The Numbers One and Zero in Northern European Textbooks

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Abstract

One and zero have always existed in arithmetic textbooks. In the modern sense they are numbers. It has not always been so. The Greek view was that a number is a multitude of units. This was often interpreted to mean that one (1) was not to be understood as a number. The zero was introduced as a part of the Hindu-Arabic numeration, originally as a symbol to designate an empty slot. It was first presented as one of the ten digits in the early 13th century. For a long time it had a special position within the group of digits and was often called an insignificant digit.

These views are reflected in Northern European writings that have influenced Icelandic arithmetic textbooks from the 13th century until the 19th. Examples from medieval times, as well as from the 17th, 18th and 19th centuries, are examined in this paper with respect to these views and in the light of contemporary cultural movements, such as the Enlightenment.

During the last decades of the 19th century mathematicians and logicians made efforts to place the definition of a number on a sound basis. No evidence has been found that these matters were discussed in Iceland, while the ancient conceptions of one and zero disappeared from Icelandic textbooks in the early 20th century.

Ideas on Zero and One and their Modernization

In our modern number sense one and zero are counted as numbers, indeed very important numbers. Both belong to the set of integers. This is a recent representation.

The ancient Greek mathematicians had a discrete view of number; that they were multiples of a unit, and Aristotle said that one is no number (Tropfke, 1980, p. 124). Euclid wrote in book 7, definitions 1 and 2:

> A **unit** is that by virtue of which each of the things that exist is called one.

> A **number** is a multitude composed of units (Heath, 1956, Vol. 2, p. 277).

Euclid made some of his proofs double; first for a number and then specially for one. A doubt arises about the unit when he says in book 7, definition 3 (Tropfke, 1980, p. 124):

> A number is **a part** of a number, the less of the greater, when it measures the greater.

Furthermore, the Greeks did not regard fractions as numbers, as a single entity, but as a relationship, a ratio between whole numbers, or in modern terms an ordered pair.

Originally the discrete view was also applied to geometrical magnitudes—length, areas and volumes. In particular the early Pythagoreans believed that any two line segments were commensurable, i.e. multiples of the same unit (Edwards, 1979, p. 6). When it became clear that this was not possible and that the diagonal of a unit square was incommensurable to its side, it led to clear distinction between the number concept and the concept of magnitude. For this reason the early development of modern mathematics, such as the calculus, was within the field of geometry. Euler, Lagrange and Cauchy attempted to substitute the principles of arithmetic for geometric intuition in the foundation of analysis. However, prior to the late 19th century the real numbers themselves were understood only in an intuitive fashion (Edwards, 1979, p. 329).

Zero and one inevitably come up in practical arithmetic. The Greek mathematicians were, however, not concerned with general calculations. Such activities were confined to their slaves (Reid, 1992, p. 5). Common arithmetic with whole numbers and fractions was exercised through the centuries, e.g. in scientific work as in astronomical measurements. Decimal fractions were invented in the 16th century, and since the 17th century mathematicians had pragmatically used irrational numbers such as $\sqrt{2}$ (Edwards, 1979, p. 329). Even $\sqrt{-1}$ had been touched upon in the 18th century (Grabiner, 1981, p. 21).

In the 19th century, efforts were made to put arithmetic on a sound foundation. Dedekind, Cantor and others published their construction of the real numbers almost simultaneously in 1872 (Edwards, 1979, p. 330).

In his book *Die Grundlage der Arithmetik*/*The Foundation of Arithmetic*, first published in 1884, G. Frege wrote that some writers defined "number" as a set or multitude or plurality, but all these views suffered from the drawback that the concept did not cover 0 and 1 (Frege, 1953, §28). Frege completed Leibnitz's (1646–1716) definition of the individual numbers by defining 0 and 1.

The Italian Giuseppe Peano's aim was similar to that of Frege, to give the number concept a sound basis, but his work was at the same time more ambitious and yet more down to earth than Frege's. Peano chose three primitive concepts, zero, number (that is, non-negative whole number), and the relationship "is the successor of," satisfying five postulates:

1. Zero is a number.
2. If *a* is a number, the successor of *a* is a number.
3. Zero is not the successor of a number.
4. Two numbers of which the successors are equal are themselves equal.
5. If a set S of numbers contains zero and also the successor of every number in S, then every number is in S.

The Peano axioms were first formulated in 1889 in *Arithmetices principia nova methodo exposita*. Here the postulational method attained a new height of precision, with no ambiguity of meaning and no concealed assumptions (Boyer et al., 1991, pp. 597–598).

Greek Ideas in Arithmetic Textbooks

An examination of early textbooks, possessed or written in Iceland through the centuries, reveals that the introduction of numbers often refers directly or indirectly to Euclid, book 7, def. 2, i.e.: "A number is a multitude, composed of units." It is also of interest to study the concept of zero, both as a digit and as a number.

We will now consider several books, largely in chronological order, to see how one and zero are treated, and investigate how understanding of the concepts has changed with time.

Algorismus

The treatise *Algorismus* is found in several Icelandic manuscripts dated from the 14th, 15th and 16th centuries. These manuscripts are copies of an older original, probably from the latter half of the 13th century (Jónsson, 1892–1896, pp. 417–424, cxxxii). The treatise explains the Hindu-Arabic number system and its algorithms. The main bulk of *Algorismus* is a translation into the Norse language of a well-known poem, written in Latin hexameter, *Carmen de Algorismo* (Steele, 1988: 72–80), by Alexander de Villa Dei (1170~1250), a French priest. *Carmen de Algorismo* is a school edition and elaboration of the Hindu-Arabic algorism. It is considered to be most closely related to the *Liber alghoarismi de practica arismetrice* (Ibn Musa Al-Khwarizmi, 1992, p. xxxi), a translation and elaboration made in the 12th century by John of Seville and Domingo Gundisalvo of Al-Khwarizmi's *Arithmetic/Kitāb al-jam´wal tafrīq bi hisāb al-Hind*.

However, *Algorismus* also contains references to other sources than *Carmen*, such as the following one:

> One is neither [even nor odd number] as it is not a number but the origin of all numbers (Jónsson, 1892–1896, p. 418).[1]

This sentence is not found in *Carmen de Algorismo.* Bekken et al. have pointed out the similarity to the statement that one is not a number in al-Khwarizmi's *Arithmetic,* which in turn refers to another book on arithmetic, most likely either Euclid's *Elements*, book 7, or *Arithmetica* by the neo-Pythagorean Nicomachus

(Bekken et al., 1985, p. 27). The citation referred to is the following from the Latin translation *Dixit Algorizmi* of al-Kwarizmi's work:

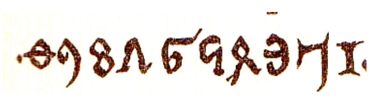
> And I have already explained in the book on algebra and almucabalah, that is on restoring and comparing, that every number is composite and every number is composed of the unit. The unit is therefore to be found in every number. And this is what is said in another book on arithmetic that the unit is the origin of all numbers and is outside numbers (Ibn Musa Al-Khwarizmi, 1992, p. 1).[2]

Clearly the unit is not counted as a number.

Algorismus has taken from *Carmen* the reference to **ten** Indian digits, including the cipher or zero, an idea which first appears in *Carmen* in European literature (Benedict, 1914, p. 122). However the cipher, called *siffra*, is allotted special attention:

> Siffra has no meaning in itself but it marks place and adds meaning to other figures (Jónsson, 1892–1896, p. 417).[3]

Even if zero is counted here as a digit it has a special position. It is for example counted last in the sequence of digits which is supposed to be read from right to left:



Frommii Arithmetica

After the introduction of Evangelical Lutheranism by the Reformation of the mid 16th century, learning and education in Iceland was confined to two cathedral schools, and from 1801 to one "learned school." Attention was rarely paid to mathematical education, except under the influence of the occasional cultural trend, such as Humanism and the Enlightenment. Both these movements were characterized by a revival of interest in Iceland's medieval literature heritage in manuscripts, written in the vernacular, and efforts to publish it in print. Publication of informative material for the public was another characteristic of these movements.

Bishop Brynjólfur Sveinsson (1605–1675) was an adherent of Humanism and promoted mathematical learning at Skálholt cathedral school in South Iceland. The cathedral's inventory in 1744 reveals *Frommii Arithmetica*, another name for *Arithmetica Danica* (Frommius, 1649), a Danish arithmetic textbook, written in Latin. This book was presumably used in teaching, as the main teacher at Sveinsson's school, Gísli Einarsson, had been a student of its author, Georgius Frommius, professor in Copenhagen. *Frommii Arithmetica* was translated into Icelandic in the 18th century (Magnússon, 1947, p. 45), but the translation has not been found. In its first chapter the definition of a number refers to Euclid, book 7, def. 2:

> Definitur Numerus ...; ex unitatibus composita multitudo, Eucl. lib. 7. Defin. 2 (Frommius, 1649, p. 2).

The digits are counted as ten, while about zero it says:

> The tenth [digit], the cypher or circulus, has no meaning for itself, but only shows the value of the following digits (Frommius, 1649, p. 2).[4]

Here again zero is meaningless and counted as the last of the ten digits.

Early 18th-Century Icelandic Manuscripts

Several Icelandic manuscripts from the early 18th century bear the name *Arithmetica*. These are arithmetic textbooks, probably translations of foreign textbooks. Their form is classical, beginning with a definition of a number, followed by an explanation of the Hindu-Arabic number system and its numeration, and the four operations in whole numbers and fractions. Some of them treat measuring and monetary units, which were the general public's most important applications of arithmetic. The (handwritten) textbook *Arithmetica. Þad er Reikningslist* is more theoretical than the others and does not mention measuring or monetary units. The author cites Euclid on the number concept. He counts the ten digits and then writes:

> Of those nine mean a certain number or magnitude for itself. The tenth one, which is cipher (zyphra) means nothing for itself while it only increases the meaning of the number in front of it to the left.
>
> Many of the learned mathematicians have maintained that 1 /*unitas*/ was not a number /*numerus*/, rather the number was a multitude of 1 /*:ex unitatibus*/ added together, Evcl: Elementa Lib. 7. Def. 2. what could not be said about 1, only supposed to be *unitatem*, as the origin and basis of all number.
>
> Others, on the other hand, maintain that 1 is to be counted as a number, as it can contain many parts of which it is composed. In addition, if 1 was not a number, another number / for example 5/ should be as many, even though 1 was removed therefrom, there though everyone understands that not that one again but 4. The latest *Arithmetici* count the digits as nine, this will be followed here (ÍB 217, 4to (1721/1750, pp. 1–2).[5]

The text bears a resemblance to *Algorismus*; "the cipher means nothing for itself" and "1 … as the origin and basis of all numbers". The quotation on the "cipher" is also found in *Frommii Arithmetica*. Furthermore, the zero is not counted as a digit in this early 18th century treatise. However, clearly the author has some doubt about not counting 1 as a number, and presents valid reasoning for his skepticism.

The multiplication table, *Tabula Pytagorica*, shows multiplication of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 by 2, 3, … 10 (p. 19).

Der Demonstrativen Rechnenkunst 1732

A comprehensive textbook, *Der Demonstrativen Rechnenkunst oder Wissenschaft gründlich und kurz zu rechnen*, by Christlieb von Clausberg, was published in Leipzig in 1732, 1748 and 1762. The book has a total of 1544 pages with detailed guidelines on calculations, useful to facilitate the work. It had considerable influence on an Icelandic textbook, as we shall see.

The book says about the number concept in §29–34:

> Each thing, as far as it is separately considered, counts **one**.[6]
>
> … When one however takes together several or many units of the same kind then a **number** emerges.[7]
>
> ... all those things, with which one finds such characteristics, are similarly a **one**, and these units taken together, constitute a **number** ...[8]
>
> … From this it is clear that the unit or **one** for itself is only the nadir or the root of the numbers, that is an accepted magnitude from which the numbers grow and are observed … ; as no **number** may be mentioned, without a certain quantity for a **one** being stated (Clausberg, 1762, pp. 14–15).[9]

We observe that the unit is the root of the numbers, as may be understood by *Dixit Algorizmi*. However, it does not say explicitly that one is not a number. Counting starts with one, two, ...

Clausberg's book does not mention the zero within the section on numbers nor does it touch upon negative values.

In §59 the digits are counted as ten, 1, 2, 3, … , 0, while §61 says:

> ... Each and every digit, when it stands alone, is valued or means as much as its name brings out, that is:
>
> 1 is called and means one
>
> 2 " " two
>
> ...
>
> and 0 is called and means nulla or nothing; and becomes therefore an **insignificant** digit and the preceding nine ones are called each of them as **significant** ones (Clausberg, 1762, p. 26).[10]

The zero is counted as a digit, but an insignificant one, while the other ones have meaning.

An Enlightenment Arithmetic Textbook, 1780

From the mid-18th century the requirements in arithmetic at the Icelandic cathedral schools were the four arithmetic operations in whole numbers and fractions. The Enlightenment movement brought increased interest in education aiming at raising the standards of Icelanders in all walks of life, in order to enhance their independence vis-à-vis foreign authorities and merchants. Iceland had belonged to the Danish realm from 1397 and Icelanders were submitted to a trade monopoly of the Danish King in 1602–1787. There were no Icelandic merchants and only one small town in the country until after the mid-19th century. Icelanders lived as subsistence farmers, and heads of household were responsible for the education of their families in the home.

Among the many books published by the proponents of the Enlightenment were a couple of arithmetic textbooks. Their purpose was to educate the "up-growing youth" and assist farmers in their trade with foreign merchants (Bjarnadóttir, 2006, pp. 74–81).

The first comprehensive printed arithmetic textbook in Icelandic: *Greinileg vegleiðsla til talnalistarinnar/A Clear Guidance to the Number Art* is written under the influence of Clausberg's textbook, *Der Demonstrativen Rechnenkunst*, as declared by the author in his foreword. Its author is one of the proponents of the Enlightenment movement in Iceland, Ólafur Olavius (1741–1788), who studied philosophy at the University of Copenhagen. In his foreword he states that the book is intended for the use of the public in exchanges with foreign merchants. Its thorough treatment of measuring- and monetary units supports that. The book is 374 pages in small format, in addition to a 28-page foreword.

It says in §2 and §3:

> **A number** is a certain multitude of units of one kind.[11]
>
> **The unit** (unitas) is the origin of number (Olavius, 1780: p. 1).[12]

This is clearly not a direct translation of Clausberg's text.

In the chapter on numeration, 0 is counted as the last of digits. That digit is said to mean nothing for itself but to influence the meaning of other digits, as defined in *Algorismus* five centuries earlier, and in *Frommii Arithmetica* a century earlier. Olavius says:

> The last digit means **nothing** alone for itself, but when some of the other nine is placed in front of it then it (that is 0) increases its value **ten** times. Therefore the nine digits are called **significant** or with a **special meaning** while **the 0 is an insignificant** digit (Olavius, 1780: p. 6).[13]

The zero is not yet a regular digit but an insignificant digit, as expressed in Clausberg's *Der Demonstrativen Rechnenkunst*, and five years later in a textbook by Stefánsson.

The multiplication table shows multiplication of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 by 2, 3, … 10 (p. 52).

A Second Enlightenment Arithmetic Textbook, 1785

In 1785 Ólafur Stefánsson (1731–1812) published his *Stutt Undirvísun í Reikníngslistinni og Algebra/A Short Teaching in the Art of Computation and Algebra*, a substantial book of 248 pages. It was immediately authorised as a textbook for the learned school. Stefánsson was a lawyer and later governor of Iceland. He said about one:

> In each quantity there are parts, each part is called a **unit** (*unitas*), and more of that kind of **units** together a **number** (*numerus*); ... (Stefánsson, 1785, p. 3).[14]

Here it is clear that the unit is not considered a number. On the digits, Stefánsson said:

> ... of the nine first ones each signifies a certain number of units, they are therefore **significant**; while the last one 0 (null or cero), which in fact is no digit, is called **insignificant**; as it for itself has no value, but increases ten times the value of the next number ahead (Stefánsson, 1785, p. 5).[15]

One and zero are not included in the multiplication table, *Tabula Pythagorica* (p. 39). Stefánsson explained that multiplication and division are inverse operations. On that occasion he treated multiplication and division with 1 and 0 specifically (p. 57–58) and noted that for example $4 \cdot 0 = 0$ and $4 : 0 = \infty$. When it came to fractions, Stefánsson called them "brot/fractions" or "brotatölur/fractional numbers" (p. 82). Later, in the introduction to algebra, Stefánsson introduced positive and negative quantities (p. 198–201) as (additive) inverses or opposites. He gives reasons for terming the negative ones quantities, and not numbers.

The 18th century came to a close without the zero being admitted to the group of significant digits, and still less into the set of numbers in Icelandic arithmetic textbooks. However, negative numbers and zero are unavoidable in the introduction to algebra. Some ambiguity was beginning to creep into the definition of the number concept.

Transition Period

Through the development of algebra new ideas about the number concept emerged. The zero and one had to be counted as numbers, initially in an intuitive fashion. Euler's *Algebra* was influential in that respect. But 19th-century intellectuals in Iceland tried to keep faith with the ancient number concept.

Euler's *Algebra*

Between Clausberg's book and the Icelandic Enlightenment books, Swiss mathematician Leonhard Euler published his *Vollständige Anleitung zu Algebra* in 1770. The book soon became well known in Europe. Written in German, within a few years it had been translated into Russian (1768–9), Dutch (1773), French (1774), Latin (1790), English (1797, 1822) and Greek (1800) (Heeffer, 2007). Euler's ideas of number are quite different from Clausberg's. In paragraph 16, chapter 2, Part I of Euler's *Algebra,* he introduced negative quantities as opposite to positive quantities, and goes on to call these quantities numbers. In paragraph 19, Euler says that

> … we obtain positive numbers by adding 1 to 0, that is to say, 1 to nothing; and by continuing always to increase thus from unity. This is the origin of the series of numbers called natural numbers; the following being the leading terms of this series:
>
> 0, +1, +2, +3, +4, +5, +6, +7, +8, +9, +10, and so on to infinity (Euler, 1822, p. 5).

Zero and one thus belong to the natural numbers. Similarly Euler added the negative numbers to the series and said

> All these numbers, whether positive or negative, have the known appellation of whole numbers, or *integers*, which consequently are either greater or less than nothing (Euler, 1822, p. 5).

Clearly, no doubt is to be found here about one and the negative numbers belonging to the number concept, but does the zero belong there?

Textbooks Used by Mathematicians

Icelandic students had to go to Copenhagen for university studies. From 1818 the regulations of the University of Copenhagen prescribed increased preparation requirements in mathematics (Bjarnadóttir, 2006, pp. 85-86). Therefore, Danish textbooks were used at Iceland's sole Learned School (later Reykjavík High School) from 1822 until the mid-1960s.

At least one 19th-century Icelander, Björn Gunnlaugsson (1788–1876), knew the works of Euler. Gunnlaugsson studied them in 1819–1820 when he was mathematics student at the University of Copenhagen with Professor C. F. Degen, who was familiar with the works of Euler, Legendre, Lagrange and Gauss (Björnsson, 1997, p. 6). Gunnlaugsson was the only mathematics teacher in the sole Learned School in Iceland from 1822 to 1862 and the first mathematics teacher in Iceland after Einarsson to study mathematics at a university. He and his choice of textbooks had great influence on Icelandic mathematics education in his time.

Gunnlaugsson followed the regulations for Danish learned schools in his teaching at the Learned School. Announcements about new textbooks were disseminated by circulars to all the learned schools in the Danish state. From school reports it is clear that Gunnlaugsson always used latest editions. He used textbooks by Georg Friedrich Ursin for a long time. Ursin's *Regnebog/Elementary Arithmetic* (3rd printing 1841) begins by counting the ten first digits and naming them, not distinguishing 1 from the other digits. About zero it says:

> 0, zero, which [is] the last [digit], when standing alone, means nothing (Ursin, 1841, p. 1).[16]

This first book covers elementary arithmetic: the four operations in whole numbers and fractions in addition to proportions in the form of *regula de tri*. One and zero do not appear in the multiplication table. Any special treatment of zero does not show up in the calculations and negative numbers are not mentioned.

Ursin's second book, *Arithmetik*, is more advanced. It says:

> To count is to collect similar quantities, one after another, to a whole. This whole, in which we do not take into account these quantities' internal position, but only their multitude, is called a number, each of the separate similar parts are called a unit. A number is thus composed of units (Ursin, 1855, p. 1).[17]

This definition does not directly exclude one from being a number, but hardly includes fractions. The digit zero denotes no units (p. 2). The multiplication table includes 1 but not 0 (p. 9), while in an attached note multiplication of 0 is defined 0, as well as multiplication by 0 (p. 10). Discussions about the thermometer and expenses exceeding income lead to the definition of negative quantities as the opposite of regular, that is, positive, quantities (p. 63). A sum of 8e – 4e + 7e – 11e equals 0 without any further comment, such as calling 0 a number or a quantity. When negative numbers come up it says that a negative number cannot be looked upon as originating from the unit, but from adding the opposite of the unit, –1, several times to itself.

C. Ramus, professor of mathematics at the University of Copenhagen, published his *Elementær Algebra/Elementary Algebra* in 1855. Gunnlaugsson used that textbook in the Icelandic Learned School 1856–1862, until his retirement. In his foreword, Ramus said that numerical quantities served to measure the quantity of similar objects, considered as units, and quantities of equal parts of the unit. (The Danish word "Mængder," here translated as "quantities," could be translated as "sets." However, the set theory had not yet been developed in 1855). In the first page of the main text, he declared the concepts of "unit"/"enhed" and "quantity"/"flerhed" to be already given. Then he said:

> Zero or nothing one accepts as a number, while as opposite to that the others are called *significant* numbers (Ramus, 1855, p. 1).[18]

Ramus felt a need for the zero as a number, but had some reservations about its validity. The zero has, however, been elevated from being an insignificant digit to an insignificant number.

In 1856 the Learned School bought *Elementær Arithmetik/Elementary Arithmetic* by A. Steen, first published in 1843, while it was introduced for teaching only in 1864 after Gunnlaugsson retired. There the natural number sequence is introduced: 1, 2, 3, … where the 1 is primarily called *the unit* (Steen, 1864, p. 8). The number sequence is extended to include 0 (p. 11), which means no quantity, while the other digits are, contrary to 0, *significant*. The sequence is expanded again to include "negative Tal," negative whole numbers (p. 11-12). Thereafter it is explained that multiplication by 0 and of 0 gives 0 (p. 15), as well as the impossibility of dividing by 0 (p. 24). Steen seems to be obliged to look upon the zero and negative whole numbers as numbers.

A 19th-Century Icelandic Arithmetic Textbook

Volcanic eruptions and other calamities brought Icelandic society to a nadir, which lasted from the mid-1780s until the 1830s. From that time there were increased efforts in public education. No arithmetic textbook was published from 1785 until 1841 when *Reikningslist: einkum handa leikmönnum/Computing Art: Mainly for Laymen* by Jón Guðmundsson (1807-1875) appeared. Guðmundsson was Gunnlaugsson's pupil. He studied law, and is not known to have studied mathematics after graduating from the learned school. His book may be considered to be a contribution in the spirit of the Enlightenment: to assist laymen in their business and daily life.

The *Computing Art/Reikningslist* is a conventional textbook on arithmetic, where negative numbers are not needed. It explains the four operations on whole numbers, named numbers, such as money, weight and time, fractions, ratio and proportions in *regula de tri*, percentages and partition of inheritance. It contains topics such as Euclid's algorithm and testing of the operations, while there are no algebraic explanations.

In the introduction it says:

> Each thing or species ... when it is considered or valued for itself, is called a **unit** ... ; if the species is increased by more of the same kind, it is called a **number**; the number signifies how many there are of the same units ... (Guðmundsson, 1841, pp. 3–4).[19]

Here we still see that the concept "number" concerns more than one unit, similar to what was seen in Stefánsson's book. However, in the continuation, Guðmundsson noted that many units, such as monetary units, may be split into smaller units. Therefore:

> Every unit may also be called number when looked to its parts (Guðmundsson, 1841, p. 4).[20]

About the zero it says:

> ... 0 (the zero) signifies nothing when it stands alone. The other nine digits are therefore altogether called **significant** or **valid digits**, to distinguish them from the zero (Guðmundsson, 1841, pp. 6–7).[21]

In spite of these definitions, zero and one are included in the subtraction table (p. 32) and the multiplication table along with the other numbers, without further explanation (p. 32). The subtraction table, however, never produces a negative number. A fraction is called "brotin tala/fractional number" and e.g. $2\frac{1}{5}$ is called "blandin tala/mixed number" (p. 14). It seems that Guðmundsson wanted to keep faith with an old number definition, while trying to modify it.

Tölvísi by Björn Gunnlaugsson

In his position as the only mathematician in Iceland, Björn Gunnlaugsson was isolated. Iceland was cut off from the outside world except during the summer months when ships could sail the north Atlantic; correspondence could take months. It seems that Gunnlaugsson did not study contemporary mathematics after he returned from Copenhagen, only philosophy (Guðmundsson, 2003, pp. 23–24), while he was well informed about new mathematics textbooks in the Danish learned-school system.

Björn Gunnlaugsson published his own book, *Tölvísi*/*Number Wisdom* in 1865. It is debatable whether it should be considered a textbook, as it was never used as such. For a mathematics book for the general public, at a time when Iceland's population numbered only about 60,000, it is ambitious. *Tölvísi* contains sections on counting and the four operations, where the author introduces algebraic operations, and fractions. This is followed by a section on the nature of the whole numbers, i.e. introduction to number theory, such as modular arithmetic, congruence, prime numbers, divisibility and Fermat's Little Theorem. The book continues with decimal fractions, periodic decimals, error bounds on all operations, powers, exponents and roots, quadratic roots, cubic roots, bi-quadratic roots and fifth roots, the binomial theorem and chain fractions. Parts of the books may be considered as a dialogue with textbook authors, such as Ursin, Ramus and Steen, where Gunnlaugsson declares that he either takes ideas from them or disagrees with them.

In his introduction, the author expressed distinction between continuous (*continuæ*) and discrete (*discretæ*) quantities:

> ... a quantity, which is used for comparing, is often called scale, ... while in the number art [arithmetic] the scale is called Unit or one and therefore a number (*numerus*) is a discrete quantity's comparison to the unit or one (Gunnlaugsson, 1865, p. 2).[22]

The number concept is allotted to discrete quantities. Gunnlaugsson counts the ten digits, referring in footnote to *Algorismus* that they were of Indian origin, but says:

> ... of which 0 (zero) means that there is no unit, but an empty space or seat (Gunnlaugsson, 1865, p. 4).[23]

However, later Gunnlaugsson extended the number sequence by adding 0 and negative quantities (pp. 24–26). He explained zero as:

> Both 0 and ∞ are ... the limits of the quantity. Zero is actually no quantity, but less than all quantities; such as ∞ is no quantity, but larger than all quantities. ... 0 or the middle itself [between positive and negative quantities] is no quantity but the origin point of quantities ... (Gunnlaugsson, 1865, p. 97).[24]

Gunnlaugsson used the word "stærð/quantity" most of the time, but occasionally it seems unavoidable to use the expression "negative numbers," mainly in connection with the number sequence. But he has, for example, doubts whether imaginary quantities are to be called quantities at all (p. 344).

Gunnlaugsson mentioned Euler a couple of times in an unpublished manuscript of the continuation of his *Tölvísi*. There he called Euler one of the recent mathematicians (compared to Leibnitz). However, Gunnlaugsson, who was philosophically inclined, was not prepared to take up Euler's intuitive stance on the number concept, preferring more ancient ideas.

The Early 20th Century

Under legislation passed in 1880, requirements were introduced for children to learn writing and arithmetic. Home education remained the rule until 1907, when legislation was passed on compulsory schooling; hence textbooks played an important role. A number of arithmetic textbooks were published in the 1860s to 1920s, both elementary for children and more advanced for adolescents and for all of those who wanted to study arithmetic. Most of them were written by theologians, teachers or other intellectuals. These books did not offer any philosophical considerations of one or zero.

The next Icelander after Gunnlaugsson to complete mathematics studies at a university was Dr. Ólafur Daníelsson, who completed his master's degree in 1904 and defended his doctoral thesis in geometry in 1909. He built up mathematics education at the Teacher Training College, established in 1908, and moved to the Reykjavík High School in 1919. His arithmetic textbook, *Reikningsbók/Arithmetic* (1906), had immense influence in Iceland in the first half of the twentieth century. Here the distinction of a unit or one from other numbers has disappeared. About the zero he says:

> ... "zero" means "nothing" and belongs to the digits (Daníelsson, 1906, p. 1).[25]

In his *Algebra* (1927), Daníelsson counted 1 and 0 as numbers as he placed them on a number line (Daníelsson, 1927, p. 9). However, his education and interest lay in geometry (Arnlaugsson and Helgason, 1996), and no evidence has been found that Daníelsson or others discussed the philosophical foundation of one and zero or of the number in general in the early 20th century.

It is noticeable that Steingrímur Arason, a primary level teacher, published a textbook together with another teacher in 1914, where the first number, denoted by pebbles, is one, in the sequence 1, 2, 3, … (Brynjólfsson and Arason, 1914, p. 1). In the second edition, published by Arason alone after he had studied teaching in the United States, zero is the first number in the sequence, denoting no pebble: 0, 1, 2, 3, … (Arason, 1928, p. 1).

Summary and Conclusions

The choice of arithmetic textbooks in Iceland was governed by its special circumstances; on one hand a society with subsistence farming and no internal trade far into the 19th century, and on the other hand requirements of the University of Copenhagen concerning preparation of the few Icelandic students. It was the elite, who had attended university, who later wrote textbooks for the general public. Among these are Olavius, Stefánsson and Guðmundsson. No mathematical literature in the vernacular was intended for the mercantile class as no such class existed. The books were aimed at farmers for use in their exchanges with foreign merchants, and youngsters preparing for the learned school or attending its first grades. This group was expected to be familiar with the ancient manuscript heritage and be fit to accept philosophical considerations. There was an overtone of nationalism, as the books served as tools for attaining independence from the foreign authorities. After the 1880 legislation on instruction in arithmetic, arithmetic textbooks were aimed at all children and adolescents, and discussions on the number concept were omitted.

The Greek definition of the number concept, that a number was a multitude of a unit, had a long life in Icelandic textbooks on arithmetic. And it is only in 1927 that zero is seen to be counted as a number in Icelandic textbooks. The Icelanders were humanists in the sense that they nurtured their national heritage preserved in the medieval manuscripts. The foundation of the number concept was laid in the *Algorismus* treatise. Those who were concerned with arithmetic in Iceland through the centuries seem to have been familiar with it. Gunnlaugsson, for example, cited it in 1865, even though it was first published in print in 1892-1896. The ancient definition of a number was not ignored, in spite of the paradoxical situation it created when extending the number system. Certainly some authors had doubts, particularly about not counting one as a number, like the unknown author of an early-18th-century manuscript.

The ancient definition of a number did not cause serious difficulties until the algebra had developed and a need for negative numbers had been established. For example the 18th-century German author Clausberg used the ancient approach to one and zero in his comprehensive non-algebraic textbook. Euler was innovative in his intuitive definition of the number concept. However, the Danish professors Ursin and Ramus, who wrote textbooks in the early and mid-19th century, either did not address the matter directly or had some reservations, especially about the zero. Euler's approach to the number concept does not seem to have been generally accepted by mathematicians. Gunnlaugsson, whose education had its foundation in the early 19th century and who knew Euler's work, did not accept the zero as a number or a quantity but considered it to be the limit of quantities. In 1865 it was too early for him to know about discussions in Europe on the logical foundation of arithmetic.

The great works on the foundation of the number concept, by Dedekind and Cantor in 1872, Frege in 1884 and Peano in 1889, were made in the period between the two Icelandic mathematicians, Gunnlaugsson and Daníelsson. During the period 1877–1919 mathematics teaching was minimal in Iceland's one learned school. Icelanders could only study mathematics at university level by going abroad, generally to Copenhagen. Those who taught mathematics at the learned school in that period had only short training in mathematics, and had a pragmatic approach to their teaching. It is not plausible that philosophical considerations of the number concept were much discussed in Iceland at that time. These matters do not seem to have concerned arithmetic textbook-writers, most of whom were pastors, busy building up public education from scratch—a more down-to-earth task than discussion of the philosophical foundation of arithmetic. Daníelsson wrote his arithmetic and algebra textbooks in 1906–1927.

His writings do not reveal any doubts about the foundations of 0 and 1, and his education in Copenhagen around the year 1900 was probably well grounded in the modern understanding.

Notes

1. Einn er hvorki [jöfn né ójöfn tala] því að hann er ekki tala heldur upphaf allrar tölu.
2. Et iam patefeci in libro algebr et almucabalah, idest restaurationis et oppositionis, quod uniuersus numerus sit compositus et quod uniuersus numerus componatur super unum. Unum ergo inuenitur in uniuerso numero. Et hoc est quod in alio libro arithmetice dicitur quia unum est radix uniuersi numeri et est extra numerum :
3. Siffra merkir ekki fyrir sig en hún gerir stað og gefur öðrum fígúrum merking.
4. Decimus [character] ciphra vel circulus, per se nihil significat, sed tantummodo sequentis numeri valorem auget.
5. Af þessum merkia Niju vissa Taulu edur Fiøllda fyrer sig siálfa. Sá Tíjundi sem er Zyphra merkir ekkert fyrer sig helldur eykur einasta Merking þeirrar Tølu sem fyrer framann hana er til vinstri Handar.

   Margir af þeim Lærdu Mathematicis hafa viliad halda að 1. / unitas / væri ej Tala /numerus/ helldur væri Talann Fiølldi af 1. /:ex unitatibus / til samans lagdur vid Evcl: Elementa Lib: 7. Def. 2. hvad um 1 kyni ej seigiast, einasta álited unitatem, sem upphaf og undirrót til allrar Tølu.

   Adrir þar í mót meina að 1 eigi ad kallast Tal, þvj hann gieti i sier innibundid marga Parta af þessum hann sie samsettur. Þar ad auki ef 1 væri ej Tal skylldi annad Tal / til dæmis 5/ vera eins margt þó 1 væri þar af tekinn, þar þó allir skilja, ad ej þann aftur utan 4. Þeir nijustu Arithmetici kalla tölustafina Niju, þessum hier verður filgt.
6. Jede Sache, in so weit sie vor sich angesehen wird, machet **Einz** aus.
7. Wenn man aber etliche oder viele einzelne von einer Art zusammen nimmt so entstehet daraus eine **Zahl**.
8. ... alle diese Dinge, bey denen man eben solche Eigenschaften findet, machen gleichfalls eine **Eins** aus, und diese Einheiten zusammen genommen, geben eine **Zahl** (§31).
9. Hieraus ist klar, dass die Unität oder **Eins** vor sich selbst nur der Nadir oder Wurzel der Zahlen ist, nemlich eine angenommene Grösse, wornach die Zahlen erwachsen und betrachtet werden (§31.32.); indem sich keine **Zahl** benennen läst, wo nicht eine gewisse Grösse für eine **Eins** angenommen wird.
10. ... Eine jede Ziffer, wenn sie allein stehet, gilt oder bedeutet so viel, wie ihr Nahme mit sich bringet, nemlich:

    1 heisset u. bedeutet Eins

    2 “ “ Twey

    ...

und 0 heisset und bedeutet Nulla oder nichts; wird dahero **unbedeutliche**, und die vorhergehenden Neune aber, jede vor sich **bedeutliche** Ziffern genannt.

11. **Tala** er áqvedinn fiøldi af einskonar einíngum.
12. **Eining** (unitas) er upphaf tølunnar.
13. Hinn sídasti tølustafr jarteinar **ekki** einn fyrir sig, en þegar settr er einnhverr af hinum níu framan hann, þá eykr han (þad er að segia 0) gilldi hans um **tíu**. Her fyrir kallast þeir níu stafir **merkiligir** edr **serþýdandi**, en **0id ómerkiligr** stafr.
14. Í serhverri stærd eru partar, hverr partr nefniz **eining** (unitas), og fleiri þesskonar **einíngar** tilsamans **tala** (numerus); ...
15. ... af níu enum fyrstu, merkir hver um sig áqvedinn einínga-fiøllda, heita þær því einnig **merkiligar**; en hin sídazta 0 (núll edr cero), sem í raun rettri er eingi tølustafr, kallaz **ómerkilig**, þar hún út af fyri sig gildir eckert, en eykr tífalldt merkíngu hinnar næztu tølu at framan.
16. 0, Nul, hvilket sidste, naar det staaer enkelt, betyder Intet.
17. At tælle er at samle flere eensartede Störrelser, den ene efter den anden, til et Hele. Dette Hele, hvorved vi ikke tage Hensyn til disse Størrelsers indbyrdes Stilling, men kun til deres Mængde, kaldes et Tal, hver af de enkelte eensartede Dele en Eenhed. Et Tal bestaaer saaledes af Eenheder.
18. Nul eller Intet lader man gjælde med som et Tal, men i Modsætning til dette kaldes de andre *betydende* Tal.
19. Sérhvørr hlutur edur tegund ... þegar hún er skodud edur metin útaf fyri sig, nefnist **eining** ... ; aukist tegundin um fleiri af sama tægi, nefnist þad **tala**; en talan géfur til kynna hvad margt sé sømu einínga ...
20. Sérhvør eining má líka nefnast tala þegar litið er til partanna sem í henni eru.
21. ... 0 (núllid) merkir ekkért, þegar þad er sér stædt. Hinir níu tølustafirnir eru því nefndir einu nafni **merkilegir** – edur **gildandi tølustafir**, til adgreiníngar frá núllinu.
22. ... stærð sem höfð er til samanburðar, nefnist opt kvarði, ... en í tölvísinni nefnist kvarðinn Eining eða einn (Eind) og þar af leiðir, að tala (numerus) er sundurlausrar stærðar samanburður við eininguna eða einn.
23. ... hvar af 0 (Núll) þýðir að þar sé engin eining, heldur autt rúm eða sæti.
24. Bæði 0 og ∞ eru ... stærðarinnar takmörk. Núll er raunar engin stærð, heldur er það minna en allar stærðir; eins er ∞ engin stærð, heldur stærra en allar stærðir. ... 0 eða miðjan sjálf [milli jákvæðra og neikvæðra stærða] er engin stærð, heldur útgangsdepill stærðanna ...
25. ... „núll" þýðir „ekkert" og er talið með tölustöfunum.

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