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Plotkin's call-by-value λ-calculus as a modal calculus

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Abstract

In the authors' previous analysis of the calling paradigms call-by-name and call-by-value through Girard's and Gödel's embeddings into the S4 modal logic, an asymmetry remains: the two paradigms are unified by the call-by-box paradigm of the modal target, but only for call-by-name can one say that the paradigm exists, up to isomorphism, inside the modal target. In this paper, we show that, pushing further the modal analysis, a symmetric situation is revealed, in that ordinary and Plotkin's λ-calculi are shown to truly co-exist inside a simple modal calculus.

Keywords: λ-calculus, call-by-name, call-by-value, S4 modal logic, Girard's embedding, Gödel's embedding
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1. Introduction

There is a correspondence between logic and functional programming languages, known as the Curry-Howard isomorphism [1]. At the basis, the correspondence links formulas, proofs and normalization in intuitionistic logic, on the one hand, with respectively types, programs and reduction in the simply-typed λ-calculus, on the other hand. Upon this basis, many other instances of correspondences enrich the isomorphism. In our work, started in [2], we are concerned with certain interpretations of intuitionistic logic into logics with modalities [3] and their connection with the fundamental calling paradigms of call-by-name and call-by-value in the λ-calculus [4].

The first connection between modal embeddings and calling paradigms was made in the context of linear logic [5, 6]. In our work, we seek the root and the deep meaning of this connection, along these general lines: one can identify a very simple modal calculus that serves as target of the modal embeddings and show that this modal target obeys a new calling paradigm, named call-by-box, that unifies call-by-name and call-by-value.

Specifically, in our previous work [2], we analyzed the computational interpretation of the traditional mappings of intuitionistic logic into modal logic S4 named after Girard and Gödel [3]. These mappings were shown to embed the ordinary (call-by-name, cbn) λ-calculus [7] and Plotkin's call-by-value (cbv) λ-calculus [4] into a very simple extension of the λ-calculus with an S4 modality. Such embeddings were even seen as a unification of the calling paradigms cbn and cbv by the new paradigm, call-by-box (cbb), found in the modal target.

In the context of linear logic [5, 6], one could already observe an asymmetry between the Girard/cbn embedding and Gödel/cbv one: the latter is said to be less interesting in [5], and has slightly weaker properties in [6]. In our work [2], that asymmetry remained: For cbn, the treatment is as neat that we may say Girard's embedding just points out an isomorphic copy of the cbn λ-calculus as a fragment of the modal target calculus. For cbv and Gödel's embedding, the results were not so satisfying.

In this paper, we investigate whether this asymmetry is inherent, or whether the modal analysis of the calling paradigms can be pushed further and reveal a hidden symmetry. We show that, by refining the modal target calculus and accordingly recasting the embeddings, we do not lose the neat treatment of cbn, but obtain similar results for cbv, that is: Gödel's embedding becomes just the indication of an isomorphic copy of Plotkin's cbv λ-calculus as a fragment of the modal target. In this sense, the ordinary and Plotkin's λ-calculi truly co-exist inside a simple modal calculus.

The slight refinement of the modal target consists in building into the untyped syntax a minimum of typing information—namely distinguishing between terms that can and cannot have a modal type. This creates in the target system two co-existing modes, the "left-first" and the "right-first", with which we can qualify the application constructor and reduction. The distinction between modes turns out to be connected to the distinction between calling paradigms. We verify that Girard's (resp. Gödel's) embedding can be recast as a mapping based on the idea of choosing the appropriate mode, translating application and reduction to left-first (resp. right-first) application and reduction. This chain of developments starts from a type distinction related to the modality of the system. In this sense, this development is still deepening the modal analysis of the calling paradigms.

Overview. Section 2 recalls basic concepts and the modal calculus $\lambda_{\mathtt{b}}$ we introduced in our previous work. Section 3 motivates the need to improve our previous treatment of the modal embeddings and starts that improvement by refining the modal target to a new calculus $\lambda_{\mathtt{b}\text{ƀ}}$. Section 4 introduces a further refinement of the modal target, named $\lambda_{\lessgtr}$, geared towards an improved treatment of the modal embeddings. Section 5 recasts the modal embeddings as mappings with $\lambda_{\lessgtr}$ as the target. Section 6 concludes.

2. Background

We recall the call-by-name and call-by-value λ-calculi, the modal calculus we introduced in our previous work, and Girard's and Gödel's embeddings.

2.1. Call-by-name and call-by-value λ-calculus

In this subsection, we briefly recall call-by-name (i.e. ordinary) λ-calculus and Plotkin's call-by-value λ-calculus [4] and fix some notation, terminology and definitions used throughout the paper.

As usual, the set of λ-terms is given by

$$M, N, P, Q ::= x \mid \lambda x.M \mid MN$$

and a **value** is a term of the form x or $\lambda x.M$. Values are ranged over by V, W.

Two reduction rules are considered

$$(\lambda x.M)N \to [N/x]M \qquad (\beta_{\mathtt{n}}) \qquad\qquad (\lambda x.M)V \to [V/x]M \qquad (\beta_{\mathtt{v}})$$

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma, x : A_1 \vdash M : A_2}{\Gamma \vdash \lambda x.M : A_1 \supset A_2} \qquad \frac{\Gamma \vdash M : A_1 \supset A_2 \quad \Gamma \vdash N : A_1}{\Gamma \vdash MN : A_2}$$

Figure 1: (Shared) typing rules of source calculi $\lambda_{\mathtt{n}}$ and $\lambda_{\mathtt{v}}$

where notation $[N/x]M$ stands for the substitution of N for x in M. As usual, $\to_{\beta_{\mathtt{n}}}$ (resp. $\to_{\beta_{\mathtt{v}}}$) denotes the compatible closure of $\beta_{\mathtt{n}}$ (resp. $\beta_{\mathtt{v}}$). Compatible closure is the closure under the term formers for λ-abstraction and application, i.e., closure under the rules:

$$\frac{M \to M'}{MN \to M'N}\ (\mu) \qquad \frac{N \to N'}{MN \to MN'}\ (\nu) \qquad \frac{M \to M'}{\lambda x.M \to \lambda x.M'}\ (\xi)$$

When we equip the λ-terms with $\to_{\beta_{\mathtt{n}}}$, we obtain the ordinary λ-calculus, or **call-by-name (cbn) λ-calculus**, here denoted $\lambda_{\mathtt{n}}$; equipping the λ-terms with $\to_{\beta_{\mathtt{v}}}$, we obtain Plotkin's **call-by-value (cbv) λ-calculus**, here denoted $\lambda_{\mathtt{v}}$. As a small illustration of the differences between the two relations, consider $M := (\lambda x.x)(yz)$; then, $M \to_{\beta_{\mathtt{n}}} yz$, but M is *irreducible* w.r.t. $\to_{\beta_{\mathtt{v}}}$, as yz is not a value. (It is an immediate observation that $\to_{\beta_{\mathtt{v}}} \subseteq \to_{\beta_{\mathtt{n}}}$.)

We briefly mention the typed version of these calculi. Types are given by:

$$A, A' ::= X \mid A \supset A'$$

Let Γ range over sets of type assignments $x : A$ with all x distinct. The typing system derives sequents of the form $\Gamma \vdash M : A$. The typing rules are given in Fig. 1. Logically, this is a presentation of intuitionistic implicational logic.

We define sub-relations of $\to_{\beta_{\mathtt{n}}}$ and $\to_{\beta_{\mathtt{v}}}$. To this end, we need the following restriction of the ν-rule:

$$\frac{N \to N'}{VN \to VN'}\ (\nu_{val})$$

Then we define: $\to_{\mathtt{w}}$ as $\beta_{\mathtt{n}}$ closed under μ and ν; $\to_{\mathtt{n}}$ as $\beta_{\mathtt{n}}$ closed under μ; $\to_{\mathtt{v}}$ as $\beta_{\mathtt{v}}$ closed under μ and ν_{val}. Furthermore, we define $\to_{\mathtt{w}}^*$ as the reflexive-transitive closure of $\to_{\mathtt{w}}$ and similarly for $\to_{\mathtt{n}}^*$ and $\to_{\mathtt{v}}^*$.

In $\to_{\mathtt{w}}$, reduction under λ's is forbidden, it is in this sense that $\to_{\mathtt{w}}^*$ is called **weak reduction**, but reduction in an application can occur both in the function

$$\frac{}{x \Rightarrow_{\mathtt{n}} x}\ VAR \quad \frac{M \Rightarrow_{\mathtt{n}} N}{\lambda x.M \Rightarrow_{\mathtt{n}} \lambda x.N}\ ABS \quad \frac{M \Rightarrow_{\mathtt{n}} M' \quad N \Rightarrow_{\mathtt{n}} N'}{MN \Rightarrow_{\mathtt{n}} M'N'}\ APP$$

$$\frac{M \to_{\mathtt{n}}^{*} \lambda x.M' \quad [N/x]M' \Rightarrow_{\mathtt{n}} P}{MN \Rightarrow_{\mathtt{n}} P}\ RDX$$

Figure 2: Standard reduction in $\lambda_{\mathtt{n}}$

position or the argument position. For instance, taking again $M := (\lambda x.x)(yz)$, we have that $\lambda y.M$ is irreducible w.r.t. $\to_{\mathtt{w}}$, but $M \to_{\mathtt{w}} yz$, and $MN \to_{\mathtt{w}}^{2} yzN'$, if $N \to_{\mathtt{w}} N'$.

Relations $\to_{\mathtt{n}}$ and $\to_{\mathtt{v}}$ are two ways of restricting $\to_{\mathtt{w}}$ to get a deterministic relation (a partial function). For instance, the term $N := (\lambda x.xy)((\lambda z.z)w)$ has two *redexes* w.r.t. $\to_{\mathtt{w}}$, but only one under each of the two restrictions, namely: $N \to_{\mathtt{n}} ((\lambda z.z)w)y$ and $N \to_{\mathtt{v}} (\lambda x.xy)w$. But note that none of the two restricted relations is contained in the other: for example, $(\lambda x.x)(yz)$ is irreducible w.r.t. $\to_{\mathtt{v}}$, but not w.r.t. $\to_{\mathtt{n}}$, and $x((\lambda y.y)z)$ is irreducible w.r.t. $\to_{\mathtt{n}}$, but not w.r.t. $\to_{\mathtt{v}}$.

We call $\to_{\mathtt{n}}^{*}$ and $\to_{\mathtt{v}}^{*}$ **call-by-name evaluation** and **call-by-value evaluation** respectively. Weak reduction and cbn evaluation make sense in $\lambda_{\mathtt{n}}$ while cbv evaluation makes sense in $\lambda_{\mathtt{v}}$.

Both $\lambda_{\mathtt{n}}$ and $\lambda_{\mathtt{v}}$ have a **standardization** theorem stating the completeness of a certain **standard reduction relation**, from which one can extract a notion of **standard reduction sequence** [4]. In the spirit of [8], [2] shows (in Corollaries 15 and 20) that the relations that the standard reduction relations in $\lambda_{\mathtt{n}}$ and $\lambda_{\mathtt{v}}$ (denoted by $\Rightarrow_{\mathtt{n}}$ and $\Rightarrow_{\mathtt{v}}$ resp.) can be axiomatized as in Figs. 2 and 3:

Theorem 1 (Standardization of $\lambda_{\mathtt{n}}$ and $\lambda_{\mathtt{v}}$).

1. *In* $\lambda_{\mathtt{n}}$, $M \to_{\beta_{\mathtt{n}}}^{*} N$ *iff* $M \Rightarrow_{\mathtt{n}} N$.
2. *In* $\lambda_{\mathtt{v}}$, $M \to_{\beta_{\mathtt{v}}}^{*} N$ *iff* $M \Rightarrow_{\mathtt{v}} N$.

Both cbn and cbv evaluations can be given alternative "big-step style" characterizations. Since in the definition of the standard reduction relations, there is no explicit closure rule for transitivity, it needs to be built into the other

$$\frac{}{x \Rightarrow_{\mathtt{v}} x}\ VAR \quad \frac{M \Rightarrow_{\mathtt{v}} N}{\lambda x.M \Rightarrow_{\mathtt{v}} \lambda x.N}\ ABS \quad \frac{M \Rightarrow_{\mathtt{v}} M' \quad N \Rightarrow_{\mathtt{v}} N'}{MN \Rightarrow_{\mathtt{v}} M'N'}\ APP$$

$$\frac{M \to_{\mathtt{v}}^* \lambda x.M' \quad N \to_{\mathtt{v}}^* V \quad [V/x]M' \Rightarrow_{\mathtt{v}} P}{MN \Rightarrow_{\mathtt{v}} P}\ RDX$$

Figure 3: Standard reduction in $\lambda_{\mathtt{v}}$

$$\frac{}{x \twoheadrightarrow_{\mathtt{n}} x}\ var \qquad \frac{}{\lambda x.M \twoheadrightarrow_{\mathtt{n}} \lambda x.M}\ abs \qquad \frac{M \twoheadrightarrow_{\mathtt{n}} M'}{MN \twoheadrightarrow_{\mathtt{n}} M'N}\ mu$$

$$\frac{M \twoheadrightarrow_{\mathtt{n}} \lambda x.M' \qquad [N/x]M' \twoheadrightarrow_{\mathtt{n}} P}{MN \twoheadrightarrow_{\mathtt{n}} P}\ rdx$$

Figure 4: Alternative characterization of cbn-evaluation in $\lambda_{\mathtt{n}}$.

rules. The new relations are named $\twoheadrightarrow_{\mathtt{n}}$ and $\twoheadrightarrow_{\mathtt{v}}$ respectively and are defined in Figs. 4 and 5.

Lemma 1 (Alternative characterizations).

1. *In* $\lambda_{\mathtt{n}}$*:* $M \to_{\mathtt{n}}^* N$ *iff* $M \twoheadrightarrow_{\mathtt{n}} N$.
2. *In* $\lambda_{\mathtt{v}}$*:* $M \to_{\mathtt{v}}^* N$ *iff* $M \twoheadrightarrow_{\mathtt{v}} N$.

Proof. The proof of each of the two items follows the same pattern. The "if" direction is by induction on the assumption. For the "only if" direction, of $\lambda_{\mathtt{v}}$ say, one does induction on $M \to_{\mathtt{v}}^* N$ and uses in the step case the lemma: if $M \to_{\mathtt{v}} P \twoheadrightarrow_{\mathtt{v}} N$, then $M \twoheadrightarrow_{\mathtt{v}} N$. The latter goes by induction on $M \to_{\mathtt{v}} P$.

2.2. The box calculus $\lambda_{\mathtt{b}}$

A simple modal λ-calculus $\lambda_{\Box}$, whose terms are generated by the grammar

$$M, N ::= x \mid \varepsilon(M) \mid \lambda x.M \mid MN \mid \mathsf{box}(N)$$

was introduced in [2] as a first vehicle to study the modal embeddings of intuitionistic logic into intuitionistic modal logic due to Girard and Gödel [3].

Types are given by the grammar:

$$A ::= X \mid B \supset A \mid B \qquad\qquad B ::= \Box A$$

$$\frac{}{x \twoheadrightarrow_{\mathsf{v}} x}\ var \qquad \frac{}{\lambda x.M \twoheadrightarrow_{\mathsf{v}} \lambda x.M}\ abs \qquad \frac{M \twoheadrightarrow_{\mathsf{v}} M'}{MN \twoheadrightarrow_{\mathsf{v}} M'N}\ mu \qquad \frac{M \twoheadrightarrow_{\mathsf{v}} V \quad N \twoheadrightarrow_{\mathsf{v}} N'}{MN \twoheadrightarrow_{\mathsf{v}} VN'}\ nu$$

$$\frac{M \twoheadrightarrow_{\mathsf{v}} \lambda x.M' \quad N \twoheadrightarrow_{\mathsf{v}} V \quad [V/x]M' \twoheadrightarrow_{\mathsf{v}} P}{MN \twoheadrightarrow_{\mathsf{v}} P}\ rdx$$

Figure 5: Alternative characterization of cbv evaluation in λ_{v}.

$$\frac{\Gamma, x : B \vdash M : A}{\Gamma \vdash \lambda x.M : B \supset A} \qquad \frac{\Gamma \vdash M : B \supset A \quad \Gamma \vdash N : B}{\Gamma \vdash MN : A} \qquad \frac{\Gamma \vdash M : A}{\Gamma \vdash \mathsf{box}(M) : \Box A}$$

$$\frac{}{\Gamma, x : B \vdash x : B} \qquad \frac{\Gamma \vdash M : \Box A}{\Gamma \vdash \varepsilon(M) : A} \qquad \frac{}{\Gamma, x : \Box A \vdash \varepsilon(x) : A}$$

Figure 6: Typing rules for the modal language $\lambda_\Box$ and λ_{b}. The rules in the first line are common to both calculi. In the second line, the first two rules belong to $\lambda_\Box$ while the third belongs to λ_{b}.

Note that in implications the antecedent must be a **boxed type** and accordingly types in contexts must be boxed. So, contexts Γ are sets of declarations $x : B$ where each x is declared at most once. The typing system derives sequents $\Gamma \vdash M : A$, and the typing rules are in Fig. 6. This system corresponds to a fragment of intuitionistic modal logic S4.

In [2], the study was then refined by introducing a sublanguage of $\lambda_\Box$, named λ_{b}, obtaining improved properties for the two embeddings. In this subsection, we will focus on various aspects of λ_{b} required by the present paper.

The grammar of terms of λ_{b} is:

$$M, N, P, Q, T ::= \varepsilon(x) \mid \lambda x.M \mid MN \mid \mathsf{box}(N)$$

Contrarily to $\lambda_\Box$, where variables x and ε correspond to distinct productions of the grammar, λ_{b} only allows these elements in the amalgamated form $\varepsilon(x)$. **Values** V are terms of the form $\varepsilon(x)$ or $\lambda x.M$. **Boxes** are terms of the form $\mathsf{box}(N)$, ranged over by $\mathcal{B}$. Types are as for $\lambda_\Box$ and the typing rules of λ_{b} are in Fig. 6.

The unique reduction rule of $\lambda_{\mathtt{b}}$ is:

$$(\lambda x.M)\mathsf{box}(N) \to [N/\varepsilon(x)]M \qquad (\beta_{\mathtt{b}})$$

where $[N/\varepsilon(x)]M$ is defined by recursion on M and all clauses are homomorphic, except for the critical clauses

$$[N/\varepsilon(x)]\varepsilon(x) = N \qquad\qquad [N/\varepsilon(x)]\varepsilon(y) = \varepsilon(y) \quad (x \neq y)$$

As usual, $\to_{\beta_{\mathtt{b}}}$ denotes the compatible closure of $\beta_{\mathtt{b}}$, i.e., the closure of $\beta_{\mathtt{b}}$ under all term formers of $\lambda_{\mathtt{b}}$.

Rule $\beta_{\mathtt{b}}$ only fires when the argument is a box. For this reason we speak of **call-by-box** (abbreviated cbb) which constitutes a **calling paradigm**, in the sense that there are companion notions of evaluation and standard reduction, and a standardization theorem holds (see below).

Several sub-relations of $\to_{\beta_{\mathtt{b}}}$ will be needed. To this end consider the closure rules:

$$\frac{M \to M'}{MN \to M'N}\ (\mu) \quad \frac{M \to M'}{M\mathcal{B} \to M'\mathcal{B}}\ (\mu_{box}) \quad \frac{N \to N'}{MN \to MN'}\ (\nu) \quad \frac{N \to N'}{VN \to VN'}\ (\nu_{val})$$

Then:

- $\to_{\mathtt{we}}$ is inductively defined by $\beta_{\mathtt{b}}$ and μ and ν.
- $\to_{\mathtt{b}>}$ is inductively defined by $\beta_{\mathtt{b}}$ and μ_{box} and ν.
- $\to_{\mathtt{b}<}$ is inductively defined by $\beta_{\mathtt{b}}$, μ and ν_{val}.

Notice: in these inductive definitions, we always close the same β-rule (hence a single calling paradigm is at stake). Relation $\to_{\mathtt{we}}$ is called **weak**—because values do not reduce—and **external**—because boxes do not reduce. The relation $\to^*_{\mathtt{we}}$ was called call-by-box evaluation in [2], but here we prefer to call it **weak-external reduction** (abbreviated w-e reduction). The w-e reduction of a given application MN consists of the interleaved w-e reduction of M and N in any order until a β_b-redex emerges at root position. The relation $\to_{\mathtt{we}}$ is non-deterministic. We may turn it into a deterministic relation by imposing

$$\frac{}{\varepsilon(x) \Rightarrow_{\mathtt{b}} \varepsilon(x)}\ VAR \qquad \frac{M \Rightarrow_{\mathtt{b}} N}{\lambda x.M \Rightarrow_{\mathtt{b}} \lambda x.N}\ ABS$$

$$\frac{M \Rightarrow_{\mathtt{b}} M' \quad N \Rightarrow_{\mathtt{b}} N'}{MN \Rightarrow_{\mathtt{b}} M'N'}\ APP \qquad \frac{M \Rightarrow_{\mathtt{b}} N}{\mathsf{box}(M) \Rightarrow_{\mathtt{b}} \mathsf{box}(N)}\ BOX$$

$$\frac{M \to^*_{\mathtt{we}} \lambda x.M' \quad N \to^*_{\mathtt{we}} \mathsf{box}(N') \quad [N'/\varepsilon(x)]M' \Rightarrow_{\mathtt{b}} P}{MN \Rightarrow_{\mathtt{b}} P}\ RDX$$

Figure 7: Standard reduction in $\lambda_{\mathtt{b}}$

either the left-first (<) or right-first (>) order of reduction in applications. In this way, we obtain two notions of **call-by-box evaluation**, namely $\to^*_{\mathtt{b}>}$ and $\to^*_{\mathtt{b}<}$. Cbn (resp. cbv) evaluation in $\lambda_{\mathtt{b}}$ will be defined later as a restriction of $\to^*_{\mathtt{b}>}$ (resp. $\to^*_{\mathtt{b}<}$).

In [2], it was shown that the calculus $\lambda_{\mathtt{b}}$ enjoys good properties like subject reduction, but for the purpose of this paper the important property to recall is standardization. Fig. 7 gives an inductive definition of the relation "M reduces in a standard way to N in $\lambda_{\mathtt{b}}$", denoted $M \Rightarrow_{\mathtt{b}} N$.

Theorem 2 (Standardization of $\lambda_{\mathtt{b}}$). *In $\lambda_{\mathtt{b}}$, $M \to^*_{\beta_b} N$ iff $M \Rightarrow_{\mathtt{b}} N$.*

2.3. Modal embeddings

In this subsection, we recall how the two modal embeddings of intuitionistic logic into modal logic S4 due to Girard and Gödel correspond to translations from $\lambda_{\mathtt{n}}$ and $\lambda_{\mathtt{v}}$ into the modal language $\lambda_{\mathtt{b}}$. We will see that these translations determine notions of call-by-name and call-by-value evaluation in $\lambda_{\mathtt{b}}$. Additionally, we will recall from [2] a collection of properties enjoyed by these translations.

Girard's translation from $\lambda_{\mathtt{n}}$ to $\lambda_{\mathtt{b}}$ is given in Fig. 8.

With respect to typing, Girard's embeddings enjoys the following property:

– $\Gamma \vdash M : A$ in $\lambda_{\mathtt{n}}$ iff $\Box\Gamma^\circ \vdash M^\circ : A^\circ$ in $\lambda_{\mathtt{b}}$.

(Here $\Box\Gamma^\circ$ is defined as $x_1 : \Box A_1^\circ, \ldots, x_n : \Box A_n^\circ$ when $\Gamma = x_1 : A_1, \ldots, x_n : A_n$.)

$$
\begin{array}{rclcrcl}
X^\circ & = & X & \qquad & x^\circ & = & \varepsilon(x) \\
(A_1 \supset A_2)^\circ & = & \Box A_1^\circ \supset A_2^\circ & & (\lambda x.M)^\circ & = & \lambda x.M^\circ \\
 & & & & (MN)^\circ & = & M^\circ \mathsf{box}(N^\circ)
\end{array}
$$

Figure 8: Translation from $\lambda_{\mathtt{n}}$ to $\lambda_{\mathtt{b}}$ ("Girard's translation")

The image of the term translation is the subset of $\lambda_{\mathtt{b}}$ terms given by the grammar

$$M, N ::= \varepsilon(x) \mid \lambda x.M \mid M\mathsf{box}(N) \tag{1}$$

Let us call this subset **Girard's image**. Due to the restricted form of arguments in applications, in Girard's image relations $\rightarrow_{\mathtt{we}}$ and $\rightarrow_{\mathtt{b}>}$ collapse to the same relation, one which can alternatively be defined as $\beta_{\mathtt{b}}$ closed under μ. This property of Girard's image we call its **indifference property** [2, 4]. The single deterministic relation on Girard's image is denoted $\rightarrow_{\mathtt{n}}$. By **call-by-name evaluation** in $\lambda_{\mathtt{b}}$ we mean $\rightarrow_{\mathtt{n}}^*$.

We recall in the next theorem the properties of Girard's translation established in [2]: preservation and reflection of reduction, evaluation, and standard reduction. These properties single out Girard's image as an isomorphic copy of $\lambda_{\mathtt{n}}$ inside $\lambda_{\mathtt{b}}$.

Theorem 3 (Properties of Girard's translation).

1. *(Reduction) $M \rightarrow_{\beta_{\mathtt{n}}} N$ in $\lambda_{\mathtt{n}}$ iff $M^\circ \rightarrow_{\beta_{\mathtt{b}}} N^\circ$ in $\lambda_{\mathtt{b}}$.*
2. *(Evaluation) $M \rightarrow_{\mathtt{n}} N$ in $\lambda_{\mathtt{n}}$ iff $M^\circ \rightarrow_{\mathtt{n}} N^\circ$ in $\lambda_{\mathtt{b}}$.*
3. *(Standard reduction) $M \Rightarrow_{\mathtt{n}} N$ in $\lambda_{\mathtt{n}}$ iff $M^\circ \Rightarrow_{\mathtt{b}} N^\circ$ in $\lambda_{\mathtt{b}}$.*

An immediate corollary of this theorem and the standardization theorem for $\lambda_{\mathtt{b}}$ is the standardization theorem for $\lambda_{\mathtt{n}}$.

Now we turn to Gödel's translation. In fact, throughout this paper we will consider a refinement of Gödel's translation, introduced in [2], which enjoys better properties. This refinement, which we still refer to as Gödel's translation

$$
\begin{array}{rclcrcl}
A^* & = & \Box A^\bullet & \qquad & V^* & = & \mathsf{box}(V^\bullet) \\
 & & & & (MN)^* & = & \mathsf{raise}(N^*)M^* \\
X^\bullet & = & X & & x^\bullet & = & \varepsilon(x) \\
(A_1 \supset A_2)^\bullet & = & \Box A_1^\bullet \supset \Box A_2^\bullet & & (\lambda x.M)^\bullet & = & \lambda x.M^*
\end{array}
$$

Figure 9: Translation from $\lambda_{\mathtt{v}}$ to $\lambda_{\mathtt{b}}$ ("Gödel's translation")

throughout this paper, is given in Fig. 9 and makes use of the abbreviation

$$\mathsf{raise}(M) := \lambda z.\varepsilon(z)M$$

The only difference compared to Gödel's original translation is in the case of application, which the latter translates by: $(MN)^* = \varepsilon(M^*)N^*$.

With respect to typing, Gödel's embeddings has the following properties:

– $\Gamma \vdash M : A$ in $\lambda_{\mathtt{v}}$ iff $\Gamma^* \vdash M^* : A^*$ in $\lambda_{\mathtt{b}}$.

– $\Gamma \vdash V : A$ in $\lambda_{\mathtt{v}}$ iff $\Gamma^* \vdash V^\bullet : A^\bullet$ in $\lambda_{\mathtt{b}}$.

(Here Γ^* is defined as $x_1 : A_1^*, \ldots, x_n : A_n^*$, when $\Gamma = x_1 : A_1, \ldots, x_n : A_n$.)

The image of the translation is contained in the subset of $\lambda_{\mathtt{b}}$ terms given by the grammar

$$M ::= \mathsf{box}(V) \mid VM \qquad V ::= \varepsilon(x) \mid \lambda x.M \tag{2}$$

For the exact image, the V in VM should be constrained to the form $\mathsf{raise}(M')$. But the subset (2) has the advantage of being closed under $\to_{\beta_b}$. Let us allow ourselves the abuse of calling (2) **Gödel's image**. Due to the restricted form of the term in function position in applications, in Gödel's image, the relations $\to_{\mathtt{we}}$ and $\to_{\mathtt{b}<}$ collapse to the same relation, one which can alternatively be defined as $\beta_{\mathtt{b}}$ closed under ν. This property of Gödel's image we call its **indifference property**. The single deterministic relation on Gödel's image is denoted $\to_{\mathtt{v}}$. By **call-by-value evaluation** of $\lambda_{\mathtt{b}}$, we mean $\to_{\mathtt{v}}^*$.

Now we recall from [2] the properties of Gödel's translation. Whereas typing is preserved and reflected, the situation with (standard) reduction is not as nice, since reflection does not hold, as Example 17 of [2] shows.

We recall in the next theorem the properties of Gödel's translation established in [2]: preservation of reduction, preservation and reflection of complete evaluation (evaluation to a value), and preservation of standard reduction.

Theorem 4 (Properties of Gödel's translation).

1. *(Reduction) If* $M \to_{\beta_{\mathtt{v}}} N$ *in* $\lambda_{\mathtt{v}}$*, then* $M^* \to^2_{\beta_{\mathtt{b}}} N^*$ *in* $\lambda_{\mathtt{b}}$*.*
2. *(Complete evaluation)* $M \to^*_{\mathtt{v}} V$ *in* $\lambda_{\mathtt{v}}$ *iff* $M^* \to^*_{\mathtt{v}} V^*$ *in* $\lambda_{\mathtt{b}}$*.*
3. *(Standard reduction) If* $M \Rightarrow_{\mathtt{v}} N$ *in* $\lambda_{\mathtt{v}}$*, then* $M^* \Rightarrow_{\mathtt{b}} N^*$ *in* $\lambda_{\mathtt{b}}$*.*

3. Refined target

3.1. Motivation

In Theorems 3 and 4, we proved properties of preservation/reflection of reduction and evaluation that can be seen as strong forms of the *translation* and *simulation* properties, in the sense defined by Plotkin in his study of cps translations [4]. For instance, regarding Girard's mapping, preservation/reflection of reduction in Theorems 3 entails the translation property: $M =_{\beta_{\mathtt{n}}} N$ in $\lambda_{\mathtt{n}}$ iff $M^\circ =_{\beta_{\mathtt{b}}} N^\circ$ in $\lambda_{\mathtt{b}}$; preservation/reflection of evaluation in Theorems 3 entails the simulation property: $Eval_{\mathtt{n}}(M) = Eval_{\mathtt{b}}(M^\circ)$, where $Eval_{\mathtt{n}}$ and $Eval_{\mathtt{b}}$ are the partial functions which, when defined, return the value calculated by cbn evaluation in $\lambda_{\mathtt{n}}$ and $\lambda_{\mathtt{b}}$ respectively. In addition, we discussed above why the image of Girard's mapping enjoys a kind of *indifference* property, also aligned with the sense given by Plotkin: cbn evaluation in the image of Girard's mapping is indistinguishable from mere weak external reduction, where evaluation of applications is subject to no constraint. Moreover, the translation, simulation, and indifference properties cooperate to yield standardization of $\lambda_{\mathtt{n}}$ as a corollary of standardization of $\lambda_{\mathtt{b}}$.

For Gödel's mapping, Theorem 4 gives weaker results: reflection of reduction is missing and reflection of evaluation only holds for complete evaluation to a value. The theorem is sufficient to extract again the simulation property, and a kind of indifference property was also observed above and is independent of these slight weaknesses of the theorem. But a full translation property is missing and standardization for $\lambda_{\mathtt{v}}$ cannot be inferred. Moreover, Theorem 3 also achieves something different in spirit: Girard's image (1) is a fragment of $\lambda_{\mathtt{b}}$ that is isomorphic to $\lambda_{\mathtt{n}}$. Theorem 4 is far from achieving anything similar for $\lambda_{\mathtt{v}}$ and Gödel's mapping.

In [2], the properties of Gödel's mapping were improved by identifying sub-relations in $\lambda_{\mathtt{b}}$ allowing reflection of (standard) reduction. Specifically, a derived β rule

$$\mathsf{raise}(\mathsf{box}(N))\mathsf{box}(\lambda x.P) \to [N/\varepsilon(x)]P \qquad (\beta_{\mathtt{b2}})$$

were introduced (notice that $\to_{\beta_{\mathtt{b2}}} \subseteq \to_{\beta}^{2}$), and a sub-relation of $\Rightarrow_{\mathtt{b}}$, named $\Rightarrow_{\mathtt{b2}}$, was inductively defined by the rules in Fig. 7, with the rule *RDX* replaced by this one:

$$\frac{N \to_{\mathtt{we}}^{*} \mathsf{box}(N') \quad M \to_{\mathtt{we}}^{*} \mathsf{box}(\lambda x.M') \quad [N'/\varepsilon(x)]M' \Rightarrow_{\mathtt{b2}} Q}{\mathsf{raise}(N)M \Rightarrow_{\mathtt{b2}} Q}\ \mathit{RDX2}$$

A form of preservation and reflection for (standard) reduction can now be obtained. The improved results are collected in the following theorem (we repeat the item for evaluation as a convenience for the reader—see the discussion that follows the theorem):

Theorem 5 (Improved properties of Gödel's translation).

1. *(Reduction)* $M \to_{\beta_{\mathtt{v}}} N$ *in* $\lambda_{\mathtt{v}}$ *iff* $M^* \to_{\beta_{\mathtt{b2}}} N^*$ *in* $\lambda_{\mathtt{b}}$.
2. *(Complete evaluation)* $M \to_{\mathtt{v}}^{*} V$ *in* $\lambda_{\mathtt{v}}$ *iff* $M^* \to_{\mathtt{v}}^{*} V^*$ *in* $\lambda_{\mathtt{b}}$.
3. *(Standard reduction)* $M \Rightarrow_{\mathtt{v}} N$ *in* $\lambda_{\mathtt{v}}$ *iff* $M^* \Rightarrow_{\mathtt{b2}} N^*$ *in* $\lambda_{\mathtt{b}}$.

An immediate corollary of this theorem and the standardization theorem for $\lambda_{\mathtt{b}}$ is the standardization theorem for $\lambda_{\mathtt{v}}$.

The proof of the third item of the previous theorem requires an addendum to the standardization theorem of $\lambda_{\mathtt{b}}$ concerning $\Rightarrow_{\mathtt{b2}}$ (see [2]). This addendum does not change the notion of standard reduction sequence that is implicitly defined by $\Rightarrow_{\mathtt{b}}$ (as argued in [2]), and for this reason, we may say the standardization theorems for $\lambda_{\mathtt{n}}$ and $\lambda_{\mathtt{v}}$ are obtained from a single standardization theorem for $\lambda_{\mathtt{b}}$, and overall an unification of $\lambda_{\mathtt{n}}$ and $\lambda_{\mathtt{v}}$ is achieved in $\lambda_{\mathtt{b}}$.

However, the reasons to be unsatisfied with Theorem 5 remain, in its treatment of Gödel's mapping, as compared to Theorem 3, in its treatment of Girard's mapping. Only the derived reduction rule $\beta_{\mathtt{b}2}$ improves the properties of Gödel's mapping and one may find this an *ad hoc* solution: is there a conceptual explanation for this second rule? In addition, $\beta_{\mathtt{b}2}$ does not yet turn Gödel's image into an isomorphic copy of $\lambda_{\mathtt{v}}$ and the indifference property of Gödel's image (hence the relation $\to_{\mathtt{v}}$ defined there) is still based on $\beta_{\mathtt{b}}$, not $\beta_{\mathtt{b}2}$. Finally, one may see the addendum to standardization of $\lambda_{\mathtt{b}}$ referred above as a separate proof of standardization, lowering the elegance of the unification achieved in $\lambda_{\mathtt{b}}$.

The question arises: Is this resisting asymmetry in the analysis of cbn and cbv through the modal embeddings by Girard and Gödel something inherent? Or is it that our analysis was not pushed far enough? We want to prove in this paper that the latter is the case. Specifically, we show that, by digging deeper into $\lambda_{\mathtt{b}}$, all the remaining defects of the treatment of cbv and Gödel's embedding are removed and a perfect and pleasing symmetry with cbn and Girard's embedding is obtained.

This digging deeper starts by refining in Subsection 3.2 the target calculus $\lambda_{\mathtt{b}}$ of the embeddings, through the distinction between terms that can have a modal type from those that cannot have a modal type, which leads to the calculus $\lambda_{\mathtt{b}\not\mathtt{b}}$. A closer look in Subsection 4.2 at this refined target calculus shows the coexistence of two modes of reduction, "left-first" and "right-first", based on which we construct yet another subsystem $\lambda_{\ltimes\!\!\!\rtimes}$. Later, in Section 5, we confirm that the two modes of reduction are strongly connected to the calling paradigms cbn and cbv. See Fig. 10 for a roadmap.

3.2. The system $\lambda_{\mathtt{b}\not\mathtt{b}}$: to be or not to be boxed

We introduce a refinement of $\lambda_{\mathtt{b}}$, named $\lambda_{\mathtt{b}\not\mathtt{b}}$, based on three ideas. First, we disallow nested boxed types, that is, types of the form $\Box\Box A$. Second, we build into the untyped version of the system a bit of type information, namely whether

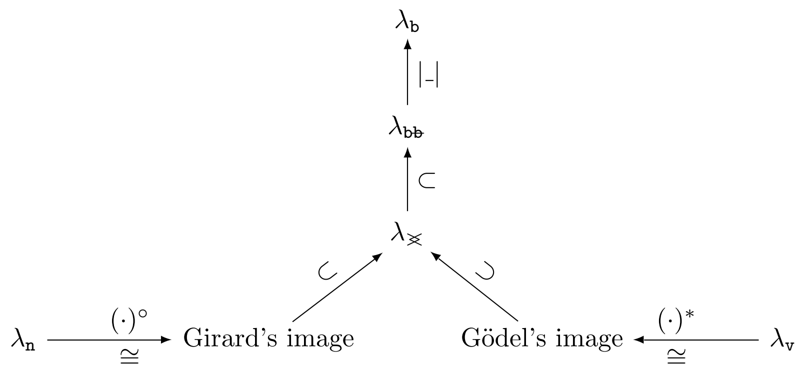

Figure 10: Digging deeper into $\lambda_{\flat}$

a given term can be typed with a boxed type or not, in the form of a syntactic distinction between "boxed" terms P and "unboxed" terms M. As we will see, this entails that application MQ is ambiguous, in the sense that it can still be boxed or unboxed (only if we knew the type of M could we disambiguate). So we are forced to split the application constructor into a boxed form and an unboxed form. A particular case of the boxed form of application will be of major importance, so important that we make it into a separate, primitive constructor—and this is the third idea.

The system is now defined with comments. Types of $\lambda_{\flat\!\!\!\flat}$ are as follows:

$$
\begin{array}{rrcl}
\text{(Types)} & A & ::= & B \mid C \\
\text{(Boxed types)} & B & ::= & \Box C \\
\text{(Unboxed types)} & C & ::= & X \mid B \supset A
\end{array}
$$

Terms of $\lambda_{\flat\!\!\!\flat}$ are defined by:

$$
\begin{array}{rrcl}
\text{(Terms)} & T & ::= & M \mid P \\
\text{(Unboxed terms)} & M, N & ::= & V \mid M @_{\not\flat} Q \\
\text{(Boxed terms)} & P, Q & ::= & \mathcal{B} \mid M @_{\flat} Q \mid QP \\
\text{(Values)} & V & ::= & \varepsilon(x) \mid \lambda x.T \\
\text{(Boxes)} & \mathcal{B} & ::= & \mathsf{box}(M)
\end{array}
$$

$$\frac{}{\Gamma, x : \Box C \vdash \varepsilon(x) : C} \qquad \frac{\Gamma, x : B \vdash T : A}{\Gamma \vdash \lambda x.T : B \supset A} \qquad \frac{\Gamma \vdash M : B \supset C \quad \Gamma \vdash Q : B}{\Gamma \vdash M @_{\not{\mathsf{b}}} Q : C}$$

$$\frac{\Gamma \vdash M : C}{\Gamma \vdash \mathsf{box}(M) : \Box C} \qquad \frac{\Gamma \vdash M : B' \supset B \quad \Gamma \vdash Q : B'}{\Gamma \vdash M @_{\mathsf{b}} Q : B} \qquad \frac{\Gamma \vdash Q : B' \quad \Gamma \vdash P : \Box(B' \supset B)}{\Gamma \vdash QP : B}$$

Figure 11: Typing rules of $\lambda_{\mathsf{b}\not{\mathsf{b}}}$.

The typing rules of $\lambda_{\mathsf{b}\not{\mathsf{b}}}$, displayed in Fig. 11, derive sequents of the forms

$$\Gamma \vdash T : A \qquad \Gamma \vdash M : C \qquad \Gamma \vdash P : B$$

where Γ is a set of declarations $x : B$. So unboxed (resp. boxed) terms are assigned unboxed (resp. boxed) types. Notice that it is coherent to consider $\varepsilon(x)$ as unboxed, because there are no nested boxed types. The term in the function (resp. argument) position of an application is a unboxed (resp. boxed) term M (resp. Q). But, if $M = \lambda x.T$, the application of M to Q should live in the syntactical class of T. Therefore the syntactical class of the application of M to Q is not determined by the syntactic classes of the M and Q, and thus we cannot tell whether the application should be unboxed or boxed. Hence we adopt the two forms $M @_{\not{\mathsf{b}}} Q$ and $M @_{\mathsf{b}} Q$. At the typing level, $M @_{\not{\mathsf{b}}} Q$ requires M of type $B \supset C$ and receives the unboxed type C, while $M @_{\mathsf{b}} Q$ requires M of type $B' \supset B$ and receives the boxed type B.

Substitution operations $[N/\varepsilon(x)]M$ and $[N/\varepsilon(x)]P$ are defined by simultaneous recursion on M and P, producing respectively an unboxed and a boxed term. Hence, substitution respects syntactic categories, in the sense that T and $[N/\varepsilon(x)]T$ are both unboxed terms, or both boxed terms. Again, the critical clause reads: $[N/\varepsilon(x)]\varepsilon(x) = N$.

The β-rule of $\lambda_{\mathsf{b}\not{\mathsf{b}}}$ is:

$$(\lambda x.T) @ \mathsf{box}(N) \to [N/\varepsilon(x)]T \qquad (\beta_{\mathsf{b}\not{\mathsf{b}}})$$

where the tag of @ (not shown) agrees with the syntactic category of T: if T is a unboxed (resp. boxed) term, then the tag is $\not{\mathsf{b}}$ (resp. b), and we may call this half of $\beta_{\mathsf{b}\not\mathsf{b}}$ $\beta_{\not\mathsf{b}}$ (resp. β_{b}). Since the syntactic category of T agrees with the syntactic category of the *contractum*, the rule $\beta_{\mathsf{b}\not\mathsf{b}}$ respects syntactic categories.

How about QP? This construction is a particular case of $M@_{\mathsf{b}}Q$ which is made primitive; and the derived typing and reduction rules relative to such particular case are also made primitive. Specifically, QP can be understood as $\mathsf{raise}(Q)@_{\mathsf{b}}P$, where $\mathsf{raise}(Q)$ abbreviates $\lambda z.\varepsilon(z)@_{\mathsf{b}}Q$, with fresh z.

Notice the typing derivation in $\lambda_{\mathsf{b}\not\mathsf{b}}$

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, z : \Box(B' \supset B) \vdash \varepsilon(z) : B' \supset B} \quad \dfrac{\Gamma \vdash Q : B'}{\Gamma, z : \Box(B' \supset B) \vdash Q : B'}\,W}{\Gamma, z : \Box(B' \supset B) \vdash \varepsilon(z)@_{\mathsf{b}}Q : B}}{\Gamma \vdash \lambda z.\varepsilon(z)@_{\mathsf{b}}Q : (\Box(B' \supset B)) \supset B} \qquad \Gamma \vdash P : \Box(B' \supset B)}{\Gamma \vdash \underbrace{(\lambda z.\varepsilon(z)@_{\mathsf{b}}Q)@_{\mathsf{b}}P}_{QP} : B}
$$

which makes use of admissibility in $\lambda_{\mathsf{b}\not\mathsf{b}}$ of the weakening rule W (proved by a simple induction over the derivation of the premiss). Accordingly, we add to $\lambda_{\mathsf{b}\not\mathsf{b}}$ the typing rule for QP shown in Fig. 11.

Observe how, in the typing rule for QP, P should be understood as the function, and Q as the argument, since the type of Q is the antecedent of the implication $B' \supset B$ seen in the type of P. Yet, when we expand QP, we obtain an application of the form $(\cdots Q \cdots)@_{\mathsf{b}}P$, and for this reason we are writing Q to the left of P.

Notice also that $\mathsf{raise}(Q)@_{\mathsf{b}}\mathsf{box}(M) \to_{\beta_{\mathsf{b}\not\mathsf{b}}} M@_{\mathsf{b}}Q$. Again, since QP is primitive, an auxiliary reduction rule for boxed functions is added:

$$
Q\mathsf{box}(M) \to M@_{\mathsf{b}}Q \qquad (O)
$$

We call this rule the *opening rule* since it opens $\mathsf{box}(M)$.

Let $R = \beta_{\mathsf{b}\not\mathsf{b}} \cup O$ and let $\to_{\mathsf{b}\not\mathsf{b}}$ denote the compatible closure of R. The relation $\to^*_{\mathsf{b}\not\mathsf{b}}$ is the full (i.e., unconstrained) reduction of $\lambda_{\mathsf{b}\not\mathsf{b}}$. Note that, as syntactic categories are preserved by the base rules $\beta_{\mathsf{b}\not\mathsf{b}}$ and O, syntactic categories are also preserved by $\to_{\mathsf{b}\not\mathsf{b}}$.

All this syntax maps trivially back to λ_{b}: there is a forgetful mapping $|\_|$ that erases the distinctions between boxed and unboxed terms, merges $M@_{\not\mathsf{b}}Q$

and $M@_{\mathtt{b}}Q$ as MQ, and expands QP. Formally:

$$
\begin{array}{rclrclrcl}
|\varepsilon(x)| &=& \varepsilon(x) & |\lambda x.T| &=& \lambda x.|T| & |\mathsf{box}(M)| &=& \mathsf{box}(|M|) \\
|M@_{\mathtt{b}}Q| &=& |M||Q| & |M@_{\not{\mathtt{b}}}Q| &=& |M||Q| & |QP| &=& (\lambda z.\varepsilon(z)|Q|)|P|
\end{array}
$$

Consider the closure rules

$$
\frac{M \to M'}{M@Q \to M'@Q}\ (\mu) \qquad \frac{Q \to Q'}{M@Q \to M@Q'}\ (\nu) \qquad \frac{Q \to Q'}{(\lambda x.T)@Q \to (\lambda x.T)@Q'}\ (\nu_\lambda)
$$

$$
\frac{P \to P'}{QP \to QP'}\ (\mu') \qquad \frac{Q \to Q'}{QP \to Q'P}\ (\nu') \qquad \frac{Q \to Q'}{Q\mathsf{box}(M) \to Q'\mathsf{box}(M)}\ (\nu'_{box})
$$

In the rules μ, ν and ν_λ, one has to choose either $\not{\mathtt{b}}$ or $\mathtt{b}$ to tag both occurrences of @ in the conclusion.

We define for $\lambda_{\mathtt{b}\not{\mathtt{b}}}$:

- $\to_{\mathtt{we}}$ is the closure of R under μ, ν, μ' and ν'.
- $\to_{\mathtt{b}}$ is the closure of R under μ, ν_λ, μ' and ν'_{box}.

The relation $\to^*_{\mathtt{we}}$ is called *weak-external (we) reduction* and the relation $\to^*_{\mathtt{b}}$ is called *call-by-box (cbb) evaluation*.

The relation $\to_{\mathtt{we}}$ is the closure of R under all closure rules but the two that allow reduction under λ-abstraction or $\mathsf{box}(\cdot)$. Because reduction under λ-abstraction (resp. $\mathsf{box}(\cdot)$) is forbidden, we use the adjective *weak* (resp. *external*). In $\to_{\mathtt{we}}$, reduction can happen freely in the components of applications $M@Q$ and QP. The relation $\to_{\mathtt{b}}$ constrains the closure rules ν and ν', that is, constrains when reduction in the argument Q is allowed.

The relation $\to_{\mathtt{we}}$ is not deterministic: for instance, in applications $M@Q$, we can freely opt for reducing M or Q if both reductions are possible. Hence, in weak-external reduction of applications $M@Q$, we can freely interleave reduction of M and Q.

The relation $\to_{\mathtt{b}}$ is a sub-relation of $\to_{\mathtt{we}}$, again a non-deterministic one ($Q\mathsf{box}(M)$ can reduce in two ways when $Q \to_{\mathtt{b}} Q'$). Nonetheless, we will see in Lemma 3 that both weak-external reduction and call-by-box evaluation are deterministic when reducing to a value or a box.

Before that, we introduce in Fig. 12 an alternative, "big-step" characterization of cbb evaluation, denoted $T \twoheadrightarrow_{\mathtt{b}} T'$. An alternative characterization of w-e reduction, denoted $T \twoheadrightarrow_{\mathtt{we}} T'$ is defined exactly in the same way, except that the rules *mu* and *nu* (resp. *mu'* and *nu'*) are replaced by *app* (resp. *app'*) where

$$\frac{M \twoheadrightarrow_{\mathtt{we}} M' \quad Q \twoheadrightarrow_{\mathtt{we}} Q'}{M@Q \twoheadrightarrow_{\mathtt{we}} M'@Q'}\ app \qquad \frac{Q \twoheadrightarrow_{\mathtt{we}} Q' \quad P \twoheadrightarrow_{\mathtt{we}} P'}{QP \twoheadrightarrow_{\mathtt{we}} Q'P'}\ app'$$

Lemma 2 (Alternative characterization). *In* $\lambda_{\mathtt{b}\flat}$*:*

1. $T \rightarrow^*_{\mathtt{we}} T'$ *iff* $T \twoheadrightarrow_{\mathtt{we}} T'$.
2. $T \rightarrow^*_{\mathtt{b}} T'$ *iff* $T \twoheadrightarrow_{\mathtt{b}} T'$.

Proof. The proof of each of the two items follows the same pattern. The "if" direction goes by induction on the assumption. For the "only if" direction, of the first item say, one does induction on $T \rightarrow^*_{\mathtt{we}} T'$ and uses in the step case the lemma: if $T \rightarrow_{\mathtt{we}} T'' \twoheadrightarrow_{\mathtt{we}} T'$, then $T \twoheadrightarrow_{\mathtt{we}} T'$. The latter goes by induction on $T \rightarrow_{\mathtt{we}} T''$. (More details in the appendix.)

Lemma 3 (Determinism). *In* $\lambda_{\mathtt{b}\flat}$*:*

1. *There is at most one* T' *such that* $T \rightarrow^*_{\mathtt{we}} T'$ *and* T' *is a value or a box.*
2. *There is at most one* T' *such that* $T \rightarrow^*_{\mathtt{b}} T'$ *and* T' *is a value or a box.*

Proof. The proofs of the two items are analogous. We profit from the alternative characterization of $\rightarrow^*_{\mathtt{we}}$ (resp. $\rightarrow^*_{\mathtt{b}}$) offered by Lemma 2 and prove instead: $T \twoheadrightarrow_{\mathtt{we}} T'$ (resp. $T \twoheadrightarrow_{\mathtt{b}} T'$) and T' is a value or a box, then T' is unique. This follows by induction on $T \twoheadrightarrow_{\mathtt{we}} T'$ (resp. $T \twoheadrightarrow_{\mathtt{b}} T'$). (More details in the appendix.)

Standard reduction of $\lambda_{\mathtt{b}\flat}$ is defined in Fig. 13.

Theorem 6 (Standardization). *In* $\lambda_{\mathtt{b}\flat}$*:* $T \rightarrow^*_{\mathtt{b}\flat} T'$ *iff* $T \Rightarrow_{\mathtt{b}\flat} T'$.

Proof. The "if" direction is a very simple induction on $T \Rightarrow_{\mathtt{b}\flat} T'$ and just uses the facts that $\rightarrow^*_{\mathtt{b}\flat}$ is reflexive, transitive and compatible, and that $\rightarrow^*_{\mathtt{b}} \subseteq \rightarrow^*_{\mathtt{b}\flat}$.

$$\frac{}{\varepsilon(x) \twoheadrightarrow_{\mathsf{b}} \varepsilon(x)}\ \mathit{var} \qquad \frac{}{\lambda x.T \twoheadrightarrow_{\mathsf{b}} \lambda x.T}\ \mathit{abs} \qquad \frac{}{\mathsf{box}(M) \twoheadrightarrow_{\mathsf{b}} \mathsf{box}(M)}\ \mathit{box}$$

$$\frac{M \twoheadrightarrow_{\mathsf{b}} M'}{M@Q \twoheadrightarrow_{\mathsf{b}} M'@Q}\ \mathit{mu} \qquad \frac{P \twoheadrightarrow_{\mathsf{b}} P'}{QP \twoheadrightarrow_{\mathsf{b}} QP'}\ \mathit{mu}'$$

$$\frac{M \twoheadrightarrow_{\mathsf{b}} \lambda x.T \quad Q \twoheadrightarrow_{\mathsf{b}} Q'}{M@Q \twoheadrightarrow_{\mathsf{b}} (\lambda x.T)@Q'}\ \mathit{nu} \qquad \frac{Q \twoheadrightarrow_{\mathsf{b}} Q' \quad P \twoheadrightarrow_{\mathsf{b}} \mathsf{box}(M)}{QP \twoheadrightarrow_{\mathsf{b}} Q'\mathsf{box}(M)}\ \mathit{nu}'$$

$$\frac{M \twoheadrightarrow_{\mathsf{b}} \lambda x.T \quad Q \twoheadrightarrow_{\mathsf{b}} \mathsf{box}(N) \quad [N/\varepsilon(x)]T \twoheadrightarrow_{\mathsf{b}} T'}{M@Q \twoheadrightarrow_{\mathsf{b}} T'}\ \mathit{rdx}$$

$$\frac{Q \twoheadrightarrow_{\mathsf{b}} Q' \quad P \twoheadrightarrow_{\mathsf{b}} \mathsf{box}(M) \quad M \twoheadrightarrow_{\mathsf{b}} M'}{QP \twoheadrightarrow_{\mathsf{b}} M'@_{\mathsf{b}}Q'}\ \mathit{rdx}'_1$$

$$\frac{Q \twoheadrightarrow_{\mathsf{b}} \mathsf{box}(N) \quad P \twoheadrightarrow_{\mathsf{b}} \mathsf{box}(M) \quad M \twoheadrightarrow_{\mathsf{b}} \lambda x.P' \quad [N/\varepsilon(x)]P' \twoheadrightarrow_{\mathsf{b}} P''}{QP \twoheadrightarrow_{\mathsf{b}} P''}\ \mathit{rdx}'_2$$

Figure 12: Alternative characterization of cbb evaluation in $\lambda_{\mathsf{b}\not\mathsf{b}}$. Proviso for rules *mu* and *nu*: the two applications in the conclusion have the same tag and, in the case of the rule *nu*, the tag agrees with the syntactic class of T (if T is an unboxed (resp. boxed) term, then the tag is $\not\mathsf{b}$ (resp. b)). Proviso for rule *rdx*: the tag of the application in the conclusion agrees with the syntactic class of T and T' (if T and T' are unboxed (resp. boxed) terms, then the tag is $\not\mathsf{b}$ (resp. b)).

The "only if" direction is proved by establishing the admissibility of the rules (1) to (8) in Fig. 14. Once this is done, the proof of the "only if" implication is by induction on $T \to^*_{\mathsf{b}\not\mathsf{b}} T'$ and follows immediately from rules (1) and (8).

The following preliminary remarks are useful:

(i) If $M \Rightarrow_{\mathsf{b}\not\mathsf{b}} \lambda x.T$, then there is T_0 such that $M \to^*_{\mathsf{b}} \lambda x.T_0$ and $T_0 \Rightarrow_{\mathsf{b}\not\mathsf{b}} T$.

(ii) If $P \Rightarrow_{\mathsf{b}\not\mathsf{b}} \mathsf{box}(M)$, then there is M_0 such that $P \to^*_{\mathsf{b}} \mathsf{box}(M_0)$ and $M_0 \Rightarrow_{\mathsf{b}\not\mathsf{b}} M$.

The proof of (1) is an easy induction on M. Then (2) follows from *RDX* and (1), and (2') follows from RDX' and (1). The proof of (3) is by induction on $T \Rightarrow_{\mathsf{b}\not\mathsf{b}} T'$. The case *RDX* requires the substitution lemma for $\lambda_{\mathsf{b}\not\mathsf{b}}$'s substitution, plus the following property of $\to_{\mathsf{b}}$: if $T \to_{\mathsf{b}} T'$, then $[N/\varepsilon(x)]T \to_{\mathsf{b}} [N/\varepsilon(x)]T'$. The rule (5) follows easily from (4), and the latter is proved by induction on $T \to_{\mathsf{b}} T'$. The rule (6) follows from preliminary remarks (i)

$$\frac{}{\varepsilon(x) \Rightarrow_{\mathsf{b}\not\mathsf{b}} \varepsilon(x)}\ VAR \qquad \frac{T \Rightarrow_{\mathsf{b}\not\mathsf{b}} T'}{\lambda x.T \Rightarrow_{\mathsf{b}\not\mathsf{b}} \lambda x.T'}\ ABS \qquad \frac{M \Rightarrow_{\mathsf{b}\not\mathsf{b}} M'}{\mathsf{box}(M) \Rightarrow_{\mathsf{b}\not\mathsf{b}} \mathsf{box}(M')}\ BOX$$

$$\frac{M \Rightarrow_{\mathsf{b}\not\mathsf{b}} M' \quad Q \Rightarrow_{\mathsf{b}\not\mathsf{b}} Q'}{M@Q \Rightarrow_{\mathsf{b}\not\mathsf{b}} M'@Q'}\ APP \qquad \frac{Q \Rightarrow_{\mathsf{b}\not\mathsf{b}} Q' \quad P \Rightarrow_{\mathsf{b}\not\mathsf{b}} P'}{QP \Rightarrow_{\mathsf{b}\not\mathsf{b}} Q'P'}\ APP'$$

$$\frac{M \to_{\mathsf{b}}^{*} \lambda x.T \quad Q \to_{\mathsf{b}}^{*} \mathsf{box}(N) \quad [N/\varepsilon(x)]T \Rightarrow_{\mathsf{b}\not\mathsf{b}} T'}{M@Q \Rightarrow_{\mathsf{b}\not\mathsf{b}} T'}\ RDX$$

$$\frac{Q \to_{\mathsf{b}}^{*} Q' \quad P \to_{\mathsf{b}}^{*} \mathsf{box}(M) \quad M@_{\mathsf{b}}Q' \Rightarrow_{\mathsf{b}\not\mathsf{b}} P'}{QP \Rightarrow_{\mathsf{b}\not\mathsf{b}} P'}\ RDX'$$

Figure 13: Standard reduction of $\lambda_{\mathsf{b}\not\mathsf{b}}$. Proviso for rule APP: the two applications in the conclusion have the same tag. Proviso for rule RDX: the tag of the application in the conclusion agrees with the syntactic class of T and T' (if T and T' are unboxed (resp. boxed) terms, then the tag is $\not\mathsf{b}$ (resp. b)).

and (ii), together with (3) and (5). The rule (6') follows from preliminary remark (ii), together with (5). Then, (7) follows easily from (6) by induction on $T \Rightarrow_{\mathsf{b}\not\mathsf{b}} (\lambda x.T')\mathsf{box}(N)$; and (7') follows easily from (6') by induction on $P \Rightarrow_{\mathsf{b}\not\mathsf{b}} Q\mathsf{box}(M)$. Finally, (8) is proved by induction on $T \Rightarrow_{\mathsf{b}\not\mathsf{b}} T'$, and uses (7) and (7').

From the proof of the "if" implication of this theorem, one extracts a notion of **standard reduction sequence**: it starts with cbb evaluation (corresponding to applications of rules RDX and RDX') of the given application $M@N$ or QP, until one decides to freeze the outer construct and do reduction inside the subexpressions (corresponding to the use of the other rules in Fig. 13).

4. Even more refined target

4.1. Motivation

In $\lambda_{\mathsf{b}\not\mathsf{b}}$, we added the redundant primitive QP that can be eliminated by the reduction rule O. But we are not interested in the normal forms w.r.t. that rule, we do not want to eliminate QP.

$$\frac{}{M \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} M}\ (1)$$

$$\frac{}{(\lambda x.T)@\mathsf{box}(N) \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} [N/\varepsilon(x)]T}\ (2) \qquad \frac{}{Q\mathsf{box}(M) \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} M@_{\mathsf{b}}Q}\ (2')$$

$$\frac{T \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} T' \quad N \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} N'}{[N/\varepsilon(x)]T \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} [N'/\varepsilon(x)]T'}\ (3)$$

$$\frac{M \to_{\mathsf{b}} N \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} P}{M \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} P}\ (4) \qquad \frac{M \to_{\mathsf{b}}^{*} N \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} P}{M \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} P}\ (5)$$

$$\frac{M \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} \lambda x.T \quad Q \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} \mathsf{box}(N)}{M@Q \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} [N/\varepsilon(x)]T}\ (6) \qquad \frac{Q \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} Q' \quad P \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} \mathsf{box}(M)}{QP \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} M@_{\mathsf{b}}Q'}\ (6')$$

$$\frac{T \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} (\lambda x.T')@\mathsf{box}(N)}{T \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} [N/\varepsilon(x)]T'}\ (7) \qquad \frac{P \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} Q\mathsf{box}(M)}{P \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} M@_{\mathsf{b}}Q}\ (7')$$

$$\frac{T \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} T' \to_{\mathsf{b}\not{\mathsf{b}}} T''}{T \Rightarrow_{\mathsf{b}\not{\mathsf{b}}} T''}\ (8)$$

Figure 14: Admissible rules of $\lambda_{\mathsf{b}\not{\mathsf{b}}}$

Indeed, in some sense QP is not eliminable, because of its role in the definition of weak-external reduction and call-by-box evaluation in $\lambda_{\mathsf{b}\not{\mathsf{b}}}$. Recall that the forgetful mapping $|\_| : \lambda_{\mathsf{b}\not{\mathsf{b}}} \to \lambda_{\mathsf{b}}$ translates reduction in the source as reduction in the target. But what is qualified as weak-external reduction or call-by-box evaluation $\lambda_{\mathsf{b}\not{\mathsf{b}}}$ is not necessarily mapped to weak-external reduction or call-by-box evaluation in the target, due to a detail. The closure rule ν' allows reduction of Q in w-e reduction of QP. But, if we map QP into λ_{b}, we obtain $(\lambda z.\varepsilon(z)|Q|)|P|$—hence the reduction of $|Q|$ is happening inside a λ-abstraction. Similarly for the closure rule ν'_{box} and the cbb evaluation of $Q\mathsf{box}(M)$. So the construction QP allows to *hide* in $\lambda_{\mathsf{b}\not{\mathsf{b}}}$ what, after translation to λ_{b}, is seen as a very particular use of reduction under λ.

In fact, in what follows, what we want is to keep the redundant QP and forbid the construction $M@_{\mathsf{b}}Q$. Such a move is: (1) sensible since the construction $M@_{\mathsf{b}}Q$ will not be needed in the image of the embeddings, as soon as we recast them as mappings into the refined target; and (2) possible since the rule O shows us that every $M@_{\mathsf{b}}Q$ can be *expanded* as QP.

4.2. The system $\lambda_{\gtrless}$: to be left-first or to be right-first

We thus introduce a system, denoted $\lambda_{\gtrless}$, which is, roughly speaking, the $@_{\mathsf{b}}$-free fragment of $\lambda_{\mathsf{b}ƀ}$. Compared to $\lambda_{\mathsf{b}ƀ}$: the grammar of types is unchanged; the grammar of terms is unchanged, except that $M@_{\mathsf{b}}Q$ is removed from the grammar of boxed terms; the typing rules stay unchanged, except that the typing rule for $M@_{\mathsf{b}}Q$ is dropped. We refrain from repeating all these definitions.

In $\lambda_{\mathsf{b}ƀ}$ or $\lambda_{\gtrless}$, an application $M@_{ƀ}Q$ is a unboxed term, like the term M in function position, so it necessarily unfolds as

$$V@_{ƀ}Q_1 \cdots @_{ƀ}Q_m \qquad (*)$$

for some $m \geq 1$, with $Q = Q_m$. In $(*)$, brackets should be restored left-first, as is suggested by the numbering of the successive arguments.

On the other hand, in $\lambda_{\mathsf{b}ƀ}$, an application $M@_{\mathsf{b}}Q$ is a boxed term, like the term Q in argument position, so it necessarily unfolds as

$$M_m@_{\mathsf{b}} \cdots M_1@_{\mathsf{b}}P_0 \qquad (**)$$

for some $m \geq 1$, with $M = M_m$, and P_0 not another $@_{\mathsf{b}}$. In $(**)$, brackets should be restored right-first. Application QP should also be "right-first" since it corresponds to a particular case of $@_{\mathsf{b}}$.

Similarly, in $\lambda_{\gtrless}$, the application QP unfolds as

$$Q_m \cdots Q_1\mathcal{B}$$

with brackets restored right-first, necessarily leading to a box, since a boxed term P_0 in $\lambda_{\gtrless}$ is either a box or an application $Q'P'$ and nothing else.

In $\lambda_{\gtrless}$, we write MQ instead of $M@_{ƀ}Q$.[1] MQ is called *left-first* application and QP is called *right-first* application, and, let us insist, they unfold as

$$VQ_1 \cdots Q_m \qquad\qquad Q_m \cdots Q_1\mathcal{B}$$

[1]This cannot cause confusion, since in $\lambda_{\gtrless}$ there is no $M@_{\mathsf{b}}Q$, and the other application QP, although also written with juxtaposition, composes two boxed terms, contrary to MQ.

These are also the general forms in $\lambda_{\gtrless}$ of unboxed and boxed terms respectively.

The β-rule of $\lambda_{\mathsf{b}\bar{\mathsf{b}}}$ can be simplified in $\lambda_{\gtrless}$ as

$$(\lambda x.M)\mathsf{box}(N) \to [N/\varepsilon(x)]M \qquad (\beta_{<})$$

where the redex is a left-first application. Right-first application QP also comes with its own β-rule in $\lambda_{\gtrless}$

$$\mathsf{box}(N)(\mathsf{box}(\lambda x.P)) \to [N/\varepsilon(x)]P \qquad (\beta_{>})$$

which can be understood in $\lambda_{\mathsf{b}\bar{\mathsf{b}}}$ as the sequence of two reduction steps

$$\mathsf{box}(N)(\mathsf{box}(\lambda x.P)) \to (\lambda x.P)@_{\mathsf{b}}\mathsf{box}(N) \to [N/\varepsilon(x)]P \tag{3}$$

In $\beta_{>}$, the argument is passed to the function located to the right.

In both β-rules, the argument must be a box, $\mathsf{box}(N)$. Following the call-by-box paradigm, this box is opened when calling the function, as happens with $\beta_{\mathsf{b}\bar{\mathsf{b}}}$ of $\lambda_{\mathsf{b}\bar{\mathsf{b}}}$, and with β_{b} of λ_{b} in the first place. In $\beta_{>}$, also the function $\lambda x.P$ is inside a box. This box is implicitly opened by the opening rule, as seen in (3).

Let $\beta_{\gtrless} := \beta_{>} \cup \beta_{<}$. As usual, $\to_{\beta_{\gtrless}}$ denotes the compatible closure of $\beta_{\gtrless}$. Similarly for $\to_{\beta_{<}}$ and $\to_{\beta_{>}}$.

If, following what we did in $\lambda_{\mathsf{b}\bar{\mathsf{b}}}$, we defined $\to_{\mathtt{we}}$ in $\lambda_{\gtrless}$ as the closure of $\beta_{\gtrless}$ under the μ and ν closure rules relative to left-first and right-first applications, we would obtain too weak a notion of weak-external reduction. In $\lambda_{\mathsf{b}\bar{\mathsf{b}}}$, the w-e reduction of a term of the form $Q\mathsf{box}(M)$ may open the box and reduce M in the hope to obtain some λ-abstraction. But in $\lambda_{\gtrless}$, since there is no rule to open $\mathsf{box}(M)$, we are tempted to complement such weak notion of w-e reduction with the possibility of reduction inside $\mathsf{box}(M)$. We prefer to never see such kind of reduction and fortunately, by resorting to the "big-step" style of definition, we have means to satisfy that preference.

Call-by-box (cbb) evaluation of $\lambda_{\gtrless}$ (notation: $T \twoheadrightarrow_{\mathsf{b}} T'$) is defined in Fig. 15. *Weak-external reduction* of $\lambda_{\gtrless}$ (notation: $T \twoheadrightarrow_{\mathtt{we}} T'$) is defined like in Fig. 15, except that: the rules *mu*< and *nu*< (resp. *mu*> and *nu*>) are replaced by the

$$\frac{}{\varepsilon(x) \twoheadrightarrow_{\mathsf{b}} \varepsilon(x)}\ \mathit{var} \qquad \frac{}{\lambda x.T \twoheadrightarrow_{\mathsf{b}} \lambda x.T}\ \mathit{abs} \qquad \frac{}{\mathsf{box}(M) \twoheadrightarrow_{\mathsf{b}} \mathsf{box}(M)}\ \mathit{box}$$

$$\frac{M \twoheadrightarrow_{\mathsf{b}} M'}{MQ \twoheadrightarrow_{\mathsf{b}} M'Q}\ \mathit{mu}{<} \qquad \frac{P \twoheadrightarrow_{\mathsf{b}} P'}{QP \twoheadrightarrow_{\mathsf{b}} QP'}\ \mathit{mu}{>}$$

$$\frac{M \twoheadrightarrow_{\mathsf{b}} \lambda x.T \quad Q \twoheadrightarrow_{\mathsf{b}} Q'}{MQ \twoheadrightarrow_{\mathsf{b}} (\lambda x.T)Q'}\ \mathit{nu}{<} \qquad \frac{Q \twoheadrightarrow_{\mathsf{b}} Q' \quad P \twoheadrightarrow_{\mathsf{b}} \mathsf{box}(M)}{QP \twoheadrightarrow_{\mathsf{b}} Q'\mathsf{box}(M)}\ \mathit{nu}{>}$$

$$\frac{M \twoheadrightarrow_{\mathsf{b}} \lambda x.M' \quad Q \twoheadrightarrow_{\mathsf{b}} \mathsf{box}(N) \quad [N/\varepsilon(x)]M' \twoheadrightarrow_{\mathsf{b}} P}{MQ \twoheadrightarrow_{\mathsf{b}} P}\ \mathit{rdx}{<}$$

$$\frac{Q \twoheadrightarrow_{\mathsf{b}} \mathsf{box}(N) \quad P \twoheadrightarrow_{\mathsf{b}} \mathsf{box}(M) \quad M \twoheadrightarrow_{\mathsf{b}} \lambda x.P' \quad [N/\varepsilon(x)]P' \twoheadrightarrow_{\mathsf{b}} P''}{QP \twoheadrightarrow_{\mathsf{b}} P''}\ \mathit{rdx}{>}$$

Figure 15: Cbb evaluation in $\lambda_{\between}$.

$$\frac{}{\varepsilon(x) \Rightarrow_{\between} \varepsilon(x)}\ \mathit{VAR} \qquad \frac{T \Rightarrow_{\between} T'}{\lambda x.T \Rightarrow_{\between} \lambda x.T'}\ \mathit{ABS} \qquad \frac{M \Rightarrow_{\between} M'}{\mathsf{box}(M) \Rightarrow_{\between} \mathsf{box}(M')}\ \mathit{BOX}$$

$$\frac{M \Rightarrow_{\between} M' \quad Q \Rightarrow_{\between} Q'}{MQ \Rightarrow_{\between} M'Q'}\ \mathit{APP}{<} \qquad \frac{Q \Rightarrow_{\between} Q' \quad P \Rightarrow_{\between} P'}{QP \Rightarrow_{\between} Q'P'}\ \mathit{APP}{>}$$

$$\frac{M \twoheadrightarrow_{\mathsf{b}} \lambda x.M' \quad Q \twoheadrightarrow_{\mathsf{b}} \mathsf{box}(N) \quad [N/\varepsilon(x)]M' \Rightarrow_{\between} P}{MQ \Rightarrow_{\between} P}\ \mathit{RDX}{<}$$

$$\frac{Q \twoheadrightarrow_{\mathsf{b}} \mathsf{box}(N) \quad P \twoheadrightarrow_{\mathsf{b}} \mathsf{box}(M) \quad M \twoheadrightarrow_{\mathsf{b}} \lambda x.P' \quad [N/\varepsilon(x)]P' \Rightarrow_{\between} P''}{QP \Rightarrow_{\between} P''}\ \mathit{RDX}{>}$$

Figure 16: Standard reduction in $\lambda_{\between}$.

following rule *app*< (resp. *app*>):

$$\frac{M \twoheadrightarrow_{\mathsf{we}} M' \quad Q \twoheadrightarrow_{\mathsf{we}} Q'}{MQ \twoheadrightarrow_{\mathsf{we}} M'Q'}\ \mathit{app}{<} \qquad \frac{Q \twoheadrightarrow_{\mathsf{we}} Q' \quad P \twoheadrightarrow_{\mathsf{we}} P'}{QP \twoheadrightarrow_{\mathsf{we}} Q'P'}\ \mathit{app}{>}$$

From the definition it follows at once that weak-external reduction can happen freely inside applications MQ and QP, and never inside a λ-abstraction or $\mathsf{box}(\cdot)$. Moreover, $\twoheadrightarrow_{\mathsf{b}} \subseteq \twoheadrightarrow_{\mathsf{we}}$ because $\twoheadrightarrow_{\mathsf{we}}$ is reflexive.

Standard reduction of $\lambda_{\between}$ is defined in Fig. 16.

According to the following result, every weak-external reduction, in the sense of λ_{bb}, that happens between two $\lambda_{\between}$-terms also holds as a weak-external reduction in the sense of $\lambda_{\between}$; and a similar conservativity holds for cbb evaluation, full and standard reduction.

Lemma 4 (Conservativity). *For all $T, T' \in \lambda_{\bowtie}$:*

1. *(Reduction)* $T \to^{*}_{\beta_{\bowtie}} T'$ *in* $\lambda_{\bowtie}$ *iff* $T \to^{*}_{\mathtt{bb}} T'$ *in* $\lambda_{\mathtt{bb}}$.
2. *(W-e reduction)* $T \twoheadrightarrow_{\mathtt{we}} T'$ *in* $\lambda_{\bowtie}$ *iff* $T \to^{*}_{\mathtt{we}} T'$ *in* $\lambda_{\mathtt{bb}}$.
3. *(Cbb evaluation)* $T \twoheadrightarrow_{\mathtt{b}} T'$ *in* $\lambda_{\bowtie}$ *iff* $T \to^{*}_{\mathtt{b}} T'$ *in* $\lambda_{\mathtt{bb}}$.
4. *(Standard reduction)* $T \Rightarrow_{\bowtie} T'$ *in* $\lambda_{\bowtie}$ *iff* $T \Rightarrow_{\mathtt{bb}} T'$ *in* $\lambda_{\mathtt{bb}}$.

PROOF. Proof of 1. For the "only if" half, first one proves the result for a single step by induction on $\to_{\beta_{\bowtie}}$, and then the result follows immediately by induction on $T \to^{*}_{\beta_{\bowtie}} T'$. For the "if" half, we prove: (a) for T_1, T_2 in $\lambda_{\mathtt{bb}}$, $T_1 \to^{*}_{\mathtt{bb}} T_2$ in $\lambda_{\mathtt{bb}}$ implies $T_1^{\bowtie} \to^{*}_{\beta_{\bowtie}} T_2^{\bowtie}$ in $\lambda_{\bowtie}$ where $(.)^{\bowtie}$ is the mapping into $\lambda_{\bowtie}$ defined homomorphically, except for $(M @_{\mathtt{b}} Q)^{\bowtie} = Q^{\bowtie} \mathsf{box}(M^{\bowtie})$; (b) the result follows once we observe that $(.)^{\bowtie}$ is invariant for terms of $\lambda_{\bowtie}$. (a) follows immediately by induction, after proving: $T_1 \to_{\mathtt{bb}} T_2$ in $\lambda_{\mathtt{bb}}$ implies $T_1^{\bowtie} = T_2^{\bowtie}$ or $T_1^{\bowtie} \to_{\beta_{\bowtie}} T_2^{\bowtie}$ in $\lambda_{\bowtie}$. The latter follows by induction on $T_1 \to_{\mathtt{bb}} T_2$.

Proof of 2. "Only if". Again, we profit from the alternative characterization of $\to^{*}_{\mathtt{we}}$ in $\lambda_{\mathtt{bb}}$ (Lemma 2) and prove instead: $T \twoheadrightarrow_{\mathtt{we}} T'$ in $\lambda_{\bowtie}$ implies $T \twoheadrightarrow_{\mathtt{we}} T'$ in $\lambda_{\bowtie}$. This is by routine induction on $T \twoheadrightarrow_{\mathtt{we}} T'$ in $\lambda_{\bowtie}$ (note that each rule for $\twoheadrightarrow_{\mathtt{we}}$ in $\lambda_{\bowtie}$ matches exactly one rule for $\twoheadrightarrow_{\mathtt{we}}$ in $\lambda_{\mathtt{bb}}$).

"If". Again, we crucially profit from the alternative characterization of $\to^{*}_{\mathtt{we}}$ in Lemma 2. Then the result follows immediately from the following lemma: if $T \twoheadrightarrow_{\mathtt{we}} T'$ in $\lambda_{\mathtt{bb}}$ and $T \in \lambda_{\bowtie}$, then: (a) if $T' \in \lambda_{\bowtie}$, then $T \twoheadrightarrow_{\mathtt{we}} T'$ in $\lambda_{\bowtie}$; and (b) if T' is a box or a λ-abstraction, then $T' \in \lambda_{\bowtie}$. This lemma follows by induction on $T \twoheadrightarrow_{\mathtt{we}} T'$ in $\lambda_{\mathtt{bb}}$. (More details in the appendix.)

Proof of 3. Repeats the structure of the proof of 2, including an analogous lemma for the "if" half, namely: for $T \in \lambda_{\bowtie}$, $T' \in \lambda_{\mathtt{bb}}$, if $T \twoheadrightarrow_{\mathtt{b}} T'$ in $\lambda_{\mathtt{bb}}$ and T' is a box or a λ-abstraction, then $T' \in \lambda_{\bowtie}$.

Proof of 4. Both the "only if" and "if" parts follow smoothly by induction on the premise, with the help of 3. Some cases of the "if" proof also make use of the lemma just mentioned in the proof of 3.

It follows that w-e reduction and cbb evaluation in $\lambda_{\between}$ are deterministic relations in the sense (and as a consequence) of Lemma 3.

Theorem 7 (Standardization). *In $\lambda_{\between}$: $T \to^*_{\beta_{\between}} T'$ iff $T \Rightarrow_{\between} T'$.*

Proof. Immediate consequence of conservativity of $\lambda_{\mathtt{b}\bar{\mathtt{b}}}$ over $\lambda_{\between}$ (specifically, parts 1 and 4 of Lemma 4) and standardization of $\lambda_{\mathtt{b}\bar{\mathtt{b}}}$ (Thm. 6): $T \to^*_{\beta_{\between}} T'$ iff $T \to^*_{\mathtt{b}\bar{\mathtt{b}}} T'$ iff $T \Rightarrow_{\mathtt{b}\bar{\mathtt{b}}} T'$ iff $T \Rightarrow_{\between} T'$.

5. Refined embeddings

The concepts of cbb evaluation and standard reduction in $\lambda_{\between}$ (Figs. 15 and 16), together with $\lambda_{\between}$'s standardization theorem, are the tools to complete the modal analysis of $\lambda_{\mathtt{n}}$ and $\lambda_{\mathtt{v}}$ (the upper layer $\lambda_{\mathtt{b}\bar{\mathtt{b}}}$ provided the modal foundation of these tools, establishing the link with $\lambda_{\mathtt{b}}$, being the source of the standardization theorem, and justifying the weak-external terminology). We head to the completion of the diagram in Fig. 10

5.1. Recasting the modal embeddings and their images

We recast the refined modal embeddings, previously seen as having images in $\lambda_{\mathtt{b}}$, as mappings having images in $\lambda_{\between}$. Girard's mapping of Fig. 8, whose target is $\lambda_{\mathtt{b}}$, can equally be seen as landing in $\lambda_{\between}$, with λ-terms mapped to unboxed terms, in particular application mapped to left-first application. The typing property of Girard's mapping, previously stated with $\lambda_{\mathtt{b}}$, holds in the same way, if stated with $\lambda_{\between}$.

The image of Girard's mapping (1), which we recall here

$$M, N ::= \varepsilon(x) \mid \lambda x.M \mid M\mathsf{box}(N)$$

is the subset of the unboxed terms of $\lambda_{\between}$ determined by a chain of restrictions: QP is eliminated (because not used in the target), so boxed terms are reduced to boxes, which are inlined in the single place where they are used (in the application MQ), which allows the elimination of boxed terms altogether, and the identification of terms T with unboxed terms. This subset is, so to speak, the left-first half of $\lambda_{\lessgtr}$ and we call it again **Girard's image**.

Gödel's mapping of Fig. 9 can be seen as landing in $\lambda_{\lessgtr}$, with values of the λ-calculus mapped to values, and λ-terms mapped to boxed terms, provided $(MN)^*$ is defined as the right-first application N^*M^*. The typing properties of Gödel's mapping, stated before with $\lambda_{\mathtt{b}}$, hold in the same way when stated with $\lambda_{\lessgtr}$.

The image of this version of Gödel's mapping is:

$$P, Q ::= \mathsf{box}(V) \mid QP \qquad\qquad V ::= \varepsilon(x) \mid \lambda x.P \tag{4}$$

Contrarily to (2), this is the exact image of the mapping. It defines a subset of boxed terms, the set determined by a chain of restrictions: MQ is eliminated (because not used in the target), so unboxed terms are identified with values and terms T are identified with boxed terms. This subset is, so to speak, the right-first half of $\lambda_{\lessgtr}$ and we call it **Gödel's image**.

5.2. An improved and symmetric unification

It is evident that the modal embeddings put the λ-terms in 1-1 correspondence with the unboxed terms in Girard's image and the boxed terms in Gödel's image (in the latter case, do not forget that QP is right-associative). Such bijections commute with substitution:

– $[N^\circ/\varepsilon(x)]M^\circ = ([N/x]M)^\circ$.

– $[V^\bullet/\varepsilon(x)]M^* = ([V/x]M)^*$ and $[V^\bullet/\varepsilon(x)]W^\bullet = ([V/x]W)^\bullet$.

Let us call these bijections **modal glasses**. They are the main tool in what follows: we will restrict full reduction, w-e reduction and cbb evaluation in $\lambda_{\lessgtr}$ to both Girard's and Gödel's images and we will put on the glasses to see the consequences for $\lambda_{\mathtt{n}}$ and $\lambda_{\mathtt{v}}$.

Let us see a first example. Take rule $\beta_{\lessgtr}$. In Girard's image, the right-first half $\beta_{>}$ does not exist because there are no right-first applications in Girard's image. The left-first half $\beta_<$ restricted to Girard's image reads

$$(\lambda x.M^\circ)\mathsf{box}(N^\circ) \to [N^\circ/\varepsilon(x)]M^\circ$$

where M, N are λ-terms. This is the same as

$$((\lambda x.M)N)^\circ \to ([N/x]M)^\circ$$

With the modal glasses we see that $\beta_<$ in Girard's image is the same as rule $\beta_{\mathtt{n}}$ in $\lambda_{\mathtt{n}}$. In Gödel's image, the left-first half $\beta_<$ does not exist because there are no left-first applications in Gödel's image. The right-first half $\beta_>$ restricted to Gödel's image reads

$$\mathsf{box}(V^\bullet)(\mathsf{box}(\lambda x.P^*)) \to [V^\bullet/\varepsilon(x)]P^*$$

where P, V are λ-terms. This is the same as

$$(\lambda x.P)^* V^* \to ([V/x]P)^*$$

With the modal glasses we see that $\beta_>$ in Gödel's image is the same as rule $\beta_{\mathtt{v}}$ in $\lambda_{\mathtt{v}}$.

By closing $\beta_<$ in Girard's image under the closure rules that make sense in that fragment, we obtain a reduction relation that, viewed through the modal glasses, looks the same as $\to_{\beta_{\mathtt{n}}}$ in $\lambda_{\mathtt{n}}$, that is: $M^\circ \to_{\beta_<} N^\circ$ iff $M \to_{\beta_{\mathtt{n}}} N$. By closing $\beta_>$ in Gödel's image under the closure rules that make sense in that fragment, we obtain a reduction relation that, seen through the modal glasses, looks the same as $\to_{\beta_{\mathtt{v}}}$ in $\lambda_{\mathtt{v}}$, that is: $M^* \to_{\beta_>} N^*$ iff $M \to_{\beta_{\mathtt{v}}} N$.

Next we move to cbb evaluation in Fig. 15. Let us first restrict this relation to Girard's image. The rules *box*, *mu>*, *nu>* and *rdx>* have no effect in this fragment. Every argument Q is a box and, since a box can reduce to itself only, the rule *nu<* becomes a particular case of the rule *mu<* and the second premiss of the rule *rdx<* can be omitted, provided $Q = \mathsf{box}(N)$. As a result, we obtain a set of rules that, seen through the modal glasses, looks the same as the set of rules in Fig. 4. This is an informal argument for: $M^\circ \twoheadrightarrow_{\mathtt{b}} N^\circ$ iff $M \twoheadrightarrow_{\mathtt{n}} N$, for M, N λ-terms. Hence, cbb evaluation in Girard's image is the same as cbn evaluation (of $\lambda_{\mathtt{n}}$-terms). We call this the **indifference property** of Girard's image.

If, instead, we restrict the rules in Fig. 15 to Gödel's image, the rules $mu<$, $nu<$ and $rdx<$ have no effect in the fragment. Every box has the form $\mathsf{box}(V)$; hence, the second and third premisses of the rule $rdx>$ can be merged since $M = \lambda x.P'$. As a result, we obtain a set of rules that, up to the modal glasses, is the same as the set of rules in Fig. 5. This is an informal argument for: $M^* \twoheadrightarrow_{\mathtt{b}} N^*$ iff $M \twoheadrightarrow_{\mathtt{v}} N$, for M, N λ-terms. Therefore, cbb evaluation in Gödel's image is the same as cbv evaluation (of $\lambda_{\mathtt{v}}$-terms). We call this the **indifference property** of Gödel's image.

Similarly, if we restrict the standard reduction relation of Fig. 16 to Girard's (resp. Gödel's) image we obtain a set of rules that, up to the modal glasses, is the same as the set of rules in Fig. 2 (resp. Fig. 3), giving an informal argument for $M^\circ \Rightarrow_{\gtrless} N^\circ$ iff $M \Rightarrow_{\mathtt{n}} N$ (resp. $M^* \Rightarrow_{\gtrless} N^*$ iff $M \Rightarrow_{\mathtt{v}} N$), for M, N λ-terms.

Let us collect the properties of Girard's mapping.

Theorem 8 (Properties of Girard's translation from $\lambda_{\mathtt{n}}$ to $\lambda_{\gtrless}$).

1. $M \to_{\beta_{\mathtt{n}}} N$ *in* $\lambda_{\mathtt{n}}$ *iff* $M^\circ \to_{\beta_<} N^\circ$ *in* $\lambda_{\gtrless}$.
2. $M \to_{\mathtt{n}}^* N$ *in* $\lambda_{\mathtt{n}}$ *iff* $M^\circ \twoheadrightarrow_{\mathtt{b}} N^\circ$ *in* $\lambda_{\gtrless}$.
3. $M \Rightarrow_{\mathtt{n}} N$ *in* $\lambda_{\mathtt{n}}$ *iff* $M^\circ \Rightarrow_{\gtrless} N^\circ$ *in* $\lambda_{\gtrless}$.

As in Theorem 3, we have the same properties of preservation and reflection, at the levels of reduction, evaluation and standard reduction.

A perfectly symmetric treatment is now obtained for Gödel's mapping.

Theorem 9 (Properties Gödel's of translation from $\lambda_{\mathtt{v}}$ to $\lambda_{\gtrless}$).

1. $M \to_{\beta_{\mathtt{v}}} N$ *in* $\lambda_{\mathtt{v}}$ *iff* $M^* \to_{\beta_>} N^*$ *in* $\lambda_{\gtrless}$.
2. $M \to_{\mathtt{v}}^* N$ *in* $\lambda_{\mathtt{v}}$ *iff* $M^* \twoheadrightarrow_{\mathtt{b}} N^*$ *in* $\lambda_{\gtrless}$.
3. $M \Rightarrow_{\mathtt{v}} N$ *in* $\lambda_{\mathtt{v}}$ *iff* $M^* \Rightarrow_{\gtrless} N^*$ *in* $\lambda_{\mathtt{b}}$.

Theorems 8 and 9 show that, at the levels of reduction, evaluation and standard reduction, the respective concepts in $\lambda_{\mathtt{n}}$ and $\lambda_{\mathtt{v}}$ are unified by the corresponding concept in $\lambda_{\asymp}$; and the theorems allow, as in [2], to extract standardization for $\lambda_{\mathtt{n}}$ and $\lambda_{\mathtt{v}}$ as an immediate consequence, this time, of standardization in $\lambda_{\asymp}$ (and indirectly in $\lambda_{\mathtt{b}\not\mathtt{b}}$).

Theorem 9 should be compared with Theorem 5. A deeper analysis of the modal target and embeddings replaced the trick of using the separate rule $\beta_{\mathtt{b2}}$ with a more conceptual solution that comprehends both cbn and cbv. Indeed, the split of two β-rules happens in $\lambda_{\mathtt{b}\not\mathtt{b}}$ for purely logical reasons: the separation of terms that can and cannot have a modal type.

A final remark. How about w-e reduction? Recall that this relation is defined by going to Fig. 15 and replacing the *mu* and *nu* rules by the *app* rules. It turns out that, in Girard's target, w-e reduction and cbb evaluation coincide because the rule *app*$<$ is cut down to the rule *mu*$<$. But in Gödel's target, w-e reduction is larger than cbb evaluation, as the rule *app*$>$ survives entirely. So, at the last minute, we discover a new reduction relation in Plotkin's $\lambda_{\mathtt{v}}$-calculus, whose significance remains to be understood. Is this a last sign of the asymmetry between cbn and cbv? It is certainly a difference between the calling paradigms, revealed by a single modal treatment of the paradigms, not a difference between two modal treatments of the paradigms.

6. Final remarks

In this paper, we reworked the call-by-box *paradigm* of [2], presenting a calculus with full reduction and a notion of evaluation linked by a standardization theorem [4]; recast the modal embeddings with that target; improved the properties of Gödel's embedding; and obtained again a unification of call-by-name and call-by-value through call-by-box, but in a somewhat clearer way: call-by-name and call-by-value coexist inside call-by-box, each of the former is just a partial view of the latter, and the modal embeddings of Girard and Gödel are reduced to "modal glasses"—isomorphisms that allow us to recognize $\lambda_{\mathtt{n}}$ and $\lambda_{\mathtt{v}}$ inside $\lambda_{\asymp}$.

The progression from $\lambda_{\mathtt{b}}$ to $\lambda_{\asymp}$ through $\lambda_{\mathtt{b}\not\mathtt{b}}$ brings a progression of conceptual distinctions. In [2], which was based on λ_{b}, we stressed the distinction between values and boxes. System $\lambda_{\mathsf{b}\not{\mathsf{b}}}$ rests on the distinction between terms that can have a boxed type from those that cannot—a distinction generalizing the distinction between boxes and values, which is still of purely logical character. System $\lambda_{\asymp}$ achieves an alignment of the distinction between boxed and unboxed terms with the "mode" distinction between right-first and left-first, which in turn is intimately related to the alternative between call-by-value and call-by-name.

Our starting point was the two modal embeddings into S4, as given in [3], one due to Gödel and the other inspired by Girard's best known embedding of intuitionistic logic into linear logic [5]. Of course in the background we have the connection between the calling paradigms and the embeddings into linear logic [6] (see [2] for more references). Another calculus subsuming call-by-name and call-by-value λ-calculi is *call-by-push-value* [9], whose origin is in denotational semantics and whose basis is the conceptual distinction between values and computations. The reworking of call-by-push-value from denotational models of linear logic in [10] is the source of the bang calculus [11, 12], which is another subsuming system and a small fragment of the language proposed in [10]. In its simplest form, the bang calculus simply adds to the λ-calculus a pair of constructors ("box" and "dereliction") inspired by the introduction and elimination constructors of the !-modality of linear logic (in this, it compares with the very simple modal language $\lambda_{\Box}$ with which we started our inquiry in [2]). The result is intended as an intermediate formalism between the subsumed calculi and linear logic proof-nets, through which the translations of intuitionistic logic into linear logic can be factorized. But, for us, the connection with linear logic is only one among several possible "instantiations" of our results; see below. In particular, our plan is to explain the calling paradigms in the simplest modal setting, showing that already a modal operator, without the need for linearity, achieves a full explanation [2].

Initially [11], the bang calculus singled out a class of "values", with properties of discardability and duplicability—hence boxes were considered values. Later

[12, 13], such a class was no longer singled out. For us, the separate disjoint classes of values and boxes are crucial, and a symptom of the final distinction between left-first and right-first modes. Since bang calculus compares with the very simple modal language $\lambda_\Box$, its study does not compare well with our treatment, which starts properly with the refined modal target λ_{b}. In fact, it is specific to our work that we are preoccupied with: the design of the modal target, guided by the goal of expressing in the best way what the modal embeddings have to say; the conceptual distinctions which the successive versions of the target introduce; and the characterization of the unifying paradigm embodied in the target, in the style of [4]. On the other hand, the study of the bang calculus includes dimensions that we did not develop, like denotational semantics [11, 12], non-idempotent intersection types [13], or applications of the unification of the calling paradigms in the form of unified development of meta-theory [14].

As said, another component of our modal treatment of the calling paradigms initiated in [2] is the idea of an instantiation of the modality of the modal target, that is, an interpretation of the modal target into another λ-calculus that can be post-composed with the modal embeddings, achieving a decomposition of a known interpretation of the λ-calculus. For instance, the mappings of the λ-calculus into the linear λ-calculus [6] are decomposed into the modal embeddings and a linear instantiation. A similar project can be tried with the mappings into call-by-push-value. This instantiation received a brief account in [2] and we hope to have the opportunity to address it fully in the future.

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Appendix A. Some proofs

Lemma 2 **(Alternative characterization).** In $\lambda_{\mathtt{b}\not{\mathtt{b}}}$:

1. $T \to_{\mathtt{we}}^{*} T'$ iff $T \twoheadrightarrow_{\mathtt{we}} T'$.
2. $T \to_{\mathtt{b}}^{*} T'$ iff $T \twoheadrightarrow_{\mathtt{b}} T'$.

Proof. The proof of each of the two items follows the same pattern. Let us illustrate the proof of the second item.

The "if" direction goes by induction on the assumption $T \twoheadrightarrow_{\mathtt{b}} T'$. The cases where the assumption is obtained by rule rdx_1' or rdx_2' both make use of the opening rule. Let us inspect the case relative to rdx_2', which requires cooperation of rule $\beta_{\mathtt{b}\not{\mathtt{b}}}$. So, consider $T = QP$, $Q \twoheadrightarrow_{\mathtt{b}} \mathsf{box}(N)$, $P \twoheadrightarrow_{\mathtt{b}} \mathsf{box}(M)$, $M \twoheadrightarrow_{\mathtt{b}} \lambda x.P'$ and $[N/\varepsilon(x)]P' \twoheadrightarrow_{\mathtt{b}} T'$ (for some Q, P, N, M, P'). By I.H., $Q \to_{\mathtt{b}}^{*} \mathsf{box}(N)$, $P \to_{\mathtt{b}}^{*} \mathsf{box}(M)$, $M \to_{\mathtt{b}}^{*} \lambda x.P'$ and $[N/\varepsilon(x)]P' \to_{\mathtt{b}}^{*} T'$. Three auxiliary facts are needed: (i) $P_0 \to_{\mathtt{b}}^{*} P_1$ implies $Q_0P_0 \to_{\mathtt{b}}^{*} Q_0P_1$; (ii) $Q_0 \to_{\mathtt{b}}^{*} Q_1$ implies $Q_0\mathsf{box}(M_0) \to_{\mathtt{b}}^{*} Q_1\mathsf{box}(M_0)$; (iii) $M_0 \to_{\mathtt{b}}^{*} M_1$ implies $M_0@_{\mathtt{b}}Q_0 \to_{\mathtt{b}}^{*} M_1@_{\mathtt{b}}Q_0$ (each of them proved by induction on the assumption, using in the step case the closure rules μ', ν_{box}' and μ, respectively). Then,

$$\begin{array}{rl} & QP \to_{\mathtt{b}}^{*} Q\mathsf{box}(M) \to_{\mathtt{b}}^{*} \mathsf{box}(N)\mathsf{box}(M) \\ \to_{\mathtt{b}} & M@_{\mathtt{b}}\mathsf{box}(N) \to_{\mathtt{b}}^{*} \lambda x.P'@_{\mathtt{b}}\mathsf{box}(N) \to_{\mathtt{b}} [N/\varepsilon(x)]P' \to_{\mathtt{b}}^{*} T', \end{array}$$

which makes use of the opening rule in the third step, and of $\beta_{\mathtt{b}\not{\mathtt{b}}}$ in the penultimate step. So, by transitivity of $\to_{\mathtt{b}}^{*}$, $T = QP \to_{\mathtt{b}}^{*} T'$. The cases relative to rules var, abs, box are immediate by reflexivity of $\to_{\mathtt{b}}^{*}$ and the cases relative to the remaining rules are analogous to the case rdx_2', but simpler.

For the "only if" direction, one does induction on $T \to_{\mathtt{b}}^{*} T'$. The base case follows by reflexivity of $\twoheadrightarrow_{\mathtt{b}}$, an easy induction on terms, for which only the axioms and rules mu, mu' are needed. The step case follows by the lemma: if $T \to_{\mathtt{b}} T'' \twoheadrightarrow_{\mathtt{b}} T'$, then $T \twoheadrightarrow_{\mathtt{b}} T'$. This lemma goes by induction on $T \to_{\mathtt{b}} T''$.

The base case $\beta_{\mathtt{b}\not{\mathtt{b}}}$ follows by rdx and reflexivity of $\twoheadrightarrow_{\mathtt{b}}$.

The base case O makes use of the two rdx' rules. Let us see the details. So, $T = Q\mathsf{box}(M)$, $T'' = M@_{\mathtt{b}}Q$ (for some Q, M). Subcase $T'' \twoheadrightarrow_{\mathtt{b}} T'$ by rule rdx:

$M \twoheadrightarrow_{\mathtt{b}} \lambda x.T_0$, $Q \twoheadrightarrow_{\mathtt{b}} \mathsf{box}(N)$ and $[N/\varepsilon(x)]T_0 \twoheadrightarrow_{\mathtt{b}} T'$ (for some x, T_0, N); this and reflexivity of $\twoheadrightarrow_{\mathtt{b}}$ allows to apply rdx_2' to conclude $T = Q\mathsf{box}(M) \twoheadrightarrow_{\mathtt{b}} T'$. Subcase $T'' \twoheadrightarrow_{\mathtt{b}} T'$ by rule nu: $T' = (\lambda x.T_0)@_{\mathtt{b}}Q'$, $M \twoheadrightarrow_{\mathtt{b}} \lambda x.T_0$ and $Q \twoheadrightarrow_{\mathtt{b}} Q'$ (for some x, T_0, Q'); this and reflexivity of $\twoheadrightarrow_{\mathtt{b}}$ allows to apply rdx_1' to conclude $T = Q\mathsf{box}(M) \twoheadrightarrow_{\mathtt{b}} (\lambda x.T_0)@_{\mathtt{b}}Q' = T'$. The subcase $T'' \twoheadrightarrow_{\mathtt{b}} T'$ by rule mu is similar to the case before (also using rdx_1' and reflexivity of $\twoheadrightarrow_{\mathtt{b}}$).

Let us now inspect the inductive case ν_{box}'. So, $T = Q\mathsf{box}(M)$, $T'' = Q'\mathsf{box}(M)$ and $Q \to_{\mathtt{b}} Q'$ (for some Q, M, Q'). Subcase $T'' \twoheadrightarrow_{\mathtt{b}} T'$ by rule rdx_1': $T' = M'@_{\mathtt{b}}Q''$, $Q' \twoheadrightarrow_{\mathtt{b}} Q''$ and $M \twoheadrightarrow_{\mathtt{b}} M'$ (for some Q'', M'); by I.H. $Q \twoheadrightarrow_{\mathtt{b}} Q''$, and we can reapply rdx_1' to conclude $T' = Q\mathsf{box}(M) \twoheadrightarrow_{\mathtt{b}} M'@_{\mathtt{b}}Q'' = T'$. The subcases where $T'' \twoheadrightarrow_{\mathtt{b}} T'$ is obtained by rule rdx_2' or rule nu' are analogous to the previous subcase (a combination of the I.H. and reapplication of the respective rule). Subcase $T'' \twoheadrightarrow_{\mathtt{b}} T'$ by rule mu': $T'' = T'$ and, since $\twoheadrightarrow_{\mathtt{b}}$ is reflexive, the I.H. gives $Q \twoheadrightarrow_{\mathtt{b}} Q'$; this and reflexivity of $\twoheadrightarrow_{\mathtt{b}}$ allow to apply nu' to conclude $T = Q\mathsf{box}(M) \twoheadrightarrow_{\mathtt{b}} Q'\mathsf{box}(M) = T'$.

Lemma 3 (Determinism). In $\lambda_{\mathtt{b}\not\mathtt{b}}$:

1. There is at most one T' such that $T \to_{\mathtt{we}}^* T'$ and T' is a value or a box.
2. There is at most one T' such that $T \to_{\mathtt{b}}^* T'$ and T' is a value or a box.

Proof. The proofs of the two items are analogous. We give some details for the first item.

We profit from the alternative characterization of $\to_{\mathtt{we}}^*$ offered in Lemma 2, and prove instead: if $T \twoheadrightarrow_{\mathtt{we}} T'$ and T' is either a value or a box, then T' is unique. This fact follows by induction on $T \twoheadrightarrow_{\mathtt{we}} T'$. Let us illustrate the case where the last step is rdx_2', hence $T = QP$, $Q \twoheadrightarrow_{\mathtt{we}} \mathsf{box}(N)$, $P \twoheadrightarrow_{\mathtt{we}} \mathsf{box}(M)$, $M \twoheadrightarrow_{\mathtt{we}} \lambda x.P'$ and $[N/\varepsilon(x)]P' \twoheadrightarrow_{\mathtt{we}} T'$ (for some Q, P, N, M, P'). Let us argue that for any value or box T'' such that $T \twoheadrightarrow_{\mathtt{we}} T''$, $T' = T''$. Picking any such T'', since it is a value or box, it must be the case that also $T \twoheadrightarrow_{\mathtt{we}} T''$ was obtained by a rdx_2'-step, with premises $Q \twoheadrightarrow_{\mathtt{we}} \mathsf{box}(N')$, $P \twoheadrightarrow_{\mathtt{we}} \mathsf{box}(M')$, $M' \twoheadrightarrow_{\mathtt{we}} \lambda x.P''$ and $[N'/\varepsilon(x)]P'' \twoheadrightarrow_{\mathtt{we}} T''$, say. Immediately, by I.H., $N = N'$ and $M = M'$.

From the latter and I.H. also $P' = P''$, hence $[N/\varepsilon(x)]P' = [N'/\varepsilon(x)]P''$. Thus, by I.H., $T' = T''$.

Lemma 4 **(Conservativity - Part 2).** For all $T, T' \in \lambda_{\between}$: $T \twoheadrightarrow_{\mathtt{we}} T'$ in $\lambda_{\between}$ iff $T \rightarrow^*_{\mathtt{we}} T'$ in $\lambda_{\mathsf{\not{bb}}}$.

Proof. Recall the lemma needed to complete the proof of the "if" direction: if $T \twoheadrightarrow_{\mathtt{we}} T'$ in $\lambda_{\mathsf{\not{bb}}}$ and $T \in \lambda_{\between}$, then: a) if $T' \in \lambda_{\between}$, then $T \twoheadrightarrow_{\mathtt{we}} T'$ in $\lambda_{\between}$; and b) if T' is a box or a λ-abstraction, then $T' \in \lambda_{\between}$. This lemma follows by induction on $T \twoheadrightarrow_{\mathtt{we}} T'$ in $\lambda_{\mathsf{\not{bb}}}$. Note that the last step of this derivation cannot be rdx'_1 when $T' \in \lambda_{\between}$ or T' is a box or a λ-abstraction. Let us inspect the case where the last step to derive $T \twoheadrightarrow_{\mathtt{we}} T'$ is rdx'_2, hence $T = QP$ for some $Q, P \in \lambda_{\between}$, and the premises are: (i) $Q \twoheadrightarrow_{\mathtt{we}} \mathsf{box}(N)$, (ii) $P \twoheadrightarrow_{\mathtt{we}} \mathsf{box}(M)$, (iii) $M \twoheadrightarrow_{\mathtt{we}} \lambda x.P'$ and (iv) $[N/\varepsilon(x)]P' \twoheadrightarrow_{\mathtt{we}} T'$ (for some $N, M, P' \in \lambda_{\mathsf{\not{bb}}}$). The I.H. relative to (i) gives $\mathsf{box}(N) \in \lambda_{\between}$ and, subsequently, $Q \twoheadrightarrow_{\mathtt{we}} \mathsf{box}(N)$ in $\lambda_{\between}$. Likewise, the I.H. relative to (ii) gives $\mathsf{box}(M) \in \lambda_{\between}$ and $P \twoheadrightarrow_{\mathtt{we}} \mathsf{box}(M)$ in $\lambda_{\between}$. So, in particular, $M \in \lambda_{\mathsf{\not{bb}}}$, and the I.H. relative to (iii) gives $\lambda x.P' \in \lambda_{\between}$ and $M \twoheadrightarrow_{\mathtt{we}} \lambda x.P'$ in $\lambda_{\between}$. Since $P', N \in \lambda_{\between}$, also $[N/\varepsilon(x)]P' \in \lambda_{\between}$, and so we can use the I.H. relative to (iv) to conclude: a) $T \twoheadrightarrow_{\mathtt{we}} T'$ in $\lambda_{\between}$ when $T' \in \lambda_{\between}$; and b) $T' \in \lambda_{\between}$ when T' is a box or a λ-abstraction.