Mathematics education in twentieth century Iceland – Ólafur Daníelsson's impact

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Abstract

Ólafur Daníelsson was a towering figure in mathematics teaching in Iceland during the first three quarters of the 20th century. His position as textbook writer, the mathematics teacher of the first 167 primary teachers, and the teacher of the first 20 cohorts of high-school mathematics-stream graduates, contributed to his superior personal influence. Furthermore, official decisions, such as a limited access to high-school education and later strict entrance examination during 30 years, where his textbooks were a required reading, made his influence last long after his retirement. In parallel with his dedicated work at building up secondary mathematics education, Daníelsson managed to keep up a scientific career, present papers at international congresses and have them published in distinguished mathematics journals.

Introduction

The first Icelandic mathematician to complete a doctoral degree was Ólafur Daníelsson (1877–1957). At the beginning of his career in 1904, the year that Iceland was granted home-rule from Denmark, there were few opportunities for a mathematician in Iceland. By the end of his career in 1941, however, Daníelsson had become an undisputed leader of mathematics education in Iceland, having trained the first primary school teachers and written influential textbooks. Daníelsson's history is the story of a strong impact of a single individual, who established mathematics education policy in the power of his education and authority. He managed also to maintain an admirable scientific profile. We shall investigate the political circumstances and the professional channels that enabled him this.

The purpose of this paper is twofold:

- to analyse the reasons for Daníelson's enormous impact on Icelandic mathematics education lasting until the 1970s, and

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- to investigate the professional circumstances of a mathematician at the outskirt of Europe in the early 1900s.

The research method is historical, i.e. a careful analysis of a range of documents. The history is told within the framework of the history of education and schools, and the general history of Iceland by referring to scholars' published works, legislation, regulations, reports and articles. Daníelsson's textbooks were analysed, their forewords as well as their mathematical content. Information about their lifetime was sought in school reports and unprinted reports on the national examination during 1946–1976 (*Skýrsla um landspróf miðskóla*). Biographical information was accessed in biographical lexicons, such as *Kennaratalið* (Kristjánsson, 1958–65). A booklet about Ólafur Daníelsson (Arnlaugsson and Helgason, 1996) is a source referred to extensively in this paper.

Background

Iceland at the turn of the 20th century

Iceland, a remote island in the North-Atlantic, was a tributary to Denmark from the fourteenth century. When the large mountainous country gained home-rule in 1904 it was nearly devoid of roads, bridges, primary schools and other products of the industrial revolution. The population was 78,000. Since 1802, there was only one Latin School, belonging to the Danish educational system. In 1877, it was too small to split into a language-history stream and a mathematics-science stream, so the language-history stream was chosen for political reasons. In the late 1800s, discussions arose whether engineering was relevant for progress in Iceland, but a lack of mathematics stream made it necessary for the students to study one extra year in Copenhagen in preparation for engineering studies. The Latin school turned into the six-year Reykjavík High School (*gymnasium*) in the Danish educational system in 1904. Iceland gained sovereignty in 1918 (Bjarnadóttir, 2006b, pp. 110–120, 131–139).

Ólafur Dan Daníelsson studies and early life

Ólafur Dan Daníelsson, a farmer's son, studied at the Latin School in Reykjavík during 1891–1897. A relative offered to provide him with financial support if he were to study engineering. However, he chose mathematics and went in 1897 to study in Copenhagen where he first had to complete the high school mathematics-science stream. In 1900, Daníelsson published his first scientific paper in the Danish journal *Nyt Tidsskrift for Matematik B*, and in 1901, he earned a gold medal for a mathematical treatise at the University of

Copenhagen (Helgason, 1996). Daníelsson completed a Mag.Scient-degree at the University of Copenhagen in 1904 to be eligible to teach in the Danish high school system. His university professors were geometers Hieronymus Georg Zeuthen (1839–1920) and Julius Petersen (1839–1910), a well-known textbook writer. Upon returning home in 1904, Daníelsson applied for the position of mathematics teacher at the Reykjavík High School, as his former teacher died that same year. However, the position went to the first Icelandic fully-educated engineer who had served in the post of National Engineer, planning the first roads and bridges in the country, but wished then for more comfortable job.

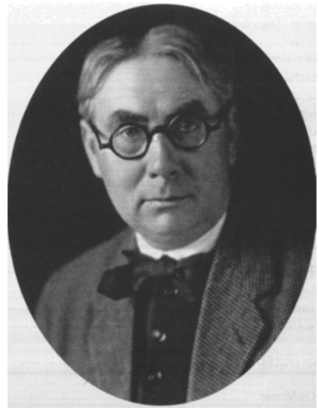

Figure 1. Ólafur Daníelsson

Daníelsson began then to prepare his doctoral thesis, which he defended in 1909: Nogle Bemærkninger om algebraiske Flader der kunne bringes til at svare entydigt til en Plan Punkt for Punkt [Several remarks on algebraic surfaces which could have one-to-one correspondence to a plane]. The thesis was an extension of earlier works by Zeuthen, and other well-known European mathematicians: Rudolph Clebsch, Guido Castelnuovo and Luigi Cremona (Helgason, 1996). Meanwhile, Daníelsson offered private lessons and composed a brief arithmetic textbook, Reikningsbók [Arithmetic], that was very elementary but with a dash of demanding problems, published in 1906 (Arnlaugsson, 1996).

Schooling in Iceland in the early 1900s

School legislation on primary schools in towns and villages and itinerary schools in the country side was enacted in 1907. A number of simple textbooks were published and their quality discussed in teacher journals. A teacher college was established in 1908 and Daníelsson was appointed as its mathematics teacher, serving until 1920. The students were mature people who might have been teaching for a number of years but who had not enjoyed much schooling themselves. The first 167 primary school teachers graduated under Daníelsson's supervision in mathematics (*Skýrsla um Kennaraskólann í Reykjavík*, 1908–1920). He revised his textbook, *Arithmetic*, for their needs and republished it in 1914.

The price of textbooks was an important factor (Bjarnadóttir, 2009b) and became even more important during the Great Depression in the 1930s. Legislation was enacted in 1937 on the establishment of a monopoly, The State Textbook Publishing House, to publish textbooks for the primary school level and distribute for free. By law, the publishing house was to offer a choice of at least two textbooks in each subject, but increasing inflation during the Second

World War and the post-war period limited the choice to a single textbook in most subjects (Bjarnadóttir, 2006a; 2009a).

Around 1930, admission to the Reykjavík High School became restricted to 25 novices a year, but remained open in the newly established Akureyri High School in Northern Iceland. A number of lower secondary schools were established around the country but without any direct connection to higher education. Following the establishment of the Republic of Iceland in 1944, new education legislation was enacted in 1946. It allowed for a national entrance examination to the high schools, to be held in the lower secondary schools. This examination standardized the syllabus for the lower secondary school level during its term in action, 1946–1976 (Bjarnadóttir, 2006b, pp. 179–234).

The restricted admission, a centralized admission examination and The State Textbook Publishing House were to create circumstances of stagnation and monopolistic textbook policy (Bjarnadóttir, 2006a; 2009a).

Daníelsson's contribution to mathematics education

A mathematics stream

In 1877, the authorities were of the opinion that there was insufficient demand for a mathematics-science stream at the Latin School and opted for a language-history stream when Latin Schools in the Danish educational system was divided into two streams. During the First World War, discussion intensified that Iceland had to have its own engineers. Daníelsson, physicist Thorkell Thorkelsson (1876–1961) and philosopher Ágúst H. Bjarnason (1875–1952), took the initiative to promote educational opportunities for students to study mathematical sciences. Bjarnason had taken his Latin-school education in Denmark and his horizon was different from his countrymen's. He wrote several articles in his journal *Iðunn* in 1917–1919. He said in agony early in 1919:

> Now Iceland has acquired sovereignty and has to stand on own feet. We Icelanders are faced with utilizing a large and difficult country and making it submissive to us. We have yet to harness our waterfalls to lead light and warmth over the country, process fertilizers from the air and run all our machines. But do we have the know-how to carry this out? Here, specialists are needed for all kinds of work; but no one knows anything in the fields that we need most, compared with the foreigners' long experience. Yet, we are bursting with arrogance, composing long poems extolling our own greatness, and we even think, some of us, that we are superior to other nations. (Bjarnason, 1919, p. 80; translation: KB)

In October 1919, a mathematics stream was finally established at the Reykjavík High School after extensive lobbyism. As the only person with university degree in mathematics, Daníelsson was appointed its main mathematics teacher. This meant a greater professional challenge for him, more prestige, less teaching and higher salary than at the Teacher College. He gained time to develop textbooks and to avail himself to research. Thorkelsson was appointed to teach the physics (Bjarnadóttir, 2006b, pp. 163–169).

The authorities had realized that it was more advantageous to prepare Icelanders for engineering studies than to hire foreign engineers who demanded higher salaries and often left after a few years with their experience. The aim of the new mathematics stream was to make the students eligible to attend the Polytechnic College in Copenhagen and study sciences at universities without spending an extra year abroad for preparation.

Textbooks

After his appointment at the Reykjavík High School, Daníelsson began his mission by choosing mathematics textbooks. For the more advanced grades, he chose the Jul.-Petersen's textbook series, written by his professor. He chose *Arithmetik og Algebra til Brug ved Gymnasiet og Realskolen* (Petersen, 1906), *Lærebog i Plangeometri for Gymnasiets sproglige og matematisk-naturvidenskabelige Linier samt Realklassen* (Petersen, 1914), *Lærebog i Trigonometri* (Hansen, 1919), *Lærebog i Differential- og Integralregning* (Hansen, 1921b), *Lærebog i analytisk Plangeometri* (Hansen, 1921a), *Lærebog i Stereometri* (Hansen, 1920) og *Tillæg til Arithmetik og Algebra* (Hansen, 1921c) (*Skýrsla um Menntaskólann í Reykjavík*, 1904–1946). C. Hansen revised professor Petersen's textbooks after his death in 1910. The more elementary textbooks of the Jul.-Petersen's series had already been in use at the Latin School since 1877 for the lower grades.

Daníelsson then began to write his own series and published four textbooks in the 1920s. He rewrote his *Arithmetic* (Daníelsson, 1906; 1914; 1920a) once more, and added *On plane geometry* (Daníelsson, 1920b), *Trigonometry* (1923) and *Algebra* (1927). The last three were the first of their kind in the vernacular. They were used at Reykjavík High School along with the advanced Danish textbooks and in Akureyri High School after it was established in 1930.

Guðmundur Arnlaugsson (1913–1996), was Daníelsson's former student. In his view (Arnlaugsson, 1996, p. 20) the textbooks were a great accomplishment, testifying to enthusiasm, optimism and craftsmanship in writing. He praised the publishers' daring and grandiosity to publish books about such extraordinary topics.

Arithmetic

Daníelsson wrote in the forewords to his first edition of his *Arithmetic*:

> This little booklet is intended … to compensate for two drawbacks which … characterize most … of our arithmetic textbooks; one is that they give no explanations at all, not even of the simplest computation methods, and

> therefore many learn the procedures by heart without understanding their reasons; and more so as many [who teach] … lack sufficient skills to explain the arithmetic down to its roots, without having for that any support from the textbooks … the other … is that their exercises are generally too easy, and each of them is most often aimed at only one computation method. The pupil can therefore guess the method without understanding the problem. (Daníelsson, 1906, p. iii; translation: KB)

This quote describes the situation in Iceland in 1906; no legislation on schools for the general public, no teacher college, and practically no educated teachers. Those who knew more tried to teach those knowing less. The *Arithmetic* was a part of a rise in educational standards during the first decades of the 20th century. It was a tiny textbook, quite elementary, but with some challenging problems and an effort to go beyond cookbook recipes, e.g. with explanations of the Euclidean algorithm for the greatest common divisor. The author dropped the explanation in the 1914 edition. He must have thought that they were premature in an elementary arithmetic textbook. The second edition was by and large written as a continuation of the first edition, adding ratio and proportions in the form of the Rule of Three, percentages and equations. In a new section on geometry, Daníelsson stated for example that the volume and the surface area of a sphere was 11/21 of the volume and surface area of the circumscribing cube, a neat simplification of more complicated formulas that those not acquainted with algebra would have had difficulties in handling.

The third edition in 1920 combined the two earlier editions. It was well suited for beginners at Reykjavík High School which many did not have solid preparation in arithmetic from primary school or home education. Danish textbooks were, however, still used there until Daníelsson's book was adopted in 1927 (*Skýrsla um Menntaskólann í Reykjavík*, 1904–1946). In the lower secondary school in Akureyri, which was working at becoming a high school, it was immediately adopted (*Skýrsla um Gagnfræðaskólann á Akureyri*, 1906–1940).

Algebra

Daníelsson's last textbook was *Kenslubók í algebru* [*A textbook in algebra*], published in 1927. His foreword bears witness to the situation in teaching advanced arithmetic at the time:

> … pupils, who have studied outside the schools, … have come up … to examination … so prepared in algebra that they have perhaps only solved the exercises, but do not know at all the basis of the symbolic language, have sometimes not had any tuition in it. (Daníelsson, 1927, p. 3–4; transl.: KB)

In this textbook Daníelsson carefully laid out the axiomatic foundation of algebra by introducing the commutative, associative and distributive laws of addition and multiplication and used them to prove various properties, such as the relations of subtraction to addition of whole numbers but without exercises.

Many students may have missed its point, and teachers found the book difficult to teach (Arnlaugsson, 1996, p. 19) but it contained an excellent collection of exercises. It was used in the two 6-year high schools. In 1946, it was together with the *Arithmetic* adopted into the syllabus for the national entrance examination of the then four year high schools to remain there into the 1970s. Later books had to adapt to Daníelsson's books, which thus became very influential through the mid-twentieth century and shaped the mathematics education of generations. Eventually they were gradually replaced from 1968 by New-Math textbooks (Bjarnadóttir, 2006b, pp. 254–268).

The *Algebra* contained a number of stories which students were expected to translate into equations. The easiest ones were of the type "think of a number". Other problems concerned the hands of the clock and their movements, and mixing, e.g. different types of wines. The classical mixing problem of the metals in the crown of King Hiero is also found. Problems with water running in pipes into cisterns and out again have their place (Daníelsson, 1927, pp. 93–114). Problems on dividing heritage are given, such as the classical problem of the man who left diamonds to his children such that the first one had one diamond and 1/7 of what was left, the second one was to have 2 diamonds and 1/7 part of what was then left, etc. Finally, all the heirs were allotted equal heritage, and the question concerned the number of heirs and diamonds (Daníelsson, 1951, p. 105). Jens Høyrup (2008) traces this problem back to Fibonacci's *Liber Abaci*, and suggests that a simple version of the problem is originally either a classical, strictly Greek or Hellenistic, or a medieval Byzantine invention, and that sophisticated versions must have been developed before Fibonacci.

Daníelsson thus knew a wealth of ancient problems. He composed his own problems too, concerning daily life of the time, such as about maids that received their salaries in boots and frocks (Daníelsson, 1951, p. 119). Those problems did not appeal to youth around the 1960s, whereas the author of this article has noticed that former students do enjoy refreshing their memory later in life by the ancient problems.

Geometry and Trigonometry

Daníelsson's textbooks *Um flatarmyndir* [*On plane geometry*] of 1920 and *Kenslubók í hornafræði* [*Trigonometry*] of 1923, however, proved to be too ambitious for his students who had little previous acquaintance with geometric concepts. Few teachers had the courage and/or capability to interpret them. Daníelsson said in his forewords:

> … some intellectuals … think that the goal of the geometry teaching is … teaching people to measure cabbage gardens or grass fields. But then a long time and a lot of work would be badly spent … it would be better to have an agronomist measure the piece of land and thus rid many of the future intellectuals of great adversity. No, the purpose … is to train the pupil in precision of his thinking and at the same time his inventiveness (Daníelsson, 1920b, pp. iii–iv; translation: KB)

The textbook *On plane geometry*, intended for novices at the six-year high school, approximately age 14, was quite theoretical. It began by a section on limits to prepare for proving the existence of irrational numbers. Next section contained a list of definitions and the postulate on a line through two points. The author admitted in his foreword that his experience was that students were relieved when that section was over. The first chapter's third section was on parallel lines, followed by exercises. Five were on computing angles, one of them in the hexadecimal system, and all exercises after that through chapter six were on proving on the basis of the definitions and theorems introduced. The following exercises up to the fifteenth and last chapter were alternatively on constructions and proving and computations by recently proved formulas, such as Heron's formula. Eventually, *On plane geometry* was transferred to the upper level of the high school. In 1937, when geometry was required at the lower level, Danish textbooks were translated (*Skýrsla um Menntaskólann í Reykjavík, 1904–1946*).

Petersen's textbook that *On plane geometry* was to replace, contained many more calculation exercises, and its exercises on proofs were fewer by far and printed in italics, so as to warn teachers and students (Petersen, 1943, pp. 75–90). Peterson's textbook was translated into Icelandic in 1943 when it could not be accessed from Denmark, even if it was hard enough. It survived in the Reykjavík School during 1877–1970 with breaks of *On plane geometry* and some New-Math experiments in the 1960s, despite notable criticism. In Petersen's eulogy in 1910 it said:

> First around the turn of the century people began to realize that the advantages of these textbooks [Peterson's series] were more obvious for the teachers than for the pupils ... the great conciseness and the omitted steps in thinking did not quite suit children. These books were excellent when the whole syllabus was to be recalled shortly before examination, but if the students were to acquire new material, one had to demand a wider form for presentation. (Hansen, 2002, p. 51; translation: KB)

In Denmark, Petersen's elementary geometry textbook was intended for the so-called *Mellemskole* [middle school] for age 12–15 (Hansen, 2002, p. 40). A reviewer wrote about the introduction to its 1905 edition:

> ... one reads between the lines the author's disgust against modern efforts, which in this country as in other places deals with making children's first acquaintance to the mathematics as little abstract as possible by letting figures and measurements of figures pave their way to understanding of the geometry's content ...
>
> Working with figures ... aids the beginner in understanding the content of the theorems, which too often has been completely lost during the effort in "training the mind". If the author knew from a daily teaching practice, how often pupils' proofs have not been a chain of reasoning but a sequence of words, he would not have formed his introduction this way ... for the middle school it [the textbook] is not suitable (Trier, 1905; translation: KB).

One might seek some explanation to the lack of success of the *On plane Geometry* to more modern theories on geometric thinking. The theory of Pierre and Dina van Hiele, developed in the late 1950s, suggests that pupils progress through levels of thought in geometry. The van Hiele model provides a framework for understanding geometric thinking (Clements, 2003, pp. 152–154). The theory is based on several assumptions: that learning is a discontinuous process characterized by qualitatively different levels of thinking; that the levels are sequential, invariant, and hierarchical, not dependent of age; that concepts, implicitly understood at one level, become explicitly understood at the next level; and that each level has its own language and way of thinking.

In the van Hiele model, level 1 is the visual level, at which pupils can recognize shapes as whole but cannot form mental images of them. At level 2, the descriptive, analytic level, pupils recognize and characterize shapes by their properties. At level 3, the abstract/relational level, students can form abstract definitions, distinguish between necessary and sufficient sets of conditions for a concept, and understand, even sometimes to provide logical arguments in the geometric domain, whereas at level 4, students can establish theorems within an axiomatic system. The van Hiele levels have proved useful in describing pupil's geometric concept development, even if they may be too broad. Pupils may possess and develop competences and knowledge at several levels simultaneously, although one level of thinking may predominate.

In the 1920s, primary schooling was underdeveloped in Iceland. The prescribed primary syllabus in geometry revolved around the area and volume of the simplest objects. Preparation for entrance to high school would rather be in languages, such as Danish and even Latin, and elementary arithmetic, but the novices had seldom met geometric concepts. They may not have been receptive for tasks suitable for van Hiele levels 3 or 4, having missed training at lower levels. But Daníelsson's ambitions lay in geometry, which he had pursued into a doctoral dissertation, and he seems to have intended to go as far as possible to share his way of thought on that topic with the students and their teachers.

The *Trigonometry* was intended for the mathematics-science stream of the upper level of the high school for only few students. They were therefore more receptive for that difficult topic than younger students were for elementary geometry. It was, however, eventually substituted by textbooks written in Danish which students had by then become accustomed to in other subjects. It seems that Daníelsson had too high ambitions in his own research subject, while his arithmetic and algebra textbooks were to survive him for decades.

Daníelsson's approach to school mathematics was strictly academic. His teaching inspired his students, at least if they showed talents and commitment. One of them said, "What especially influenced us was his enthusiasm and respect for mathematics" (Arnlaugsson, 1996). He explained arithmetic in an intelligible way, supported by proofs if he thought it would be useful, but only allowed space for initiative and creativity in his verbal exercises.

Advanced education

The University of Iceland, established in 1911, only offered studies in theology, medicine and law, in addition to studies of Icelandic history and literature. There were no science subjects, so that the Icelandic Literary Society, established in 1816, and the Iceland's Society of Engineers, established in 1912, and their journals, became important platforms to enhance scientific knowledge in the country. Daníelsson (1913; 1921; 1922) wrote in 1913 an article on geometry and the specific relativity theory in *Skírnir*, the journal of the Icelandic Literary Society, in 1921 in the *Journal of Iceland's Society of Engineers* on the same topic and in 1922 in *Skírnir* on the general theory of relativity.

Daníelsson thus reached far beyond the schools where he taught, in his effort to educate his countrymen. He began training his students in scientific thought while Iceland was still rural and self-study was common. One of his first high school students, Leifur Ásgeirsson (1903–1990), studied by distance learning only, mailing his solutions to his master. Ásgeirsson completed his doctoral degree in mathematics in Göttingen, Germany, in 1933 under the supervision of Richard Courant (Birnir et al., 1998). He published an article in *Mathematische Annalen*, Vol. 113, in 1937: *Über eine Mittelwertseigenschaft von Lösungen homogener linearer partieller Differentialgleichungen 2. Ordnung mit konstanten Koeffizienten*. This theorem became attached to Ásgeirsson's name and called Ásgeirsson's Mean Value Theorem. It concerns an ultra-hyperbolic differential equation in a neighborhood of a convex compact set (Hörmander, 2001).

Ásgeirsson left Göttingen in 1933 to become the headmaster of a lower secondary school in the countryside of North-Iceland. When the WWII isolated Iceland from the continent of Europe, Ásgeirsson was appointed as professor at the newly established engineering department of the University of Iceland. Other teachers at the department were also former students of Daníelsson or of his students, for example Arnlaugsson. Sigurður Helgason (1927–), professor emeritus at Massachusetts Institute of Technology, was initially Ásgeirsson's student (Birnir et al., 1998). Both Ásgeirsson and Helgason proposed internationally known theorems, named after them. Daníelsson was thus the mathematical ancestor of a new series of mathematical scientists, as well as a number of excellent mathematics teachers.

Daníelsson's scientific work

Research papers

In the 1920s, Daníelsson increased his efforts into research of algebraic geometry. He attended Scandinavian mathematical congresses and wrote numerous papers in academic journals:

- In the Danish *Matematisk Tidsskrift*: In 1926, 1940, 1945, 1948.
- In *Mathematische Annalen*: Vol. 102 (1930), 109 (1934), 113 (1937), 114 (1937).

Daníelsson attended the Sixth Scandinavian Congress of Mathematicians in Copenhagen in 1925 and the seventh congress in Oslo in 1927, and presented papers in both congresses. His first paper, "En Lösning af Malfattis problem" [A solution of Malfatti's Problem] was published in *Matematisk Tidsskrift* (Daníelsson, 1926). Next, he published a paper in *Matematische Annalen*: "Über korrespondierende Punkte der Steinerschen Fläche vierter Ordnung und die Hauptpunkte derselben" (Daníelsson, 1930), the last one of 44 articles by Einstein, van der Waerden, von Neumann, Landau, Ore, and Kolmogoroff among others. Later, the article "Über orientierbare und nicht orientierbare algebraische Flächen" (Daníelsson, 1937) was number six of a similar number of authors. The same issue contained the dissertation of Leifur Ásgeirsson on differential equations of second order. The journals and mathematicians' congresses were the threads that linked Daníelsson and Ásgeirsson to the mathematical community on the continent, even if they had to earn their livelihood by secondary school teaching.

Smaller works

All his life, Daníelsson was interested in elementary geometry, as by his expression "it is hard to find simpler and neater tasks than artful mathematics problems" (Daníelsson, 1914, foreword). His last paper was published in the *Journal of the Icelandic Society of Engineers* in 1946 and in *Matematisk Tidsskrift* in 1948. The papers contained the following theorem which Daníelsson in his isolation did not know that had been proved by others earlier (Helgason, 1996):

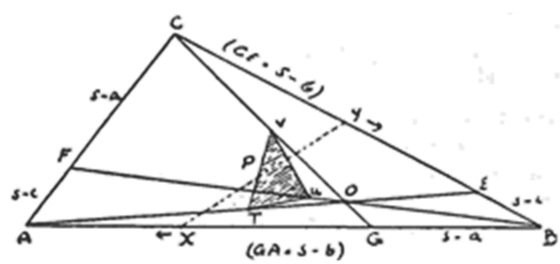

Figure 2

> The locus of the "point of bisection", P_1 of a transversal XY, bisecting the perimeter of a given triangle is the perimeter of a given triangle ABC [see Fig. 2], is the perimeter of another triangle TUV, whether the triangles are considered in a Euclidean space or a hyperbolic one. … The vertices of the triangle TUV are the points of bisection of the straight lines drawn from the vertices of the triangle ABC and which bisect its perimeter. (Daníelsson, 1946, p. 70)

Daníelsson chose the points E, F and G, so that AE divides the side a into s–b and s–c, (s is half the perimeter of ABC, and a, b and c the sides opposite the angles A, B and C). BF and CG divide b and c similarly, so by Ceva's theorem AE, BF and CG pass through the same point, O.

Now the point X moves from G to A and point Y moves from C to E in such a way that CY = GX. The transversal XY must then always halve the perimeter of the triangle ABC since both EA and CG do so. As the series of points X and Y are congruent, the locus of the point of bisection P_1 is a straight line from V to T. Similarly, if X moves from A to F and Y from E to B in such a way that AX = EY, the bisecting point of XY moves along the line TU and it continues moving from U to V when X moves from F to C, and Y from B to G.

Daníelsson continued to claim that proving various other rules respecting the triangle TUV was easy in the case of triangle ABC being a Euclidian one, for example that the sides of the triangle TUV are perpendiculars to the bisecting lines of the angles A, B and C (Daníelsson, 1946, pp. 69–71).

Discussion

Daníelsson's extensive influence may be attributed to several factors. His strong personality and firm belief in mathematics as a superb science made him an excellent champion for mathematics education. Moreover, he was the mathematics teacher of the first 167 primary school teachers in Iceland. His former students propagated Daníelsson's vision and interpretation of school mathematics, such as primary school teacher Elías Bjarnason (1927–29) who composed a simplified version for primary school level of Daníelsson's own *Arithmetic* for adolescents (Schiöth, 2008). Bjarnason's textbook was chosen as the sole arithmetic textbook for all children, 10–14 year old, during the 1940s to 1970s by the monopoly State Textbook Publishing House. Bjarnason's textbook may have been considered as the most suitable preparation for secondary level schooling due to its compatibility with Daníelsson's *Arithmetic*. Similarly, the first secondary school mathematics teachers were those who had studied at the mathematics stream at the Reykjavík High School as there was no training of secondary school teachers at the University of Iceland until the 1950s (Bjarnadóttir, 2006b, pp.189–191, 431).

Another factor was that the Reykjavík High School was the sole school of its kind until 1930. When another high school was established in North Iceland, Daníelsson became protector of its mathematics-science stream and the students graduated under his supervision**.** Admission to Reykjavík High School became restricted in 1929, which created strong competition. Following this, new lower secondary schools were established in the 1930s around the country for the common people who had not had the opportunity to attend school after age fourteen. Daníelsson's *Arithmetic* was adopted in more and more of these schools to enable their most promising students to transfer to one of the upper grades of the six year Reykjavík High School (Bjarnadóttir, 2013).

In 1946, the two six-year high schools were reduced to four-year upper secondary schools, and a national entrance examination was implemented in a number of lower secondary schools as a precondition for admission to the upper secondary schools. In deference to the hitherto dominant Reykjavík High School, the mathematics syllabus for the entrance examination was taken from the former second grade of this school, which included Daníelsson's *Arithmetic* and *Algebra* textbooks. The syllabus and its exam remained in place from 1946 to 1976, although alternatives to Daníelsson's textbooks were gradually phased in, especially after 1968 when the New Math had been introduced (Bjarnadóttir, 2006b, pp. 179–268).

Daníelsson retired from teaching in 1941 and died in 1957. His influence spanned nearly seven decades, from 1906 when he published his first textbook and 1908 when he began teaching at Iceland's Teacher College, until 1976 when his textbooks were removed from the reading list of the national entrance examination. His legacy as a dedicated mathematician is unquestioned. Mathematics education in Iceland was shaped by his vision.

The conditions in Iceland at this time, such as the restricted access to one of its two high schools, national isolation during the two world wars, the great depression in between, the monopoly of the State Textbook Publishing House, and the national entrance examination with its syllabus defined by a booklist dominated by Daníelsson's textbooks, created circumstances where discussion gradually faded out (Bjarnadóttir, 2006a; 2009a). Many generations of teachers did not know other mathematics textbooks than those by Daníelsson or built on his ideas. As they were so genuine and flawless they were not debated and no discussion took place until long after his death.

One may read from the foreword of the 1906 edition of the *Arithmetic* that Daníelsson was concerned with understanding mathematics as most textbook-writers have been (Bjarnadóttir, 2007). It is not likely that Daníelsson studied the pedagogical theories of Pestalozzi and his followers on primary teaching even if they were favoured in Denmark in the first decades of the twentieth century while he stayed there (Hansen, 2009). He might rather have known the theories by Felix Klein, who was interested in mathematics teaching at the border of high schools and universities (Schubring, 2008), but that was not an issue in 1906 in Iceland which had no university teaching in mathematics. Daníelsson's target group for his *Arithmetic* was different from that of textbook writers in countries with whom Daníelsson was acquainted, in Denmark and Germany. Daníelsson was a pioneer and he had to stick to his own ideas about mathematics learning and teaching.

Daníelsson declared in his first book in 1906 that he wanted to remedy the shortcomings of other textbooks where students learned the procedures by rote without understanding their reason. He tried to provide explanations in the first edition of the book, at least of some topics, but he dropped them in later editions, probably feeling that they did not reach his audience. His teacher student, Bjarnason (1927–29) adopted the procedures without explanation. This developed into the situation where the State Textbook Publishing House, which was established to ensure equal access for all to textbooks, ended up advocating a single unexplained procedure for each topic. Thus, the diversity in approaches of the early twentieth century eventually faded away.

Conclusions

Ólafur Daníelsson was the right person at the right time when educational authorities finally decided to meet the demands for preparation for technical education. The Great War had disturbed sailing contacts to Copenhagen where the preparation had been sought, so a mathematics stream was established at the Reykjavík High School and entrusted to the hands of Daníelsson. His education and attitude to mathematics was such that he made great demands and offered no compromises. He was the only mathematician in the country for a quarter of a century and had few to consult with. He set himself high goals and achieved them. Higher technical education had to wait until the isolation of World War II, when Ásgeirsson was called back from his rural lower secondary school teaching.

In time, Daníelsson's position at the Reykjavík High School made that influences from his views were felt far outside the school itself. Primary schools and lower secondary schools adapted to the requirements of the school due to the restricted access, both before and during the period of the national examination. It took an international reform movement in the 1960s and 1970s, the New Math, to turn the general attention away from Daníelsson's influence towards different kinds of approach to mathematics teaching in Iceland.

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