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The $T\overline{T}$ deformation and zeta functions in 3D gravity

Rahul Poddar

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Dissertation submitted in partial fulfillment of a
Philosophiae Doctor degree in Physics

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Abstract

This thesis explores two topics in three-dimensional gravity, the $T\overline{T}$ deformation and zeta functions of three-dimensional quotient manifolds.

The $T\overline{T}$ deformation is an irrelevant deformation of a two-dimensional translationally invariant quantum field theory which is well-behaved in the UV. We consider the case of two-dimensional holographic Warped Conformal Field theories (WCFTs), which are dual to gravity in three-dimensional anti-de Sitter (AdS_3) spacetime with Compere, Song, Strominger (CSS) boundary conditions. WCFTs are non-relativistic quantum field theories which have a Virasoro$\times\mathrm{U}(1)$ Kac-Moody symmetry algebra. We compute the boundary conditions and asymptotic symmetry algebra for a $T\overline{T}$ deformed WCFT. We find that the $\mathrm{U}(1)$ Kac-Moody algebra survives the deformation if one allows the boundary metric to transform appropriately under the asymptotic symmetries, however the Virasoro sector becomes highly deformed and is no longer chiral.

The Selberg zeta function is defined by the Euler product over prime geodesics on a hyperbolic quotient manifold. It provides a simpler way to compute functional determinants of kinetic operators compared to traditional means. We introduce a new construction of a zeta function, which generalizes the Selberg zeta function to non-hyperbolic quotient manifolds. We employ our generalization to quotients of three-dimensional Warped AdS_3 and three-dimensional flat spacetime. We find that the zeroes of the zeta function accurately predicts the quasi-normal mode spectrum in these non-hyperbolic cases, providing evidence for the proposed construction of the zeta function.

Ágrip

Ritgerð þessi fjallar um tvö viðfangsefni í þyngdarfræði í þrívíðu tímarúmi. Annars vegar svonefnda $T\bar{T}$-bjögun og hins vegar zetaföll fyrir þrívíð deildarúm.

$T\bar{T}$-bjögun á tvívíðu skammtasviðslíkani með hliðrunarsamhverfu er víkjandi bjögun, í skilningi endurstöðlunar, sem leiðir ekki til ósamleitni á stuttum lengdarkvarða. Í fyrri hluta rigerðarinnar er sjónum beint að $T\bar{T}$-bjögun á undinni hornrækinni sviðsfræði, sem hefur þyngdarfræðilega framsetningu í þrívíðu tímarúmi með neikvæðan heimsfasta. Undin hornrækin sviðsfræði hefur Virasoro$\times$U(1) Kac-Moody samhverfualgebru. Kannað er hvaða áhrif $T\bar{T}$-bjögun slíkrar kenningar hefur á jaðarskilyrði og samhverfu í tilsvarandi þyngdarfræðilíkani. Helstu niðurstöður eru að U(1) samhverfan er óbreytt en Virasoroalgebran bjagast og er ekki lengur hendin.

Selberg zetafall er skilgreint sem Eulermargfeldi yfir frumgagnvegi á breiðgerðu deildarúmi. Með hjálp þess er mun auðveldara að reikna róf hreyfiorkuvirkja í skammtasviðlíkönum heldur en með hefðbundnum aðferðum. Í seinni hluta ritgerðarinnar er kynnt til sögunnar nýtt zetafall sem er alhæfing á Selberg zetafallinu fyrir fleiri deildarúm en þau sem eru breiðgerð. Nýja zetafallið er ákvarðað annars vegar fyrir deildarúm af undnu þrívíðu tímarúmi með neikvæðan heimsfasta og hins vegar fyrir deildarúm af flötu þrívíðu tímarúmi. Í báðum tilvikum fæst samsvörun milli núllstöðva zetafallsins og tvinngilds eiginrófs á viðkomandi deildarúmi, sem er í samræmi við eldri niðurstöður fyrir breiðgerð deildarúm.

To my parents, Mahua and Gautam.

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A Selberg zeta function for warped AdS_3 black holes
Journal of High Energy Physics, Vol. 2023, Article number 49.
`https://doi.org/10.1007/JHEP01(2023)049`.
I contributed to all of the calculations and wrote a majority of the bulk of the paper.

Paper II: **Rahul Poddar**, 2023,
$T\overline{T}$-Deformations of Holographic Warped CFTs.
Physical Review D, 108 (2023) 10, 105016.
`https://doi.org/10.1103/PhysRevD.108`.
I performed all the calculations and wrote the paper.

Paper III: Arjun Bagchi, Cynthia Keeler, Victoria Martin, **Rahul Poddar**, 2024,
A generalized Selberg zeta function for flat space cosmologies.
Journal of High Energy Physics, Vol. 2024, Article number 66.
`https://doi.org/10.1007/JHEP04(2024)066`
I contributed to all of the calculations and wrote a most of the bulk of the paper.

Abbreviations

SENS School of Engineering and Natural Sciences

UoI University of Iceland

QFT Quantum Field Theory

CFT Conformal Field Theory

n**D** n Dimensions

RG Renormalization Group

OPE Operator Product Expansion

WCFT Warped Conformal Field Theory

CSS Compere, Song, and Strominger

AdS$_3$ three-dimensional Anti-de Sitter space

FSC Flat Space Cosmology

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1 Introduction

Modern physics is arguably the study of symmetry. When we reduce the world around us to systems we can model mathematically, we impose many simplifications to maximize the symmetry of the system, which we can use to make things tractable. We have made tremendous progress in human history in utilizing this approach to model our world around us, but there is still much we can not explain. We do not live in a highly symmetric universe, as is evident by the world around us. A maximally symmetric universe wouldn't have the complexity required to admit intricate structures like life. It is therefore important to test the tools we have developed in less symmetric systems, while still looking for analytical control.

There are many approaches to study systems away from symmetry. In weakly coupled quantum field theory, like quantum electrodynamics we can use perturbation theory, where we perturbatively add interactions between fields. This has been an extremely successful approach, yielding a remarkably precise and accurate theory to describe the interactions between light and matter. However perturbation theory can not be completely accurate since it is an approximation by definition.

Unfortunately, we have full analytic control over only the most symmetric systems. Examples include free quantum field theories on maximally symmetric spacetimes, conformally invariant quantum field theories, and gravity in a constant negatively curved spacetime. Another approach we can take is to look for suitable deformations away from these maximally symmetric systems which allow us to explore less symmetric systems but still have analytic control. This is the subject of this thesis.

The $T\overline{T}$ deformation is a deformation of two-dimensional quantum field theories which posses translation invariance. A $T\overline{T}$ deformed theory is an interacting theory and is not conformally invariant, so we have moved away from the maximally symmetric case already. However, it is a deformation that is constructed out of conserved currents, which allow it to preserve structures in the undeformed theory, like the structure of the Hilbert space for example. It is in a peculiar class of deformations known as irrelevant deformations which typically is ill-behaved in the high energy regime. However, the $T\overline{T}$ deformation is well behaved in the UV despite being irrelevant.

In this thesis we will study the $T\overline{T}$ deformation, and then apply it to an already "deformed" theory, a warped CFT. This theory is not deformed by any operator, but we have reduced the symmetries by only allowing scale invariance along a particular coordinate, thereby removing conformal and Lorentz (rotation) invariance in the theory. We will use tools from bottom-up holography and 3D gravity to perform analytic calculations here.

Another aspect we will explore in this thesis is free fields on curved manifolds. In particular, the object of interest is the functional determinant of the scalar Laplacian on various manifolds. This can be computed in highly symmetric cases using techniques like the heat kernel method. Evaluating the heat kernel in other manifolds seems to be a dauntingly difficult task, and likely not analytically possible in most cases.

In this thesis we introduce the Selberg zeta function, a geometric cousin to the Riemann zeta function. It can be expressed as the Euler product over prime geodesics on quotients of the hyperbolic plane. Prime geodesics are closed geodesics in a quotient manifold which come back to themselves after only one trip around the quotient. It turns out that this zeta function is intimately related to functional determinants of Laplacians on quotient manifolds. If one can write down the (generalized) Selberg zeta function for an arbitrary quotient manifold, then one has access to the functional determinant of the Laplacian on that manifold, which is related to 1-loop quantum corrections to the partition function.

We introduce such a generalized construction of a Selberg-like zeta function which can work on a general quotient manifold, and test our construction on three dimensional flat spacetime, and the highly non-symmetric three-dimensional warped AdS. We are able to compute quantities like functional determinants and the quasi-normal mode spectrum analytically, providing evidence for our construction.

The outline of thesis is as follows.

In chapter 2 we review key aspects of three dimensional gravity. In particular, we go over the Fefferman-Graham expansion of asymptotically anti-de Sitter spacetimes, and show how three dimensions is special. We also go over how to construct three dimensional orbifolds by observing the quotient structure at the boundary. We finish the chapter with a review of an alternative set of boundary conditions known as the Compere, Song, and Strominger (CSS) boundary conditions.

In chapter 3 we review some properties of the $T\overline{T}$ operator and the corresponding deformation. We sketch the proofs for calculating the deformed energy spectrum and partition function. We derive the deformed boundary conditions for a $T\overline{T}$ deformed holographic CFT. We briefly review Paper II [1], where we focus on $T\overline{T}$ deformations holographic WCFTs, and compute the deformed symmetry algebra using holographic methods.

In chapter 4 we first list some properties a zeta function ought to have, and show how the Riemann zeta function satisfies these properties. Then we explore the Selberg zeta function in its original context and its generalization to higher dimensions. We then explore the zeta function for the BTZ black hole, and show how it is related to the functional determinant of the scalar Laplacian on the BTZ black hole background. In Paper III [2] we propose a generalized construction of a Selberg-like zeta function for quotients of non-hyperbolic manifolds by taking a product over representations of the global isometry group. We show evidence for this construction by looking at quotients of Warped AdS in Paper I [3] and flat spacetime in Paper III [2].

We conclude in chapter 5. Future directions for both the $T\overline{T}$ program and the Selberg-zeta program are also provided.

This thesis is based on the following three papers written during my doctoral studies at the University of Iceland, which are displayed on page 39.

I "*A Selberg zeta function for warped* AdS_3 *black holes*" JHEP **01** (2023) 049
V. L. Martin, R. Poddar, A. Þórarinsdóttir

II "$T\overline{T}$ *Deformations of Holographic Warped CFTs*" Phys. Rev. D **108**, 105016
R. Poddar

III "*A generalized Selberg zeta function for flat space cosmologies*" JHEP **04** (2024) 066
A. Bagchi, C. Keeler, V. L. Martin, R. Poddar

2 Gravity in three dimensions

In this chapter we will briefly go over some aspects of gravity with a negative cosmological constant in 2+1 dimensions.

2.1 Einstein gravity

The action for pure Einstein gravity in $d \geq 2$ is given by

$$S_{\text{Grav}} = \frac{1}{16\pi G_N} \int_{\mathcal{M}} d^{d+1}x \sqrt{|g|}\,(R - 2\Lambda) - \frac{1}{8\pi G_N} \int_{\partial\mathcal{M}} d^d y \sqrt{|h|} K, \tag{2.1}$$

where R is the Ricci scalar, K is the trace of the extrinsic curvature of the boundary. We have included the Gibbons-Hawking-York boundary term to ensure a well-defined Dirichlet boundary value problem. It should be noted that one can add terms involving the intrinsic curvature of the boundary manifold to the extrinsic curvature as counterterms to ensure finiteness of the on-shell action [4]. The negative cosmological constant we will consider is given by

$$\Lambda = -\frac{d(d-1)}{2\ell^2}, \tag{2.2}$$

where we are in $d+1$ dimensions, and ℓ is a length scale. The only dynamical field here is the metric $g_{\mu\nu}$, whose equations of motion are

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{2.3}$$

where $T_{\mu\nu}$ is the stress-energy of any matter present. We will be considering only vacuum solutions, so $T_{\mu\nu} = 0$. Negatively curved vacuum solutions are known as Anti-de Sitter or AdS spacetimes, and have the metric

$$ds^2 = \ell^2 \left(\frac{dr^2}{r^2} + r^2 \eta_{ab} dx^a dx^b \right), \tag{2.4}$$

where the length scale ℓ can be identified as the AdS radius, and $\eta = \text{diag}(-1, 1, \ldots, 1)$ is the flat metric along the d transverse directions. On specifying to 2+1 dimensions, we find that Einstein gravity has no local dynamical degrees of freedom, and is purely topological. However, despite the lack of dynamical gravity, the phase space of 3 dimensional gravity is rich and interesting.

2.2 The Fefferman-Graham expansion

To find the phase space of 3D gravity with general Dirichlet boundary conditions is itself a challenging task. We can solve an easier problem by specifying a class of boundary conditions, and finding the set of solutions satisfying these boundary conditions for three-dimensional Anti-de Sitter space (AdS_3).

To do so, let us consider the Fefferman-Graham gauge [5] of a negatively curved space, which specifies the metric in the rr and rz^a components,

$$ds^2 = \ell^2 \frac{dr^2}{r^2} + g_{ab}(r, z) dz^a dz^b, \tag{2.5}$$

where r is the radial coordinate. We will use Latin indices $a, b, \ldots$ to label the two transverse directions z^a, and Greek indices $\mu, \nu \ldots$ to denote the three dimensional coordinates x^μ which include r and z^a. The boundary of the spacetime is located at $r \to \infty$.

One can expand $g_{ab}(r, z)$ around the boundary in even powers of the radial coordinate r,

$$g_{ab} = \ell^2 r^2 \left(g^{(0)}_{ab} + r^{-2} g^{(2)}_{ab} + r^{-4} g^{(4)}_{ab} + \ldots \right). \tag{2.6}$$

The equations of motion provide strong constraints on this expansion. When the Weyl tensor vanishes, the expansion truncates at $g^{(4)}$, and there are no higher order terms [6]. In three dimensions, the Weyl tensor vanishes identically, so this is always true for 3D Einstein gravity. The equations of motion also give us

$$g^{(4)}_{ab} = \frac{1}{4} g^{(2)}_{ac} g^{(2)}_{db} g^{(0)cd}, \tag{2.7}$$

where indices are raised and lowered by $g^{(0)}$. The curvature of the boundary metric can also be worked out from the equations of motion, and is

$$\mathrm{Tr}\big((g^{(0)})^{-1} g^{(2)} \big) = \frac{\ell^2}{2} R^{(0)}. \tag{2.8}$$

Thus, it is enough to specify $g^{(0)}$ and $g^{(2)}$ to determine the full bulk metric. In other words, we need to specify the metric and its radial derivative at the boundary.

2.3 Asymptotically AdS_3 spacetimes

Here we will explore the famous Brown-Henneaux boundary conditions for asymptotically AdS_3 spacetimes [7]. These boundary conditions are such that the boundary manifold is flat, but the bulk metric has specific fall off conditions. Let us consider the boundary manifold to be the complex plane $\mathbb{C}$, with metric $ds^2 = dz d\bar{z}$. Thus, we have

$$g^{(0)} = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}. \tag{2.9}$$

The "vacuum" solution with these boundary conditions is well known, it is AdS_3 in the Poincaré patch,

$$ds^2 = \ell^2 \left(\frac{dr^2}{r^2} + r^2 dz d\bar{z} \right). \tag{2.10}$$

To generate the rest of the phase space of solutions with Brown-Henneaux boundary conditions, we have to find the transformations which preserve the Fefferman-Graham gauge (2.5) and this particular boundary metric. In other words, we need a vector field ξ such that the Lie derivative of the metric with respect to that vector field vanishes for the radial components, and the transverse components asymptotically. Preserving the radial components of the metric we have

$$\mathcal{L}_\xi g_{r\mu} = 0 \implies \xi = rf(z,\bar{z})\partial_r + \left(V^a(z,\bar{z}) - \int \frac{g^{ab}}{r}\partial_b f(z,\bar{z})dr\right)\partial_a. \tag{2.11}$$

Imposing Dirichlet boundary conditions on the metric implies that the vector field ξ preserves transverse components asymptotically, so we have

$$\lim_{r\to\infty}\mathcal{L}_\xi g_{ab} = 0 \implies \partial_{\bar{z}}V^z(z,\bar{z}) = 0, \partial_z V^{\bar{z}}(z,\bar{z}) = 0, f(z,\bar{z}) = -\frac{1}{2}\left(V'(z) + \bar{V}'(\bar{z})\right), \tag{2.12}$$

where $V \equiv V^z, \bar{V} \equiv V^{\bar{z}}$, since they satisfy the Cauchy-Riemann conditions. A vector field which satisfies such boundary conditions is called an “asymptotic Killing vector.” Such vector fields preserve the asymptotics, but change the bulk metric. So, we can flow in the phase space of solutions with the asymptotic Killing vectors. We find that the general solution for asymptotically AdS$_3$ with Brown-Henneaux boundary conditions is [8]

$$\frac{ds^2}{\ell^2} = \frac{dr^2}{r^2} + \frac{1}{r^2}\left(r^2 dz + \frac{1}{k}\bar{L}(\bar{z})d\bar{z}\right)\left(r^2 d\bar{z} + \frac{1}{k}L(z)dz\right), \tag{2.13}$$

where $k = \frac{\ell}{4G_N}$. This is the full classical phase space, parameterized by the holomorphic and anti-holomorphic functions $L(z)$, $\bar{L}(\bar{z})$ respectively.

The general solution (2.13) encompasses the famous BTZ black hole solution [9]. For the BTZ black hole, the functions L and $\bar{L}$ are constants, and are in terms of the mass M and angular momentum J of the black hole,

$$L = \frac{M\ell + J}{2}, \quad \bar{L} = \frac{M\ell - J}{2}. \tag{2.14}$$

We can also derive the variation of these parameterizing functions in the flow in phase space. Using

$$\mathcal{L}_\xi g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial L(z)}\delta_\xi L(z) + \frac{\partial g_{\mu\nu}}{\partial \bar{L}(\bar{z})}\delta_\xi \bar{L}(\bar{z}), \tag{2.15}$$

we find that L and $\bar{L}$ transform like the stress-energy tensor for a CFT, with an infinitesimal Schwarzian derivative

$$\begin{aligned}\delta_\xi L(z) &= V(z)L'(z) + 2V'(z)L(z) - \frac{k}{2}V'''(z),\\ \delta_\xi \bar{L}(\bar{z}) &= \bar{V}(\bar{z})\bar{L}'(\bar{z}) + 2\bar{V}'(\bar{z})\bar{L}(\bar{z}) - \frac{k}{2}\bar{V}'''(\bar{z}).\end{aligned} \tag{2.16}$$

In a CFT, the coefficient of the triple derivative term is $-\frac{c}{12}$, so we have the famous result

$$c = \frac{3\ell}{2G_N}. \tag{2.17}$$

Hence we can conclude that the asymptotic symmetry algebra of AdS_3 consists of two commuting copies of the Virasoro algebra with this central charge,

$$L(z) = \sum_n L_n z^{-n-2} + \frac{c}{24}\delta_{n,0}, \quad [L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}n(n^2-1)\delta_{m,-n}. \tag{2.18}$$

This is one of many hints of the AdS_3/CFT_2 correspondence, which states that semi-classical asymptotically AdS_3 gravity is dual to a strongly coupled quantum CFT. This correspondence holds in the regime of large central charge c, or as (2.17) shows, for large $\frac{\ell}{G_N}$.

2.4 Quotients of AdS_3

The Ricci scalar of the general asymptotically AdS_3 solution (2.13) turns out to be the same as that of empty AdS_3,

$$R = -\frac{6}{\ell^2}, \tag{2.19}$$

which is a constant. Hence we see that (2.13) is still *locally* AdS_3. To construct locally AdS_3 solutions, we can consider orbifolds, or quotients with a discrete subgroup of the global isometry group of AdS_3.

We can study the quotient by studying the action on the boundary. The flat torus is a quotient of the complex plane $\mathbb{T}^2 \sim \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, such that $z \sim z+1$ and $z \sim z+\tau$.

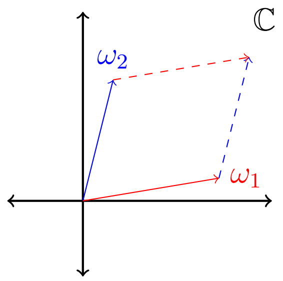

Figure 2.1. A torus as a discrete quotient of the complex plane. The blue lattice vector ω_2 is identified with the dashed vector, and similarly for the red lattice vector ω_1.

Here $\tau = \tau_1 + i\tau_2 = \frac{\omega_2}{\omega_1}$ is the modular parameter of the torus, and takes values on the upper half plane $\mathbb{H}^2$. Equivalent tori are related by fractional linear transforms of the form

$$\tau \to \frac{a\tau+b}{c\tau+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}). \tag{2.20}$$

Therefore, all inequivalent tori are parameterized by $\tau \in \mathbb{H}^2/\mathrm{PSL}_2(\mathbb{Z})$, and the fundamental domain of this quotient is typically given by $|\tau_1| \leq \frac{1}{2}, |\tau| \geq 1$. The group $\mathrm{PSL}_2(\mathbb{Z})$ is also known as the modular group, and $\mathrm{PSL}_2(\mathbb{Z})$ transformations are known as modular transformations. It is a discrete group with two generators,

$$\mathcal{T} : \tau \to \tau + 1, \quad \mathcal{S} : \tau \to -\frac{1}{\tau}. \tag{2.21}$$

To understand how the quotient acts on the bulk, we have to understand how the quotient on the boundary manifold extends into the bulk. Quotients must be generated along an isometry

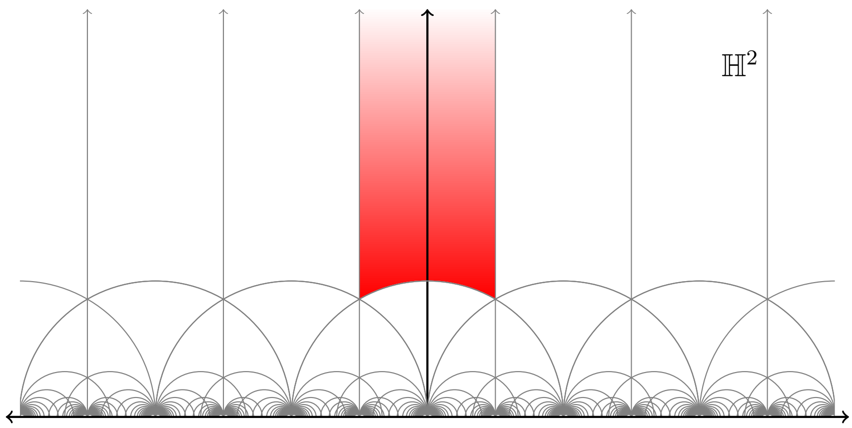

Figure 2.2. This is the tessellation of $\mathbb{H}^2$ *by* $\mathrm{PSL}_2(\mathbb{Z})$*. Each enclosed region is equivalent to another by a* $\mathrm{PSL}_2(\mathbb{Z})$ *transformation. The fundamental domain of* $\mathbb{H}^2/\mathrm{PSL}_2(\mathbb{Z})$ *is shaded in red*

for the resulting manifold to be sensible. In other words, Killing vectors are the generators of the isometry group. The discrete subgroup with which the quotient is performed must also be generated by the same generators. For example, the quotient manifold $S^1 \sim \mathbb{R}/\mathbb{Z}$ can be thought of as being generated by the group element $e^{a\partial_x}$ where a is the circumference of the circle S^1.

There are six isometries of AdS_3, since it is a maximally symmetric manifold in three dimensions. In the Poincaré patch (2.10) coordinates (with $r = 1/y$), they are

$$\begin{aligned} L_{-1} &= -\partial_z, \quad L_0 = -\left(z\partial_z + \frac{1}{2}y\partial_y\right), \quad L_1 = -(z^2\partial_z + y^2\partial_{\bar{z}} + yz\partial_y), \\ \bar{L}_{-1} &= -\partial_{\bar{z}}, \quad \bar{L}_0 = -\left(\bar{z}\partial_{\bar{z}} + \frac{1}{2}y\partial_y\right), \quad \bar{L}_1 = -(y^2\partial_z + \bar{z}^2\partial_{\bar{z}} + y\bar{z}\partial_y), \end{aligned} \tag{2.22}$$

which obey the $\mathfrak{sl}_2(\mathbb{R}) \times \mathfrak{sl}_2(\mathbb{R})$ Lie algebra,

$$[L_m, L_n] = (m-n)L_{m+n}, \quad [\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n}. \tag{2.23}$$

Taking any linear combination of these isometry generators is a valid isometry with respect to which one can perform a quotient. The BTZ black hole, for instance, the quotient in Boyer-Lindquist coordinates is the identification $\phi \sim \phi + 2\pi$ [10] (cf. (4.13)). So the quotient generator must be a 2π translation in the azimuthal ϕ direction. The specific linear combination is,

$$2\pi\partial_\phi = 2\pi i L_0 \tau - 2\pi i \bar{L}_0 \bar{\tau} = 2\pi i \left(-\frac{|r_-|}{\ell}(L_0 - \bar{L}_0) + i\frac{r_+}{\ell}(L_0 + \bar{L}_0)\right), \tag{2.24}$$

where $\tau_1 = -|r_-|/\ell, \tau_2 = r_+/\ell$, and

$$r_\pm = \sqrt{2G_N\ell(M\ell + J)} \pm \sqrt{2G_N\ell(M\ell - J)} \tag{2.25}$$

are the inner and outer horizon radii respectively. The Euclidean BTZ black hole requires the Wick rotations $J \to -iJ_E$ and $r_- \to i|r_-|$. Once again, we see that the modulus τ parameterizes the boundary torus of the Euclidean BTZ black hole, characterizing the quotient, this time in terms of physical parameters M and J_E. The Lorentzian BTZ black hole does not have a boundary torus but a boundary cylinder, since time can not be compactified in this case, however the quotient $\phi \sim \phi + 2\pi$ is still parameterized by M and J.

2.5 The CSS boundary conditions

CSS [11] found new boundary conditions for AdS_3 which do not have the Virasoro$\times$Virasoro asymptotic symmetry group, but have a Virasoro$\times\mathrm{U}(1)$ Kac-Moody asymptotic symmetry group. The difference here is that instead of imposing Dirichlet boundary conditions at infinity, they imposed “Dirichlet-Neumann” boundary conditions. In other words, they specified part of the metric at infinity, and also specified part of the derivative of the metric at infinity. This allows for symmetry transformations that do not preserve the full boundary metric, which can be useful for describing special holographic systems like warped CFTs.

The CSS boundary conditions can be expressed in terms of the Fefferman-Graham expansion

$$g^{(0)} = \begin{pmatrix} P'(z) & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad g^{(2)}_{\bar z\bar z} = \frac{\Delta}{k}, \tag{2.26}$$

where $k = \frac{\ell}{4G_N}$, and Δ is a constant specifying the derivative of part of the metric at infinity. The holomorphic function $P(z)$ is left undetermined, since the Dirichlet-Neumann boundary conditions require part of the boundary metric to be unspecified.

Solving Einstein’s equations (2.3) with these boundary conditions, we obtain the full bulk metric with CSS boundary conditions [11],

$$\begin{aligned} \frac{ds^2}{\ell^2} &= \frac{dr^2}{r^2} + \frac{\Delta}{k}d\bar z^2 - \left(r^2 + \frac{2\Delta P'(z)}{k} + \frac{\Delta L(z)}{k^2r^2}\right)dzd\bar z \\ &+ \left(r^2P'(z) + \frac{(L(z)+\Delta P'(z))^2}{k} + \frac{\Delta L(z)P'(z)}{k^2r^2}\right)dz^2. \end{aligned} \tag{2.27}$$

Here both $L(z)$ and $P(z)$ are unspecified holomorphic functions which parameterize the phase space of AdS_3 with CSS boundary conditions.

Let us verify that the asymptotic symmetry algebra with these boundary conditions is Virasoro$\times\mathrm{U}(1)$ Kac-Moody. Using (2.11) and imposing the boundary condition

$$\lim_{r\to\infty} \mathcal{L}_\xi g_{ab} = \begin{cases} 0 & (a,b) \neq (z,z), \\ r^2\delta_\xi P'(z) & (a,b) = (z,z), \end{cases} \tag{2.28}$$

we have the asymptotic symmetry generator,

$$\xi(V) = -\frac{r}{2}V'(z)\partial_r + \left(V(z) + \frac{k\Delta V''(z)}{2(k^2r^4 - \Delta L(z))}\right)\partial_z + \frac{k(kr^2 + \Delta P'(z))V''(z)}{2(k^2r^4 - \Delta L(z))}\partial_{\bar z}. \tag{2.29}$$

Computing the flow in phase space,

$$\mathcal{L}_\xi g_{\mu\nu} = \partial_{L(z)} g_{\mu\nu} \delta_\xi L(z) + \partial_{P'(z)} g_{\mu\nu} \delta_\xi P'(z), \tag{2.30}$$

we find

$$\begin{aligned} \delta_\xi L(z) &= V(z)L'(z) + 2V'(z)L(z) - \frac{k}{2}V'''(z), \\ \delta_\xi P'(z) &= P'(z)V'(z) + V(z)P''(z). \end{aligned} \tag{2.31}$$

Here we see that the variation of L under ξ has the cubic derivative of V, the infinitessimal part of the Schwarzian derivative, transforming as the boundary stress-energy tensor, and therefore its modes satisfy the Virasoro algebra with $c = 3\ell/2G$. We also see that P' transforms as a weight one conformal primary.

Imposing Neumann boundary conditions on the asymptotic symmetry generator η we have,

$$\eta(\sigma) = \sigma(z)\partial_{\bar{z}}, \tag{2.32}$$

which generates the flow

$$\delta_\eta L(z) = 0, \quad \delta_\eta P(z) = -\sigma(z), \tag{2.33}$$

whose modes satisfy the $\mathrm{U}(1)$ Kac-Moody algebra,

$$P(z) = \sum_n P_n z^{-n-1}, \qquad [P_m, P_n] = -2\Delta m \delta_{m,-n}. \tag{2.34}$$

We can make the algebra factorize into commuting Virasoro and U(1) Kac-Moody algebras if we consider the following linear combination of these asymptotic symmetry generators,

$$\begin{aligned} \zeta(V) \equiv \xi(V) - \eta(VP') = -\frac{r}{2}V'(z)\partial_r &+ \left(V(z) + \frac{k\Delta V''(z)}{2(k^2r^4 - \Delta L(z))}\right)\partial_z \\ &+ \left(P'(z)V(z) + \frac{k(kr^2 + \Delta P'(z))V''(z)}{2(k^2r^4 - \Delta L(z))}\right)\partial_{\bar{z}}. \end{aligned} \tag{2.35}$$

This is a true asymptotic Killing vector, in the sense that it satisfies $\lim_{r\to\infty} \mathcal{L}_\zeta g_{ab} = 0$. The flow in phase space under this generator is

$$\delta_\zeta L(z) = V(z)L'(z) + 2V'(z)L(z) - \frac{k}{2}V'''(z), \quad \delta_\zeta P(z) = 0. \tag{2.36}$$

Therefore, the charge algebra of ζ and η form a factorized Virasoro and U(1) Kac-Moody algebra.

3 The $T\overline{T}$ deformation

The $T\overline{T}$ deformation has become a very interesting and active area of research in recent years. Coarse graining a Quantum Field Theory (QFT), or going to a lower energy (IR) regime is known as a Renormalization Group (RG) flow. A scale invariant (relativistic) QFT usually has an enhanced symmetry known as a conformal symmetry, and is known as a Conformal Field Theory (CFT). CFTs are fixed points of the RG flow. Typically, deformations of a QFT trigger a RG flow, usually from a theory describing the high energy regime (a UV theory) to an IR theory. Deformations of QFTs can be classified into three broad categories, with the $T\overline{T}$ deformation falling into the "irrelevant" category.

- **Relevant**: These deformations are well behaved and trigger the RG flow from the UV to an IR fixed point, and have the effect of "coarse-graining" the UV or short distance behaviour of a theory.
- **Marginal**: These deformations do not trigger any RG from the UV to the IR, but keep a theory at a fixed point. If there is a continuous submanifold of fixed points, then a marginal deformation can allow you to move in this submanifold, such that the resulting theory is still conformal.
- **Irrelevant**: Irrelevant deformations are deformations which trigger a "reverse" RG flow up from the IR to the UV. These are usually not well defined, and cannot be renormalized, needing infinitely many counterterms. Theories deformed by an irrelevant deformation usually are ill-behaved and are divergent in the UV.

Before we discuss the $T\overline{T}$ operator and the corresponding deformation in more detail, let us list some of the interesting features a $T\overline{T}$-deformed field theory possesses.

1. The deformation preserves the original global symmetries of the undeformed theory.
2. The deformed finite size energy spectrum can be computed in closed form [12].
3. Momenta are undeformed
4. The deformation preserves integrability of the undeformed theory [13].
5. The deformed S-matrix is gravitationally dressed [13].
6. The deformation can be interpreted as coupling the undeformed theory to a two-dimensional dilaton gravity theory [14, 15].
7. The deformed theory shares many features with non-critical bosonic string theory, for example the action of N deformed free bosons is a Nambu-Goto action in $N + 2$ dimensional target space in light-cone gauge[16].
8. The deformed torus partition function can be explicitly worked out [17].

3.1 The $T\overline{T}$ operator

It turns out that the $T\overline{T}$ deformation, despite being irrelevant, produces a well behaved theory in the UV. To show this, let us first properly define the $T\overline{T}$ operator, and compute its expectation value. The $T\overline{T}$ operator is simply defined as the determinant of the stress-energy tensor of a 2D QFT [12],

$$\mathcal{O}_{T\overline{T}} = \det T_{\mu\nu}. \tag{3.1}$$

The determinant in two dimensions is quadratic, so to define this composite operator properly, one has to use point splitting and take the coincident limit, and show that the only leading term in the Operator Product Expansion (OPE),

$$\mathcal{O}_i(z_1)\mathcal{O}_j(z_2) = \sum_k C_{ij}^k(z_1 - z_2)\mathcal{O}_k(z_2) \tag{3.2}$$

is the $\mathcal{O}_{T\overline{T}}$ operator, with the rest of the subleading terms being derivative operators $\partial\mathcal{O}$. On the complex plane $\mathbb{C}$ ($ds^2 = dzd\bar{z}$), the $T\overline{T}$ operator takes the form

$$\mathcal{O}_{T\overline{T}} = \lim_{z'\to z}\Big(T_{zz}(z,\bar{z})T_{\bar{z}\bar{z}}(z',\bar{z}') - T_{z\bar{z}}(z,\bar{z})T_{\bar{z}z}(z',\bar{z}')\Big). \tag{3.3}$$

By using conservation of the stress-energy tensor

$$\partial_\mu T^{\mu\nu} = 0, \tag{3.4}$$

and translation invariance, one can show that the OPE takes the form

$$T_{zz}(z,\bar{z})T_{\bar{z}\bar{z}}(z',\bar{z}') - T_{z\bar{z}}(z,\bar{z})T_{z\bar{z}}(z',\bar{z}') = \mathcal{O}_{T\overline{T}}(z,\bar{z}) + \sum_i C_i^\mu(z - z', \bar{z} - \bar{z}')\partial_\mu\mathcal{O}_i(z,\bar{z}). \tag{3.5}$$

Taking the vacuum expectation value on both sides shows that the only surviving term on the right hand side is the operator $\mathcal{O}_{T\overline{T}}$, since translation invariance implies $\langle\partial_\mu\mathcal{O}\rangle = \partial_\mu\langle\mathcal{O}\rangle = 0$. Further more, again using translation invariance and conservation of the stress-energy tensor, one can show that the expectation value of the $T\overline{T}$ operator is a constant and factorizes into a product of one-point functions,

$$\langle\mathcal{O}_{T\overline{T}}\rangle = \langle T_{zz}\rangle\langle T_{\bar{z}\bar{z}}\rangle - \langle T_{z\bar{z}}\rangle\langle T_{\bar{z}z}\rangle. \tag{3.6}$$

Similarly, the expectation value factorizes for a general energy eigenstate $|n\rangle$

$$\langle n|\mathcal{O}_{T\overline{T}}|n\rangle = \langle n|T_{zz}|n\rangle\ \langle n|T_{\bar{z}\bar{z}}|n\rangle - \langle n|T_{z\bar{z}}|n\rangle\ \langle n|T_{\bar{z}z}|n\rangle. \tag{3.7}$$

Considering the cylinder $\mathbb{R}^2/\mathbb{Z}$, i.e. $x \sim x + R, t \in \mathbb{R}$, the expectation values of the components of the stress-energy tensor can be written in terms of the energy, momentum and radius

$$\langle n|T_{tt}|n\rangle = -\frac{E_n}{R}, \quad \langle n|T_{tx}|n\rangle = \frac{iP_n}{R}, \quad \langle n|T_{xx}|n\rangle = -\frac{\partial E_n}{\partial R}, \tag{3.8}$$

which implies, after performing the coordinate transformation $z = x + it$,

$$\langle n|\mathcal{O}_{T\overline{T}}|n\rangle = -\frac{1}{R}\left(E_n\frac{\partial E_n}{\partial R} + \frac{P_n^2}{R}\right). \tag{3.9}$$

To recap, we have computed the expectation value of the $T\overline{T}$ operator in a general eigenstate, and found that it is a constant and can be computed from the energy and momentum of the state. Since it is a constant (i.e. independent of position), but is constructed from dynamical operators, namely the stress-energy tensor, it suggests that this is a topological property of the QFT.

3.2 Deforming a QFT

Now, let us consider deforming a QFT with the $T\overline{T}$ operator. We can define a flow equation for the action [13],

$$\partial_\lambda S^{(\lambda)} = -\int_{\mathcal{M}} d^2 z \mathcal{O}^{(\lambda)}_{T\overline{T}}, \tag{3.10}$$

where λ here is the deformation parameter. Note that the stress-energy tensor and therefore the $T\overline{T}$ operator depends on the deformation parameter λ, and thus to find the deformed action, one needs to integrate over the flow instead of just adding the operator like a usual deformation. At leading order in small λ,

$$S(\lambda) \approx S(0) - \lambda \int_{\mathcal{M}} d^2 z \mathcal{O}_{T\overline{T}}(0) + O(\lambda^2),$$

where we see that λ has length dimension 2, since the $T\overline{T}$ operator is an irrelevant operator with mass dimension 4. The Euclidean partition function $Z = \int [\mathcal{D}\phi] e^{-S}$ therefore has the following logarithmic derivative

$$\partial_\lambda \ln Z^{(\lambda)} = \frac{1}{Z^{(\lambda)}} \int [\mathcal{D}\phi](-\partial_\lambda S) e^{-S} = \int d^2 z \, \langle \mathcal{O}_{T\overline{T}} \rangle = \mathrm{vol} \, \langle \mathcal{O}_{T\overline{T}} \rangle \,. \tag{3.11}$$

Since we showed in the previous section that the expectation value of the $T\overline{T}$ operator is a constant, the integral over the manifold is just the volume of the manifold vol. If we then consider the Euclidean torus at finite temperature, i.e. $x \sim x + R$, $\tau \sim \tau + \beta$, such that $Z = \mathrm{Tr}_{\mathcal{H}} \, e^{-\beta H}$, we have

$$\partial_\lambda \ln Z^{(\lambda)} = \frac{1}{Z^{(\lambda)}} \mathrm{Tr}_{\mathcal{H}}(-\beta \partial_\lambda H e^{-\beta H}) = -\beta \partial_\lambda E_n = \beta R \, \langle \mathcal{O}_{T\overline{T}} \rangle \,. \tag{3.12}$$

Using (3.12) and (3.9), we can write down the flow equation for the finite size spectrum,

$$\partial_\lambda E_n(R, \lambda) = E_n(R, \lambda) \frac{\partial E_n(R, \lambda)}{\partial R} + \frac{P_n(R)^2}{R}. \tag{3.13}$$

This differential equation arises in fluid dynamics and is known as the inviscid Burgers' equation [18]. We also find that the momenta P_n are unchanged by the deformation.

For a CFT on a cylinder, the R dependence in the energy and momenta is quite straightforward, and is given by,

$$E_n(R, 0) = \frac{1}{R}\left(h_n + \bar{h}_n - \frac{c}{12}\right), \quad P_n(R) = \frac{1}{R}(h_n - \bar{h}_n), \tag{3.14}$$

where $h_n, \bar{h}_n$ are the conformal dimensions of the state, and c is the central charge of the CFT. Solving the flow equation for the deformed spectrum with the CFT initial conditions, the solution for the deformed energy spectrum is

$$E_n(R, \lambda) = \frac{R}{\lambda}\left(\sqrt{1 + \frac{2\lambda E_n(R, 0)}{R} + \frac{\lambda^2 P_n(R)^2}{R^2}} - 1\right). \tag{3.15}$$

This spectrum also appears (up to a constant shift) in the unit winding sector of a closed string in a $c + 2$ dimensional target space, with string tension $\frac{1}{\lambda}$ [19]. Note that the spectrum is real for $\lambda > 0$, given $\lambda \leq \frac{6R^2}{c}$ for a unitary CFT, but shows Hagedorn behaviour in the UV limit [20]. This is a signature for non-locality of the deformed theory. For $\lambda < 0$, the spectrum is real for $E_n(R, 0) \leq -\frac{R}{2\lambda}\left(1 + \frac{\lambda^2 P_n(R)^2}{R^2}\right)$.

Let us define a dimensionless deformation parameter,

$$\mu = \frac{\lambda}{\pi R^2}, \tag{3.16}$$

where the π is for later convenience. One can expand (3.15) in small and large μ respectively to obtain,

$$\begin{aligned} E_n(R,\lambda) &= E_n - (E_n^2 - P_n^2)\frac{\pi R \mu}{2} + E_n(E_n^2 - P_n^2)\frac{\pi^2 R^2 \mu^2}{2} + \mathcal{O}\left(\mu^3\right), \\ E_n(R,\lambda) &= \text{sgn}(\mu)\left(P_n + \frac{E_n - P_n}{P_n \pi R \mu} + \frac{E_n^2 - P_n^2}{2P_n^3 \pi^2 R^2 \mu^2} + \mathcal{O}\left(\mu^{-3}\right)\right), \end{aligned} \tag{3.17}$$

where the right hand side contains only undeformed quantites. From here, we can see that the deformed spectrum reduces to the undeformed spectrum as expected when $\mu \to 0$, and the first order term is the determinant of the undeformed stress-energy tensor. Similarly, we see that momenta dominate the spectrum at large μ.

3.3 The deformed partition function

Let us consider the partition function of a CFT on a torus with modulus τ. The partition function is modular invariant, i.e. invariant under (2.20), and is a function only of the modular parameter, usually expressed in terms of the nome $q = e^{2\pi i \tau}$,

$$Z(\tau, \bar{\tau}) = \text{Tr}_{\mathcal{H}}\, q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} = \sum_n e^{2\pi i \tau_1 P_n R - 2\pi \tau_2 E_n R}. \tag{3.18}$$

Using the observation that the derivative of the deformed partition function,

$$\partial_\mu Z(\tau, \bar{\tau}|\mu) = \sum_n -2\pi\tau_2 R\big(\partial_\mu E_n(R,\mu)\big) e^{2\pi i \tau_1 P_n R - 2\pi\tau_2 E_n(R,\mu) R}, \tag{3.19}$$

one can derive the flow equation for the torus partition function by replacing by the derivative $\partial_\mu \to -2\pi\tau_2 E_n'$, and using (3.13) along with some algebra,

$$\left(1 - \tau_2 \partial_\tau \partial_{\bar{\tau}} - \frac{\mu}{2}\left(\partial_{\tau_2} - \frac{1}{\tau_2}\right)\right)\partial_\mu Z(\tau, \bar{\tau}|\mu) = 0. \tag{3.20}$$

Note that although a CFT partition function can be holomorphically factorized into characters, this property is lost for the deformed partition function due to the τ_2 dependence in the flow equation, which explicitly breaks holomorphicity. This shows that a chiral CFT can not be $T\overline{T}$ deformed.

One can find a solution to (3.20) in terms of an integral kernel,

$$Z(\tau, \bar{\tau}|\mu) = \frac{\tau_2}{\pi\mu}\int_{\mathbb{H}^2} \frac{d^2\sigma}{\sigma_2^2} e^{-\frac{|\tau-\sigma|^2}{\mu\sigma_2}} Z(\tau, \bar{\tau}). \tag{3.21}$$

This can be derived in many other ways as well, including coupling the CFT to a 2D dilaton gravity theory on the torus [14, 15], or by considering two worldsheet scalars with unit winding [21, 22, 23]. An important check is to perform the integral and recover the deformed energy spectrum. This is indeed possible, by noticing that the σ_1 integral is a Gaussian integral, and the σ_2 integral is an integral representation of the modified Bessel function of the second kind.

An important observation to make is that the deformed partition function is also modular invariant. Let us briefly define modular forms. A (holomorphic) modular form of weight k is a function that transforms in the following way under a modular transformation,

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^k f(\tau). \tag{3.22}$$

A non-holomorphic modular form of weight (k_1, k_2) can similarly be defined,

$$g\left(\frac{a\tau+b}{c\tau+d},\frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right)=(c\tau+d)^{k_1}(c\bar{\tau}+d)^{k_2}g(\tau,\bar{\tau}). \tag{3.23}$$

Note for example, τ_2 is a non-holomorphic form of weight $(-1,-1)$,

$$\tau_2 \to |c\tau+d|^{-2}\tau_2, \tag{3.24}$$

and similarly the dimensionless deformation parameter $\mu = \frac{\lambda}{\pi R^2}$ is also a non-holomorphic form of weight $(-1,-1)$, while λ is constant, since the circumference transforms as a non-holomorphic modular form of weight $(\frac{1}{2},\frac{1}{2})$, i.e. $R \to |c\tau+d|R$. Combining these results, we see that the $T\overline{T}$ deformed partition function (3.21) is modular invariant, given that the original partition function is modular invariant, which is true for CFTs. In fact, the $T\overline{T}$ deformation is the unique deformation which changes only the energy spectrum and preserves modular invariance of the original CFT partition function [24].

3.4 Holographic $T\overline{T}$ deformed CFTs

Here we will briefly review the derivation of the mixed boundary conditions derived in [25] using the variational principle. The $T\overline{T}$ deformation is a double-trace deformation of the boundary CFT, which according to the AdS/CFT dictionary should only change the asymptotic behaviour of the bulk fields, including the metric [26, 27].

The approach taken here will be to assume that the $T\overline{T}$ deformation is manifest as a transformation of the background metric, justified by the random geometry [17] and flat JT gravity [14, 15] proposals.

A variation of the metric in the action sources the stress-energy tensor, while the action is deformed by the $T\overline{T}$ operator. If we impose that the variation and deformation commute, we have

$$\delta(\partial_\lambda S)=\partial_\lambda(\delta S) \implies \delta(\sqrt{\gamma}\mathcal{O}_{T\overline{T}})=\partial_\lambda(\sqrt{\gamma}T_{ab}\delta^{ab}), \tag{3.25}$$

where γ is the "dynamical" background metric of the $T\overline{T}$ deformed CFT. Solving this set of differential equations we can compute the $T\overline{T}$ deformed metric and stress-energy tensor, purely in terms of undeformed quantities,

$$\begin{aligned} \partial_\lambda \gamma_{ab}(\lambda) &= \gamma_{ab} - 2\lambda \hat{T}_{ab} + \lambda^2 \hat{T}_{ac}\hat{T}_{bd}\gamma^{cd}, \\ \partial_\lambda T_{ab} &= \hat{T}_{ab} - \lambda \hat{T}_{ac}\hat{T}_{bd}\gamma^{cd}, \end{aligned} \tag{3.26}$$

where $\hat{T}_{ab} = T_{ab} - \gamma_{ab}T^c_c$ is the trace-reversed stress-energy tensor. Note that the deformed metric is quadratic in λ. Now consider an AdS_3 bulk geometry that is dual to this $T\overline{T}$ deformed CFT by imposing these boundary conditions. Comparing (3.26) with the Fefferman-Graham expansion for AdS_3 (2.6), and using the fact that the AdS/CFT dictionary tells us that the boundary stress-energy tensor in terms of the Fefferman-Graham expansion is given by

$$\hat{T}_{ab} = \frac{k}{2\pi} g^{(2)}_{ab}, \tag{3.27}$$

where $k = \frac{\ell}{4G_N}$, we have the deformed boundary metric and stress-energy tensor in terms of the terms in Fefferman-Graham expansion

$$\begin{aligned} \gamma_{ab}(\lambda) &= \ell^2 \left(g^{(0)}_{ab} - \left(2\lambda\frac{k}{2\pi}\right) g^{(2)}_{ab} + \left(2\lambda \frac{k}{2\pi}\right)^2 g^{(4)}_{ab} \right), \\ \hat{T}_{ab}(\lambda) &= \frac{k}{2\pi}\left(g^{(2)}_{ab} - \left(2\lambda\frac{k}{2\pi}\right) g^{(4)}_{ab}\right). \end{aligned} \tag{3.28}$$

An important observation to make is that this is exactly the Fefferman-Graham expansion for AdS_3, provided we make the identification $r_c = \sqrt{-\frac{\pi}{k\lambda}}$. Hence we can interpret this result in two ways:

1. $T\overline{T}$ deformed holographic CFTs are dual to AdS_3 with "mixed" boundary conditions given by (3.28) [25].
2. $T\overline{T}$ deformed holographic CFTs are dual to AdS_3 with Dirichlet boundary conditions, where the boundary is located at $r_c = \sqrt{-\frac{\pi}{k\lambda}}$ instead of infinity [28].

Note that the second interpretation can only work for pure AdS_3 gravity with no matter, since the Fefferman-Graham expansion only truncates when there are no matter fields in the bulk. Also, note that the second interpretation also requires $\lambda < 0$, which we have seen puts conditions on how large the energy can be, by computing the deformed spectrum (3.15). However, provided these two conditions are satisfied, both interpretations are equivalent. In fact, if one computes the Brown-York stress-energy tensor [29] with the appropriate counterterms for AdS_3 [4] on the hypersurface of constant $r = r_c$, we recover the deformed stress-energy tensor (3.28).

A third way to interpret the action of the $T\overline{T}$ deformation is by studying the solution to the metric flow equations (3.26). Since the deformed metric depends on the stress tensor, one can interpret this as the coordinates of the manifold being dependent on the fields in the QFT.

For example, if we start on the complex plane with $ds^2 = dzd\bar{z}$, the deformed metric is

$$ds^2 = (dz - 2\lambda\bar{L}(\bar{z})d\bar{z})(d\bar{z} - 2\lambda L(z)dz), \tag{3.29}$$

where L and $\bar{L}$ are the holomorphic and anti-holomorphic components of the CFT stress tensor, (and also the functions parameterizing the AdS_3 phase space in (2.13)). If we were to write this metric in explicitly flat coordinates $ds^2 = dZd\bar{Z}$, we have the relations

$$Z = z - 2\lambda \int \bar{L}(\bar{z})d\bar{z}, \quad \bar{Z} = \bar{z} - 2\lambda \int L(z)dz, \tag{3.30}$$

where it is clear that the dynamical coordinates are no longer chiral in terms of the undeformed coordinates. These dynamical coordinates can also be derived through the flat JT gravity proposal [14, 15].

One can find the asymptotic symmetry algebra which preserves the $T\overline{T}$-deformed boundary conditions (3.28), which corresponds to the symmetry algebra of a $T\overline{T}$-deformed holographic CFT. This has been worked out in [25, 30]. In the notation of [25], the asymptotic Killing vectors read

$$\begin{aligned} \xi_L &= \rho f'(z)\partial_\rho + \left(f(z) + \frac{\ell^2(\rho-\rho_c)\rho\bar{\mathcal{L}}f''(z)}{2(1-\rho^2\mathcal{L}\bar{\mathcal{L}})}\right)\partial_Z \\ &\quad - \left(\frac{\ell^2(\rho-\rho_c)\rho f''(z)}{2(1-\rho^2\mathcal{L}\bar{\mathcal{L}})} + \rho_c \int^{z(Z,\bar{Z})} \mathcal{L}(z')f'(z')dz'\right)\partial_{\bar{Z}}, \\ \xi_R &= \rho \bar{f}'(\bar{z})\partial_\rho + \left(\bar{f}(\bar{z}) + \frac{\ell^2(\rho-\rho_c)\rho\bar{\mathcal{L}}\bar{f}''(\bar{z})}{2(1-\rho^2\mathcal{L}\bar{\mathcal{L}})}\right)\partial_{\bar{Z}} \\ &\quad - \left(\frac{\ell^2(\rho-\rho_c)\rho \bar{f}''(\bar{z})}{2(1-\rho^2\mathcal{L}\bar{\mathcal{L}})} + \rho_c \int^{\bar{z}(Z,\bar{Z})} \bar{\mathcal{L}}(\bar{z}')\bar{f}'(\bar{z}')d\bar{z}'\right)\partial_Z. \end{aligned} \tag{3.31}$$

These asymptotic Killing vectors have been written in the dynamical coordinates defined in (3.30), the radial coordinate $\rho = r^2$, and the radial cutoff $\rho_c = r_c^2$. It is interesting to see how these boundary conditions fall into the class of boundary conditions of AdS_3 studied in [31]. If they do, which one expects since they study the most general asymptotic AdS_3 boundary conditions, one could interpret the $T\overline{T}$ deformed boundary conditions as a change of slicing of the state space, or a change of the state dependence of the free functions in the asymptotic Killing vectors.

3.5 Holographic $T\overline{T}$ deformed Warped CFTs

This subsection is a brief overview of Paper II [1].

The $T\overline{T}$ deformation only requires the QFT to be translationally invariant, not Lorentz invariant. To explore the deformation of a QFT which is not Lorentz invariant, we explore the deformation of a two-dimensional Warped Conformal Field Theory (WCFT). WCFTs are translationally invariant QFTs which have scale invariance in only a particular coordinate, for example the spatial coordinate. Choosing a specific coordinate breaks Lorentz (or rotational) invariance. However, there is still enough symmetry for a two-dimensional WCFT to possess a Virasoro$\times$U(1) Kac-Moody symmetry algebra.

On the complex plane, finite warped symmetry transformations take the form [32, 33]

$$z \to f(z), \quad \bar{z} \to \bar{z} + g(z). \tag{3.32}$$

It is clear from here that WCFTs do not couple to standard (pseudo-)Riemannian manifolds but rather to manifolds that are locally invariant under warped symmetry transformations (3.32).

Such manifolds are called "warped geometries" [34] and are a variant of Newton-Cartan geometries. Another strategy is to allow for the background manifold to transform under the warped symmetry transformations (3.32). A natural way to implement this is by employing the CSS [11] boundary conditions for AdS_3. These boundary conditions are "Dirichlet-Neumann" boundary conditions as opposed to the Dirichlet boundary conditions of Brown and Henneaux, cf. section 2.5. These boundary conditions allow for the transformations (3.32) by including the degree of freedom in translating the $\bar{z}$ coordinate as an unspecified holomorphic function in the boundary metric.

Let us write down the solution to the flow equations (3.26) with the CSS boundary conditions as the initial condition, which we work out in [1],

$$\gamma_{ij}(\lambda)dz^i dz^j = -\Big(d\bar{z} + (\lambda L(z) - P'(z))dz\Big)\Big(dz + \lambda\Delta(d\bar{z} - P'(z)dz)\Big). \tag{3.33}$$

We also must allow for translations along a new global Killing vector in the transverse coordinates by promoting it to an "asymptotic Killing vector",

$$\eta(\lambda; h) = -h(u,v)(\lambda\Delta\partial_u + \partial_v). \tag{3.34}$$

The dynamical coordinates analogous to (3.30) are readily computed

$$\begin{aligned}
du - \lambda\Delta\,\mathrm{d}(h(u,v)) &= dz + \lambda\Delta\,\mathrm{d}(\bar{z} - P(z))\,,\\
dv - \mathrm{d}(h(u,v)) &= d\bar{z} + (\lambda L(z) - P'(z))dz,\\
dz &= \frac{du - \lambda\Delta dv}{1 - \lambda^2\Delta L},\\
d\bar{z} &= \frac{dv - \lambda L du + (du + \lambda\Delta dv)P'(z)}{1 - \lambda^2\Delta L} - (\mathrm{d}(h(u,v))\,,
\end{aligned} \tag{3.35}$$

where d is the exterior derivative, and we have used (3.34) to introduce the degree of freedom h. However, noting that both h and P are arbitrary functions of u, v and correspond to the same degree of freedom in the bulk, we can make the gauge choice $h = P$.

And therefore we can also write the full deformed bulk metric

$$\begin{aligned}
ds^2 = \ell^2\frac{dr^2}{r^2} + \frac{\ell^2}{k^2r^2(1-\lambda^2\Delta L)^2}&\Big(\big(kr^2(\lambda^2\Delta L - 1)\partial_u h - (1 + \lambda k r^2)L\big)du\\
&+ \big(kr^2\partial_v h\left(\lambda^2\Delta L - 1\right) + \lambda\Delta L + kr^2\big)dv\Big)\times\\
&\Big(\big(\Delta\partial_u h\left(\lambda^2\Delta L - 1\right) - \lambda\Delta L - kr^2\big)du + \Delta\big(\partial_v h\left(\lambda^2\Delta L - 1\right) + \lambda k r^2 + 1\big)dv\Big).
\end{aligned} \tag{3.36}$$

Using the same techniques outlined in section 2.3, we can compute the flow in the phase space to obtain the symmetry algebra. We can first test the Neumann boundary conditions, using the asymptotic Killing vector (3.34), we have

$$\delta_{\eta(\lambda;\sigma)}L = 0, \quad \delta_{\eta(\lambda;\sigma)}h = \sigma. \tag{3.37}$$

This shows that we retain the $\mathrm{U}(1)$ Kac-Moody algebra.

When we impose Dirichlet boundary conditions on the deformed metric we obtain the flow along an asymptotic Killing vector ξ,

$$
\begin{aligned}
&\delta_\xi h = 0,\\
&\delta_\xi L = \frac{1}{2\Theta^2 h''}\Bigg(\Big(2\lambda k L' L_m^2 L_p(2\lambda\Delta h' - 1)h'' + 3kL_m^3 L_p^2 (h'')^2\Big)W'' - k\Theta L_m^3 L_p h'' W'''\\
&\qquad + X'\Big(- 2\lambda^2 k(L')^2 L_m^2 h''\big(1 - 3\lambda\Delta h' + 2\lambda^2\Delta^2 (h')^2\big) - 6kL_m^3 L_p^2 (h'')^3\\
&\qquad + h''\big(- \Theta L_m(\lambda k L''(1 - 2\lambda\Delta h') + 4\lambda L^2(1 - \lambda\Delta h')\\
&\qquad + L(-\lambda^3 k\Delta L'' - 2(2 - \lambda^4 k\Delta^2 L'')h')) + 6k\Theta L_m^3 L_p h'''\big)\\
&\qquad - L'\big(2\Theta^3 + \lambda k L_m^2(-7 + 8\lambda\Delta h' - \lambda^2\Delta L(1 - 8\lambda\Delta h'))(h'')^2\\
&\qquad + 2\lambda k\Theta L_m^2(1 - 2\lambda\Delta h')h'''\big) - k\Theta^2 L_m^3 h''''\Big)\\
&\qquad + W'\Big(L'\big(2\Theta^2 L_p + 2\lambda k L_m^2 L_p(1 - 2\lambda\Delta h')h''' + k\Theta L_m^3 L_p h''''\big) - 3kL_m^3 L_p^2 h'' h'''\Big)\Bigg),
\end{aligned}
\tag{3.38}
$$

where primes denote derivatives with respect to u and

$$
\begin{aligned}
\Theta &= \partial_u h - \lambda L(1 - \lambda\Delta\partial_u h),\\
L_m &= 1 - \lambda^2\Delta L, \quad L_p = 1 + \lambda^2\Delta L.
\end{aligned}
\tag{3.39}
$$

This results in a non-linearly deformed non-chiral Virasoro algebra, which can be checked by taking the deformation parameter to zero carefully. Thus, we see that momenta and spin one currents remain untouched by the $T\overline{T}$ deformation, but the energy spectrum is deformed and the corresponding deformed Virasoro algebra loses chirality.

This result strengthens and extends the proposals of [28, 25, 30] to the case of bottom-up holography where the boundary theory is not a conformal field theory, but instead a non-relativistic theory.

4 The Selberg zeta function and quasi-normal modes

4.1 Properties of zeta functions

Let us start this chapter by asking the question, "*what is a zeta function?*" This is not a question with a definite answer, but mathematicians can list some general properties a zeta function ought to have (see [35] for a more thorough discussion).

1. A zeta function may be expressed as a Dirichlet series,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}. \tag{4.1}$$

2. A zeta function may be expressed as product over "primes," also known as an Euler product

$$\zeta(s) = \prod_{p \text{ “prime”}} \frac{1}{1 - p^{-s}}. \tag{4.2}$$

3. A zeta function can be analytically continued to the complex plane, and may satisfy a functional equation on the complex plane.
4. The location of the zeroes of a zeta function may encode some interesting properties.

The archetype of a zeta function is the Riemann zeta function, for which all of the above properties hold. The Riemann zeta function can be defined as the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{4.3}$$

which converges for $\operatorname{Re} s > 1$. The Euler product form is

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}, \tag{4.4}$$

which can be derived from (4.3) in a straightforward manner. Riemann's work was to show that the zeta function satisfied the following functional equation,

$$\Lambda(s) = \Lambda(1-s), \quad \Lambda(s) \equiv \frac{1}{\pi^{\frac{s}{2}}} \Gamma\left(\frac{s}{2}\right) \zeta(s), \tag{4.5}$$

with which one can show that the analytic continuation of the zeta function can be expressed as an integral

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx. \tag{4.6}$$

Thus, one can show that the Riemann zeta function is a meromorphic function with a simple pole at $s = 1$ where the series diverges, which also implies that there are infinitely many primes.

The famous Riemann hypothesis is that all the non-trivial zeroes of the Riemann zeta function lie on the line $\operatorname{Re} s = \frac{1}{2}$, which encodes the distribution of prime numbers.

4.2 The Selberg zeta function

The Selberg zeta function is a geometric cousin of the Riemann zeta function, where the primes are "prime geodesics" on a quotient manifold. Selberg was interested in the spectra of Laplacians on discrete quotients Γ of Riemann surfaces. He derived a trace formula to study characters of unitary representations of a Lie group G on $L^2(G/\Gamma)$, and defined a zeta function constructed by the eigenvalues of the Laplacian, which now bears his name [36].

For the Riemann surface $\mathbb{H}^2/\Gamma$, where Γ is a compact discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$, Selberg's zeta function for $\operatorname{Re} s > 1$ of the Euler product form is given by,

$$Z_{\mathbb{H}^2/\Gamma}(s) = \prod_p \prod_{k=1}^{\infty} (1 - e^{-\ell(p)(s+k)}). \tag{4.7}$$

Here p denotes the set of "primitive" or prime closed geodesics in $\mathbb{H}^2/\Gamma$, which are the analogues to the prime numbers in the Riemann zeta function. A closed geodesic is a curve such that the beginning and end of the curve coincide in the fundamental domain of the quotient. A primitive geodesic is a geodesic which is closed due to the quotient, and only goes around the quotient once, just like a prime number only has 1 and itself as its factors. The function $\ell(p)$ denotes the length of the prime geodesic. The Selberg trace formula then relates the spectrum of the Laplacian on $\mathbb{H}^2/\Gamma$ to the set of lengths $\ell(p)$.

We can think about the product over prime geodesics as a product over the conjugacy classes of primitive hyperbolic elements of Γ. A hyperbolic element γ of $\mathrm{SL}_2(\mathbb{R})$ is an element whose fixed points are in $\mathbb{R} \cup \{\infty\}$, while a primitive element $\gamma_\star$ is the unique element such that $\gamma^n = \gamma_\star$ implies $n = \pm 1$. One can then show that a primitive hyperbolic element is one that generates its centralizer in Γ. In other words, there is a bijective map between primitive hyperbolic conjugacy classes and prime geodesics on $\mathbb{H}^2/\Gamma$.

In addition to having an Euler product form, the Selberg zeta function also has the other properties which makes it a zeta function. There exists an analytic continuation to the complex plane as a meromorphic function that satisfies a functional equation similar to that of the Riemann zeta function,

$$Z_{\mathbb{H}^2/\Gamma}(1-s) = e^{-\chi(\mathbb{H}^2/\Gamma)\int_0^{s-\frac{1}{2}} \pi x \tan(\pi x) dx} Z_{\mathbb{H}^2/\Gamma}(s), \tag{4.8}$$

where χ is the Euler characteristic. Furthermore, the non-trivial zeroes of the Selberg zeta function correspond to the discrete spectrum of the Laplacian on $\mathbb{H}^2/\Gamma$, which satisfies the analogue of the Riemann hypothesis for $\mathbb{H}^2/\Gamma$ compact. Using the Selberg zeta function and trace formula, one can prove the "prime geodesic theorem," which shows that the density of prime geodesics has a similar form to the density of prime numbers.

The connection of the Selberg zeta function to physics is through the relationship to the spectrum of the Laplacian. Schematically, the Selberg trace formula allows one to equate the Selberg zeta function to the determinant of the Laplacian

$$Z_{\mathbb{H}^2/\Gamma} \propto \det\big(\Delta_{\mathbb{H}^2/\Gamma}\big)^{\frac{1}{2}}. \tag{4.9}$$

For a more rigorous statement and proof, see [37].

The natural question to ask is if one can generalize this to higher dimensions, and other quotient manifolds. The higher dimensional generalization for $\mathbb{H}^N/\Gamma$ was carried out by Patterson [38], where Γ is a Kleinian group and N is an arbitrary positive integer > 2. A Kleinian group is simply a discrete subgroup of the orientation preserving isometry group of $\mathbb{H}^N$. Patterson defined his generalization of the Selberg zeta function as

$$Z_{\mathbb{H}^N/\Gamma}(s) = \prod_{k_1,\ldots,k_{N-1}\geq 0}\left[1-\left(\prod_{i=1}^{N-1}\alpha_i^{k_i}\right)e^{-\left(s+\sum_{i=1}^{N-1}k_i\right)\ell(p)}\right], \tag{4.10}$$

where α_i's are the eigenvalues of the group element of the $\mathrm{O}(N)$ subgroup of Γ, and $\ell(p)$ is the length of the single prime geodesic.

4.3 Relationship with the BTZ black hole

Let us consider the BTZ black hole, which is indeed the quotinet of a higher dimensional hyperbolic manifold by a Kleinian group.

To see how this works, let us take a closer look at the BTZ black hole as a quotient manifold, introduced in section 2.4.

Starting with the Euclidean Poincaré patch coordinates for $\mathbb{H}^3$

$$ds^2 = \frac{\ell^2}{z^2}(dx^2+dy^2+dz^2), \tag{4.11}$$

we can recover the exterior ($r > r_+$) of the spinning Euclidean BTZ black hole through the coordinate transformation

$$\begin{aligned} x &= \sqrt{\frac{r^2-r_+^2}{r^2+|r_-|^2}}\cos\left(\frac{r_+\tau}{\ell^2}+\frac{|r_-|\phi}{\ell}\right)\exp\left(\frac{r_+\phi}{\ell}-\frac{|r_-|\tau}{\ell^2}\right),\\ y &= \sqrt{\frac{r^2-r_+^2}{r^2+|r_-|^2}}\sin\left(\frac{r_+\tau}{\ell^2}+\frac{|r_-|\phi}{\ell}\right)\exp\left(\frac{r_+\phi}{\ell}-\frac{|r_-|\tau}{\ell^2}\right),\\ z &= \sqrt{\frac{r_+^2+|r_-|^2}{r^2+|r_-|^2}}\exp\left(\frac{r_+\phi}{\ell}-\frac{|r_-|\tau}{\ell^2}\right). \end{aligned} \tag{4.12}$$

The resulting metric for the BTZ black hole is

$$ds^2 = \frac{(r^2-r_+^2)(r^2+|r_-|^2)}{\ell^2 r^2}d\tau^2 + \frac{\ell^2 r^2}{(r^2-r_+^2)(r^2+|r_-|^2)}dr^2 + r^2\left(d\phi - \frac{r_+|r_-|}{\ell r^2}d\tau\right). \tag{4.13}$$

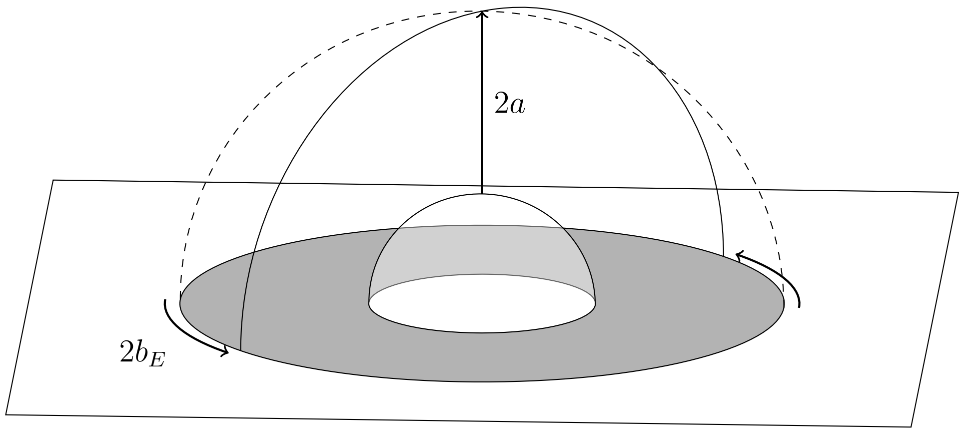

Figure 4.1. The fundamental domain of $\mathbb{H}^3/\mathbb{Z}$ is the region between the inner and outer hemispheres, which are identified. The outer hemisphere is rotated by an angle $2b_E$. The plane drawn is the $z = 0$ plane corresponding to the boundary.

In order for the metric to be regular everywhere, we need to make $\phi \sim \phi + 2\pi n$, with $n \in \mathbb{Z}$. This identification is generated by the Kleinian group Γ. To identify this group, let us study the action of an element of this group $\gamma_n \in \Gamma$ on the coordinates of $\mathbb{H}^3$,

$$\gamma_n \cdot (x, y_E, z) = (x', y'_E, z'). \tag{4.14}$$

Explicitly working out the action of $\phi \sim \phi + 2\pi n$ using (4.12), we can see

$$\gamma_n \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} e^{2a} & 0 & 0 \\ 0 & e^{2a} & 0 \\ 0 & 0 & e^{2a} \end{pmatrix}^n \begin{pmatrix} \cos 2b_E & -\sin 2b_E & 0 \\ \sin 2b_E & \cos 2b_E & 0 \\ 0 & 0 & 1 \end{pmatrix}^n \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{4.15}$$

where $a = \pi r_+/\ell$ and $b_E = \pi |r_-|/\ell$. We can now write down the quotient group Γ

$$\Gamma = \left\{ \gamma_n = \gamma^n \middle| \gamma = \begin{pmatrix} e^{2a} & 0 & 0 \\ 0 & e^{2a} & 0 \\ 0 & 0 & e^{2a} \end{pmatrix} \begin{pmatrix} \cos 2b_E & -\sin 2b_E & 0 \\ \sin 2b_E & \cos 2b_E & 0 \\ 0 & 0 & 1 \end{pmatrix}, n \in \mathbb{Z} \right\}, \tag{4.16}$$

where the $n = 1$ element, γ, is the primitive element. One can show that this consists of only hyperbolic elements. It is clear that $\Gamma \sim \mathbb{Z}$, so this quotient is often denoted as $\mathbb{H}^3/\mathbb{Z}$.

Another way to obtain the quotient group is by exponentiating the generators which generate the quotient. As we saw earlier in section 2.4, the generator $2\pi\partial_\phi$ (2.24) produced the translation $\phi \to \phi + 2\pi$, and thus the group Γ can also be written as repeated 2π translations along ϕ,

$$\Gamma = \left\{ \gamma_n = \gamma^n \middle| \gamma = e^{2\pi\partial_\phi}, n \in \mathbb{Z} \right\}. \tag{4.17}$$

We can choose the fundamental domain of this quotient to be the region (see figure 4.1)

$$\mathcal{F}_{BTZ} = \left\{ (x, y, z) \in \mathbb{H}^3 \middle| 1 \leq \sqrt{x^2 + y^2 + z^2} \leq e^{2a} \right\}. \tag{4.18}$$

The prime geodesic is the geodesic at $x = y = 0$, labeled in the figure 4.1 with length $2a$, since the rest of the closed geodesics will not trace over themselves.

To write down the Patterson-Selberg zeta function, we have to identify the eigenvalues of the $\mathrm{O}(3)$ rotation part of the quotient, and the length of the prime geodesic corresponding to the primitive element. We found that the length of the prime geodesic $\ell(p) = 2a$, and the $\mathrm{O}(3)$ matrix has eigenvalues $e^{\pm 2ib_E}$, as is clear from (4.16). We can therefore write the (Patterson-)Selberg zeta function for the BTZ black hole from (4.10) [39, 40],

$$Z_\Gamma(s) = \prod_{k_1,k_2=0}^{\infty} \left[1 - e^{2ib_E k_1} e^{-2ib_E k_2} e^{-2a(k_1+k_2+s)}\right]. \tag{4.19}$$

It was noticed in [39] that the Selberg zeta function for the BTZ black hole can be used to reproduced many physical quantities of the black hole, including quantum corrections to the effective action [41], free energy, and entropy [42, 43].

This relationship was made more precise in [44], where the connection between two procedures of computing the 1-loop scalar partition function, i.e. the heat kernel method [41] and the "quasi-normal mode method" [45]. The 1-loop partition function for a scalar field of mass m is given by the functional determinant of the corresponding kinetic operator,

$$Z^{(1)}(m) = \left(\det\left(-\nabla^2 + m^2\right)\right)^{-\frac{1}{2}}. \tag{4.20}$$

Using the heat kernel method, one solves the relevant heat equation, and compute the logarithm of the functional determinant, i.e. the 1-loop effective action, up to an infinite volume factor,

$$\begin{aligned} (\partial_t + \nabla_x^2)K(t;x,y) = 0, \quad K(0;x,y) = \delta(x,y), \\ -\frac{1}{2}\log\det\left(-\nabla^2 + m^2\right) = \frac{1}{2}\int_0^\infty \frac{dt}{t}\int d^3x\sqrt{g}K(t;x,y), \end{aligned} \tag{4.21}$$

where t is the "heat" parameter. In [45] it was shown that the 1-loop partition function for bosonic fields in black hole backgrounds and other fixed temperature spacetimes can be written as a product over the zeroes and poles in the scaling dimension of the scalar field Δ, up to an overall factor of an entire function $e^{Q(\Delta)}$, as a result of Weierstrass factorization,

$$Z^{(1)}(\Delta) = e^{Q(\Delta)} \frac{\prod_{\Delta_0}(\Delta - \Delta_0)^{d_0}}{\prod_{\Delta_p}(\Delta - \Delta_p)^{d_p}}, \tag{4.22}$$

where d_0, d_p are the degeneracies of the zeroes Δ_0 and poles Δ_p respectively. For the scalar in particular, there are no zeroes, but the poles occur on solutions to the equation of motion $-\nabla^2 + \Delta_p(\Delta_p - d) = 0$ for $d+1$ space-time dimensions. This occurs not only for when the field is on-shell but when the quasi-normal modes $\omega_\star$ in the Lorentzian signature coincide with the Matsubara (thermal) frequencies in the Euclidean signature [45].

Quasi-Normal modes are oscillatory solutions to the scalar equations of motion which are damped as they fall into the black hole event horizon, and vanish at infinity. These occur only at specific values of the oscillation frequency, which are collectively called the quasi-normal mode spectrum. Matsubara frequencies are also specific values of frequencies of a scalar field such that it is well behaved (i.e. single valued) along the compact euclidean time direction. Hence they are also referred to as thermal frequencies.

Both approaches yield the (regulated) 1-loop scalar partition function,

$$Z^{(1)} = \prod_{k_1,k_2=0}^{\infty} \frac{1}{1 - q^{k_1+\Delta/2}\bar{q}^{k_2+\Delta/2}}, \tag{4.23}$$

where $q = e^{2\pi i\tau}$, τ being the modulus of the boundary torus. Recall (2.24), for the boundary torus of the BTZ black hole, $\tau_1 = -|r_-|/\ell$ and $\tau_2 = r_+/\ell$. Substituting τ for $a = \pi r_+/\ell$ and $b_E = \pi|r_-|\ell$, we can write the 1-loop scalar partition function as

$$Z^{(1)} = \prod_{k_1,k_2=0}^{\infty} \left(1 - e^{2ib_E k_1} e^{-2ib_E k_2} e^{-2a(k_1+k_2+\Delta)}\right)^{-1}, \tag{4.24}$$

which, on comparing to the Selberg zeta function for the BTZ black hole (4.19), we see the remarkable result

$$Z^{(1)} = \frac{1}{Z_{\mathbb{H}^3/\Gamma}(s)}\bigg|_{s=\Delta} = \det\left(-\nabla^2 + m^2\right)^{-\frac{1}{2}}. \tag{4.25}$$

Therefore the zeroes of the Selberg zeta function on $\mathbb{H}^3/\Gamma$ correspond to the poles of the 1-loop scalar partition function. This means that for $Z_{\mathbb{H}^3/\Gamma}(s_\star) = 0$, there is a condition on the quasi-normal mode spectrum. In fact, the condition is simply

$$s_\star = \Delta \iff \omega_\star = \omega_n, \tag{4.26}$$

where Δ is the scaling dimension of the scalar, $\omega_\star$ is a quasi-normal mode frequency and ω_n is a Matsubara frequency. This relationship was worked out in [44].

4.4 Generalizing the Selberg zeta function

In Paper III [2], we propose a generalized Selberg-like zeta function which may be used to describe functional determinants on quotients of manifolds which are not necessarily $\mathbb{H}^3$. The resulting orbifold should contain only one prime geodesic, corresponding to the analog of the primitive hyperbolic element of the analogous Kleinian group. The definition uses highest weight scalar representations (scalar primaries) of the isometry group, and takes the infinite product over their descendants.

$$\zeta(s)_{\mathcal{M}/\Gamma}(s) = \prod_{\text{descendants}} \langle 1 - \gamma \rangle_{\text{scalar primary of weight } s} \tag{4.27}$$

Let us reproduce the BTZ result here. The isometry group of AdS_3 is $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$, and fields on a manifold must be representations of the global isometry group of the manifold. Representations of $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})$ are labeled by the conformal dimensions h and $\bar{h}$, as their L_0 eigenvalues,

$$L_0 \left|h, \bar{h}\right\rangle = h \left|h, \bar{h}\right\rangle, \quad \bar{L}_0 \left|h, \bar{h}\right\rangle = \bar{h} \left|h, \bar{h}\right\rangle. \tag{4.28}$$

A highest weight representation, or a primary state is annihilated by the lowering operators $L_1, \bar{L}_1$

$$L_1 \left|h, \bar{h}\right\rangle = 0, \quad \bar{L}_1 \left|h, \bar{h}\right\rangle = 0, \tag{4.29}$$

and their descendants are generated by the action of raising operators $L_{-1}, \bar{L}_{-1}$

$$\left(L_{-1}\right)^{k_2} \left|h, \bar{h}\right\rangle = \left|h, \bar{h}, k_2\right\rangle, \quad \left(\bar{L}_{-1}\right)^{k_1} \left|h, \bar{h}\right\rangle = \left|h, \bar{h}, k_1\right\rangle, \tag{4.30}$$

with $(k_1, k_2) \in \mathbb{Z}_{\geq 0}$.

In terms of the $\mathrm{SL}_2(\mathbb{R})$ generators and the modulus for the boundary torus $\tau = \tau_1 + i\tau_2$ we have the group element γ in terms of the generator of the quotient (2.24):

$$\gamma = e^{2\pi\partial_\phi} = e^{2\pi i((L_0-\bar{L}_0)\tau_1+(L_0+\bar{L}_0)i\tau_2)} = q^{L_0}\bar{q}^{\bar{L}_0}, \tag{4.31}$$

where $q = e^{2\pi i\tau}$. Now let us consider a primary scalar ($h = \bar{h}$) field of scaling dimension $\Delta = h + \bar{h} = 2h = 2\bar{h}$ and its descendants, the eigenvalues of the $\mathrm{SL}_2(\mathbb{R})$ generators will be

$$\begin{aligned} L_0\left|h,\bar{h},k_1,k_2\right\rangle &= (h+k_2)\left|h,\bar{h},k_1,k_2\right\rangle = \left(\frac{\Delta}{2}+k_2\right)\left|h,\bar{h},k_1,k_2\right\rangle \\ \bar{L}_0\left|h,\bar{h},k_1,k_2\right\rangle &= \left(\bar{h}+k_1\right)\left|h,\bar{h},k_1,k_2\right\rangle = \left(\frac{\Delta}{2}+k_1\right)\left|h,\bar{h},k_1,k_2\right\rangle. \end{aligned} \tag{4.32}$$

Plugging (4.32) into the zeta function prescription (4.27), we recover the Selberg zeta function for the BTZ black hole reported in [39, 40], after identifying Δ with the parameter s:

$$\begin{aligned} \zeta_{\mathbb{H}^3/\mathbb{Z}}(s) &= \prod_{k_1,k_2=0}^{\infty}\left(1-e^{2\pi i((k_1-k_2)\tau_1+(k_1+k_2+s)i\tau_2)}\right) \\ &= \prod_{k_1,k_2=0}^{\infty}\left(1-e^{2ib_E(k_1-k_2)-2a(k_1+k_2+s)}\right) \\ &= Z_{\mathbb{H}^3/\mathbb{Z}}(s). \end{aligned} \tag{4.33}$$

4.5 Non-hyperbolic examples

4.5.1 Flat space cosmologies

We explore quotients of three-dimensional flat space $\mathbb{R}^{2,1}$ to form spacetimes known as a Flat Space Cosmology (FSC). The isometry group is generated by the Killing vectors

$$\begin{gathered} \mathfrak{L}_{\pm1} = Y(-\partial_X \mp \partial_T) + (X \mp T)\partial_Y, \quad \mathfrak{L}_0 = (X\partial_T + T\partial_X), \\ \mathfrak{M}_{\pm1} = G(\partial_T \pm \partial_X), \quad \mathfrak{M}_0 = G\partial_Y. \end{gathered} \tag{4.34}$$

These obey the BMS_3 algebra without central extension

$$[\mathfrak{L}_m,\mathfrak{L}_n] = (m-n)\mathfrak{L}_{m+n}, \quad [\mathfrak{L}_m,\mathfrak{M}_n] = (m-n)\mathfrak{M}_{m+n}, \quad [\mathfrak{M}_m,\mathfrak{M}_n] = 0. \tag{4.35}$$

The quotient is taken with respect to the group element

$$\gamma = e^{2\pi\partial_\phi} = e^{2\pi i(\mathfrak{L}_0\eta + i\mathfrak{M}_0\rho)}, \tag{4.36}$$

where η and ρ are the modular parameters of the flat space quotient.

Since fields in flat space must be representations of the global symmetry group, we can label a field with their mass m and spin j:

$$\mathfrak{M}_0\left|m,j\right\rangle = m\left|m,j\right\rangle, \quad \mathfrak{L}_0\left|m,j\right\rangle = j\left|m,j\right\rangle. \tag{4.37}$$

A primary scalar has spin $j = 0$.

We can use (4.27) to obtain the zeta function for FSC, after identifying the mass m with the complex parameter s:

$$\zeta_{\mathbb{R}^3/\mathbb{Z}}(s) = \prod_{j=0}^{\infty} \left(1 - e^{2\pi i(\eta j + i\rho s)}\right), \tag{4.38}$$

where $\eta = \tilde{r}_+$ and $\rho = -r_0/G$ are the physical parameters in the FSC.

The test is whether this reproduces the known result for the functional determinant of the scalar field kinetic operator on quotients of flat space. In fact, it does. The 1-loop scalar partition function on flat space was initially calculated in [46],

$$\begin{aligned} Z^{\text{1-loop}}_{\text{flat, scalar}}(m) &= \exp\left(\sum_{n=1}^{\infty} \frac{e^{-2\pi\rho m n}}{n|(1-e^{2\pi i\eta n})|^2}\right) \\ &= \exp\left(\sum_{n=1}^{\infty}\sum_{j=0}^{\infty} \frac{1}{n}(e^{-2\pi\rho m} e^{2\pi i\eta j})^n\right) \\ &= \prod_{j=0}^{\infty} \frac{1}{1-e^{2\pi i(\eta j + i m\rho)}}. \end{aligned} \tag{4.39}$$

Therefore, we see that the generalized Selberg zeta function constructed for $\mathbb{R}^3/\mathbb{Z}$ reproduces the regularized scalar 1-loop partition function:

$$\zeta_{\mathbb{R}^3/\mathbb{Z}}(s) = (\det \nabla^2_{\text{flat, scalar}})^{\frac{1}{2}} = \prod_{j=0}^{\infty}\left(1 - e^{2\pi i(is\rho + j\eta)}\right) = \left(Z^{\text{1-loop}}_{\text{flat, scalar}}(m)\right)^{-1}\Bigg|_{m=s}. \tag{4.40}$$

4.5.2 Warped AdS quotients

Let us briefly introduce warped AdS. It is a warped Hopf fibration of AdS_2, and is not a solution to pure Einstein gravity, but to topological massive gravity in three dimensions. Spacelike warped AdS_3 has the metric [47]:

$$ds^2 = \frac{L^2}{\nu^2+3}\left(-\cosh^2\sigma d\tau^2 + d\sigma^2 + \frac{4\nu^2}{\nu^2+3}(du + \sinh\sigma d\tau)^2\right), \tag{4.41}$$

where ν is the warping parameter. Setting ν to 1 recovers AdS_3. For $\nu^2 > 1$ we have spacelike *stretched* warped AdS_3 and for $\nu^2 < 1$ it is called *squashed*. The Killing vectors for spacelike warped AdS_3 are

$$\begin{aligned} J_2 &= 2\partial_u \\ \tilde{J}_1 &= 2\sin\tau\tanh\sigma\partial_\tau - 2\cos\tau\partial_\sigma + 2\sin\tau\,\mathrm{sech}\,\sigma\partial_u \\ \tilde{J}_2 &= -2\cos\tau\tanh\sigma\partial_\tau - 2\sin\tau\partial_\sigma - 2\cos\tau\,\mathrm{sech}\,\sigma\partial_u \\ \tilde{J}_0 &= 2\partial_\tau. \end{aligned} \tag{4.42}$$

The tilded vectors form an $SL_2(\mathbb{R})$ algebra, while J_2 is a $U(1)$ generator. Spacelike stretched warped AdS black holes are quotients of spacelike stretched warped AdS_3 [48], along the Killing vector

$$\partial_{\tilde{\phi}} = \pi L(T_L J_2 - T_R \tilde{J}_2), \tag{4.43}$$

where

$$T_L = \frac{\nu^2+3}{8\pi L}\left(\rho_+ + \rho_- - \frac{\sqrt{(\nu^2+3)\rho_+\rho_-}}{\nu}\right), \qquad T_R = \frac{\nu^2+3}{8\pi L}(\rho_+ - \rho_-). \tag{4.44}$$

The metric of the black hole is thus given by

$$\begin{aligned} \frac{ds^2}{L^2} &= d\tilde{t}^2 + \frac{d\rho^2}{(\nu^2+3)(\rho-\rho_+)(\rho-\rho_-)} + \left(2\nu\rho - \sqrt{\rho_+\rho_-(\nu^2+3)}\right) d\tilde{t}d\tilde{\phi} \\ &\quad + \frac{\rho}{4}\left(3(\nu^2-1)\rho + (\nu^2+3)(\rho_+ + \rho_-) - 4\nu\sqrt{\rho_+\rho_-(\nu^2+3)}\right) d\tilde{\phi}^2, \end{aligned} \tag{4.45}$$

where $\phi \sim \phi + 2\pi$ due to the quotient, and $\rho_\pm$ are the radii of the inner and outer event horizons.

We compute the Selberg zeta function for this quotient in Paper I. We recover the form of the original Selberg zeta function for the BTZ black hole, but modified for the Lorentzian signature, where the $\mathrm{O}(3)$ rotation becomes a $\mathrm{SO}(2,1)$ boost,

$$Z_{\mathrm{WAdS}_3/\Gamma}(s) = \prod_{k_1,k_2=0}^{\infty}\left[1 - e^{2bk_1}e^{-2bk_2}e^{-2a(k_1+k_2+s)}\right]. \tag{4.46}$$

We find the values of the parameters a and b to be

$$\begin{aligned} a &= \frac{\pi r_+(\nu^2+3)(2\nu r_+ - r_-\sqrt{\nu^2+3})}{8L\nu(r_+ - r_-)} \\ b &= \frac{\pi r_-(\nu^2+3)(-2\nu r_- + r_+\sqrt{\nu^2+3})}{8L\nu(r_+ - r_-)}. \end{aligned} \tag{4.47}$$

where

$$\rho_\pm = \frac{r_\pm^2}{r_+ - r_-}. \tag{4.48}$$

To confirm that this is the correct Selberg zeta function, we compute the thermal frequencies and correctly reproduce the quasi-normal mode frequency spectrum. Thus we have ample supporting evidence for our the construction of a generalized Selberg-like zeta function (4.27) for non-hyperbolic orbifolds.

5 Conclusions

In this thesis we have explored of two main ideas, the generalization of $T\overline{T}$ deformation to non-relativistic holographic field theories and the construction of a generalized Selberg zeta function for non-hyperbolic quotient manifolds. We do this in the spirit of pushing our tools in theoretical and mathematical physics into areas with less symmetry to find the limits of these tools.

The $T\overline{T}$ deformation is a very interesting deformation which has been widely discussed in the recent literature. It is an irrelevant deformation of two-dimensional quantum field theories which is well behaved in the UV and preserves integrability, unlike most other irrelevant deformations. Some features of a $T\overline{T}$-deformed theory are that it is a non-local theory, it preserves the global symmetries of the undeformed theory, has a gravitationally dressed scattering matrix and the finite size energy spectrum can be worked out in closed form [12, 13, 16]. It shares many features with bosonic string theory, such as a Hagedorn behaviour, similar energy spectrum and a two-dimensional gravity interpretation [14, 15, 22].

The $T\overline{T}$ deformation of a theory with a bulk holographic dual can be interpreted in more than one way. Since it is a double-trace deformation, one can work out the deformed boundary conditions of the bulk. The resulting boundary conditions can be interpreted as mixed boundary conditions [25] or as Dirichlet boundary conditions at a finite radial distance [28] when the deformation parameter is negative.

We explored the $T\overline{T}$ deformation of holographic warped CFTs which are dual to AdS_3 with CSS boundary conditions [1]. We compute the deformed boundary conditions and obtain the "dynamical" coordinates of the $T\overline{T}$ deformed WCFT. Using the deformed boundary conditions we also obtain the metric of the bulk AdS_3 dual, with which we can compute the asymptotic symmetry algebra. The AdS/CFT dictionary then tells us that the asymptotic symmetry algebra is the symmetry algebra of the QFT on the boundary, which in this case is the $T\overline{T}$ deformed WCFT.

We find that many of the features we see in the $T\overline{T}$ deformations of holographic CFTs carry over to the WCFT case. We find that the $\mathrm{U}(1)$ Kac-Moody algebra is undeformed by the deformation, which is consistent since it is generated by spin 1 currents. We also find that the deformed Virasoro algebra loses chirality, just as one sees in the case of a $T\overline{T}$ deformed holographic CFT. We compute the deformed Virasoro algebra and see that it is deformed in a highly non-linear fashion.

In addition to looking at $T\overline{T}$ deformations, we explore the construction of zeta functions of three dimensional orbifolds. Just like the Riemann zeta function which is an Euler product over prime numbers, the Selberg zeta function is expressed as an Euler product over prime geodesics. The Selberg zeta function was originally constructed for discrete hyperbolic quotients of $\mathbb{H}^2$ [36]. It was later generalized for quotients of $\mathbb{H}^N$ with Kleinian groups by [38], which was used by [39, 40] to construct the zeta function for the BTZ black hole.

We generalize the construction of the Patterson-Selberg zeta function to quotients of non-hyperbolic manifolds, such as Warped AdS and three dimensional flat space [2]. We propose a construction of a zeta function using an Euler product over scalar representations of the global symmetry algebra. We verify this by two methods. In the flat space quotient, we reproduce the functional determinant of the scalar Laplacian which was originally computed using the heat kernel on flat space [46]. In both the flat space and Warped AdS quotients, we reproduce the quasi-normal modes on these spacetimes [3] by computing the zeroes of the zeta function, and imposing the condition that the zeroes of the zeta function correspond to the appropriate scaling dimension to reproduce the quasi-normal mode spectrum due to the relationship of the zeta function with the functional determinant.

For future directions to the $T\overline{T}$ program, it will be very interesting to develop methods to study $T\overline{T}$ deformed correlation functions, entanglement and Rényi entropies, and other related quantities non-perturbatively, using techniques involving, holography, the flow equation for the partition function, and the coupling to dilaton gravity. These quantities have been worked out perturbatively already, however it is important to understand them non-perturbatively because of the drastic change of behaviour in the UV caused by the deformation, such as introducing non-locality.

There are also other deformations which fall into the same class as the $T\overline{T}$ deformation, such as the $T\overline{J}$, $J\overline{T}$, and $J\overline{J}$ deformations, all of which are built using conserved currents in the theory. It will be interesting to extend the work done here to the rest of these deformations. In particular, it was shown in [49] that a $J\overline{T}$ deformed holographic CFT has a Virasoro$\times\mathrm{U}(1)$ Kac-Moody algebra, similar to the WCFTs studied here.

The Selberg zeta program also has many interesting future directions. In particular, the immediate follow up is to prove the conjectured construction of the zeta function we provide, and find where the construction could fail. Another interesting follow up would be to provide a construction for quotient groups with multiple primitive hyperbolic elements, i.e. orbifolds with multiple prime geodesics. This could lead to constructions which provide higher order corrections in the genus expansion of a partition function.

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Publications

Paper I

A Selberg zeta function for warped AdS_3 black holes

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A Selberg zeta function for warped AdS_3 black holes

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Abstract: The Selberg zeta function and trace formula are powerful tools used to calculate kinetic operator spectra and quasinormal modes on hyperbolic quotient spacetimes. In this article, we extend this formalism to non-hyperbolic quotients by constructing a Selberg zeta function for warped AdS_3 black holes. We also consider the so-called self-dual solutions, which are of interest in connection to near-horizon extremal Kerr. We establish a map between the zeta function zeroes and the quasinormal modes on warped AdS_3 black hole backgrounds. In the process, we use a method involving conformal coordinates and the symmetry structure of the scalar Laplacian to construct a warped version of the hyperbolic half-space metric, which to our knowledge is new and may have interesting applications of its own, which we describe. We end by discussing several future directions for this work, such as computing 1-loop determinants (which govern quantum corrections) on the quotient spacetimes we consider, as well as adapting the formalism presented here to more generic orbifolds.

Keywords: Black Holes, Differential and Algebraic Geometry, Space-Time Symmetries

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Contents

1 Introduction

By studying only the quotient structure of certain spacetimes, it is possible to learn much about dynamics and quantum corrections on those backgrounds. A principal tool to accomplish this is the Selberg zeta function, a cousin of the Riemann zeta function in which prime numbers are replaced by prime geodesics on a hyperbolic spacetime $\mathbb{H}^n/\Gamma$, where Γ is a discrete subgroup of SL(2,$\mathbb{R}$) [1–3]. For example, for $\mathbb{H}^2/\Gamma$ the Selberg zeta function is of the form

$$Z_\Gamma(s) = \prod_p \prod_{n=0}^{\infty} \left(1 - N(p)^{-s-n}\right), \tag{1.1}$$

where the product p is over the conjugacy classes of prime geodesics, and $N(p)$ is a function of geodesic length [4]. The Selberg zeta function and trace formula are of significant physical interest as a tool to compute spectra of kinetic operators (and thus quantum corrections) on hyperbolic quotient manifolds. A brief list of applications includes quantum chaos [5, 6], quantum JT gravity [7], torsion and topological invariants [8–10], heat kernels and regularized one-loop determinants [11–15], quantum corrections to black hole entropy [16, 17], quasinormal modes [15, 18], and even band theory [19].

So far, the Selberg zeta function formalism has been limited to hyperbolic quotient spacetimes. A notable example is the Euclidean BTZ black hole [20], which is a quotient of hyperbolic 3-space $\mathbb{H}^3$ by the discrete subgroup $\Gamma \sim \mathbb{Z}$ [21]. Perry and Williams [3] constructed a Selberg zeta function for the BTZ black hole from this quotient structure, and proved a corresponding trace formula. In that work, the action of the quotienting group Γ is studied through a set of "conformal coordinates" that transform the BTZ metric in Boyer-Lindquist-like coordinates to the Poincaré patch. It was discussed in [3, 18] that the zeroes of the Selberg zeta function $\{s_*|Z_\Gamma(s_*) = 0\}$ can be mapped to the BTZ quasinormal modes. This was worked out explicitly in [15], where it was shown that tuning the Selberg zeroes s_* to the conformal dimension of the field in question was equivalent to equating the field's quasinormal modes to the thermal (Matsubara) frequencies:

$$s_* = \Delta \quad \leftrightarrow \quad \omega_{QN} = \omega_n. \tag{1.2}$$

This analysis was extended to fields of general spin in [22].

In this work, we construct a Selberg zeta function for warped AdS_3 black holes, extending the work of [3] beyond hyperbolic quotients for the first time. Warped AdS_3 black holes are quotients of warped AdS_3 [23], with quotienting group $\Gamma \sim \mathbb{Z}$ realized by the discrete identification $\phi \to \phi + 2\pi n$, with n an integer. The isometry group is $\mathrm{SL}(2,\mathbb{R})\times\mathrm{U}(1)$. Warped AdS_3 geometries arise in many different contexts in theoretical physics, such as topological massive gravity (see [23] and references therein), lower-spin gravity [24] (in the context of warped black holes, see [25]), and in near-horizon extremal [26] and near-region non-extremal [27] Kerr. Thus it is very desirable to calculate quantum corrections on these backgrounds, and it is very likely that our formalism will help facilitate this calculation. This is addressed further in the discussion section.

Like [3], we build the Selberg zeta function using the quotient structure of warped AdS_3 black holes. However, unlike [3], we do not have a straightforward way to analyze the group action under $\phi \to \phi + 2\pi n$, because we do not have a warped version of the Poincaré patch as a target metric with which to construct conformal coordinates. We solve this problem in section 3, where we use a trick from the hidden conformal symmetry program to exploit the symmetries of the scalar Laplacian to construct a set of conformal coordinates. Namely, we build conformal coordinates such that the $\mathrm{SL}(2,\mathbb{R})\times\mathrm{U}(1)$ quadratic Casimir $\mathcal{H}^2 + \lambda H_0^2 \propto \nabla^2$. Using the resulting conformal coordinate transformation, we are able to derive a warped version of the upper half-space metric that to our knowledge is new. This metric might have interesting applications of its own, such as the study of $T\overline{T}$ deformations of warped conformal field theories. We give more details about this is the in the discussion section.

We also consider the self-dual solutions reported in [23] and further studied in [28–30]. These are the warped analogue of the self-dual solutions for AdS_3 presented in [31], and as explained there these solutions are interesting to study in their own right. They are formed by a different quotient, $\tilde{\theta} \to \tilde{\theta} + 2\pi\tilde{\alpha}n$, which we describe.

This article is organised as follows. In section 2 we review previous results that will be helpful for the reader. In 2.1 we largely follow [3] in demonstrating how to construct the Selberg zeta function for the case of the BTZ black hole, ending with mapping the Selberg zeroes to the BTZ quasinormal modes as done in [15]. In 2.2 we review the quotient structure of warped AdS_3 black holes, largely following [23]. In section 3 we exploit the symmetries of the Klein-Gordon operator to build a set of conformal coordinates. These allow us to create both a warped Poincaré patch metric as well as the warped Selberg zeta function. In section 4 it is necessary to calculate several quantities on warped quotient backgrounds (namely the conformal weight Δ, the thermal frequencies ω_n and the quasinormal mode frequencies ω_*) before constructing the map between the Selberg zeroes and quasinormal mode frequencies in section 5. In section 6 we summarize our results and discuss several future directions.

2 Review

Here we review material that we will need throughout the rest of this work. We begin by reviewing how, in the case of the BTZ black hole, quotient structure can be used to construct a Selberg zeta function whose zeroes are mapped to the quasinormal modes on the BTZ background. We then review the warped AdS_3 black hole geometry and its quotient structure. In section 3 we put these two elements together and create a Selberg zeta function for warped AdS_3 black holes.

2.1 Constructing the Selberg zeta function for $\mathbb{H}^3/\Gamma$

We begin with the BTZ black hole metric in Boyer-Lindquist-like coordinates

$$ds^2 = -\frac{(r^2-r_+^2)(r^2-r_-^2)}{L^2r^2}dt^2 + \frac{L^2r^2}{(r^2-r_+^2)(r^2-r_-^2)}dr^2 + r^2\left(d\phi - \frac{r_+r_-}{Lr^2}dt\right)^2 \tag{2.1}$$

where L is the AdS radius, and the outer and inner horizons r_+ and r_- are related to the black hole's mass M and angular momentum J by $r_+^2 + r_-^2 = L^2M$ and $r_+r_- = JL/2$. The Euclidean[1] BTZ black hole is obtained by $t \to -i\tau$, $J \to -iJ_E$ and $r_- \to -i|r_-|$.

It is well-known [21] that the BTZ black hole metric can be mapped to the Poincaré patch metric through a set of discontinuous coordinate transformations, valid in regions $r > r_+$, $r_+ > r > r_-$ and $r_- > r$. For concreteness we will focus on the coordinate

[1]The authors of [3] work in Euclidean signature because the existence of well-defined Selberg zeta functions has been shown specifically for hyperbolic quotients. When we adapt this formalism to warped AdS_3 black holes in section 3, we take the liberty of working in Lorentzian signature.

transformation valid for $r > r_+$:

$$\begin{aligned} x &= \sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}} \cos\left(\frac{r_+\tau}{L^2} + \frac{|r_-|\phi}{L}\right) \exp\left(\frac{r_+\phi}{L} - \frac{|r_-|\tau}{L^2}\right) \\ y_E &= \sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}} \sin\left(\frac{r_+\tau}{L^2} + \frac{|r_-|\phi}{L}\right) \exp\left(\frac{r_+\phi}{L} - \frac{|r_-|\tau}{L^2}\right) \\ z &= \sqrt{\frac{r_+^2 - r_-^2}{r^2 - r_-^2}} \exp\left(\frac{r_+\phi}{L} - \frac{|r_-|\tau}{L^2}\right). \end{aligned} \tag{2.2}$$

Under the coordinate transformation (2.2), the Euclidean version of the BTZ metric (2.1) becomes

$$ds^2 = \frac{L^2}{z^2}(dx^2 + dy_E^2 + dz^2). \tag{2.3}$$

We will refer to coordinates like (x, y_E, z) in (2.2) as conformal coordinates, and we will now see that they provide an avenue to analyze the quotient structure of $\mathbb{H}^3/\Gamma$ through a group theoretic lens.

The identification $\phi \sim \phi + 2\pi$ allows for the BTZ black hole to be understood as a quotient of AdS_3 by a discrete subgroup $\Gamma \sim \mathbb{Z}$ of the isometry group SL(2,$\mathbb{R}$)×SL(2,$\mathbb{R}$). We can study the group action of the single generator $\gamma \in \Gamma$ by taking $\phi \to \phi + 2\pi n$ in (2.2). This will map a point $(x, y_E, z) \in \mathbb{H}^3$ to another point (x', y_E', z')

$$\gamma^n \cdot (x, y_E, z) = (x', y_E', z') \tag{2.4}$$

through

$$\begin{aligned} x' &= e^{2\pi n r_+/L}(x \cos(2\pi n|r_-|/L) - y_E \sin(2\pi n|r_-|/L)) \\ y_E' &= e^{2\pi n r_+/L}(y_E \cos(2\pi n|r_-|/L) + x \sin(2\pi n|r_-|/L)) \\ z' &= e^{2\pi n r_+/L} z. \end{aligned} \tag{2.5}$$

By inspecting (2.5), it is clear that the group action can be understood as a dilation and a rotation in $\mathbb{R}^2$:

$$\gamma \begin{pmatrix} x \\ y_E \\ z \end{pmatrix} = \begin{pmatrix} e^{2a} & 0 & 0 \\ 0 & e^{2a} & 0 \\ 0 & 0 & e^{2a} \end{pmatrix} \begin{pmatrix} \cos 2b_E & -\sin 2b_E & 0 \\ \sin 2b_E & \cos 2b_E & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y_E \\ z \end{pmatrix}, \tag{2.6}$$

where $a = \pi r_+/L$ and $b_E = \pi|r_-|/L$.

We can recast this in the language of [21], in which they write the generator $\gamma \in \Gamma$ as the Killing vector

$$\partial_\phi = \frac{r_+}{L} J_{12} + \frac{|r_-|}{L} J_{03}. \tag{2.7}$$

Here J_{12} and J_{03} are two of the six generators of the SL(2,$\mathbb{R}$)×SL(2,$\mathbb{R}$) isometry group, and they are presented in terms of $\mathbb{H}^3$ embedding coordinates in appendix A. In Poincaré coordinates (2.3), it is evident that these generate dilations $J_{12} = -(x\partial_x + y_E\partial_{y_E} + z\partial_z)$ and rotations $J_{03} = -y_E\partial_x + x\partial_{y_E}$. In terms of the parameters a and b_E, this generator is

$$2\pi\partial_\phi = 2aJ_{12} + 2b_E J_{03}. \tag{2.8}$$

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From the terms a and b_E that parametrize the dilation and rotation, Perry and Williams [3] constructed the Selberg zeta function for the BTZ black hole

$$Z_\Gamma(s) = \prod_{k_1,k_2=0}^{\infty} \left[1 - e^{2ib_E k_1} e^{-2ib_E k_2} e^{-2a(k_1+k_2+s)}\right] . \tag{2.9}$$

In later sections of this article, it will be convenient to consider a Lorentzian version of the zeta function (2.9), which we develop now. To connect to the Lorentzian case we write $y_E \to iy$, $|r_-| \to ir_-$ and $b_E \to -ib$. The group action now consists of a boost in $\mathbb{R}^{1,1}$ and a dilation

$$\gamma \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} e^{2a} & 0 & 0 \\ 0 & e^{2a} & 0 \\ 0 & 0 & e^{2a} \end{pmatrix} \begin{pmatrix} \cosh 2b & -\sinh 2b & 0 \\ -\sinh 2b & \cosh 2b & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{2.10}$$

and the Selberg zeta function is the same as (2.9) but with $ib_E = b$

$$Z_\Gamma(s) = \prod_{k_1,k_2=0}^{\infty} \left[1 - e^{2bk_1} e^{-2bk_2} e^{-2a(k_1+k_2+s)}\right] . \tag{2.11}$$

The exponential part is real in the Lorentzian case because the "rotation" part of the quotient is no longer compact.

The zeroes of the zeta function (2.11) occur when the exponent is equal to $\pm 2\pi ik$ for $k \in \mathbb{Z}$. The sign of k differs in the left and right quasinormal modes, hence we have $+$ for the right modes and $-$ for the left modes.

$$s_* = -(k_1 + k_2) + \frac{b}{a}(k_1 - k_2) \pm \frac{i\pi k}{a}. \tag{2.12}$$

Performing a change of basis, the integers k_1 and k_2 are written in terms of the radial quantum number j and the thermal integer n in the following way [3]

$$\begin{aligned} n \geq 0: &\quad k_1 + k_2 = 2j + n \quad k_1 - k_2 = \mp n, \\ n < 0: &\quad k_1 + k_2 = 2j - n \quad k_1 - k_2 = \mp n. \end{aligned} \tag{2.13}$$

For $n \geq 0$ the zeroes are

$$s_* = -(2j + n) - \frac{bn}{a} - \frac{i\pi k}{a} \qquad s_* = -(2j + n) + \frac{bn}{a} + \frac{i\pi k}{a} \tag{2.14}$$

and for $n < 0$

$$s_* = -(2j - n) - \frac{bn}{a} - \frac{i\pi k}{a} \qquad s_* = -(2j - n) + \frac{bn}{a} + \frac{i\pi k}{a}. \tag{2.15}$$

2.2 Warped AdS_3 quotients

In order to describe warped AdS_3 black holes as quotients, we begin by reviewing the global spacetime warped AdS_3 itself. Non-warped AdS_3 can be expressed as a Hopf fibration of AdS_2 like so:

$$ds^2 = \frac{L^2}{4}(-\cosh^2 \sigma d\tau^2 + d\sigma^2 + (du + \sinh \sigma d\tau)^2), \tag{2.16}$$

where u is the fibered coordinate and we refer to (τ, u, σ) as global fibered coordinates. Spacelike[2] warped AdS_3 is obtained by warping the fiber length [32]:

$$ds^2 = \frac{L^2}{\nu^2+3}\left(-\cosh^2\sigma d\tau^2 + d\sigma^2 + \frac{4\nu^2}{\nu^2+3}(du + \sinh\sigma d\tau)^2\right). \tag{2.17}$$

When $\nu^2 > 1$ the spacetime is called *stretched* and for $\nu^2 < 1$ it is called *squashed*. The Killing vectors for spacelike warped AdS_3 are

$$\begin{aligned} J_2 &= 2\partial_u \\ \tilde{J}_1 &= 2\sin\tau\tanh\sigma\partial_\tau - 2\cos\tau\partial_\sigma + 2\sin\tau\,\mathrm{sech}\,\sigma\partial_u \\ \tilde{J}_2 &= -2\cos\tau\tanh\sigma\partial_\tau - 2\sin\tau\partial_\sigma - 2\cos\tau\,\mathrm{sech}\,\sigma\partial_u \\ \tilde{J}_0 &= 2\partial_\tau. \end{aligned} \tag{2.18}$$

The tilded vectors form an SL(2,$\mathbb{R}$) algebra, while J_2 is a U(1) generator.

2.2.1 Spacelike stretched warped AdS_3 black holes

We first consider spacelike stretched warped AdS_3 solutions, which are black holes [23]. In direct analogy to the BTZ black hole, one can take a quotient of warped AdS_3 to find such black hole solutions. The metric for the spacelike warped AdS_3 black hole is

$$\begin{aligned} \frac{ds^2}{L^2} &= d\tilde{t}^2 + \frac{d\rho^2}{(\nu^2+3)(\rho-\rho_+)(\rho-\rho_-)} + \left(2\nu\rho - \sqrt{\rho_+\rho_-(\nu^2+3)}\right)d\tilde{t}d\tilde{\phi} \\ &\quad + \frac{\rho}{4}\left(3(\nu^2-1)\rho + (\nu^2+3)(\rho_+ + \rho_-) - 4\nu\sqrt{\rho_+\rho_-(\nu^2+3)}\right)d\tilde{\phi}^2, \end{aligned} \tag{2.19}$$

where $\rho \in [0,\infty)$, $\tilde{t} \in \mathbb{R}$, $\tilde{\phi} \in [0, 2\pi)$, and $\tilde{\phi} \sim \tilde{\phi} + 2\pi$. The coordinate transformation from (2.17) to (2.19) is given in (A.6). For $\nu^2 < 1$ this spacetime has closed timelike curves for large r, but not for $\nu^2 > 1$. It is important to note that this metric is not asymptotically spacelike warped AdS_3 [33]. This can be seen by taking the large ρ limit of the metric (2.19):

$$\frac{ds^2}{L^2} = d\tilde{t}^2 + \frac{d\rho^2}{(\nu^2+3)\rho^2} + 2\nu\rho\, d\tilde{t}d\tilde{\phi} + \frac{3}{4}(\nu^2-1)\rho^2 d\tilde{\phi}^2. \tag{2.20}$$

The angular coordinate $\tilde{\phi}$ is compact in the asymptotic black hole spacetime (2.20), but it is not in unquotiented spacelike warped AdS_3 (see [33] for more details).

For the spacelike stretched warped AdS_3 black hole, the quotient is along the vector:

$$\partial_{\tilde{\phi}} = \pi L(T_L J_2 - T_R \tilde{J}_2), \tag{2.21}$$

where

$$T_L = \frac{\nu^2+3}{8\pi L}\left(\rho_+ + \rho_- - \frac{\sqrt{(\nu^2+3)\rho_+\rho_-}}{\nu}\right), \qquad T_R = \frac{\nu^2+3}{8\pi L}(\rho_+ - \rho_-). \tag{2.22}$$

[2] For information on timelike and null analogues, see [23].

For $\nu = 1$, this reduces to the BTZ black hole in a rotated frame $(\tilde{t}, \rho, \tilde{\phi})$. In order for quantities like T_L and T_R to reduce to their more recognizable BTZ values, it will be convenient to go to the non-rotated frame (t, r, ϕ):

$$\tilde{t} = \frac{r_+ - r_-}{L^2} t, \quad \tilde{\phi} = \frac{L\phi - t}{L^2}, \quad \rho = \frac{r^2}{r_+ - r_-}, \quad \rho_\pm = \frac{r_\pm^2}{r_+ - r_-}. \tag{2.23}$$

The metric in these coordinates is

$$\begin{aligned} ds^2 = &\frac{1}{4L^2(r_+-r_-)^2}\Big(3r^4(\nu^2-1)+r^2\Big((r_+^2+r_-^2)(\nu^2-8\nu+3)-4r_+r_-\nu(\sqrt{\nu^2+3}-4)\Big) \\ &+4(r_+-r_-)^2\Big(r_+^2+r_-^2+r_+r_-(\sqrt{\nu^2+3}-2)\Big)\Big)dt^2+\frac{4L^2r^2}{(r^2-r_+^2)(r^2-r_-^2)(\nu^2+3)}dr^2 \\ &-\frac{1}{2L(r_+-r_-)^2}\Big(3r^4(\nu^2-1)+r^2\Big((r_+^2+r_-^2)(\nu^2-4\nu+3)-4r_+r_-\nu\Big(\sqrt{\nu^2+3}-2\Big)\Big) \\ &+2r_+r_-(r_+-r_-)^2\sqrt{\nu^2+3}\Big)dtd\phi+\frac{r^2\left(3r^2(\nu^2-1)-4r_+r_-\nu\sqrt{\nu^2+3}+(r_+^2+r_-^2)(\nu^2+3)\right)}{4(r_+-r_-)^2}d\phi^2. \end{aligned} \tag{2.24}$$

This reduces exactly to the BTZ metric (2.1) when $\nu = 1$. The left and right temperatures in these BTZ-like coordinates are

$$T_L = \frac{\nu^2+3}{8\pi L}\left(\frac{\nu(r_+^2+r_-^2) - r_+r_-\sqrt{\nu^2+3}}{\nu(r_+-r_-)}\right), \qquad T_R = \frac{\nu^2+3}{8\pi L}(r_+ + r_-). \tag{2.25}$$

These reduce to the BTZ left and right temperatures when $\nu = 1$

$$T_L = \frac{r_+ - r_-}{2\pi L}, \quad T_R = \frac{r_+ + r_-}{2\pi L}. \tag{2.26}$$

2.2.2 Warped self-dual solutions

As previously mentioned, spacelike warped AdS_3 for $\nu^2 < 1$ has closed timelike curves (CTCs) for large r. Taking a different quotient from (2.21) by identifying along the J_2 isometry results in no such curves [23] and we have the so-called self-dual solutions that are a discrete quotient of warped AdS_3 in analogous to the quotients studied in [31]. Self-dual solutions are not black holes in the strictest sense but they can be regarded as such because they have killing horizons in the $(\tilde{t}, \rho, \tilde{\phi})$ coordinates and no CTCs. They also have similar thermodynamic behaviours, like an entropy that obeys the Cardy formula, as discussed in [23, 28]. Self-dual solutions for warped dS_3 black holes are studied in [34].

For the self-dual solutions mentioned in [23] the same analysis can be done as for the warped solutions that we will study but with the identification in $\tilde{t}$ instead of $\tilde{\phi}$. We instead adopt a different coordinate system, and look at the self-dual solutions studied in [28] with identification $\tilde{\theta} \sim \tilde{\theta} + 2\pi\tilde{\alpha}$. The metric is given by

$$ds^2 = \frac{L^2}{\nu^2+3}\left(-(\tilde{\rho}-\tilde{\rho}_+)(\tilde{\rho}-\tilde{\rho}_-)d\tilde{\tau}^2 + \frac{d\tilde{\rho}^2}{(\tilde{\rho}-\tilde{\rho}_+)(\tilde{\rho}-\tilde{\rho}_-)} + \frac{4\nu^2}{\nu^2+3}\left(\tilde{\alpha}d\tilde{\theta} + \frac{2\tilde{\rho}-\tilde{\rho}_+-\tilde{\rho}_-}{2}d\tilde{\tau}\right)^2\right). \tag{2.27}$$

It will be convenient to redefine the radial coordinate to a BTZ-like radial coordinate $\tilde{r}$:

$$\tilde{\rho} = \frac{\tilde{r}^2}{\tilde{r}_+ - \tilde{r}_-}, \quad \tilde{\rho}_\pm = \frac{\tilde{r}_\pm^2}{\tilde{r}_+ - \tilde{r}_-}. \tag{2.28}$$

In these coordinates, the line element of the self-dual warped solution is

$$ds^2 = \frac{L^2}{\nu^2+3}\left(-\frac{(\tilde{r}^2-\tilde{r}_-^2)(\tilde{r}^2-\tilde{r}_+^2)\,d\tilde{\tau}^2}{(\tilde{r}_- - \tilde{r}_+)^2} + \frac{4\tilde{r}^2 d\tilde{r}^2}{(\tilde{r}^2-\tilde{r}_-^2)(\tilde{r}^2-\tilde{r}_+^2)} + \frac{\nu^2}{\nu^2+3}\left(2\tilde{\alpha}d\tilde{\theta} + \frac{-2\tilde{r}^2+\tilde{r}_-^2+\tilde{r}_+^2}{\tilde{r}_- - \tilde{r}_+}d\tilde{\tau}\right)^2\right). \tag{2.29}$$

The left and right temperatures in these coordinates are

$$T_L = \frac{\tilde{\alpha}}{2\pi L}, \qquad T_R = \frac{\tilde{r}_+ + \tilde{r}_-}{4\pi L}. \tag{2.30}$$

The authors of [28] comment that in the extremal limit the right temperature T_R vanishes. The reason that this is not apparent in (2.30) is that the coordinate transformation (2.28) is singular in the extremal limit, and thus (2.30) holds only in the non-extremal case.

3 A Selberg zeta function for warped AdS$_3$ black holes

Our first step in constructing the Selberg zeta function for warped AdS$_3$ black holes is to identify a suitable set of "conformal coordinates" (analogous to equation (2.2) for the BTZ black hole) that will allow us to interpret the identification $\phi \sim \phi + 2\pi$ group-theoretically. It is difficult to find a suitable coordinate transformation from the perspective of the metric, since in the warped case we do not have a target Poincaré-like metric in mind. That is, for the case of the spacelike warped AdS$_3$ black hole, we seek a coordinate transformation between (2.24) and a warped version of (2.3) (the latter of which we do not have). However, we will now see that we can make progress by borrowing a trick from the hidden conformal symmetry program of [27] and others, by studying instead the symmetry structure of the scalar wave equation. We outline our general approach below before moving on to the specific examples of spacelike warped AdS$_3$ black holes in section 3.1 and the self-dual solutions in section 3.2.

Consider again the Poincaré patch metric of AdS$_3$:

$$ds^2 = \frac{L^2}{z^2}(dw^+dw^- + dz^2). \tag{3.1}$$

The isometry group of (3.1), SL(2, $\mathbb{R}$)× SL(2, $\mathbb{R}$), is generated by six Killing vectors:

$$\begin{aligned} &H_1 = i\partial_+, \quad H_0 = i\left(w^+\partial_+ + \frac{1}{2}z\partial_z\right), \quad H_{-1} = i((w^+)^2\partial_+ + w^+z\partial_z - z^2\partial_-), \\ &\bar{H}_1 = i\partial_-, \quad \bar{H}_0 = i\left(w^-\partial_- + \frac{1}{2}z\partial_z\right), \quad \bar{H}_{-1} = i((w^-)^2\partial_- + w^-z\partial_z - z^2\partial_+). \end{aligned} \tag{3.2}$$

For the case of warped AdS$_3$, the warp parameter $\nu \neq 1$ in (2.17) breaks the unbarred SL(2,$\mathbb{R}$) symmetry down to a U(1), and thus the Killing vectors of warped AdS$_3$ are $(H_0, \bar{H}_0, \bar{H}_{\pm 1})$. Our goal in this section is to construct a locally spacelike warped AdS$_3$ metric in terms of these coordinates, which will be our "warped Poincaré patch."

Consider for now a massless scalar field Φ on one of our backgrounds of interest (either the spacelike warped AdS_3 black hole or the self-dual solution). In either case, one can show that the solutions of the Klein-Gordon equation are hypergeometric functions (as we will see for the massive scalar case in section 4). Since hypergeometric functions transform in representations of SL(2,$\mathbb{R}$), we conclude that the Klein-Gordon operator is related to the SL(2,$\mathbb{R}$) quadratic Casimir:

$$\begin{aligned}\mathcal{H}^2 &= -\bar{H}_0^2 + \frac{1}{2}(\bar{H}_1\bar{H}_{-1} + \bar{H}_{-1}\bar{H}_1)\\ &= \frac{1}{4}(z^2\partial_z^2 - z\partial_z) + z^2\partial_+\partial_- \,.\end{aligned} \tag{3.3}$$

In particular, we would like for the Klein-Gordon operator to be proportional to the entire SL(2,$\mathbb{R}$)×U(1) Casimir:

$$(\mathcal{H}^2 + \lambda H_0^2)\Phi \propto \nabla^2\Phi \tag{3.4}$$

where the coefficient λ of the U(1) generator is yet to be determined.

Our job now is to find a coordinate transformation

$$\begin{aligned} w^+ &= \sqrt{\frac{x-\frac{1}{2}}{x+\frac{1}{2}}}e^{\alpha\phi+\beta t},\\ w^- &= \sqrt{\frac{x-\frac{1}{2}}{x+\frac{1}{2}}}e^{\gamma\phi+\delta t},\\ z &= \sqrt{\frac{1}{x+\frac{1}{2}}}e^{1/2((\alpha+\gamma)\phi+(\beta+\delta)t)},\end{aligned} \tag{3.5}$$

such that (3.4) holds. In (3.5), x is related to the BTZ-like radial coordinate r in (2.24) via $x = \frac{r^2-1/2(r_+^2+r_-^2)}{r_+^2-r_-^2}$. For the field ansatz

$$\Phi = R(x)e^{i(k\phi-\omega t)}, \tag{3.6}$$

we find that the quadratic Casimir constructed from the conformal coordinates (3.5) and generators (3.2) is

$$\begin{aligned}&(\mathcal{H}^2+\lambda H_0^2)\Phi \\ &= \left(\partial_x\left(x^2-\frac{1}{4}\right)\partial_x + \frac{(\omega(\alpha+\gamma)+k(\beta+\delta))^2}{4\left(x-\frac{1}{2}\right)(\beta\gamma-\alpha\delta)^2} - \frac{(\omega(\alpha-\gamma)+k(\beta-\delta))^2}{4\left(x+\frac{1}{2}\right)(\beta\gamma-\alpha\delta)^2} + \lambda\frac{(k\delta+\gamma\omega)^2}{(\beta\gamma-\alpha\delta)^2}\right)\Phi.\end{aligned} \tag{3.7}$$

Comparing this Casimir to the Klein-Gordon operator will allow us to solve for the conformal coordinate parameters $(\alpha,\beta,\gamma,\delta)$. With those in hand, we will be able to interpret the identification $\phi \to \phi + 2\pi$ group-theoretically, in analogy to [3], and build a Selberg zeta function for warped black holes. As a nice bonus, we find a warped version of the Poincaré patch metric (3.1), which reduces to (3.1) when $\nu = 1$.

Before we move on to specific examples in the next subsections, some comments about (3.7) are in order. Hidden conformal symmetry was studied in warped AdS_3 black holes in [35] and in self-dual warped solutions in [30]. In both cases, the authors throw away a term in the Klein-Gordon operator, making a "near-region limit" or "soft hair" argument, in analogy to [27]. We show below that the terms they throw away are nothing more than the U(1) piece corresponding to our λ term in (3.7). We argue that there is no need to take such a limit to discard such terms, as they too are built into the symmetry structure of the Casimir.

3.1 Warped AdS_3 black holes

The Klein-Gordon operator of the warped AdS_3 black hole background (2.24) is proportional to the quadratic Casimir (3.7) with proportionality constant:

$$\mathcal{H}^2 + \lambda H_0^2 = \frac{L^2}{\nu^2+3}\nabla^2. \tag{3.8}$$

This proportionality constant is determined by comparing the radial derivative pieces.

In x coordinates, the Klein-Gordon operator with the proportionality constant now reads:

$$\frac{L^2}{3+\nu^2}\nabla^2\Phi = \left(\partial_x\left(x^2-\frac{1}{4}\right)\partial_x + \frac{P}{4\left(x-\frac{1}{2}\right)} + \frac{Q}{4\left(x+\frac{1}{2}\right)} + S\right)\Phi \tag{3.9}$$

where

$$\begin{aligned}
P &= \frac{4L^2\left(k\left(2(\nu-1)r_+^2 - 2r_-^2 - r_+r_-\left(\sqrt{\nu^2+3}-4\right)\right) - L\omega r_+\left(2\nu r_+ - r_-\sqrt{\nu^2+3}\right)\right)^2}{(r_+-r_-)^2(r_++r_-)^2(\nu^2+3)^2},\\
Q &= -\frac{4L^2\left(k\left(-2r_+^2 + 2(\nu-1)r_-^2 - r_+r_-(\sqrt{\nu^2+3}-4)\right) - L\omega r_-\left(2\nu r_- - r_+\sqrt{\nu^2+3}\right)\right)^2}{(r_+-r_-)^2(r_++r_-)^2(\nu^2+3)^2},\\
S &= \frac{3L^2(\nu^2-1)(k-L\omega)^2}{(\nu^2+3)^2(r_+^2-r_-^2)^2}.
\end{aligned} \tag{3.10}$$

As mentioned previously, in [35] these coefficients are derived in the $(\tilde{t},\rho,\tilde{\phi})$ coordinates of (2.19), but the U(1) term, S, is thrown away in the spirit of the hidden conformal symmetry program in a certain limit of their eigenvalue ω. There is no need to throw away this piece, as it is part of the symmetry group and does not cause any obstruction in building the conformal coordinates. In fact, the U(1) piece will help us choose the proper solution branch, as we will see.

Equating the coefficients of k and ω in P and Q to those in the $x \pm \frac{1}{2}$ poles in the Casimir (3.7), one can solve for $(\alpha,\beta,\gamma,\delta)$. Due to the squares, there are 4 branches of solutions, and we pick the ones which reduce to those found for BTZ when ν is set to 1 as reported in appendix B.[3]

[3]Had we not known what they reduce to, we can use the U(1) term to eliminate 2 of the four branches, and then we pick the positive branch for α.

They are:

$$\begin{aligned}
\alpha &= \frac{(\nu^2+3)\left(\nu(r_+^2+r_-^2)-r_+r_-\sqrt{\nu^2+3}\right)}{4L\nu(r_+-r_-)}, \\
\beta &= \frac{(r_+-r_-)(\nu^2+3)}{2L^2\nu} - \frac{(\nu^2+3)\left(\nu(r_+^2+r_-^2)-r_+r_-\sqrt{\nu^2+3}\right)}{4L^2\nu(r_+-r_-)}, \\
\gamma &= \frac{1}{4L}\left(\nu^2+3\right)(r_++r_-), \\
\delta &= -\frac{1}{4L^2}\left(\nu^2+3\right)(r_++r_-).
\end{aligned} \tag{3.11}$$

Note that $\alpha = 2\pi T_L$ and $\gamma = 2\pi T_R$ as defined in (2.25). This also allows us to determine λ to be $3(\nu^2-1)/4\nu^2$.

The conformal coordinates that we have built yield a warped version of the AdS_3 Poincaré patch. In these conformal coordinates, the previously slightly terrible metric (2.24) turns out to be quite simple:

$$ds^2 = \frac{4L^2}{(\nu^2+3)^2z^2}\left((\nu^2+3)dw^+dw^- + 4\nu^2dz^2 + \frac{3(\nu^2-1)w^+}{z^2}(dw^{-2}+2zdw^-dz)\right). \tag{3.12}$$

This reduces to the Poincaré patch (3.1) for $\nu = 1$, unlike the Poincaré coordinates presented in [23]. The Killing vectors of this metric are:

$$\begin{aligned}
&\bar{H}_1 = i\partial_-, \qquad \bar{H}_0 = i\left(w^-\partial_- + \frac{1}{2}z\partial_z\right), \qquad \bar{H}_{-1} = i(-z^2\partial_+ + w^{-2}\partial_- + w^-z\partial_z), \\
&\qquad\qquad\quad H_0 = i\left(w^+\partial_+ + \frac{1}{2}z\partial_z\right),
\end{aligned} \tag{3.13}$$

which are a subset of (3.2), as expected. The killing vectors corresponding to rotation in the $w^\pm$ plane and dilation are

$$\begin{aligned}
H_0 - \bar{H}_0 &= i(w^+\partial_+ - w^-\partial_-), \\
H_0 + \bar{H}_0 &= i(w^+\partial_+ + w^-\partial_- + z\partial_z)
\end{aligned} \tag{3.14}$$

respectively. Using the inverse transformation of (3.5), we can also see that the quotient is generated by the group element

$$e^{-i4\pi^2(T_R\bar{H}_0+T_LH_0)} = e^{2\pi\partial_\phi}, \tag{3.15}$$

as discussed in [27]. Comparing this with [23] and (2.21), we see

$$H_0 = \frac{i}{2}J_2, \quad \bar{H}_0 = -\frac{i}{2}\tilde{J}_2. \tag{3.16}$$

Now that we have the warped black hole quotient structure and a set of suitable conformal coordinates in hand, we can repeat the analysis of Perry and Williams (outlined in section 2.1) to build our Selberg zeta function for warped AdS_3 black holes.

We begin by switching to the coordinates $x = (w^+ + w^-)/2$ and $y = (w^+ - w^-)/2$. Under the transformation $\phi \to \phi + 2\pi$, we see that the coordinates once again transform in the same way as (2.10):

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} e^{2a} & 0 & 0 \\ 0 & e^{2a} & 0 \\ 0 & 0 & e^{2a} \end{pmatrix} \begin{pmatrix} \cosh 2b & -\sinh 2b & 0 \\ -\sinh 2b & \cosh 2b & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{3.17}$$

where we have

$$\begin{aligned} a &= \frac{\pi r_+(\nu^2+3)(2\nu r_+ - r_-\sqrt{\nu^2+3})}{8L\nu(r_+ - r_-)} \\ b &= \frac{\pi r_-(\nu^2+3)(-2\nu r_- + r_+\sqrt{\nu^2+3})}{8L\nu(r_+ - r_-)}. \end{aligned} \tag{3.18}$$

The prime geodesic on the quotient spacetime is the curve which remains invariant under the rotation, which is the $x = y = 0$ line. The length of this prime geodesic ℓ is related to the dilation parameter by $\ell = 2a$.

Finally, the Selberg zeta function for warped quotients that we propose is (2.11), where in the case of warped AdS_3 black holes the parameters a and b are (3.18). We will see in section 5 that the zeroes of this zeta function are successfully mapped to the quasinormal modes on the warped AdS_3 black hole backgrounds. It should be noted that, in principle, the structure of the Selberg zeta function for warped quotients could have been more complicated (cf. the function $N(p)$ in equation (1.1)) than in the BTZ case, or indeed it need not have existed at all. We suspect that for more complicated orbifolds $N(p)$ may need to be constructed with greater care.

3.2 Warped self-dual solutions

Following the same procedure as in section 3.1 we construct conformal coordinates for the self-dual solutions (2.29), using the coordinate transformation

$$\begin{aligned} w^+ &= \sqrt{\frac{x-\frac{1}{2}}{x+\frac{1}{2}}}\, e^{\alpha\tilde{\theta}+\beta\tilde{\tau}}, \\ w^- &= \sqrt{\frac{x-\frac{1}{2}}{x+\frac{1}{2}}}\, e^{\gamma\tilde{\theta}+\delta\tilde{\tau}}, \\ z &= \sqrt{\frac{1}{x+\frac{1}{2}}}\, e^{1/2((\alpha+\gamma)\tilde{\theta}+(\beta+\delta)\tilde{\tau})}. \end{aligned} \tag{3.19}$$

The Klein-Gordon operator for this background is proportional to the quadratic Casimir (3.7) and has the same proportionality constant as the warped AdS_3 black hole

$$\mathcal{H}^2 + \lambda H_0^2 = \frac{L^2}{\nu^2+3}\nabla^2. \tag{3.20}$$

Explicitly, in x coordinates and using the ansatz $\Phi = e^{i(k\tilde{\theta}-\omega\tilde{\tau})}R(\tilde{r})$, we have:

$$\frac{L^2}{3+\nu^2}\nabla^2\Phi = \left(\partial_x\left(x^2-\frac{1}{4}\right)\partial_x + \frac{P}{4\left(x-\frac{1}{2}\right)} + \frac{Q}{4\left(x+\frac{1}{2}\right)} + S\right)\Phi \tag{3.21}$$

where

$$
\begin{aligned}
P &= \left(\frac{k}{\tilde{\alpha}} + \frac{2\omega}{(\tilde{r}_+ + \tilde{r}_-)}\right)^2, \\
Q &= -\left(\frac{k}{\tilde{\alpha}} - \frac{2\omega}{(\tilde{r}_+ + \tilde{r}_-)}\right)^2, \\
S &= \frac{3k^2\left(\nu^2-1\right)}{4\tilde{\alpha}^2\nu^2}.
\end{aligned}
\tag{3.22}
$$

Again we equate the coefficients k and ω in (3.22) to the ones in (3.7) and solve for $(\alpha, \beta, \gamma, \delta)$. The U(1) term eliminates two of the four branches and we choose the positive remaining one

$$
\alpha = \tilde{\alpha}, \qquad \beta = \gamma = 0, \qquad \delta = \frac{1}{2}(\tilde{r}_+ + \tilde{r}_-). \tag{3.23}
$$

Note that the coefficient of the U(1) term is the same as for the warped AdS_3 black hole: $\lambda = 3\left(\nu^2-1\right)/4\nu^2$. The parameters (3.23) are related to the temperatures (2.30) via $\alpha = 2\pi L T_L$ and $\delta = 2\pi L T_R$. The conformal coordinates (3.19) with these values of $(\alpha, \beta, \gamma, \delta)$ yield the same warped Poincaré patch as (3.12). From the conformal coordinates we calculate the coefficients a and b that appear in the quotient. Again we switch to $x = (w^+ + w^-)/2$ and $y = (w^+ - w^-)/2$ and under $\tilde{\theta} \to \tilde{\theta} + 2\pi\tilde{\alpha}$ the coordinates transform in the same way as (2.10), and we obtain

$$
a = \frac{\pi\tilde{\alpha}}{2}, \qquad b = -\frac{\pi\tilde{\alpha}}{2}. \tag{3.24}
$$

The quotient is generated by the Killing vector

$$
\partial_{\tilde{\theta}} = \pi L T_L J_2 = \frac{\tilde{\alpha}}{2} J_2, \tag{3.25}
$$

which now can be expressed in terms of the warped Poincaré patch generators:

$$
\partial_{\tilde{\theta}} = -i\tilde{\alpha} H_0. \tag{3.26}
$$

4 Scalar fields on $WAdS_3$ black hole backgrounds

In order to map the zeroes of our Selberg zeta function to the quasinormal modes of the warped AdS_3 black holes in section 5, we need to first study several aspects of a massive scalar field propagating on these backgrounds. We begin with the conformal weight Δ in section 4.1, followed by the thermal (Matsubara) frequencies ω_n in section 4.2 and finally quasinormal modes ω_* in section 4.3.

4.1 Conformal weights of the scalar field

Here we compute the conformal weight of highest weight representations of the symmetry algebra of the black hole spacetime. We use the relationship between the quadratic Casimir and the Laplacian of the massive scalar [36]

$$
(\mathcal{H}^2 + \lambda H_0^2)\Phi = \frac{L^2}{\nu^2+3} m^2 \Phi. \tag{4.1}
$$

We can express the SL(2,ℝ) Casimir in terms of the SL(2,ℝ) conformal weight

$$\mathcal{H}^2\Phi = h(h-1)\Phi, \tag{4.2}$$

and we know from (3.9) that the U(1) generator acting on the scalar field is

$$\lambda H_0^2\Phi = S(k,\omega)\Phi. \tag{4.3}$$

Combining the two we have

$$h(h-1) = \frac{L^2}{\nu^2+3}m^2 - S. \tag{4.4}$$

Solving for $\Delta = 2h$ and choosing the positive branch, we obtain the conformal weight of the scalar field:

$$\Delta = 1 + \sqrt{1 + 4\left(\frac{m^2L^2}{\nu^2+3} - S\right)}. \tag{4.5}$$

Note that S depends on the choice of coordinates. In the $(\tilde{t},\rho,\tilde{\phi})$ coordinates of (2.19), we have

$$S = \frac{3L^2(\nu^2-1)}{\nu^2+3}\omega^2 \tag{4.6}$$

which reproduces the result in [37]. In the coordinates of (t,r,ϕ) (2.24), we have

$$S = \frac{3L^2(\nu^2-1)}{\nu^2+3}\left(\frac{k-L\omega}{r_+^2-r_-^2}\right)^2 \tag{4.7}$$

and in the $(\tilde{\tau},\tilde{r},\tilde{\theta})$ coordinates of (2.29) for the self-dual solution, we have

$$S = \frac{3k^2\left(\nu^2-1\right)}{4\tilde{\alpha}^2\nu^2}. \tag{4.8}$$

Equation (4.8) is the same as the one computed for spacelike stretched warped AdS_3 with $\tilde{\alpha} = 1$, which reproduces the result in [23].

Notice that for large values of ω or k in (4.6), (4.7) and (4.8) the conformal weight becomes complex. The asymptotic behaviour of the solutions ($\sim r^{-\Delta/2}$) implies that these solutions are travelling waves. The fact that these becomes complex indicate that these spacetimes have superradiant behaviour [36]. Since $\mathrm{Re}(\Delta) > 0$ the outgoing wave solutions are always normalizable [38]. Note that for the negative branch of (4.4) for small enough values of S we have $\mathrm{Re}(\Delta_-) < 0$ which renders the solutions non-normalizable. We only consider the positive branch since the Selberg zeta function can be expressed a sum over eigenvalues of normalizable functions via the Selberg trace formula.[4]

[4]We thank the referee for bringing this fact to our notice.

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4.2 Thermal frequencies

A procedure for calculating thermal frequencies in rotating spacetimes was outlined in [39]. Due to translation symmetry in both t and ϕ, we can use the following ansatz for Φ:

$$\Phi(t,r,\phi) = e^{-i\omega t+ik\phi} f(r). \tag{4.9}$$

Using this ansatz in the equation of motion, we obtain a second order differential equation in r, which we examine around the outer horizon r_+:

$$A(r)f(r) + B(r)(r-r_+)f'(r) + (r-r_+)^2 f''(r) = 0. \tag{4.10}$$

For the spacelike warped AdS_3 black hole, we have

$$\begin{aligned} A &= \frac{16r^2L^2}{((r+r_+)(r^2-r_-^2))^2}\left(k^2 g_{tt} + k\omega g_{t\phi} + \omega^2 g_{\phi\phi}\right) \\ B &= \frac{3r^4 - r_+^2 r_-^2 - r^2(r_+^2 + r_-^2)}{r(r^2-r_-^2)(r+r_+)}. \end{aligned} \tag{4.11}$$

To construct the indicial equation, we must input a power series solution around r_+ for A, B and f, like so:

$$A = \sum_{n=0}^{\infty} a_n (r-r_+)^n, \quad B = \sum_{n=0}^{\infty} b_n (r-r_+)^n, \quad f = (r-r_+)^{\alpha} \sum_{n=0}^{\infty} f_n (r-r_+)^n. \tag{4.12}$$

We only need the leading terms of A and B for the indicial equation, which after some algebra turn out to be:

$$\begin{aligned} a_0 &= \left(\frac{L(2r_-^2 - 2r_+^2(\nu-1) + r_+ r_-(\sqrt{\nu^2+3}-4))k + Lr_+\left(2\nu r_+ - r_-\sqrt{\nu^2+3}\right)\omega}{(r_+-r_-)^2(r_++r_-)(\nu^2+3)}\right)^2, \\ b_0 &= 1. \end{aligned} \tag{4.13}$$

If we set $\alpha = in/2$ with n an integer, we can solve the indicial equation for ω in terms of n, obtaining the thermal frequencies:

$$\omega_n = \frac{in(r_+-r_-)^2(r_++r_-)(\nu^2+3) - 2Lk(2r_-^2 + 2r_+^2(1-\nu) + r_+r_-(\sqrt{\nu^2+3}-4))}{2L^2 r_+\left(2\nu r_+ - r_-\sqrt{\nu^2+3}\right)}. \tag{4.14}$$

If we set ν to 1, we recover the thermal frequencies for the BTZ black hole [39]:

$$\omega_n = \frac{Lkr_- + in(r_+^2 - r_-^2)}{L^2 r_+}, \tag{4.15}$$

which are also known as the Matsubara frequencies. In the $(\tilde{t}, \rho, \tilde{\phi})$ coordinates of (2.19), this result becomes

$$\omega_n = \frac{-4k - in(\rho_+ - \rho_-)(\nu^2+3)}{4\rho_+\nu - 2\sqrt{\rho_+\rho_-(\nu^2+3)}}. \tag{4.16}$$

Repeating the analysis in the self-dual solution (2.29) we get

$$\omega_n = -\frac{(\tilde{r}_- + \tilde{r}_+)(k - i\tilde{\alpha}n)}{2\tilde{\alpha}}. \tag{4.17}$$

4.3 Quasinormal modes

To compute quasinormal modes, we must study the equation of motion of the massive scalar field in the black hole background

$$(\nabla^2 - m^2)\Phi(t, r, \phi) = 0. \tag{4.18}$$

These have been computed in the "rotated coordinates" ($(\tilde{t}, \rho, \tilde{\phi})$ as in (2.19)) before [37, 40], but here we present the calculation and results in the BTZ-like coordinates ((t, r, ϕ) as in (2.24)), in which the quasinormal modes reduce to those of the BTZ black hole [41] when the warping parameter ν is set to 1.

4.3.1 Warped AdS$_3$ black holes

Here we follow the procedure described in [41]. The structure of the equation of motion is clear when we make this coordinate change:

$$\zeta = \frac{r^2 - r_+^2}{r^2 - r_-^2}, \tag{4.19}$$

and we will use the same ansatz as (4.9): $\Phi(t, \zeta, \phi) = e^{-i\omega t + ik\phi} f(\zeta)$. In these coordinates we have

$$\zeta(1-\zeta)f''(\zeta) + (1-\zeta)f'(\zeta) + \left(\frac{A}{\zeta} + B + \frac{C}{1-\zeta}\right) f(\zeta) = 0, \tag{4.20}$$

where

$$\begin{aligned}
A &= \frac{L^2\left(k\left(2(\nu-1)r_+^2 - 2r_-^2 - \left(\sqrt{\nu^2+3}-4\right) r_+ r_-\right) - L r_+ \omega\left(2\nu r_+ - r_-\sqrt{\nu^2+3}\right)\right)^2}{(r_- - r_+)^4 (r_- + r_+)^2 (\nu^2+3)^2}, \\
B &= -A(r_+ \leftrightarrow r_-), \\
C &= \frac{3L^2(\nu^2-1)(k - L\omega)^2}{(\nu^2+3)^2 (r_- - r_+)^2} - \frac{L^2 m^2}{\nu^2+3}.
\end{aligned} \tag{4.21}$$

The solution to (4.20) is

$$\begin{aligned}
&f(\zeta) = \zeta^\alpha (1-\zeta)^\beta {}_2F_1(a, b, c; \zeta), \\
&\quad a = \alpha + \beta + i\sqrt{-B}, \qquad b = \alpha + \beta - i\sqrt{-B}, \quad c = 2\alpha + 1.
\end{aligned} \tag{4.22}$$

The exponents α and β describe the behaviour of the scalar field near the outer horizon and at infinity respectively:

$$\alpha^2 = -A, \quad \beta = \frac{1}{2}\left(1 - \sqrt{1-4C}\right). \tag{4.23}$$

Notice that β can be expressed in terms of the conformal weight of the scalar field (4.5) and (4.7) as

$$\beta = 1 - \frac{\Delta}{2}. \tag{4.24}$$

We require that the solution is ingoing at the horizon[5] and vanishes at infinity, so we set the following boundary conditions:[6]

$$c-a=-j, \quad c-b=-j, \quad j\in\mathbb{Z}_{\geq 0}. \tag{4.25}$$

Here we can solve these boundary conditions for ω in terms of k and Δ. We note that Δ does depend on ω, but we keep it implicit since the matching of the Selberg zeta function zeroes s_* to the quasinormal modes does not depend upon the intrinsic structure of Δ, as we will see. Thus we obtain

$$\omega_L^* = \frac{k}{L} - i\frac{(\Delta+2j)\left(\nu^2+3\right)(r_+-r_-)}{4L^2\nu}, \tag{4.26}$$

and

$$\omega_R^* = -\frac{k}{L}\left(\frac{2\left(r_+-r_-\right)^2}{\nu(r_+^2+r_-^2)-r_+r_-\sqrt{\nu^2+3}}-1\right) - \frac{i\left(\nu^2+3\right)(r_++r_-)\left(r_+-r_-\right)^2(\Delta+2j)}{4L^2\left(\nu(r_+^2+r_-^2)-r_+r_-\sqrt{\nu^2+3}\right)}. \tag{4.27}$$

We recover the BTZ quasinormal modes as calculated in [41] upon setting ν to 1.

4.3.2 Warped self-dual solutions

We compute the quasinormal modes for the self-dual solutions and reproduce previous results [29, 30] in $(\tilde{\tau},\tilde{\rho},\tilde{\theta})$ coordinates. Using the same ansatz: $\Phi(\tilde{\tau},\zeta,\tilde{\theta}) = e^{-i\omega\tilde{\tau}+ik\tilde{\theta}}f(\zeta)$ and change of coordinates

$$\zeta = \frac{\tilde{r}^2-\tilde{r}_+^2}{\tilde{r}^2-\tilde{r}_-^2}, \tag{4.28}$$

we reach the same radial equation as before (4.20), where

$$\begin{aligned} A &= \left(\frac{k}{2\tilde{\alpha}}+\frac{\omega}{\tilde{r}_++\tilde{r}_-}\right)^2, \\ B &= -\left(\frac{k}{2\tilde{\alpha}}-\frac{\omega}{\tilde{r}_++\tilde{r}_-}\right)^2, \\ C &= \frac{3k^2\left(\nu^2-1\right)}{4\tilde{\alpha}^2\nu^2}-\frac{L^2m^2}{\nu^2+3}. \end{aligned} \tag{4.29}$$

Following the procedure done in section 4.3.1, with the corresponding Δ for the self-dual solution, we obtain the quasinormal modes. The boundary conditions only permit one solution, the right ingoing mode:

$$\omega_R^* = -\frac{1}{4}i(\tilde{r}_++\tilde{r}_-)(\Delta+2j). \tag{4.30}$$

[5]Modes that are outgoing at the horizon are sometimes called antiquasinormal. These can be treated similarly, but here we just stick to the ingoing modes for simplicity.

[6]To see how to derive these conditions look at equation (15) in [41].

5 Mapping the Selberg zeroes to quasinormal modes

Now we take all the results from the previous sections and show that tuning the zeroes of the Lorentzian Selberg zeta function s_* to the conformal weights Δ, we recover the condition that the corresponding quasinormal modes must be tuned to the thermal frequencies, in the spirit of [15].

For simplicity, we only consider the ingoing (quasinormal) modes, and so we only need to consider the case with $n \geq 0$ in the change of bases listed in (2.13). It should be noted that the change of basis (2.13) did not (to our knowledge) necessarily have to be the same in the warped case. Nevertheless, this change of basis turns out to be consistent in mapping the Selberg zeroes to the warped quasinormal modes. Since the basis change (2.13) is a result from scattering theory on the Euclidean BTZ background [3], the fact that (2.13) still works in the warped case probably tells us that warping the fiber length does not significantly change the poles of the scattering matrix.

5.1 Warped AdS_3 black holes

The procedure outlined in [15] was to express $s_* - \Delta$ in terms of the corresponding quasinormal mode and BTZ thermal frequencies. Here we will go in the opposite direction and construct s_* from the quasinormal modes, the warped black hole's thermal frequencies and the conformal weight. We will then solve for a and b used in the quotient, and show that they coincide with (3.18) in the warped black hole case and (3.24) in the self-dual case. That is, we write

$$s_{*L} = \Lambda_L^{-1}(\omega_L^* - \omega_n) + \Delta, \quad s_{*R} = \Lambda_R^{-1}(\omega_R^* - \omega_n) + \Delta, \tag{5.1}$$

where Λ_L and Λ_R are proportionality constants, which can be determined by eliminating Δ on the right hand side of the equations (since the quasinormal modes depend upon Δ but the zeroes s_* should not explicitly depend upon Δ). Plugging everything that we calculated in previous sections into the right hand side of the equations, we get:

$$\frac{\omega_L^* - \omega_n}{\Lambda_L} + \Delta = -2j - \left(\frac{8iL\nu(r_+ - r_-)}{r_+(\nu^2+3)(2r_+\nu - \sqrt{\nu^2+3})}\right)k - \left(\frac{2\nu(r_+^2 - r_-^2)}{r_+(2\nu r_+ - r_-\sqrt{\nu^2+3})}\right)n, \tag{5.2}$$

and

$$\frac{\omega_R^* - \omega_n}{\Lambda_R} + \Delta = -2j + \left(\frac{8iL\nu(r_+ - r_-)}{r_+(\nu^2+3)(2r_+\nu - \sqrt{\nu^2+3})}\right)k - \left(\frac{2((r_+^2 - r_-^2)\nu - r_+r_-\sqrt{\nu^2+3})}{r_+(2\nu r_+ - r_-\sqrt{\nu^2+3})}\right)n, \tag{5.3}$$

where

$$\Lambda_L = \frac{i(r_+ - r_-)(\nu^2+3)}{4L^2\nu}, \quad \Lambda_R = \frac{i(r_+ - r_-)^2(r_+ + r_-)(\nu^2+3)}{4l^2((r_+^2 + r_-^2)\nu - r_+r_-\sqrt{\nu^2+3})}. \tag{5.4}$$

Equating (5.2) and (5.3) to (2.14) (reproduced here for convenience)

$$s_{*L} = -2j - \left(1 + \frac{b}{a}\right)n - \frac{i\pi k}{a}, \quad s_{*R} = -2j - \left(1 - \frac{b}{a}\right)n + \frac{i\pi k}{a}, \tag{5.5}$$

and solving for a and b, we obtain

$$a = \frac{\pi r_+(\nu^2+3)(2\nu r_+ - r_-\sqrt{\nu^2+3})}{8L\nu(r_+ - r_-)},$$
$$b = \frac{\pi r_-(\nu^2+3)(-2\nu r_- + r_+\sqrt{\nu^2+3})}{8L\nu(r_+ - r_-)}, \tag{5.6}$$

which reproduces the values obtained from the quotient (3.18). This shows that the proposed Selberg zeta function zeroes for warped AdS_3 black holes are successfully mapped to the quasinormal modes.

5.2 Warped self-dual solutions

For the self-dual solutions the left quasinormal mode vanishes, and thus we only match one of the zeroes, s_{*R}, to the remaining quasinormal mode. Along the lines of (5.1), we write

$$s_{*R} = \Lambda_R^{-1}(\omega_R^* - \omega_n) + \Delta, \tag{5.7}$$

and we find that the proportionality constant is

$$\Lambda_R = -\frac{i}{4}(\tilde{r}_+ + \tilde{r}_-). \tag{5.8}$$

Plugging everything into the right hand side of (5.7) (using (4.5), (4.8), (4.17), (4.30) and (5.8)), we find

$$s_{*R} = -2(j+n) - \frac{2ik}{\tilde{\alpha}}. \tag{5.9}$$

Comparing (5.9) and (5.5) allows us to solve for a and b:

$$a = \frac{\pi\tilde{\alpha}}{2}, \qquad b = -\frac{\pi\tilde{\alpha}}{2}. \tag{5.10}$$

This matches our previous result from the quotient (3.24). Thus, just as in the warped black hole case in the previous section, the zeroes of our proposed Selberg zeta function for the self-dual solutions are successfully mapped to the quasinormal modes.

6 Discussion

Using the quotient structure of warped AdS_3 black holes [23], we have constructed a Selberg zeta function for this spacetime in the spirit of [3], providing an example that extends the work of [3] beyond hyperbolic quotients. We have shown that the zeroes of this zeta function are mapped to the scalar quasinormal modes on the warped AdS_3 black hole background [37, 40], in exactly the same way as in the non-warped (BTZ) case [15, 18]. We repeat this analysis for the warped self-dual solutions reported in [23, 28], which are obtained from different quotients than the spacelike warped AdS_3 black holes and are interesting in their own right.

Along the way, we develop a method of constructing a warped version of the AdS_3 Poincaré patch metric, using the symmetry structure of the Klein-Gordon equation and a familiar ansatz for conformal coordinates [27, 42]. To the best of our knowledge, our metric (3.12) does not appear in the literature. This metric could be of interest in the context of the warped AdS/warped CFT correspondence [24]. The conformal boundary of the metric (3.12) seems to have the correct properties (such as a degenerate metric) required by the geometry on which a boundary warped CFT lives, which is not manifest in other coordinate systems.

Perhaps the most immediate and intriguing direction for future work is to connect the Selberg zeta function to 1-loop determinants on the warped AdS_3 black hole background, in a similar fashion as in [15]. With this connection in hand, it might be possible to study quantum corrections on the near-horizon extremal Kerr (NHEK) geometry (which is warped AdS_3) [26], as well as in other contexts where warped AdS_3 appears (see the Introduction for more details).

Furthermore, it is extremely tantalyzing to explore whether this Selberg quotient formalism can be related to the recent and very interesting connection between black hole quasinormal modes and quantum Seiberg-Witten curves [43, 44]. Both this work and [43] fix a geometric lens on the problem of computing spectral data, and a connection between them seems both likely and fruitful to persue. Indeed, such a connection was already hinted at in [45].

Hyperbolic quotient spacetimes occur in several other interesting physical systems in which quantum corrections are of interest. One notable example is AdS wormholes (see for example [46]). In fact, it was recently shown in [47] that certain k-boundary wormholes were constructed by quotienting AdS by the discrete group $\mathbb{Z}_k$. Since the quotient group that we consider is isomorphic to $\mathbb{Z}$, these wormholes provide a physically interesting and tractable arena to study more complicated quotienting groups. A second notable example in which hyperbolic quotient geometries arise is in the calculation of holographic entanglement Rényi entropies [48]. In that work, one difficulty was calculating the generators of the quotienting Schottky group Γ, and most of the time the authors relied on an expansion in small cross-ration. Perhaps now that we have more experience in calculating generators of quotient groups from the bulk perspective, this problem can be revisited with more success. The main challenge in that endeavor would be making a meaningful connection to the boundary CFT data.

There is strong evidence that the Selberg trace formula also has a different and interesting physical interpretation: it is likely the formal like between two well-known methods for computing functional determinants, namely the heat kernel method (see for example [12]) and the quasinormal mode method [49]. This can be seen from how the Selberg trace formula is generally derived from the Selberg zeta function. As explained in [3], one considers two different representations of the Selberg zeta function: the Euler product (2.9) and the Hadamard product

$$Z_\Gamma(s) = e^{Q(s)} \prod_{s_*} (1 - s/s_*) e^{s/s_* + 1/2(s/s_*)^2 + 1/3(s/s_*)^3}, \tag{6.1}$$

where Z_Γ is meromorphic and $Q(s)$ is an entire function. The Selberg trace formula is obtained by taking the logarithmic derivative of (2.9) and (6.1) and equating them. To extend this to warped AdS_3 quotients, the non-trivial interplay between the asymptotics of Δ, ω and k may be important to interpret through the lens of [49], since this is used to determine the UV counterterms of the partition function, like for example in [50]. We leave this interesting and potentially insightful direction for future work.

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A Coordinate transformations

Here we list some useful coordinate transformations between some of the metrics used in this work.

- **Embedding coordinates.** The hyperbolic 3-space can be written in terms of its embedding

$$ds^2 = -dU^2 + dV^2 + dX^2 + dY^2 \tag{A.1}$$

with the constraint

$$-U^2 + V^2 + X^2 + Y^2 = -L^2. \tag{A.2}$$

The following transformation

$$x = \frac{Y}{U+X}, \qquad y_E = \frac{V}{U+X}, \qquad z = \frac{L}{U+X}, \tag{A.3}$$

is used to write the $\mathbb{H}_3$ line element in the Poincaré patch coordinates (2.3). The isometry generators of (A.1) are

$$J_{AB} = X_B \partial_A - X_A \partial_B. \tag{A.4}$$

We can write these explicitly in embedding coordinates and Poincaré coordinates:

$$\begin{aligned}
J_{01} &= V\partial_U + U\partial_V = -xy_E\partial_x + \frac{1}{2}(x^2 - y_E^2 + z^2 + 1)\partial_{y_E} - y_E z\partial_z, \\
J_{02} &= X\partial_V - V\partial_X = xy_E\partial_x - \frac{1}{2}(x^2 - y_E^2 + z^2 - 1)\partial_{y_E} + y_E z\partial_z, \\
J_{03} &= Y\partial_V - V\partial_Y = -y_E\partial_x + x\partial_{y_E}, \\
J_{12} &= X\partial_U + U\partial_X = -x\partial_x - y_E\partial_{y_E} - z\partial_z, \\
J_{13} &= Y\partial_U + U\partial_Y = -\frac{1}{2}(x^2 - y_E^2 - z^2 - 1)\partial_x - xy_E\partial_{y_E} - xz\partial_z, \\
J_{23} &= Y\partial_X - X\partial_Y = \frac{1}{2}(1 - x^2 + y_E^2 + z^2)\partial_x - xy_E\partial_{y_E} - xz\partial_z.
\end{aligned} \tag{A.5}$$

- **The stretched warped AdS$_3$ black hole.** The coordinate transformation between warped AdS$_3$ in global fibred coordinates (2.17) and the warped black hole in rotated coordinates (2.19) is:

$$\begin{aligned}\tau &= \tan^{-1}\left[\frac{2\sqrt{(\rho-\rho_+)(\rho-\rho_-)}}{2\rho-\rho_+-\rho_-}\sinh\left(\frac{1}{4}(\rho_+-\rho_-)(\nu^2+3)\tilde{\phi}\right)\right],\\ \sigma &= \sinh^{-1}\left[\frac{2\sqrt{(\rho-\rho_+)(\rho-\rho_-)}}{\rho_+-\rho_-}\cosh\left(\frac{1}{4}(\rho_+-\rho_-)(\nu^2+3)\tilde{\phi}\right)\right],\\ u &= \frac{\nu^2+3}{4\nu}\left[2\tilde{t}+\left(\nu(\rho_++\rho_-)-\sqrt{\rho_+\rho_-(\nu^2+3)}\right)\tilde{\phi}\right]\\ &\quad+\coth^{-1}\left[\frac{2\rho-\rho_+-\rho_-}{\rho_+-\rho_-}\coth\left(\frac{1}{4}(\rho_+-\rho_-)(\nu^2+3)\tilde{\phi}\right)\right].\end{aligned} \tag{A.6}$$

This transformation is valid for $\rho > \rho_+$ [51]. For $\rho < \rho_+$ we instead have [23]:

$$\begin{aligned}u &= \frac{\nu^2+3}{4\nu}\left[2\tilde{t}+\left(\nu(\rho_++\rho_-)-\sqrt{\rho_+\rho_-(\nu^2+3)}\right)\tilde{\phi}\right]\\ &\quad-\tanh^{-1}\left[-\frac{2\rho-\rho_+-\rho_-}{\rho_+-\rho_-}\coth\left(\frac{1}{4}(\rho_+-\rho_-)(\nu^2+3)\tilde{\phi}\right)\right].\end{aligned} \tag{A.7}$$

- **The warped self-dual solution.** The coordinate transformation between warped AdS$_3$ in global fibred coordinates (2.17) and the self-dual solution (2.27) is [28]:

$$\begin{aligned}\tau &= \tan^{-1}\left[\frac{2\sqrt{(\tilde{\rho}-\tilde{\rho}_+)(\tilde{\rho}-\tilde{\rho}_-)}}{2\tilde{\rho}-\tilde{\rho}_+-\tilde{\rho}_-}\sinh\left(\frac{\tilde{\rho}_+-\tilde{\rho}_-}{2}\tilde{\tau}\right)\right],\\ \sigma &= \sinh^{-1}\left[\frac{2\sqrt{(\tilde{\rho}-\tilde{\rho}_+)(\tilde{\rho}-\tilde{\rho}_-)}}{2\tilde{\rho}-\tilde{\rho}_+-\tilde{\rho}_-}\cosh\left(\frac{\tilde{\rho}_+-\tilde{\rho}_-}{2}\tilde{\tau}\right)\right],\\ u &= \tilde{\alpha}\tilde{\theta}+\tanh^{-1}\left[\frac{2\tilde{\rho}-\tilde{\rho}_+-\tilde{\rho}_-}{\tilde{\rho}_+-\tilde{\rho}_-}\coth\left(\frac{\tilde{\rho}_+-\tilde{\rho}_-}{2}\tilde{\tau}\right)\right].\end{aligned} \tag{A.8}$$

B BTZ conformal coordinates

In this appendix we illustrate how our method for obtaining conformal coordinates works in the case of the BTZ black hole. We reproduce the Lorentzian version of (2.2), as expected.

The BTZ metric is given by

$$ds^2 = -\frac{r^2-r_+^2-r_-^2}{L^2}dt^2+\frac{L^2r^2}{(r^2-r_+^2)(r^2-r_-^2)}dr^2+r^2d\phi^2-2\frac{r_+r_-}{L}dtd\phi. \tag{B.1}$$

For the BTZ black hole, we find the Klein-Gordon operator is proportional to the quadratic Casimir as:

$$\mathcal{H}^2 = \frac{L^2}{4}\nabla^2. \tag{B.2}$$

The Klein-Gordon equation with field ansatz (3.6) takes the form

$$\left(\partial_x\left(x^2-\frac{1}{4}\right)\partial_x+\frac{P_{\text{BTZ}}}{4\left(x-\frac{1}{2}\right)}+\frac{Q_{\text{BTZ}}}{4\left(x+\frac{1}{2}\right)}\right)\Phi=0, \tag{B.3}$$

where $x = \frac{r^2 - 1/2(r_+^2 + r_-^2)}{r_+^2 - r_-^2}$ and

$$P_{\rm BTZ} = \frac{L^2(kr_- - \omega L r_+)}{(r_+^2 - r_-^2)^2}, \qquad Q_{\rm BTZ} = -\frac{L^2(kr_+ - \omega L r_-)}{(r_+^2 - r_-^2)^2}. \tag{B.4}$$

When equating the coefficients k and ω of (3.7) to those in (B.4), one has 4 branches to chose from. The branch which yields the correct transformation to the Poincaré patch (3.1) is the following:

$$\begin{aligned} \frac{\alpha+\gamma}{\beta\gamma-\alpha\delta} &= \frac{L^2 r_+}{r_+^2 - r_-^2}, & \frac{\beta+\delta}{\beta\gamma-\alpha\delta} &= -\frac{L r_-}{r_+^2 - r_-^2}, \\ \frac{\alpha-\gamma}{\beta\gamma-\alpha\delta} &= -\frac{L^2 r_-}{r_+^2 - r_-^2}, & \frac{\beta-\delta}{\beta\gamma-\alpha\delta} &= \frac{L r_+}{r_+^2 - r_-^2}. \end{aligned} \tag{B.5}$$

On solving these set of equations, we have

$$\alpha = L\beta = \frac{r_+ - r_-}{L}, \qquad \gamma = -L\delta = \frac{r_+ + r_-}{L}. \tag{B.6}$$

Again, we see that these are related to the left and right temperatures $\alpha = 2\pi T_L$ and $\gamma = 2\pi T_R$. Plugging these values into (3.5), one obtains the coordinate transformation from the Boyer-Lindquist coordinates of the BTZ black hole to the Poincaré patch

$$\begin{aligned} w^+ &= \sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}} e^{\frac{1}{L^2}(r_+ - r_-)(t + L\phi)}, \\ w^- &= \sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}} e^{\frac{1}{L^2}(r_+ + r_-)(L\phi - t)}, \\ z &= \sqrt{\frac{r_+^2 - r_-^2}{r^2 - r_-^2}} e^{\frac{1}{L^2}(L r_+ \phi - r_- t)}. \end{aligned} \tag{B.7}$$

These are nothing more than the Lorentzian version of (2.2).

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Paper II

$T\overline{T}$ deformations of holographic warped CFTs

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$T\bar{T}$ deformations of holographic warped CFTs

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We explore $T\bar{T}$ deformations of warped conformal field theories (WCFTs) in two dimensions as examples of $T\bar{T}$ deformed nonrelativistic quantum field theories. WCFTs are quantum field theories with a Virasoro × U(1) Kac-Moody symmetry. We compute the deformed symmetry algebra of a $T\bar{T}$ deformed holographic WCFT, using the asymptotic symmetries of AdS_3 with $T\bar{T}$ deformed Compére, Song, and Strominger boundary conditions. The U(1) Kac-Moody symmetry survives provided one allows the boundary metric to transform under the asymptotic symmetry. The Virasoro sector remains but is now deformed and no longer chiral.

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I. INTRODUCTION

A warped conformal field theory (WCFT) is a quantum field theory with an $SL(2,\mathbb{R})\times U(1)$ global symmetry in two dimensions, which breaks Lorentz invariance. Such QFTs have translation invariance, but scaling invariance is restricted to only one coordinate. Finite warped symmetry transformations take the form [1,2]

$$z \to f(z), \qquad \bar{z} \to \bar{z} + g(z). \tag{1.1}$$

However, despite not being Lorentz invariant, this class of two-dimensional quantum field theories still possesses an infinite-dimensional symmetry algebra, namely, a Virasoro $\times$ U(1) Kac-Moody current algebra. WCFTs are interesting, as they appear in a number of holographic systems with an $SL(2,\mathbb{R})\times U(1)$ symmetry, such as warped AdS_3 [3], the near-horizon geometry of extremal rotating black holes [4,5], and AdS_3 with Dirichlet-Neumann boundary conditions [6]. Holographic WCFTs have passed a number of consistency checks, such as a Cardy formula [2], holographic entanglement entropy [7–10], and one-loop determinants [11].

Since WCFTs are nonrelativistic, they do not couple to standard (pseudo-)Riemannian manifolds. One approach is to couple WCFTs to "warped geometries" [3], a variant of Newton-Cartan geometries. These geometries can be found at the boundary of warped AdS_3 spacetimes. Unfortunately, these geometries have certain pathologies, such as a degenerate metric, which make some calculations untenable. Another way to couple WCFT to a background manifold is to allow the manifold to transform with the warped symmetry transformations. Holographically, this requires relaxing Dirichlet boundary conditions of the bulk metric to boundary conditions which allow for asymptotic symmetry transformations to transform the boundary metric under the warped symmetry transformation of the boundary WCFT. The Dirichlet-Neumann boundary conditions of Compére, Song, and Strominger (CSS) [6] do exactly this. This approach bypasses the need for a warped geometry with degenerate metrics, and we can work with conventional techniques.

Two-dimensional translationally invariant quantum field theories admit a class of solvable irrelevant deformations built from conserved currents, most notable of which is the $T\bar{T}$ deformation [12,13]. The $T\bar{T}$ operator is defined by the determinant of the energy-momentum tensor of the quantum field theory, and the deformed action obeys the following flow equation:

$$\partial_\lambda S_{\mathrm{QFT}}(\lambda) = -\frac{1}{2}\int d^2x\sqrt{\gamma}\mathcal{O}^{(\lambda)}_{T\bar{T}},$$
$$\mathcal{O}_{T\bar{T}} = \det T = \frac{1}{2}\epsilon^{\mu\rho}\epsilon^{\nu\sigma}T_{\mu\nu}T_{\rho\sigma}, \tag{1.2}$$

where the deformation parameter λ is the coupling to the $T\bar{T}$ operator $\mathcal{O}_{T\bar{T}}$. This operator is defined using point splitting, which in the coincident limit produces a well-defined local operator up to total derivatives. The expectation value of $\mathcal{O}_{T\bar{T}}$ turns out to be a constant, and from this one can derive the flow of energy eigenstates of the quantum field theory defined on a cylinder of radius R:

$$\frac{\partial E_n}{\partial\lambda} = E_n\frac{\partial E_n}{\partial R} + \frac{P_n^2}{R}. \tag{1.3}$$

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Even though the energy eigenvalues are changed, the Hilbert space remains undeformed, since there is a one-one correspondence between the states of the original and deformed theory. Similarly, other observables can be calculated in the deformed theory, for example, the deformed Lagrangian, partition function, two-two scattering matrices, correlation functions, etc. [14–21].

There are various ways to interpret how the $T\bar{T}$ deformation acts on holographic CFTs. One proposal by [22] is to impose Dirichlet boundary conditions for the bulk metric at a finite radius. Another proposal given by [23] is to treat the $T\bar{T}$ deformation as a double-trace deformation, which will deform the asymptotic behavior of the bulk fields [24,25]. This approach agrees with the cutoff AdS proposal when both are valid but has the advantage of working when there are bulk matter fields and also for either sign of the deformation parameter, which the former does not. More recently, there has also been the "glue-on AdS holography" proposal [26] which also agrees with [23] for the positive sign of the deformation parameter in the absence of matter fields.

Using the mixed boundary conditions and Dirichlet boundary conditions at finite radius, the asymptotic symmetry algebra of the bulk dual to $T\bar{T}$ deformed holographic CFT was calculated in [23,27,28]. Despite losing conformal invariance, the asymptotic symmetry algebra turns out to still have a Virasoro × Virasoro structure. However, either the central charge becomes state dependent, or one loses the holomorphic factorization of the symmetry algebra, which can also be expressed as a nonlinear deformation of the standard Virasoro algebra.

In this work, we explore $T\bar{T}$ deformations of WCFTs from a holographic perspective. To establish what a $T\bar{T}$ deformation of a WCFT is, one must first define what the energy-momentum tensor of a WCFT is. Canonically, for WCFTs defined on a warped geometry, energy-momentum tensors are not symmetric, and a determinant is harder to define, since the metric is degenerate and noninvertible. The energy-momentum tensor turns out to be a tensor with components being a chiral stress tensor and a U(1) current. For WCFTs dual to warped AdS_3, it is also not possible to use the Fefferman-Graham expansion to compute the energy-momentum tensor for the same reason; the boundary metric is not invertible. However, if we study WCFTs dual to AdS_3 with CSS boundary conditions, for the price of a boundary metric which is not invariant under warped transformations, we have an invertible metric, a symmetric energy-momentum tensor, and a conventional definition for a determinant. Given these considerations, it is possible to propose a definition for a $T\bar{T}$ deformed WCFT which is dual to AdS_3 with $T\bar{T}$ deformed CSS boundary conditions.

This paper is organized as follows. We first briefly review the mixed boundary conditions of [23] in Sec. II A. Then, in Sec. II B, we review the CSS boundary conditions and derive the Virasoro × U(1) Kac-Moody algebra. In Sec. III, we derive the $T\bar{T}$ deformed CSS boundary conditions to compute the deformed symmetry algebra of a $T\bar{T}$ deformed WCFT. We will see that if one imposes deformed boundary conditions equivalent to Dirichlet boundary conditions at the radial cutoff surface, we recover a deformed Virasoro algebra, but we lose the U(1) Kac-Moody algebra. However, if we allow the Dirichlet-Neumann boundary conditions to remain at the cutoff surface, which is what the mixed boundary conditions suggests is the correct approach, we recover an undeformed Kac-Moody symmetry. We then conclude and discuss future directions in Sec. IV.

II. REVIEW

A. Mixed boundary conditions from $T\bar{T}$

We begin by briefly reviewing the mixed boundary conditions derived in [23] from the variational principle. The variation of the boundary QFT action with respect to the boundary metric sources the energy-momentum tensor of the QFT and of the bulk dual. The flow of the variation of the QFT action is equal to the variation of the deformation which generates the flow. So we have

$$\partial_\lambda \delta S = \delta \partial_\lambda S,$$
$$\partial_\lambda \left(\frac{1}{2} \int_{\partial\mathcal{M}} d^2x \sqrt{\gamma} T^{(\lambda)}_{ij} \delta\gamma^{(\lambda)ij} \right) = \delta \int_{\partial\mathcal{M}} d^2x \sqrt{\gamma} \mathcal{O}^{(\lambda)}_{T\bar{T}}. \quad (2.1)$$

From this, we can compute the flow equations for the boundary metric and the energy-momentum tensor with respect to the deformation parameter λ. Expressing the equations in terms of the trace-reversed energy-momentum tensor $\hat{T}_{ij} = T_{ij} - \gamma_{ij} T^i_i$, we have

$$\partial_\lambda \gamma_{ij} = -2\hat{T}_{ij},$$
$$\partial_\lambda \hat{T}_{ij} = -\hat{T}_{il}\hat{T}_j{}^l,$$
$$\partial_\lambda (\hat{T}_{il}\hat{T}_j{}^l) = 0. \quad (2.2)$$

Solving these equations, we can express the deformed metric and energy-momentum tensor in terms of the undeformed metric and energy-momentum tensor:

$$\gamma_{ij}(\lambda) = \gamma_{ij} - 2\lambda\hat{T}_{ij} + \lambda^2 \hat{T}_{ik}\hat{T}_{jl}\gamma^{kl},$$
$$\hat{T}_{ij}(\lambda) = \hat{T}_{ij} - \lambda\hat{T}_{ik}\hat{T}_{jl}\gamma^{kl}, \quad (2.3)$$

where everything on the right-hand side is undeformed quantities. The new deformed quantities are now the new boundary conditions for the bulk fields. To see this, let us consider pure Einstein gravity.

For pure Einstein gravity in three dimensions, the Fefferman-Graham expansion of the metric truncates at second order in $1/r^2$ [29]:

$$ds^2 = l^2\frac{dr^2}{r^2} + g_{ab}dz^a dz^b$$
$$= l^2\frac{dr^2}{r^2} + l^2r^2\left(g_{ab}^{(0)} + \frac{g_{ab}^{(2)}}{r^2} + \frac{g_{ab}^{(4)}}{r^4}\right)dz^a dz^b, \quad (2.4)$$

using which we can now express the boundary energy-momentum tensor in terms of the Fefferman-Graham expansion:

$$\hat{T}_{ab} = \frac{k}{2\pi}g_{ab}^{(2)}, \quad (2.5)$$

where $k = \frac{l}{4G_N}$. For pure gravity, we also have

$$g_{ab}^{(4)} = \frac{1}{4}g_{ac}^{(2)}g_{db}^{(2)}g_{(0)}^{cd}. \quad (2.6)$$

Therefore, we can express the deformed boundary metric and energy-momentum tensor in terms of the Fefferman-Graham expansion:

$$\gamma_{ab}(\lambda) = l^2\left(g_{ab}^{(0)} - \left(2\lambda\frac{k}{2\pi}\right)g_{ab}^{(2)} + \left(2\lambda\frac{k}{2\pi}\right)^2 g_{ab}^{(4)}\right),$$
$$\hat{T}_{ab}(\lambda) = \frac{k}{2\pi}\left(g_{ab}^{(2)} - \left(2\lambda\frac{k}{2\pi}\right)g_{ab}^{(4)}\right). \quad (2.7)$$

Equating this to the Fefferman-Graham expansion (2.4), it is easy to see that the deformed boundary metric can be thought of as being placed at a finite radius $r_c = \sqrt{-\frac{\pi}{k\lambda}}$. Indeed, it turns out that the Brown-York energy-momentum tensor (with the appropriate counterterm) evaluated at this surface reproduces the deformed energy-momentum tensor derived here. This makes it clear that, in pure gravity, the mixed boundary conditions and imposing Dirichlet boundary conditions at $r_c = \sqrt{-\frac{\pi}{k\lambda}}$ are equivalent.[1]

For a derivation of the mixed boundary conditions from the Chern-Simons formulation of 3D gravity, see Ref. [30].

B. CSS boundary conditions

Examples of constructing a holographic bulk dual to a WCFT are either warped AdS_3 or AdS_3 with CSS boundary conditions. We shall use the CSS boundary conditions, since they are amenable to the mixed boundary conditions from the $T\bar{T}$ deformation.

Expressing the metric in Fefferman-Graham gauge (2.4), we have the following Dirichlet-Neumann boundary conditions for the metric [6]:

$$g^{(0)} = \begin{pmatrix} P'(z) & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \qquad g_{\bar{z}\bar{z}}^{(2)} = \frac{\Delta}{k}, \quad (2.8)$$

where $k = \frac{l}{4G_N}$ and Δ is a constant. These falloff conditions are chiral, with $P(z)$ being an undetermined holomorphic function. This is to accommodate (1.1), which shifts $P'(z)$, and, hence, we must leave it undetermined. Note that this is unlike the warped geometry in [3], where the warped geometry metric is invariant under (1.1).

One can compute the full bulk metric with the CSS boundary conditions by taking the Fefferman-Graham expansion (2.4) to be

$$\frac{ds^2}{l^2} = \frac{dr^2}{r^2} + \frac{\Delta}{k}d\bar{z}^2 - \left(r^2 + \frac{2\Delta P'(z)}{k} + \frac{\Delta L(z)}{k^2r^2}\right)dzd\bar{z}$$
$$+\left(r^2P'(z) + \frac{(L(z)+\Delta P'(z))^2}{k} + \frac{\Delta L(z)P'(z)}{k^2r^2}\right)dz^2. \quad (2.9)$$

Here, both $L(z)$ and $P(z)$ are undetermined holomorphic functions and parametrize the phase space of AdS_3 with CSS boundary conditions. A special case is the Banãdos-Teitelboim-Zanelli black hole when $P'(z) = L'(z) = 0$ [6,31].

The asymptotic symmetries of this metric are interesting, as they differ from the usual product of $SL(2,\mathbb{R})$ algebras despite being locally AdS_3. To compute asymptotic Killing vectors, we first require that they preserve radial gauge:

$$\mathcal{L}_\xi g_{r\mu} = 0. \quad (2.10)$$

This fixes the asymptotic Killing vector ξ to have the form

$$\xi = rf(z,\bar{z})\partial_r + \left(V^a(z,\bar{z}) - \int\frac{g^{ab}}{r}\partial_b f(z,\bar{z})dr\right)\partial_a. \quad (2.11)$$

Evaluating this for the CSS metric (2.9), we get

$$\xi^r = rf(z,\bar{z}),$$
$$\xi^z = V^z(z,\bar{z}) - \frac{k(\partial_z f(z,\bar{z}) + (kr^2+\Delta P'(z))\partial_{\bar{z}}f(z,\bar{z}))}{k^2r^4 - \Delta L(z)},$$
$$\xi^{\bar{z}} = V^{\bar{z}}(z,\bar{z}) - \frac{k}{k^2r^4-\Delta L(z)}((kr^2+\Delta P'(z))\partial_z f(z,\bar{z})$$
$$+((2kr^2+\Delta P'(z))P'(z) + L(z))\partial_{\bar{z}}f(z,\bar{z})). \quad (2.12)$$

If we impose Dirichlet boundary conditions at infinity,

$$\lim_{r\to\infty}\mathcal{L}_\xi g_{\mu\nu} = 0, \quad (2.13)$$

we get conditions on the undetermined functions in ξ:

$$\partial_{\bar{z}}V^a(z,\bar{z}) = 0, \qquad f(z,\bar{z}) = -\frac{1}{2}\partial_z V^z(z,\bar{z}),$$
$$V^{\bar{z}}(z,\bar{z}) = P'(z)V^z(z,\bar{z}), \quad (2.14)$$

[1]We will be absorbing the factor of π into the normalization of λ and $\mathcal{O}_{T\bar{T}}$ from now on to avoid clutter in the equations.

and so we can write our asymptotic killing vector [where $V(z) \equiv V^z(z, \bar{z})$]

$$\xi(V) = -\frac{1}{2}V'(z)\partial_r + \left(V(z) + \frac{k\Delta V''(z)}{2(k^2r^4 - \Delta L(z))}\right)\partial_z + \left(P'(z)V(z) + \frac{k(kr^2 + \Delta P'(z))}{2(k^2r^4 - \Delta L(z))}V''(z)\right)\partial_{\bar{z}}. \tag{2.15}$$

Asymptotic Killing vectors generate flow in the phase space; i.e.,

$$\mathcal{L}_\xi g_{\mu,\nu} = \partial_{L(z)}g_{\mu\nu}\delta_\xi L(z) + \partial_{P'(z)}g_{\mu\nu}\delta_\xi P'(z). \tag{2.16}$$

From this, we can compute δL and δP. It turns out that ξ transforms only $L(z)$ and reproduces the infinitesimal Schwarzian transformation:

$$\delta_\xi L(z) = V(z)L'(z) + 2V'(z)L(z) - \frac{k}{2}V'''(z), \quad \delta_\xi P = 0. \tag{2.17}$$

To transform $P(z)$, we cannot allow the asymptotic killing vector to satisfy Dirichlet boundary conditions at infinity (2.13), since warped symmetry requires changing the boundary metric. The "asymptotic Killing vector"[2] which generates transformations in $P(z)$ is

$$\eta(\sigma) = \sigma(z)\partial_{\bar{z}}, \tag{2.18}$$

and the transformations of the parametrizing functions are

$$\delta_\eta L = 0, \qquad \delta_\eta P(z) = -\sigma(z). \tag{2.19}$$

Note that η also generates the warped symmetry transformation $\bar{z} \to \bar{z} + \sigma(z)$.

We can use the Fefferman-Graham expansion to compute the boundary energy-momentum tensor

$$T_{ab} = \frac{k}{2\pi}(g^{(2)}_{ab} - g^{kl}_{(0)}g^{(2)}_{kl}g^{(0)}_{ab}) = \frac{1}{2\pi}\begin{pmatrix} L(z) + \Delta P'(z)^2 & -\Delta P'(z) \\ -\Delta P'(z) & \Delta \end{pmatrix}. \tag{2.20}$$

At this point, it should be stated that this energy-momentum tensor is not the canonical energy-momentum tensor for a warped CFT. For a warped CFT defined on a manifold with warped geometry, the energy-momentum tensor is not symmetric, since symmetry of the energy-momentum tensor is a result of Lorentz invariance. However, a warped CFT dual to AdS_3 with CSS boundary conditions is not defined on a manifold with warped geometry. Rather, the manifold is not invariant under warped transformations, but for that price we gain the symmetry of the energy-momentum tensor.

The conserved charges corresponding to the asymptotic Killing vectors are

$$Q_{\xi(f)} = \frac{1}{2\pi}\int_{\partial\Sigma} d\phi n^a T_{ab}\xi^b = \frac{1}{4\pi^2}\int_0^{2\pi} d\phi f(z)L(z),$$
$$Q_{\eta(f)} = \frac{1}{2\pi}\int_{\partial\Sigma} d\phi n^a T_{ab}\eta^b = \frac{\Delta}{4\pi^2}\int_0^{2\pi} d\phi f(z)(P'(z) - 1), \tag{2.21}$$

where $\partial\Sigma$ is at $r \to \infty$, $t = \frac{z+\bar{z}}{2}$ constant, $\phi = \frac{z-\bar{z}}{2} \in (0, 2\pi)$, and $n = \partial_t = \partial_z + \partial_{\bar{z}}$.

We can now also compute the charge algebra, using the Dirac brackets of Einstein gravity:

$$\{Q_{\zeta_1(f)}, Q_{\zeta_2(g)}\} = \delta_{\zeta_1(f)}Q_{\zeta_2(g)}. \tag{2.22}$$

So we have

$$\{Q_{\xi(f)}, Q_{\xi(g)}\} = \delta_{\xi(f)}Q_{\xi(g)} = \frac{1}{4\pi^2}\int_0^{2\pi} d\phi g(z)\delta_{\xi(f)}L(z) = \frac{1}{4\pi^2}\int_0^{2\pi} g(z) \times \left(f(z)L'(z) + 2f'(z)L(z) - \frac{k}{2}f'''(z)\right). \tag{2.23}$$

Expanding the functions in modes,

$$f(z) = \sum_n f_n e^{inz}, \qquad g(z) = \sum_m g_m e^{imz},$$
$$L(z) = \sum_p L_p e^{-ipz}. \tag{2.24}$$

Replacing Dirac brackets with commutators, we obtain the Virasoro algebra

$$[L_m, L_n] = (m - n)L_{m+n} - \frac{k}{2}n^3\delta_{m,-n}. \tag{2.25}$$

Note that equating $\frac{k}{2} = \frac{c}{12}$ gives the familiar $c = 6k = \frac{3l}{2G_N}$. Similarly, we obtain a U(1) Kac-Moody algebra from the commutator of the charges Q_η:

$$[P_m, P_n] = m\Delta\delta_{m,-n}. \tag{2.26}$$

Note that the Virasoro and Kac-Moody algebra is factorized in this basis. This is presented in this form in [32], which

[2]The quotes are to indicate that, since this vector does not satisfy Dirichlet boundary conditions, it is technically not an asymptotic Killing vector, but, since it generates flows in the phase space, it will continue to be referred to as such later in this paper.

also gives the relation between this and the algebra presented in [6].

III. $T\bar{T}$ DEFORMED CSS BOUNDARY CONDITIONS

To compute the $T\bar{T}$ deformed bulk metric corresponding to the $T\bar{T}$ deformed boundary WCFT, we first compute the deformed boundary metric using (2.7):

$$\gamma_{ij}(\lambda)dz^i dz^j = -(d\bar{z} + (\lambda L(z) - P'(z))dz) \times (dz + \lambda\Delta(d\bar{z} - P'(z)dz)). \tag{3.1}$$

This metric is flat, so we express it in explicitly flat coordinates with indices a and b:

$$\gamma_{ab}(\lambda)du^a du^b = -dudv. \tag{3.2}$$

Equating the two, we can calculate the state-dependent coordinate transformation for a $T\bar{T}$ deformed WCFT, analogous to the ones introduced in [33,34]:

$$\begin{aligned} du &= dz + \lambda\Delta(d\bar{z} - P'(z)dz), \\ dv &= d\bar{z} + dz(\lambda L(z) - P'(z)), \\ dz &= \frac{du - \lambda\Delta dv}{1 - \lambda^2\Delta L(z)}, \\ d\bar{z} &= \frac{(P'(z) - \lambda L(z))du + (\lambda\Delta P'(z) - 1)dv}{1 - \lambda^2\Delta L(z)}. \end{aligned} \tag{3.3}$$

Furthermore, we can use the flow equations to compute the full bulk metric dual to the $T\bar{T}$ deformed WCFT:

$$\frac{ds^2}{l^2} = \frac{dr^2}{r^2} + \frac{(du(\lambda\Delta L + kr^2) - \Delta dv(\lambda kr^2 + 1))}{k^2r^2(\lambda^2\Delta L - 1)^2} \times (du(\lambda kr^2 + 1)L - dv(\lambda\Delta L + kr^2)), \tag{3.4}$$

where $L \equiv L(u, v) = L(z)$. Note that, on doing so, we lose the $P(z)$ degree of freedom, since this is equivalent to imposing Dirichlet boundary conditions at the constant radial surface $r_c = \sqrt{-\frac{1}{k\lambda}}$. If we are to impose Dirichlet-Neumann boundary conditions on this surface, we can recover the U(1) degree of freedom. To do so, we have to perform the transformation

$$u \to u - \lambda\Delta P(u, v), \qquad v \to v - P(u, v). \tag{3.5}$$

This is the analog of the warped symmetry transformation but now in the state-dependent coordinates. We will explore both types of boundary conditions, starting with the simpler case of only imposing Dirichlet boundary conditions.

A. Asymptotic Killing vectors I: Dirichlet boundary conditions

We will first compute the $T\bar{T}$ deformed asymptotic symmetries which preserve the deformed boundary conditions, which is equivalent to imposing Dirichlet boundary conditions at the radial cutoff surface.

Preserving radial gauge (2.10), we see that the asymptotic Killing vector in the deformed spacetime has the form

$$\begin{aligned} \xi^r(\lambda) &= rf(u, v), \\ \xi^u(\lambda) &= V^u(u, v) - \frac{k}{k^2r^4 - \Delta L}(\Delta(2\lambda kr^2 + \lambda^2\Delta L + 1)\partial_u f \\ &\quad + (\lambda\Delta L(2 + k\lambda r^2) + kr^2)\partial_v f), \\ \xi^v(\lambda) &= V^v(u, v) - \frac{k}{k^2r^4 - \Delta L}(L(2\lambda kr^2 + \lambda^2\Delta L + 1)\partial_v f \\ &\quad + (\lambda\Delta L(2 + k\lambda r^2) + kr^2)\partial_u f). \end{aligned} \tag{3.6}$$

It will be convenient to define

$$\begin{aligned} W^u(u, v) &= V^u(u, v) + k\lambda\partial_{\bar{z}} f(u, v), \\ W^v(u, v) &= V^v(u, v) + k\lambda(\partial_z f(u, v) + P'(u, v)\partial_{\bar{z}} f(u, v)), \end{aligned} \tag{3.7}$$

where, using (3.3), the derivatives in z and $\bar{z}$ are

$$\partial_{\bar{z}} = \lambda\Delta\partial_u + \partial_v, \qquad \partial_z = \partial_u + \lambda L(u, v)\partial_v - P'(u, v)\partial_{\bar{z}}. \tag{3.8}$$

In terms of W^a, the mixed boundary condition, or, equivalently, the Dirichlet boundary condition at $r = r_c$,

$$\mathcal{L}_{\xi(\lambda)} g_{\mu\nu}(\lambda)|_{r=r_c} = 0, \tag{3.9}$$

constrains the functions in $\xi(\lambda)$ to obey

$$\begin{aligned} f(u, v) &= -\frac{1}{2}\left(\frac{1 - \lambda^2\Delta L}{1 + \lambda^2\Delta L}\right)(\partial_u W^u + \partial_v W^v), \\ W^u &= -\left(\frac{\lambda\Delta}{1 + \lambda^2\Delta L}\right)(\partial_u W^u + \partial_v W^v), \\ W^v &= -\left(\frac{\lambda L}{1 + \lambda^2\Delta L}\right)(\partial_u W^u + \partial_v W^v). \end{aligned} \tag{3.10}$$

It turns out that this is not enough to solve for δL. In the undeformed case (2.15), the functions in the asymptotic Killing vector are all holomorphic functions, so we apply the holomorphicity property in the deformed case as well:

$$\partial_{\bar{z}} W^a(u, v) = 0, \qquad \partial_{\bar{z}} L(u, v) = 0. \tag{3.11}$$

Combining the previous two equations, we get the conditions

$$
\begin{aligned}
f(u,v) &= -\frac{1}{2}(1-\lambda^2\Delta L(u,v))\partial_u W^u(u,v),\\
\partial_v L(u,v) &= -\lambda\Delta\partial_u L(u,v),\\
\partial_v W^u(u,v) &= -\lambda\Delta\partial_u W^u(u,v),\\
\partial_a W^v(u,v) &= -\lambda L(u,v)\partial_a W^u(u,v).
\end{aligned}
\tag{3.12}
$$

We can use these equations to eliminate v derivatives of all the functions and all derivatives of W^v.

Now we have enough information to be able to solve for δL. To do so, we solve

$$\mathcal{L}_{\xi(\lambda)}g_{\mu\nu}(\lambda) = \partial_{L(u,v)}g_{\mu\nu}\delta_\xi L(u,v). \tag{3.13}$$

There are three equations, but, with the relations in (3.12), all three equations become identical, and the r dependence drops out. Solving for δL, we get

$$
\begin{aligned}
\delta_\xi L(u,v) = &(W^u - \lambda\Delta W^v)L'\\
&+\frac{1}{2}(4L+\lambda^2 k\Delta(1-\lambda^2\Delta L)L'')(1-\lambda^2\Delta L)W^{u\prime}\\
&+\lambda^2 k\Delta(1-\lambda^2\Delta L)^2 L' W^{u\prime\prime} - \frac{k}{2}(1-\lambda^2\Delta L)^3 W^{u\prime\prime\prime},
\end{aligned}
\tag{3.14}
$$

where $' = \partial_u$. When $\lambda \to 0$, we recover (2.17). Note that this depends on two arbitrary functions W^u and W^v.

1. *Deformed charge algebra*

To compute the symmetry algebra of the $T\bar{T}$ deformed holographic WCFT, we must compute the conserved charge algebra of the dual spacetime. We first compute the deformed boundary energy-momentum tensor using the flow equations, which also coincides with the AdS_3 Brown-York energy-momentum tensor evaluated on the constant radial surface $r = r_c$:

$$T_{ij}^{(\lambda)} = -\frac{l}{2\pi}\begin{pmatrix} \frac{L}{1-\lambda^2\Delta L} & \frac{1+\lambda k+\lambda^2\Delta L}{\lambda^2 k(1-\lambda^2\Delta L)} \\ \frac{1+\lambda k+\lambda^2\Delta L}{\lambda^2 k(1-\lambda^2\Delta L)} & \frac{\Delta}{1-\lambda^2\Delta L}\end{pmatrix}. \tag{3.15}$$

Conserved charges are defined with respect to a constant time coordinate t, which is defined in terms of u and v by

$$u = t+\phi, \qquad v = t-\phi. \tag{3.16}$$

Since L is holomorphic in z, we can express the t derivative in terms of the ϕ derivative:

$$\partial_t L = \frac{1+\lambda\Delta}{1-\lambda\Delta}\partial_\phi L. \tag{3.17}$$

So we can express the u derivatives of holomorphic functions only in ϕ derivatives as well:

$$\partial_u = \frac{1}{2}(\partial_t + \partial_\phi) = \frac{1}{1-\lambda\Delta}\partial_\phi. \tag{3.18}$$

For constant t, we can now eliminate W^v in (3.14), using Eqs. (3.12) and (3.18):

$$\partial_u W^v = -\lambda L(\phi)\partial_u W^u \Rightarrow \partial_\phi W^v = -\lambda L(\phi)\partial_\phi W^u. \tag{3.19}$$

Integrating over ϕ, we have

$$W^v(\phi) = \int^\phi d\phi' L(\phi')\partial_{\phi'}W^u(\phi'). \tag{3.20}$$

Now we can label the variation of the conserved charges with only one arbitrary function W^u. Using (2.21) but with the deformed energy-momentum tensor, the conserved charge is

$$Q_f = \frac{l}{4\pi^2}\int_0^{2\pi} d\phi f(\phi)\frac{\Delta - L(\phi)}{1-\lambda^2\Delta L(\phi)}. \tag{3.21}$$

We can now compute the charge algebra:

$$
\begin{aligned}
\{Q_W, Q_f\} &= \delta_W Q_f\\
&= \frac{l}{4\pi^2}\int_0^{2\pi} d\phi f\\
&\quad\times\left(\frac{-\delta_W L}{1-\lambda^2\Delta L} + \frac{\Delta - L}{(1-\lambda^2\Delta L)^2}(\lambda^2\Delta\delta_W L)\right).
\end{aligned}
\tag{3.22}
$$

Using (3.14) and (3.20), substituting $f(\phi) = e^{im\phi}$ and $W(\phi) = e^{in\phi}$, and removing ϕ derivatives from L using integration by parts, we have

$$
\begin{aligned}
\{Q_W, Q_f\} = \delta_W Q_f = &\frac{l(1+\lambda\Delta)}{8\pi^2(1-\lambda\Delta)^2}\int_0^{2\pi} d\phi \frac{1}{1-\lambda^2\Delta L(\phi)}\Big[2in^3 k(1-\lambda\Delta)^3 e^{i(m+n)\phi}(1-\lambda^2\Delta L(\phi))\\
&+ e^{im\phi}L(\phi)\Big(2mn\lambda\Delta(1-\lambda\Delta)^2\int^\phi e^{in\phi'}L(\phi')d\phi' - ie^{in\phi}(n\lambda^2 k\Delta(1-\lambda^2\Delta L(\phi))\\
&\times(n^2\lambda\Delta(3-\lambda\Delta(3-\lambda\Delta)) - m^2)) - 2(1-\lambda\Delta)^2((m-n) - 2n\lambda\Delta L(\phi))\Big)\Big].
\end{aligned}
\tag{3.23}
$$

Since L is not independent of t, only the zero modes are conserved in time. In this choice of basis of functions and Fourier modes, the central charge term is state dependent. This is similar to what was found in [23] for a $T\bar{T}$ deformed CFT. It is straightforward to verify that, on taking the $\lambda \to 0$ limit and expressing L in Fourier modes, one recovers the Virasoro algebra.

B. Asymptotic Killing vectors II: Dirichlet-Neumann boundary conditions

If we want to impose the same boundary conditions at the radial cutoff in the $T\bar{T}$ deformed metric as the undeformed Dirichlet-Neumann CSS boundary conditions at infinity of the undeformed metric, we have to find a global Killing vector which corresponds to translations on the boundary. It is easy to verify that $\lambda\Delta\partial_u + \partial_v$ is such a global Killing vector of (3.4). To generate transformations in the boundary metric, we then promote this global Killing vector to an "asymptotic Killing vector" analogous to (2.18):

$$\eta(\lambda;\sigma) = -\sigma(u,v)(\lambda\Delta\partial_u + \partial_v). \tag{3.24}$$

To introduce the $P(z)$ degree of freedom back into the metric (3.4), one can make the coordinate transformation (3.5):

$$u \to u - \lambda\Delta h(u,v), \qquad v \to v - h(u,v), \tag{3.25}$$

which is generated by the asymptotic Killing vector (3.24) as the analog to the warped transformation $\bar{z} \to \bar{z} - h(z)$. The state-dependent coordinate transformation is now

$$\begin{aligned} du - \lambda\Delta \mathrm{d}(h(u,v)) &= dz + \lambda\Delta \mathrm{d}(\bar{z} - P(z)),\\ dv - \mathrm{d}(h(u,v)) &= d\bar{z} + (\lambda L(z) - P'(z))dz,\\ dz &= \frac{du - \lambda\Delta dv}{1-\lambda^2\Delta L},\\ d\bar{z} &= \frac{dv - \lambda L du + (du + \lambda\Delta dv)P'(z)}{1-\lambda^2\Delta L}\\ &\quad - (\mathrm{d}(h(u,v)), \end{aligned} \tag{3.26}$$

where d is the exterior derivative. Note that, since both h and P are arbitrary functions of (u,v), we can choose the gauge where $h = P$. The coordinate transformation now becomes much simpler:

$$\begin{aligned} du &= dz + \lambda\Delta\bar{z}, \qquad dv = d\bar{z} + \lambda L(z)dz,\\ dz &= \frac{du - \lambda\Delta dv}{1-\lambda^2\Delta L}, \qquad d\bar{z} = \frac{dv - \lambda L du}{1-\lambda^2\Delta L}. \end{aligned} \tag{3.27}$$

The metric now reads

$$\begin{aligned} ds^2 = l^2\frac{dr^2}{r^2} &+ \frac{l^2}{k^2r^2(1-\lambda^2\Delta L)^2}((kr^2(\lambda^2\Delta L - 1)\partial_u h - (1+\lambda kr^2)L)du + (kr^2\partial_v h(\lambda^2\Delta L - 1) + \lambda\Delta L + kr^2)dv)\\ &\times((\Delta\partial_u h(\lambda^2\Delta L - 1) - \lambda\Delta L - kr^2)du + \Delta(\partial_v h(\lambda^2\Delta L - 1) + \lambda kr^2 + 1)dv). \end{aligned} \tag{3.28}$$

This metric still has the asymptotic Killing vector (3.24), and when $\sigma = 1$ it is a global Killing vector. Computing the flow in phase space generated by (3.24), we have

$$\mathcal{L}_{\eta(\lambda,\sigma)}g_{\mu\nu}(\lambda; L(u,v), \partial_u h(u,v)) = \partial_L g_{\mu\nu}\delta L + \partial_h g_{\mu\nu}\delta h, \tag{3.29}$$

which, on solving, we see that we recover the undeformed U(1) symmetry:

$$\delta L = 0, \qquad \delta h = \sigma. \tag{3.30}$$

Now we will see if on performing the warped transformation (3.25) we lose the deformed Virasoro symmetry (3.14). The vector field which preserves radial gauge is

$$\begin{aligned} \xi^r &= rf(u,v),\\ \xi^u &= V^u(u,v) + \frac{k}{(\Delta L - k^2r^4)(1-\lambda\Delta\partial_u h - \partial_v h)^2}[\partial_v f(\Delta\lambda L(\Delta\lambda\partial_u h(-2\partial_v h + k\lambda r^2 + 1)\\ &\quad + (1-\lambda kr^2)\partial_v h - \lambda kr^2 - 2) + \Delta\partial_u h(\partial_v h - \lambda kr^2 - 1) + kr^2(\partial_v h - 1) + \lambda^3\Delta^2L^2\partial_v h(\lambda\Delta\partial_u h - 1))\\ &\quad - \Delta\partial_u f(2(\lambda kr^2 + 1)(\lambda^2\Delta L - 1)\partial_v h + (\partial_v h)^2(\lambda^2\Delta L - 1)^2 + \lambda^2\Delta L + 2\lambda kr^2 + 1)],\\ \xi^v &= V^v(u,v) + \frac{k}{(\Delta L - k^2r^4)(1-\lambda\Delta\partial_u h - \partial_v h)^2}[\partial_u f(-\partial_v h(\Delta\lambda^2 L - 1)(\Delta\lambda L + kr^2)\\ &\quad + \Delta\partial_u h(\Delta\lambda^2 L - 1)(\partial_v h(\Delta\lambda^2 L - 1) + k\lambda r^2 + 1) - \lambda\Delta L(\lambda kr^2 + 2) - kr^2)\\ &\quad + \partial_v f(2\partial_u h(\lambda^2\Delta L - 1)(\Delta\lambda L + kr^2) - \Delta(\partial_u h)^2(\Delta\lambda^2 L - 1)^2 - L(\Delta\lambda^2 L + 2k\lambda r^2 + 1))]. \end{aligned} \tag{3.31}$$

To have solutions which preserve the mixed boundary conditions, we are required to impose holomorphicity of h in $z, \bar{z}$ coordinates:

$$\partial_{\bar{z}} h = 0 \Rightarrow \partial_v h = -\lambda\Delta\partial_u h. \tag{3.32}$$

To simplify the equations, one can introduce the following definitions:

$$\begin{aligned} W^u &= V^u - \lambda^3 k\Delta^2(1-\lambda^2\Delta L)\partial_u f \partial_u h, \\ W^v &= V^v + \lambda k(1-\lambda^2\Delta L)\partial_u f(1-\lambda\Delta\partial_u h), \\ X &= W^u - \lambda\Delta W^v. \end{aligned} \tag{3.33}$$

To compute the variation of the functions L and h, it will be necessary to impose holomorphicity in z for the functions W^a, X, and f. We can now compute the variation of the metric which preserves the mixed boundary conditions or, equivalently, impose Dirichlet boundary conditions at the constant radial surface:

$$\mathcal{L}_{\xi(\lambda,\sigma)} g_{\mu\nu}(\lambda; L(u,v), \qquad \partial_u h(u,v))|_{r=r_c} = 0. \tag{3.34}$$

This is a set of three equations; however, only two are linearly independent. The conditions we get from solving the above equations are

$$\begin{aligned} f(u,v) &= -\frac{1}{2}\left(\frac{1-\lambda^2\Delta L}{1+\lambda^2\Delta L}\right)X'(u,v), \\ X(u,v) &= \frac{W^v(u,v)}{h''(u,v)} - \frac{h'(u,v)-\lambda L(1-\lambda\Delta h'(u,v))X'(u,v)}{h''(u,v)(1+\lambda^2\Delta L)}, \end{aligned} \tag{3.35}$$

where $' \equiv \partial_u$.

We are now in a position to compute the flow of the metric in phase space generated by this vector field subject to the above constraints:

$$\mathcal{L}_{\xi(\lambda,\sigma)} g_{\mu\nu}(\lambda; L(u,v), \partial_u h(u,v)) = \partial_L g_{\mu\nu}\delta L_\xi + \partial_h g_{\mu\nu}\delta h_\xi. \tag{3.36}$$

As before, this set of three equations subject to the constraints reduces to two equations and removes any dependence on the radial coordinate r. The variations of the functions h and L are

$$\begin{aligned} \delta_\xi h &= 0, \\ \delta_\xi L &= \frac{1}{2\Theta^2 h''}((2\lambda k L' L_m^2 L_p(2\lambda\Delta h' - 1)h'' + 3kL_m^3 L_p^2(h'')^2)W'' - k\Theta L_m^3 L_p h'' W''' \\ &\quad + X'(-2\lambda^2 k(L')^2 L_m^2 h''(1 - 3\lambda\Delta h' + 2\lambda^2\Delta^2(h')^2) - 6kL_m^3 L_p^2(h'')^3 + h''(-\Theta L_m(\lambda k L''(1-2\lambda\Delta h') \\ &\quad + 4\lambda L^2(1-\lambda\Delta h') + L(-\lambda^3 k\Delta L'' - 2(2-\lambda^4 k\Delta^2 L'')h')) + 6k\Theta L_m^3 L_p h''') - L'(2\Theta^3 \\ &\quad + \lambda k L_m^2(-7 + 8\lambda\Delta h' - \lambda^2\Delta L(1-8\lambda\Delta h'))(h'')^2 + 2\lambda k\Theta L_m^2(1-2\lambda\Delta h')h''') - k\Theta^2 L_m^3 h'''') \\ &\quad + W'(L'(2\Theta^2 L_p + 2\lambda k L_m^2 L_p(1-2\lambda\Delta h')h''' + k\Theta L_m^3 L_p h'''') - 3kL_m^3 L_p^2 h'' h''')), \end{aligned} \tag{3.37}$$

where

$$\begin{aligned} \Theta &= \partial_u h - \lambda L(1-\lambda\Delta\partial_u h), \qquad W = W^v, \\ L_m &= 1-\lambda^2\Delta L, \qquad L_p = 1+\lambda^2\Delta L. \end{aligned} \tag{3.38}$$

We see that we still preserve a deformed Virasoro generator and do not generate a transformation in the U(1) generator. However, the algebra produced the modes of the charges will not be closed, as the variation of L depends on the U(1) generator h. We will not compute a charge algebra for this, since it is not illuminating but, in principle, can be computed using the same procedure outlined in the previous section.

Let us compare the results of this section with Sec. III A. We find that the $T\bar{T}$ deformation does not affect the spin 1 currents, and, therefore, the deformed theory should retain whatever Kac-Moody algebra the undeformed theory has. This suggests that the results in Sec. III A are only a special case of this section, where the bulk dual is dual to a state with zero momentum in the boundary deformed WCFT.

IV. DISCUSSION

In this paper, we computed the $T\bar{T}$ deformed generators of a warped CFT, using holographic techniques developed in [23,27]. Previously, holographic $T\bar{T}$ techniques have been used to compute $T\bar{T}$ deformations of holographic CFTs. Since the $T\bar{T}$ deformation is a double-trace deformation, the boundary conditions of the holographic bulk dual are modified. For the $T\bar{T}$ deformation, this can be interpreted as imposing Dirichlet boundary conditions at a finite radial surface for the bulk metric. However, when considering a holographic WCFT dual to AdS_3, one has to employ the CSS boundary conditions [6], which are Dirichlet-Neumann boundary conditions for the bulk metric.

We, therefore, computed the $T\bar{T}$ deformed CSS boundary conditions, by imposing either only Dirichlet boundary conditions at the cutoff radial surface or Dirichlet-Neumann boundary conditions at the same surface. Using this, we computed the $T\bar{T}$ deformed asymptotic symmetry algebra for both cases and found that, for a $T\bar{T}$

deformed holographic WCFT, the U(1) Kac-Moody generators are not affected, but the Virasoro generators are deformed in a nonlinear way. In fact, when considering the Dirichlet-Neumann boundary conditions at the finite radial surface, we see that the deformed Virasoro generator will no longer create a closed algebra with itself, but the full deformed asymptotic algebra is still closed. This suggests that the symmetry algebra of the $T\bar{T}$ deformed WCFT still contains the U(1) Kac-Moody algebra, which follows from the fact that the $T\bar{T}$ deformation preserves translation invariance.

A natural question to ask now is which of the two boundary conditions corresponds to the correct $T\bar{T}$ flow of the boundary field theory. Since the $T\bar{T}$ deformation does not effect spin 1 currents, the deformed theory should not lose the U(1) Kac-Moody algebra, which suggests that the second approach yields the correct deformed theory.

This result strengthens and extends the proposals of [22,23,27] to the case of an example of bottom-up holography where the boundary theory is not a conformal field theory but instead a nonrelativistic theory. It will be interesting to explore how the holographic $T\bar{T}$ dictionary extends to other examples of holography and, in particular, non-AdS holography.

There are many directions one can take from here. Another starting point for a bulk dual to a nonrelativistic QFT would be a $J\bar{T}$ deformation of a CFT dual to AdS_3 with a U(1) Chern-Simons matter field, to generate CSS-like boundary conditions [35]. Since warped CFTs can also be formulated as dual to modified gravity theories with a warped AdS bulk, it would also be interesting to use the Chern-Simons formalism of holographic $T\bar{T}$ [30] to compute the $T\bar{T}$ deformations of WCFT dual to warped AdS_3 as a solution to lower spin gravity [3] or as a solution to massive gravity [36,37]. Stepping away from holography, it would be interesting to compute the $T\bar{T}$ deformed WCFT partition function and explore the deformations of other nonrelativistic QFTs such as the quantum Lifshitz model in $2 + 1$D, which will require understanding $T\bar{T}$ deformations in higher dimensions.

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Paper III

A generalized Selberg zeta function for flat space cosmologies

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A generalized Selberg zeta function for flat space cosmologies

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Abstract: Flat space cosmologies (FSCs) are time dependent solutions of three-dimensional (3D) gravity with a vanishing cosmological constant. They can be constructed from a discrete quotient of empty 3D flat spacetime and are also called shifted-boost orbifolds. Using this quotient structure, we build a new and generalized Selberg zeta function for FSCs, and show that it is directly related to the scalar 1-loop partition function. We then propose an extension of this formalism applicable to more general quotient manifolds $\mathcal{M}/\mathbb{Z}$, based on representation theory of fields propagating on this background. Our prescription constitutes a novel and expedient method for calculating regularized 1-loop determinants, without resorting to the heat kernel. We compute quasinormal modes in the FSC using the zeroes of a Selberg zeta function, and match them to known results.

Keywords: Space-Time Symmetries, Differential and Algebraic Geometry, Cosmological models

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Contents

1 Introduction

Three dimensional (3D) spacetime has been a very useful playground for understanding aspects of gravity and especially quantum gravity. The lack of propagating degrees of freedom in 3D Einstein gravity was, in the early days, thought of as an indication of the triviality of the theory. The discovery of the Bañados-Teitelboim-Zanelli (BTZ) black holes in 3D Anti-de Sitter spacetimes (AdS_3) [1] made it clear that 3D gravity was rich with structure but more analytically tractable than its higher dimensional avatars, making it a particularly useful tool in understanding elusive quantum aspects.

Non-extremal BTZ black holes of generic mass and angular momentum make up the most general zero mode solutions with Brown-Henneaux [2] boundary conditions. BTZ black holes can be understood geometrically as orbifolds of AdS_3 [3]. For asymptotically flat spacetimes in 3D, a very similar story exists. The most generic zero mode solutions with boundary conditions outlined in [4] are cosmological solutions with mass and angular momentum [5] called Flat Space Cosmologies (FSCs). FSCs can also be understood as orbifolds of flat space. Specifically the FSCs correspond to the shifted-boost orbifold of 3D flat space. The connection between the non-extremal BTZs and FSCs is actually more profound. Minkowski spacetimes can of course be reached by an infinite radius limit of AdS spacetimes. One can take a similar infinite radius limit on BTZ black holes. This rather straightforward exercise is made interesting by the fact that there are no black holes in 3D flat space. What the limit does is take the outer horizon of the non-extremal BTZ to infinity while keeping the inner horizon at a finite value [5, 6]. We thus end up with a spacetime which is the inside of the original black hole. The radial and temporal directions are flipped and hence we have a cosmological spacetime instead of a black hole.

3D quotient manifolds (such as the BTZ black hole and FSC spacetime) are ideal for studying quantum corrections in the presence of horizons. The 3D aspect is important, because gravity is known to be 1-loop exact in this context [7, 8]. The quotient aspect is also important: in recent years, a technique has been developed to expediently calculate functional determinants of kinetic operators from the spacetime quotient structure alone [9–11]. In this technique, one utilizes the generators of the quotient group to build a Selberg-like zeta function $Z_\Gamma(s)$, which is directly related to the regularized scalar 1-loop partition function. For example, the 1-loop partition function for a complex scalar field of mass m is

$$Z^{(1)}(\Delta) = \int \mathcal{D}\phi e^{-\int \phi^*(-\nabla^2+m^2)\phi} \propto \det\left(-\nabla^2 + m^2\right)^{-1}, \tag{1.1}$$

where Δ is the conformal dimension of the scalar field (for AdS_3, we have $\Delta = 1+\sqrt{1+L^2m^2}$). Traditional methods of computing (1.1) include the heat kernel (for example, [8]), which one must then regulate. Alternatively, using the Selberg technique, we directly obtain

$$Z^{(1)}_{\text{reg}}(\Delta) = \frac{1}{Z_\Gamma(\Delta)}. \tag{1.2}$$

The original Selberg zeta function is defined for the quotient manifold $\mathbb{H}^2/\Gamma$ [12], where Γ is a discrete subgroup of $\text{SL}_2(\mathbb{R})$, the isometry group of $\mathbb{H}^2$. Selberg introduced his zeta function while studying his trace formula for characters of representations of a Lie group G on the space of square-integrable functions on G/Γ, where Γ is a discrete subgroup of G. The inputs for the function are details of the quotient, in particular the lengths and the number of primitive geodesics, for example when $\mathbb{H}^2/\Gamma$ is geometrically finite, the Selberg zeta function is defined as

$$Z_{\mathbb{H}^2/\Gamma}(s) = \prod_p \prod_{n=0}^{\infty} (1 - e^{-(s+n)\ell(p)}), \tag{1.3}$$

where p denotes the conjugacy classes of primitive geodesics, $\ell(p)$ is the length of the primitive geodesic p, and Γ consists of the identity and hyperbolic elements only.

There is a beautiful number-theoretic aspect to the Selberg approach as well: the zeros of Selberg zeta function correspond to the quasinormal modes (QNMs) of the field in question. Thus, Selberg technique can be viewed as an extension of the work of Denef, Hartnoll and Sachdev [13], who used the Weierstrass factorization theorem to cast $Z^{(1)}$ as a product over its zeros and poles. The extension is that, with the Selberg method, we seem to get the overall factor $e^{\text{Pol}(\Delta)}$ for free, without resorting to the heat kernel.

The Selberg technique was introduced in [9] in the context the BTZ black hole, using the results of mathematicians Perry and Williams [14]. The approach was then extended to higher-dimensional hyperbolic quotients in [15] and for higher spin fields in [10]. An important extension to this program appeared in [11], where a Selberg zeta function was built and defined for a non-hyperbolic quotient manifold (namely, the warped AdS_3 black hole). In addition to the physical results, the work [11] is of mathematical interest, potentially signalling the validity of the celebrated Selberg trace formula beyond hyperbolic quotients. Emboldened by the warped AdS_3 result, in this work we endeavor to construct a Selberg zeta function for flat space quotients, using the FSC as a prototypical example mirroring the BTZ solution.

In both the contexts of the BTZ black hole and the FSC spacetime, we are further able to recast the Selberg zeta function using representation theory. We conjecture a general prescription for computing for the Selberg zeta function on general smooth quotient manifolds $\mathcal{M}/\Gamma$, where $\Gamma \sim \mathbb{Z}$:

$$\zeta_{\mathcal{M}/\Gamma}(s) = \prod_{\text{descendants}} \langle 1-\gamma \rangle_{\text{scalar primary of weight } s} \tag{1.4}$$

where $\gamma \in \Gamma$. We are optimistic that this expression provides a tantalizing arena to further study the physical meaning of the so-called non-standard representations the appear in the calculation of Wilson spools [16, 17]. We will descibe this connection more fully in the Discussion section.

The seminal analysis of Brown and Henneaux [2] revealed that the asymptotic symmetries of 3D asymptotically AdS spacetimes enhanced from the isometry algebra $so(2,2)$ to two copies of the Virasoro algebra

$$\begin{aligned}
[\mathcal{L}_n, \mathcal{L}_m] &= (n-m)\mathcal{L}_{n+m} + \frac{c}{12} n^2(n-1)\delta_{n+m,0}, \\
[\bar{\mathcal{L}}_n, \bar{\mathcal{L}}_m] &= (n-m)\bar{\mathcal{L}}_{n+m} + \frac{\bar{c}}{12} n^2(n-1)\delta_{n+m,0}, \quad [\mathcal{L}_n, \bar{\mathcal{L}}_m] = 0.
\end{aligned} \tag{1.5}$$

In the above, c and $\bar{c}$ are the central charges, which for Einstein gravity is given by

$$c = \bar{c} = \frac{3L}{2G}, \tag{1.6}$$

where L is the AdS radius and G the Newton's constant. The symmetries are of course that of a 2D conformal field theory and the Brown-Henneaux analysis is often look upon as a precursor to the AdS/CFT correspondence. Due to the underlying infinite dimensional symmetries, AdS_3/CFT_2 has become a favourite testing ground for the holographic principle. BTZ black holes are dual to thermal states on the field theory side and have played (and continue to play) a starring role in lower dimensional AdS holography.

Barnich and Compere [18] showed that an analysis similar to Brown and Henneaux in 3D asymptotically flat spacetime led to what is called the 3D Bondi-van der Burg-Metzner-Sachs (BMS_3) algebra.

$$\begin{aligned}
[\mathfrak{L}_n, \mathfrak{L}_m] &= (n-m)\mathfrak{L}_{n+m} + \frac{c_{\mathfrak{L}}}{12}\delta_{n+m,0}\, n^2(n-1), \\
[\mathfrak{L}_n, \mathfrak{M}_m] &= (n-m)\mathfrak{M}_{n+m} + \frac{c_{\mathfrak{M}}}{12}\delta_{n+m,0}\, n^2(n-1), \quad [\mathfrak{M}_n, \mathfrak{M}_m] = 0.
\end{aligned} \tag{1.7}$$

The central terms can again be computed and for Einstein gravity they are given by

$$c_{\mathfrak{L}} = 0, \quad c_{\mathfrak{M}} = \frac{3}{G}. \tag{1.8}$$

Taking a cue out of the discussion above about holography in AdS, a line of research on the construction of holography for flat spacetimes has emerged which currently is called Carrollian holography. The principle claim is that the holographic dual of 3D asymptotically flat spacetimes is a 2D quantum field theory which has the infinite dimensional BMS_3 algebra as its symmetries [19, 20]. The word Carrollian is reflective of the fact that the 2D field theories with this symmetry can be obtained in a Carrollian limit of 2D relativistic CFTs when one takes the speed of light to zero [21, 22]. In the above context, this is the fact that the BMS_3 algebra can be obtained from an Inönü-Wigner contraction of the two copies of Virasoro algebra:

$$\mathfrak{L}_n = \mathcal{L}_n - \bar{\mathcal{L}}_{-n}, \quad \mathfrak{M}_n = \frac{1}{L}\left(\mathcal{L}_n + \bar{\mathcal{L}}_{-n}\right), \quad L \to \infty. \tag{1.9}$$

Here the inverse of the AdS radius plays the role of the speed of light in the boundary theory and the $L \to \infty$ limit is analogous to the $c_l \to 0$ limit, where c_l is the speed of light. More generally, the BMS_{d+1} algebra which one can obtain from the asymptotic analysis of d-dimensional flat spacetimes and the Conformal $\mathrm{Carroll}_d$ algebra that arises in the $c_l \to 0$ limit of relativistic conformal algebra in $(d-1)$- dimensions are isomorphic [21, 22]. The proposal that Carrollian CFTs are putative duals of asymptotically flat spacetimes thus passes the first and most obvious check of holography, the matching of bulk and boundary symmetries. Carrollian holography is a co-dimension one holographic dual and has recently also been used in the more physically interesting 4D asymptotically flat spacetimes, starting with [23, 24]. The Carrollian approach is to be contrasted to the Celestial approach which advocates a co-dimension two holographic dual [25, 26]. A comparison between the two approaches has been recently discussed in [27].

This line of enquiry has led to some interesting successes in the 3D bulk — 2D boundary case, including the matching of the bulk entropy to a BMS-Cardy analysis [6], matching of entanglement entropy between the bulk and boundary theories [28, 29], matching of stress-tensor correlations [30]. In a manner analogous to the BTZ black hole, the FSC solutions play a central role in the construction of holography for 3D flatspace. The entropy of the FSC is reproduced by the above mentioned BMS-Cardy analysis, which uses a Carrollian version of modular transformation [6]. One can also look at a matching of structure constants of the 2D Carrollian CFTs with an analysis of a one-point function probe in the background of a FSC. This was done in [31] following methods similar to the AdS story [32]. The BMS-Cardy and the one-point analysis has been generalised to FSCs with extra U(1) symmetries in [33]. There is a natural generalisation to Carrollian torus two point functions and connections to bulk quasi-normal modes (QNM) in [34].

Our investigations in this paper would address the question of QNM of the FSC solution from a completely different point of view, using the method of the Selberg zeta function which we alluded to above and go on to describe in more detail below. The most obvious way forward in computing say scalar QNM, i.e. by solving the Klein-Gordon equation in the FSC background, runs into problems as one now needs to put boundary conditions on the cosmological horizon as opposed to the black hole horizon. We choose to circumvent this problem by appealing to the quotient structure of the FSC and generalizing the construction of the Selberg-zeta function to these orbifolds of 3D flatspace. We will see that we will reproduce results of FSC QNM of [34] derived earlier using Carroll modular transformation techniques.

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This paper is organized as follows. In section 2, we review the two necessary topics for this work: the Selberg zeta function in the context of BTZ black holes and the relevant aspects of the FSC spacetime. In section 3 we derive the Selberg zeta function for FSC spacetimes in two ways: (1) using representation theory as in equation (1.4), and (2) from the quotient group action, as reviewed in section 2.1. In section 4, we show that our FSC Selberg zeta function does indeed reproduce the correct scalar 1-loop partition function. In section 5, we calculate the zeros of the FSC Selberg zeta function, and compare them to the FSC QNMs that were calculated in [34]. In section 6 we review our results and discuss future directions.

2 Review

We review the two main ideas that we will need to construct a Selberg-like zeta function for FSC spacetimes. In section 2.1, we review how a Selberg zeta function was built for the Euclidean BTZ black hole ($\mathbb{H}^3/\mathbb{Z}$) [14]. In section 2.2, we review the geometry and quotient structure of our spacetime of interest: flat space cosmologies ($\mathbb{R}^3/\mathbb{Z}$) [5, 6, 35].

2.1 Selberg zeta function for $\mathbb{H}^3/\mathbb{Z}$

In this section, we review how to construct the Selberg zeta function for the Euclidean BTZ black hole, which has quotient structure $\mathbb{H}^3/\mathbb{Z}$. Many of the ideas that we will discuss in this section were first presented in [14].

We begin with the BTZ black hole metric in Boyer-Lindquist-like coordinates

$$ds^2 = -\frac{(r^2-r_+^2)(r^2-r_-^2)}{L^2r^2}dt^2 + \frac{L^2r^2}{(r^2-r_+^2)(r^2-r_-^2)}dr^2 + r^2\left(d\phi - \frac{r_+r_-}{Lr^2}dt\right)^2, \tag{2.1}$$

where L is the AdS radius, and the outer and inner horizons, r_+ and r_-, are related to the black hole's mass M and angular momentum J:

$$r_\pm = \sqrt{2GL(LM+J)} \pm \sqrt{2GL(LM-J)}. \tag{2.2}$$

The Euclidean BTZ black hole is obtained from (2.1) via the transformations $t \to -i\tau$, $J \to -iJ_E$ and $r_- \to -i|r_-|$. The Euclidean BTZ black hole metric can be built from the Poincaré patch metric

$$ds^2 = \frac{L^2}{z^2}(dx^2 + dy_E^2 + dz^2) \tag{2.3}$$

through a set of discontinuous coordinate transformations, valid in regions $r > r_+$, $r_+ > r > r_-$ and $r_- > r$ [3]. For concreteness we will focus on the coordinate transformation valid for $r > r_+$:

$$\begin{aligned}
x &= \sqrt{\frac{r^2-r_+^2}{r^2-r_-^2}}\cos\left(\frac{r_+\tau}{L^2} + \frac{|r_-|\phi}{L}\right)\exp\left(\frac{r_+\phi}{L} - \frac{|r_-|\tau}{L^2}\right)\\
y_E &= \sqrt{\frac{r^2-r_+^2}{r^2-r_-^2}}\sin\left(\frac{r_+\tau}{L^2} + \frac{|r_-|\phi}{L}\right)\exp\left(\frac{r_+\phi}{L} - \frac{|r_-|\tau}{L^2}\right)\\
z &= \sqrt{\frac{r_+^2-r_-^2}{r^2-r_-^2}}\exp\left(\frac{r_+\phi}{L} - \frac{|r_-|\tau}{L^2}\right).
\end{aligned} \tag{2.4}$$

We can now view the coordinate transformation (2.4) through a group theoretic lens. The identification $\phi \sim \phi + 2\pi$ allows for the BTZ black hole to be understood as a quotient of AdS_3 by a discrete subgroup $\Gamma \sim \mathbb{Z}$ of the isometry group $SL_2(\mathbb{R})\times SL_2(\mathbb{R})$. We can study the group action of the single generator $\gamma \in \Gamma$ by taking $\phi \to \phi + 2\pi n$ in (2.4). This will map a point $(x, y_E, z) \in \mathbb{H}^3$ to another point (x', y'_E, z')

$$\gamma^n \cdot (x, y_E, z) = (x', y'_E, z') \tag{2.5}$$

through

$$\begin{aligned} x' &= e^{2\pi n r_+/L}(x\cos(2\pi n|r_-|/L) - y_E \sin(2\pi n|r_-|/L)) \\ y'_E &= e^{2\pi n r_+/L}(y_E\cos(2\pi n|r_-|/L) + x \sin(2\pi n|r_-|/L)) \\ z' &= e^{2\pi n r_+/L} z. \end{aligned} \tag{2.6}$$

By inspecting (2.6), it is clear that the group action can be understood as a dilation and a rotation:

$$\gamma \begin{pmatrix} x \\ y_E \\ z \end{pmatrix} = \begin{pmatrix} e^{2a} & 0 & 0 \\ 0 & e^{2a} & 0 \\ 0 & 0 & e^{2a} \end{pmatrix} \begin{pmatrix} \cos 2b_E & -\sin 2b_E & 0 \\ \sin 2b_E & \cos 2b_E & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y_E \\ z \end{pmatrix}, \tag{2.7}$$

where $a = \pi r_+/L$ and $b_E = \pi|r_-|/L$. The matrix γ has three eigenvalues: e^{2a} and $e^{2a\pm 2ib_E}$. The authors of [14] used these eigenvalues to construct the Selberg zeta function of the BTZ black hole (which is explained in detail in section 3.2):

$$Z_\Gamma(s) = \prod_{k_1,k_2=0}^{\infty} \left[1 - e^{-(2a-2ib_E)k_1} e^{-(2a+2ib_E)k_2} e^{-2as}\right]. \tag{2.8}$$

In [9], it was shown explicitly that the function $Z_\Gamma(s)$ is directly related to the *regularized* 1-loop partition function for a complex scalar field, obtained in a rather simple way from the spacetime quotient structure alone. As such, the authors of [9] also showed that the zeros of $Z_\Gamma(s)$ are mapped to the quasinormal modes of a scalar field propagating on a BTZ background, as expected from the work of Denef, Hartnoll and Sachdev [13].

We end this section with two comments. First, another useful way to write the zeta function (2.8) is

$$Z_\Gamma(s) = \prod_{k_1,k_2=0}^{\infty} \left[1 - q^{k_2+s/2}\bar{q}^{k_1+s/2}\right], \tag{2.9}$$

where $q = e^{2\pi i\tau}$, $\tau = \tau_1 + i\tau_2$, $a = \pi\tau_2$ and $b_E = -\pi\tau_1$. This has the benefit of having a symmetric contribution of s. Second, it is worth mentioning that the quotient generator $\gamma \in \Gamma$ can be written as a linear combination of embedding generators [3], using the conventions in appendix A and (2.6)

$$\partial_\phi = \frac{r_+}{L} J_{U,Y} - \frac{|r_-|}{L} J_{X,V_E}. \tag{2.10}$$

Here $J_{A,B}$ are the isometries in the embedding space which preserve the AdS_3 hyperboloid, defined in (A.3). In Poincaré coordinates (2.3), it is evident that these generate dilations

$J_{U,Y} = x\partial_x + y_E\partial_{y_E} + z\partial_z$ and rotations $J_{X,V_E} = y_E\partial_x - x\partial_{y_E}$. Writing (2.10) in terms of the $SL_2(\mathbb{R})$ generators $\mathcal{L}_n$ and $\bar{\mathcal{L}}_n$ using (A.11) and performing a Wick rotation, we have

$$2\pi\partial_\phi = 2\pi i\left(-\frac{|r_-|}{L}(\mathcal{L}_0 - \bar{\mathcal{L}}_0) + i\frac{r_+}{L}(\mathcal{L}_0 + \bar{\mathcal{L}}_0)\right). \tag{2.11}$$

Identifying these coefficients with $\tau_1 = -b_E/\pi = -|r_-|/L$ and $\tau_2 = a/\pi = r_+/L$, we have

$$2\pi\partial_\phi = 2\pi i((\mathcal{L}_0 - \bar{\mathcal{L}}_0)\tau_1 + i(\mathcal{L}_0 + \bar{\mathcal{L}}_0)\tau_2). \tag{2.12}$$

This form of writing the quotient generator is useful in generalizing the Selberg zeta function to other quotient manifolds, as seen in section 3.1.

2.2 Flat space cosmology

As discussed in the introduction, flat space cosmologies can be obtained by taking a particular large L limit of the BTZ black hole as well as an orbifold of 3d flat spacetimes [5, 6]. In this section, we review the limiting construction and the quotient structure of flat space cosmologies: $\mathbb{R}^3/\mathbb{Z}$ [5, 6, 35]. We will also discuss briefly properties of these solutions and their importance in the context of flatspace holography.

We begin by consider again the non-extremal BTZ black hole metric (2.1):

$$ds^2_{\text{BTZ}} = \left(8GM - \frac{r^2}{L^2}\right)dt^2 + \frac{dr^2}{-8GM + \frac{r^2}{L^2} + \frac{16G^2J^2}{r^2}} - 8GJdtd\phi + r^2d\phi^2. \tag{2.13}$$

To obtain a flat space quotient geometry from (2.13), one rescales the horizons and sends the AdS radius to infinity

$$r_+ \to L\sqrt{8GM} = L\hat{r}_+, \qquad r_- \to \sqrt{\frac{2G}{M}}|J| = r_0, \qquad \frac{L}{G} \to \infty, \tag{2.14}$$

where Newton's constant in the last expression is included to make the limit dimensionless[1]. The result is the flat metric

$$ds^2_{\text{FSC}} = \hat{r}_+^2dt^2 - \frac{r^2\,dr^2}{\hat{r}_+^2(r^2 - r_0^2)} + r^2d\phi^2 - 2\hat{r}_+r_0dtd\phi. \tag{2.15}$$

This spacetime is known as a flat space cosmology (FSC), because the radial coordinate is now timelike, and thus the horizon r_0 is a timelike (i.e. cosmological) horizon. The limit (2.14) essentially pushes the outer horizon r_+ of the BTZ black hole to infinity, so that we are now inside the black hole, and the BTZ inner horizon r_- becomes the cosmological horizon r_0. The FSC metric is a solution to the vacuum Einstein equations with no cosmological constant.

Just as the BTZ black hole is a quotient of empty AdS_3, the FSC is a quotient of empty 3D Minkowski space. That is, for the Minkowski metric

$$ds^2 = -dT^2 + dX^2 + dY^2, \tag{2.16}$$

[1]It is important to note here that this rescaling of the radius with G does not effect the structure of the asymptotic symmetry algebra (1.7) except changing $c_{\mathfrak{M}}$ from $3/G$ to 3.

there are two coordinate transformations (either valid for $r > r_0$ or $r < r_0$) that build the FSC spacetime. The transformation valid for $r > r_0$,

$$T = \sqrt{\frac{r^2 - r_0^2}{\hat{r}_+^2}} \cosh\left(\hat{r}_+\phi\right), \quad X = \sqrt{\frac{r^2 - r_0^2}{\hat{r}_+^2}} \sinh\left(\hat{r}_+\phi\right), \quad Y = r_0\phi - \hat{r}_+ t, \tag{2.17}$$

reproduces the FSC metric (2.15). Under the action $\phi \sim \phi + 2\pi$, the Minkowski coordinates transform as $X^\pm \sim e^{\pm 2\pi\hat{r}_+} X^\pm$ and $Y \sim Y + 2\pi r_0$, where $X^\pm \equiv X \pm T$.
The Euclidean FSC solution is constructed by performing the Wick rotation [35]:

$$\hat{r}_+ = -i\tilde{r}_+, \qquad t = i\tau, \qquad T = -iT_E. \tag{2.18}$$

The quotient $\phi \to \phi + 2\pi$ is generated by the following Killing vector [6]:

$$2\pi\partial_\phi = 2\pi\big(r_0\partial_Y + \hat{r}_+(X\partial_T + T\partial_X)\big) = 2\pi\big(r_0\partial_Y + \tilde{r}_+(X\partial_{T_E} - T_E\partial_X)\big), \tag{2.19}$$

written in terms of the Lorentzian and Euclidean coordinates. In terms of the BMS generators ((A.14), after performing a Wick rotation), the quotient generator reads

$$2\pi\partial_\phi = 2\pi i\big(\mathfrak{L}_0\eta + i\mathfrak{M}_0\rho\big), \tag{2.20}$$

where $\eta = \tilde{r}_+,\ \rho = -\frac{r_0}{G}$ are the modular parameters of the flat space quotient [6]. One can also start with the quotient structure of the non-extremal BTZ and take the limit directly on the generator of the quotient to obtain the above.
FSCs have many interesting properties some of which we briefly mention below.

- ⋆ The thermodynamics of FSC solutions have been studied in the literature. One can derive a first law of thermodynamics for FSC and this has peculiar negative signs [6]. The reason behind this is the cosmological horizon can be thought of as descending from the inner horizon of the non-extremal BTZ, which itself comes with a peculiar "wrong-sign" first law [36, 37].
- ⋆ As mentioned in the introduction, one can compute the entropy of the FSC as the area of the cosmological event horizon. This can then be matched with a Cardy-like state counting computation in the boundary 2D Carrollian CFT [6] [2]. (See also [38].) This could be thought of as a first validation of the Carrollian holography programme. Logarithmic corrections of the FSC entropy were addressed in [39]. The BMS-Cardy formula was re-derived from the limit in [40, 41].
- ⋆ There are phase transitions between hot flat space and the FSC solutions that can be thought of as analogue of the Hawking-Page phase transitions in AdS_3 [35]. These cosmological phase transitions takes a time independent solution of 3d Einstein gravity to a time dependent one.

[2]The initial works use Galilean CFTs instead of Carrollian CFTs. In $D = 2$, the Galilean and Carrollian (conformal) algebras are isomorphic and hence computationally there is no distinction between the two. It was later appreciated that calling the algebra Carrollian is more appropriate since the isomorphism, in its most primitive version, does not hold beyond $D = 2$ and the connection between Carrollian CFTs and asymptotic symmetries of flat spacetimes extend to all dimensions.

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⋆ There are generalisations of FSCs to include additional U(1) charges and higher spins [42–44].

Apart from the above, FSCs have been used in other holographic contexts e.g. in studies of entanglement [28, 29, 45, 46] and chaos [47] in flat spacetimes, for verifying the construction of an asymptotic formula for BMS structure constants derived from modular properties of the BMS torus one-point function [31], and similar studies for torus two-point functions [34]. In general, FSCs can be thought of as the bulk duals of thermal states in the 2d Carrollian CFT. So, in conclusion, FSCs play a vital role in understanding various aspects of gravity in 3d asymptotically flat spacetimes and especially are important in understanding holography in this set-up.

3 Deriving the generalized Selberg zeta function for $\mathbb{R}^3/\mathbb{Z}$

In this section, we derive a Selberg-like zeta function for the FSC spacetime in two different ways:

1. a novel method using representation theory
2. direct examination of the quotient group action, as done for the BTZ black hole in section 2.1.

In both cases, the Selberg-like zeta function we build is precisely related to the regularized 1-loop complex scalar partition function [48] via:

$$Z_\Gamma(\Delta) = \frac{1}{Z^{(1)}_{\text{reg}}(\Delta)}, \tag{3.1}$$

where Δ labels the representation of the field. Thus our formalism constitutes a new and easy way to derive 1-loop partition functions of scalar fields (as well as higher spin fields [9]) on quotient manifolds.

3.1 Selberg-like zeta function from representation theory

We propose the following construction for a Selberg-like zeta function generalized to non-hyperbolic quotient manifolds $\mathcal{M}/\Gamma$, where $\Gamma \sim \mathbb{Z}$, motivated by representation theory:[3]

$$\zeta_{\mathcal{M}/\Gamma}(s) = \prod_{\text{descendants}} \langle 1-\gamma \rangle_{\text{scalar primary of weight } s} \tag{3.2}$$

where $\gamma \in \Gamma$. In the examples we consider, the quotient Γ is generated by the identification of the angular coordinate $\phi \sim \phi + 2\pi$, so the discrete subgroup

$$\Gamma = \{\gamma^n | n \in \mathbb{Z}\} \subset \text{Isom}(\mathcal{M}) \tag{3.3}$$

is generated by $\gamma = e^{2\pi\partial_\phi}$. Therefore, the zeta function will be expressed as

$$\zeta_{\mathcal{M}/\Gamma}(s) = \prod_{\text{descendants}} \left\langle 1 - e^{2\pi\partial_\phi} \right\rangle_{\text{scalar primary of weight } s} \tag{3.4}$$

[3]Since we consider only $\Gamma \sim \mathbb{Z}$, Γ has only one generator. For more general subgroups of $SL_2(\mathbb{C})$, equation (3.2) will also contain a product over generators γ.

in the cases we consider. We will now demonstrate that this construction reproduces the Selberg zeta function for the BTZ black hole, as well as the scalar 1-loop partition function for FSCs [48].

3.1.1 BTZ black hole

Fields in AdS_3 form representations of the asymptotic symmetries of AdS_3, i.e. two copies of the Virasoro algebra. We will restrict our attention on the global subgroup of the algebra, which forms the AdS isometry group $SL_2(\mathbb{R})\times SL_2(\mathbb{R})$. The generators of the isometry group are $\mathcal{L}_n, \bar{\mathcal{L}}_n$, where $n = \pm 1, 0$ (for explicit definitions of $(\mathcal{L}_n, \bar{\mathcal{L}}_n)$, please see appendix A, and in particular (A.11)).

All states $\left|h, \bar{h}\right\rangle$ of the above theory are labelled by their conformal dimensions $(h, \bar{h})$:

$$\mathcal{L}_0 \left|h, \bar{h}\right\rangle = h \left|h, \bar{h}\right\rangle, \quad \bar{\mathcal{L}}_0 \left|h, \bar{h}\right\rangle = \bar{h} \left|h, \bar{h}\right\rangle. \tag{3.5}$$

A primary state $\left|h, \bar{h}\right\rangle_p$ with respect to $SL_2(\mathbb{R})\times SL_2(\mathbb{R})$ (and a "quasi"-primary with respect to the two copies of the Virasoro algebra) is further annihilated by both $\mathcal{L}_1$ and $\bar{\mathcal{L}}_1$:

$$\mathcal{L}_1 \left|h, \bar{h}\right\rangle_p = 0, \quad \bar{\mathcal{L}}_1 \left|h, \bar{h}\right\rangle_p = 0. \tag{3.6}$$

We will drop the subscript p on the primary state from now on. The descendants of this primary state can be written as

$$(\mathcal{L}_{-1})^{k_2} \left|h, \bar{h}\right\rangle = \left|h, \bar{h}, k_2\right\rangle, \quad \left(\bar{\mathcal{L}}_{-1}\right)^{k_1} \left|h, \bar{h}\right\rangle = \left|h, \bar{h}, k_1\right\rangle, \tag{3.7}$$

with $(k_1, k_2) \in \mathbb{Z}_{\geq 0}$.

In terms of the $SL_2(\mathbb{R})$ generators and the modular parameter for the boundary torus $\tau = \tau_1 + i\tau_2$ we have the group element γ in terms of the generator of the quotient (2.12):

$$\gamma = e^{2\pi\partial_\phi} = e^{2\pi i((\mathcal{L}_0 - \bar{\mathcal{L}}_0)\tau_1 + (\mathcal{L}_0 + \bar{\mathcal{L}}_0)i\tau_2)} = q^{\mathcal{L}_0}\bar{q}^{\bar{\mathcal{L}}_0}, \tag{3.8}$$

where $q = e^{2\pi i\tau}$. If we consider a primary scalar field of conformal dimension $\Delta = h + \bar{h} = 2h = 2\bar{h}$ and its descendents, the eigenvalues of the $SL_2(\mathbb{R})$ generators will be

$$\begin{aligned}
\mathcal{L}_0 \left|h, \bar{h}, k_1, k_2\right\rangle &= (h + k_2) \left|h, \bar{h}, k_1, k_2\right\rangle = \left(\frac{\Delta}{2} + k_2\right) \left|h, \bar{h}, k_1, k_2\right\rangle \\
\bar{\mathcal{L}}_0 \left|h, \bar{h}, k_1, k_2\right\rangle &= \left(\bar{h} + k_1\right) \left|h, \bar{h}, k_1, k_2\right\rangle = \left(\frac{\Delta}{2} + k_1\right) \left|h, \bar{h}, k_1, k_2\right\rangle.
\end{aligned} \tag{3.9}$$

Plugging (3.9) into the zeta function prescription (3.2), we recover the Selberg zeta function for the BTZ black hole reported in [14], after identifying Δ with the parameter s:

$$\begin{aligned}
\zeta_{\mathbb{H}^3/\mathbb{Z}}(s) &= \prod_{k_1,k_2=0}^{\infty} \left(1 - e^{2\pi i((k_1-k_2)\tau_1 + (k_1+k_2+s)i\tau_2)}\right) \\
&= \prod_{k_1,k_2=0}^{\infty} \left(1 - e^{2ib_E(k_1-k_2) - 2a(k_1+k_2+s)}\right).
\end{aligned} \tag{3.10}$$

Recall that, for the BTZ black hole, the parameters a, b take the values $a = \pi r_+/L$, $b_E = \pi|r_-|/L$ as discussed in section 2.1.

3.1.2 Flat space cosmology

Now we apply the prescription (3.2):

$$\zeta_{\mathbb{R}^3/\Gamma}(s) = \prod_{\text{descendants}} \langle 1-\gamma\rangle_{\text{scalar primary of weight } s} \tag{3.11}$$

to the FSC spacetime, where $\Gamma \sim \mathbb{Z}$. The group element γ in terms of the BMS_3 generators is

$$\gamma = e^{2\pi\partial_\phi} = e^{2\pi i(\mathfrak{L}_0\eta + i\mathfrak{M}_0\rho)}, \tag{3.12}$$

where η and ρ are the modular parameters of the flat space quotient defined in the section 2.2.

Since fields in flat space must be representations of the global symmetry group, we can label a field with their mass m and spin $k_1 - k_2$:

$$\mathfrak{M}_0 |m, k_1-k_2\rangle = m\,|m, k_1-k_2\rangle\,, \quad \mathfrak{L}_0 |m, k_1-k_2\rangle = (k_1-k_2)\,|m, k_1-k_2\rangle\,. \tag{3.13}$$

A primary scalar field will therefore have spin $k_1 - k_2 = 0$. We have labelled descendents with $k_1 - k_2$ because of $\mathfrak{L}$'s relationship with the $\text{SL}_2(\mathbb{R})$ $\mathcal{L}$'s (A.13) and the alternate construction of the zeta function discussed in the following section 3.2. Now we can plug this into (3.11) and obtain the zeta function for FSC from just considering the quotient, once again identifying the mass m with the parameter s:

$$\zeta_{\mathbb{R}^3/\mathbb{Z}}(s) = \prod_{k_1,k_2=0}^{\infty} \left(1 - e^{2\pi i(\eta(k_1-k_2)+i\rho s)}\right), \tag{3.14}$$

where $\eta = \tilde{r}_+$ and $\rho = -r_0/G$ for FSC.

As a check, we will see that this is precisely related to the 1-loop scalar partition function on FSC [48], and will obtain it as a careful limit from the BTZ zeta function below.

3.2 FSC zeta function from quotient group action

Another useful way to construct the zeta function is by looking at the eigenvalues of the quotient group element as an action on the coordinates (as in (2.7)). We begin by revisiting this technique for the BTZ black hole, including some further details regarding how this construction works in terms of prime geodesics, in section 3.2.1. We then use this approach to build the Selberg zeta function for FSC in section 3.2.2.

3.2.1 BTZ black hole

Let us consider the group action of $\phi \sim \phi + 2\pi$ on $\mathbb{H}^3$, as constructed in (2.7):

$$\gamma \begin{pmatrix} x \\ y_E \\ z \end{pmatrix} = \begin{pmatrix} e^{2a} & 0 & 0 \\ 0 & e^{2a} & 0 \\ 0 & 0 & e^{2a} \end{pmatrix} \begin{pmatrix} \cos 2b_E & -\sin 2b_E & 0 \\ \sin 2b_E & \cos 2b_E & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y_E \\ z \end{pmatrix}. \tag{3.15}$$

The eigenvalues of γ are $e^{2a}, e^{2(a\pm ib_E)}$.

It will be important to define the prime geodesic for the following discussion. The quotient manifold $\mathcal{M}/\Gamma$ will have closed geodesics, which are geodesics that are periodic due to the action of the quotient group. A prime geodesic is a geodesic which only requires a singular action of the group to trace out its path. In this example, the geodesic going radially outward on the z axis is the prime geodesic, since it is invariant under the rotation in the x, y plane. The length of the prime geodesic is therefore given by the eigenvalue e^{2a}.

The zeta function can be constructed by an Euler product over the eigenvalues of γ. The weights of the eigenvalues are non-negative integers, except for the eigenvalue corresponding to the length of the prime geodesic, whose weight is the argument of the zeta function, and is not restricted to integers. The resulting zeta function is therefore

$$\zeta_{\mathbb{H}^3/\mathbb{Z}}(s) = \prod_{k_1,k_2=0}^{\infty} \left(1 - e^{2(a+ib)k_1} e^{2(a-ib)k_2} e^{2as}\right), \tag{3.16}$$

which reproduces the zeta function for $\mathbb{H}^3/\mathbb{Z}$. This method has the advantage of not constructing representations of fields on the manifold, but the previous approach gives a direct relationship to the 1-loop partition function of the scalar field on the corresponding background.

3.2.2 Flat space cosmology

Similarly, let us construct the zeta function for $\mathbb{R}^3/\mathbb{Z}$. The group action on the coordinates of flat space can be expressed as

$$\gamma \begin{pmatrix} T_E \\ X \\ Y \end{pmatrix} = \begin{pmatrix} \cos(2\pi\eta) & \sin(2\pi\eta) & 0 \\ -\sin(2\pi\eta) & \cos(2\pi\eta) & 0 \\ 0 & 0 & e^{-2\pi\rho G\partial_Y} \end{pmatrix} \begin{pmatrix} T_E \\ X \\ Y \end{pmatrix}. \tag{3.17}$$

To calculate the eigenvalue of the transformation corresponding to the translation along Y, it will be convenient to perform the coordinate transformation $Y = \log Z$. The flat space metric is

$$ds^2 = dT_E^2 + dX^2 + \frac{dZ^2}{Z^2}, \tag{3.18}$$

and hence translations along Y are now scale transformations along Z:

$$Y \to Y + a \implies Z \to e^a Z. \tag{3.19}$$

The group action can be expressed as a matrix acting on the coordinates

$$\gamma \begin{pmatrix} T_E \\ X \\ Z \end{pmatrix} = \begin{pmatrix} \cos(2\pi\eta) & \sin(2\pi\eta) & 0 \\ -\sin(2\pi\eta) & \cos(2\pi\eta) & 0 \\ 0 & 0 & e^{-2\pi G\rho} \end{pmatrix} \begin{pmatrix} T_E \\ X \\ Z \end{pmatrix}, \tag{3.20}$$

which allows us to calculate the eigenvalues of γ as $e^{-2\pi\rho}, e^{\pm 2\pi i\eta}$. The prime geodesic is along the Y axis, which means the length of the prime geodesic is the eigenvalue $e^{-2\pi G\rho}$, and will be weighted by the argument of the zeta function. The zeta function, following the prescription described before for the $\mathbb{H}^3/\mathbb{Z}$ case can now be constructed:

$$\zeta_{\mathbb{R}^3/\mathbb{Z}} = \prod_{k_1,k_2=0}^{\infty} \left(1 - e^{-2\pi\rho s} e^{2\pi i\eta k_1} e^{-2\pi i\eta k_2}\right) = \prod_{k_1,k_2=0}^{\infty} \left(1 - e^{2\pi i(\eta(k_1-k_2)+i\rho s)}\right), \tag{3.21}$$

which reproduces the zeta function constructed via our representation theory method (3.14).

4 The 1-loop partition function and the Selberg zeta function

It was shown in [9][4] that the regularized 1-loop partition function of a real, massive scalar field propagating on a BTZ black hole background is directly related to the BTZ Selberg zeta function of [14]:

$$Z^{\text{1-loop}}_{\text{regularized}}(\Delta) = \frac{1}{Z_\Gamma(\Delta)}. \tag{4.1}$$

In the above expression, $\Delta = 1 + \sqrt{1+m^2L^2}$, the function $Z_\Gamma(s)$ was presented in (2.8), and the subscript "regularized" is to differentiate it from the full, divergent 1-loop partition function arising from the heat kernel calculation [8]:

$$Z^{\text{1-loop}}(\Delta) = \text{Vol}\left(\mathbb{H}^3/\mathbb{Z}\right) + Z^{\text{1-loop}}_{\text{regularized}}(\Delta). \tag{4.2}$$

Equation (4.1) makes it clear that the poles of $Z^{\text{1-loop}}_{\text{regularized}}$ correspond to the zeros of Z_Γ.

The 1-loop partition function of a real, massive scalar field propagating on an FSC background was calculated by [48]. See also [49]. In that case, equation (4.2) is replaced by

$$Z^{\text{1-loop}}(m) = \text{Vol}\left(\mathbb{R}^3/\mathbb{Z}\right) + Z^{\text{1-loop}}_{\text{regularized}}(m), \tag{4.3}$$

where m is the scalar mass. In this section, we employ our generalized Selberg zeta function (3.2) to obtain the interesting result

$$Z^{\text{1-loop}}_{\text{regularized}}(m) = \frac{1}{\zeta(m)}, \tag{4.4}$$

in direct analogy to the BTZ black hole. Thus, equation (4.2) provides a remarkably easy way to compute the regularized 1-loop partition function, eschewing heat kernel techniques. From equation (B.7), the 1-loop scalar partition function is

$$Z^{\text{1-loop}}_{\text{flat, scalar}}(m) = (\det \nabla^2_{\text{flat, scalar}})^{-\frac{1}{2}} = \exp\left(\sum_{n=1}^{\infty} \frac{e^{-2\pi mn\rho}}{n|(1-e^{2\pi in\eta})|^2}\right). \tag{4.5}$$

This can be rewritten to fit the form of the zeta function,

$$\begin{aligned} Z^{\text{1-loop}}_{\text{flat, scalar}}(m) &= \exp\left(\sum_{n=1}^{\infty} \frac{e^{-2\pi\rho mn}}{n|(1-e^{2\pi i\eta n})|^2}\right) \\ &= \exp\left(\sum_{n=1}^{\infty}\sum_{k_1,k_2=0}^{\infty} \frac{1}{n}(e^{-2\pi\rho m}e^{2\pi i\eta(k_1-k_2)})^n\right) \\ &= \prod_{k_1,k_2=0}^{\infty} \frac{1}{1-e^{2\pi i(\eta(k_1-k_2)+im\rho)}}. \end{aligned} \tag{4.6}$$

We see that the generalized Selberg zeta function constructed for $\mathbb{R}^3/\mathbb{Z}$ reproduces the regularized scalar 1-loop partition function:

$$\zeta_{\mathbb{R}^3/\mathbb{Z}}(s) = (\det \nabla^2_{\text{flat, scalar}})^{\frac{1}{2}} = \prod_{k_1,k_2=0}^{\infty}\left(1-e^{2\pi i(is\rho+(k_1-k_2)\eta)}\right) = \left(Z^{\text{1-loop}}_{\text{flat, scalar}}(m)\right)^{-1}\Big|_{m=s}. \tag{4.7}$$

[4]For example, see equation (47) of [9].

5 The Selberg zeros as quasinormal modes

It was shown in [9] that the zeros of the BTZ Selberg zeta function give the BTZ quasinormal modes. That is, given the zeros $s_\star$ (defined by $Z_\Gamma(s_\star) = 0$), the following statements are equivalent:

$$s_\star = \Delta \qquad \leftrightarrow \qquad \omega_{QN} = \omega_n, \tag{5.1}$$

where Δ is the conformal dimension of the field, ω_n are the thermal Matsubara frequencies defined by regularity at the horizon, and ω_{QN} are the QNMs. As discussed in [9], equation (5.1) constitutes an alternative method of finding quasinormal modes: given $(s_\star, \omega_n, \omega_{QN})$, we can use (5.1) to determine ω_{QN} (up to an overall function). In this section, we first calculate the thermal frequencies in FSC spacetimes, and then move on to study FSC QNMs via (5.1).

5.1 Thermal frequencies in FSC

Thermal frequencies are calculated by demanding regularity of the Euclidean metric at the horizon, and then imposing the conditions on to the scalar field solution [50]. The FSC metric is

$$ds^2 = -\frac{1}{\hat{r}_+}\frac{r^2}{r^2 - r_0^2}dr^2 + \frac{\hat{r}_+^2(r^2 - r_0^2)}{r^2}dt^2 + r^2\left(d\phi - \frac{\hat{r}_+ r_0}{r^2}dt\right)^2. \tag{5.2}$$

Performing the Wick rotation:

$$t = i\tau, \quad \hat{r}_+ = -i\tilde{r}_+, \tag{5.3}$$

we get the Euclidean FSC metric

$$ds^2 = \frac{1}{\tilde{r}_+}\frac{r^2}{r^2 - r_0^2}dr^2 + \frac{\tilde{r}_+^2(r^2 - r_0^2)}{r^2}d\tau^2 + r^2\left(d\phi - \frac{\tilde{r}_+ r_0}{r^2}d\tau\right)^2. \tag{5.4}$$

We now require the FSC metric to not have any conical singularities near the horizon. The near horizon metric can be explored using the coordinate $r^2 = r_0^2 + \epsilon\rho^2$ for small ϵ, which yeilds

$$ds^2 = r_0^2\left(d\phi - \frac{\tilde{r}_+}{r_0}d\tau\right)^2 + \frac{\epsilon}{\tilde{r}_+^2}\left(d\rho^2 + \tilde{r}_+^2\rho^2 d\phi^2\right) + \mathcal{O}(\epsilon^2). \tag{5.5}$$

For there to be no conical singularities in the subleading term, we have to go around the subleading term in ϕ, while keeping the transverse direction $\phi - \frac{\tilde{r}_+}{r_0}\tau$ fixed. Therefore,

$$\phi \sim \phi + \frac{2\pi}{\tilde{r}_+}, \quad \tau \sim \tau + \frac{2\pi r_0}{\tilde{r}_+^2}. \tag{5.6}$$

The scalar field ansatz can be written as

$$\Phi(t, r, \phi) = e^{i(\omega t - k\phi)} f(r). \tag{5.7}$$

Performing the Wick rotation (5.3) and applying the regularity conditions at the horizon (5.6) to the scalar field, we have the condition

$$e^{\frac{2\pi}{\tilde{r}_+^2}(i\tilde{r}_+ k + r_0\omega)} = 1 \tag{5.8}$$

so that the scalar field is also regular at the horizon. Solving the above condition for ω, we obtain the thermal frequencies in the spacetime:

$$\omega_n = i\frac{\tilde{r}_+}{r_0}(n\tilde{r}_+ - k), \quad n \in \mathbb{Z}. \tag{5.9}$$

5.2 Quasinormal modes in FSC

We would now like to implement the condition:[5]

$$s_\star = m \implies \omega_n = \omega_{QN}. \tag{5.10}$$

That is, tuning the Selberg zeros $s_\star$ to the mass m is equivalent to equating the thermal frequencies to the quasinormal modes. We can write this condition as

$$s_\star - m + Q\left(\omega_n - \omega_{QN}\right) = 0, \tag{5.11}$$

where Q is an undetermined function. For the BTZ case, we know how to determine Q: we know that the BTZ QNMs should not contain the thermal integer n, so we can choose Q to eliminate n. In the FSC case it is not so simple, because (in analogy to de Sitter QNMs) we expect ω_{QN} to contain thermal and angular quantum numbers (but *not* radial ones).

Note that equation (3.14) is the Euclidean Selberg zeta for FSC. The zeros are given by the condition

$$is_\star\rho + (k_1 - k_2)\eta = \ell \tag{5.12}$$

with $l \in \mathbb{Z}$, yielding

$$s_\star = \frac{\ell - \eta(k_1 - k_2)}{i\rho}. \tag{5.13}$$

Since in the exponent of the zeta function we have the vector field generating the quotient ∂_ϕ, we should identify ℓ with the angular quantum number k, so that $\ell = \pm k$. The zeros become

$$s_\star = \frac{\pm k - \eta(k_1 - k_2)}{i\rho}. \tag{5.14}$$

The values of ρ and η for FSC are

$$\rho = -\frac{r_0}{G}, \quad \eta = \tilde{r}_+, \tag{5.15}$$

as calculated in section 2.2. The thermal frequencies calculated by imposing regularity at the horizon are (5.9)

$$\omega_n = i\frac{\tilde{r}_+}{r_0}(n\tilde{r}_+ - k). \tag{5.16}$$

Substituting equations (5.14), (5.15) and (5.16) into (5.11), we get:

$$\begin{aligned}
\omega &= \omega_n - \frac{1}{Q}\left(s_\star - m\right) \\
&= \frac{i\tilde{r}_+}{r_0}(n\tilde{r}_+ - k) - \frac{1}{Q}\left(-G\frac{\pm k - \tilde{r}_+(k_1 - k_2)}{ir_0} - m\right) \\
&= \frac{m}{Q} \mp i\frac{k}{Qr_0}(G \pm Q\tilde{r}_+) + i\frac{\tilde{r}_+}{Qr_0}\left(G(k_1 - k_2) + Qn\tilde{r}_+\right).
\end{aligned} \tag{5.17}$$

[5] Note that in flat space $\Delta = m$.

Since we expect the FSC quasinormal modes to make sense as a limit from the BTZ quasinormal modes, we choose that the coefficient of k to be zero, since the angular quantum number does not appear in the BTZ quasinormal modes. This implies $Q = \mp G/\tilde{r}_+$. Thus, the quasinormal mode reads

$$\omega = \mp\tilde{r}_+ \left(\frac{m}{G} + i((k_1 - k_2) \mp n)\frac{\tilde{r}_+}{r_0}\right) . \tag{5.18}$$

Taking inspiration from the BTZ case again, we can make the identification: $k_1 - k_2 = \pm n$, which recovers the leading quasinormal mode in the FSC spacetime as computed in [34].

$$\omega = \mp\frac{m}{G}\tilde{r}_+ = \mp i\frac{m}{G}\hat{r}_+. \tag{5.19}$$

These correspond to the poles of a 2-point function of temporaly separated probes in the FSC boundary theory [34].

We end this section with a technical note: we have called the modes we study "quasinormal modes" due to their derivation from the BTZ quasinormal modes. However, unlike the BTZ modes, our modes are purely imaginary. Thus they differ from the normal modes of thermal AdS (which are purely real), as well as from the quasinormal modes of BTZ (which have both real and imaginary parts). In fact these modes may be more properly termed something like "evanescent modes" since they are purely decaying/growing, in analogy with the evanescent waves of electromagnetism. Regardless, since these modes still correspond to the zeroes of the zeta function, and thus to poles of the 1-loop partition function, we have chosen to stick with the terminology "quasinormal mode" throughout the bulk of the paper.

6 Discussion

In this work, we have built a Selberg-like zeta function for flat space quotients $\mathbb{R}^3/\mathbb{Z}$, with the FSC as an interesting and concrete example. We showed that this zeta function correctly gives the FSC scalar 1-loop partition function and the dominant QNM. These results extend the previously established Selberg method of calculating 1-loop determinants beyond negatively-curved spacetimes. In addition, we have reinterpreted how to construct the Selberg zeta function based on representation theory of fields propagating on a given background, giving us more insight into building the Selberg zeta function in more general quotient scenarios $\mathcal{M}/\Gamma$.

There are many future directions for this work. Most straightforwardly, it would be interesting to apply the Selberg formalism to study quantum effects in other quotient spacetimes such as k-boundary wormholes [51–54], and those appearing in the context of holographic entanlgement entropy [55, 56]. It would be interesting to see the implications to corrections to black hole entropy using this formalism [57].

Perhaps the most intriguing quotients of all are the Lens spaces $S^3/\mathbb{Z}_p$ (see for example the de Sitter Farey Tail [58]). This is because de Sitter admits a so-called non-standard represntation of matter, which turns out to be more closely related to quasinormal modes [59]. Further, these non-standard representations are instrumental in the recently constructed Wilson spools [16, 17]. Although the non-standard representations appear to be physically relevant, their precise physical description remains unknown. It would be interesting to use (3.1) to gain more insight into these non-standard representations. This work is currently under way.

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Getting back to the specific case of 3D flat spacetimes and the FSC solutions, while we have matched up the QNM answer (5.19) that was obtained in [34] for temporally separated probes, [34] also obtained an answer for spatially separated modes which was more reminiscent of answers from a 2D CFT calculation. This was then curiously obtained from BTZ QNM as the sub-leading term in a $L \to \infty$ expansion. It would be of interest to see if there is a way to also obtain this curious result from our Selberg zeta function.

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A Generators and embedding

In this appendix, we review some of the details of [5]. We describe how 3D flat space can be embedded in $\mathbb{R}^{2,2}$ as a limit of AdS_3. We then work out the six isometry generators $J_{A,B}$ in a series of useful coordinate systems, ultimately defining the Virasoro and BMS_3 generators.

We embed the hyperboloid

$$-U^2 + X^2 + Y^2 - V^2 = -L^2 \tag{A.1}$$

in $\mathbb{R}^{2,2}$ with the metric

$$ds^2 = -dU^2 + dX^2 + dY^2 - dV^2. \tag{A.2}$$

The isometries of (A.2) that preserve the hyperboloid are

$$J_{A,B} = X_A \partial_B - X_B \partial_A, \tag{A.3}$$

where (A, B) take values (U, V, X, Y).

To embed flat space as a limit of AdS_3 in $\mathbb{R}^{2,2}$, we parametrize U and V with the compact coordinate T/L by employing the following embedding:

$$U = \sqrt{L^2 + X^2 + Y^2} \cos\left(\frac{T}{L}\right), \qquad V = \sqrt{L^2 + X^2 + Y^2} \sin\left(\frac{T}{L}\right). \tag{A.4}$$

To obtain AdS_3, we should consider the universal covering space $T/L \in \mathbb{R}$, where the metric is

$$ds^2 = -\frac{(L^2+X^2+Y^2)}{L^2}dT^2 + \frac{L^2(dX^2+dY^2)+(XdY-YdX)^2}{L^2+X^2+Y^2}. \tag{A.5}$$

It is clear to see that in the limit $L \to \infty$ we recover three dimensional flat space:

$$ds^2 = -dT^2 + dX^2 + dY^2. \tag{A.6}$$

The rotation generators (A.3) in these "flat" coordinates are

$$\begin{aligned}
J_{U,X} &= \frac{XL}{\sqrt{L^2+X^2+Y^2}}\sin\left(\frac{T}{L}\right)\partial_T - \sqrt{L^2+X^2+Y^2}\cos\left(\frac{T}{L}\right)\partial_X,\\
J_{U,Y} &= \frac{YL}{\sqrt{L^2+X^2+Y^2}}\sin\left(\frac{T}{L}\right)\partial_T - \sqrt{L^2+X^2+Y^2}\cos\left(\frac{T}{L}\right)\partial_Y,\\
J_{X,Y} &= X\partial_Y - Y\partial_X,\\
J_{U,V} &= -L\,\partial_T,\\
J_{X,V} &= \sqrt{L^2+X^2+Y^2}\sin\left(\frac{T}{L}\right)\partial_X + \frac{XL}{\sqrt{L^2+X^2+Y^2}}\cos\left(\frac{T}{L}\right)\partial_T,\\
J_{Y,V} &= \sqrt{L^2+X^2+Y^2}\sin\left(\frac{T}{L}\right)\partial_Y + \frac{YL}{\sqrt{L^2+X^2+Y^2}}\cos\left(\frac{T}{L}\right)\partial_T.
\end{aligned} \tag{A.7}$$

We now present the generators $J_{A,B}$ in Poincaré patch coordinates

$$ds^2 = \frac{L^2(dx^2-dy^2+dz^2)}{z^2}. \tag{A.8}$$

The Poincaré coordinates in terms of the embedding coordinates are:

$$x = \frac{X}{U+Y}, \quad y = \frac{-V}{U+Y}, \quad z = \frac{L}{U+Y}, \quad u = y+x, \quad v = y-x. \tag{A.9}$$

The generators (A.3) in Poincaré coordinates are

$$\begin{aligned}
J_{U,X} &= \frac{1}{2}\Big((-1+u^2-z^2)\partial_u + (1-v^2+z^2)\partial_v + (u-v)z\partial_z\Big)\\
J_{U,Y} &= u\partial_u + v\partial_v + z\partial_z\\
J_{X,Y} &= \frac{1}{2}\Big((-1-u^2+z^2)\partial_u + (1+v^2-z^2)\partial_v - (u-v)z\partial_z\Big)\\
J_{U,V} &= \frac{1}{2}\Big((1+u^2+z^2)\partial_u + (1+v^2+z^2)\partial_v + (u+v)z\partial_z\Big)\\
J_{X,V} &= -u\partial_u + v\partial_v\\
J_{Y,V} &= \frac{1}{2}\Big((-1+u^2+z^2)\partial_u + (-1+v^2+z^2)\partial_v + (u+v)z\partial_z\Big).
\end{aligned} \tag{A.10}$$

If we take a particular set of linear combinations of the generators (A.10), we obtain 6 generators that form two commuting copies of the $\mathrm{SL}_2(\mathbb{R})$ algebra. Explicitly, one can construct

$$\begin{aligned}
\mathcal{L}_{-1} &= -\partial_u = \frac{1}{2}(J_{U,X} - J_{U,V} + J_{X,Y} + J_{Y,V}) \\
\mathcal{L}_0 &= -\left(u\partial_u + \frac{1}{2}z\partial_z\right) = \frac{1}{2}(J_{X,V} - J_{U,Y}) \\
\mathcal{L}_1 &= -\left(u^2\partial_u + z^2\partial_v + uz\partial_z\right) = \frac{1}{2}(-J_{U,X} - J_{U,V} + J_{X,Y} - J_{Y,V}) \\
\bar{\mathcal{L}}_{-1} &= -\partial_v = \frac{1}{2}(-J_{U,X} - J_{U,V} - J_{X,Y} + J_{Y,V}) \\
\bar{\mathcal{L}}_0 &= -\left(v\partial_v + \frac{1}{2}z\partial_z\right) = \frac{1}{2}(-J_{U,Y} - J_{X,V}) \\
\bar{\mathcal{L}}_1 &= -\left(z^2\partial_u + v^2\partial_v + vz\partial_z\right) = \frac{1}{2}(J_{U,X} - J_{U,V} - J_{X,Y} - J_{Y,V})
\end{aligned} \tag{A.11}$$

which obey the commutation relations

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}, \quad [\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] = (m-n)\bar{\mathcal{L}}_{m+n}, \quad [\mathcal{L}_n, \bar{\mathcal{L}}_m] = 0, \tag{A.12}$$

where (m,n) take values $(0,\pm 1)$. For the Euclidean generators, one has to Wick rotate $V \to V_E = iV$, $\partial_V \to \partial_{V_E} = -i\partial_V$ and $y \to y_E = iy$. In the linear combinations of the embedding space generators above, this implies one must replace $J_{A,V} \to -iJ_{A,V_E}$.

The BMS$_3$ algebra (the algebra satisfied by the asymptotic symmetry group of 3D flat space) can be recovered as a limit of (A.12) by making the following definitions

$$\mathfrak{L}_n = \lim_{\epsilon\to 0} \mathcal{L}_n - \bar{\mathcal{L}}_{-n}, \quad \mathfrak{M}_n = \lim_{\epsilon\to 0} \epsilon(\mathcal{L}_n + \bar{\mathcal{L}}_{-n}), \quad \epsilon = \frac{G}{L}. \tag{A.13}$$

The BMS$_3$ generators in T, X, Y coordinates read

$$\begin{aligned}
\mathfrak{L}_{\pm 1} = Y(-\partial_X \mp \partial_T) + (X \mp T)\partial_Y, \quad \mathfrak{L}_0 = (X\partial_T + T\partial_X), \\
\mathfrak{M}_{\pm 1} = G(\partial_T \pm \partial_X), \quad \mathfrak{M}_0 = G\partial_Y.
\end{aligned} \tag{A.14}$$

These obey the BMS$_3$ algebra without central extension

$$[\mathfrak{L}_m, \mathfrak{L}_n] = (m-n)\mathfrak{L}_{m+n}, \quad [\mathfrak{L}_m, \mathfrak{M}_n] = (m-n)\mathfrak{M}_{m+n}, \quad [\mathfrak{M}_m, \mathfrak{M}_n] = 0. \tag{A.15}$$

B FSC partition function as a limit of the BTZ partition function

We will see that when we carefully take this limit, the partition function poles do not accumulate into a branch cut. This allows us to construct a meromorphic Selberg-like zeta function for FSC (for scalars, it is essentially the FSC partition function itself). This will in turn allow us to predict FSC QNMs which are essentially guaranteed to be correct via an appropriate application of the Denef-Hartnoll-Sachdev method of computing QNMs (which is now applicable since we've established meromorphicity) [13].

First, let us label the representations of the global BMS algebra as a limit of the $SL_2(\mathbb{R})\times SL_2(\mathbb{R})$ algebra, as discussed in the appendix (A.13),

$$\mathfrak{L}_n = \mathcal{L}_n - \bar{\mathcal{L}}_{-n}, \quad \mathfrak{M}_n = \lim_{\frac{G}{L}\to 0} \frac{G}{L}(\mathcal{L}_n + \bar{\mathcal{L}}_{-n}), \tag{B.1}$$

where $\mathfrak{L}$ are the diffeomorphisms of the spatial circle at null infinity, and $\mathfrak{M}$ are super-translations, and we take the dimensionless limit $\frac{G}{L} \to 0$, and $\mathcal{L}, \bar{\mathcal{L}}$ are Virasoro generators. Fields are representations of the global symmetry algebra are labelled by the zero modes of these two sets of generators.

$$\mathfrak{L}_0 \left|j, m\right\rangle = j \left|j, m\right\rangle, \quad \mathfrak{M}_0 \left|j, m\right\rangle = m \left|j, m\right\rangle. \tag{B.2}$$

We can write these eigenvalues in terms of the conformal dimensions $h, \bar{h}$ of the bulk field with the appropriate limit:

$$j = h - \bar{h}, \quad m = \lim_{\frac{G}{L} \to 0} \frac{G}{L}(h + \bar{h}). \tag{B.3}$$

Now let us consider the functional determinant of the scalar laplacian on AdS_3:

$$-\log \det \nabla^2_{\mathrm{AdS,\ scalar}} = 2 \sum_{n=1}^{\infty} \frac{q^{nh} \bar{q}^{n\bar{h}}}{n|1 - q^n|^2}, \tag{B.4}$$

where $h = \bar{h}$ since this is a scalar field and therefore has spin $j = 0$, $2h = \Delta$, and $q = e^{2\pi i \tau}$, $\tau = \tau_1 + i\tau_2$. To take the flat space limit, we have to take a limit in the modular parameter:

$$\eta = \frac{\tau + \bar{\tau}}{2} = \tau_1, \quad \frac{G}{L}\rho = \left(\frac{\tau - \bar{\tau}}{2i}\right) = \tau_2. \tag{B.5}$$

Now, rewriting the 1-loop scalar partition function in terms of the new modular parameter and the eigenvalues of the BMS algebra, we have

$$\begin{aligned} -\log \det \nabla^2_{\mathrm{AdS,\ scalar}} &= 2 \sum_{n=1}^{\infty} \frac{e^{\pi i(\eta + i\epsilon\rho) n \frac{m}{\epsilon}} e^{-\pi i(\eta - i\epsilon\rho) n \frac{m}{\epsilon}}}{n(1 - e^{2\pi i n(\eta + \epsilon\rho)})(1 - e^{-2\pi i n(\eta - \epsilon\rho)})}, \\ &= 2 \sum_{n=1}^{\infty} \frac{e^{-2\pi m n \rho}}{n(1 - e^{2\pi i n(\eta + \epsilon\rho)})(1 - e^{-2\pi i n(\eta - \epsilon\rho)})}. \end{aligned} \tag{B.6}$$

where $\epsilon = \frac{G}{L}$. We can easily take the limit $\epsilon \to 0$ now, to obtain

$$-\log \det \nabla^2_{\mathrm{flat,\ scalar}} = 2 \sum_{n=1}^{\infty} \frac{e^{-2\pi m n \rho}}{n|(1 - e^{2\pi i n \eta})|^2}, \tag{B.7}$$

which is the functional determinant of the scalar Laplacian derived in [48] using the heat kernel method, in which they use the notation $(\eta, \rho) \to \left(\frac{\theta}{2\pi}, \frac{\beta}{2\pi}\right)$.

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