Royaumont's aftermath in Iceland – Motion geometry, transformations and groups

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Abstract

In the Meran Program in 1905 and at the Royaumont Seminar in 1959, among the main themes were transformation geometry, including motion geometry, and group theory. Those themes entered some Danish mathematics textbooks and the English School Mathematics Project's textbook series, both used in Iceland around 1970. One of the arguments for including group theory in school mathematics was that its structure corresponded to structures in the minds of children. Eventually, the emphasis on motion in geometry subordinated the structure of the transformation groups. These ideas proved short-lived in Iceland, they coincided with a great expansion of the school system, students were unaccustomed to studying textbooks in English, available teachers were not receptive, and mathematical analysis was considered neglected. In later applications, geometric transformations have become the basis of a large industry: animations in motion pictures and games.

Keywords: motion geometry, transformation, group theory, School Mathematics Project

Introduction

At a seminar on new thinking in school mathematics, held in Royaumont in 1959, proposals on geometry teaching awoke great attention and wide influence, criticising the present teaching of Euclidean geometry. There was the substitution of an algebraic approach for the traditional deductive geometry (OEEC, 1961, pp. 35-46). Secondly there was an approach known as motion geometry, mapping geometry or reflection geometry, which has as a basic principle the idea of transformation of the plane, following Felix Klein's Erlangen Programme. Geometry instruction was to be made dynamic, while instruction of Euclid was considered static (OEEC, 1961, pp. 76-77).

Both approaches appeared in textbooks that were used in Iceland around 1970. One series with roots in the Royaumont seminar, was the English School Mathematics Project, SMP. There was a particular emphasis on structure, mainly in the ideas about groups and transformations, in the early SMP textbooks, *Book T* and *Book T4*, which proved short-lived, but they made impact on further curricular development in England, and also in Iceland. Also, Danish textbooks, used in Iceland, presented axiomatic geometry of isometric transformations.

In the following we shall discuss the fate of the SMP textbooks, and how teaching and using motion geometry has evolved. We shall also look into what purpose it served to introduce the group concept in school mathematics.

Barbin, E., Bjarnadóttir, K., Furinghetti, F., Karp, A., Moussard G., A., Prytz, J., & Schubring, G. (Eds.) (2020). *"Dig where you stand" 6. Proceedings of the sixth International Conference on the History of Mathematics Education.* Münster: WTM-Verlag.

Felix Klein

Klein's Erlangen Programme

In his inaugural lecture at the University of Erlangen in 1872, *Vergleichende Betrachtungen über neuere geometrische Forschungen* [A Comparative Review of Recent Researches in Geometry], Felix Klein elucidated the importance of the term *group* for the classification of geometries and presented his unified way of looking at geometries. Klein's basic idea was that each geometry could be characterized by a group of transformations which preserve elementary properties of the given geometry. In this way, Klein unified isometric, similar, affine and projective geometry in one system. The choice of distinct transformations groups leads to distinct geometries (Trkovská, 2007, p. 253).

The Meran curriculum

From Klein´s direct initiative, a programme of restructuring subject matter at secondary schools was formulated in 1905 in Merano, South-Tyrol. A prime objective was to close the gap between school and university mathematics education. To this end, the reformers introduced the function concept as the central theme in school mathematics (Krüger, 2019, p. 36). This programme called for general functional thinking, and bringing groups of geometric transformations into the subject matter at secondary schools (Trkovská, 2007, p. 252).

The motto 'education in functional thinking' in the Meran curriculum not only refers to the subject-related modernization of teaching mathematics, but also incorporated educational principles that were central in public debates at that time (Krüger, 2019, p. 36), as underlined in its introduction:

> It is necessary […] to bring the course of teaching more closely in line with the natural process of mental development than has formerly been, to take preliminary mental representations everywhere into account [and] to establish organic connections between new insights and previous knowledge. (Gutzmer, 1908; translation by Krüger, 2019, p. 37)

In this preface, a psychological principle was described, as the students' mental development should be taken into account with regard to teaching mathematics. Special emphasis was placed on the role of previous knowledge and mental representations of mathematical concepts (Krüger, 2019, p. 37).

By introducing the concept of moving geometric objects (*Prinzip der Bewegung*), the Meran reformers turned against the traditional Euclidean method of teaching geometry that had dominated mathematical education during the 19th century. The Euclidean method was criticised at that time, as it was considered inappropriate for the students' mental development. It was described as "stiff" and "lifeless" on two accounts. Firstly, it followed a stiff pattern of definition, theorem, and proof. Secondly, the Euclidian method was seen as a stiff mathematical proof technique making use of congruencies. The intent of focussing constantly on functional thinking was to invigorate geometry teaching. In the years that followed, many teaching materials that realised the demand for movement by mechanical techniques or animated illustrations were developed. Others came up with mathematical films (Krüger, 2019, p. 43).

Functional thinking did not mean teaching the concept of function as we understand it today: a relation between sets that associates to every element of a first set exactly one element of the second set. Rather, it focusses on a specific kinematic mental capability that can be described by investigating change, variability, and movement. 'Education in the habit of functional thinking' was considered as a certain ability to perceive and analyse the variability of quantities and their functional dependencies. It was regarded as a didactical principle for teaching mathematics, and focussed on the concentration and unification of different branches of school mathematics (Krüger, 2019, pp. 35, 49).

The Royaumont meeting

The most radical changes in school mathematics at the meeting in Royaumont in November 1959, proposed by Professor Jean Dieudonné for deductive geometry, mainly concerned with the substitution of an algebraic approach for the traditional one. The axioms of linear algebra and their consequences were to be developed both from the algebraic and the geometric point of view. At the age of 15, emphasis should be on linear transformations, their various types and the groups they form. Matrices of order 2 would appear in a natural way in this development (OEEC, 1961, p. 43).

Another avenue of reform concerned motion geometry, also known as mapping geometry or reflection geometry, all of which have their basic principle the idea of transformations of the plane, following Klein's Erlangen Programme. More than half the secondary schools in Germany were experimenting with this approach to the teaching of geometry in the 1950s. Dr. Otto Botsch, Oberstudiendirektor of Helmholtz-Gymnasium, Heidelberg, West-Germany, an author of a textbook on this geometry and a guest speaker at Royaumont, presented the main developments and purposes of this instruction programme (OEEC, 1961, p. 76).

Dr. Botsch claimed that Euclid was a prefabricated house and its instruction static. The aim was to make instruction dynamic. This could not be done by giving pupils a systematically ordered catalogue of tasks to accomplish which is essentially what was done in teaching Euclid. According to Felix Klein, who gave a 'leitmotiv' for geometry based on the theory of groups, one could not proceed downward in school mathematics from highly generalized geometrical transformations to simple congruencies. One must proceed in the opposite order, starting with translations, rotations and reflections, and proceeding step by step to a more generalized application of groups of transformations. With a suitable set of postulates, a geometry of reflection could be developed. Translations in space lead to a geometry of vectors, which initially can be limited to basic operations. One could proceed to study the properties of a group and then show that the set of transformations establishing congruencies form a group (OEEC, 1961, pp. 76-79).

Dr. Botsch's address provoked much discussion and controversial debate. Worries were expressed about amalgamating these discoveries into *some* deductive system. This system could be dynamic with respect to learning, rather than the static recital of theorems. The study of an explicit axiomatic treatment of vectorial space in two dimensions must be preceded by an implicit axiomatic treatment of some sort. Good axiomatic systems also existed, e.g. by professor Choquet. The discussion was influenced by the experience of teachers on the one hand, and the mathematicians' lack of teaching experience on the other (Schubring, 2013; OEEC, 1961, pp. 80-81).

Royaumont's aftermath

After the Royaumont seminar, ideas began to disseminate among teacher educators, teachers and teacher students, paving the way into mathematics textbooks. Gustave Choquet, a guest speaker at Royaumont (OEEC, 1961, pp. 63-67) published an article on modern mathematics and teaching already in 1960 in the journal *Mathematics Teaching* (Choquet, 1960), translated into English from an article in the journal *L'Enseignement des Sciences*. There he stated a twofold aim of mathematics teaching; A: certain things that every future technician, engineer and scientist must know, and B: train the pupils to think correctly and logically. He claimed that a film is particularly well suited to portray that aspect of modern mathematics now to be called dynamic. The mathematics taught must be adapted to the developing mental structures of the pupils. He said:

> Psychologists in general, and Piaget in particular, have underlined the fact that they [the structures equivalence relations, order relations and groups] correspond to structures in the mind of children. It is not in any way astonishing, since mathematicians are only children that have grown up, that one finds in germ in the minds of children the very structures that will appear important to them and whose use will give them satisfaction when adult. (Choquet, 1960, p. 13)

This quote reflects the assumed importance of the group-concept in teaching mathematics to children and adolescents. Choquet also discussed Euclidean geometry. Since Hilbert's time the Euclidean axiom set had been known to be incomplete, and where it was completed the result was complicated and artificial. Choquet outlined the demands to meet the requirements of a set of axioms: It should be small in number, simple, yet strong, directly derived from the concrete. Improving the mathematical language was also a most urgent reform.

In 1964, Choquet published his proposal for a new set of axioms in his book *L'Enseignement de la Géometrie*, translated into English (Choquet, 1969). The book was written for teachers of mathematics at secondary schools, student teachers, lovers of mathematics and for secondary school pupils (Choquet, 1969, p. 5). There he presented an axiomatization based on vectors, in four sets of basic axioms: axioms of incidence, of order, for affine structure, and for metric structure, and, as a variation, metrically based axiomatization, replacing axioms for affine structure with folding axioms (Choquet, 1969, pp. 122-128).

Already in 1965, Choquet's book was referred to in the Danish textbook, *Matematik 65* by Christiansen and Lichtenberg (1965, pp. 312-347), where the metrically based variation of Choquet's axioms was presented. The axioms were of incidence, of order, for metric structure, and of folding. The folding axiom could, by suitable practical interpretation, correspond to the usual geometrical transformations, reflections, rotations, and translations; the reflections generating the other isometries.

Modern mathematics became quite widespread in Denmark and consequentially in Iceland as upper secondary school mathematics teachers had traditionally been educated in Denmark. Svend Bundgaard, professor at Aarhus University, and a guest speaker at Royaumont, was an avid enthusiast of modern mathematics. For his influence, it was presented at all school levels in Iceland. Christiansen and Lichtenberg's (1965) textbook was read by secondary-school-mathematics-teacher-students, who were only three a year in Iceland at that time. For some years, a Danish textbook in geometry (Andersen, Bo, Nielsen & Damgaard Sørensen, 1963) was used for a few years as a first course in upper secondary schools (MR, 1965-1973). The transformations reflection, rotation and translations were used in presenting axiomatic geometry.

In England, The School Mathematics Project, SMP, was founded in 1961. The SMP began to produce textbooks, the *Book T* and *Book T4* to begin with. These books were also used in Iceland at the secondary school level.

Development of the English School Mathematics Project

The objective of the School Mathematics Project, SMP, was to devise radically new mathematics courses, with accompanying syllabuses and examinations, which would reflect, more adequately than did the traditional syllabuses, the up-to-date nature and usages of mathematics (Howson, 1970, p. iii). Barry Cooper (1985) wrote a critical account of the SMP's development in the book *Renegotiating Secondary School Mathematics*, *a Study of Curriculum Change and Stability*.

According to Bryan Thwaites, professor of theoretical mechanics and the leader of the SMP, the source of development of the SMP can be traced back to 1957, when a conference was convened in Oxford for the purpose of bringing together, for the first time, those who taught mathematics in schools and those who used mathematics in real life (Cooper, 1985, p. 91). Thwaites, said in retrospect in 1977 that traditional school mathematics had remained static for decades. The contents of school mathematics had no relevance to applications in the world at large, and the coming computer revolution (Cooper, 1985, p. 238).

Exchanges between mathematicians, mathematics teachers and representatives from industry at several conferences created a basis for support from industry for reform of school mathematics. Initially, the emphasis was on applied mathematics. On the other hand, the leadership of *The Association of Teachers of Mathematics*, ATM, one of the two influential teachers' associations, many of whose members were supporters of Piaget's ideas on psychology, campaigned for the introduction of post-1800 algebraic ideas into syllabuses, but also for pedagogical changes, legitimising both elements of its mission in terms of improving the child's "understanding" of mathematics (Cooper, 1985, p. 151).

The established English tradition of public examinations at ages 16+, "O" level, and 18+, "A" level for entering university, has had an enormous effect on the implementation of the mathematics curriculum in English secondary schools. All secondary school curriculum development must be underscored by suitable examinations, and the School Mathematics Project (SMP) worked from its early days with university examining bodies to ensure that public examinations were developed which reflect and support curricular aims (Little, 1993).

Originally *Book T*, first published in 1964, and *Book T4*, first published in 1965, covered the syllabus for a three-year course up to O-level examination in SMP mathematics, that is for 13-16-year-old grammar school pupils. By 1967, *Books 1-5*, starting at 11+, were leading to the same O-level examination, replacing *Book T* and *Book T4* which were then regarded obsolete. *Advanced Mathematics Books 1-4* covered the syllabus for A-level examination in SMP Mathematics. By 1965, it had become clear from experience in comprehensive schools that the mathematical content of the SMP texts was suitable for a wider range of pupil than originally anticipated but the presentation needed adaptation. Thus it was decided to produce a new series, *Books A-H*, which could serve as a secondary school course, starting at the age of 11+. That series was considered inappropriate for official examination (Howson, 1970, iii).

In his study, Barry Cooper (1985, p. 239) concluded that the proposed syllabus for O- and A-levels could be seen as to have presented a compromise between the demands for redefinition by modern algebraists and an alliance of applied mathematicians and employers. At the O-level one found set theory and symbolism, the study of number bases, the linear algebra of vectors and matrices, transformational geometry (replacing the Euclidian variety) and probability theory. Representing the latter, there was statistics, and linear programming with its focus on "models" of situations.

SMP Book T and Book T4

The introductory chapter to geometry in *Book T* is devoted to simple transformations. It refers to Klein, that all geometry could be thought about by considering the way geometrical figures can change and fixing on properties which do not change. Simple changes are made to geometric figures and attention drawn to properties that remain unchanged, such as points situated on a reflection axis in reflection, a rotational centre in rotation, and distance, angle and direction of all lines in translation (Howson, 1964, pp. 45-83).

Another chapter is devoted to shearing where shape is not retained although straight lines remain straight. Areas and volumes are invariant. A chapter contains enlargement. Furthermore, there are chapters on introduction to numbers, sets and inequalities, the slide rule, displaying data, percentages and proportions, graphs and relationships, coordinates, trigonometry, binary arithmetic and loci (Howson, 1964, pp. v-ix).

One of the main features of *Book T* was motion geometry. In *Book T4* the transformations reflection, rotation, and translation were studied further, in particular the ways in which they combine. Work on Cartesian co-ordinates was consolidated prior to the introduction of vectors and to build the bridge between algebra and geometry. Examples of group tables were implemented, while groups were not yet defined. Matrices, the link between algebra and geometry, were introduced, followed by a chapter where matrices were used to cast new light on transformations, trigonometry and simultaneous equations. It also provided a source of examples for work on algebraic structure. Then there were chapters on statistics, linear programming, probability, and practical arithmetic on everyday economics (Howson, 1966, pp. v-vii).

More precisely, a set of eight transformations, closed under the rule of combining two transformations, were studied, repeated by a study of eight 2 x 2 matrices producing one of the transformations each (fig. 1) in addition to shearing. This was generalized, leading to an algebraic structure, comparison of structures and the definition of isomorphism, and finally to the definition of a group (Howson, 1966, pp. 27-69, 113-143, 289-309).

TABLE 2. COMBINATION OF THE EIGHT TRANSFORMATIONS OF TABLE I

Applied to	I	M_x	M_y	M_a	M_b	Q_1	H	Q_3
I	I	M_x	M_y	M_a	M_b	Q_1	H	Q_3
M_x	M_x	I	H	Q_3	Q_1	M_b	M_y	M_a
M_y	M_y	H	I	Q_1	Q_3	M_a	M_x	M_b
M_a	M_a	Q_1	Q_3	I	H	M_x	M_b	M_y
M_b	M_b	Q_3	Q_1	H	I	M_y	M_a	M_x
Q_1	Q_1	M_a	M_b	M_y	M_x	H	Q_3	I
H	H	M_y	M_x	M_b	M_a	Q_3	I	Q_1
Q_3	Q_3	M_b	M_a	M_x	M_y	I	Q_1	H

M_x = reflection in Ox
M_y = reflection in Oy
M_a = reflection in $x = y$
M_b = reflection in $x+y = 0$

Q_1 = quarter-turn about O
H = half-turn about O
Q_3 = three-quarter-turn about O
I = identity.

Fig. 1. Combination of eight transformations (Howson, 1966, p. 54).

NUMBER

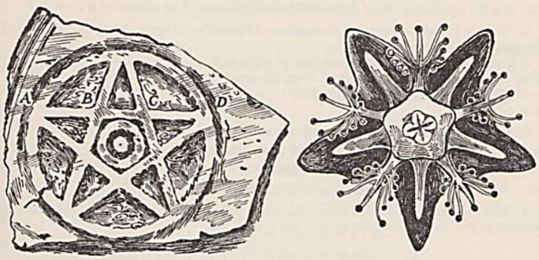

Fig. 1

There is divinity in odd numbers, either in nativity, chance or death.
WILLIAM SHAKESPEARE, *The Merry Wives of Windsor*

Fig. 2. The title figure of Chapter 1, Number, in *Book T* (Howson, 1964, p. 1).

One cannot leave the *Books T* and *T4* without admiring their layout, illustrations and references to literature, increasing the cultural value of the textbook series. Each introduction to a chapter is accompanied by a drawing and a reference to literature, such as by T.S. Eliot, Charles Dickens, William Shakespeare, Bertrand Russel, Alexander Pope, Christopher Marlowe, and Lewis Carroll.

The *SMP Advanced Mathematics* of 1967 was intended to follow straight on from *Books 1* to *5*, or *Books T* and *T4*, which by then were to be regarded as obsolete, and its four volumes were intended to lead up to the A-level examination. The content of the series focused on analysis and algebraic framework in the first volume, while in later volumes more applied topics, such as mechanics and statistics were treated (Howson, 1970, pp. iii, vii-x). The *SMP Advanced Mathematics* never reached the success of the lower level syllabus.

Throughout the whole SMP curriculum, one of the main departures from tradition lay in an increased emphasis on structure. Therefore, it was desirable to examine one special structure in detail; with axioms explicitly stated and some formal deductions made from them; for that purpose, elementary group theory was considered to be the most suitable area for study (Howson, 1970, p. vii).

The editor of the SMP-textbook series, Geoffrey Howson, who joined the SMP-group in 1962, said in retrospect in 1976 that the abstract approach to algebra that was entering the universities, was felt as the way ahead – structure. There was particular emphasis on structure in the early SMP books, e.g. *Book T4* had a section on isomorphism. Howson felt that they were in fact more successful with matrices, which might even have been, more realistically, the mathematics of the future. The material on sets was taken over from American work, and they were not so committed on that (Cooper, 1985, p. 238).

SMP's Books T and T4 in Icelandic schools

The upper secondary school system in Iceland had remained stable and actually run into stagnation since the turn of the 20th century. There was one school until 1930, later four in 1965 and six in 1970, preparing pupils for university, comparable to Danish and German gymnasia or English grammar schools. Traditionally, the schoolyear was short, eight months, and students graduated with university entrance examination at the age of 20. There were vocational colleges for training teachers and nurses, and technical schools, teaching crafts.

In the 1960s, the need for more educational options rose sharply. The school system ran into flux. Continuation departments to lower secondary schools were established in the late 1960s in order to offer some general education for those who were not admitted to the grammar schools, serving as a bridge into vocational and a new technical college. Those departments developed into comprehensive schools, merging with technical schools, and offering preparations for university entrance as well as for vocational colleges. Two new grammar schools were established in Reykjavík. Simultaneously, news about school mathematics reforms reached Iceland. Textbooks at the upper secondary schools had until that time been of Danish origin, some of which were up to

100-year-old, such as *Lærebog i den elementære Plangeometri* [*Textbook in the elementary plane geometry*] by Julius Petersen (1943), first used in 1877, lasting until 1971.

Menntaskólinn í Reykjavík, MR, Reykjavík Grammar School, published an annual school report every year since its establishment in 1846, reporting covered pages in textbooks. The information on textbook use is based on the reports (MR, 1965-1973). First an American New Math textbook (Allendoerfer & Oakley, 1963) was used at MR's mathematical line, while in 1968 the *SMP Book T4* was adopted in the second year, to be continued in the third year, followed by *SMP Advanced Mathematics* in the third and fourth year. The programme of the SMP series was run through three year-groups, at the three grammar schools in Reykjavík, later supplemented by translated drafts, eventually to be replaced by a Swedish series. The progress was slow. According to a school report, combined transformations were worked on only once. The pupils needed time for calculus. Ultimately only a few sections of *Book 3* were read in the fourth year. *Book 4* was not reached. The two new schools in Reykjavík followed MR in the main features, as the former mathematics teachers in MR had been appointed headmasters in the new schools.

In the first years of the continuation departments from 1968, the syllabus contained *Book T*. The content taken was sets, inequalities, transformations and geometry of the plane, statistics, trigonometry, the coordinate system, and graphs. Notes to teachers said that the textbook expositions were well suited to stimulate free discussion about the topics under the teacher's supervision. It was desirable to nurture the basic concepts themselves so well that methods and operations built on them would appear natural and self-evident to the students. For the second year the basis was *Book T4* with transformations, trigonometry, statistics and probability, geometry in the plane, and coordinates, and as selective topics: matrices and vectors, practical items from algebra, geometry in two and three dimensions, and modern algebra. This developed into translated extracts from both books (Skólarannsóknir, 1969, 1970).

The syllabuses for both years of the continuation departments reflect the wish to introduce practical and up-to-date mathematics to this new group of pupils who were heading towards some vocational education, something that could be useful for anybody and was not available in Icelandic language, such as statistics, the coordinate system and graphs, etc. But the foreign language was an obstacle, and the basic concepts of sets demanded much time and effort.

One notices that both *Book T* and *Book T4* had been declared obsolete in 1967. Still, the series was implemented in Icelandic upper secondary schools in 1968. Iceland is situated at the border of Europe. It took time for ideas to be transmitted. The first television station was opened in 1966 and people were not as used to listen to or read English as they are today, even if it was taught at school. Mail and travel was still mainly by sea, which also was changing rapidly.

The grammar schools operated according to regulation no. 3/1937 for the Reykjavík Grammar School, MR, a translation of a Danish regulation of 1903. The mathematics teachers of that school had the authority to select a syllabus that they felt suitable. In 1968, they were in position to decide to work with the SMP-material. One notices that *Book T* was written for the 13 to 14-year age level and *Book T4* for the 15 to 16 year age level. University entrance age in Iceland at 20 was two years behind that of England. A difference in maturity may have been expected to bridge the language problems.

The fate of SMP textbooks in England and Iceland

Criticism in England

The English school system in the 1960s was highly stratified. The teachers who wrote the original SMP-texts came from grammar schools and other selective schools where the pupils were heading for university studies. The content and the vocabulary of the textbooks series was aimed at them. Broadly speaking, in the 1950s, two versions of secondary school mathematics were being taught to two different categories of pupil, largely in different types of schools, by teachers who, again broadly speaking, had been educated in two different types of post-school institution: university and teacher training college (Cooper, 1985, p. 63).

After the initial success of the SMP-material, it was produced for more differing groups of pupils than the grammar-school pupils. The very success of SMP, in ensuring its diffusion 'downwards', brought a curricular selection originally developed within the prestigious sector, and only modified by teacher-writers experienced with 'less-able' children, into contact with many teachers whose subject and pedagogical perspectives derived from within a different culture in which criteria for selecting mathematical content differed considerably. Some examples of criticism from teachers are: "with the average pupil and downward, math is all about getting a job […] I avoid the modern sort of rubbish". "I am not favourably impressed by the SMP books. They're too wordy". "Modern maths is not very relevant, except perhaps for the top grammar school pupils". "I've no time for the SMP books at all […] SMP doesn't have enough examples for ordinary kids". "Any modern topics are strictly for entertainment value only [for the bottom 30 per cent]". (Cooper, 1985, pp. 257-258). This, and a dissatisfaction of members of various university disciplines, where it was either felt that geometrical sense could not be acquired from the syllabus, or it looked like the curriculum was made by someone only interested in applied mathematics and not the pure side, ensured that SMP would be continuously subject to criticism (Cooper, 1985, pp. 251, 266).

Griffiths and Howson (1974) who suggested the possibility of a relationship between 'ability', as measured by the distinction between the numbered and lettered series, and social class, expressed:

> With all types of pupil, the final teaching language may have to take account of their *social* language; it is no good using the language of mandarins to children of factory workers […] For example, the early SMP texts *T* and *T*4 were written in the language of mathematics specialists, intent on getting mathematics right. These books were rewritten in the language of grammar school boys, and the resulting books 1 to 5 were

> again rewritten (with modifications) in the language of 'CSE' children as books *A-H* (Griffiths and Howson, 1974, pp. 340-341).

The relationship between CSE[1] pupils and the children of factory workers seems to emerge from this piece. There is also a question about what grammar school girls were to learn. Or did they not attend grammar schools?

Problems in Iceland

No documentation exists on criticism on the SMP-series in Iceland. In the small community of only few schools, exchanges of views were verbal. The sources are interviews and inquiries to one university lecturer and three upper secondary school teachers, taken in the 2000s on events 30-45 years earlier. The American textbook (Allendoerfer & Oakley, 1963) had been in use for four years in the leading *Menntaskólinn í Reykjavík*, MR, before the SMP-series. The fact that none of the three Reykjavík grammar schools with supporting headmasters, and efforts to translate at least parts of the texts, continued the experiment with the Anglo-Saxon modern mathematics curricula, witnesses that it did not work.

There were problems acquiring teachers who would be familiar with the new policy and were not too stuck in the old methods. The teachers might be engineers, physicists or even geologists. The two mathematics teachers who had promoted the new syllabus, had been appointed as headmasters in the two new grammar schools in Reykjavík, and only a few teacher students were in training at that time. Even top pupils in those schools had significant problems with the language (Halla B. Baldursdóttir, personal communication).

When the pupils studying this new Anglo-Saxon syllabus reached the University of Iceland, the same grammar school teachers at MR, now headmasters, who also had taught the introductory mathematical analysis and linear algebra courses for two decades, were not there anymore, and could not promote the new approach within the University. The new syllabus was felt providing too little calculus preparation for the mathematical analysis (J. R. Stefánsson, personal communication). There was also perceived lack of skills (Skarphéðinn Pálmason, personal communication).

Historically, mathematics was first taught at university level in 1940 in Iceland, then essentially a pre-industrialized country. The goal was definitely to produce engineers, producing mathematicians was a distant dream, and preparing for the mathematics analysis was considered vital. Earlier, and after 1973, freshmen at the University had studied Nordic textbooks, which were somewhat easier for Icelandic pupils and had more conventional syllabus.

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[1] The Certificate of Secondary Education (CSE) was a subject specific qualification family, awarded in both academic and vocational fields in England, Wales, and Northern Ireland, set in the years 1965 to 1987 inclusive. The CSE was introduced to provide a set of qualifications available to a broader range of schoolchildren and distinct from the GCE, General Certification of Education, (O-Levels), that were mostly aimed at pupils at grammar and independent schools (rather than secondary modern schools).

The SMP-textbooks were also difficult to cope with for the pupils in the continuation departments who had less academic preparation than the pupils at the grammar schools which had turned away from them. One would expect that this would be the case here too. However, the syllabus in the continuation departments continued to be mainly an extract of the SMP books translated into Icelandic, with set-theoretical notation. Probably it was their practical orientation with graphs, statistics and probability that made this syllabus to be considered advantageous for the vocational training that the pupils were preparing for. The SMP-series was more practically oriented than the Nordic textbooks in the grammar schools. There wasn't either a university to demand any specific background.

The role of algebraic geometry in the computer world

Bryan Thwaites, cited earlier (Cooper, 1985, p. 238), proved far-sighted in his comment about the coming computer revolution. A leader of a successful computer-game project, Eve Online (H. V. Pétursson, personal communication), graduating in computing science in the 1990s, expressed the importance of linear algebra and algebraic mappings in his work in creating adventure world in computers:

> I was studying linear algebra at the University of Iceland as a computer science student. It was taught very abstract, operations on n x m matrices, etc. Then I took one course in engineering on operations research which was nice and contains a lot of linear algebra. When I began to do computer graphics to create a rendering engine for Eve Online, which is linear algebra and transformations, then for the first time I made a passionate connection with linear algebra, and cursed much not having been shown this practical application which would have made me much more enthusiastic about the all the abstract concepts. The basic point is to connect the syllabus to something that the kids are interested in. Fifty years ago, there existed no computer graphics – now it exists and it is used everywhere. The syllabus must be updated to the reality of today. To watch a professor for many hours using chalk on a blackboard – deduce proofs in a completely abstract way might have been fine at some point […] It´s just not effective today, people's brains are trained for so much more and we are not mobilising them by these archaic methods or connecting with their passion. At least, you did not do that when I was at school and I had to learn most of this by myself after I graduated.

What did Dr. Botsch say? One should not proceed downward from a highly generalized topic. One must proceed step by step to a more generalized application. This applies to freshmen at universities as well as to younger pupils.

Discussion

In the early 20th century, current geometry teaching was considered stiff, lifeless and static, instead of dynamic. Motion geometry, functional thinking and dynamic mathematics, even performed by mathematical films and animated illustrations, were expected to bring life into school mathematics. This emerged in the Merano reform, the Royaumont seminar, and by Prof. Choquet. The authors of *Books T* and *T4* presented simple transformations first geometrically in *Book T*, calling it motion geometry, referring to Klein's Erlangen Programme, and by matrices in *Book T4*. Thus the two geometry approaches in Royaumont, Dieudonné's linear algebra and groups, and Botsch's motion geometry, merged in the SMP series.

However, the dynamics did not reach a permanent place in secondary-school-curriculum. At the bottom, the best intentions of reforms did not manage to turn a lifeless and static curriculum into something dynamic. In the modern mathematics textbooks *Book T* and *Book T*4 the transformations led up to the concept of a group, and the emphasis was ultimately on algebraic structure of isomorphism and group theory, and not to motion or dynamics. The term motion geometry was mentioned, while the emphasis on motion subordinated the structure of the transformations.

In the times that followed, applied mathematicians were concerned about the mathematics of the future in the coming computer age. Progress in application has indeed been in motion geometry, in the animation industry where functional thinking has blossomed, but university teaching did not support it, at least not in 1990s Iceland. School mathematics textbooks and university teaching of the 1900s may have been too conservative at times, too occupied with axiomatic structure, to pay attention to the relevance of applications.

Both reform movements, in Merano and Royaumont, were concerned with pupils' mental development. The search for ways to bring the course of teaching more closely in line with the natural process of mental development led to appreciation of Piaget's theories, now seriously criticized, that structures, such as in groups, correspond to structures in the mind of children. This seems to have led to overemphasis on group structure, a blind passage, which pushed out more conventional topics, such as calculus, a preparation for higher technical studies, and did not meet the expectations of pure mathematicians either.

The SMP material embraced the various paradigms of both pure and applied mathematics, as well as elements of traditional school mathematics (Cooper, 1985, p. 275). Compared to the Nordic modern mathematics syllabus for grammar schools, the English SMP material was more related to applications, and consequently, the SMP material in Iceland was used primarily for upper secondary level streams preparing for vocational colleges (Bjarnadóttir, 2006, pp. 323, 341). Set-theoretical notation was regarded necessary but an obstacle.

Teachers were an important factor. For experienced teachers it can require great effort to break their habits in switching to a different syllabus with a new thinking. This also applies to university teachers. They usually expect a certain bulk of knowledge and may not appreciate a different kind of knowledge from their new students. Therefore, there was a certain advantage to recruit at both levels inexperienced teachers who had been trained in the new policy. However, the supply of such teachers was limited, and ultimately too few had the background knowledge and enthusiasm to implement the new ideas and thinking, so different from what traditionally had prevailed.

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