model (2.4) takes a similar form in first-order variables,

$$S_{\rm M} = \int d^2\sigma\left[\Pi\cdot\dot{X} - \frac{\gamma_{\sigma\tau}}{\gamma_{\sigma\sigma}}\Pi\cdot X' - \frac{\sqrt{-\gamma}}{2\,T_{\rm eff}\,\gamma_{\sigma\sigma}}\left(\Pi_A\Pi^A + T_{\rm eff}^2\left(X'^A X'_A + 2\lambda_\sigma^{A'}X'_{A'}\right)\right)\right]\,, \tag{2.25}$$

but there are significant differences between the two models. The longitudinal part of the magnetic Carroll action retains the standard Polyakov form while for the CGP action the

Carroll string	$\dot{X}^A = \frac{\partial H}{\partial \Pi_A}$	$\dot{\Pi}^A = -\frac{\partial H}{\partial X_A}$	$\dot{X}^{A'} = \frac{\partial H}{\partial \Pi_{A'}}$	$\dot{\Pi}^{A'} = -\frac{\partial H}{\partial X_{A'}}$
CGP [20]	$T_{\text{eff}}\dot{X}^A = \Pi^A$	$\dot{\Pi}^A = 0$	$\dot{X}^{A'} = 0$	$\dot{\Pi}^{A'} = T_{\text{eff}}\, X''^{A'}$
Magnetic model	$T_{\text{eff}}\dot{X}^A = \Pi^A$	$\dot{\Pi}^A = T_{\text{eff}}\, X''^A$	$\dot{X}^{A'} = 0$	$\dot{\lambda}^{A'}_\tau = \lambda'^{A'}_\sigma$
Chiral model	$T_{\text{eff}}\dot{X}^A = \Pi^A$	$\dot{\Pi}^A = T_{\text{eff}}\, X''^A$	$\dot{X}^{A'} = X'^{A'}$	$\dot{\lambda}^{A'}_+ = \lambda'^{A'}_+$

Table 1. Dynamical equations for three different Carroll string models in conformal gauge.

longitudinal part $T_{\text{eff}}X'^A X'_A$ is absent and in the transverse fields of the magnetic Carroll model are constrained by the auxiliary fields $\lambda^{A'}_\sigma$. Notably, neither model has a term quadratic in transverse momenta $\Pi^{A'}\Pi_{A'}$.

The Hamilton equations governing the dynamics of the two models in a conformal gauge are listed in table 1. For completeness, we also include the dynamical equations (also in a conformal gauge) of the chiral Carroll model studied in section 3 below. The field equations place strong restrictions on the dynamics in all three models. In particular, there is no transverse motion at all allowed in the CGP and magnetic Carroll models. The chiral model is less constrained in this respect and supports left-moving transverse modes. Longitudinal momentum is constant in the CGP Carroll model but the magnetic and the chiral Carroll models do not have that restriction.

2.3 Symmetries of the magnetic Carroll model

The magnetic Carroll action (2.4) is invariant under infinitesimal symmetry transformations of the form

$$\delta X^A = K^A + \Lambda^A{}_{A'}X^{A'} + \Lambda^A{}_B X^B\,, \tag{2.26a}$$

$$\delta X^{A'} = K^{A'} + \Lambda^{A'}{}_{B'}X^{B'}\,, \tag{2.26b}$$

$$\delta\lambda^{A'}_a = -\Lambda_{B'}{}^{A'}\lambda^{B'}_a - \Lambda_A{}^{A'}\partial_a X^A\,, \tag{2.26c}$$

where K^A and $K^{A'}$ denote longitudinal and transverse target space translations respectively, $\Lambda^A{}_B$ and $\Lambda^{A'}{}_{B'}$ denote a longitudinal boost and a transverse rotation[2] respectively, and $\Lambda^A{}_{A'}$ denotes a Carroll boost from the longitudinal sector to the transverse sector. The corresponding Noether charges are

$$\begin{aligned} P_A &= \int d\sigma\, \Pi_A\,, & P_{A'} &= \int d\sigma\, \Pi_{A'}\,,\\ M_{AB} &= 2\int d\sigma\, \Pi_{[A}X_{B]}\,, & M_{A'B'} &= 2\int d\sigma\, \Pi_{[A'}X_{B']}\,,\\ C_{AB'} &= \int d\sigma\, \Pi_A X_{B'}\,. \end{aligned} \tag{2.27}$$

The charges form a closed algebra with the non-vanishing commutators given by

$$[P_{A'}\,, C_{AB'}] = -\delta_{A'B'}P_A\,, \tag{2.28a}$$

$$[M_{AB}\,, C_{CA'}] = 2\eta_{C[B}\,C_{A]A'}\,, \tag{2.28b}$$

[2] Here both parameters are skew-symmetric, i.e. $\Lambda_{AB} = -\Lambda_{BA}$ and $\Lambda_{A'B'} = -\Lambda_{B'A'}$.

$$[M_{A'B'}\,,C_{AC'}] = -2\delta_{C'[A'}\,C_{|A|B']}\,, \tag{2.28c}$$
$$[M_{A'B'}\,,P_{C'}] = 2\delta_{C'[B'}P_{A']}\,, \tag{2.28d}$$
$$[M_{AB}\,,P_C] = 2\eta_{C[B}P_{A]}\,, \tag{2.28e}$$
$$[M_{AB}\,,M_{CD}] = 4\,\eta_{[C[B}M_{A]D]}\,, \tag{2.28f}$$
$$[M_{A'B'}\,,M_{C'D'}] = 4\,\delta_{[C'[B'}M_{A']D']}\,. \tag{2.28g}$$

This algebra was proposed in [20], where it was obtained as a contraction of the Poincaré algebra and referred to as the *string Carroll algebra*. We adopt the same terminology here.

The magnetic Carroll action (2.4) is also invariant under an infinitesimal dilatation with scale parameter ℓ that acts only on the transverse sector,

$$\delta X^{A'} = \ell\, X^{A'}\,, \qquad \delta\lambda_a^{A'} = -\ell\,\lambda_a^{A'}\,. \tag{2.29}$$

The corresponding Noether charge is

$$D = \frac{1}{2\pi\alpha'_{\text{eff}}}\int d\sigma\,\lambda_\tau^{A'}X_{A'} = \int d\sigma\,\Pi^{A'}X_{A'}\,, \tag{2.30}$$

and the non-vanishing commutators with the rest of the symmetry generators are

$$[D,\,P_{A'}] = P_{A'}\,, \qquad [D,\,C_{AA'}] = -C_{AA'}\,. \tag{2.31}$$

This is reminiscent of the additional longitudinal dilatation symmetry found in non-relativistic string theory [26], which gives rise to an additional gauge field in the gauging procedure for the string Newton-Cartan algebra. Here, the transverse dilatation symmetry (2.29) is expected to be associated with an extra gauge field when gauging the string Carroll algebra (2.28).

The non-relativistic string in flat spacetime admits an infinite dimensional symmetry algebra, which contains the string Newton-Cartan algebra as a finite dimensional subalgebra [26, 27]. A similar story plays out in the CGP Carroll string model [20] and also in the Carroll string models considered here. The magnetic and chiral Carroll models admit infinite dimensional symmetries, which contain as finite dimensional subalgebras the string Carroll algebra and the extended string Carroll algebra (defined in section 3.3 below), respectively. As it turns out, the symmetries of the magnetic Carroll model are a subset of those of the chiral Carroll model and therefore we postpone the discussion of the infinite dimensional symmetries until we have introduced the chiral Carroll model in section 3.

3 Chiral Carroll string model

In this section, we consider a novel Carroll limit on target space. The construction involves the same asymmetric scaling of longitudinal and transverse coordinates as in the magnetic Carroll model but this time around an extra set of auxiliary fields is introduced that allows us to impose a worldsheet chirality on the transverse coordinate fields. The resulting model realises a non-trivial extension of string Carroll symmetry and the transverse dynamics turn out to be less constrained than in the magnetic Carroll model.

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3.1 Relativistic origin of the chiral Carroll model

Our starting point is once again the Polyakov action (2.1) with a rescaling of the coordinate fields as in (2.2) but now we use two sets of auxiliary fields $\lambda_{aA'}$ and $\chi_{aA'}$ to rewrite the action,[3]

$$\begin{aligned} S_{\mathrm{P}} &= -\frac{1}{4\pi\alpha'_{\mathrm{eff}}}\int d^2\sigma\sqrt{-\gamma}\gamma^{ab}\left[\partial_a X^A\partial_b X_A + \frac{\alpha'}{\alpha'_{\mathrm{eff}}}\partial_a X^{A'}\partial_b X_{A'}\right] \\ &= -\frac{1}{4\pi\alpha'_{\mathrm{eff}}}\int d^2\sigma\sqrt{-\gamma}\Big[\gamma^{ab}\partial_a X^A\partial_b X_A + 2\,\Gamma_+^{ab}\lambda_{aA'}\partial_b X^{A'} \\ &\qquad + \frac{2\alpha'_{\mathrm{eff}}}{\alpha'}\Gamma_-^{ab}\chi_{aA'}\partial_b X^{A'} - 2\left(\frac{\alpha'_{\mathrm{eff}}}{\alpha'}\right)^2\gamma^{ab}\lambda_a^{A'}\chi_{bA'}\Big]\,, \end{aligned} \tag{3.1}$$

where we have introduced the chiral projection operators

$$\Gamma_\pm^{ab} = \frac{1}{2}\left(\gamma^{ab} \pm \epsilon^{ab}\right)\,, \tag{3.2}$$

with ϵ^{ab} the Levi-Civita tensor. We employ the conventions $\epsilon^{ab} = \bar{\epsilon}^{ab}/\sqrt{-\gamma}$, $\epsilon_{ab} = \bar{\epsilon}_{ab}\sqrt{-\gamma}$, where $\bar{\epsilon}^{ab}$ denotes the Levi-Civita symbol, $\bar{\epsilon}^{01} = +1 = -\bar{\epsilon}^{10}$. We can now readily take the limit $\alpha' \to \infty$ to obtain a *chiral Carroll string model*,

$$S_{\mathrm{C}} = -\frac{1}{4\pi\alpha'_{\mathrm{eff}}}\int d^2\sigma\sqrt{-\gamma}\left(\gamma^{ab}\partial_a X^A\partial_b X_A + 2\,\Gamma_+^{ab}\lambda_a^{A'}\partial_b X_{A'}\right)\,. \tag{3.3}$$

Upon adopting the conformal gauge (2.5) the chiral Carroll action reduces to

$$S_{\mathrm{C}} = \frac{1}{\pi\alpha'_{\mathrm{eff}}}\int d^2\sigma\left(\partial_+ X^A\partial_- X_A + \lambda_+^{A'}\partial_- X_{A'}\right)\,. \tag{3.4}$$

Compared to the magnetic model (2.6) we have effectively projected out one worldsheet chirality of the transverse fields $X^{A'}$ and $\lambda_{aA'}$.[4] The new auxiliary fields $\chi_{aA'}$ do not appear in the final action in the $\alpha' \to \infty$ limit but they nevertheless play an important role in implementing the projection onto a single worldsheet chirality. In conformal gauge the equations of motion obtained from the chiral Carroll action reduce to

$$\partial_+\partial_- X^A = \partial_-\lambda_+^{A'} = \partial_- X^{A'} = 0\,. \tag{3.5}$$

This is a subset of the field equations of the magnetic model (2.8) and therefore the dynamics is now less restrictive than before. In particular, the transverse fields $X^{A'}$ can have non-trivial dependence on the σ^+ direction in the chiral model, in contrast to the magnetic model

[3] The equations of motion for $\lambda_a^{A'}$ and $\chi_a^{A'}$ implied by (3.1) are covariantly given by:

$$\chi_a^{A'} = \left(\frac{\alpha'}{\alpha'_{\mathrm{eff}}}\right)^2\gamma_{ab}\Gamma_+^{bc}\partial_c X^{A'}\,, \qquad \lambda_a^{A'} = \left(\frac{\alpha'}{\alpha'_{\mathrm{eff}}}\right)\gamma_{ab}\Gamma_-^{bc}\partial_c X^{A'}\,.$$

[4] We have chosen to retain the left-moving fields $\lambda_{+A'}$. We could of course just as well have projected onto the other chirality.

where the transverse fields are only allowed to take constant values. The energy momentum tensor constraints are now given by

$$\begin{aligned} 0 &= \partial_+ X^A \partial_+ X_A + \lambda_+^{A'} \partial_+ X_{A'} \,, \\ 0 &= \partial_- X^A \partial_- X_A \,. \end{aligned} \tag{3.6}$$

By implementing the lightcone gauge condition

$$X^+ = x^+ + \alpha'_{\text{eff}} P^+ \tau \,, \tag{3.7}$$

and inserting it into the constraints we find that

$$\begin{aligned} \partial_+ X^- &= \frac{1}{\alpha'_{\text{eff}} P^+} \lambda_+^{A'} \partial_+ X_{A'} \,, \\ \partial_- X^- &= 0 \,. \end{aligned} \tag{3.8}$$

From the latter it follows that X^- is also chiral and hence the mode expansion for X^- can be written in the following form

$$X^-(\sigma^-, \sigma^+) = X_L^-(\sigma^+) = x^- + \frac{1}{2}\alpha'_{\text{eff}} P^- \sigma^+ + i\sqrt{\frac{\alpha'_{\text{eff}}}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^- e^{-in\sigma^+} \,. \tag{3.9}$$

However, since X^- is periodic in σ this then implies that $P^- = 0$ i.e. that $E = P_1$. Therefore, although we have transverse excitations, they do not contribute to the energy. Now consider the transverse fields which are left-moving and periodic in σ,

$$\begin{aligned} X^{A'}(\sigma^+) &= x^{A'} + i\sqrt{\frac{\alpha'_{\text{eff}}}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^{A'} e^{-in\sigma^+} \,, \\ \lambda_+^{A'}(\sigma^+) &= \sqrt{\frac{\alpha'_{\text{eff}}}{2}} \sum_n \lambda_n^{A'} e^{-in\sigma^+} \,, \end{aligned} \tag{3.10}$$

where we note in particular that the transverse momenta $P^{A'}$ are related to the zero modes of the auxiliary fields $\lambda^{A'}$ via

$$P^{A'} = \int_0^{2\pi} d\sigma \Pi^{A'} = \frac{1}{\sqrt{2\alpha'_{\text{eff}}}} \lambda_0^{A'} \,. \tag{3.11}$$

As usual, the longitudinal α_n^- are obtained by inserting the mode expansions into the constraint in (3.8). We find that for $n \neq 0$ we have

$$\alpha_n^- = \frac{1}{P^+ \sqrt{2\alpha'_{\text{eff}}}} \sum_m \lambda_m^{A'} \alpha_{n-m}^{B'} \delta_{A'B'} \,, \tag{3.12}$$

where we have additionally defined $\alpha_0^{A'} = 0$ and $\alpha_0^- = 0$ for notational convenience. The zero mode of the constraint then implies that

$$\sum_n \lambda_n^{A'} \alpha_{-n}^{B'} \delta_{A'B'} = 0 \,, \tag{3.13}$$

which says the total left-moving excitation level must be zero after all.

In order to have non-zero string excitations one needs to compactify a transverse direction, say $A' = D-1$, such that

$$X^{D-1}(\sigma+2\pi) = X^{D-1}(\sigma) + 2\pi w\mathcal{R}\,, \tag{3.14}$$

where w is the winding number. We now have a zero mode and a quantised momentum in the compact direction,

$$\alpha_0^{D-1} = \sqrt{\frac{2}{\alpha'_{\text{eff}}}}w\mathcal{R}\,, \qquad P^{D-1} = \frac{n}{\mathcal{R}}\,, \tag{3.15}$$

and the two constraint equations in (3.8) become

$$E = P_1\,, \qquad \sum_{n>0} \lambda_n^{A'}\alpha_{-n}^{B'}\delta_{A'B'} = -wn\,, \tag{3.16}$$

where we can interpret the second equation as a level matching condition that determines the left-moving excitation level in terms of the winding number.

3.2 Canonical formulation

The canonical conjugate momenta for the longitudinal and transverse worldsheet fields of the chiral Carroll action (3.3) are given by

$$\Pi_A = \frac{\delta\mathcal{L}_{\text{C}}}{\delta\dot{X}^A} = -T_{\text{eff}}\sqrt{-\gamma}\gamma^{a\tau}\partial_a X_A\,, \qquad \Pi_{A'} = \frac{\delta\mathcal{L}_{\text{C}}}{\delta\dot{X}^{A'}} = -T_{\text{eff}}\sqrt{-\gamma}\,\Gamma_+^{a\tau}\lambda_{aA'}\,. \tag{3.17}$$

The canonical Hamilton density $\mathcal{H}_{\text{M}} = \Pi_\mu\dot{X}^\mu - \mathcal{L}_{\text{M}}$ can be written as a sum of a longitudinal and transverse part,

$$\begin{aligned}\mathcal{H}_{\text{C}}^{\parallel} &= \Pi_A\partial_\tau X^A + \frac{1}{2}T_{\text{eff}}\sqrt{-\gamma}\gamma^{ab}\partial_a X^A\partial_b X_A \\ &= \frac{\sqrt{-\gamma}}{2\,T_{\text{eff}}\,\gamma_{\sigma\sigma}}\left(\Pi_A\Pi^A + T_{\text{eff}}^2\,\partial_\sigma X_A\partial_\sigma X^A\right) + \frac{\gamma_{\sigma\tau}}{\gamma_{\sigma\sigma}}\,\Pi_A\partial_\sigma X^A\,, \end{aligned}\tag{3.18}$$

$$\begin{aligned}\mathcal{H}_{\text{C}}^{\perp} &= \Pi_{A'}\partial_\tau X^{A'} + T_{\text{eff}}\sqrt{-\gamma}\gamma^{ab}\lambda_a^{A'}\partial_b X_{A'} \\ &= \frac{(\gamma_{\sigma\tau}+\epsilon_{\sigma\tau})}{\gamma_{\sigma\sigma}}\,\Pi_{A'}\partial_\sigma X^{A'}\,. \end{aligned}\tag{3.19}$$

The longitudinal part is given by the same Polyakov expression as we found for the magnetic model in (2.18). The transverse part differs from the magnetic model in that it does not involve the auxiliary fields at all and incorporates chirality through the anti-symmetric Levi-Civita tensor ϵ_{ab}. The Hamilton equations for the chiral model are

$$\dot{\Pi}_A - \partial_\sigma\left(\frac{\gamma_{\sigma\tau}}{\gamma_{\sigma\sigma}}\Pi_A + \frac{T_{\text{eff}}\sqrt{-\gamma}}{\gamma_{\sigma\sigma}}\,\partial_\sigma X_A\right) = 0\,, \tag{3.20}$$

$$\dot{X}^A - \frac{\gamma_{\sigma\tau}}{\gamma_{\sigma\sigma}}\partial_\sigma X^A - \frac{\sqrt{-\gamma}}{T_{\text{eff}}\,\gamma_{\sigma\sigma}}\,\Pi_B\,\eta^{AB} = 0\,, \tag{3.21}$$

$$\dot{\Pi}_{A'} - \partial_\sigma\left(\frac{(\gamma_{\sigma\tau}+\epsilon_{\sigma\tau})}{\gamma_{\sigma\sigma}}\,\dot{\Pi}_{A'}\right) = 0\,, \tag{3.22}$$

$$\dot{X}^{A'} - \frac{(\gamma_{\sigma\tau}+\epsilon_{\sigma\tau})}{\gamma_{\sigma\sigma}}\,X'^{A'} = 0\,. \tag{3.23}$$

In the conformal gauge (2.5), these Hamilton equations reproduce (3.5) and they are also listed in table 1 above for comparison with the CGP and magnetic Carroll models. The field equations of the chiral Carroll model only require the transverse fields $X^{A'}$ and their conjugate momenta to be constant along the σ^- worldsheet direction. Therefore, unlike the CGP and magnetic Carroll models, the chiral model allows for transverse excitations moving in one direction along the string.

3.3 Symmetries of the chiral Carroll model

The chiral Carroll action (3.3) is invariant under infinitesimal symmetry transformations on the target space of the form

$$\begin{aligned}\delta X^A &= \Lambda^A{}_{A'}X^{A'} + \Lambda^A{}_B X^B + K^A\,,\\ \delta X^{A'} &= \Lambda^{A'}{}_{B'}X^{A'} + K^{A'}\,,\\ \delta\lambda_{aA'} &= -\Lambda^{B'}{}_{A'}\lambda_{aB'} - \Lambda^A{}_{A'}\partial_a X_A\,,\end{aligned} \tag{3.24}$$

where $\{K^A, K^{A'}, \Lambda^A{}_B, \Lambda^{A'}{}_{B'}, \Lambda^A{}_{A'}\}$ denote respectively longitudinal and transverse translations, longitudinal and transverse rotations, and Carroll string boosts. The corresponding Noether charges are given by

$$\begin{aligned} P_A &= \int d\sigma\,\Pi_A\,, & P_{A'} &= \int d\sigma\,\Pi_{A'}\,,\\ M_{AB} &= 2\int d\sigma\,\Pi_{[A}X_{B]}\,, & M_{A'B'} &= 2\int d\sigma\,\Pi_{[A'}X_{B']}\,,\\ C_{AB'} &= \int d\sigma\,(\Pi_A X_{B'} - T_{\text{eff}}\,X_A\partial_\sigma X_{B'})\,, \end{aligned} \tag{3.25}$$

where we note that the extra term in the Carroll string boosts $C_{AB'}$ compared with (2.27) arises from a total derivative in the symmetry invariance of the action. The non-vanishing commutators are

$$[Z_{A'B'}, P_{C'}] = 2\delta_{C'[A'}Z_{B']} \tag{3.26a}$$
$$[M_{A'B'}, Z_{C'}] = 2\delta_{C'[A'}Z_{B']}, \tag{3.26b}$$
$$[M_{A'B'}, Z_{C'D'}] = 4\delta_{[C'[A'}Z_{B']D']}, \tag{3.26c}$$
$$[M_{AB}, P_C] = 2\eta_{C[B}P_{A]}, \tag{3.26d}$$
$$[C_{AB'}, P_C] = -\eta_{CA}Z_{B'}, \tag{3.26e}$$
$$[C_{AB'}, P_{C'}] = \delta_{C'B'}P_A, \tag{3.26f}$$
$$[M_{A'B'}, P_{C'}] = -2\delta_{C'[A'}P_{B']}, \tag{3.26g}$$
$$[M_{AB}, M_{CD}] = 4\eta_{[C[A}M_{B]D]} \tag{3.26h}$$
$$[M_{A'B'}, M_{C'D'}] = 4\delta_{[C'[A'}M_{B']D']} \tag{3.26i}$$
$$[M_{AB}, C_{CD'}] = -2\eta_{C[A}C_{B]D'}, \tag{3.26j}$$
$$[M_{A'B'}, C_{CD'}] = -2C_{C[A'}\delta_{B']D'}, \tag{3.26k}$$
$$[C_{AB'}, C_{CD'}] = \eta_{CA}Z_{B'D'}, \tag{3.26l}$$

where we have defined additional charges,

$$Z_{A'} = T_{\text{eff}} \int d\sigma\, \partial_\sigma X_{A'}\,, \qquad\qquad Z_{A'B'} = Z_{[A'B']} = 2\,T_{\text{eff}} \int d\sigma\, X_{[A'}\, \partial_\sigma X_{B']}\,. \tag{3.27}$$

With these extensions included, we have a closed algebra that we refer to as the *extended string Carroll algebra*. This algebra can also be derived from an İnönü-Wigner contraction of the coadjoint Poincaré algebra as we will explicitly show in subsection 3.5 below.

Finally, we mention that the algebra (3.26) can be further enlarged by an emergent scaling symmetry D as in (2.30). This adds the following non-vanishing commutators

$$[D, P_{A'}] = P_{A'}\,, \quad [D, Z_{A'}] = -Z_{A'}\,, \quad [D, C_{AA'}] = -C_{AA'}\,, \quad [D, Z_{A'B'}] = -2Z_{A'B'}\,. \tag{3.28}$$

In the next subsection, we will see how the scaling symmetry arises from an infinite dimensional extension of (3.26).

3.4 Infinite dimensional symmetries of the chiral Carroll action

The chiral Carroll action (3.3) is in fact invariant under an infinite dimensional symmetry,[5]

$$\delta X^A = f^A(X^{A'}) + \Omega^A{}_B(X^{A'})X^B\,, \tag{3.29a}$$

$$\delta X^{A'} = \xi^{A'}(X^{B'})\,, \tag{3.29b}$$

$$\delta\lambda_{aA'} = -\partial_a X_A \partial_{A'} f^A - \partial_a X_A \partial_{A'} \Omega^A{}_B X^B - \lambda_{aB'}\partial_{A'}\xi^{B'}\,, \tag{3.29c}$$

where $f^A, \Omega^A{}_B$ and $\xi^{A'}$ are arbitrary functions that depend only on the transverse coordinates $X^{A'}$ and $\Omega^A{}_B$ is anti-symmetric, $\Omega_{AB} = -\Omega_{BA}$. In the specific case of

$$f^A(X^{A'}) = K^A + \Lambda^A{}_{A'}X^{A'}\,, \quad \Omega^A{}_B(X^{B'}) = \Lambda^A{}_B\,, \quad \xi^{A'}(X^{B'}) = K^{A'} + \Lambda^{A'}{}_{B'}X^{B'}\,, \tag{3.30}$$

we recover the target space symmetries (3.24) discussed above. Note that under the infinitesimal transformation $\delta X^A = f^A(X^{A'})$ the action is quasi-symmetric since under such a transformation we obtain a total derivative term,

$$\delta S_{\text{C}} = \int d^2\sigma\, \partial_a \left(T_{\text{eff}}\sqrt{-\gamma}\,\epsilon^{ab}\,\partial_{A'} f^A X_A \partial_b X^{A'}\right)\,, \tag{3.31}$$

which gives an additional contribution to the conserved charges associated with f^A. Now consider expanding the functions $f^A, \Omega^A{}_B$ and $\xi^{A'}$ in a Taylor series of the transverse coordinates,

$$f^A(X^{A'}) = a^A + a^A{}_{A'}X^{A'} + a^A{}_{A'B'}X^{A'}X^{B'} + \ldots = a^A{}_{A'_1A'_2\cdots A'_n}X^{A'_1A'_2\cdots A'_n}, \tag{3.32a}$$

$$\Omega^A{}_B(X^{A'}) = \omega^A{}_B + \omega^A{}_{BA'}X^{A'} + \omega^A{}_{BA'B'}X^{A'}X^{B'} + \ldots = \omega^A{}_{BA'_1A'_2\cdots A'_n}X^{A'_1A'_2\ldots A'_n}, \tag{3.32b}$$

$$\xi^{A'}(X^{B'}) = \varphi^{A'} + \varphi^{A'}{}_{B'}X^{B'} + \varphi^{A'}{}_{B'C'}X^{B'}X^{C'} + \ldots = \varphi^{A'}{}_{A'_1A'_2\cdots A'_n}X^{A'_1A'_2\ldots A'_n}, \tag{3.32c}$$

[5]These symmetry transformations can be derived through the conservation of a generator of canonical symmetry transformations G, given by

$$G = \int d\sigma \left(\xi^A\Pi_A + \xi^{A'}\Pi_{A'} + \psi_a^{A'}\Pi_{aA'} + \Lambda\pi_e + \gamma\pi_\mu\right)$$

such that $\delta X^A = \{G, X^A\} = \xi^A, \delta X^{A'} = \{G, X^{A'}\} = \xi^{A'}$, and $\delta\lambda_a^{A'} = \{G, \lambda_a^{A'}\} = \psi_a^{A'}$, where the functions $\xi^A, \xi^{A'}, \psi_a^{A'}, \Lambda$, and γ satisfy differential equations obtained from $dG/d\tau = 0$.

where we have introduced the short hand notation $X^{A'_1 A'_2\cdots A'_n} = X^{A'_1}X^{A'_2}\cdots X^{A'_n}$. For each of the functions $f^A, \Omega^A{}_B$ and $\xi^{A'}$, there is an infinite tower of conserved charges given by

$$\xi^{A'}: \qquad K^{(n)}_{A'A'_1A'_2\cdots A'_n} = \int d\sigma\, \Pi_{A'} X_{A'_1A'_2\cdots A'_n}\,, \tag{3.33a}$$

$$f^A: \qquad L^{(n)}_{AA'_1A'_2\cdots A'_n} = \int d\sigma\, \left(\Pi_A X_{A'_1A'_2\cdots A'_n} - T_{\text{eff}}\, X_A \partial_\sigma \left(X_{A'_1\cdots A'_n}\right)\right)\,, \tag{3.33b}$$

$$\Omega^A{}_B: \qquad M^{(n)}_{ABA'_1A'_2\cdots A'_n} = 2\int d\sigma\, \Pi_{[A} X_{B]} X_{A'_1A'_2\cdots A'_n}\,. \tag{3.33c}$$

where the index $n \geq 0$ denotes the number of additional transverse indices. For instance, we recover the charges in (3.25) as follows,

$$P_{A'} = K^{(0)}_{A'}\,, \qquad P_A = L^{(0)}_A\,, \qquad M_{AB} = M^{(0)}_{AB}\,, \qquad C_{AB'} = L^{(1)}_{AB'}\,, \qquad M_{A'B'} = 2K^{(1)}_{[A'B']}\,. \tag{3.34}$$

The infinite tower of charges in (3.33) satisfies an infinite dimensional Lie algebra,

$$\begin{aligned}
\left[K^{(n)}_{A'A'_1\cdots A'_n}, K^{(m)}_{B'B'_1\cdots B'_m}\right] &= \sum_{k=1}^{n} \delta_{B'A'_k} K^{(n+m-1)}_{A'A'_1\cdots A'_{k-1}A'_{k+1}\cdots A'_n B'_1\cdots B'_m} \\
&\quad - \sum_{k=1}^{m} \delta_{A'B'_k} K^{(n+m-1)}_{B'A'_1\cdots A'_n B'_1\cdots B'_{k-1}B'_{k+1}\cdots B'_m}\,, && (3.35a)\\
\left[K^{(n)}_{A'A'_1\cdots A'_n}, L^{(m)}_{BB'_1\cdots B'_m}\right] &= -\sum_{k=1}^{m} \delta_{A'B'_k} L^{(m+n-1)}_{BA'_1\cdots A'_n B'_1\cdots B'_{k-1}B'_{k+1}\cdots B'_m}\,, && (3.35b)\\
\left[K^{(n)}_{A'A'_1\cdots A'_n}, M^{(m)}_{BCB'_1\cdots B'_m}\right] &= -\sum_{k=1}^{m} \delta_{A'B'_k} M^{(n+m-1)}_{BCA'_1\cdots A'_n B'_1\cdots B'_{k-1}B'_{k+1}\cdots B'_m}\,, && (3.35c)\\
\left[L^{(n)}_{AA'_1\cdots A'_n}, L^{(m)}_{BB'_1\cdots B'_m}\right] &= \eta_{AB}\, Z^{(n,m)}_{A'_1\cdots A'_n B'_1\cdots B'_m}\,, && (3.35d)\\
\left[L^{(n)}_{AA'_1\cdots A'_n}, M^{(m)}_{BCB'_1\cdots B'_m}\right] &= 2\eta_{A[B} L^{(n+m)}_{C]A'_1\cdots A'_n B'_1\cdots B'_m}\,, && (3.35e)\\
\left[M^{(n)}_{ABA'_1\cdots A'_n}, M^{(m)}_{CDB'_1\cdots B'm}\right] &= 4\eta_{[C[B} M^{(n+m)}_{A]D]A'_1\cdots A'_n B'_1\cdots B'_m}\,, && (3.35f)
\end{aligned}$$

where we have again introduced a set of extension generators,

$$Z^{(n,m)}_{A'_1\cdots A'_n B'_1\cdots B'_m} = T_{\text{eff}} \int d\sigma\, \left(X_{A'_1\cdots A'_n}\partial_\sigma \left(X_{B'_1\cdots B'_m}\right) - X_{B'_1\cdots B'_n}\partial_\sigma \left(X_{A'_1\cdots A'_m}\right)\right)\,. \tag{3.36}$$

In the specific case of

$$Z^{(0,1)}_{A'} = T_{\text{eff}} \int d\sigma \partial_\sigma X_{A'}\,, \qquad Z^{(1,1)}_{A'B'} = T_{\text{eff}} \int d\sigma X_{[A'}\partial_\sigma X_{B']}\,, \tag{3.37}$$

we recover the extensions defined in (3.27). The only additional non-vanishing commutators that arise from adding these extensions are given by

$$\begin{aligned}
\left[Z^{(n,m)}_{A'_1\cdots A'_n B'_1\cdots B'_m}, K^{(\ell)}_{C'C'_1\cdots C'_\ell}\right] &= \sum_{k=1}^{m} \delta_{C'B'_k}\, Z^{(m-1,l+n)}_{B'_1\cdots B'_{k-1}B'_{k+1}\cdots B'_m C'_1\cdots C'_l A'_1\cdots A'_n} \\
&\quad + \sum_{k=1}^{n} \delta_{C'A'_k}\, Z^{(n-1,l+m)}_{A'_1\cdots A'_{k-1}A'_{k+1}\cdots A'_n C'_1\cdots C'_l B'_1\cdots B'_m}\,. && (3.38)
\end{aligned}$$

The magnetic Carroll model (2.4) also has an infinite dimensional symmetry algebra which is simply the subalgebra of (3.35) obtained by setting the extensions (3.36) to zero.

The symmetry algebras of both the chiral and magnetic Carroll models include the following generators,

$$D \equiv \delta^{A'B'} K^{(1)}_{A'B'} = \int d\sigma\, \Pi_{A'} X^{A'} , \tag{3.39a}$$

$$K_{A'} \equiv 2\delta^{B'C'} K^{(2)}_{B'C'A'} - \delta^{B'C'} K^{(2)}_{A'B'C'} = \int d\sigma \left(2\Pi_{B'} X^{B'} X_{A'} - \Pi_{A'} X^{B'} X_{B'}\right) . \tag{3.39b}$$

The former corresponds to spatial dilatations and the latter corresponds to spatial special conformal transformations also found for the CGP Carroll string [20]. Furthermore, we can construct a longitudinal conformal transformation generator

$$K_A = \delta^{A'B'} L^{(2)}_{A\,A'B'} = \int d\sigma \left(\Pi_A X_{A'} X^{A'} - T_{\text{eff}} X_A \partial_\sigma \left(X^{A'} X_{A'}\right)\right) . \tag{3.40}$$

The three generators in (3.39) and (3.40) are somewhat similar to the generators of the conformal Carroll algebra given by [28, 29],

$$\tilde{D} = \int d\sigma \left(\Pi^A X_A + \Pi^{A'} X_{A'}\right) , \tag{3.41a}$$

$$\tilde{K}_A = \int d\sigma X^{A'} X_{A'} \Pi_A , \tag{3.41b}$$

$$\tilde{K}_{A'} = \int d\sigma \left(2X_{A'}(X^A \Pi_A + X^{B'} \Pi_{B'}) - (X^{B'} X_{B'})\Pi_{A'}\right) . \tag{3.41c}$$

but there are important differences. In particular, the longitudinal parts of $\tilde{D}$ and $\tilde{K}_{A'}$ are absent from our spatial special conformal transformations in equation (3.39) and we have an additional term associated with the quasi-symmetry in comparing $\tilde{K}_A$ with (3.40).

3.5 Extended Carroll algebra from the coadjoint Poincaré algebra

In this section, we derive the extended string Carroll algebra (3.26) through an İnönü-Wigner contraction of the coadjoint Poincaré algebra. The procedure is as follows: Let us consider an extension of the Poincaré algebra enlarged by the antisymmetric two-tensor $\hat{Z}_{\mu\nu}$ and the vector $\hat{Z}_\mu$ (see, for example, [30]), which consists of the following non-vanishing relativistic brackets[6]

$$\left[\hat{M}_{\mu\nu}, \hat{M}_{\rho\sigma}\right] = 4\eta_{[\rho[\mu} \hat{M}_{\nu]\sigma]} , \tag{3.42}$$

$$\left[\hat{M}_{\mu\nu}, \hat{P}_{\rho}\right] = 2\eta_{\rho[\mu} \hat{P}_{\nu]}, \tag{3.43}$$

$$\left[\hat{M}_{\mu\nu}, \hat{Z}_{\rho}\right] = 2\eta_{\rho[\mu} \hat{Z}_{\nu]}, \tag{3.44}$$

$$\left[\hat{M}_{\mu\nu}, \hat{Z}_{\rho\sigma}\right] = 4\eta_{[\rho[\mu} \hat{Z}_{\nu]\sigma]} , \tag{3.45}$$

$$\left[\hat{Z}_{\mu\nu}, \hat{P}_{\rho}\right] = 2\eta_{\rho[\mu} \hat{Z}_{\nu]} , \tag{3.46}$$

along with the following vanishing brackets

$$\left[\hat{P}_\mu, \hat{P}_\nu\right] = \left[\hat{Z}_\mu, \hat{P}_\nu\right] = \left[\hat{Z}_\mu, \hat{Z}_\nu\right] = \left[\hat{Z}_{\mu\nu}, \hat{Z}_\rho\right] = \left[\hat{Z}_{\mu\nu}, \hat{Z}_{\rho\sigma}\right] = 0 . \tag{3.47}$$

[6]The hatted generators will be the ones associated to the relativistic algebra.

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In order to obtain the extended string Carroll algebra from the algebra above, we scale the generators in the following manner

$$\hat{P}_A = \omega\, P_A\,, \qquad\qquad \hat{Z}_A = \omega\, Z_A, \tag{3.48}$$

$$\hat{P}_{A'} = P_{A'} + \frac{\omega^2}{2}\, Z_{A'}\,, \qquad\qquad \hat{Z}_{A'} = \omega\, Z_{A'}\,, \tag{3.49}$$

$$\hat{M}_{AB} = M_{AB}\,, \qquad\qquad \hat{Z}_{AB} = Z_{AB}\,, \tag{3.50}$$

$$\hat{M}_{A'B'} = M_{A'B'}\,, \qquad\qquad \hat{Z}_{A'B'} = \omega\, Z_{A'B'}\,, \tag{3.51}$$

$$\hat{M}_{AB'} = \omega\, C_{AB'} - \frac{\omega^2}{2} Z_{AB'}\,, \qquad\qquad \hat{Z}_{AB'} = \omega\, Z_{AB'}\,, \tag{3.52}$$

where we introduced longitudinal indices $A = 0, 1$ and transverse indices $A' = 2, \dots, D-1$, and ω a dimensionless parameter. These scaling choices are inspired by Galilean results of 'case 2b' appearing in [30]. The contraction is carried out by taking the $\omega \to \infty$ limit, and we obtain the extended string Carroll algebra (3.26) supplemented with the following additional brackets

$$[M_{AB}, Z_C] = 2\eta_{C[A} Z_{B]}\,, \qquad\qquad [Z_{AB'}, P_{C'}] = -\delta_{C'B'} Z_A\,, \tag{3.53}$$

$$[M_{AB}, Z_{CD}] = 4\eta_{[C[A} Z_{B]D]}\,, \qquad\qquad [Z_{AB}, P_C] = 2\eta_{C[A} Z_{B]}\,, \tag{3.54}$$

$$[M_{AB}, Z_{CD'}] = 2\eta_{C[A} Z_{B]D'}\,, \qquad\qquad [C_{AB'}, Z_{CD}] = -2\eta_{A[C} Z_{D]B'}\,, \tag{3.55}$$

$$[M_{A'B'}, Z_{CD'}] = 2Z_{C[A'}\delta_{B']D'}\,, \qquad\qquad [M_{AB}, Z_{CD'}] = 2\eta_{A[C} Z_{B]D'}\,. \tag{3.56}$$

Therefore, the resulting algebra can be written as $\mathcal{A} \oplus \mathcal{I}$ such that

$$[\mathcal{A}, \mathcal{I}] \subseteq \mathcal{I}\,, \qquad [\mathcal{A}, \mathcal{A}] \subseteq \mathcal{A}\,, \qquad [\mathcal{I}, \mathcal{I}] \subseteq \mathcal{I}\,, \tag{3.57}$$

where the subalgebra $\mathcal{A}$ and ideal $\mathcal{I}$ are

$$\mathcal{A} = \{M_{AB}, M_{A'B'}, C_{AB'}, P_A, P_{A'}, Z_{A'}, Z_{A'B'}\}\,, \quad \text{and} \quad \mathcal{I} = \{Z_{AB'}, Z_{AB}, Z_A\}\,. \tag{3.58}$$

Taking the quotient of $\mathcal{A}/\mathcal{I}$ we recover the extended string Carroll algebra (3.26).

4 Discussion

We have presented two inequivalent ways of taking a Carroll limit of the flat target space of the Polyakov action, yielding two Carroll string models that we refer to as magnetic and chiral. Both models make use of auxiliary fields that render the action finite in the Carroll limit. In the chiral model a second set of auxiliary fields projects out one worldsheet chirality, leaving only left-moving string excitations.

The magnetic model, described in section 2, has a global symmetry algebra on the target space that coincides with the string Carroll algebra found for the CGP Carroll string model of Cardona et al. in [20]. The magnetic model and the CGP model also share the feature that the Carroll strings can carry arbitrary transverse momentum but do not move in the transverse directions, reminiscent of Carroll particles.

The dynamics of the chiral model, described in section 3, is less constrained and allows arbitrary left-moving transverse string excitations, which does not have a counterpart when studying Carroll particles. The global symmetry of the chiral Carroll model is an extension of the string Carroll algebra, enlarged by a vector generator Z_A and an anti-symmetric generator Z_{AB}. We show how this extended string Carroll algebra can be obtained from a contraction of the coadjoint Poincaré algebra.

We further show how the global symmetries of chiral Carroll string model are embedded in an infinite dimensional symmetry algebra, which contains the infinite dimensional symmetry of the magnetic Carroll string as a subalgebra. An analogous infinite dimensional symmetry structure was found previously for strings with Galilean symmetry in [26, 27].

Here we have only considered closed strings and we required at least one of the transverse directions to be compactified in order to have non-trivial (left-moving) transverse string excitations in the chiral Carroll model. As a result, open strings will not have transverse modes in the models and it remains an open question whether open Carroll strings can have interesting dynamics via some other construction.

We considered a flat target space throughout. The authors of [31, 32] considered a series expansion for a curved target space in the Galilean limit and were able to recover, amongst others, the Gomis-Ooguri model. It would be curious to see if our Carroll string models can be obtained in an analogous Carroll expansion and explore how the models couple to curved spacetime. Finally, relaxing the relativistic symmetry on the worldsheet might lead to interesting interplay between the target space and the worldsheet, yielding novel string spectra.

Acknowledgments

We thank Oscar Fuentealba, Joaquim Gomis, Troels Harmark, Jelle Hartong, Emil Have, Niels Obers, Gerben Oling, and Ziqi Yan for discussions and insightful comments. Research supported by the Icelandic Research Fund Grant 228952-053 and by the University of Iceland Research Fund. LT would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the program "Black Holes: Bridges Between Number Theory and Holographic Quantum Information" where work on this paper was undertaken. This work was supported by EPSRC grant no EP/K032208/1. WS acknowledges support of the Simons Foundation through the INI-Simons Postdoctoral Fellowship.

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Paper II

The spectrum of a quantum Lifshitz black hole in two dimensions

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JHEP 02 (2025) 214

The thesis author contributed to the derivation of the effective boundary description for the Lifshitz like two dimensional dilaton model, the computation of the logarithmic correction to the entropy and the corresponding density of states, as well as to the interpretation of the results and the writing of the manuscript.

Published for SISSA by Springer

Received: *September 20, 2024*
Revised: *January 14, 2025*
Accepted: *February 4, 2025*
Published: *February 28, 2025*

The spectrum of a quantum Lifshitz black hole in two dimensions

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Abstract: We examine the low-energy spectrum of a four-dimensional near-extremal black hole that arises as a solution to a low energy effective theory of heterotic string theory. The effective two-dimensional gravitational description exhibits features of Lifshitz symmetry, which break the usual $\mathrm{SL}(2,\mathbb{R})$ invariance down to $\mathrm{U}(1)$. For this effective two-dimensional gravitational description, we derive a one-dimensional Schwarzian-like action that inherits the $\mathrm{U}(1)$ symmetry. The Schwarzian-like description allows us to compute a logarithmic correction to the entropy through a saddle-point approximation of the two-dimensional partition function. This logarithmic correction modifies the density of states, lifting the delta-function divergence at extremality, and removes the exponential ground-state degeneracy seen in the semiclassical analysis. Furthermore, the prefactor of the logarithmic term is $\frac{1}{2}$, rather than $\frac{3}{2}$ found for the $\mathrm{SL}(2,\mathbb{R})$ invariant description, indeed reflecting having fewer symmetries.

Keywords: 2D Gravity, Black Holes, Models of Quantum Gravity

ArXiv ePrint: 2408.15336

https://doi.org/10.1007/JHEP02(2025)214

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1 Introduction

In the extremal limit of a Reissner-Nordström black hole the temperature vanishes while the Bekenstein-Hawking entropy remains finite and exponentially large suggesting that an extremal black hole has a large number of ground states compared to the degrees of freedom. Phenomenologically this large entropy at zero temperature is already in tension with Nernst's third law of thermodynamics, which suggests that both the entropy at zero temperature as well as the ground state degeneracy are expected to be small. Page suggested in [1] that a way to resolve this tension was by showing that for extremal black holes all states are non-degenerate while they are separated by exponentially tiny energy gaps.

There is a second puzzle referred to as the mass-gap puzzle which was pointed out in [2] and further elaborated in [3]. Namely, as the black hole approaches extremality the wavelength of an emitted quanta of Hawking radiation becomes large compared to the size of the black hole. In fact for low values of the temperature the typical emitted quanta of Hawking radiation, E_{Hawking}, exceeds the thermodynamically accessible energy of the black hole above extremality, E_{BH},

$$E_{\text{BH}} \sim T^2, \qquad E_{\text{Hawking}} \sim T\,. \tag{1.1}$$

The intersection of the two curves E_{BH} and E_{Hawking}, implies the presence of a conjectured gap above extremality, E_{gap}, below which the semi-classical thermodynamics no longer applies (see Figure 1) since even emitting a single quanta of Hawking radiation would lead to inconsistencies in the semi-classical analysis of the black hole thermodynamics.

Several different aspects of these two[1] puzzles have been studied throughout the years and many resolutions have been proposed for determining the energy levels of a charged black hole.

[1]It has been suggested from a random-matrix model prescription that a large gap will appear whenever there is a mechanism producing a high degree of ground state degeneracy [5, 6].

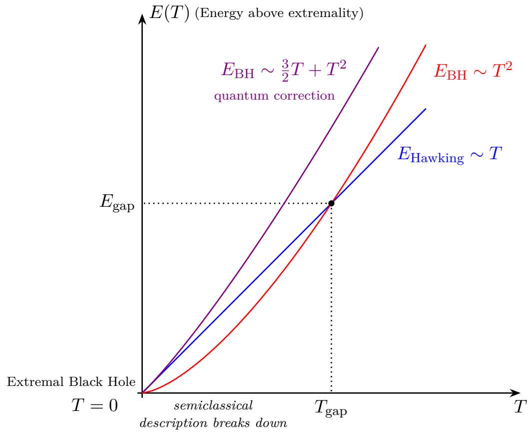

Figure 1. The thermodynamically accessible energy of the black hole above extremality scales with temperature as $E_{\rm BH} \sim T^2$, while according to Wien's displacement law, a typical quanta of Hawking radiation scales linearly with temperature, $E_{\rm Hawking} \sim T$. The semi-classical description breaks down for $T \leq T_{\rm gap}$ where $(T_{\rm gap}, E_{\rm gap})$ marks the intersection of the two curves. The quantum logarithmic correction (purple line) modifies the energy above extremality to $E_{\rm BH} \sim \frac{3}{2}T + T^2$, resolving the thermodynamics at small temperatures [4].

Early work on microstate counting for supersymmetric extremal black holes by Strominger and Vafa [7] showed that it was possible to reproduce the correct Bekenstein-Hawking entropy from a microscopic prescription and seemed to imply that there indeed was a large ground state degeneracy present at extremality. Furthermore Maldacena and Susskind [8] provided D-brane constructions with gaps between extremal and near-extremal states. Maldacena and Strominger [9] later provided a gravitational derivation of the proposed mass-gap scale purely from supersymmetric Einstein-Maxwell gravity without retorting to string theory arguments. The existence of mass-gap in the case of supersymmetric black holes which preserve two supercharges or more is strongly supported by a vast gallery of examples, see e.g. [10–14]. Notably, in the case of $\mathcal{N} = 1$ supersymmetric black holes do not exhibit a gapped spectrum [15]. Deformations of $\mathcal{N} = 2$ SYK are also able to close the spectral gap [16].

Recently it has been argued that the existence of a mass-gap is an artefact of supersymmetry rather than a generic feature of non-supersymmetric black holes [4, 17]. Logarithmic corrections to the entropy of extremal black holes imply that by including quantum corrections for the gravitational path integral the spectrum and dynamics of near-extremal black holes is modified such that at the expected gap energy scale there is a continuous spectrum of states [4, 10, 13, 18–21]. The techniques for carrying out these computations were developed by Sen and collaborators in [22–25] for the application to supersymmetric black holes.

Alternatively, viewing the mass-gap as an artefact of supersymmetry is supported in large part by the extensive work carried out in highlighting the relationship between near-extremal Reissner-Nordström black holes and the Schwarzian theory, which we elaborate on a bit further. In the near-extremal limit of a Reissner-Nordström black hole the metric factorises into $\mathrm{AdS}_2 \times S^2$ and by dimensionally reducing the Einstein-Maxwell action on a transverse two-sphere, an effective two-dimensional description is obtained which captures some of the dynamics of the four-dimensional black hole [26, 27].[2] In this case, this two-dimensional effective theory turns out to be Jackiw-Teitelboim (JT) gravity [29, 30].[3] In the full gravitational path integral for JT gravity the dilaton can be integrated out [33, 34] and by imposing wiggling boundary conditions the path integral reduces to that of the Schwarzian theory [35–37]. The path integral of the Schwarzian theory is one-loop exact and hence the result represents the full gravitational path integral for JT gravity [38] and can be used to compute $\log(T)$ corrections. Note that this two-dimensional description does not capture all $\log(T)$ corrections that can arise in the four-dimensional partition function, as it ignores modes that are related to the two-sphere and the gauge field. This Schwarzian theory also arises in the low energy effective limit of SYK [39–42], which gives rise to a holographic duality between JT gravity and the low energy dynamics of the SYK model, see e.g. [33].

The purpose of this paper is to consider the mass-gap puzzle for non-supersymmetric Lifshitz black holes that arise in a low energy effective description of string theory and compute the logarithmic corrections to the entropy. In Lifshitz theories the metric displays an anisotropic scale invariance between the temporal and spatial coordinates in stark contrast to the relativistic scale invariance [43, 44], reducing the number of possible isometries. We aim to explicitly investigate the notion that the $\log(T)$ corrections to the black hole entropy depend on the amount of symmetry emerging near the horizon in the small temperature limit, as has been argued in, e.g., [4, 10]. In passing we derive a one-dimensional boundary description of the effective two-dimensional theory of gravity that we will employ, which explicitly breaks conformal symmetry and as such can be the starting point of studying non-conformal incarnations of low-dimensional holography.

The rest of the paper is organised as follows. In section 2 we show the intricate connection between a four-dimensional Lifshitz black hole and a one-dimensional Lifshitz-like boundary action which shares similar features to that of the Schwarzian. This is done in the following manner; in subsection 2.1 we introduce the four-dimensional Lifshitz black hole action and study its near-horizon geometry. We show that the geometry factorises into $\mathrm{Lif}_{2,\alpha} \times S^2$ where $\mathrm{Lif}_{2,\alpha}$ is a Lifshitz-like spacetime that is notably asymptotically AdS_2 for Lifshitz parameter $0 < \alpha < 1$. We find that the thermodynamically accessible energy of the black hole above extremality scales with temperature as $E_{\mathrm{BH}} \sim T^{1+\frac{1}{\alpha}}$. This leads to the same mass-gap puzzle for these Lifshitz black holes as we highlighted previously in (1.1) for near-extremal Reissner-Nordström black holes. In subsection 2.2 we dimensionally reduce the four-dimensional model and study its two-dimensional counterpart. We summarise the main features of the resulting semi-classical thermodynamics in subsection 2.3. In subsection 2.4 we derive a Lifshitz-like boundary action which captures the same semi-classical thermodynamics as the two-dimensional Lifshitz-like dilaton model. In section 3 we compute the quantum corrected partition function and the corresponding density of states. We compute the 1-loop *approximation* to the quantum corrected partition function with a saddle point approximation in subsection 3.1 and derive the corresponding density of states in subsection 3.2 while highlighting the implications of our findings for the mass-gap puzzle for non-supersymmetric black holes which break conformal symmetries.

[2]For a nice review of this dimensional reduction we refer the reader to section 6 of [28].
[3]For a more detailed review of JT gravity we refer the reader to [31, 32].

2 A Lifshitz-like dilaton gravity model

We start with considering the decoupling limit of the four-dimensional Lifshitz model under consideration in subsection 2.1 and are interested in computing loop corrections. In order to compute loop corrections in the decoupling limit, we turn to the effective two-dimensional description that arises in the decoupling limit, of which we study the geometry and thermodynamics in subsections 2.2 and 2.3, respectively. Finally, we derive a boundary description for the two-dimensional Lifshitz-like model in subsection 2.4, which will allow us to compute a loop correction.

2.1 The four-dimensional model

In this subsection we consider the following four-dimensional action presented in [45, 46] which was derived in from the low energy limit of a dimensional reduction of a ten-dimensional heterotic supergravity theory using a suitable truncation of the spectrum in [47],

$$I = \int d^4x\sqrt{-g}\, e^{-2\phi}\left[R + 4\,(\nabla\phi)^2 - \frac{2}{3}(\nabla\psi)^2 - F^2 - e^{2(\phi-\frac{q}{3}\psi)}F^2\right]. \tag{2.1}$$

We will refer to ϕ as dilaton, ψ as compacton, and $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ the Maxwell field strength, with q a coupling constant. The equations of motion for the action presented in (2.1) are given by

$$R + 4\nabla^2\phi - 4(\nabla\phi)^2 - \frac{2}{3}(\nabla\psi)^2 - F^2 = 0, \tag{2.2a}$$

$$\nabla^2\psi - 2\nabla^\mu\phi\nabla_\mu\psi + \frac{q}{2}e^{2(\phi-\frac{q}{3}\psi)}F^2 = 0, \tag{2.2b}$$

$$\nabla_\mu\left[\left(e^{-2\phi} + e^{-\frac{2q}{3}\psi}\right)F^{\mu\nu}\right] = 0, \tag{2.2c}$$

$$\begin{aligned} &R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi - 2\nabla^2\phi\, g_{\mu\nu} + 2(\nabla\phi)^2 g_{\mu\nu} - \frac{2}{3}\nabla_\mu\psi\nabla_\nu\psi \\ &+\frac{1}{3}(\nabla\psi)^2 g_{\mu\nu} + \frac{1}{2}\left(1+e^{2(\phi-\frac{q}{3}\psi)}\right)F^2 g_{\mu\nu} - 2\left(1+e^{2(\phi-\frac{q}{3}\psi)}\right)F_{\mu\rho}F_\nu{}^\rho = 0, \end{aligned} \tag{2.2d}$$

where the first corresponds to varying the dilaton, the second with varying the compacton, the third corresponds with varying the Maxwell-field A_μ and the last equation corresponds to varying the metric. An exact magnetically charged black hole solution is given by [45, 46]

$$\begin{aligned} ds^2 &= -\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{-1+\alpha}{1+\alpha}}dt^2 + \left(1-\frac{r_+}{r}\right)^{-1}\left(1-\frac{r_-}{r}\right)^{-1}dr^2 + r^2 d\Omega^2\,, \\ \phi(r) &= -\frac{1}{2(1+\alpha)}\ln\left(1-\frac{r_-}{r}\right), \quad \psi(r) = \frac{3}{q}\left[\phi(r) - \frac{1}{2}\ln(2\alpha)\right], \quad A = Q_m\cos\theta\, d\phi\,, \end{aligned} \tag{2.3}$$

where we defined $\alpha = \frac{3}{2q^2} \geq 0$ and the two parameters r_+ and r_- correspond to outer and inner horizons respectively and can be related to the magnetic monopole charge Q_m and the mass M of the black hole by the following relation

$$M = \frac{1}{2}r_+ + \frac{1}{2}\frac{\alpha}{1+\alpha}r_-\,, \quad Q_m^2 = \frac{1}{2}\frac{r_+r_-}{1+\alpha}. \tag{2.4}$$

By investigating the Hawking temperature

$$T = \frac{1}{4\pi r_+}\left(1-\frac{r_-}{r_+}\right)^{\frac{\alpha}{1+\alpha}}, \tag{2.5}$$

we conclude that $r_0 = r_+ = r_-$ leads to an extremal black hole.

Decoupling regime. Let us investigate the near-extremal decoupling limit for $0 < \alpha < 1$. By fixing the charge to its extremal value

$$Q_0 = \frac{r_0}{\sqrt{2(1+\alpha)}} \,. \tag{2.6}$$

The horizons $r_\pm(T)$ have the following small temperature expansion at lowest order

$$r_\pm(T) = r_0 \pm \frac{r_0}{2}\,(4\pi r_0 T)^{1+\frac{1}{\alpha}} + \dots\,, \tag{2.7}$$

from which we find that the mass near extermality is given by the following expression to the lowest order in temperature

$$M(T) = M_0 + \frac{T^{1+\frac{1}{\alpha}}}{M_{\text{gap}}}\,, \tag{2.8}$$

where the extremal mass and the mass gap are respectively given by

$$M_0 := \frac{1+2\alpha}{1+\alpha}\frac{r_0}{2}\,, \qquad \frac{1}{M_{\text{gap}}} := \frac{1}{1+\alpha}\frac{r_0}{4}(4\pi r_0)^{1+\frac{1}{\alpha}}\,. \tag{2.9}$$

Let us now study the decoupling limit of the metric. On top of using the small temperature expansion for r_+ and r_- from equation (2.7), to ensure Schwarzschild gauge at leading order we introduce new coordinates $\tilde{t}$ and $\tilde{r}$ such that

$$t = \frac{\tilde{t}}{2\pi T}\,, \qquad r = r_+ + r_0\left[\left(\frac{\tilde{r}}{\tilde{r}_h}\right)^{1+\frac{1}{\alpha}} - 1\right](4\pi r_0 T)^{1+\frac{1}{\alpha}}\,, \tag{2.10}$$

where the horizon location in the new radial coordinate is indicated by the value

$$\tilde{r}_h = 2r_0^2\left(1+\frac{1}{\alpha}\right)\,. \tag{2.11}$$

In the small-temperature regime, $T \ll 1$, the new radial coordinate satisfies $\frac{\tilde{r}}{\tilde{r}_h} \sim \frac{1}{4\pi r_0 T} \gg 1$, consistent with the near-horizon approximation. In these coordinates, the metric reads[4]

$$ds^2 = -F(\tilde{r})d\tilde{t}^2 + \frac{d\tilde{r}^2}{F(\tilde{r})} + r_0^2 d\Omega_2^2 + \mathcal{O}(T^{1+\frac{1}{\alpha}})\,, \tag{2.12}$$

where

$$F(\tilde{r}) = \frac{2\alpha\tilde{r}_h}{1+\alpha}\left[\left(\frac{\tilde{r}}{\tilde{r}_h}\right)^2 - \left(\frac{\tilde{r}}{\tilde{r}_h}\right)^{1-\frac{1}{\alpha}}\right]\,. \tag{2.13}$$

[4] Although not required for the analysis presented here, the next order correction is given by

$$\frac{ds^2}{(4\pi r_0 T)^{1+\frac{1}{\alpha}}} = -\frac{2\alpha^2\tilde{r}_h}{(1+\alpha)^2}\left[3\left(\frac{\tilde{r}}{\tilde{r}_h}\right)^2 - \left(\frac{\tilde{r}}{\tilde{r}_h}\right)^{1-\frac{1}{\alpha}} - 2\left(\frac{\tilde{r}}{\tilde{r}_h}\right)^{3+\frac{1}{\alpha}}\right]d\tilde{t}^2$$
$$+ \frac{1}{2r_0^2}\left[\frac{1}{2} - \left(\frac{\tilde{r}}{\tilde{r}_h}\right)^{1+\frac{1}{\alpha}}\right]\left[\left(\frac{\tilde{r}}{\tilde{r}_h}\right)^2 - \left(\frac{\tilde{r}}{\tilde{r}_h}\right)^{1-\frac{1}{\alpha}}\right]^{-1} d\tilde{r}^2 + r_0^2\left[-1 + 2\left(\frac{\tilde{r}}{\tilde{r}_h}\right)^{1+\frac{1}{\alpha}}\right]d\Omega^2\,.$$

Notice that we cannot turn off the black hole and its temperature is fixed at $\tilde{T} = 1/(2\pi)$, which is to be expected for the rescaled time coordinate $\tilde{t}$. When focussing on the $\tilde{r} - \tilde{t}$ plane the Ricci scalar $R_{\tilde{r}-\tilde{t}}$ reads

$$R_{\tilde{r}-\tilde{t}} = \frac{2\alpha}{1+\alpha}\frac{1}{\tilde{r}_h}\left[-2 + \frac{1-\alpha}{\alpha^2}\left(\frac{\tilde{r}_h}{\tilde{r}}\right)^{1+\frac{1}{\alpha}}\right], \tag{2.14}$$

which implies that $\alpha = 1$ yields a constant negatively valued Ricci scalar and the effective two-dimensional theory is JT gravity. For $0 < \alpha < 1$ the Ricci scalar is running, but asymptotically always a patch of AdS_2. At extremality, we find the following factorisation of space

$$ds^2 = \mathrm{Lif}_{2,\alpha} \times S^2\,, \tag{2.15}$$

where S^2 refers to a two-sphere and $\mathrm{Lif}_{2,\alpha}$ represents a two-dimensional Lifshitz-like spacetime with dynamical critical exponent α, which we will discuss in more details in the next section. For now it is relevant to highlight the fact that away from $\alpha = 1$ the Ricci scalar runs.

The low temperature puzzle. The available thermodynamic energy above extremality is now given by $E_{\mathrm{BH}}(T) = M(T) - M_0 = \frac{1}{M_{\mathrm{gap}}}T^{1+\frac{1}{\alpha}}$. We therefore run into the same mass-gap puzzle as highlighted in the introduction for Reissner-Nordström black holes [2]. For low enough values of the temperature the typical emitted quanta of Hawking radiation, $E_{\mathrm{Hawking}} \sim T$ starts to exceed the available energy of the black hole above extremality, E_{BH},

$$E_{\text{4d black hole}} \sim T^{1+\frac{1}{\alpha}}\,, \quad E_{\mathrm{Hawking}} \sim T\,. \tag{2.16}$$

When these energies become of comparable size, the amount of energy released through Hawking radiation becomes significant compared to the energy available to the black hole. As such the assumption of equilibrium is invalid and for low enough temperatures the saddle should receive large semi-classical corrections.

It was in [25] where such corrections were computed through considering a set of normalisable zero modes at extremality, composed of the asymptotic Killing vectors. The resulting outcome is an extra term $\sim \log T$ that enters the free energy and modifies the low energy behaviour, averting the above scenario. To apply the method of [25], we require the computation of asymptotic Killing vectors for the $\mathrm{Lif}_{2,\alpha}$ spacetime we find at extremality. It turns out that we are unable to do this, so instead we will compute the logarithmic correction through the boundary description of the two-dimensional $\mathrm{Lif}_{2,\alpha}$ theory.

2.2 The two-dimensional model

The effective two-dimensional description of the four-dimensional model presented in (2.1) is obtained by performing a reduction over a constant two-sphere and integrating out the Maxwell field [45]. Specifically, we insert the ansatz, $\psi(r) = \frac{3}{q}\phi(r) - \frac{3}{2q}\,\mathrm{Log}(2\alpha)$, for the compacton into the action and reduce over a sphere of constant radius $r_0 = r_+ = r_-$, which corresponds to the extremal radius of the black hole. By factorizing the spacetime metric as

$$g_{\mu\nu}dx^\mu dx^\nu = \tilde{g}_{ab}d\tilde{x}^a d\tilde{x}^b + r_0^2 d\Omega_2^2\,, \tag{2.17}$$

we obtain the intermediate two-dimensional action

$$I_{2\mathrm{d}} = 4\pi r_0^2 \int d^2x\sqrt{-\tilde{g}}\, e^{-2\phi}\left[\tilde{R} + 4(1-\alpha)(\tilde{\nabla}\phi)^2 + \frac{2}{r_0^2} - 2(1+2\alpha)\frac{Q_0^2}{r_0^4}\right], \tag{2.18}$$

where ϕ is the dilaton, $\tilde{R}$ is the two-dimensional Ricci-scalar and $Q_0 = r_0/\sqrt{2(1+\alpha)}$ represents the extremal value of the magnetic monopole charge. By defining the energy scale

$$\lambda = \frac{1}{2r_0\sqrt{1+\alpha}}, \tag{2.19}$$

we can express the action in its final form (dropping the cumbersome tilde notation)

$$I_{\mathrm{CM}} = 4\pi r_0^2 \int d^2x\sqrt{-g}\, e^{-2\phi}\left[R + 4(1-\alpha)(\nabla\phi)^2 + 4\lambda^2\right]. \tag{2.20}$$

While this $4\pi r_0^2$ prefactor naturally arises from the dimensional reduction process and reflects the relationship between the 4D and 2D Newton's constants (G_4 and G_2), we will instead use the form of the action introduced by Lemos and Sá [48], written as[5]

$$I_{\mathrm{Lif}} = \frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{-g}\, e^{-2\phi}\left[R + 4(1-\alpha)(\nabla\phi)^2 + 4\lambda^2\right]. \tag{2.22}$$

We will refer to this as the Lifshitz-like dilaton model, motivated by the relation between entropy and temperature, $S \sim T^{1/\alpha}$, which we will establish in (2.33). While in this paper we derive this model from the dimensional reduction of (2.1), an alternative derivation was presented in [50]. There, the model arises by considering the near-horizon region of a Lifshitz black hole constructed using bottom-up methods by Taylor [51, 52], with a large number of transverse dimensions. This connection with Lifshitz black hole models provides another reason for referring to it as Lifshitz-like. Let us now describe the significance of the dimensionless parameter α, which determines the asymptotics of the solutions:

- $\alpha = 1$*:* the action reduces to the Jackiw-Teitelboim (JT) model [29, 30], which describes asymptotically AdS_2 spacetimes and plays a central role in near-horizon black hole physics and holography.

- $\alpha = 0$*:* the action corresponds to the Callan-Giddings-Harvey-Strominger (CGHS) model [53], which admits asymptotically flat solutions. These solutions are not the focus of our study and will not be considered further.

- $0 < \alpha < 1$*:* the model admits asymptotically AdS_2 solutions but with modified scaling properties, leading to a Lifshitz-like behavior. This range of α interpolates between the JT and CGHS models and is the primary focus of our analysis.

[5] By introducing $\Phi = e^{-2\phi}$ and Weyl rescaling the metric $g_{ab} \to \Phi^{\alpha-1} g_{ab}$ we can eliminate the kinetic term

$$I_{\mathrm{W\text{-}Lif}} = \frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{-g}\left(R\Phi + 4\lambda^2\Phi^\alpha\right), \tag{2.21}$$

This action has been studied in [49], but does not allow for the asymptotically AdS_2 solutions we aim to study.

Varying the action in (2.22) with respect to the metric and dilaton yields respectively

$$\nabla_\mu \nabla_\nu \phi - 2\alpha \nabla_\mu \phi \nabla_\nu \phi - \left(\nabla^2 \phi - (1+\alpha)(\nabla\phi)^2 + \lambda^2 \right) g_{\mu\nu} = 0\,, \tag{2.23a}$$

$$R + 4(1-\alpha)\nabla^2\phi - 4(1-\alpha)(\nabla\phi)^2 + 4\lambda^2 = 0\,. \tag{2.23b}$$

The general solution for $0 < \alpha < 1$ in Schwarzschild gauge is given by

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)}\,, \quad f(r) = (ar)^2 - b(ar)^{1-\frac{1}{\alpha}}\,, \quad \phi(r) = -\frac{1}{2\alpha}\ln(ar)\,, \tag{2.24}$$

where the constant a is just a convenient redefinition of λ given by

$$a = \frac{2\alpha\lambda}{\sqrt{1+\alpha}} \geq 0\,, \tag{2.25}$$

and b is a free parameter which, from a higher-dimensional perspective, parametrizes the departure of the solution from extremality [46]. In the context of the dimensional reduction, the parameters a and b can be read off from (2.13) in terms of the extremal radius r_0 of the higher-dimensional black hole, where $a = \frac{\alpha}{r_0(1+\alpha)}$ and $b = (2r_0)^{1+\frac{1}{\alpha}}$. For this section, however, we will treat b as an unconstrained free parameter, reflecting the freedom inherent in the two-dimensional dilaton model. Defining the horizon radius r_h implicitly for the two-dimensional model by $f(r_h) = 0$, we have additionally the following relation $b = (ar_h)^{1+\frac{1}{\alpha}}$.

We can relate the free parameter b to the ADM mass M of the solution [46, 48, 54].

$$M = \frac{ab}{2\alpha}\,. \tag{2.26}$$

Additionally it may be verified by using the equations of motion (2.23a) that

$$\xi^\mu = \epsilon^{\mu\nu}\nabla_\nu(e^{-2\alpha\phi})\,, \tag{2.27}$$

is always a Killing vector of the metric which can be used to compute the aforementioned ADM mass [55]. In Schwarzschild gauge this becomes $\xi = -a\partial_t$. The existence of this Killing vector for a dynamical dilaton already indicates how the $\mathrm{SL}(2,\mathbb{R})$ isometries are broken down to U(1) and hence from AdS_2 to nearly-AdS_2 in the case of JT gravity [36].

The Ricci scalar of the metric for $0 < \alpha < 1$ is given by

$$R = -2a^2 - 2aM\left(1 - \frac{1}{\alpha}\right)(ar)^{-(1+\frac{1}{\alpha})}\,. \tag{2.28}$$

From this Ricci scalar we see that the solution for large radial values is asymptotically AdS_2. We may additionally defer from this that for $b = 0$, i.e., $M = 0$ the solution is pure AdS_2.

An important difference to highlight between $\alpha = 1$ (JT gravity) and $0 < \alpha < 1$ is how a dynamical dilaton affects the isometries. For JT one can toggle $\exp\phi = 0$, independent of the metric, such that the bulk reflects pure AdS_2 solutions. Turning on the dilaton breaks down the $\mathrm{SL}(2,\mathbb{R})$ symmetry to U(1), so-called nearly AdS_2 [36].

To conclude this section we highlight the Lifshitz-like nature of this dilaton model. The line element presented in (2.24) is invariant under the following Lifshitz scale transformation

$$t \to L^{-1}t\,, \quad r \to Lr\,, \quad b \to L^{1+\frac{1}{\alpha}}b\,. \tag{2.29}$$

This anisotropic scale invariance between the temporal and spatial coordinates is in stark contrast to the conformal scale invariance and is a general hallmark of Lifshitz theories [43, 44]. Furthermore, under the Lifshitz scale transformation the action (2.22) transforms as

$$I_{\rm Lif} \to L^{\frac{1}{\alpha}} I_{\rm Lif}\,, \tag{2.30}$$

mainly as an artefact of the dilaton scaling as $e^{-2\phi} \to L^{\frac{1}{\alpha}} e^{-2\phi}$. For $\alpha \neq 1$ the action is therefore not invariant but rather a *similarity* of the action [56]. This is also sometimes referred to as a *hyperscaling violation* of the action [57, 58]. The hyperscaling violation exponent $\frac{1}{\alpha}$ therefore naturally shows up in the thermodynamical analysis of the model. In fact the Lifshitz scale tranformation in equation (2.29) implies that the inverse temperature changes as $\beta \to L^{-1}\beta$. Therefore the temperature changes as $T \to LT$ according to (2.29) and this in turn means that the entropy scales as $S \to L^{\frac{1}{\alpha}} S$. Since the entropy is a function of the temperature we have shown by scaling arguments that entropy will be proportional to $S \sim T^{\frac{1}{\alpha}}$ which we will formally derive in the next section.

2.3 Thermodynamics of the two-dimensional bulk theory

The thermodynamical properties of the solution given by (2.23a) have previously been analysed in [46, 50, 54]. In this section we summarise succinctly these thermodynamical properties. Let us first supplement the bulk action (2.22) with the corresponding Gibbons-Hawking-York boundary term which is needed to make the gravitational variational problem well-defined. Additionally we derive a counterterm that renders the Euclidean on-shell action finite. Including both these boundary terms the action may be written as

$$\begin{aligned} I_{\rm Lif} = &-\frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\, e^{-2\phi}\left[R + 4(1-\alpha)(\nabla\phi)^2 + 4\lambda^2\right] \\ &- \int_{\partial\mathcal{M}} du\sqrt{h}\, e^{-2\phi}\left[K - \frac{2\lambda}{\sqrt{1+\alpha}}\right], \end{aligned} \tag{2.31}$$

where K denotes the trace of the extrinsic curvature and $h = h_{uu}$ is the induced metric on the boundary parameterised by the coordinate u. The free energy, F, is then given by

$$F = \frac{\mathcal{I}_{\rm Lif}}{\beta} = -\alpha M = -\frac{2\pi\alpha}{1+\alpha}\left(\frac{\pi a}{\alpha\lambda^2}\right)^{\frac{1}{\alpha}} T^{1+\frac{1}{\alpha}}\,, \tag{2.32}$$

where $\beta = \frac{1}{T}$ denotes the inverse of the Hawking temperature of the black hole and M denotes the ADM mass (2.26). The entropy can then be computed from the relationship

$$S(T) = -\frac{\partial F}{\partial T} = 2\pi\left(\frac{\pi a}{\alpha\lambda^2}\right)^{\frac{1}{\alpha}} T^{\frac{1}{\alpha}}\,. \tag{2.33}$$

This equation highlights the Lifshitz-like nature of the model due to the scaling relation between temperature and entropy.

For future reference we rewrite the entropy in terms of the ADM mass

$$S(M) = 2\pi\left(\frac{a}{2\alpha}\right)^{-\frac{1}{1+\alpha}} M^{\frac{1}{1+\alpha}}\,. \tag{2.34}$$

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Through the first law, $F = E - ST$, we can now verify that indeed energy equals to the ADM mass, $E = M$. We are in particular able to verify the following identity

$$TS = (1+\alpha)E\,. \tag{2.35}$$

which is a two-dimensional incarnation of the Smarr formula [59].

2.4 Deriving the boundary description of the two-dimensional model

In this section we will compute the boundary description of the Lifshitz-like model that reflects the U(1) symmetry this model has. We note that in $\alpha = 1$ case, the usual Schwarzian boundary theory is modified by turning on an explicit source, breaking down the symmetries to U(1).

Let us review the idea behind the Schwarzian computation in JT gravity so that we have a better grip on performing the equivalent computation for the Lifshitz-like model. The Schwarzian description arises from considering a Euclidean disc of the AdS_2 vacuum and studying the reparametrisation freedom of the disc. The $\mathrm{SL}(2,\mathbb{R})$ freedom of the Schwarzian reflects the existence of a vacuum in which the metric equals AdS_2, which is $\mathrm{SL}(2,\mathbb{R})$ invariant, and where the dilaton is constant. If the constant dilaton were running, say dependent on the radial coordinates, then $\mathrm{SL}(2,\mathbb{R})$ gets spontaneously broken to U(1). In the case of JT, all metric solutions are always AdS_2 due to the Ricci scalar always equalling the same constant on-shell. For that reason one can perform a coordinates transformation to go between the AdS_2 vacuum patch and the 'black hole' patch once the Schwarzian is obtained. In the Lifshitz-like model we have access to two distinct solutions, AdS_2, and a black hole patch where the Ricci scalar is running. The AdS_2 vacuum patch and black hole patch are only related through a large diffeomorphism. Contrasting the JT case, even the AdS_2 vacuum has to be supported by a running dilaton. As such, U(1) symmetry is the maximally attainable symmetry in both cases. When performing the boundary computation we will start from the black hole patch. Let us first rewrite the bulk as

$$\begin{aligned}
I_{\text{bulk}} &= -\frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\,e^{-2\phi}\left(R + 4(1-\alpha)(\nabla\phi)^2 + 4\lambda^2\right)\\
&= -\frac{1}{2}\int_{\mathcal{M}} d^2x\sqrt{g}\,e^{-2\phi}\left(8(1-\alpha)(\nabla\phi)^2 - 4(1-\alpha)\nabla^2\phi\right)\\
&= 2(1-\alpha)\int_{\mathcal{M}} d^2x\sqrt{g}\,\nabla_\mu\left(g^{\mu\nu}e^{-2\phi}\nabla_\nu\phi\right)\\
&= 2(1-\alpha)\int_{\partial\mathcal{M}} dt\sqrt{h}\,e^{-2\phi}n^\mu\nabla_\mu\phi\,,
\end{aligned} \tag{2.36}$$

where to go to the second line we used the dilaton equation of motion and to go to the third line we used $e^{-2\phi}\nabla_\mu\nabla_\nu\phi = \nabla_\mu\left(e^{-2\phi}\nabla_\nu\phi\right) + 2e^{-2\phi}\nabla_\mu\phi\nabla_\nu\phi$. In the last line we introduced the unit outward normal n^μ on $\partial\mathcal{M}$. We flesh these manipulations out in great detail in order to highlight that at no stage did we need to neglect any additional boundary terms. Finally, let us additionally remark that the contribution from evaluating this bulk piece on the horizon vanishes. With this in mind we may rewrite the entire action in the following boundary form

$$I_{\text{Lif}} = -\int_{\partial\mathcal{M}} dt\sqrt{h}\,e^{-2\phi}\left(-2(1-\alpha)n^\mu\nabla_\mu\phi + K - \frac{a}{\alpha}\right)\,. \tag{2.37}$$

We will now wick rotate the t coordinate and consider an enclosed boundary parametrised by some periodic coordinate u

$$u \sim u + u_{\max}\,, \quad t(u + u_{\max}) = t(u) + \gamma\,. \tag{2.38}$$

Note that when $\gamma \neq \beta$, where β is the inverse Hawking temperature, we have a conical singularity. The metric induced on the boundary h_{uu} is given by

$$h_{uu}(t(u), r(u)) = f(r(u))(t'(u))^2 + \frac{(r'(u))^2}{f(r(u))} = \frac{1}{\epsilon^2}\,, \tag{2.39}$$

where $|\epsilon| \ll 1$ and where we chose $1/\epsilon^2$ as we expect the leading order to coincide with AdS_2 behaviour. With f we denote the emblackening factor presented in (2.24). It will turn out towards the end of the computation that we need to rescale $t(u)$, γ and the black hole mass parameter b in order to retain a non-trivial solution for $\alpha \neq 1$. From (2.39) we solve

$$r(u) = \frac{1}{a\epsilon t'(u)} + \mathcal{O}(\epsilon^3)\,, \tag{2.40}$$

where we point out that in $f(r)$ for large radial values $(ar)^2$ will dominate $b(ar)^{1-\frac{1}{\alpha}}$ since $0 < \alpha < 1$. Let us now turn to computing the extrinsic curvature as a function of u. For a curve $(t(u), r(u))$ the normal is given by

$$n_\mu = \frac{1}{\sqrt{h_{uu}}}(-r'(u), t'(u))\,. \tag{2.41}$$

Introducing the vielbein $e^\mu_u = (t'(u), r'(u))$ we can evaluate the extrinsic curvature as

$$K = h^{uu}K_{uu} = h^{uu}e^\mu_u e^\nu_u(\nabla_\mu n_\nu)\,, \quad \Gamma^t_{tr} = -\Gamma^r_{rr} = -\frac{1}{2}\partial_r f(r)\,, \quad \Gamma^r_{tt} = \frac{1}{2}f(r)\partial_r f(r)\,. \tag{2.42}$$

We can now present

$$K = a + \frac{1}{a}\{t,u\}\epsilon^2 + E(t'(u)\epsilon)^{1+\frac{1}{\alpha}} + \mathcal{O}(\epsilon^4)\,, \tag{2.43}$$

$$\sqrt{h} = \frac{1}{\epsilon} + \frac{\epsilon}{2a^2}\left(\frac{t''(u)}{t'(u)}\right)^2 - \frac{\alpha E}{a}t'(u)^{1+\frac{1}{\alpha}}\epsilon^{\frac{1}{\alpha}} + \mathcal{O}(\epsilon^3)\,, \tag{2.44}$$

$$e^{-2\phi} = (ar(u))^{\frac{1}{\alpha}} = \left(\frac{1}{\epsilon t'(u)}\right)^{\frac{1}{\alpha}} + \mathcal{O}(\epsilon^{4-\frac{1}{\alpha}})\,, \tag{2.45}$$

$$-2(1-\alpha)n^\mu\nabla_\mu\phi = \left(\frac{1}{\alpha} - 1\right)a - \frac{1-\alpha}{2a\alpha}\left(\frac{t''(u)}{t'(u)}\right)^2\epsilon^2 - E(1-\alpha)(t'(u)\epsilon)^{1+\frac{1}{\alpha}} + \mathcal{O}(\epsilon^4)\,, \tag{2.46}$$

where we introduced the Schwarzian derivative

$$\{t,u\} = \frac{t'''(u)}{t'(u)} - \frac{3}{2}\left(\frac{t''(u)}{t'(u)}\right)^2\,, \tag{2.47}$$

and used the explicit on-shell solution of the dilaton ϕ in (2.24) to obtain an explicit expression for $n^\mu\nabla_\mu\phi$ arising from the bulk term (2.36). Putting this all together we find that the full action becomes

$$I_{\text{Lif}} = -\int du\left(\alpha E t'(u) + \frac{\epsilon^{1-\frac{1}{\alpha}}}{at'(u)^{\frac{1}{\alpha}}}\{t,u\}_\alpha\right) + \mathcal{O}(\epsilon^{3-\frac{1}{\alpha}})\,, \tag{2.48}$$

where we defined the *deformed Schwarzian*

$$\{t,u\}_\alpha := \left(\frac{t''(u)}{t'(u)}\right)' - \frac{1}{2\alpha}\left(\frac{t''(u)}{t'(u)}\right)^2 = \{t,u\} - \frac{1}{2}\left(\frac{1}{\alpha}-1\right)\left(\frac{t''(u)}{t'(u)}\right)^2 . \tag{2.49}$$

Note that for $\alpha = 1$ this coincides with the usual Schwarzian derivative. This deformed Schwarzian retains shift symmetries and rescaling symmetries, but no inversion symmetry.

For $0 < \alpha < 1$ there is always a divergence present in equation (2.48) at order $\epsilon^{1-\frac{1}{\alpha}}$. To remedy the divergence in (2.48), we need to rescale the black hole energy E and the coordinate t such that

$$E \to \epsilon^{1-\alpha} E\,, \qquad t(u) \to \epsilon^{\alpha-1} t(u)\,, \tag{2.50}$$

which is motivated by requiring the on-shell value of (2.48) when $t(u) = u$ to be finite. In combination with the above we choose u_{max} and γ introduced in (2.38) such that

$$u \sim u + \beta\,, \quad t(u+\beta) = t(u) + \beta\,. \tag{2.51}$$

With these rescalings we now obtain

$$K = a + \left(\frac{1}{a}\{t,u\} + Et'(u)^{1+\frac{1}{\alpha}}\right)\epsilon^2 + \mathcal{O}(\epsilon^4)\,, \tag{2.52}$$

$$\sqrt{h} = \frac{1}{\epsilon} + \mathcal{O}(\epsilon) \tag{2.53}$$

$$e^{-2\phi} = \frac{1}{t'(u)^{\frac{1}{\alpha}}}\frac{1}{\epsilon} + \mathcal{O}(\epsilon^3)\,, \tag{2.54}$$

$$-2(1-\alpha)n^\mu\nabla_\mu\phi = \left(\frac{1}{\alpha}-1\right)a - (1-\alpha)\left(Et'(u)^{1+\frac{1}{\alpha}} + \frac{1}{2a\alpha}\left(\frac{t''(u)}{t'(u)}\right)^2\right)\epsilon^2 + \mathcal{O}(\epsilon^4)\,. \tag{2.55}$$

In this case the action is finite and taking the limit $\epsilon \to 0$ we obtain the following action

$$I_{\text{dSch}}[t] = -\int du \frac{1}{t'(u)^{\frac{1}{\alpha}}}\left(\alpha Et'(u)^{1+\frac{1}{\alpha}} - \frac{1-\alpha}{2a\alpha}\left(\frac{t''(u)}{t'(u)}\right)^2 + \frac{1}{a}\{t,u\}\right) \tag{2.56}$$

$$= -\alpha E\int t'(u)du - \frac{1}{a}\int du \frac{1}{t'(u)^{\frac{1}{\alpha}}}\{t,u\}_\alpha\,, \tag{2.57}$$

which we will refer to as the deformed Schwarzian action (dSch). This boundary description reproduces the thermodynamics of I_{Lif} when we take the saddle point approximation $t(u) = u$. We will study its loop correction in the next section.

Let us touch upon the $\alpha = 1$ case and contrast it to the canonical JT result (see e.g. [36]), to emphasise the influence of choosing boundary condition (2.45) rather than leaving it as an unspecified source. For $\alpha = 1$ we establish

$$I^{\alpha=1}_{\text{dSch}}[t] = -\int du\, \overline{\phi}(u)\left[\frac{1}{a}\{t,u\} + E(t'(u))^2\right]\,, \quad \overline{\phi}(u) = \frac{1}{t'(u)}\,, \tag{2.58}$$

where it is important to recall that we have already employed the black hole coordinate patch. By adopting the coordinate transformation to go to the Poincaré disc

$$t = \sqrt{\frac{2}{aE}}\arctan\tau(u)\,, \tag{2.59}$$

we recover

$$I^{\alpha=1}_{\rm dSch}[\tau] = -\frac{1}{a}\int du\; \overline{\phi}(u)\,\{\tau,u\}\;, \quad \overline{\phi}(u) = \sqrt{\frac{aE}{2}}\frac{1+\tau(u)^2}{\tau'(u)}\,. \tag{2.60}$$

Let us now contrast this last results with the canonical JT outcome. We could alternatively have started with $\alpha = 1$ from the JT bulk (2.21) with corresponding Gibbons-Hawking-York term and counter term. We could then go through the usual procedure of deriving a boundary description, with this time requiring the boundary value $e^{-2\phi} \sim \phi_r(u)/\epsilon$, where $\phi_r(u)$ will play the role of a boundary source. If we adopt a Poincaré patch with time coordinate τ we find the Schwarzian action

$$I_{\rm Schwarzian} \sim \int du\phi_r(u)\{\tau,u\}\,, \tag{2.61}$$

where the equations of motion for τ imply

$$\phi_r(u) = \frac{\alpha+\gamma\tau(u)+\delta\tau(u)^2}{\tau'(u)}\,, \tag{2.62}$$

with α, γ, δ integration constants. To obtain the more familiar form of the Schwarzian action one can rescale u such that $\phi_r(u)$ can be chosen a constant and effectively drops out of the action and the $\mathrm{SL}(2,\mathbb{R})$ symmetry remains intact. However, in order to obtain (2.60) we have chosen the integration constants α, γ, δ such that $\phi_r(u) = \overline{\phi}(u)$, which is turning on a source that breaks the $\mathrm{SL}(2,\mathbb{R})$ symmetry.

3 Quantum black holes and density of states

Our goal is to use a saddle point approximation to estimate the partition function of the Lifshitz-like dilaton model. After the approximation is implemented, some aspects of this computation mirror computationally what happens in the determination of the JT gravity partition function as a boundary Schwarzian, but we emphasise that we *approximate* the path integral, whereas for the JT gravity it has been argued in [38] that the path integral of the JT Schwarzian theory is one-loop exact. For this reason, let us posit the essence of the JT gravity approach. We start with specifying a regulated Euclidean AdS_2 disc and the boundary conditions for the fields at its edge, in particular for the dilaton. Performing the path integral over the dilaton ϕ fixes the Ricci scalar to be a constant and reduces the action purely to boundary terms described by the Schwarzian action, $S_{\rm Schwarzian}[t]$. Given the correct boundary conditions for the dilaton, treating its boundary conditions as an external source denoted by ϕ_r, the Schwarzian action retains $\mathrm{SL}(2,\mathbb{R})$ reparametrisation invariance through the boundary mode t, which leaves an imprint on the path integral measure over $[Dt]$. We can summarise the above words into

$$Z_{\rm JT} = \int [\mathcal{D}g_{ab}][\mathcal{D}\phi]e^{-I_{\rm JT}[g_{ab},\phi]} = \int_{\mathrm{Diff}(S^1)/\mathrm{SL}(2,\mathbb{R})}[\mathcal{D}t]e^{-I_{\rm Schwarzian}[t,\phi_r]}\,. \tag{3.1}$$

With the notation $\mathrm{Diff}(S^1)/\mathrm{SL}(2,\mathbb{R})$ we mean that we integrate over full diffeomorphisms on the circle modulo the $\mathrm{SL}(2,\mathbb{R})$ global symmetry of the Schwarzian, which reflect the isometries of AdS_2.

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For the Lifshitz-like theory we consider, integrating out the dilaton does not yield a simple result. As such we consider a saddle approximation in which we take the on-shell value for the dilaton and fix the Ricci scalar using equation of motion (2.23). Fixing only the Ricci scalar allows us to study a boundary mode provided we use the on-shell value of the dilaton the dilaton ϕ in (2.24) to obtain an expression for $n^\mu \nabla_\mu \phi$. For us this fixes ϕ_r which, contrasting the standard JT case, does *not* function as an external source but has specified dependence on t'. As shown in the previous section, we obtain a deformed Schwarzian action, which has U(1) global symmetry. We emphasise that within the approach we adopt, we are neglecting any possible contribution from the fact that contrasting the JT-setup, the dilaton cannot be integrated out exactly.

We summarise this approach as

$$Z_{\text{Lif}} = \int [\mathcal{D}g_{ab}][\mathcal{D}\phi] e^{-I_{\text{Lif}}[g_{ab},\phi]} \approx \int_{\text{Diff}(S^1)/\text{U}(1)} [\mathcal{D}t] e^{-I_{\text{deformed Schwarzian}}[t,\phi_r]} \,. \tag{3.2}$$

To evaluate the saddle approximation above we first consider fluctuations around the saddle point of the deformed Schwarzian action in subsection 3.1. In that section we also specify the measure and evaluate the partition function. In subsection 3.2 we study the resulting density of states and comment on these results.

3.1 Computing the partition function of a quantum black hole

In order to evaluate the saddle-point approximated path integral, we need to work out the action and measure and evaluate the combination.

Perturbative expansion of the deformed Schwarzian action. The equations of motion resulting from the action

$$I_{\text{dSch}}[t] = -\alpha E \int t'(u) du - \frac{1}{a} \int du \frac{1}{t'(u)^{\frac{1}{\alpha}}} \{t, u\}_\alpha \ , \tag{3.3}$$

obtained in (2.56), are given by

$$\frac{d}{du} \left(\frac{\{t,u\}_\alpha}{t'(u)^{1+\frac{1}{\alpha}}} \right) = 0 \,. \tag{3.4}$$

A solution is then in particular $t(u) = u$, which satisfies the ranges (2.51), in which case we find that the classical on-shell action gives the same free energy as expected from (2.32)

$$I_{\text{dSch}}[t = u] = -\alpha\beta M = \beta F \,. \tag{3.5}$$

Let us now consider fluctuations around the saddle point of the deformed Schwarzian action

$$t(u) = u + \epsilon(u) \,, \tag{3.6}$$

where $\epsilon(u+\beta) = \epsilon(u)$. At quadratic order we then find the following

$$\begin{aligned} I_{\text{dSch}}[t = u + \epsilon(u)] &= -\alpha\beta M + \frac{1}{2a\alpha} \int_0^\beta du \Big((1+2\alpha)\epsilon''(u)^2 + 2(1+\alpha)\epsilon'(u)\epsilon'''(u) \Big) + \mathcal{O}(\epsilon^3) \\ &= -\alpha\beta M + \frac{1}{2a\alpha} \int_0^\beta du \, \epsilon''(u)^2 + \mathcal{O}(\epsilon^3) \,, \end{aligned} \tag{3.7}$$

where we performed integration by parts. Following the approach of [33], let us now expand $\epsilon(u)$ in Fourier modes

$$\epsilon(u) = \frac{\beta}{2\pi} \sum_{n\neq 0} \epsilon_n e^{-\frac{2\pi}{\beta} inu}\,, \tag{3.8}$$

where $\epsilon_n = \epsilon_n^{(R)} + i\epsilon_n^{(I)}$. Demanding that $\epsilon(u)$ is real therefore leads to the two following conditions $\epsilon_n^{(R)} = \epsilon_{-n}^{(R)}$ and $\epsilon_n^{(I)} = -\epsilon_{-n}^{(I)}$. Then we find that the action can be written as

$$I_{\mathrm{dSch}}[t = u + \epsilon(u)] = -\alpha\beta E(\beta) + \frac{2\pi^2}{\beta\alpha a} \sum_{n\neq 0} n^4 \left(\epsilon_n^{(R)}\epsilon_n^{(R)} + \epsilon_n^{(I)}\epsilon_n^{(I)}\right). \tag{3.9}$$

The corresponding propagator for this action can be computed by carefully treating the U(1) zero mode corresponding to $n = 0$ and is given by

$$\langle \epsilon(u)\epsilon(0)\rangle = \frac{\beta\alpha a}{2\pi} \left(-\frac{1}{24}(u-\pi)^4 + \frac{\pi^2}{12}(u-\pi)^2 - \frac{7\pi^4}{360}\right). \tag{3.10}$$

Obtaining the measure. In order to evaluate the one-loop correction to the saddle-point approximation in (3.2) we first need to specify the measure $[\mathcal{D}t]_{\mathrm{Lif}}$ which the path integral will be computed with. Let us first recall the outlines of the calculation of the measure for JT before specifying the form for our Lifshitz-like dilaton model.

Due to the fact that Jackiw-Teitelboim gravity is $\mathrm{SL}(2,\mathbb{R})$ invariant the measure $[\mathcal{D}t]_{\mathrm{JT}}$ explicitly does not involve any modes which give rise to Killing vectors corresponding to the $\mathrm{SL}(2,\mathbb{R})$ isometries of AdS_2. The measure for JT has been computed by [34, 38] and is

$$[\mathcal{D}t]_{\mathrm{JT}} = \prod_u \frac{dt(u)}{t'(u)} = \prod_{n\geq 2} 4\pi n(n-1)(n+1) d\epsilon_n^{(R)} d\epsilon_n^{(I)}\,. \tag{3.11}$$

where we used the saddle point approximation $t(u) = u+\epsilon(u)$ with $\epsilon(u)$ given by equation (3.8). Let us emphasise again that the product is taken for $n \geq 2$ since $n = -1, n = 0$ and $n = +1$ specifically correspond to removed zero modes due to the $\mathrm{SL}(2,\mathbb{R})$ isometries transformations of the bulk solution. Due to the fact that in the Lifshitz-like model we consider, the dynamical dilaton breaks the $\mathrm{SL}(2,\mathbb{R})$ symmetry down into U(1) where the only Killing vectors corresponding to the U(1) isometries of AdS_2 we are lead to consider the following measure

$$[\mathcal{D}t]_{\mathrm{Lif}} = \prod_{n\geq 1} 4\pi n d\epsilon_n^{(R)} d\epsilon_n^{(I)}\,. \tag{3.12}$$

A few remarks are in order. Let us first note that the most drastic change in the evaluation of the partition function now comes from the product also including the contribution from the $n = \pm 1$ modes, since the classical solution of the deformed Schwarzian only has one zero mode that can be removed, corresponding to U(1) isometry. Let us also remark that the exact power of n does not matter so much in the evaluation of the path integral since it only changes the numerical constant in front of the partition function and not its thermodynamical behaviour. Here we have chosen the lowest degree polynomial.

Evaluating fluctuations around saddle points. Having stipulated the form of the measure for the Lifshitz-like dilaton gravity model in equation (3.12) along with the quadratic action in equation (3.9) we may now compute the one-loop saddle-point corrected partition function of the Lifshitz-like dilaton models as follows

$$Z_Q(\beta) \approx \int_{\text{Diff}(S^1)/\text{U}(1)} [\mathcal{D}t]_{\text{Lif}} \exp(\alpha E(\beta) \int t'(u) du + \frac{1}{a} \int du \frac{\{t,u\}_\alpha}{t'(u)^{\frac{1}{\alpha}}}) \tag{3.13}$$

$$= e^{\alpha\beta E(\beta)} \prod_{n\geq 1} 4\pi n \int d\epsilon_n^{(R)} d\epsilon_n^{(I)} e^{-\frac{2\pi^2 n^4}{\beta\alpha a}\left(\epsilon_n^{(R)}\epsilon_n^{(R)} + \epsilon_n^{(I)}\epsilon_n^{(I)}\right)} \tag{3.14}$$

$$= e^{\alpha\beta E(\beta)} \prod_{n\geq 1} \frac{2\alpha\beta a}{n^3} = \frac{1}{4\pi^{3/2}\sqrt{a\alpha\beta}} e^{\alpha\beta E(\beta)} . \tag{3.15}$$

where in the last step we used zeta function regularisation, namely that

$$\prod_{n\geq 1} \frac{2\alpha\beta a}{n^3} = \exp[\lim_{\epsilon\to 0} \partial_\epsilon \left((2\alpha\beta a)^\epsilon \zeta(3\epsilon)\right)] = \frac{1}{4\pi^{3/2}\sqrt{a\alpha\beta}} , \tag{3.16}$$

where $\zeta(3\epsilon) := \sum_{n\geq 1} n^{-3\epsilon}$.

The quantum corrected free energy F_Q reads

$$-\beta F_Q := \log Z(\beta) = \frac{\alpha C}{\beta^{\frac{1}{\alpha}}} + \frac{1}{2}\log\frac{1}{a\beta} + \dots , \quad C := \left(\frac{2\pi}{1+\alpha}\right)^{\frac{1+\alpha}{\alpha}} \left(\frac{2\alpha}{a}\right)^{\frac{1}{\alpha}} , \tag{3.17}$$

where the dots denote temperature independent constants. The logarithm is due to taking into account the loop corrections. We note that the factor $1/2$ reflects the single generator of the symmetry group. In the $\text{SL}(2,\mathbb{R})$ case, for example, this factor is $3/2$, reflecting its three generators. Furthermore, as we lower the temperature, from $\beta \sim 1/a$ onwards we expect quantum corrections to take over. In fact, solving $F_Q(\beta_*) = 0$ yields the temperature $\beta_* > 0$ at which one expect non-perturbative effects to the saddle point approximation to arise.

3.2 Density of states

The density of states classifies the energy spectrum of a system, such as the quantum black hole we are studying. We can obtain the density of states through the partition function. The partition function Z is related to the density of states ρ in the following manner

$$Z(\beta) = \int_0^\infty dE\, \rho(E) e^{-\beta E} , \tag{3.18}$$

in other words, $Z(\beta)$ is the Laplace transform of the density of states, $\rho(E)$. Traditionally we may compute the density of states from the partition function $Z(\beta)$ by an inverse Laplace transform

$$\rho(E) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} Z(\beta) e^{\beta E} d\beta , \tag{3.19}$$

where the integral is performed along a vertical line in the complex plane and the constant $c \in \mathbb{R}$ is a sufficiently large so that we may enclose all poles of the partition function with a semi-circle in the left half-plane. In practice this can often be evaluated with the Cauchy residue theorem.

Classical spectrum. To contrast the quantum spectrum, let us first study the classical black hole spectrum by considering just the saddle point approximation:

$$Z_{\mathrm{cl}}(\beta) \approx e^{-\mathcal{I}_{\mathrm{classical}}} = e^{\alpha\beta E(\beta)} = e^{\alpha C \beta^{-\frac{1}{\alpha}}} . \tag{3.20}$$

Here we have made use of the thermodynamical relationship from equation (2.32) and the defintion of C in (3.17). By expanding the exponential function in equation (3.20) in a formal series and taking the inverse Laplace transform of each term it follows that the classical density of states is given by

$$\rho_{\mathrm{cl}}(E) = \delta(E) + \frac{1}{E}\sum_{n=1}^{\infty} \frac{\left(\alpha C E^{\frac{1}{\alpha}}\right)^n}{n!\,\Gamma[\frac{n}{\alpha}]} , \tag{3.21}$$

where notably the δ function arises as the inverse Laplace transform of the first term in the formal series. For general values of α we are unable to simplify the formal series further. However, for $\alpha = 1$ we find

$$\rho_{\mathrm{cl}}^{\alpha=1}(E) = \delta(E) + \sqrt{\frac{\alpha C}{E}}\, I_1(2\sqrt{\alpha C E}) , \tag{3.22}$$

where I_1 denotes a modified Bessel function of the first kind, see e.g. [6] for a coinciding result in the context of super symmetric Schwarzian theories.

To get some insights into the expression for $\rho_{\mathrm{cl}}(E)$ for general α, we will investigate the large and small energy regimes. In the large energy approximation $CE^{\frac{1}{\alpha}} \gg 1$ we have

$$\rho_{\mathrm{cl}}(E) \approx \frac{C^{\frac{\alpha}{2(1+\alpha)}}}{E^{\frac{1+2\alpha}{2+2\alpha}} \sqrt{2\pi(1+\frac{1}{\alpha})}} e^{(1+\alpha) C^{\frac{\alpha}{1+\alpha}} E^{\frac{1}{1+\alpha}}} \sim e^{S(E)} , \tag{3.23}$$

where $S(E)$ denotes the Bekenstein-Hawking entropy, as is expected for large enough black holes. Let us now explore the small energy regime of (3.21). In the small energy approximation $CE^{\frac{1}{\alpha}} \ll 1$ we find that

$$\rho_{\mathrm{cl}}(E) \approx \delta(E) + \frac{\alpha C E^{\frac{1}{\alpha}-1}}{\Gamma[\frac{1}{\alpha}]} . \tag{3.24}$$

Despite the fact that the density of states appearing above blows up in the low energy limit then this does not mean that it is pathological. What matters more is whether or not the total concentration of available states in the system between any two energies E_1 and E_2, where $E_1 < E_2$, is finite, i.e. whether

$$N_{\mathrm{cl}}^{\mathrm{tot}}(E_1, E_2) = \int_{E_1}^{E_2} \rho_{\mathrm{cl}}(E)\, dE < \infty , \tag{3.25}$$

which is the case for all values of $0 < \alpha \leq 1$. In the low energy approximation of the density of states of the classical saddle (3.24), we furthermore point out that the power of E is always positive. As such, away from $\alpha = 1$, the behaviour of the density of states is such that there is a fall-off as $E \to 0$. Finally, it is significant that for $0 < \alpha \leq 1$

$$N_{\mathrm{cl}}^{\mathrm{tot}}(E_1 = 0, E_2 \to 0) \to 1 , \tag{3.26}$$

because reinstating the extremal entropy S_0 of the higher dimensional black hole reveals that the ground state of the classical saddle has a large degeneracy of $\exp S_0$. This is expected to disappear when corrections are added, for non-super symmetric theories.

Quantum corrected spectrum. Let us now contrast density of states of the classical saddle with the one-loop corrected partition function. By taking the inverse Laplace transform of the one-loop corrected partition function given by equation (3.15) we find

$$\rho_{\mathrm{Q}}(E)=\frac{1}{4\pi^{3/2}\sqrt{a\alpha E}}\sum_{n=0}^{\infty}\frac{(\alpha C E^{\frac{1}{\alpha}})^n}{n!\,\Gamma[\frac{1}{2}+\frac{n}{\alpha}]}\,. \tag{3.27}$$

For the case of $\alpha=1$ we find the closed expression

$$\rho_{\mathrm{Q}}^{\alpha=1}(E)=\frac{\cosh 2\sqrt{CE}}{4\pi^2\sqrt{aE}}\,, \tag{3.28}$$

which we will discuss further in the next subsubsection and contrast this result with the more usual JT result where the $\mathrm{SL}(2,\mathbb{R})$ remains intact. For general values of α we can gain more insights into the density of states of the quantum black hole through approximations. We consider the large energy approximation $CE^{\frac{1}{\alpha}}\gg 1$, for which we find that the quantum effects are washed out and the classical saddle point answer is recovered:

$$\rho_{\mathrm{Q}}(E)\approx\frac{1}{4\sqrt{2}\,\pi^2\sqrt{(1+\alpha)aE}}e^{(1+\alpha)C^{\frac{\alpha}{1+\alpha}}E^{\frac{1}{1+\alpha}}}\sim\rho_{\mathrm{cl}}(E)\,, \tag{3.29}$$

which reinforces that the quantum corrections do not influence the high energy regime.

To get a better idea of the quantum mechanical effects we now consider the small energy regime. In the small energy limit for $CE^{\frac{1}{\alpha}}\ll 1$ this means that

$$\rho_Q(E)\approx\frac{1}{4\pi^2\sqrt{a\alpha E}}\,. \tag{3.30}$$

Contrasting the classical saddle point (3.24), the density of states in the quantum case rapidly grows for small energies instead of tapering off towards zero. However, when integrating this density we find that finite behaviour for small energies. In, fact we find

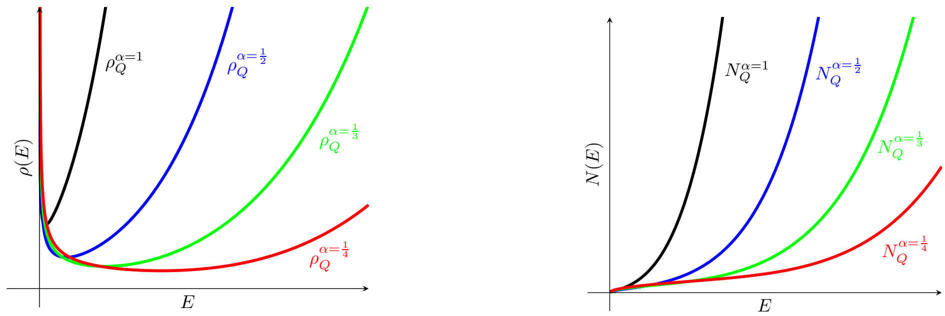
$$N_{\mathrm{Q}}^{\mathrm{tot}}(E_1,E_2)=\int_{E_1}^{E_2}\rho_{\mathrm{Q}}(E)\,dE\,,\quad N_{\mathrm{Q}}^{\mathrm{tot}}(E_1=0,E_2\to 0)\to 0\,, \tag{3.31}$$

for all $0<\alpha\leq 1$. This means that even if we were te reinstate the extremal degeneracy $\exp S_0$, the ground state will have no degeneracy when quantum corrected (see Figure 2 and Figure 3).

The low energy behaviour of the dispersion relation can be reproduced from simple free theories. Namely, let us consider a D-dimensional theory with a free Lifshitz-type dispersion relation $E\sim|k|^z$, where k is the momentum. The density of states then becomes

$$\rho(E):=\frac{d\Omega(k(E))}{d|k|}\sim\frac{1}{\sqrt{E}}\,,\quad\Leftrightarrow\quad z=2(D-1)\,, \tag{3.32}$$

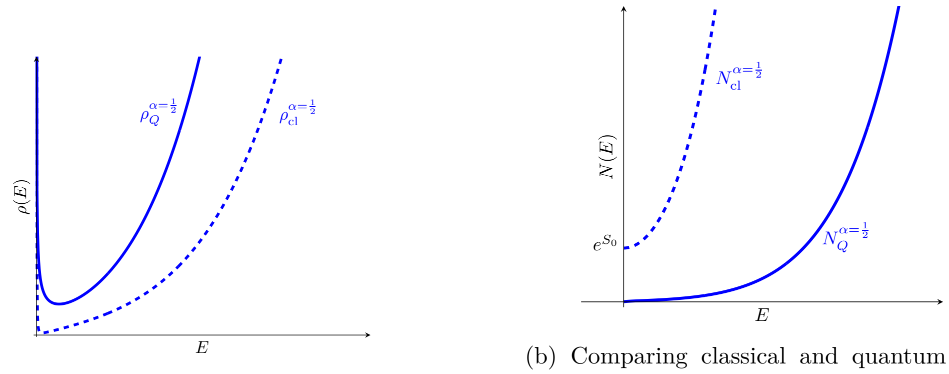
where we used that the total number of states contained in a sphere in momentum space scales as $\Omega(k)\sim|k|^{D-1}$. We find that this value of z satisfies $D\leq z+1$, which for models of Lifshitz models is reported to behave differently from $D<z+1$, see e.g. [60, 61]. We point out that in particular the behaviour of the density of states matches that of a $1+1$-dimensional theory with a non-relativistic dispersion relation.



(a) The quantum corrected density of states $\rho_Q(E)$ for different values of α according to equation (3.27). Note the small energy dependence $\rho_Q \sim \frac{1}{\sqrt{E}}$ independent of α.

(b) The number of states, $N(E)$ below a certain energy E for different values of the parameter α according to equation (3.31). Note the lack of an exponentially large ground state.

Figure 2. Highlighting the low energy behaviour of the quantum corrected density of states.



(a) Comparing the classical density of states (3.20) to the quantum corrected (3.27) for the specific value $\alpha = 1/2$.

(b) Comparing classical and quantum corrected expressions for $\alpha = 1/2$. Note the absence of a degenerate ground state for the quantum correction.

Figure 3. Comparing the classical and quantum corrected density of states.

Comparison with the usual JT gravity approach It is important to stress that $\alpha = 1$ corresponds to a boundary description of JT gravity with a source turned on that breaks the $\mathrm{SL}(2,\mathbb{R})$ down to $\mathrm{U}(1)$ for the classical saddle. This contrasts the usual approach in which such a source is not considered and the classical saddle retains its $\mathrm{SL}(2,\mathbb{R})$ invariance. In the usual approach the density of states of the quantum corrected saddle becomes proportional to a hyperbolic sine of energy [33], which goes to zero as energy vanishes. The resulting conclusion is that there is a very low degeneracy for these black holes at low energy. In our case, for any α, the opposite is true, as for low energy the density of state grows with the inverse square root of energy, see (3.30).

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Furthermore, let us summarize succinctly that for $\alpha = 1$ we found that the partition function and density of states were respectively given by

$$Z_Q^{\alpha=1}(\beta) = \frac{1}{4\pi^{3/2}\sqrt{a\beta}} e^{C\beta^{-1}}, \qquad \rho_{\mathrm{Q}}^{\alpha=1}(E) = \frac{\cosh 2\sqrt{CE}}{4\pi^2\sqrt{aE}} . \tag{3.33}$$

This result coincides with the trumpet result for JT in equation (126) of Saad-Shenker-Stanford [33] where notably the $\mathrm{SL}(2,\mathbb{R})$ symmetry is also broken down to U(1). This same form has also appeared in the context of $\mathcal{N} = 1$ supersymmetric JT gravity.[6]

4 Discussion and outlook

In this paper we have computed a one-loop correction to the partition function of 4d near-extremal charged Lifshitz black holes that arise in low energy effective actions of string theory. This has been accomplished by dimensionally reducing the four-dimensional black hole to an effective two-dimensional dilaton theory of gravity which is asymptotically AdS_2 and arises in the near-extremal charge limit of the Lifshitz black hole. By imposing wiggling boundary conditions we derived a boundary action formulation that resembles the Schwarzian action, with the key exception that our boundary action does not retain the $\mathrm{SL}(2,\mathbb{R})$, but breaks it down to U(1). A key difference with the JT case is that we perform a saddle point approximation of the dilaton to obtain the boundary description. We employed the boundary action in order to compute the quantum corrected partition function of these Lifshitz black holes. Our analysis shows that for small temperatures the quantum one-loop corrected partition function is given by

$$Z_{\mathrm{BH}}(T) \sim \sqrt{T} e^{S_0} + \text{higher order terms in } T \, , \tag{4.1}$$

where S_0 is the extremal entropy. This indicates that including quantum corrections to the black hole entropy removes the exponential ground-state degeneracy observed in the semiclassical analysis. This corresponds with the results reported for Reissner-Nordström and Kerr-Newman black holes in recent work [13, 18–20], except that we find that the prefactor is $\sqrt{T}$ instead of $T^{3/2}$. This difference in prefactor stems from the lack of symmetries present at the horizon of these Lifshitz black holes, which merely retain U(1) symmetries instead of the full $\mathrm{SL}(2,\mathbb{R})$ symmetries. There is an intricate connection between the amount of symmetries present in the near-extremal region of the black hole and the prefactor of the one-loop corrected partition function.

The one-loop corrected partition function $Z_{\mathrm{BH}}(T)$, which we computed vanishes in the low temperature limit, indicating that the corresponding density of states approaches zero as energy decreases, thereby removing the previously observed exponential ground-state degeneracy. The expression that we obtained for the quantum one-loop corrected density of states in equation (3.27) behaves as $\rho_Q(E) \sim 1/\sqrt{E}$ for small energies indicating that the quantum corrected density of states blows up for small energies, i.e. $\rho(E) \to \infty$ as $E \to 0$. This same low energy behaviour of the density of states is encountered in several non-relativistic theories, the most elementary being the one-dimensional infinite potential well. Note that

[6] See e.g. equation (3.46) of [38], (6.4) of [62], (5.4)-(5.8) of [63], or (2.25) of [6].

integrating over a small band of energy smoothes out this blowing up behaviour and the total number of states does go to zero as $E \to 0$, as highlighted pictorially in Figure 3. In contrast the classical spectrum clearly exhibits an exponentially large ground state degeneracy.

In our approach we of course do not capture the full four-dimensional near-extremal one-loop corrections: we have ignored vector and gauge modes, which our two-dimensional approach fails to capture (we only capture the tensor mode). It would be interesting to see if we could capture those other contributions using the recently developed methods for computing the logarithmic corrections to the black hole entropy in [13, 18–20, 22–25], which rely on the fact that the normalised basis of zero mode deformations of AdS_2 is well known and is utilised to simplify the computation of the logarithmic correction. We were unable to perform the same feat for Lif_2 due to the technical difficulty of obtaining the zero modes in that spacetime, which we leave to future work. Another potential contribution to the path integral that we neglected are possible bulk fluctuation contributions.

As a conclusion we note that it would be interesting to pursue whether there exists a matrix model dual these Lifshitz-like dilaton theories, similar to JT gravity. This seems to be a non-trivial task, as the action does not appear to be a deformation of JT gravity [64]. Finally, in the case of the boundary description of JT gravity it has been argued that the partition function of double-scaled SYK for low energies agrees with the Schwarzian result [33, 62, 65, 66]. It would be interesting to construct a quantum mechanical dual to our constructed boundary description that breaks down $\mathrm{SL}(2,\mathbb{R})$ to $\mathrm{U}(1)$.

Acknowledgments

We thank Alex Frenkel, Friðrik Freyr Gautason, and Diego Hidalgo for discussions and insightful comments. WS thanks Matt Blacker, Alejandra Castro, and Chiara Toldo for collaboration on related topics. MBH and WS acknowledge the Icelandic Research Fund Grant 228952-053. WS also acknowledges support of the Simons Foundation through the INI-Simons Postdoctoral Fellowship.

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

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Paper III

Quantum Fluctuations and Newton-Cartan Geometry for Non-Relativistic de Sitter space

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Preprint, accepted for publication in the Journal of High Energy Physics (JHEP).

The thesis author contributed to the analysis of the non-relativistic de Sitter boundary action, the one-loop partition function, and the construction of the associated Newton-Cartan bulk geometry. The thesis author also contributed to the derivations, interpretation of the results, and writing of the manuscript.

Quantum Fluctuations and Newton-Cartan Geometry for Non-Relativistic de Sitter space

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Abstract: We study a non-relativistic realisation of two-dimensional de Sitter gravity both from its boundary and bulk description with the goal of learning about de Sitter space and paving the way for extending the holographic duality into a non-relativistic direction. On the boundary side, we analyse the Schwarzian-type boundary action associated with non-relativistic de Sitter gravity and evaluate its one-loop partition function in order to compute its quantum fluctuations. Rather than relying on the coadjoint-orbit construction, we derive the path integral measure directly from the action using the Ostrogradsky formalism. We find a temperature-dependent prefactor scaling as T^2, of which the power agrees with the counting of the four global symmetry generators present. On the bulk side, we construct the corresponding torsionless Newton-Cartan geometry and show that it satisfies the equations of motion of a non-relativistic JT-like action and uplift the geometry to a three-dimensional Lorentzian geometry.

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1 Introduction

It is difficult to overstate the implications of the initial discovery of the Anti-de Sitter/conformal field theory (AdS/CFT) duality on our understanding of physics, with a central role being played by $\mathcal{N} = 4$ Super Yang-Mills, on one side of the duality, and type II-B supergravity on an AdS background on the other side [1]. More recently, our understanding of the duality has been significantly extended in the realm of two-dimensional gravity, which can be considered the lowest-dimensional realisation of the AdS/CFT correspondence and, in this manner, facilitates the study of quantum gravity due to its technical manageability. In this context, the bulk gravity model containing an AdS background is governed by Jackiw-Teitelboim (JT) gravity theory [2, 3], where the so-called near-AdS_2 dynamics is controlled by a boundary action involving the Schwarzian derivative. It was found that he same action also governs the low-energy regime of the $(0 + 1)$-dimensional Sachdev-Ye-Kitaev (SYK) model [4–11]. Some features of the SYK model, such as the infrared "nearly" conformal invariance and chaotic behaviour, see e.g. [4], suggest the possible existence of a gravitational dual model in AdS_2 spacetime.

Amid the intense efforts to extend the AdS/CFT duality to a more general notions of holography that also encompasses different spacetimes, most of that effort has focused on ensuring the *relativistic* symmetries in the bulk. Going beyond this relativistic bulk context, there has been an emergence of explorations of non-Lorentzian geometries [12–16] and non-Lorentzian string theories [17–22]. These research topics, initially fuelled by the desire to learn about non-relativistic quantum field theories in the context of condensed matter, see e.g. [23], grew into a field of its own graduating beyond its original scope. The term non-Lorentzian here is used as an umbrella term for both the non-relativistic ("speed of light to infinity") and Carrollian ("speed of light to zero") limits (see Figure 1). Particularly, the non-relativistic limit can be complemented with the cosmological constant vanishing $\Lambda \to 0$, so that the dS or AdS algebra becomes the Newton-Hooke algebra [24–26], the analogue of the Galilei algebra in the presence of a universal cosmological repulsion or attraction constant, respectively. The idea of considering the non-relativistic version of de Sitter space was explored for its possible application to non-relativistic cosmology in [24, 27] and it was also considered in the context of M- or String theory where a modification of the matrix model is obtained using the corresponding symmetry algebra [28]. In the present paper, we will utilise an centrally extended version of the so-called Newton-Hooke symmetry algebra that can be obtained with a central generator. This central generator can be thought of as the analogue of the Bargmann generator, which allows for the representation of 'massive' states. In its two-dimensional incarnation will refer to it as the Extended de Sitter Galilean (EdS-G) symmetry algebra following [29].

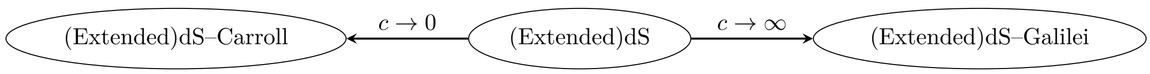

Figure 1. The figure shows two kinematical contractions of de Sitter (dS) symmetry algebra. Here c denotes the contraction parameter, which may be interpreted heuristically as the speed of light. The extended version refers to the possibility of adding Abelian generators to the dS algebra, thereby yielding extended versions of the non-Lorentzian algebras.

The present paper is devoted to investigating the EdS-G gravity, which allows us to address holography in a controlled non-relativistic de Sitter setting as well as learning about de Sitter space. We will explore both the boundary and bulk realisation of this non-relativistic version of de Sitter gravity.

On the boundary side we will use the BF formalism in two dimensions, a first-order gauge theory formulation of a gravity theory that realises EdS-G isometries, for which a boundary formulation was derived in [30] – see also [29] for studies of non-Lorentzian realisations of JT. It was shown that the asymptotic structure of this Galilean gravity is governed by a so-called non-relativistic Schwarzian theory, which exhibits an infinite-dimensional symmetry algebra given by the twisted Virasoro algebra. Inspired by the computation of the one-loop contribution to the boundary description of the relativistic

Schwarzian, see [31] for a comprehensive review, we compute the one-loop contribution to the non-relativistic Schwarzian theory and find that it correctly reflects the number of generators of EdS-G. On the bulk side we present a Newton-Cartan incarnation that geometrises the EdS-G algebra and we show that its solves the equations of motion of a non-relativistic version of JT.

The structure of the paper is as follows: We compute one-loop fluctuations to the boundary theory governing the dynamics of a two-dimensional EdS-G gravity theory in Section 2. We continue with a further presentation of a Newton-Cartan formulation of the gravitational bulk in Section 3. In the last Section 4, we relate our computations with recent developments in the fields and discuss future prospects.

2 The boundary description of non-relativistic de Sitter space

We study the Schwarzian-type effective boundary action associated to a non-relativistic bulk theory whose symmetries are governed by the two-dimensional Galilean de Sitter algebra. This action was derived in [30] using a first-order formulation in terms of BF variables, and it was shown that this governs the boundary dynamics of an EdS-G gravity theory. We begin by reviewing such a derivation. Because the Galilean de Sitter algebra is isomorphic to the Carrollian Anti-de Sitter algebra, a similar boundary action also appears in Carrollian settings of [29], although the bulk interpretation is different. This section is organised in the following manner. We first introduce the action in section 2.1, we then analyse the symmetries of the action in section 2.2, determine its classical saddle in section 2.3, and finally evaluate its one-loop contribution in section 2.4.

2.1 The Galilean de Sitter boundary action

Let us start by considering the two-dimensional de Sitter space symmetry algebra $\mathfrak{so}(1,2)$ whose only non-vanishing commutation relations are

$$[B,\tilde{H}]=\tilde{P}\,,\qquad [B,\tilde{P}]=\tilde{H}\,,\qquad [\tilde{H},\tilde{P}]=-\tilde{\Lambda}\,B\,. \tag{2.1}$$

Here B generates Lorentzian boosts, $\tilde{P}$ spatial translations and $\tilde{H}$ time translations. Here $\tilde{\Lambda}$ denotes the cosmological constant. The tilde differentiates with generators that we introduce after contraction. Restricting to a positive value of $\tilde{\Lambda}$, one possible Inönü–Wigner contraction in which one morally takes the speed of light to infinity while simultaneously rescaling $\tilde{\Lambda}$ leads to the non-relativistic EdS-G algebra, which is sometimes referred to as the Newton-Hooke$^2_+$ algebra [32], given by[1]

$$[G,H]=P\,,\qquad [G,P]=M\,,\qquad [H,P]=-\Lambda\,G\,, \tag{2.2}$$

[1]Notice that despite of the absence of the speed of light c in the symmetry algebra (2.1), a careful incorporation of c factors modifies the commutator between momenta as $[\tilde{H},\tilde{P}]=c^2\,\tilde{\Lambda}\,B$. Then, going through the contraction process, it is necessary to take simultaneously $c\to\infty$ and $\tilde{\Lambda}\to 0$ limits; otherwise, we would have to delete the boost B from the algebra. The origin of c factors arises from an analysis of the equations governing a non-relativistic cosmological model, based on geodesics in de Sitter space [33]. In this paper we use units in which $c=1$.

where G denotes the Galilean boost, H an P spacetime translational generator, and M a central generator. The EdS-G algebra is the analogue of the Bargmann algebra in the presence of a universal cosmological repulsion or attraction $\tilde{\Lambda} = 1/\ell^2$. Here ℓ stands for the characteristic scale. Given this algebra, we want to formulate a bulk and boundary theory that realises the algebra. We will now focus on engineering the boundary theory.

Within this limit, we propose to formulate the gravity model. For later purposes, it is convenient to introduce the conformal basis

$$\mathcal{D} = \ell H \,, \qquad \mathcal{H} = \ell P + G \,, \qquad \mathcal{K} = \ell P - G \,, \qquad \mathcal{Z} = \ell M \,. \tag{2.3}$$

In this basis, the algebra (2.2) becomes

$$[\mathcal{H}, \mathcal{D}] = \mathcal{H} \,, \qquad [\mathcal{K}, \mathcal{D}] = -\mathcal{K} \,, \qquad [\mathcal{H}, \mathcal{K}] = 2\mathcal{Z} \,, \tag{2.4}$$

with a non-degenerate invariant bilinear form, given by

$$\langle \mathcal{D}, \mathcal{D} \rangle = c_0 \,, \qquad \langle \mathcal{D}, \mathcal{Z} \rangle = c_1 \,, \qquad \langle \mathcal{H}, \mathcal{K} \rangle = -2c_1 \,. \tag{2.5}$$

Here c_0 and $c_1 \neq 0$ are two arbitrary constants. Notice that the algebra (2.4) is exactly the extended two-dimensional version of the Poincaré algebra (also known as the Maxwell algebra). The EdS-G algebra (2.2) was explicitly realised in [30], where both in BF formalism and metric formalism an EdS-G gravity model was constructed along with the boundary action. In the remainder of this subsection we recap results relevant to the current work. In this context, the construction of the boundary action for a two-dimensional Galilean gravity theory can be interpreted as the action principle for a particle moving on the group manifold. The procedure goes as follows. Consider, the left-invariant Maurer-Cartan form Ω associated to the (2.2), which is built from the group element

$$U = e^{s\mathcal{Z}} e^{\rho\mathcal{H}} e^{y\mathcal{K}} e^{u\mathcal{D}} \,, \tag{2.6}$$

where $\{s, \rho, y, u\}$ are coordinates of the group manifold. Then, we compute the associated (left-invariant) Maurer–Cartan one-form given by

$$\Omega = U^{-1} dU = (ds + 2y\, d\rho)\, \mathcal{Z} + e^{u} d\rho\, \mathcal{H} + e^{-u} dy\, \mathcal{K} + du\, \mathcal{D} \,. \tag{2.7}$$

Using the invariant bilinear form (2.5), we construct the action principle as

$$S[s, \rho, y, u] = \int dt \left\langle \Omega, \Omega \right\rangle^{\star} = \int dt \Big(c_0\, u'^2 + c_1 \Big(u's' + 2y\, \rho' - 2\rho' y' \Big) \Big) \,. \tag{2.8}$$

with prime derivative with respect to the worldline parameter t of the particle, and $\star$ denotes the pull-back action. This action principle depends on four dynamical fields. Like in $(2+1)$-gravity theory with the reduction from WZNW model to Liouville theory [34], here it is also possible to further reduce the number of dynamical fields by imposing the following inverse Higgs constraints (IHC) [17, 35]

$$\Omega_{\mathcal{Z}} = 0 \,, \qquad \Omega_{\mathcal{H}} = 1 \,, \tag{2.9}$$

from which it follows that $y = -\frac{1}{2}e^u s'$ and $\rho' = e^{-u}$. Then, the boundary action becomes

$$S[s,u] = \int dt \Big(c_0 {u'}^2 + c_1 (s'u' + s'') \Big) \,. \tag{2.10}$$

At the same time, the reduced Maurer–Cartan form or gauge connection becomes

$$a = \Omega\big|_{\mathrm{IHC}} = \mathcal{H} - \frac{1}{2}(s'' + s'u')\,\mathcal{K} + u'\,\mathcal{D}\,. \tag{2.11}$$

As an extra definition, it is convenient to perform the field redefinition $u = v - \log(v')$, so that the reduced action takes the form

$$S[s,v] = \int dt \left[c_0 \left(v'^2 - 2v'' + \left(\frac{v''}{v'} \right)^2 \right) + 2c_1 \left(s'' + s' \left(v' - \frac{v''}{v'} \right) \right) \right]. \tag{2.12}$$

This one-dimensional action principle is invariant under the EdS-G (global) symmetries (2.2), and governs the asymptotic dynamics of a two-dimensional EdS-G gravity model. As we will see below, it also admits an invariance under the warped Virasoro algebra. In fact, by considering the one-form gauge connection

$$a(t) = a_t(t)\,dt\,, \qquad a_t(t) = \mathcal{H} + \mathcal{L}(t)\,\mathcal{K} + \mathcal{T}(t)\,\mathcal{D}\,, \tag{2.13}$$

the infinite-dimensional symmetry appears from examining the gauge symmetries preserving this structure for a. Here the functions $\mathcal{L}$ and $\mathcal{T}$ are two arbitrary functions of t only related to the previous fields as follows

$$\mathcal{L}(t) = -\frac{1}{2}\left(s'' + s' \left(v' - \frac{v''}{v'} \right) \right), \qquad \mathcal{T}(t) = v' - \frac{v''}{v'}\,, \tag{2.14}$$

These relations arise naturally from studying the boundary dynamics of the BF model in a two-dimensional bulk. Concretely, the relations are supported through the equation $a = U^{-1}dU$. We find that the gauge symmetry transformation laws that preserve the structure of a, demand that

$$\delta\mathcal{L} = \sigma\mathcal{L}' + 2\mathcal{L}\sigma' + \mathcal{T}\chi' + \chi''\,, \tag{2.15}$$
$$\delta\mathcal{T} = \sigma'\mathcal{T} + \sigma\mathcal{T}' - \sigma''\,, \tag{2.16}$$

with χ and σ two independent gauge parameters which are functions of t. These are precisely the transformation laws underlying the warped Virasoro structure of the theory [36]. Furthermore, the algebra of the canonical generators can be obtained from the identity $\{Q_{\epsilon_1}, Q_{\epsilon_2}\} = \delta_{\epsilon_2} Q_{\epsilon_1}$ together with the transformation laws in (2.15). In this case, we have two canonical generators associated with the two gauge symmetry parameters and their equal-time Poisson brackets are given by

$$i\{\mathcal{L}_m\,, \mathcal{L}_n\} = (m-n)\mathcal{L}_{m+n} + c\,n^3 \delta_{n+m,0}\,, \tag{2.17a}$$
$$i\{\mathcal{L}_m\,, \mathcal{T}_n\} = -n\mathcal{T}_{m+n} + i\kappa\, m^2 \delta_{n+m,0}\,, \tag{2.17b}$$
$$i\{\mathcal{T}_m\,, \mathcal{T}_n\} = 0\,, \tag{2.17c}$$

with the central charges $c = 2\pi c_1$ and $\kappa = 4\pi c_0$. After the redefinitions $\mathcal{T}_m \to \mathcal{T}_m - i\kappa\delta_m$ and $\mathcal{L}_m \to \mathcal{L}_m + \frac{c}{2}\delta_m$, we recognise this as the standard warped Virasoro algebra (see [37] for an overview). Therefore, we have shown that the asymptotic structure of the EdS-G gravity model is governed by the warped Virasoro algebra with two different central charges. The global part of the symmetry algebra (2.17) corresponding to (2.2) is built on four generators $\{L_0, L_1, P_1, P_{-1}\}$ that preserve the classical vacuum configuration

$$\mathcal{L} = 0\,, \qquad \text{and} \qquad \mathcal{T} = \text{constant}\,. \tag{2.18}$$

Having made contact with the warped Schwarzian theory in [37], let us briefly explain in which respects it differs from the present Galilean de Sitter boundary theory. On the one hand, the warped Schwarzian theory is formulated in terms of coadjoint orbits of the warped Virasoro group and the associated Kirillov–Kostant–Souriau symplectic form [38–40]. The relation should, however, be understood at the level of asymptotic symmetry structure rather than as an identification of the two boundary actions. In reference [37], the theory is constructed directly from the coadjoint orbit, whereas here the boundary action arises from a non-relativistic de Sitter bulk construction after imposing inverse Higgs constraints. Thus, although the asymptotic symmetry algebra is the same, the bulk origin of the two theories is different. We will also derive the symplectic form entering the path integral measure directly from the effective action by introducing Ostrogradsky canonical momenta, see e.g. [41]. To the best of our knowledge, this action-based derivation of the symplectic form has not been emphasised in the literature. As a consistency check, Appendix A shows that the same method reproduces the familiar reduced symplectic structure of the Schwarzian theory. We then revisit the computation of [37] in Appendix B using our formalism. This reproduces the corresponding one-loop result and clarifies in what sense the present Galilean de Sitter boundary action is closely related to, but not identical to, the warped Schwarzian theory.

2.2 Isometries of the Galilean de Sitter action

In order to understand the structure of the path integral and, in particular, the origin of zero modes in the one-loop analysis, we first determine the global symmetries of the boundary action. Having reviewed the derivation of the Galilean de Sitter action in equation (2.12), we recall that it is given by

$$I_{\text{EdS-G}} = \int dt \left[c_0 \left(v'^2 - 2v'' + \left(\frac{v''}{v'}\right)^2 \right) + 2c_1 \left(s'' + s' \left(v' - \frac{v''}{v'} \right) \right) \right]. \tag{2.19}$$

We now observe that the action is invariant under the finite transformations

$$s(t) \mapsto s(t) + a + be^{-v(t)}\,, \qquad v(t) \mapsto -\log\big(c + de^{-v(t)}\big)\,, \tag{2.20}$$

where a, b, c, d are arbitrary constants. The corresponding infinitesimal isometries which leave the action invariant are given by

$$\delta s = \epsilon_a + \epsilon_b e^{-v(t)}\,, \qquad \delta v = \epsilon_c + \epsilon_d\, e^{v(t)}\,, \tag{2.21}$$

where $\{\epsilon_a, \epsilon_b, \epsilon_c, \epsilon_d\}$ are infinitesimal parameters, which will be used to construct the corresponding Noether charges. These four independent transformations generate the global part of the warped Virasoro symmetry algebra (2.17) and will later be associated with the non-Gaussian sector of the path integral. Because the action contains higher-derivative terms, the corresponding Noether charges must be constructed using the generalised Ostrogradsky formalism. We compute the generalised Ostrogradsky momenta as

$$p_1^{(s)} = \frac{\partial L}{\partial s'} - \frac{d}{dt}\left(\frac{\partial L}{\partial s''}\right) = 2c_1\left(v' - \frac{v''}{v'}\right), \tag{2.22}$$

$$p_1^{(v)} = \frac{\partial L}{\partial v'} - \frac{d}{dt}\left(\frac{\partial L}{\partial v''}\right) = 2c_1 s' + 2c_0 v' + \frac{2}{v'}\left[c_1 s'' + c_0\left(\frac{v''^2}{v'^2} - \frac{v'''}{v'}\right)\right], \tag{2.23}$$

$$p_2^{(s)} = \frac{\partial L}{\partial s''} = 2c_1\,, \tag{2.24}$$

$$p_2^{(v)} = \frac{\partial L}{\partial v''} = -\frac{2}{v'}\left[c_1 s' + c_0\left(v' - \frac{v''}{v'}\right)\right]. \tag{2.25}$$

From these, we may compute the four associated charges, obtaining

$$Q[\epsilon] = \int dt\left[p_1^{(s)}\,\delta s + p_1^{(v)}\,\delta v + p_2^{(s)}\,\delta s' + p_2^{(v)}\,\delta v'\right], \tag{2.26}$$

where δs and δv are the infinitesimal symmetry variations given in (2.21). Substituting these variations into the charge formula yields

$$Q_a = Q[\epsilon_a] = 2c_1 \int dt\left(v' - \frac{v''}{v'}\right), \tag{2.27}$$

$$Q_b = Q[\epsilon_b] = -2c_1 \int dt \frac{v''}{v'} e^{-v(t)}\,, \tag{2.28}$$

$$Q_c = Q[\epsilon_c] = 2\int \frac{dt}{v'}\left[c_1 s'' + c_0\left(\frac{v''^2}{v'^2} - \frac{v'''}{v'}\right)\right], \tag{2.29}$$

$$Q_d = Q[\epsilon_d] = 2\int \frac{dt\, e^{v(t)}}{v'}\left[c_1 s'' + c_0\left(v'' + \frac{v''^2}{v'^2} - \frac{v'''}{v'}\right)\right]. \tag{2.30}$$

Evaluating the Poisson brackets of the charges directly, we find that Q_a, Q_b, Q_c, Q_d realise the conformal Galilean algebra in (2.4). More explicitly, the non-vanishing brackets are

$$\{Q_c, Q_b\} = -Q_b\,, \qquad \{Q_c, Q_d\} = Q_d\,, \qquad \{Q_d, Q_b\} = Q_a\,. \tag{2.31}$$

Identifying $Q_c = \mathcal{D}$, $Q_b = \mathcal{H}$, $Q_d = \mathcal{K}$, and $Q_a = 2\mathcal{Z}$, we recover the algebra (2.4). This confirms that the conserved charges of the boundary theory realise the conformal Galilean algebra in its global symmetries. As we will see in the next sections, these symmetries are directly responsible for the presence of zero modes in the quadratic fluctuation analysis, which in turn require a separate treatment in the path integral.

2.3 Equations of motion and saddle point

We now determine the classical saddle of the action and analyse quadratic fluctuations around it, which will form the basis of the one-loop computation in the next section. To this end, we decompose the fields into background configurations and fluctuations

$$v = \overline{v} + \delta v\,, \qquad s = \overline{s} + \delta s\,, \tag{2.32}$$

where the barred quantities denote the background fields and the quantities accompanied by δ denote the fluctuations. The background fields are taken to satisfy the boundary conditions

$$\bar{v}(t+\beta) = v(t) - 2\pi i\,, \qquad \bar{s}(t+\beta) = \bar{s}(t)\,, \tag{2.33}$$

while the fluctuations are β-periodic. Notice that from now on, we have implicitly promoted the coordinate t to be cyclic. The appearance of the imaginary shift in $\bar{v}$ might not be surprising, at least on an algebraic level, when we consider the very similar actions presented in [29, 37], which are invariant under the Extended AdS-Carroll algebra. This algebra is isomorphic to EdS-G by interchanging the spacetime translational generators as $P \leftrightarrow H$. The actions there, in essence, have $v \to iv$ as compared to the EdS-G action. In the next section, we will require the fluctuations around the saddle-point solution of v to be real; as such, v' will turn out to receive only a complex shift due to the boundary conditions above. We believe that this is reminiscent of the choices of complex contours in discussions surrounding the treatment of the boundary theory of two-dimensional *relativistic* de Sitter [42, 43].

Having specified the boundary conditions, we now expand the action to first order in the fluctuations. Requiring the first variation to vanish, then yields the equations of motion for the background fields. We obtain

$$I^{(1)}_{\text{EdS}-\text{G}} = -2\int dt \left\{ \partial_t \left[c_1 \left(\overline{s}' + \frac{\overline{s}''}{\overline{v}'} \right) + c_0 \left(\overline{v}' - \frac{\overline{v}'''}{\overline{v}'^2} - \frac{\overline{v}''^2}{\overline{v}'^3} \right) \right] \delta v + c_1 \partial_t \left(\overline{v}' - \frac{\overline{v}''}{\overline{v}'} \right) \delta s \right\}. \tag{2.34}$$

From this, we may read off the equations of motion, which are given by

$$\delta v : \qquad \partial_t \left[c_1 \left(\overline{s}' + \frac{\overline{s}''}{\overline{v}'} \right) + c_0 \left(\overline{v}' - \frac{\overline{v}'''}{\overline{v}'^2} - \frac{\overline{v}''^2}{\overline{v}'^3} \right) \right] = 0\,, \tag{2.35}$$

$$\delta s : \qquad \partial_t \left(\overline{v}' - \frac{\overline{v}''}{\overline{v}'} \right) = 0\,. \tag{2.36}$$

The equations can be solved sequentially: first, one determines $\bar{v}$ from the second equation, and then substitutes this result into the first equation to determine $\bar{s}$. The solutions are then given by

$$\begin{aligned} e^{-\overline{v}} &= k_1 e^{k_2 t} + k_3\,, \\ \overline{s}' &= k_4 e^{\overline{v}/c_1} + k_5\,, \end{aligned} \tag{2.37}$$

with $k_1, ..., k_5$ integration constants. Imposing the periodicity constraints in (2.33) we choose the saddle point to be

$$\overline{s}(t) = 0\,, \qquad \overline{v}(t) = -\frac{2\pi i}{\beta} t\,, \tag{2.38}$$

in agreement with the values appearing in (2.18), which are invariant under the global conformal Galilean subalgebra (see Appendix C). Constant shifts in $\bar{s}$ and $\bar{v}$ have been omitted, since the action depends only on derivatives of these fields. Continuing with higher orders in the expansion, the action to second order in the fluctuations gives

$$I^{(2)}_{\text{EdS}-\text{G}} = \int dt \Big[2c_1\, \delta s'\, \delta v' + c_0 \delta v'^2 + \frac{2c_1}{\bar{v}'}\, \delta v'\, \delta s'' + \frac{c_0}{\bar{v}'^2} \delta v''^2 - \frac{2c_1 \bar{s}' \bar{v}''}{\bar{v}'^3} \delta v'^2 + \frac{3c_0 \bar{v}''^2}{\bar{v}'^4} \delta v'^2 + \frac{2c_1 \bar{s}'}{\bar{v}'^2}\, \delta v'\, \delta v'' - \frac{4c_0 \bar{v}''}{\bar{v}'^3}\, \delta v'\, \delta v'' \Big] . \tag{2.39}$$

To fully account for the quadratic fluctuations, we must evaluate the path integral over the fluctuation modes, which we will do in the next section. In addition, the path integral measure is fixed as the Pfaffian of the associated symplectic form, defined as

$$\Omega = \int_0^\beta dt \Big(dp_v \wedge dv + dp_s \wedge ds \Big) . \tag{2.40}$$

There the generalized momenta p_v and p_s are defined, using the Ostrogradsky formalism, through

$$p_w = \frac{\partial \mathcal{L}}{\partial w'} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial w''} \right) , \tag{2.41}$$

with $\mathcal{L} = \mathcal{L}(w, w', w'')$ the Lagrangian corresponding to the action in (2.19), which depends at most on second derivatives, and where w denotes either v or s. Applying this to the quadratic fluctuation action, we obtain the corresponding generalised Ostrogradsky momenta

$$p_{\delta s} = \frac{2c_1}{\bar{v}'} \left(\left(\bar{v}' - \tfrac{\bar{v}''}{\bar{v}'} \right) \delta v' - \delta v'' \right) , \tag{2.42}$$

$$p_{\delta v} = \frac{2c_1}{\bar{v}'} \left(\bar{v}' \delta s' + \delta s'' - \tfrac{\bar{s}''}{\bar{v}'} \delta v' \right) + \frac{2c_0}{\bar{v}'^2} \left(\left(\bar{v}'^2 - \tfrac{3\bar{v}''^2}{\bar{v}'^2} + \tfrac{2\bar{v}'''}{\bar{v}'} \right) \delta v' + \tfrac{2\bar{v}''}{\bar{v}'} \delta v'' - \delta v''' \right) . \tag{2.43}$$

Using these expressions, one finds that the symplectic form is given by

$$\Omega^{(2)}_{\text{EdS}-\text{G}} = 2c_1 \int \frac{dt}{\bar{v}'} \Big[2\bar{v}' d\delta v' \wedge d\delta s + d\delta v' \wedge d\delta s' + d\delta s'' \wedge d\delta v - \tfrac{\bar{s}''}{\bar{v}'} d\delta v' \wedge d\delta v \Big] + 2c_0 \int \frac{dt}{\bar{v}'^2} \Big[\bar{v}'^2 d\delta v' \wedge d\delta v - d\delta v''' \wedge d\delta v + d\delta v'' \wedge d\delta v' \Big] . \tag{2.44}$$

We are primarily interested in evaluating these expressions on the saddle (2.38), namely the quadratic action (2.39) and the symplectic form (2.44). Substituting the saddle (2.38) into these expressions, we obtain

$$I^{(0)}_{\text{EdS}-\text{G}} = -\frac{4\pi^2 c_0}{\beta} , \tag{2.45}$$

$$I^{(2)}_{\text{EdS}-\text{G}} = 2c_1 \int dt \Big[\delta s' \delta v' + i \tfrac{\beta}{2\pi} \delta v' \delta s'' \Big] + c_0 \int dt \Big[\delta v'^2 - \left(\tfrac{\beta}{2\pi} \right)^2 \delta v''^2 \Big] , \tag{2.46}$$

$$\Omega^{(2)}_{\text{EdS}-\text{G}} = 4c_1 \int dt \Big[d\delta v' \wedge d\delta s + i \tfrac{\beta}{2\pi} d\delta s'' \wedge d\delta v \Big] \quad 2c_0 \int dt \Big[d\delta v' \wedge d\delta v + \left(\tfrac{\beta}{2\pi} \right)^2 d\delta v''' \wedge d\delta v \Big] , \tag{2.47}$$

where $I^{(1)}_{\text{EdS}-\text{G}} = 0$ by the equations of motion.

2.4 Evaluating the one-loop path integral

In this section, we evaluate the one-loop contribution to the boundary path integral for the Galilean de Sitter gravity theory. We will do this in the boundary description. *Exactly* equating a gravitational path integral to the path integral of a quantum mechanical system in one dimension – its boundary description – is highly non-trivial, but well-studied in JT gravity [44]. This can be understood in the following way: since the dilaton in JT gravity acts as a Lagrange multiplier, one can perform integration over the dilaton at the level of the path integral, which fixes the curvature of the geometry and only leaves boundary fluctuations. We will return later to the bulk formulation of non-relativistic JT gravity in section 3, where we will argue that an analogous mechanism is at work for Galilean de Sitter gravity, which can be cast into a JT-like action formulation with a dilaton that again acts as a Lagrange multiplier and may be integrated out. The relevant path integral we want to solve becomes

$$\mathcal{Z}_{\text{EdS-G}} = \int_{\mathcal{M}} [\mathcal{D}s]\,[\mathcal{D}v]\, e^{-I_{\text{EdS-G}}[s,v]}\,. \tag{2.48}$$

Given the boundary equations we impose, the space $\mathcal{M}$ we integrate over consists of all configurations of s and v subject to the boundary conditions (2.33), and with EdS-G symmetry modded out. Plugging in the fluctuations, this yields

$$\mathcal{Z}_{\text{EdS-G}} = e^{-I^{(0)}_{\text{EdS-G}}[\bar{s},\bar{v}]} \int_{\mathcal{M}} [\mathcal{D}\delta s]\,[\mathcal{D}\delta v]\, e^{-I^{(2)}_{\text{EdS-G}}[\delta s,\delta v]}\,. \tag{2.49}$$

As we have seen, the theory (2.19) possesses a set of global symmetries generated by the EdS-G algebra (2.2). These will play a crucial role in evaluating the path integral, as they give rise to zero modes. The evaluation of the path integral proceeds in three steps. First, we expand the fluctuations in Fourier modes, which diagonalise the quadratic action. Second, we evaluate the resulting Gaussian integrals for the non-zero modes. Third, we treat the zero modes separately, as they are associated with the global symmetries identified in Section 2.2.

To implement the first step, we expand the fluctuations in Fourier modes as follows

$$\delta v(t) = \sum_{n\in\mathbb{Z}} \left(v_n^{(R)} + i v_n^{(I)}\right) e^{\frac{2\pi}{\beta} int}\,, \qquad \delta s(t) = \sum_{n\in\mathbb{Z}} \left(s_n^{(R)} + i s_n^{(I)}\right) e^{\frac{2\pi}{\beta} int}\,, \tag{2.50}$$

where we will require the real (R) and imaginary parts (I) to satisfy

$$w_n = w_n^{(R)} + i w_n^{(I)}\,, \quad w_{-n}^{(R)} = w_n^{(R)}\,, \quad w_{-n}^{(I)} = -w_n^{(I)}\,, \tag{2.51}$$

and w is a placeholder for s and v. Plugging the Fourier expansion into the second-order on-shell action (2.46) and symplectic form (2.47) yields

$$I^{(2)}_{\text{EdS}-\text{G}} = -\frac{4\pi^2}{\beta} \sum_{n\in\mathbb{Z}} \left(2c_1 n^2(n-1)\delta s_n \delta v_{-n} + c_0 n^2(n^2-1)\delta v_n \delta v_{-n}\right), \tag{2.52}$$

$$\Omega^{(2)}_{\text{EdS}-\text{G}} = 4\pi i \sum_{n\in\mathbb{Z}} \left(2c_1 n(n-1) d\delta v_n \wedge d\delta s_{-n} - c_0 n(n^2-1) d\delta v_n \wedge d\delta v_{-n}\right). \tag{2.53}$$

We have used $\int_0^\beta dt e^{i(m-n)t} = \beta\,\delta_{mn}$. Using the real and imaginary parts of the Fourier coefficients, we may rewrite the expressions entirely in terms of positive modes, in which case we find

$$
\begin{aligned}
I^{(2)}_{\text{EdS}-\text{G}} = \frac{16\pi^2}{\beta} c_1 \delta s_{-1} \delta v_1 - \frac{8\pi^2}{\beta} c_0 \sum_{n\geq 2} n^2(n^2-1)\Big[\delta v_n^{(R)} \delta v_n^{(R)} + \delta v_n^{(I)} \delta v_n^{(I)}\Big] \\
- \frac{16\pi^2}{\beta} c_1 \sum_{n\geq 2} n^2 \Big[\delta s_n^{(R)} \delta v_n^{(R)} + \delta s_n^{(I)} \delta v_n^{(I)}\Big] \\
- \frac{16\pi^2 i}{\beta} c_1 \sum_{n\geq 2} n^3 \Big[\delta s_n^{(R)} \delta v_n^{(I)} - \delta s_n^{(I)} \delta v_n^{(R)}\Big] .
\end{aligned}
\tag{2.54}
$$

For the symplectic form, we get that

$$
\begin{aligned}
\Omega^{(2)}_{\text{EdS}-\text{G}} = 16\pi i c_1 d\delta v_{-1} \wedge d\delta s_1 + 16\pi i c_1 \sum_{n\geq 2} n^2 \Big[d\delta v_n^{(R)} \wedge d\delta s_n^{(R)} + d\delta v_n^{(I)} \wedge d\delta s_n^{(I)}\Big] \\
- 16\pi c_1 \sum_{n\geq 2} n \Big[d\delta v_n^{(I)} \wedge d\delta s_n^{(R)} - d\delta v_n^{(R)} \wedge d\delta s_n^{(I)}\Big] \\
+ 32\pi c_0 \sum_{n\geq 2} n(n^2-1) d\delta v_n^{(I)} \wedge d\delta v_n^{(R)} .
\end{aligned}
\tag{2.55}
$$

We have explicitly separated the $n = 1$ mode, as it requires special treatment and is associated with the global symmetries identified in Section 2.2. For $n \geq 2$, the quadratic fluctuation integral is Gaussian and can therefore be evaluated using

$$
\int_{\mathbb{R}^4} e^{-\frac{1}{2} x^T A_n x} = \sqrt{\frac{(2\pi)^4}{\det A_n}} . \tag{2.56}
$$

For each mode $n \geq 2$, the quadratic action may be written in the form $\frac{1}{2} x^T A_n x$, where the fluctuation vector is $x = \big(\delta v_n^{(R)}, \delta v_n^{(I)}, \delta s_n^{(R)}, \delta s_n^{(I)}\big)$. For $n \geq 2$, the matrix A_n takes the form

$$
A_n = \frac{16\pi^2 n^2}{\beta} \begin{pmatrix} c_0(n^2-1) & 0 & 2c_1 & 2ic_1 n \\ 0 & c_0(n^2-1) & -2ic_1 n & 2c_1 \\ 2c_1 & 2ic_1 n & 0 & 0 \\ -2ic_1 n & 2c_1 & 0 & 0 \end{pmatrix} . \tag{2.57}
$$

The corresponding symplectic form may likewise for $n \geq 2$ be represented by an 4×4 antisymmetric matrix Ω_n

$$
\Omega_n = 16\pi i \begin{pmatrix} 0 & -2ic_0 n(n^2-1) & c_1 n^2 & -c_1 in \\ 2ic_0 n(n^2-1) & 0 & c_1 in & c_1 n^2 \\ -c_1 n^2 & -c_1 in & 0 & 0 \\ c_1 in & -c_1 n^2 & 0 & 0 \end{pmatrix} . \tag{2.58}
$$

From these expressions, we obtain

$$
\sqrt{\frac{(2\pi)^4}{\det A_n}} = \frac{\beta^2}{256\pi^2 c_1^2 n^4 (n^2-1)} , \qquad \text{Pf}(\Omega_n) = 256\pi^2 c_1^2 n^2 (n^2-1) . \tag{2.59}
$$

Hence we find

$$Z(\beta) = e^{\frac{4\pi^2 c_0}{\beta}} \mathcal{B}(\beta) \prod_{n\geq 2} \mathrm{Pf}(\Omega_n) \sqrt{\frac{(2\pi)^4}{\det A_n}} = e^{\frac{4\pi^2 c_0}{\beta}} \mathcal{B}(\beta) \prod_{n\geq 2} \frac{\beta^2}{n^2} \,. \tag{2.60}$$

Before detailing $\mathcal{B}(\beta)$, we evaluate the infinite product using zeta-function regularisation. This relies on the analytic continuation of the Riemann zeta function,

$$\zeta(s) = \sum_{n\geq 1} n^{-s}, \qquad \zeta'(s) = -\sum_{n\geq 1} n^{-s} \log n \,. \tag{2.61}$$

Applying this to the product in (2.60), we obtain

$$\prod_{n\geq 2} \frac{\beta^2}{n^2} = \exp\left(\sum_{n\geq 2} (2\log\beta - 2\log n)\right) = \exp\Big(2\log(\beta)\,(\zeta(0)-1) + 2\zeta'(0)\Big) = \frac{1}{2\pi\beta^3}\,, \tag{2.62}$$

where we used the regularized values $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$. The non-Gaussian contribution from the $n = 1$ modes is captured by $\mathcal{B}(\beta)$, which we compute as follows

$$\begin{aligned}\mathcal{B}(\beta) = 16\pi i\, c_1 \Bigg(&\iint d\delta v_1^{(R)} d\delta s_1^{(R)} + \iint d\delta v_1^{(I)} d\delta s_1^{(I)} + i \iint d\delta v_1^{(R)} d\delta s_1^{(I)} - i \iint d\delta v_1^{(I)} d\delta s_1^{(R)} \Bigg) \\ &\times \exp\left(\frac{16\pi^2}{\beta} c_1 \Big(\delta s_1^{(R)} \delta v_1^{(R)} + \delta s_1^{(I)} \delta v_1^{(I)} - i\Big(\delta s_1^{(I)} \delta v_1^{(R)} - \delta s_1^{(R)} \delta v_1^{(I)}\Big)\Big)\right) = 4\beta\,, \end{aligned}\tag{2.63}$$

where in order to evaluate the preceding integral we introduced a regulator $-i\epsilon|v_1|^2$, with $\mathrm{Im}(\epsilon) < 0$ to ensure convergence. The integral is first computed at finite ϵ, after which the $\epsilon \to 0$ limit is taken, noting that the final result is regulator-independent.

Collecting all contributions, we obtain the final expression for the partition function

$$\mathcal{Z}_{\text{EdS-G}}(\beta) = \frac{2}{\pi\beta^2} \exp\left(\frac{4\pi^2 c_0}{\beta}\right). \tag{2.64}$$

This corresponds to the main result of this section. The overall scaling $\mathcal{Z}_{\text{EdS-G}}(\beta) \sim \beta^{-2}$ is consistent with the general expectation that each global symmetry generator contributes a factor of $\beta^{-1/2}$. Since the EdS-G algebra possesses four generators, this results in the observed β^{-2} behaviour. This is a characteristic feature of Schwarzian-type theories, in which exponential growth arises from the classical saddle, and the quantum fluctuations provide a temperature-dependent prefactor determined by the number of global symmetry generators.

Performing an inverse Laplace transform, we obtain the density of states (dos) as a function of the energy E,

$$\mathcal{D}_{\text{dos}}(E) = \frac{\sqrt{E}}{\pi^2\sqrt{c_0}}\, I_1\Big(4\pi\sqrt{c_0 E}\Big)\,, \tag{2.65}$$

where I_1 is the modified Bessel function of the first kind. In particular, for small energies $E \to 0$, this behaves as

$$\mathcal{D}_{\rm dos}(E) = \frac{2}{\pi} E + \mathcal{O}(E^2)\,, \tag{2.66}$$

which shows that the density of states vanishes linearly at low energies. The linear behaviour at low energies also reflects how quantum fluctuations resolve the naive semiclassical gap in the spectrum and indicate the presence of low-energy states in the quantum theory.

3 A bulk Newton-Cartan description of Galilean de Sitter

In the previous section, we developed a boundary description of non-relativistic de Sitter space and evaluated its one-loop contribution. The goal of this section is to complement that analysis by constructing the corresponding bulk description and elucidating its geometric structure. A particular novelty is that we work out a non-relativistic equivalent of JT gravity and show that the geometry corresponding to EdS-G satisfies its equations of motion. We begin in section 3.1 by deriving the two-dimensional Newton–Cartan geometry associated with the bulk gauge connection. In section 3.2 we show that this geometry realises the EdS-G algebra (2.2) as its isometry algebra, and in section 3.3 we provide a relativistic uplift to three dimensions. We then turn in section 3.4 to a complementary second-order description, showing that the same Newton–Cartan geometry arises from a non-relativistic JT-type action. Finally, in section 3.5 we compare this second-order formulation with the underlying first-order gauge-theoretic description. Taken together, these results establish a bulk description whose geometric and symmetry structures precisely match those of the boundary dynamics studied in section 2.

3.1 Bulk Newton–Cartan geometry from the gauge connection

To construct a bulk description of Galilean de Sitter gravity, we work in a two-dimensional Newton–Cartan framework. Newton–Cartan geometry, originally introduced by Élie Cartan [45, 46], provides a geometric formulation of non-relativistic gravity analogous to the role played by Riemannian geometry in general relativity. In particular, it is the natural geometric language for theories with Galilean symmetry. We refer to [47] for an excellent modern review of non-relativistic gravity, including its history, formulation, and recent developments.

In two dimensions, the Newton–Cartan data consist of a temporal one-form τ_μ, a spatial vielbein e_μ together with its inverse e^μ, and a $U(1)$ mass gauge field m_μ. More generally, Newton–Cartan geometry may also be torsionful, depending on whether the clock form τ_μ is closed. Such torsionful geometries play an important role in more general non-relativistic settings and can account for effects such as non-relativistic gravitational lensing [48].

Our starting point is the boundary gauge field derived in Section 2. In the gauge (2.13), it takes the form

$$a = \mathcal{H} + \mathcal{L}(t)\mathcal{K} + \mathcal{T}(t)\mathcal{D}\,. \tag{3.1}$$

To reinterpret this in Newton–Cartan variables, we pass from the basis $\{\mathcal{H}, \mathcal{D}, \mathcal{K}, \mathcal{Z}\}$ to the basis adapted to the EdS-G algebra,

$$H = \frac{1}{\ell}\mathcal{D}\,, \qquad P = \frac{1}{2\ell}(\mathcal{H} + \mathcal{K})\,, \qquad G = \frac{1}{2}(\mathcal{H} - \mathcal{K})\,, \qquad M = \frac{1}{\ell}\mathcal{Z}\,. \tag{3.2}$$

We then introduce the radial group element

$$b(r) = e^{-r\mathcal{H}}\,, \tag{3.3}$$

and construct the bulk connection in radial gauge according to the permissible gauge transformation

$$A(t,r) = b^{-1}(r)db(r) + b^{-1}(r)a(t)b(r)\,. \tag{3.4}$$

Besides that, expanding the one-form A in the extended Galilean basis (2.19) as

$$A(t,r) = \tau H + eP + \omega G + mM\,, \tag{3.5}$$

allows us to read off the Newton–Cartan fields directly from the coefficients of the generators, obtaining

$$\tau_\mu dx^\mu = \ell\,\mathcal{T}(t)\,dt\,, \tag{3.6}$$

$$e_\mu dx^\mu = \ell\Big(1 + \mathcal{L}(t) + r\mathcal{T}(t)\Big)dt - \ell\,dr\,, \tag{3.7}$$

$$m_\mu dx^\mu = 2\ell r\mathcal{L}(t)\,dt\,, \tag{3.8}$$

$$\omega_\mu dx^\mu = \Big(1 - \mathcal{L}(t) + r\mathcal{T}(t)\Big)dt - dr\,. \tag{3.9}$$

These gauge fields define the two-dimensional Newton–Cartan geometry associated with the bulk gauge connection $A(t,r)$. It is convenient to introduce the inverse temporal vector v^μ and the inverse spatial vielbein e^μ, which together with τ_μ and e_μ satisfy the following orthogonality and completeness relations of Newton–Cartan geometry

$$v^\mu\tau_\mu = 1\,, \qquad \tau_\mu e^\mu = 0\,, \qquad e_\mu v^\mu = 0\,, \qquad e_\mu e^\mu = 1\,, \qquad \delta^\mu{}_\nu = v^\mu\tau_\nu + e^\mu e_\nu\,, \tag{3.10}$$

which we can solve for the contravariant objects, yielding

$$v^\mu\partial_\mu = \frac{1}{\ell\mathcal{T}(t)}\Big(\partial_t + (1 + \mathcal{L}(t) + r\mathcal{T}(t))\partial_r\Big)\,, \qquad e^\mu\partial_\mu = -\frac{1}{\ell}\,\partial_r\,. \tag{3.11}$$

They allow us to construct the degenerate spatial metric and its inverse,

$$h_{\mu\nu} = e_\mu e_\nu\,, \qquad h^{\mu\nu} = e^\mu e^\nu\,. \tag{3.12}$$

For the general connection above, the spatial metric becomes

$$h_{\mu\nu}dx^\mu dx^\nu = \ell^2\Big(1 + \mathcal{L}(t) + r\mathcal{T}(t)\Big)^2 dt^2 - 2\ell^2\Big(1 + \mathcal{L}(t) + r\mathcal{T}(t)\Big)dt\,dr + \ell^2 dr^2\,, \tag{3.13}$$

and $h^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{\ell^2}\partial_r^2$ which are required to satisfy the following completeness relation

$$h^{\mu\nu}\tau_\nu = 0\,, \qquad h_{\mu\nu}v^\nu = 0\,, \qquad h^{\mu\rho}h_{\rho\nu} = \delta^\mu{}_\nu - v^\mu\tau_\nu\,. \tag{3.14}$$

This completes the Newton–Cartan data associated with the bulk gauge connection. In the next subsection, we show that this geometry realises the EdS-G algebra as its isometry algebra.

3.2 Isometries of the Newton–Cartan geometry

In this subsection we restrict the geometry to constant values of $\mathcal{L}(t) = \mathcal{L}_0$ and $\mathcal{T}(t) = \mathcal{T}_0$, for simplicity. The Newton–Cartan geometry we consider, therefore, takes the form

$$\tau_\mu dx^\mu = \ell \mathcal{T}_0 \, dt \,, \tag{3.15}$$

$$e_\mu dx^\mu = \ell \Big(1 + \mathcal{L}_0 + r\mathcal{T}_0 \Big) dt - \ell \, dr \,, \tag{3.16}$$

$$m_\mu dx^\mu = 2\ell r \mathcal{L}_0 \, dt \,. \tag{3.17}$$

What makes the above truly a Newton–Cartan geometry is the fact that, in addition to the Galilean boost-invariant quantities $h^{\mu\nu}$ and τ_μ, one can construct the following boost-invariant combinations [49, 50]

$$\hat{v}^\mu = v^\mu - h^{\mu\nu} m_\nu \,, \tag{3.18}$$

$$\hat{h}_{\mu\nu} = h_{\mu\nu} - \tau_\mu m_\nu - \tau_\nu m_\mu \,. \tag{3.19}$$

The presence of the gauge field m_μ captures the Bargmann symmetry of this two-dimensional model. Let us emphasise this point clearly by examining the isometries. Given the Killing vector ξ, the objects $\{h^{\mu\nu}, \tau_\mu\}$ transform as usual spacetime tensors by a Lie derivative, namely

$$\mathcal{L}_\xi h^{\mu\nu} = 0 \,, \qquad \mathcal{L}_\xi \tau_\mu = 0 \,. \tag{3.20}$$

Notice that these objects of the Newton-Cartan geometry are Galilean boost-invariant, so their transformation laws do not exhibit any gauge parameter. For the rest of the tensors built upon from these, it is possible to group both diffeomorphisms ξ and gauge symmetries λ in one symmetry transformation $\delta_\epsilon \equiv \mathcal{L}_\xi + \delta_\lambda$, generating the following symmetry transformations for the remaining tensors

$$\mathcal{L}_\xi v^\mu = h^{\mu\nu} \lambda_\nu \,, \qquad \mathcal{L}_\xi h_{\mu\nu} = 2\lambda_{(\mu} \tau_{\nu)} \qquad \mathcal{L}_\xi m_\mu = \partial_\mu \sigma + \lambda_\mu \,, \tag{3.21}$$

where σ parametrises the $U(1)$ gauge parameter of m_μ while λ_μ denotes the Galilei boost parameters. Notice that we used the completeness relations (3.10) to derive these expressions and they also show that the combinations introduced in (3.18) are indeed boost invariant.

Let us now turn to working out the symmetries. Given the Newton-Cartan geometry in (3.15), we insert the ansatz

$$\xi = \xi^t(t,r)\partial_t + \xi^r(t,r)\partial_r \,, \qquad \lambda_\mu = \lambda_\mu(t,r) \,, \qquad \sigma = \sigma(t,r) \,, \tag{3.22}$$

into (3.20)–(3.21) and find a coupled system of partial differential equations. The first two constraints, $\mathcal{L}_\xi h^{\mu\nu} = 0$ and $\mathcal{L}_\xi \tau_\mu = 0$ from (3.20) together imply that

$$\partial_r \xi^t = 0 \,, \qquad \partial_r \xi^r = 0 \,, \qquad \partial_t \xi^t = 0 \,. \tag{3.23}$$

so these already imply that $\xi^t = \text{const}$ and $\xi^r = \xi^r(t)$, while the remaining conditions in (3.21) determine the gauge parameters λ_μ and σ in terms of ξ^μ. Solving the full system leads to

$$
\begin{aligned}
\xi &= \frac{b_0}{\mathcal{T}_0}\partial_t + \Big(b_1\cosh(\mathcal{T}_0 t) + b_2\sinh(\mathcal{T}_0 t)\Big)\,\partial_r\,,\\
\lambda &= \ell(b_1-b_2)(\cosh(\mathcal{T}_0 t)-\sinh(\mathcal{T}_0 t))\Big((\mathcal{T}_0 r+\mathcal{L}_0+1)\partial_t - \partial_r\Big)\,,\\
\sigma &= b_3 + \frac{\ell}{\mathcal{T}_0}\Big((b_1+b_2)\mathcal{L}_0 e^{\mathcal{T}_0 t} + (b_1-b_2)(\mathcal{T}_0 r+1)e^{-\mathcal{T}_0 t}\Big)\,,
\end{aligned}
\tag{3.24}
$$

where b_0, b_1, b_2, b_3 are integration constants. To define the extended algebra we take the tuples (ξ_i, σ_i) for $i = 0, 1, 2, 3$ where for each i we set all $b_j = 0$ except b_i. Now, we define the commutator by

$$
[(\xi_i,\sigma_i),(\xi_j,\sigma_j)] := \Big([\xi_i,\xi_j], \xi_i(\sigma_j) - \xi_j(\sigma_i)\Big)\,. \tag{3.25}
$$

This bracket arises from combining diffeomorphisms with $U(1)$ gauge transformations and corresponds to the semi-direct product structure of spacetime and the Bargmann symmetries. Thus, we find (with some suitable rescaling of the constants b_i) that

$$
\begin{aligned}
H &= (\xi_0,\sigma_0) = \left(\frac{1}{\ell\mathcal{T}_0}\partial_t, 0\right)\,,\\
G &= (\xi_1,\sigma_1) = \left(-2\ell\cosh(\mathcal{T}_0 t)\partial_r,\; -\frac{2\ell^2}{\mathcal{T}_0}\Big[\mathcal{L}_0 e^{\mathcal{T}_0 t} + (1+r\mathcal{T}_0)e^{-\mathcal{T}_0 t}\Big]\right)\,,\\
P &= (\xi_2,\sigma_2) = \left(2\sinh(\mathcal{T}_0 t)\partial_r,\; \frac{2\ell}{\mathcal{T}_0}\Big[\mathcal{L}_0 e^{\mathcal{T}_0 t} - (1+r\mathcal{T}_0)e^{-\mathcal{T}_0 t}\Big]\right)\,,\\
M &= (\xi_3,\sigma_3) = (0, 4\ell^2)\,.
\end{aligned}
\tag{3.26}
$$

Applying the definition in (3.25) we see that the generators satisfy the EdS-G algebra given in (2.2) when identifying $\tilde{\Lambda} = 1/\ell^2$.

3.3 Relativistic uplift

A standard way to relate Newton–Cartan geometry to a higher-dimensional relativistic spacetime is through a null uplift [47, 51]. This provides a useful dictionary between the two-dimensional Newton–Cartan data and higher-dimensional Lorentzian quantities. Concretely, the uplift is defined by the metric ansatz

$$
ds^2 = h_{\mu\nu}dx^\mu dx^\nu + 2\tau_\mu dx^\mu(du - m_\nu dx^\nu)\,, \tag{3.27}
$$

where u is a newly introduced null coordinate. Restricting the Newton–Cartan geometry in (3.15) to the constant sector $\mathcal{L}(t) = \mathcal{L}_0$ and $\mathcal{T}(t) = \mathcal{T}_0$, a direct computation shows that the uplifted three-dimensional metric is Ricci flat and takes the form

$$
ds^2 = \ell^2\Big[(1+\mathcal{L}_0+r\mathcal{T}_0)^2 - 4r\mathcal{L}_0\mathcal{T}_0\Big]dt^2 - 2\ell^2(1+\mathcal{L}_0+r\mathcal{T}_0)\,dt\,dr + \ell^2 dr^2 + 2\ell\mathcal{T}_0\,dt\,du\,. \tag{3.28}
$$

It is straightforward to verify that ∂_u is null with respect to (3.28) and that all metric components are independent of u. The uplift therefore takes the standard Bargmann form associated with null uplifts of Newton–Cartan geometry [47, 51]. Considering the three-dimensional Lorentzian geometry, we find the following Killing vectors

$$
\begin{aligned}
H &= \frac{1}{\ell \mathcal{T}_0}\,\partial_t\,, \\
G &= -2\ell\cosh(\mathcal{T}_0 t)\,\partial_r - \frac{2\ell^2}{\mathcal{T}_0}\Big(\mathcal{L}_0 e^{\mathcal{T}_0 t} + (1+r\mathcal{T}_0)e^{-\mathcal{T}_0 t}\Big)\partial_u\,, \\
P &= 2\ell\sinh(\mathcal{T}_0 t)\,\partial_r + \frac{2\ell^2}{\mathcal{T}_0}\Big(\mathcal{L}_0 e^{\mathcal{T}_0 t} - (1+r\mathcal{T}_0)e^{-\mathcal{T}_0 t}\Big)\partial_u\,, \\
M &= 4\ell^2\,\partial_u\,,
\end{aligned}
\tag{3.29}
$$

which we again recognise as the EdS-G generators from equation (2.2). Comparing (3.29) with the two-dimensional Newton–Cartan generators in (3.26), one sees that the higher-dimensional isometries decompose into ordinary spacetime diffeomorphisms on (t, r) together with translations along the null direction u. From the lower-dimensional viewpoint, the latter are precisely the $U(1)$ gauge transformations of the mass gauge field m_μ.

3.4 Non-relativistic Jackiw-Teitelboim gravity from an action principle

Having identified the bulk Newton–Cartan geometry associated with the gauge connection, we now ask whether the same structure can be obtained from a second-order action principle. In this subsection, we show that this is indeed the case: the geometry constructed above arises as a solution of a non-relativistic JT-type model formulated directly in Newton–Cartan variables. This provides a complementary second-order description of the bulk theory and sets the stage for the comparison with the first-order gauge-theoretic formulation in Section 3.5. The corresponding action, proposed in [30], takes the form[2]

$$
I_{\mathrm{NR-JT}} = \kappa \int_{\mathcal{M}} d^2x\, E\,\phi\Big(R^{\mathrm{NR}} - \frac{2}{\ell^2}\Big)\,. \tag{3.31}
$$

where the measure form is $E = -\epsilon^{\mu\nu}\tau_\mu e_\nu$ and the non-relativistic Ricci scalar and spin connection are defined as

$$
R^{\mathrm{NR}} = 4v^{[\mu}e^{\nu]}\partial_\mu\omega_\nu\,, \qquad \omega_\nu = 2v^{[\alpha}e^{\beta]}(\partial_\alpha(e_\beta)e_\nu - \partial_\alpha(m_\beta)\tau_\nu)\,. \tag{3.32}
$$

The action contains four dynamical fields $\{\tau_\mu, e_\mu, m_\mu, \phi\}$ with the spin-connection ω_μ depending on them and therefore not being a dynamical field in this formalism. The expression of the spin-connection comes from solving the field equations in the first-order formalism algebraically. Moreover, this NR-JT action (3.31) can be understood as arising from a non-relativistic $\frac{1}{c^2}$ expansion of relativistic JT gravity, in close analogy with the non-relativistic expansion of general relativity studied in [14]. In the present formulation,

[2] An equivalent form of the action, which is simply a rewriting of (3.31) in terms of one-form variables, is

$$
I[\tau, e, m, \phi] = \kappa \int d^2x\,\epsilon^{\mu\nu}\phi\Big(\partial_\mu\omega_\nu - \frac{1}{\ell^2}\tau_\mu e_\nu\Big)\,. \tag{3.30}
$$

The equations of motion for this equivalent formulation are presented in appendix D.

the expression for $R^{\rm NR}$ should be viewed as a convenient scalar capturing the geometric structure of the theory, rather than as the Ricci scalar constructed from a standard non-relativistic connection. In particular, we do not attempt to express $R^{\rm NR}$ in terms of a curvature scalar constructed from a non-relativistic connection such as the torsionless Newton–Cartan connection compatible with τ_μ and $h^{\mu\nu}$ (see e.g. [47]). We leave this for future work.

Let us also remark that the action is invariant under a Galilean-de Sitter boost given

$$\delta_{\rm EdS-G}\phi = 0\,, \quad \delta_{\rm EdS-G}\tau_\mu = 0\,, \quad \delta_{\rm EdS-G}e_\mu = \lambda_G\tau_\mu\,, \quad \delta_{\rm EdS-G}m_\mu = \lambda_G e_\mu\,, \tag{3.33}$$

with $\lambda_G = \lambda_G(t,r)$ the Galilean gauge parameter boost. We used the condition $d\tau = 0$ to prove this invariance.

Varying the action with respect to ϕ, τ_μ, e_μ, and m_μ yields the equations of motion

$$\delta\phi: \qquad R^{\rm NR} - \frac{2}{\ell^2} = 0\,, \tag{3.34}$$

$$\delta m_\mu: \qquad \partial_\alpha\Big(\tau^{[\alpha}e^{\mu]}\tau_\nu\partial_\beta(E\,\phi\,\tau^{[\beta}e^{\nu]})\Big) = 0\,, \tag{3.35}$$

$$\delta\tau_\mu: \qquad E\,\phi R^{\rm NR}\tau^\mu = \partial_\rho\Big(4E\phi\tau^{[\rho}e^{\nu]}\Big)\omega_\nu\tau^\mu + \tau^{[\alpha}e^{\beta]}\partial_\alpha(m_\beta)\partial_\nu\Big(8E\phi\tau^{[\nu}e^{\mu]}\Big)\,, \tag{3.36}$$

$$\begin{aligned}\delta e_\mu: \qquad E\phi R^{\rm NR}e^\mu = {}& \partial_\rho\Big(4E\phi\tau^{[\rho}e^{\nu]}\Big)\omega_\nu e^\mu + \partial_\alpha\Big(\tau^{[\alpha}e^{\mu]}e_\nu\partial_\beta\Big(8E\phi\tau^{[\beta}e^{\nu]}\Big)\Big) \\ & + \partial_\nu\Big(8E\phi\tau^{[\mu}e^{\nu]}\Big)\tau^{[\alpha}e^{\beta]}\partial_\alpha(e_\beta)\,,\end{aligned} \tag{3.37}$$

where in order to derive the equations of motion we made use of the orthogonality and completeness relations of the vielbein given in (3.10). In particular, starting from $\tau_\mu\tau^\nu + e_\mu e^\nu = \delta^\nu_\mu$ we find after contracting with τ^μ and e^μ, the following identities for the variations [52]

$$\delta v^\nu = -v^\mu v^\nu\delta\tau_\mu - e^\nu v^\mu\delta e_\mu\,, \tag{3.38}$$

$$\delta e^\nu = -v^\nu e^\mu\delta\tau_\mu - e^\mu e^\nu\delta e_\mu\,, \tag{3.39}$$

along with the variation of the spin current, given by

$$\delta\omega_\nu = -\omega_\nu\tau^\rho\delta\tau_\rho + \omega_\nu e_\rho\delta e^\rho + 2\tau^{[\alpha}e^{\beta]}(\partial_\alpha(\delta e_\beta)e_\nu + \partial_\alpha(e_\beta)\delta e_\nu - \partial_\alpha(m_\beta)\delta\tau_\nu - \partial_\alpha(\delta m_\beta)\tau_\nu)\,. \tag{3.40}$$

It is instructive to verify that the Newton–Cartan geometry in (3.15), obtained previously from the gauge connection, also solves the equations of motion above, precisely when constraining to the saddle point values we found in (2.38).

3.5 The relationship between the first and second order formulations

Having shown that the Newton–Cartan geometry introduced above arises from a second-order non-relativistic JT-type action, we now compare this description with the first-order gauge theory formulation underlying the boundary construction. Our goal is to show how the bulk fields appearing in the Newton–Cartan description are related to the variables of the first-order formalism, and to clarify in what sense the second-order equations reproduce only part of the dynamics of the full first-order theory.

The BF action principle for Galilean de Sitter gravity in the first-order formalism is [30]

$$I_{\rm BF} = c_1 \int_{\mathcal{M}} \Big[\phi R(G) + \frac{1}{\ell^2}\Big(\eta R(M) + \zeta R(H) - \rho R(P)\Big)\Big] + \frac{c_0}{\ell^2}\int_{\mathcal{M}} \eta R(H)\,. \tag{3.41}$$

Here c_0 and c_1 are the two arbitrary constants parametrising the invariant bilinear form in (2.5), expressed in the Galilean basis and the two-form curvatures are given by

$$R(H) = d\tau\,, \qquad R(P) = de + \omega\tau\,, \qquad R(G) = d\omega - \frac{1}{\ell^2}\tau e\,, \qquad R(M) = dm + \omega e\,. \tag{3.42}$$

The fields $\{\eta,\zeta,\rho,\phi\}$ are scalars, while $\{\tau,e,\omega,m\}$ are one-forms. In this formalism, ω_μ is still a dynamical field. A general variation of (3.41) produces the boundary term

$$\Theta = c_1 \int_{\partial\mathcal{M}} \Big(\phi\delta\omega + \frac{1}{\ell^2}\Big(\eta\delta m + \zeta\delta\tau - \rho\delta e\Big)\Big) + \frac{c_0}{\ell^2}\int_{\partial\mathcal{M}} \eta\delta\tau\,. \tag{3.43}$$

To obtain a well-defined variational principle, we impose boundary conditions at the asymptotic de Sitter boundary, such that the scalar fields are proportional to the temporal components of the gauge fields,

$$\eta = \tau_t\,, \qquad \rho = e_t\,, \qquad \phi = \omega_t\,, \qquad \zeta = m_t\,, \tag{3.44}$$

We conclude that the action must then in addition to (3.41) be supplemented by the boundary term

$$I_\partial = -\frac{c_1}{2}\int dt\Big(\omega_t^2 + \frac{1}{\ell^2}(2\tau_t m_t - e_t^2)\Big) - \frac{c_0}{2\ell^2}\int dt\,\tau_t^2\,. \tag{3.45}$$

Now that the variational principle is well defined, variation of the action (3.41) with respect to $\{\eta,\rho,\phi,\zeta\}$ imposes the flatness constraints

$$R(H) = 0\,, \qquad R(P) = 0\,, \qquad R(G) = 0\,, \qquad R(M) = 0\,, \tag{3.46}$$

for the curvatures defined in (3.42). One may solve algebraically these flatness constraints in (3.46) to determine the spin connection ω in terms of the other fields m,τ and e finding

$$\omega_\mu = \frac{1}{E}\,\epsilon^{\alpha\beta}\big(\partial_\alpha(m_\beta)\,\tau_\mu - \partial_\alpha(e_\beta)\,e_\mu\big)\,, \tag{3.47}$$

where $E = -\epsilon^{\mu\nu}\tau_\mu e_\nu$. Note that this was precisely the form we used to define the second-order formulation in (3.32). The variation with respect to $\{\tau,e,\omega,m\}$ gives

$$d\eta = 0\,, \quad d\rho + \omega\eta - \tau\phi = 0\,, \quad d\phi - \frac{1}{\ell^2}(\tau\rho - e\eta) = 0\,, \quad d\zeta + \omega\rho - e\phi = 0\,. \tag{3.48}$$

Solving the curvature constraints (3.42) subject to the boundary conditions (3.44), we obtain

$$\tau = \ell\mathcal{T}(t)\,dt\,, \tag{3.49a}$$

$$e = \ell\big(-dr + (r\mathcal{T}(t) + 1 + \mathcal{L}(t))\,dt\big)\,, \tag{3.49b}$$

$$\omega = -dr + \big(r\mathcal{T}(t) + 1 - \mathcal{L}(t)\big)\,dt\,, \tag{3.49c}$$

$$m = 2\ell r\mathcal{L}(t)dt\,. \tag{3.49d}$$

where $\mathcal{T}(t)$ and $\mathcal{L}(t)$ are for now arbitrary functions. Substituting the temporal components of the gauge fields in (3.49) into the boundary term (3.45), and using the expressions for $\mathcal{T}$ and $\mathcal{L}$ in (2.14), we recover the Schwarzian-like action (2.12). Furthermore, the scalar equations imply additional constraints on the auxiliary fields. From $d\eta = 0$ we obtain $\eta = \eta_0$ with η_0 constant. Using the solution (3.49) in (3.48), one finds

$$\phi(t,r) = \frac{\eta_0}{\ell}\, r + \phi_0(t)\,, \qquad \rho(t,r) = \eta_0\, r + \rho_0(t)\,, \tag{3.50}$$

for arbitrary functions $\phi_0(t)$ and $\rho_0(t)$. Substituting these expressions back into (3.48) then yields

$$\eta_0(1 - \mathcal{L}(t)) - \ell\mathcal{T}(t)\phi_0(t) + \frac{d\rho_0(t)}{dt} = 0\,, \tag{3.51}$$

$$\eta_0(1 + \mathcal{L}(t)) - \mathcal{T}(t)\rho_0(t) + \ell\frac{d\phi_0(t)}{dt} = 0\,. \tag{3.52}$$

Combining the last two equations, we obtain

$$\left(\frac{d^2}{dt^2} - \frac{d\big(\log(\mathcal{T})\big)}{dt}\,\frac{d}{dt} + V(\phi_0, \mathcal{L}, \mathcal{T})\right)\phi_0(t) = 0\,, \tag{3.53}$$

with the potential

$$V(\phi_0, \mathcal{L}, \mathcal{T}) = \frac{\eta_0}{\ell}\left(\mathcal{T}(1 - \mathcal{L}) + \frac{d\mathcal{L}}{dt} - \frac{d\big(\log(\mathcal{T})\big)}{dt} - \mathcal{L}\,\frac{d\big(\log(\mathcal{T})\big)}{dt}\right) - \mathcal{T}^2\,\phi_0\,. \tag{3.54}$$

We emphasise that neither ρ nor η appears in the second-order action (3.31). A comparison with the first-order formalism suggests that the second-order description corresponds to a partially reduced sector of the full first-order theory, in which part of the auxiliary scalar structure has already been solved for or fixed. In particular, the first-order scalar fields ρ and η do not appear as independent variables in the second-order action, indicating that their dynamics are encoded only implicitly through the reduced variables. We emphasise that the equation (3.53) is reproduced in the second-order formalism, and therefore provides a kinematical equation for the scalar field ϕ. However, it is subject to the constraining equation (3.36) coming from variation with respect to τ_μ in the second-order formalism, suggesting that the gauge field τ_μ could be interpreted as a Lagrange multiplier in this formalism.[3]

4 Discussion

In this paper, we considered the two-dimensional Galilean version of de Sitter space, of which the symmetries are captured by the extended de Sitter-Galilei (EdS-G) algebra. We studied aspects of both its bulk and boundary realisation.

On the boundary side, we employed a Schwarzian-like action to compute one-loop contributions. We report that these contributions capture quantum fluctuations that

[3]We thank Professor Jorge Zanelli for pointing this suggestion out.

dominate the low-energy density of states and at low temperatures we found the partition function to be domintated by $\mathcal{Z}_{\text{EdS-G}}(\beta) \sim \beta^{-2}$. Here this 2 correctly reflects the factors of 1/2 that are added corresponding to the four generators of the EdS-G algebra. To arrive at this result, we derived the symplectic form entering the path integral measure directly from the effective action by introducing Ostrogradsky canonical momenta. To the best of our knowledge, this action-based derivation of the symplectic form has not been emphasised in the literature. We also found that the logarithmic contribution is not accompanied by an imaginary phase, in agreement with corresponding two-dimensional relativistic computations [42, 43]. This contrasts with the relativistic result from four dimensions [53, 54], which may be attributed to the intrinsically two-dimensional nature of the present computation. It should also be pointed out that central charge c_1 drops out of our final result – this is potentially a side-effect of choosing our vacuum in (2.38). We furthermore assumed, in our computation, that the answer is one-loop exact. In light of the close relation to the warped Schwarzian theory [37], this appears plausible, although it certainly warrants further investigation. An exciting direction for future work would be to construct a SYK-like realisation of this model. A possible source of inspiration in this direction is [36], where related ideas were explored in the context of flat space holography and the complex SYK model and its corresponding Carrollian counterpart.

On the bulk side we show that once the EdS-G algebra is geometrised into a Newton-Cartan language, that one can construct a JT-like action of which this geometry solves the equations of motion. We note in particular that, upon restricting to the torsionless sector with $d\tau = 0$, the equation obtained by varying the JT-like action with respect to τ_μ involves only first derivatives. It would be interesting to understand whether this JT-like action reflects a deeper Lagrange-multiplier role for τ_μ, or whether it is simply a feature of the torsionless truncation. We also leave a careful Hamiltonian analysis of this action for future discussion. A natural next step for future study is to study bulk fluctuations around the here studied background, along lines similar to recent analyses of logarithmic contributions and Lichnerowicz-type zero modes in near-extremal black holes [53–56]. The goal would be to use these zero modes to verify our prediction regarding the exponent factor of β^{-2} entering in the partition function. Finally, we think this setup may also aid the exploration of studying thermal gravitational settings in a non-Lorentzian setting.

We want to end by emphasising that in this paper we have contributed towards paving the way for extending the holographic duality in low dimensions beyond relativistic symmetries. In conjunction we also believe our results contribute to shaping tools for further unwrapping the mysteries of de Sitter spacetime.

Acknowledgments

We thank Hamid Afshar, Patricio-Salgado Rebolledo, and Jorge Zanelli for fruitful comments. MH is supported by the Icelandic Research Fund under grant 228952-053 and the University of Iceland Doctoral Grant HEI2025-96517. DH is supported by the Icelandic Research Fund 2511228-051. DH also thanks the Institute for Theoretical Physics of Utrecht University for its hospitality, where part of the project was carried out, and, in particular, Professor Stefan Vandoren and Huaxuan Zeng for fruitful discussions on related topics. MH and DH also thank the Nordita Institute for the kind hospitality in which part of this project was undertaken. MH thanks Chalmers tekniska högskola for the kind hospitality in which part of this project was undertaken. The work of WS is supported by Starting Grant 2023-03373 from the Swedish Research Council and the Olle Engkvists Stiftelse.

A Ostrogradsky symplectic form for the Schwarzian theory

In this appendix, we show that the symplectic structure of the Schwarzian theory [44, 57–59] can be recovered directly from Ostrogradsky variables.

We begin with the Schwarzian theory

$$I_{\mathrm{Sch}}[f] = -\int d\tau \left(\mathrm{Sch}(f,\tau) + \frac{1}{2} f'^2\right) . \tag{A.1}$$

We now expand to second order with $f = \bar{f} + \delta f$ deriving the

$$I^{(1)}_{\mathrm{Sch}} = \int d\tau\, \partial_\tau \left(\bar{f}' - \frac{\bar{f}''^2}{\bar{f}'^3} + \frac{\bar{f}'''}{\bar{f}'^2}\right)\delta f\,, \tag{A.2}$$

$$I^{(2)}_{\mathrm{Sch}} = \int \frac{d\tau}{\bar{f}'^2}\left[-\frac{1}{2}\bar{f}'^2\delta f'^2 + \frac{3}{2}\frac{\bar{f}''^2}{\bar{f}'^2}\delta f'^2 - 2\frac{\bar{f}''}{\bar{f}'}\delta f'\delta f'' + \frac{1}{2}\delta f''^2\right] . \tag{A.3}$$

Now, we compute the Ostrogradsky conjugate momenta to δf to find

$$p_{\delta f} = \frac{\partial L^{(2)}_{\mathrm{Sch}}}{\partial(\delta f')} - \frac{d}{d\tau}\left(\frac{\partial L^{(2)}_{\mathrm{Sch}}}{\partial(\delta f'')}\right) = \frac{2}{\bar{f}'^2}\left[\mathrm{Sch}\left(\tanh\tfrac{\bar{f}}{2},\tau\right)\delta f' + \tfrac{\bar{f}''}{\bar{f}'}\delta f'' - \tfrac{1}{2}\delta f'''\right] . \tag{A.4}$$

Hence, the symplectic form which we will use to compute the measure is given by

$$\begin{aligned}\Omega^{(2)}_{\mathrm{Sch}} &= \int_0^\beta d\tau\, dP_{\delta f} \wedge d(\delta f) \\ &= \int_0^\beta \frac{2\, d\tau}{\bar{f}'^2}\left[\mathrm{Sch}\left(\tanh\frac{\bar{f}}{2},\tau\right) d\delta f' \wedge d\delta f + \frac{\bar{f}''}{\bar{f}'} d\delta f'' \wedge d\delta f - \frac{1}{2} d\delta f''' \wedge d\delta f\right].\end{aligned} \tag{A.5}$$

The last term can be integrated by parts to find

$$\Omega^{(2)}_{\mathrm{Sch}} = \int_0^\beta \frac{d\tau}{\bar{f}'^2}\left[d\delta f'' \wedge d\delta f' + 2\,\mathrm{Sch}(\tanh\tfrac{\bar{f}}{2},\tau) d\delta f' \wedge d\delta f\right] . \tag{A.6}$$

Once evaluated on the classical solution $\bar{f}(\tau) = \frac{2\pi}{\beta}\tau$ this gives the same as equation (114) of [59] and equation (2.8) of [44].

B One-loop analysis for the warped Schwarzian action

In this appendix, we revisit the quadratic fluctuations of the warped Schwarzian action of [37]. Our aim is to show that the one-loop partition function can be obtained directly from the Ostrogradsky symplectic form, without appealing to the coadjoint orbit formulation.

Starting from equation (5.5) of [37], we expand the fields in Fourier modes and rewrite the result as a sum over positive modes only. This gives

$$\begin{aligned}
S^{(2)} &= \int_0^\beta d\tau\left[\frac{c}{24}\left(\epsilon''^2-\left(\frac{2\pi}{\beta}\right)^2\epsilon'^2\right)+\frac{k}{4}(\sigma'+\alpha\epsilon')^2-\kappa\left(\epsilon''+\frac{2\pi i}{\beta}\epsilon'\right)(\sigma'+\alpha\epsilon')\right]\\
&= \frac{4\pi^2}{\beta}\sum_n\left[\frac{c}{24}\left(\frac{2\pi}{\beta}\right)^2 n^2(n^2-1)\epsilon_n\epsilon_{-n}+\frac{k}{4}n^2\tilde{\sigma}_n\tilde{\sigma}_{-n}-\frac{2\pi i}{\beta}\kappa n^2(n+1)\epsilon_n\tilde{\sigma}_{-n}\right]\\
&= \frac{8\pi^2}{\beta}\sum_{n\geq 2}\frac{c}{24}\left(\frac{2\pi}{\beta}\right)^2 n^2(n^2-1)|\epsilon_n|^2-\frac{8\pi^2}{\beta}i\sum_{n\geq 1}\frac{2\pi i}{\beta}\kappa n^3\left(\epsilon_n^{(I)}\tilde{\sigma}_n^{(R)}-\epsilon_n^{(R)}\tilde{\sigma}_n^{(I)}\right)\\
&\quad+\frac{8\pi^2}{\beta}\sum_{n\geq 1}\frac{k}{4}n^2|\tilde{\sigma}_n|^2-\frac{8\pi^2}{\beta}\sum_{n\geq 1}\frac{2\pi i}{\beta}\kappa n^2\left(\epsilon_n^{(R)}\tilde{\sigma}_n^{(R)}+\epsilon_n^{(I)}\tilde{\sigma}_n^{(I)}\right).
\end{aligned}\tag{B.1}$$

We introduced $\tilde{\sigma}=\sigma+\alpha\epsilon$ to simplify the resulting expressions. Note that the $n=1$ mode is degenerate, since the determinant of the corresponding quadratic form vanishes. This sector must therefore be treated separately in the path integral. We now turn to the symplectic structure associated with the quadratic action. Using the Ostrogradsky prescription, the quadratic symplectic form is constructed, where the canonical momenta are given by

$$p_\epsilon=\frac{\partial\mathcal{L}^{(2)}}{\partial\epsilon'}-\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}^{(2)}}{\partial\epsilon''}\right)=\frac{1}{2}k\alpha\tilde{\sigma}'+\kappa\sigma''-\frac{c\pi^2}{3\beta^2}\epsilon'-\frac{4\pi i\alpha\kappa}{\beta}\epsilon'-\frac{2\pi i}{\beta}\kappa\sigma'-\frac{1}{12}c\epsilon''',\tag{B.2}$$

$$p_\sigma=\frac{\partial\mathcal{L}^{(2)}}{\partial\sigma'}-\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}^{(2)}}{\partial\sigma''}\right)=\frac{1}{2}k\tilde{\sigma}'-\kappa\epsilon''-\frac{2\pi i}{\beta}\kappa\epsilon'.\tag{B.3}$$

Therefore we find

$$\begin{aligned}
\Omega^{(2)} &= \int_0^\beta d\tau\left[\tfrac{1}{2}k\,d\tilde{\sigma}'\wedge d\tilde{\sigma}-\tfrac{c}{12}\left(d\epsilon'''\wedge d\epsilon+\left(\tfrac{2\pi}{\beta}\right)^2 d\epsilon'\wedge d\epsilon\right)+2\kappa\,d\sigma'\wedge d\epsilon-\tfrac{4\pi i}{\beta}\kappa\,d\tilde{\sigma}'\wedge d\epsilon\right]\\
&= 2\pi i\sum_n\left[\tfrac{1}{2}kn\,d\tilde{\sigma}_n\wedge d\tilde{\sigma}_{-n}+\tfrac{c}{12}\left(\tfrac{2\pi}{\beta}\right)^2 n(n^2-1)\,d\epsilon_n\wedge d\epsilon_{-n}+2i\kappa\left(\tfrac{2\pi}{\beta}\right)n(n-1)\,d\tilde{\sigma}_n\wedge d\epsilon_{-n}\right]\\
&= 4\pi k\sum_{n\geq 1}n\,d\tilde{\sigma}_n^{(R)}\wedge d\tilde{\sigma}_n^{(I)}+4\beta\sum_{n\geq 2}\frac{c}{12}\left(\frac{2\pi}{\beta}\right)^3 n(n^2-1)\,d\epsilon_n^{(R)}\wedge d\epsilon_n^{(I)}\\
&\quad-4\beta\kappa\sum_{n\geq 1}\left(\frac{2\pi}{\beta}\right)^2 n^2\left(d\tilde{\sigma}_n^{(R)}\wedge d\epsilon_n^{(R)}+d\tilde{\sigma}_n^{(I)}\wedge d\epsilon_n^{(I)}\right)\\
&\quad+4\beta\kappa i\sum_{n\geq 1}\left(\frac{2\pi}{\beta}\right)^2 n\left(d\tilde{\sigma}_n^{(I)}\wedge d\epsilon_n^{(R)}-d\tilde{\sigma}_n^{(R)}\wedge d\epsilon_n^{(I)}\right).
\end{aligned}\tag{B.4}$$

With respect to the basis $x=\left\{\tilde{\sigma}_n^{(R)},\tilde{\sigma}_n^{(I)},\epsilon_n^{(R)},\epsilon_n^{(I)}\right\}$ we may write

$$\Omega_n^{(2)}=\begin{pmatrix}0 & A_n & -nB_n & -iB_n\\ -A_n & 0 & iB_n & -nB_n\\ nB_n & -iB_n & 0 & C_n\\ iB_n & nB_n & -C_n & 0\end{pmatrix},\qquad \begin{aligned}A_n&=4\pi k\,n,\\ B_n&=4\kappa\beta\left(\frac{2\pi}{\beta}\right)^2 n,\\ C_n&=\frac{c\beta}{3}\left(\frac{2\pi}{\beta}\right)^3 n(n^2-1).\end{aligned}\tag{B.5}$$

$$I_n^{(2)} = \begin{pmatrix} D_n & 0 & -iE_n & nE_n \\ 0 & D_n & -nE_n & -iE_n \\ -iE_n & -nE_n & F_n & 0 \\ nE_n & -iE_n & 0 & F_n \end{pmatrix}, \qquad \begin{aligned} D_n &= \frac{2\pi^2}{\beta} kn^2, \\ E_n &= \frac{8\pi^3}{\beta^2} kn^2, \\ F_n &= \frac{c\pi^2}{3\beta}\left(\frac{2\pi}{\beta}\right)^2 n^2(n^2-1)\,. \end{aligned} \tag{B.6}$$

For modes with $n \geq 2$ the quadratic fluctuation integral is a standard Gaussian. The case $n = 1$ is special, since the corresponding integral is not Gaussian and the symplectic form is degenerate, and must therefore be treated separately. We denote its contribution by $\mathcal{B}_1(\beta)$. Thus the one loop partition function takes the form

$$Z(\beta) = \mathcal{B}_1(\beta) \prod_{n\geq 2} \mathrm{Pf}\Big(\Omega_n^{(2)}\Big) \sqrt{\frac{(2\pi)^4}{\det I_n^{(2)}}} = 2\beta \prod_{n\geq 2} \frac{4\beta^2}{n^2} = \frac{1}{8\pi\beta^2}\,. \tag{B.7}$$

In the last step, we used the zeta-function regularisation, and in the preceding step, we used the following identities. For $n \geq 2$, we have

$$\mathrm{Pf}(\Omega_n^{(2)}) = \frac{32\pi^4}{3\beta^2} n^2(n^2-1)(ck - 24\kappa^2)\,, \tag{B.8}$$

$$\sqrt{\det I_n^{(2)}} = \frac{32\pi^6}{3\beta^4} n^4(n^2-1)(ck - 24\kappa^2)\,. \tag{B.9}$$

The remaining contribution comes from the $n = 1$ mode. In this case, we evaluate

$$\mathcal{B}_1(\beta) = 2\pi i\left[k\int d\tilde{\sigma}_1 d\tilde{\sigma}_1^* + 4i\kappa\left(\tfrac{2\pi}{\beta}\right)^2 \int d\epsilon_1 d\tilde{\sigma}_1^*\right] e^{\frac{4\pi^2 i}{\beta^2}\left[\frac{k\beta}{2}|\tilde{\sigma}_1|^2 - 4\pi i\kappa\epsilon_1^*\tilde{\sigma}_1\right]} = 2\beta\,. \tag{B.10}$$

The first term gives 2β, while the remaining four mixed contributions evaluate to $\pm 4\pi i$ and cancel pairwise in the sum. This agrees with the result obtained in [37].

C The saddle from global conformal Galilean invariance

In this appendix, we give an alternative, algebraic characterisation of the saddle used in the one-loop analysis. The key observation is that the global conformal Galilean de Sitter algebra arises as a finite-dimensional subalgebra of the warped Virasoro algebra governing the transformations of $\mathcal{L}(t)$ and $\mathcal{T}(t)$. Requiring $\mathcal{L}(t)$ and $\mathcal{T}(t)$ to be invariant under this global subalgebra then singles out precisely the constant background used in the main text.

The global conformal Galilean-de Sitter algebra can be identified as a finite-dimensional subalgebra of the warped Virasoro algebra introduced in (2.17) and given by

$$i\,\{L_m\,, L_n\} = (m-n)L_{n+m} + c\,(n^3 - n)\,\delta_{n+m,0}\,, \tag{C.1a}$$

$$i\,\{L_m\,, P_n\} = -n\,P_{m+n} + i\kappa\,(m^2 - m)\,\delta_{m+n,0}\,, \tag{C.1b}$$

$$i\,\{P_n\,, P_m\} = 0\,. \tag{C.1c}$$

To describe the asymptotic dynamics of Newton-Cartan geometry in $(1+1)$-spacetime dimensions, we use the conformal Galilean-de Sitter algebra given by

$$[\mathcal{H}\,, \mathcal{D}] = \mathcal{H}\,, \qquad [\mathcal{K}\,, \mathcal{D}] = -\mathcal{K}\,, \qquad [\mathcal{H}\,, \mathcal{K}] = 2\,\mathcal{Z}\,. \tag{C.2}$$

We want to show that (C.2) is contained as a subalgebra in (C.1). In effect, looking at the set $\{L_0\,, L_1\,, P_{-1}\}$, we observe that, (using the commutation relations (C.1)), they satisfy the non-vanishing Poisson bracket relations

$$i\,\{L_0\,,L_1\} = -L_1\,, \qquad i\,\{L_0\,,P_{-1}\} = P_{-1}\,, \qquad i\,\{L_1\,,P_{-1}\} = P_0\,. \tag{C.3}$$

So, promoting $i\{\,,\} \to [\,,]$ and making the identification $L_0 = \mathcal{D}\,,\ L_1 = \mathcal{H}\,,\ P_{-1} = \mathcal{K}\,,\ P_0 = 2\mathcal{Z}$, we recover (C.2) as a subalgebra of (C.1).

The gauge parameters $\sigma(t)$ and $\chi(t)$ admit the mode expansions

$$\sigma(t) = \sum_{n\in\mathbb{Z}} e^{\frac{2\pi i n t}{\beta}}\sigma_n\,, \qquad \chi(t) = \sum_{n\in\mathbb{Z}} e^{\frac{2\pi i n t}{\beta}}\chi_n \tag{C.4}$$

For the global subalgebra identified above, corresponding to the charges L_0, L_1, P_{-1}, the relevant modes are

$$\left\{\sigma_0\,, e^{\frac{2\pi i t}{\beta}}\sigma_1\right\} \qquad \left\{\chi_0\,, e^{-\frac{2\pi i t}{\beta}}\chi_{-1}\right\}. \tag{C.5}$$

We now consider the transformation laws for $\mathcal{L}$ and $\mathcal{T}$ in (2.15),

$$\delta_{(\sigma,\chi)}\mathcal{L} = \sigma\mathcal{L}' + 2\mathcal{L}\sigma' + \mathcal{T}\chi' + \chi''\,, \tag{C.6a}$$

$$\delta_{(\sigma,\chi)}\mathcal{T} = \sigma'\mathcal{T} + \sigma\mathcal{T}' - \sigma''\,. \tag{C.6b}$$

We analyse the allowed values of $\mathcal{T}$ and $\mathcal{L}$ that satisfy conformal Galilean invariance. This provides an algebraic characterisation of the background around which the one-loop fluctuation analysis is performed. Evaluating the invariance conditions $\delta\mathcal{L} = 0$ and $\delta\mathcal{T} = 0$ for each of the global modes (σ, χ) listed on the left gives the corresponding constraints on $\mathcal{T}(t)$ and $\mathcal{L}(t)$.

$$\begin{aligned}
(\sigma_0,\, e^{-\frac{2\pi i t}{\beta}}\chi_{-1}): &\qquad \mathcal{T}(t) = A, &\qquad \mathcal{L}(t) &= -\frac{e^{-\frac{2\pi i t}{\beta}}}{\beta\sigma_0}(A\beta + 2i\pi)\chi_{-1} + B\,,\\
(e^{\frac{2\pi i t}{\beta}}\sigma_1,\, \chi_0): &\qquad \mathcal{T}(t) = -\frac{2\pi i}{\beta} + C\,e^{-\frac{2\pi i t}{\beta}}, &\qquad \mathcal{L}(t) &= D\,e^{-\frac{4\pi i t}{\beta}}\,,\\
(e^{\frac{2\pi i t}{\beta}}\sigma_1,\, e^{-\frac{2\pi i t}{\beta}}\chi_{-1}): &\qquad \mathcal{T}(t) = -\frac{2\pi i}{\beta} + E\,e^{-\frac{2\pi i t}{\beta}}, &\qquad \mathcal{L}(t) &= -\frac{E\chi_{-1}}{\sigma_1}e^{-\frac{6\pi i t}{\beta}} + F\,e^{-\frac{4\pi i t}{\beta}}\,.
\end{aligned} \tag{C.7}$$

Therefore, all these conditions are compatible if $A = -2\pi i/\beta$, $B = C = D = E = F = 0$, which implies that the arbitrary functions take the form

$$\mathcal{T}(t) = -\frac{2\pi i}{\beta}\,, \qquad \mathcal{L}(t) = 0\,. \tag{C.8}$$

The conditions (C.8) in terms of the Goldstone fields v and s translate to

$$v(t) = -\frac{2i\pi t}{\beta}\,, \qquad s(t) = 0\,. \tag{C.9}$$

This is precisely the constant configuration for $\mathcal{L}$ and $\mathcal{T}$ selected by invariance under the global Galilean de Sitter algebra.

D Equations of motion for the non-relativistic JT action

In this appendix, we present the equations of motion for the second-order non-relativistic JT action written entirely in terms of lower-index Newton–Cartan variables. This formulation is equivalent to the one presented in the main text, but its variation is somewhat simpler since the action is expressed directly in terms of the one-forms τ_μ, e_μ, and m_μ. We have explicitly verified that the resulting equations of motion are equivalent to those obtained from the formulation used in the main text.

The action is

$$I[\tau, e, m, \phi] = \kappa \int d^2x\, \epsilon^{\mu\nu} \phi \left(\partial_\mu \omega_\nu - \frac{1}{\ell^2} \tau_\mu e_\nu \right), \tag{D.1}$$

where the spin connection is given by

$$\omega_\mu = \frac{1}{E} \epsilon^{\alpha\beta} \Big(\partial_\alpha m_\beta \tau_\mu - \partial_\alpha e_\beta e_\mu \Big), \tag{D.2}$$

and $E = -\epsilon^{\nu\rho} \tau_\nu e_\rho$. If one furthermore wishes to impose the torsionless condition $d\tau = 0$, this may be implemented by introducing a Lagrange multiplier ζ and adding the term

$$\frac{\mu}{\ell^2} \int d^2x\, \zeta\, \epsilon^{\mu\nu} \partial_\mu \tau_\nu \tag{D.3}$$

to the action. Variation with respect to ζ then imposes $\epsilon^{\mu\nu}\partial_\mu \tau_\nu = 0$, which is equivalent to $d\tau = 0$. We now vary the action with respect to the dynamical fields. For simplicity, boundary terms arising from integrations by parts will be omitted.

$$\begin{aligned}
\delta\phi : &\quad \epsilon^{\mu\nu} \Big(\partial_\mu \omega_\nu - \frac{1}{\ell^2} \tau_\mu e_\nu \Big) = 0\,, \\
\delta m_\beta : &\quad \epsilon^{\alpha\beta} \epsilon^{\mu\nu} \partial_\alpha \left(\frac{1}{E} \partial_\mu \phi \tau_\nu \right) = 0\,, \\
\delta \tau_\mu : &\quad \Big(\epsilon^{\alpha\nu} \epsilon^{\mu\beta} e_\beta \omega_\alpha + \epsilon^{\alpha\beta} \epsilon^{\mu\nu} \partial_\alpha m_\beta \Big) \partial_\nu \phi - \frac{E}{\ell^2} \epsilon^{\mu\nu} e_\nu \phi = 0\,, \\
\delta e_\mu : &\quad \left(\epsilon^{\beta\nu} \epsilon^{\alpha\mu} \tau_\alpha \omega_\beta - \epsilon^{\mu\nu} \epsilon^{\alpha\beta} \partial_\alpha e_\beta \right) \partial_\nu \phi + E \epsilon^{\beta\nu} \epsilon^{\alpha\mu} \partial_\alpha \left(\frac{1}{E} \partial_\nu \phi\, e_\beta \right) + \frac{E}{\ell^2} \epsilon^{\mu\nu} \tau_\nu \phi = 0\,.
\end{aligned} \tag{D.4}$$

These equations are equivalent to (3.34) obtained from the formulation presented in the main text.

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