Rögnvaldur G. Möller

**Abstract.** Various connections between the theory of permutation groups and the theory of topological groups are described. These connections are applied in permutation group theory and in the structure theory of topological groups.

Mathematics Subject Classification (2000). Primary 22D05, 20B07, 05C25; Secondary 20E06, 20E08.

Keywords. Totally disconnected locally compact groups, permutation groups, graphs.

### Introduction

The aim of this paper is to discuss various links between permutation groups, graphs and topological groups. The action of a group on a set can be used to define a topology on the group, called the *permutation topology*. The earliest references for this topology are the paper [46] by Maurer and the paper [38] by Karrass and Solitar. This topology opens up the possibility of applying concepts and results from the theory of topological groups in permutation group theory. One can also go the other way and apply ideas from permutation group theory to problems about topological groups. In particular, some simple constructions of graphs, that are commonly used in permutation group theory, can be applied.

In the first section we discuss the languages we use, when working with graphs and permutation groups.

In Section 2 we look at the definition of the permutation topology and consider applications of the theory of topological groups to questions about permutation groups. The main result in this section is a theorem of Schlichting from [62]. This is a theorem about permutation groups, but Schlichting's proof uses notions from functional analysis and result of Iwasawa [37] about topological groups. Here we present a proof using concepts from permutation group theory and Iwasawa's Theorem.

In the third and fourth sections we discuss applications of techniques and ideas from permutation groups theory and graph theory to the theory of topological groups.

In Section 3 Willis' structure theory of totally disconnected locally compact groups is in the limelight. Willis' paper [76] helped spark a new interest in totally disconnected locally compact groups. Later work by Willis and others has shown that the concepts of the theory have many applications and are open to various interpretations. Most of the material in this section comes from the paper [50], where the basics of Willis' theory are given a graph theoretic interpretation. In the fourth and last section we discuss an analogue of a Cayley graph, called a rough Cayley graph, that one can construct for a compactly generated, totally disconnected, group. The rough Cayley graph is defined in Section 4.1. In that section, it is also shown that this graph is a quasi-isometry invariant of the group. In the latter parts of Section 4, it is shown how one uses rough Cayley graphs, by developing an analogue of the theory of ends of groups and the theory of groups with polynomial growth.

There are various other topics, that should be discussed in a survey like this. The study of random walks on groups and graphs is another place where graphs, permutations and topological groups meet. The book by Woess [84] is an excellent introduction to this field. Another meeting place for graphs, permutations and topological groups is the theory of groups acting on trees. In particular, one could mention the theory of harmonic analysis and representation theory of groups acting on trees, see the book by Figà-Talamanca and Nebbia [27] and the theory of tree lattices, see the book by Bass and Lubotzky [4]. Then there is the topic of generic elements and subgroups, see the papers [7, 8] by Bhattacharjee and the paper [2] by Abert and Glasner. And then I have not even mentioned the manifold appearances of our trio of graphs, permutations and topology in model theory. Describing all these topics would have meant a book length paper.

### 1. Languages for graphs and permutation groups

**1.1.** A language for graphs. We will discuss both *undirected graphs*, or just *graphs*, and *directed graphs*, called in this paper *digraphs*.

Our undirected graphs are without loops and multiple edges. Thus one can think of a (undirected) graph  $\Gamma$  as an ordered pair  $(V\Gamma, E\Gamma)$  where  $V\Gamma$  is a set and  $E\Gamma$  is a set of two element subsets of  $V\Gamma$ . The elements of  $V\Gamma$  are called *vertices* and the elements of  $E\Gamma$  are called *edges*.

Vertices  $\alpha$  and  $\beta$  in a graph  $\Gamma$  are said to be *neighbours*, or *adjacent*, if  $\{\alpha, \beta\}$  is an edge in  $\Gamma$ . The *valency* of a vertex is the number of its neighbours. A graph, all of whose vertices have finite valency, is called *locally finite*. For vertices  $\alpha$  and  $\beta$  of the graph, a *walk* of length n from  $\alpha$  to  $\beta$  is a sequence  $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$  of vertices, such that  $\alpha_i$  and  $\alpha_{i+1}$  are adjacent for  $i = 0, 1, \ldots, n - 1$ . A walk, all of whose vertices are distinct, is called a *path*. A *ray* in a graph is a sequence  $\alpha_0, \alpha_1, \ldots$  of distinct vertices such that  $\alpha_i$  is adjacent to  $\alpha_{i+1}$  for all *i*. A graph is connected if for any two vertices  $\alpha$  and  $\beta$  there is a walk from  $\alpha$  to  $\beta$ . Let  $d(\alpha, \beta)$  denote the length of a shortest walk from a vertex  $\alpha$  to a vertex  $\beta$ . For a connected graph, the function *d* is a metric on its set of vertices. Let *A* be a set of vertices of  $\Gamma$ . The *subgraph of*  $\Gamma$  *spanned by A* is the graph whose vertex set is *A*, and whose edge set is the set of all edges in  $\Gamma$  whose end vertices are both in *A*. We say that a set of vertices *A* is *connected*, if the subgraph spanned by *A* is connected. The *connected components* (or just *components*) of a graph are the maximal connected sets of vertices.

We define a digraph  $\Gamma$  to be a pair  $(V\Gamma, E\Gamma)$  where  $V\Gamma$  is a set and  $E\Gamma$  is a set of

ordered pairs of distinct elements of  $V\Gamma$ , i.e.  $E\Gamma \subseteq (V\Gamma \times V\Gamma) \setminus \{(\alpha, \alpha) \mid \alpha \in V\Gamma\}$ . An edge  $(\alpha, \beta)$  may be thought of as an "arrow" starting in  $\alpha$  and ending in  $\beta$ . The *in-valency* of a vertex  $\alpha$  in  $\Gamma$  is the number of edges of the type  $(\beta, \alpha)$  (number of edges going "in to"  $\alpha$ ) and the *out-valency* of a vertex  $\alpha$  is the number of edges  $(\alpha, \beta)$  that go "out of"  $\alpha$ . A digraph is said to be *locally finite* if every vertex has finite in- and out-valencies.

One can "forget" the directions of edges in  $\Gamma$  and define an undirected graph  $\overline{\Gamma}$  with the same vertex set and two vertices  $\alpha$  and  $\beta$  adjacent if and only if  $(\alpha, \beta)$  or  $(\beta, \alpha)$  is an edge in the digraph  $\Gamma$ .

A walk in a digraph  $\Gamma$  is a sequence  $\alpha_0, \ldots, \alpha_n$  such that  $(\alpha_i, \alpha_{i+1})$  or  $(\alpha_{i+1}, \alpha_i)$ is an edge in  $\Gamma$  for  $i = 0, \ldots, n-1$ . (That is to say,  $\alpha_0, \ldots, \alpha_n$  is a walk in the undirected graph  $\overline{\Gamma}$  associated to  $\Gamma$ .) An arc, more specifically an *n*-arc, is a sequence  $\alpha_0, \ldots, \alpha_n$  of distinct vertices such that  $(\alpha_i, \alpha_{i+1})$  is an edge in  $\Gamma$  for  $i = 0, \ldots, n-1$ . Arcs are sometimes referred to as directed paths. The set of descendants of a vertex  $\alpha$  is defined as the set

 $\operatorname{desc}(\alpha) = \{ \beta \in V\Gamma \mid \text{there exists an arc from } \alpha \text{ to } \beta \}.$ 

The set  $\operatorname{desc}_k(\alpha)$  is defined as the set of all vertices  $\beta$  such that the shortest arc from  $\alpha$  to  $\beta$  has length k. The set of *ancestors* of a vertex  $\alpha$ , denoted  $\operatorname{anc}(\alpha)$ , is defined as the set of all vertices  $\beta$  such that  $\alpha \in \operatorname{desc}(\beta)$ , i.e.  $\beta \in \operatorname{anc}(\alpha)$  if and only if there exists an arc from  $\beta$  to  $\alpha$ .

Finally, we review the definition of a *Cayley graph* of a group. Let G be a group and S a subset of G. The (undirected) Cayley graph Cay(G, S) of G with respect to S has G as the vertex set and  $\{g, h\}$  is an edge if h = gs or  $h = gs^{-1}$  for some s in S. The Cayley graph Cay(G, S) is connected if and only if S generates G. The left regular action of G on itself gives a transitive action of G as a group of graph automorphisms on Cay(G, S).

It is common to define the Cayley graph of a group G with respect to a subset S of G as the digraph with vertex set G and set of directed edges the collection of all ordered pairs (g, gs) for g in G and s in S. In these notes, however, it is convenient to think of Cayley graphs as being undirected.

**1.2.** A language for permutation groups. In this section G is a group acting on a set  $\Omega$ . The image of a point  $\alpha \in \Omega$  under an element  $g \in G$  will be written  $g\alpha$ . The group of all permutations of a set  $\Omega$  is called the *symmetric group on*  $\Omega$  and is denoted by  $Sym(\Omega)$ .

The action of G on  $\Omega$  induces a natural homomorphisms  $G \to \text{Sym}(\Omega)$ . If this homomorphism is injective, then the group G is said to act *faithfully* on  $\Omega$ . A group G which acts faithfully on a set  $\Omega$  can be regarded as a subgroup of  $\text{Sym}(\Omega)$ , and then we say that G is a *permutation group* on  $\Omega$ .

An action is said to be *transitive*, if for every two points  $\alpha, \beta \in \Omega$  there is some element  $g \in G$  such that  $g\alpha = \beta$ . For a point  $\alpha \in \Omega$ , the subgroup

$$G_{\alpha} = \{g \in G \mid g\alpha = \alpha\}$$

is called the *stabilizer* in G of the point  $\alpha$ . The *pointwise stabilizer*  $G_{(\Delta)}$  of a subset  $\Delta$  of  $\Omega$  is defined as the subgroup of all the elements in G that fix every element of  $\Delta$ , that is,

$$G_{(\Delta)} = \{g \in G \mid g\delta = \delta \text{ for every } \delta \in \Delta\} = \bigcap_{\delta \in \Delta} G_{\delta}$$

The setwise stabilizer  $G_{\{\Delta\}}$  of  $\Delta$  is defined as the subgroup consisting of all elements of G that leave  $\Delta$  invariant, that is,

$$G_{\{\Delta\}} = \{g \in G \mid g\Delta = \Delta\}.$$

The *G*-orbit, or, simpler orbit, of a point  $\alpha$  is the set  $\{g\alpha \mid g \in G\}$ .

Suppose U is a subgroup of G. The group G acts on the set G/U of right cosets of U, and this action is transitive. The image of a coset hU under an element  $g \in G$ is (gh)U. Conversely, if G acts transitively  $\Omega$  and  $\alpha$  is a point in  $\Omega$ , then  $\Omega$  can be identified with  $G/G_{\alpha}$ . Here "identified" means that there is a bijective map  $\theta : \Omega \to G/G_{\alpha}$  such that for every  $\omega \in \Omega$  and every element  $g \in G$  we have  $\theta(g\omega) = g\theta(\omega)$ .

The orbits of stabilizers of points in  $\Omega$  are called *suborbits*, that is, the suborbits are sets of the form  $G_{\alpha\beta}$  where  $\alpha, \beta \in \Omega$ . Orbits of G on the set of ordered pairs of elements from  $\Omega$  are called *orbitals*. When G is transitive on  $\Omega$  one can, for a fixed point  $\alpha \in \Omega$ , identify the suborbits of  $G_{\alpha}$  with the orbitals: the suborbit  $G_{\alpha\beta}\beta$  is identified with the orbital  $G(\alpha, \beta)$ . The number of elements in a suborbit  $G_{\alpha\beta}\beta$ , often called the *length* of the suborbit, is given by the index  $|G_{\alpha} : G_{\alpha} \cap G_{\beta}|$ . The number of elements in the orbit  $U\beta$  of a subgroup U is equal to the index  $|U: U \cap G_{\beta}|$ .

Next, we define a digraph, whose vertex set is  $\Omega$  and whose edge set is the union of *G*-orbitals, and call it the *directed orbital graph of the action*. The action of *G* on its vertex set  $\Omega$  induces an action of *G* as a group of automorphisms on the directed orbital graph  $\Gamma$ , because if  $(\alpha, \beta)$  is an edge in  $\Gamma$  then  $(g\alpha, g\beta)$  is in the same orbital as  $(\alpha, \beta)$  and therefore also an edge in  $\Gamma$ . Similarly, we define the *undirected orbital graph* of a *G*-action on a set  $\Omega$  as a graph whose vertex set is  $\Omega$  and whose edge set is the union of *G*-orbits on the set of two element subsets of  $\Omega$ .

A block of imprimitivity for G is a subset  $\Delta$  of  $\Omega$  such that for every  $g \in G$ , either  $g\Delta = \Delta$  or  $\Delta \cap (g\Delta) = \emptyset$ . The existence of a non-trivial proper block of imprimitivity  $\Delta$  (non-trivial means that  $|\Delta| > 1$  and proper means that  $\Delta \neq \Omega$ ) is equivalent to the existence of a non-trivial proper G-invariant equivalence relation  $\sim$  on  $\Omega$ . If there is no non-trivial proper G-invariant equivalence relation on  $\Omega$ we say that G acts primitively on  $\Omega$ . In most books on permutation groups it is shown that, if G acts transitively on  $\Omega$  then G acts primitively on  $\Omega$  if and only if  $G_{\alpha}$  is a maximal subgroup of G for every  $\alpha \in \Omega$ . Part of the proof of this fact is to show that if  $G_{\alpha} < H < G$  then  $H\alpha$  is a non-trivial proper block of imprimitivity. A further useful fact is that if N is a normal subgroup of G, then the orbits of N on  $\Omega$  are blocks of imprimitivity for G.

Recent books covering this material are [9], [12] and [18].

### 2. The permutation topology

**2.1. Definition of the permutation topology.** Let G be a group acting on a set  $\Omega$ . The action can be used to introduce a topology on G. The topology of a topological group is completely determined by a neighbourhood basis of the identity element. The *permutation topology* on G is defined by choosing as a neighbourhood basis of the identity the family of pointwise stabilizers of finite subsets of  $\Omega$ , i.e. a neighbourhood basis of the identity is given by the family of subgroups

 $\{G_{(\Phi)} \mid \Phi \text{ is a finite subset of } \Omega\}.$ 

A sequence  $(g_i)$  of elements in G has an element  $g \in G$  as a limit if and only if for every  $\alpha \in \Omega$  there is a number N (depending on  $\alpha$ ) such that  $g_n \alpha = g\alpha$  for every  $n \geq N$ . There are other ways to define the permutation topology. Think of  $\Omega$  as having the discrete topology and elements of G as maps  $\Omega \to \Omega$ . Then the permutation topology is equal to the topology of pointwise convergence, and it is also the same as the compact-open topology.

The basic idea is that two permutations g and h are "close" to each other if they agree on "many" points. If the set  $\Omega$  is countable, then the permutation topology can be defined by a metric, here is one way to do that.

Enumerate the points in  $\Omega$  as  $\alpha_1, \alpha_2, \ldots$ . Take two elements  $g, h \in G$ . Let n be the smallest number such that  $g\alpha_n \neq h\alpha_n$  or  $g^{-1}\alpha_n \neq h^{-1}\alpha_n$ . Set  $d(g,h) = 1/2^n$ . Then d is a metric on G that induces the permutation topology.

From the definition of the permutation topology we can immediately characterize open subgroups in G:

A subgroup of G is open if and only if it contains the pointwise stabilizer of some finite set of points.

Various properties of the action of G on  $\Omega$  are reflected by properties of this topology on G.

The permutation topology on G is Hausdorff if and only if the action of G on  $\Omega$  is faithful. Moreover, G is totally disconnected if and only if the action is faithful.

*Remark.* In general we do not assume that our topological groups are Hausdorff, but note that totally disconnected groups are always Hausdorff.

We say that a group G acting on  $\Omega$  is a *closed* if it is image in Sym( $\Omega$ ) is a closed subgroup, where Sym( $\Omega$ ) has the permutation topology. Closed permutation groups can be characterized in the following way, see [11, Section 2.4].

**Proposition 2.1.** A permutation group G on a set  $\Omega$  is closed if and only if G is the full automorphism group of some first order structure on  $\Omega$ .

A first order structure on  $\Omega$  is a collection of:

- constants that belong to  $\Omega$ ,
- functions defined on  $\Omega$  and taking their values in  $\Omega$ ,

• relations defined on  $\Omega$ .

It is easy to show that the automorphism group of such a structure is closed. To prove the converse, one uses the concept of the *canonical relational structure* on  $\Omega$  such that for  $n = 1, 2, \ldots$  we get one *n*-ary relation for each orbit of *G* on *n*-tuples of  $\Omega$ . If *G* is a closed permutation group on  $\Omega$ , then *G* is the full automorphism group of this structure.

For those actions that are not faithful, we say that G is closed in the permutation topology if the image of G in  $Sym(\Omega)$  under the natural homomorphism is closed. This condition is equivalent to the condition that stabilizers of points are closed subgroups of G.

Compactness (note that in this work the term *compact* does not include the Hausdorff condition) has a natural interpretation in the permutation topology. A subset of a topological space is said to be *relatively compact* if it has compact closure. The following lemma slightly generalizes a result by Woess for automorphism groups of locally finite connected graphs, and the proof is the same as Woess' proof.

**Lemma 2.2.** ([83, Lemma 1 and Lemma 2]) Let G be a group acting transitively on a set  $\Omega$  and endow G with the permutation topology. Assume that G is closed in the permutation topology and that all suborbits are finite.

(i) The stabilizer  $G_{\alpha}$  of a point  $\alpha \in \Omega$  is compact.

(ii) A subset A of G is relatively compact in G if and only if the set  $A\alpha$  is finite for every  $\alpha$  in  $\Omega$ .

Furthermore, if A is a subset of G and  $A\alpha$  is finite for some  $\alpha \in \Omega$  then  $A\alpha$  is finite for every  $\alpha$  in  $\Omega$ .

Proof. (i) Let K denote the kernel of the action of G on  $\Omega$ . Every open neighbourhood of the identity will contain K and thus K is compact. Because  $G_{\alpha}$  is closed we see that  $K \cap G_{\alpha}$  is compact. In order to show that  $G_{\alpha}$  is compact it is thus enough to show that  $G_{\alpha}/(K \cap G_{\alpha})$  is compact. The group  $G_{\alpha}/(K \cap G_{\alpha})$  acts faithfully on  $\Omega$  and the quotient topology on  $G_{\alpha}/(K \cap G_{\alpha})$  is the same as the permutation topology induced by the action on  $\Omega$ . We may thus assume that  $G_{\alpha}$  acts faithfully on  $\Omega$ .

Let  $(\Omega_i)_{i \in I}$  denote the family of  $G_{\alpha}$ -orbits on  $\Omega$ . Let  $H_i$  denote the permutation group induced by  $G_{\alpha}$  on  $\Omega_i$ . Since, each  $\Omega_i$  is finite, then each group  $H_i$  is finite and discrete in permutation topology. Set  $H = \prod_{i \in I} H_i$ , the full Cartesian product. Note that  $\Omega$  is the disjoint union of the  $\Omega_i$ 's and that H has a natural action on  $\Omega$ . The permutation topology on H induced by the action on  $\Omega$  is the same as the product topology. Thinking of  $G_{\alpha}$  and H as subgroups of  $\text{Sym}(\Omega)$  we see that  $G_{\alpha}$ is a closed subgroup of the compact subgroup H and is thus compact.

(ii) Suppose that  $A^-$ , the closure of A in G, is compact. Then for a point  $\alpha \in \Omega$  there is a finite open covering of  $A^-$  by sets of the type  $gG_{\alpha}$ ; that is to say, we can find  $g_1, \ldots, g_n \in G$  such that  $A^- \subseteq \bigcup_{i=1}^n g_i G_{\alpha}$ . Then  $A\alpha \subseteq \{g_1\alpha, \ldots, g_n\alpha\}$ .

Conversely, suppose that  $A\alpha = \{\alpha_1, \ldots, \alpha_n\}$ . Let  $g_i$  be an element in A such that  $g_i\alpha = \alpha_i$ . Then  $A \subseteq \bigcup_{i=1}^n g_i G_\alpha$ . The latter set is compact, so the closure of A is compact.

Lemma 2.2 implies that if G is a closed transitive permutation group on a countable set  $\Omega$  such that all suborbits are finite then G, with the permutation topology, is a locally compact, totally disconnected group. In particular, the automorphism group of a locally finite, transitive graph is a locally compact, totally disconnected group.

A subgroup H in a topological group G is said to be cocompact if G/H is a compact space. This concept has also a natural interpretation in terms of the permutation topology.

**Lemma 2.3.** ([55, Proposition 1], cf. [50, Lemma 7.5]) Let G be a group acting transitively on a set  $\Omega$ . Assume that G is closed in the permutation topology and all suborbits are finite. Then a subgroup H of G is cocompact if and only if H has finitely many orbits on  $\Omega$ .

*Proof.* Suppose first that H is cocompact. This means that both the spaces of right and left cosets of H in G are compact. Let X denote the set of right cosets of H in G. The quotient map  $\pi : G \to X$  is open. The family of cosets  $\{gG_{\alpha}\}_{g\in G}$  is an open covering of G and hence  $\{\pi(gG_{\alpha})\}_{g\in G}$  is an open covering of X. Since X is compact, there is a finite subcovering  $\pi(g_1G_{\alpha}), \ldots, \pi(g_nG_{\alpha})$  of X. Then  $G = Hg_1G_{\alpha} \cup \ldots \cup Hg_nG_{\alpha}$  and therefore  $\Omega = H(g_1\alpha) \cup \ldots \cup H(g_n\alpha)$ .

Conversely, suppose that H has only finitely many orbits on  $\Omega$ , say there are elements  $g_1, \ldots, g_n$  such that  $\Omega = H(g_1\alpha) \cup \ldots \cup H(g_n\alpha)$ . Then  $G = Hg_1G_\alpha \cup \ldots \cup Hg_nG_\alpha$  and  $X = \pi(g_1G_\alpha) \cup \ldots \cup \pi(g_nG_\alpha)$ . Each of the sets  $\pi(g_1G_\alpha)$  is compact. Hence X, the set of right cosets of H in G, is compact, because it is a union of finitely many compact sets.

Ideas from permutation group theory and the permutation topology can be applied to the study of a topological group G. For an open subgroup U we set  $\Omega = G/U$ , the space of left cosets. The stabilizers in G of points in  $\Omega$  are conjugates of U, and thus also open subgroups of G. The stabilizer of a finite set  $\Phi = \{\alpha_1, \ldots, \alpha_n\}$  of points is just the intersection of the open subgroups  $G_{\alpha_1}, \ldots, G_{\alpha_n}$  and is thus open in G. From this we see that the permutation topology coming from the action of G on  $\Omega$  is contained in the topology on G. If the topological group G is assumed to be a totally disconnected and locally compact, then we can choose U to be a compact open subgroup of G (by a theorem of van Dantzig [14]). In particular, the stabilizers in G of points in  $\Omega$  are all compact (they are conjugates of U). The second part of Lemma 2.2 above also holds for the action of G on  $\Omega$ , so a subset A of G has compact closure if and only if  $A\alpha$  is finite for every  $\alpha \in \Omega$ . This implies that  $|G_{\alpha}\beta| < \infty$  for all points  $\alpha$  and  $\beta$  in  $\Omega$ . This is because  $|G_{\alpha}\beta| = |G_{\alpha}: G_{\alpha} \cap G_{\beta}|$  and this index is finite because  $G_{\alpha} \cap G_{\beta}$  is an open subgroup of the compact group  $G_{\alpha}$ .

**2.2.** Suborbits and the modular function. In this section the general assumption will be that G is a closed permutation group acting on a set  $\Omega$  and that all suborbits of the group G are finite.

Under the above assumptions the group G is a locally compact, totally disconnected group. In this section we want to interpret the modular function on G in terms of the action of G on  $\Omega$ . The connection between suborbits and the modular function can be seen from the following argument due to Schlichting [61], see also [70].

Let  $\mu$  be a right Haar-measure on G. Define the modular function  $\Delta$  so that if A is a measurable set then  $\mu(gA) = \Delta(g)\mu(A)$ .

**Lemma 2.4.** ([61, Lemma 1], cf. [70, Theorem 1])) Let G be a closed, transitive permutation group on a set  $\Omega$ . Assume furthermore that all suborbits of G are finite. Let  $\Delta$  denote the modular function on G. If h is an element in G with  $h\alpha = \beta$  then

$$\Delta(h) = \frac{|G_{\beta}\alpha|}{|G_{\alpha}\beta|} = \frac{|G_{\alpha}: G_{\alpha} \cap h^{-1}G_{\alpha}h|}{|G_{\alpha}: G_{\alpha} \cap hG_{\alpha}h^{-1}|}.$$

*Proof.* Then, with  $\mu$  denoting the right Haar-measure on G,

$$|G_{\beta}\alpha| = |G_{\beta}: G_{\alpha} \cap G_{\beta}|$$
  
=  $\mu(G_{\beta})/\mu(G_{\alpha} \cap G_{\beta})$   
=  $\mu(hG_{\alpha}h^{-1})/\mu(G_{\alpha} \cap G_{\beta})$   
=  $\Delta(h)\mu(G_{\alpha})/\mu(G_{\alpha} \cap G_{\beta})$   
=  $\Delta(h) |G_{\alpha}: G_{\alpha} \cap G_{\beta}|$   
=  $\Delta(h) |G_{\alpha}\beta|$ .

And we see that  $\Delta(h) = |G_{\beta}\alpha|/|G_{\alpha}\beta|$ .

*Remark.* Let G be a locally compact, Hausdorff group with modular function  $\Delta$ . Assume G acts transitvely on a set  $\Omega$  such that the stabilizers of points are compact open subgroups of G. The calculation in the proof of Lemma 2.4 is also valid in this case and thus the conclusion in Lemma 2.4 holds also.

For an orbital  $A = G(\alpha, \beta)$  we define the *paired orbital* as the orbital  $A^* = G(\beta, \alpha)$ . This pairing of orbitals also gives us a pairing of suborbits, where the suborbit  $G_{\alpha}\beta$  is paired to a suborbit  $G_{\alpha}\gamma$  where  $\gamma$  is a point in  $\Omega$  such that  $(\alpha, \gamma)$  is in  $G(\beta, \alpha)$ . Denote by  $\Gamma$  the directed orbital graph for the orbital  $G(\alpha, \beta)$ . The size of the suborbit  $G_{\alpha}\beta$  is the out-valency of  $\Gamma$  and the size of the paired suborbit  $G_{\alpha}\gamma$  is the in-valency of  $\Gamma$ . Using Lemma 2.4, we obtain the following proposition.

**Proposition 2.5.** ([70, Theorem 1]) Let G be a closed, transitive permutation group on a set  $\Omega$ . Assume that all suborbits of G are finite. Then the lengths of paired suborbits are always equal if and only if G is unimodular.

Using that the modular function is a homomorphism, we obtain the following.

**Corollary 2.6.** Let G be a closed, transitive permutation group on a set  $\Omega$ . Let  $K = \ker(\Delta)$ . Then

(a) If α ∈ Ω then Gα ≤ K.
(b) Let G' be the derived group of G. Then G' ≤ K.
(c) If g is an torsion element of G then g ∈ K.

*Proof.* Follows directly from the definitions and the fact that G/K is isomorphic to a multiplicative subgroup of the positive real numbers.

**Corollary 2.7.** Let G be a closed, transitive permutation group on a set  $\Omega$ . Assume that all suborbits of G are finite. If G is also primitive, then paired suborbits have equal length.

*Proof.* Let  $K = \ker(\Delta)$ . Then, because G is primitive and K is normal in G, we know that either  $K = \{e\}$  or K is transitive. If  $K = \{e\}$  then G is abelian and must in fact be trivial. If K is transitive then the result of Lemma 2.4 implies that lengths of paired suborbits are equal.

For a long time it was an open question, posed by Peter M. Neumann, whether one could have a group acting primitively on a set  $\Omega$  with a finite suborbit paired to an infinite one. Such examples were constructed by David M. Evans in [26].

Consider a connected graph  $\Gamma$  and assume that G is a closed subgroup of Aut $(\Gamma)$  that acts transitively on  $\Gamma$  and that all suborbits are finite. For convenience we think of each undirected edge  $\{\alpha, \beta\}$  in  $\Gamma$  as consisting of two directed edges  $(\alpha, \beta)$  and  $(\beta, \alpha)$ . Each directed edge  $e = (\alpha, \beta)$  will be labeled by a number

$$\Delta_e = \frac{|G_\beta \alpha|}{|G_\alpha \beta|}.$$

Observe that  $\Delta_{(\alpha,\beta)} = \Delta_{(\beta,\alpha)}^{-1}$ . Furthermore, note that if g is an element of G such that  $g\alpha = \beta$  and  $e = (\alpha, \beta)$  then  $\Delta(g) = \Delta_e$ . Suppose g is an element of G and  $g\alpha = \gamma$  and that there is a vertex  $\beta$  in X such that  $(\alpha, \beta)$  and  $(\beta, \gamma)$  are edges in X. Find elements  $g_1$  and  $g_2$  in G such that  $g_1\alpha = \beta$  and  $g_2\beta = \gamma$ . Then  $g\alpha = g_2g_1\beta$  and from the formula above for the modular function we can deduce that  $\Delta(g) = \Delta(g_2g_1)$ . We also find that

$$\Delta(g) = \Delta(g_2g_1) = \Delta(g_2)\Delta(g_1) = \Delta_{(\beta,\gamma)}\Delta_{(\alpha,\beta)}.$$

This can be extended to directed walks of arbitrary length, so that if  $g\alpha = \beta$  then we take a directed walk from  $\alpha$  to  $\beta$ , enumerate the edges in the walk as  $e_1, \ldots, e_k$ and then

$$\Delta(g) = \Delta_{e_1} \cdots \Delta_{e_k}.$$

Hence the labeled graph completely describes the modular function on G. This idea can be found in the paper [3] by Bass and Kulkarni.

Suppose now not only that all the suborbits of G are finite but also that there is a finite upper bound m on their length. In that case, Lemma 2.4 implies that the image of the modular function is a bounded set. The image of the modular function is a bounded subgroup of the multiplicative group of positive real numbers and is thus the trivial subgroup. Hence G is unimodular. We have deduced the following unpublished result of Praeger.

**Corollary 2.8.** Let G be a closed, transitive permutation group on a set  $\Omega$  such that all suborbits of G are finite. If there is a finite bound on the length of suborbits, then paired suborbits have equal length.

The next result, also due to Praeger [59], uses the modular function to infer information about graph structure. The *directed integer graph*  $\mathbf{Z}$  has the set of integers as a vertex set and the edge set is the set of all ordered pairs (n, n + 1).

**Theorem 2.9.** ([59]) Let  $\Gamma$  be an infinite, connected, vertex and edge transitive directed graph with finite but unequal in- and out-valence. Then there is a graph epimorphism  $\varphi$  from  $\Gamma$  to the directed integer graph  $\mathbf{Z}$ . For each  $i \in \mathbf{Z}$ , the inverse image  $\varphi^{-1}(i)$  is infinite.

Proof. Write  $G = \operatorname{Aut}(\Gamma)$ . Let  $q = d^-/d^+$  where  $d^+$  is the out-valence of  $\Gamma$  and  $d^$ is the in-valence of  $\Gamma$ . Consider an edge  $(\alpha, \beta)$  in  $\Gamma$ . Because G acts transitively on the edges of  $\Gamma$  we can conclude that  $|G_{\alpha}\beta| = d^+$  and  $|G_{\beta}\alpha| = d^-$  and hence if  $g\alpha = \beta$  then  $\Delta(g) = d^-/d^+$ . Using the graph  $\Gamma$  to calculate the modular function in a similar way as described above we conclude that for every element  $g \in G$  there is an integer i such that  $\Delta(g) = q^i$ . Hence, if K denotes the kernel of the modular function then  $G/K = \mathbb{Z}$ . Fix a vertex  $\alpha_0$  in  $\Gamma$  and define a map  $\varphi : V\Gamma \to \mathbb{Z}$  so that  $\varphi(\beta) = i$  if there is an element g in G such that  $g\alpha_0 = \beta$  and  $\Delta(g) = q^i$ . It is clear that the choice of g is immaterial. From the way one uses  $\Gamma$  to calculate the modular function we see that if  $(\alpha, \beta)$  is an edge in  $\Gamma$  then  $\varphi(\beta) = \varphi(\alpha) + 1$  which implies that  $\varphi$  is a homomorphism from  $\Gamma$  to the directed integer graph.

Assume now, seeking contradiction, that  $\varphi^{-1}(i)$  is finite for some *i*. Note that the fibers of  $\varphi$  are just the orbits of the kernel of the modular homomorphism and are thus blocks of imprimitivity for *G*. Hence, all the fibers of  $\varphi$  have the same cardinality, say *k*. The number of edges going out of  $\varphi^{-1}(0)$  is  $d^+k$  and the number of edges going into  $\varphi^{-1}(1)$  is  $d^-k$ . But, both these numbers should be equal to the number of edges going from  $\varphi^{-1}(0)$  to  $\varphi^{-1}(1)$  and because we are assuming that  $d^- \neq d^+$  we have a contradiction.

(A similar proof of Praegers result is in a paper by Evans [25].)

Remark. In the next section highly arc transitive digraphs are discussed. A digraph  $\Gamma$  satisfying the conditions in Theorem 2.9 need not be highly arc transitive, but it is easy to show that if  $d^+$  and  $d^-$  are coprime, then  $\Gamma$  must be highly arc transitive.

The next result we discuss, is a remarkable theorem of Schlichting [62].

**Theorem 2.10.** ([62]) Let G be a group acting transitively on a set  $\Omega$ . Then there is a finite bound on the sizes of suborbits of G if and only if there is a G-invariant equivalence relation  $\sim$  on  $\Omega$  with finite classes, such that the stabilizers of points in the action of G on  $\Omega/\sim$  are finite.

While this is a theorem about permutation groups, Schlichting's proof utilizes various concepts from functional analysis and a theorem of Iwasawa [37, Theorem 1]. Later Theorem 2.10 was rediscovered by Bergman and Lenstra [10], who gave a group theoretical/combinatorial proof.

The theorem of Iwasawa that Schlichting uses in his proof of Theorem 2.10 is about the relationship between the classes [IN] and [SIN] of topological groups.

**Definition 2.11.** A locally compact group is said to be in the class [IN] if there is a compact neighbourhood K of the identity (i.e. K contains an open set containing the identity) that is invariant under conjugation by elements in G.

A locally compact group is said to be in the class [SIN] if every neighbourhood of the identity contains a compact neighbourhood K of the identity that is invariant under conjugation by elements in G.

Let us start by relating these two properties to the permutation topology.

**Proposition 2.12.** Let G be a transitive permutation group on a set  $\Omega$  and assume that all suborbits are finite. Then G with the permutation topology is in the class [IN] if and only if there is a finite bound on the sizes of suborbits.

Proof. First assume that G is in the class [IN]. Suppose  $\alpha, \beta \in \Omega$ . We want to find a constant upper bound, independent of  $\alpha$  and  $\beta$ , for the size of the suborbit  $G_{\alpha}\beta$ . Let K be a compact neighbourhood of the identity that is invariant under conjugation. Since K contains an open neighbourhood of the identity we can find a finite set  $\Phi$  such that  $G_{(\Phi)} \subseteq K$ . Choose a point  $\gamma$  in  $\Omega$ . Because K is compact,  $|K\gamma| = m < \infty$ . Let k be an upper bound for the indices  $|G_{\delta} : G_{(\Phi)}| \leq k$  with  $\delta \in \Phi$ . Find an element  $f \in G$  such that  $f\beta = \gamma$  and set  $\alpha' = f\alpha$ . Whence  $|G_{\alpha}\beta| = |G_{\alpha'}\gamma|$ . Then we find an element h such that  $\alpha' \in h\Phi$ . Note that  $G_{(h\Phi)} = hG_{(\Phi)}h^{-1}$ . Since K is invariant under conjugation and  $G_{(\Phi)} \subseteq K$ , we conclude that  $G_{(h\Phi)} \subseteq K$ . Therefore  $|G_{(h\Phi)}\alpha| \leq |K\alpha| = m$ . We also know that  $|G_{\alpha'} : G_{(h\Phi)}| \leq k$  and thus  $|G_{\alpha}\beta| = |G_{\alpha'}\gamma| \leq k|G_{(h\Phi)}\gamma| \leq km$ . Hence km is an upper bound for the size of suborbits of G.

Conversely, assume that there is an upper bound m on the sizes of suborbits. For a finite subset  $\Phi$  of  $\Omega$  we let  $m(\Phi)$  denote the size of the largest orbit of  $G_{(\Phi)}$ . We choose a finite subset  $\Phi$  such that  $m_0 = m(\Phi)$  is as small as possible. Define  $A = \bigcup_{g \in G} gG_{(\Phi)}g^{-1}$ . Clearly, the set A is open and invariant under conjugation by elements of G. We claim that A is relatively compact. By Lemma 2.2(ii), we only need to show that  $A\alpha$  is finite for some  $\alpha \in \Omega$ . Choose  $\alpha \in \Omega$  such that  $|G_{(\Phi)}\alpha| = m(\Phi)$ . For this  $\alpha$  we will show that  $|A\alpha| \leq m$ . Arguing by contradiction, suppose  $|A\alpha| \geq n > m(\Phi)$ . Take  $f_1, \ldots, f_n \in A$  such that the elements  $f_1\alpha, \ldots, f_n\alpha$  are all distinct. Since  $f_1, \ldots, f_n \in A = \bigcup_{g \in G} gG_{(\Phi)}g^{-1}$ , we can find elements  $g_1, \ldots, g_n \in G$  such that  $f_i \in g_iG_{(\Phi)}g_i^{-1} = G_{(g_i\Phi)}$ . Write  $\Phi_i = g_i \Phi$  and set  $E = \{f_1 \alpha, \ldots, f_n \alpha\} \cup \Phi_1 \cup \cdots \cup \Phi_n$ . Then  $m(E) = m_0$ . Let  $\Delta$  denote a  $G_{(E)}$ -orbit of size  $m_0$ . Note that  $\Delta$  is also a  $G_{(\Phi_i)}$ -orbit. Choose an element  $\delta \in \Delta$ . There is for each  $i = 1, \ldots, n$  an element  $h_i \in G_{(E)}$  such that  $h_i \delta = f_i \delta$  and therefore  $h_i^{-1} f_i \in G_{\delta}$ . But  $h_i^{-1} f_i \alpha = f_i \alpha$  and we can conclude that  $|G_{\delta}\alpha| \geq n > m$  contrary to assumptions. (The above argument is related to the proof of Theorem 2.10 by Bergmann and Lenstra [10], but this version is from a lecture course given by Peter M. Neumann in Oxford 1988–1989.)

**Proposition 2.13.** Let G be a transitive permutation group on a set  $\Omega$  and assume that all suborbits are finite. Then G with the permutation topology is in the class [SIN] if and only G is discrete (i.e. the stabilizers of points are finite).

*Proof.* It is obvious that if G is discrete then G is in [SIN].

Let us now assume that G is in [SIN]. Then, if  $\alpha$  denotes a point in  $\Omega$ , the open subgroup  $G_{\alpha}$  contains a compact neighbourhood K of the identity that is invariant under conjugation. Then for  $g \in G$  we see that  $K = gKg^{-1} \subseteq gG_{\alpha}g^{-1} = G_{g\alpha}$ and thus  $K \subseteq \bigcap_{g \in G} gG_{\alpha}g^{-1} = \bigcap_{g \in G} G_{g\alpha}$ . Because G is assumed to be transitive we conclude that K fixes every point of  $\Omega$ . But we are also assuming that G acts faithfully on  $\Omega$  so  $K = \{e\}$  and G is discrete.

The theorem of Iwasawa mentioned above says that if a locally compact group is in the class [IN] then there is a compact normal subgroup N such that G/N is in the class [SIN].

Proof. (Theorem 2.10) Assume that there is a finite upper bound on the sizes of suborbits of G. By Proposition 2.12 the group G is in the class [IN]. Iwasawa's Theorem gives us a compact, normal subgroup N of G such that G/N is in [SIN]. The orbits of the normal subgroup N are the classes of a G-invariant equivalence relation  $\sim$  on  $\Omega$  and because N is compact these classes are all finite. The group H = G/N certainly is in [SIN], but it is not certain that H acts faithfully on  $\Omega' = \Omega/ \sim$  so we can not apply Proposition 2.13 directly. Note that if  $\alpha \in \Omega'$  then  $H_{\alpha}$  is an open subgroup of H. Since H is in [SIN] there is a compact invariant neighbourhood neighbourhood K contained in  $H_{\alpha}$ . As in the proof of Proposition 2.13 we conclude that K is contained in the kernel of the action of H on  $\Omega'$ . Thus the kernel N' of the action of H on  $\Omega'$  is an open subgroup of H. The group H/N' can be regarded as a permutation group on  $\Omega'$ . From this we conclude that the permutation topology on H/N' is discrete, which implies that the stabilizer in H/N' of a point  $\alpha \in \Omega'$  must be finite.

The proof of the other direction is left to the reader.

Schlichting's Theorem implies the following general result about totally disconnected, locally compact groups.

**Corollary 2.14.** Let G be a totally disconnected locally compact group.

(i) The group G has a compact open normal subgroup if and only if there is a compact open subgroup U and a number  $m_U$  such that  $|U: U \cap gUg^{-1}| \leq m_U$  for all  $g \in G$ .

(ii) If there is such a number  $m_U$  for one compact open subgroup U then there is a number  $m_V$  for any compact open subgroup V such that  $|V: V \cap gVg^{-1}| \le m_V$ for all  $g \in G$ .

*Proof.* (i) If G contains a compact open normal subgroup N, then we can take U = N and  $m_N = 1$ .

Conversely, assume that U is a compact open subgroup and there is a number  $m_U$  such that  $|U: U \cap gUg^{-1}| \leq m_U$  for all  $g \in G$ . Put  $\Omega = G/U$ . Take a point  $\alpha$  in  $\Omega$  such that  $U = G_{\alpha}$ . Note that if  $g \in G$  and  $\beta = g\alpha$  then  $|G_{\alpha}\beta| = |G_{\alpha}: G_{\alpha} \cap G_{\beta}| = |U: U \cap gUg^{-1}| \leq m_U$ . Thus there is a finite upper bound on the sizes of suborbits and we can apply Schlichting's Theorem, which provides us with a G-invariant equivalence relation  $\sim$  on  $\Omega$  with finite classes such that the stabilizer of a  $\sim$ -class acts like a finite group on  $\Omega' = \Omega/\sim$ . This in turn implies that N, the kernel of the action of G on  $\Omega'$ , is a open normal subgroup of G. If  $\alpha \in \Omega$  then  $N\alpha$  is contained in the  $\sim$ -class of  $\alpha$  and thus  $N\alpha$  is finite. Therefore N is also compact.

(ii) From the proof of statement (i) we get the existence of a compact open normal subgroup N. Consider the action of G on the set  $\Omega = G/V$ . The orbits of N on  $\Omega$  are all finite and give us the equivalence classes of a G-invariant equivalence relation  $\sim$ . Let k be the number of element in a  $\sim$ -class. The group G acts on  $\Omega' = \Omega / \sim$  and the stabilizers of points in  $\Omega'$  act like finite groups on  $\Omega'$ . Thus the sizes of suborbits in the action of G on  $\Omega'$  are bounded above by some number l. Let  $\tilde{\alpha}$  denote the  $\sim$ -class of an element  $\alpha \in \Omega$ . For elements  $\alpha$  and  $\beta$  in  $\Omega$  we see that  $|G_{\alpha}\beta| \leq |G_{\tilde{\alpha}}\beta| \leq k|G_{\tilde{\alpha}}\tilde{\beta}| = kl$ . From this it follows that  $|V: V \cap gVg^{-1}| \leq kl$ .  $\square$ 

**2.3. The theorems of Trofimov.** As already mentioned, the automorphism group of a locally finite, connected graph with the permutation topology is locally compact and totally disconnected. In this section we will discuss three theorems of Trofimov (see [68], [69], [71]). The conclusions in them all resemble the conclusion in Schlichting's Theorem, but none of the theorems is proved by referring to Schlichting's Theorem. We start with the earliest of these three theorems. First, we explain the terminology used.

An automorphism g of a connected graph  $\Gamma$  is said to be *bounded* if there is a constant c such that  $d_{\Gamma}(\alpha, g\alpha) \leq c$  for all vertices  $\alpha$  in  $\Gamma$ .

**Theorem 2.15.** ([68]) Let  $\Gamma$  be a locally finite, transitive graph and  $B(\Gamma)$  be the subgroup of bounded automorphisms. The following assertions are equivalent:

(i) The subgroup  $B(\Gamma)$  is transitive.

(ii) There is an equivalence relation  $\sim$  with finite equivalence classes on the vertex set of  $\Gamma$  such that  $B(\Gamma)$  acts on  $\Gamma$  like a finitely generated free abelian group.

In a connected graph  $\Gamma$  we define the ball of radius n with center in a vertex  $\alpha$  as the set  $B_n(\alpha) = \{\beta \in V\Gamma \mid d(\alpha, \beta) \leq n\}$ . The bounded automorphisms of  $\Gamma$  are

related to topological properties of Aut( $\Gamma$ ) via the following result of Woess. With a later application in mind, Woess' result is stated for metric spaces rather than just graphs. A (closed) ball of radius r with center  $\alpha$  in a metric space X as the set  $B_r(\alpha) = \{\beta \in X \mid d_X(\alpha, \beta) \leq r\}$ . An isometry g of a metric space X is said to be bounded if there is a constant c such that  $d_X(\alpha, g\alpha) \leq c$  for all points  $\alpha$  in X. Recall also that an element g in a topological group G is called an FC<sup>-</sup>-element if the conjugacy class of g has compact closure.

**Lemma 2.16.** (Cf. [83, Lemma 4]) Suppose G is a topological group acting transitively by isometries on a metric space X. Assume furthermore that the stabilizer in G of a point in X is a compact open subgroup and that for every value of n the ball  $B_n(\alpha)$  is finite. An element  $g \in G$  is bounded if and only if g is an  $FC^-$ -element of G.

*Proof.* Suppose  $g \in G$  acts as a bounded isometry on X. Find a number M such that  $d(g\alpha, \alpha) \leq M$  for every  $\alpha \in X$ . For  $h \in G$ , write  $g^h = hgh^{-1}$ . Set  $g^G = \{g^h \mid h \in G\}$ . It is clear that  $d(g^h\alpha, \alpha) = d(gh^{-1}\alpha, h^{-1}\alpha) \leq M$  for every  $\alpha \in X$ . We see that the set  $g^G\alpha$  is finite and by Lemma 2.2(ii) the conjugacy class  $g^G$  has compact closure.

Conversely, suppose that the conjugacy class  $g^G$  has compact closure. Then, for every  $\alpha \in X$  the set  $g^G \alpha$  is finite. Take a number M such that  $d(g^h \alpha, \alpha) \leq M$  for every  $h \in G$ . Take some  $\beta \in X$ . Choose  $h \in G$  so that  $\beta = h^{-1}\alpha$ . Then  $d(g\beta,\beta) = d(gh^{-1}\alpha,h^{-1}\alpha) = d(g^h\alpha,\alpha) \leq M$ . So g acts on  $\Gamma$  as a bounded automorphism.

This connection with topological notions can be used to give a short proof of Theorem 2.15, where only elementary results from the theory of topological groups are used, see [49].

A locally finite graph  $\Gamma$  is said to have *polynomial growth* if the number of vertices of  $\Gamma$  in  $B_n(\alpha)$  is bounded above by a polynomial in n. It is easy to see that this property does not depend on the choice of the vertex  $\alpha$ . A finitely generated group G is said to have polynomial growth if its Cayley graph with respect to a finite generating set has polynomial growth (the choice of generating sets is immaterial, since having polynomial growth is a quasi-isometry invariant).

The second theorem of Trofimov related to Schlichting's Theorem is the following:

**Theorem 2.17.** ([69, Theorem 2]) Suppose  $\Gamma$  is a connected, locally finite graph with polynomial growth, and G is a group that acts transitively on  $\Gamma$ . Then there is a G-invariant equivalence relation  $\sim$  with finite classes on the vertex set of  $\Gamma$ such that the quotient of G by the kernel of the induced action on  $\Gamma/\sim$  is a finitely generated, virtually nilpotent group with finite stabilizers for vertices of  $\Gamma/\sim$ .

It should be noted that Trofimov proves an even stronger result [69, Theorem 1], since he shows that it is possible to find an equivalence relation  $\sim$  as described in Theorem 2.17 such that the stabilizer of a vertex in Aut( $\Gamma/\sim$ ) is finite.

The theorem of Trofimov can be seen as a graph theoretical version of Gromov's celebrated theorem characterizing finitely generated groups with polynomial growth, see [30]. Indeed, Trofimov uses Gromov's Theorem in his proof. A version of Gromov's theorem for topological groups has been proved by Losert in [42]. Woess in [83] used Losert's version of Gromov's Theorem from [42] to give a short proof of Theorem 2.17.

We will be returning to polynomial growth and Trofimov's result in Section 4.4.

There is a third theorem of Trofimov's with a similar feel to it as the two theorems stated above. This theorem involves the concept of an *o*-automorphisms of a graph. An automorphism g of a connected graph  $\Gamma$  is called an *o*-automorphism if

$$\max\{d(\beta, g\beta) \mid \beta \in V\Gamma, d(\alpha, \beta) \le n\} = o(n),$$

where  $\alpha$  is a fixed vertex. It is easy to show that this property does not depend on the choice of the vertex  $\alpha$ . It is also easy to prove that the *o*-automorphisms form a normal subgroup of Aut( $\Gamma$ ).

**Theorem 2.18.** ([71, Corollary 1]) Suppose  $\Gamma$  is a connected, locally finite graph and G is a group that acts transitively on  $\Gamma$ . Then the following are equivalent: (i)  $G \leq o(\operatorname{Aut}(\Gamma))$ 

(ii) There is a G-invariant equivalence relation  $\sim$  on the vertex set of  $\Gamma$  such that the equivalence classes of  $\sim$  are finite and if K denotes the kernel of the action of G on the equivalence classes then G/K is a finitely generated nilpotent group acting regularly on  $\Gamma/\sim$ .

Trofimov's proofs of these three theorems are long and difficult. The proofs mentioned above of the first two theorems, are short, but admittedly, in the proof of Theorem 2.17 the results from the theory of topological groups used are highly non-trivial. It would be interesting to find a topological interpretation of the concept of an *o*-automorphism. Possibly that could lead to a shorter proof of Theorem 2.18.

### 3. The scale function and tidy subgroups

The theory of locally compact groups is the part of the theory of topological groups that has widest appeal and most applications in other branches of mathematics. When looking at locally compact groups there are the connected groups on one end of the spectrum and the totally disconnected groups on the other end.

The fundamental result in the theory of locally compact totally disconnected groups is the theorem of van Dantzig [14] that such a group must always contain a compact open subgroup. A big step towards a general theory was taken in the paper [76] by Willis. The fundamental concepts of Willis's theory are the scale function and tidy subgroups. **Definition 3.1.** Let G be a locally compact totally disconnected group and x an element in G. For a compact open subgroup U in G define

$$U_{+} = \bigcap_{i=0}^{\infty} x^{i} U x^{-i}$$
 and  $U_{-} = \bigcap_{i=0}^{\infty} x^{-i} U x^{i}$ .

Say U is *tidy* for g if

(TA) 
$$U = U_+U_- = U_-U_+$$

and

(TB)  $U_{++} = \bigcup_{i=0}^{\infty} x^i U_+ x^{-i}$  and  $U_{--} = \bigcup_{i=0}^{\infty} x^{-i} U_- x^i$  are both closed in G.

Let G be a locally compact totally disconnected group. The *scale function* on G is defined as

 $\mathbf{s}(x) = \min\{|U: U \cap x^{-1}Ux| : U \text{ a compact open subgroup of } G\}.$ 

The connection between the scale function and tidy subgroups is described in the following theorem due to Willis.

**Theorem 3.2.** ([79, Theorem 3.1]) Let G be a totally disconnected, locally compact group and  $g \in G$ . Then  $\mathbf{s}(g) = |U : U \cap g^{-1}Ug|$  if and only if U is tidy for g.

*Remark.* Instead of stating our results for totally disconnected, locally compact groups, we could phrase our results for locally compact groups, that contain a compact, open subgroup.

Now on to something completely different.

#### 3.1. Highly arc transitive digraphs.

**Definition 3.3.** A digraph  $\Gamma$  is called *s*-arc transitive if the automorphism group acts transitively on the set of *s*-arcs. If  $\Gamma$  is *s*-arc transitive for all numbers  $s \ge 0$  then  $\Gamma$  is said to be highly arc transitive.

We also say that a group  $G \leq \operatorname{Aut}(\Gamma)$  acts highly arc transitively on  $\Gamma$  if G acts transitively on the s-arcs in  $\Gamma$  for all s.

The definition of highly arc transitive digraphs occurs first in the paper [13] by Cameron, Praeger and Wormald. Similar conditions, both for directed and undirected graphs, have been studied by various authors in various contexts.

Let us start by looking at several examples.

*Examples.* (i) Let  $\Gamma$  be a directed tree with constant in- and out-valencies. Clearly  $\Gamma$  is highly arc transitive.

(ii) Let  $\Gamma$  be a digraph with the set **Q** of rational numbers as a vertex set and  $(\alpha, \beta)$  an edge in  $\Gamma$  if and only if  $\alpha > \beta$ . Again it is clear that  $\Gamma$  is a highly arc transitive digraph.

(iii) (Cf. [50, Example 1]) Let  $T_1$  denote the regular directed tree in which every vertex has in-valency 1 and out-valency q. Let  $L = \ldots, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \ldots$  be a directed line in  $T_1$ . Define

 $H = \{h \in \operatorname{Aut}(T_1) \mid \text{there is a number } i \text{ such that } h\alpha_i = \alpha_i \}.$ 

If  $h \in H$  and h fixes some  $\alpha_i$  then h also fixes all vertices  $\alpha_j$  with j < i. One can also see that the orbits of H are infinite and each orbit contains precisely one vertex from L. The orbits of H are called *horocycles*. The horocycles could also be defined without reference to the automorphism group. Then we could define two vertices  $\alpha$  and  $\beta$  to be in the same *horocycle* if there is a number n such that the unique path in  $\Gamma$  from  $\alpha$  to  $\beta$  starts by going backwards along n arcs and then going forward along n arcs. Let  $C_i$  denote the horocycle containing  $\alpha_i$ .

For each  $i \in \mathbf{Z}$  take r-1 copies  $S_i^1, \ldots, S_i^{r-1}$  of  $T_1$  and let  $\psi_i^j : S_i^j \to T_1$  be an isomorphism. The preimage of the horocycle  $C_i$  is a horocycle  $B_i^j$  in  $S_i^j$ . When restricted,  $\psi_i^j$  defines an isomorphism between the digraphs spanned by desc $(B_i^j)$ and desc( $C_i$ ). Use this partial isomorphism to identify the vertices in desc( $B_i^j$ ) with the vertices in desc( $C_i$ ). Do this for every i and you get a new digraph  $T_2$ . The digraph  $T_2$  is far from being a tree, but if  $\alpha$  is a vertex in  $T_2$  then the digraph spanned by  $\operatorname{desc}(\alpha)$  is a rooted infinite directed q-ary tree. The vertices in  $T_2$  that did belong to  $T_1$  now all have out-valency equal to q, and in-valency equal to r. Look at the part of  $S_i^j$  that did not get identified with vertices in  $T_1$ . This part is a union of horocycles, at each horocycle in it we glue r-1 new copies of  $T_1$  in the same fashion. Do this for each i and each horocycle in  $S_i^j$ , not belonging to  $T_1$ , and get a digraph  $T_3$ . Continuing in the same fashion we construct a sequence  $T_1 \subseteq T_2 \subseteq T_3 \subseteq \ldots$  of digraphs. In the end we get a digraph  $DL(q,r) = \bigcup T_i$ . In this digraph every vertex has in-valency equal to r and out-valency equal to q. If  $\alpha$  is a vertex in DL(q, r), then the subdigraph spanned by desc( $\alpha$ ) is an infinite rooted directed q-ary tree and the subgraph spanned by  $anc(\alpha)$  is a rooted tree, such that all edges are directed towards the root and the in-valency of every vertex is r and the out-valency is 1. Clearly DL(q, r) is highly arc transitive.

The digraphs DL(q, r) are a directed versions of the Diestel-Leader graphs (defined in [17]) that have been studied by various authors. Woess [83] asked if every locally finite transitive graph is quasi-isometric to some Cayley graph. It was conjectured by Diestel and Leader that if  $q \neq r$  then the graph DL(q, r) is not quasi-isometric to any Cayley graph. This conjecture was proved by Eskin, Fisher and Whyte in [23].

An optimist would hope to find a general classification of locally finite, highly arc transitive graphs, but it seems very implausible that any such classifications is possible. But, there is a particular class of highly arc transitive digraphs where one can give a precise description of their structure. Surprisingly enough this particular class can be used to probe the secrets of Willis' theory.

First, we state two simple lemmata from the paper [13] by Cameron, Praeger and Wormald. We prove the second one, because it is natural to apply the permutation topology on Aut( $\Gamma$ ) in the proof.

**Lemma 3.4.** ([13, Proposition 3.10]) Let  $\Gamma$  be a connected, highly arc transitive digraph with finite out-valency. Suppose  $\Gamma$  is not a directed cycle. If  $\alpha$  and  $\beta$  are vertices in  $\Gamma$  and there is a directed path of length n from  $\alpha$  to  $\beta$ , then every directed path from from  $\alpha$  to  $\beta$  has length n. Furthermore,  $\Gamma$  has no directed cycles.

**Lemma 3.5.** Let  $\Gamma$  be a locally finite, highly arc transitive digraph. Take two directed lines  $L_1 = \ldots, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2 \ldots$  and  $L_2 = \ldots, \beta_{-1}, \beta_0, \beta_1, \beta_2 \ldots$  in  $\Gamma$  then there is a an automorphism g of  $\Gamma$  such that  $g\alpha_i = \beta_i$  for all i.

*Proof.* Write  $G = \operatorname{Aut}(\Gamma)$  and note that G is locally compact. Using the property that  $\Gamma$  is highly arc transitive, we can find an element  $g_i \in \operatorname{Aut}(\Gamma)$  such that  $g_i \alpha_j = \beta_j$  for all  $j \in \{-i, \ldots, i\}$ . The sequence  $(g_i)_{i \in \mathbb{N}}$  is contained in the set  $g_1 G_{\alpha_0}$ , which is compact in the permutation topology on  $\operatorname{Aut}(\Gamma)$ . Hence this sequence has a convergent subsequence that converges to an element g in  $\operatorname{Aut}(\Gamma)$  which has the desired property.

**Proposition 3.6.** ([51, Lemma 3]) Let  $\Gamma$  be a locally finite, highly arc transitive digraph and L a directed line in  $\Gamma$ . Then the subdigraph  $\Gamma_L$  spanned by desc(L), is highly arc transitive and has more than one end.

Proof. Write  $L = \ldots, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \ldots$  Consider s-arcs  $\beta_0, \ldots, \beta_s$  and  $\gamma_0, \ldots, \gamma_s$ in  $\Gamma_L$ . The vertices  $\beta_0$  and  $\gamma_0$  will have a common ancestor  $\alpha_{i_0}$  on the line L. Now we can extend the s-arcs to infinite lines  $L_\beta = \ldots, \beta_{-1}, \beta_0, \beta_1, \beta_2 \ldots$  and  $L_\gamma = \ldots, \gamma_{-1}, \gamma_0, \gamma_1, \gamma_2 \ldots$  that both contain the directed ray  $\ldots, \alpha_{i_0-2}, \alpha_{i_0-1}, \alpha_{i_0}$ . Then we can find an element  $g \in G$  such that  $g\beta_i = \gamma_i$  for all i and because gmaps the ray  $\ldots, \alpha_{i_0-1}, \alpha_{i_0}$  into L we can see that  $g(\operatorname{desc}(L)) = \operatorname{desc}(L)$ , i.e. the subdigraph  $\Gamma_L$  is invariant under g. Whereupon we conclude that  $\Gamma_L$  is highly arc transitive.

Let  $\beta'$  be a vertex in  $\Gamma_L$ . Since  $in_{\Gamma_L}(\beta')$  is finite, there clearly is a number i such that  $in_{\Gamma_L}(\beta') \subseteq desc(\alpha_i)$ . Let k be the length of a directed path from  $\alpha_i$  to  $\beta'$  (by Lemma 3.4 all directed paths from  $\alpha_i$  to  $\beta'$  have the same length). Making use of arc transitivity we conclude that if  $\beta \in desc_k(\alpha_0)$ , then there is an element  $g \in Aut(\Gamma_L)$  such that  $g(\alpha_i) = \alpha_0$  and  $g(\beta') = \beta$ . Therefore we see that if  $(\gamma, \beta)$  is an arc in  $\Gamma_L$  (i.e.  $\gamma \in in_F(\beta)$ ) then  $\gamma \in desc(\alpha_0)$ . More precisely,  $\gamma \in desc_{k-1}(\alpha_0)$ . This is so because, if  $\alpha_0, \gamma_1, \ldots, \gamma_l, \gamma$  is a directed path from  $\alpha_0$  to  $\gamma$  then  $\alpha_0, \gamma_1, \ldots, \gamma_l, \gamma, \beta$  is a directed path from  $\alpha_0$  to  $\alpha$  and thus has length k.

Set  $A = \bigcup_{i \ge k} \operatorname{desc}_i(\alpha_0)$  and  $A^* = VF \setminus A$ . Suppose  $(\gamma, \beta)$  is an arc from  $A^*$  to A. Now  $\beta \in \operatorname{desc}_l(\alpha_0)$  for some  $l \ge k$ . Then  $\gamma \in \operatorname{desc}_{l-1}(\alpha_0)$ , by the choice of k. Obviously l = k and  $(\gamma, \beta)$  is an arrow from  $\operatorname{desc}_{k-1}(\alpha_0)$  to  $\operatorname{desc}_k(\alpha_0)$ .

A priori, there is also the possibility that some arc  $(\beta, \gamma)$  in F goes from A to  $A^*$ . But on closer look, this is impossible, because then  $\beta$  would be in desc<sub>l</sub>( $\alpha_0$ ) for some  $l \geq k$  and thus  $\gamma \in \text{desc}_{l+1}(\alpha_0) \subseteq A$ , and therefore  $\gamma \in A$ , contradicting the assumption that  $\gamma \in A^*$ .

The only arcs between A and  $A^*$  are going from  $\operatorname{desc}_{k-1}(\alpha_0)$  to  $\operatorname{desc}_k(\alpha_0)$ . The set of such arcs is clearly finite (because  $\operatorname{desc}_{k-1}(\alpha_0)$  is finite and the out-valency of vertices in  $\Gamma$  is finite), and by removing them, we split  $\Gamma$  up into components. The two sets  $\{\ldots, \alpha_0, \alpha_1, \ldots, \alpha_{k-1}\}$  and  $\{\alpha_k, \alpha_{k+1}, \ldots\}$  will belong to different components, so we have at least two infinite components. Hence  $\Gamma$  has more than one end.

The structure of digraphs like  $\Gamma_L$  in the above proposition is described in the following theorem.

**Theorem 3.7.** ([51, Theorem 1]) Let  $\Gamma$  be a locally finite, highly arc transitive digraph. Suppose that there is a line  $L = \ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots$  such that  $V\Gamma = \text{desc}(L)$ .

Then there exists a surjective homomorphism  $\phi: \Gamma \to T$  where T is a directed tree with in-valency 1 and finite out-valency. The automorphism group of  $\Gamma$  has a natural action on T as a group of automorphisms such that  $\phi(g\alpha) = g\phi(\alpha)$  for every  $g \in \operatorname{Aut}(\Gamma)$  and every vertex  $\alpha$  in  $\Gamma$ . This action of  $\operatorname{Aut}(\Gamma)$  on T is highly arc transitive. Furthermore, the fibers  $\phi^{-1}(\alpha), \alpha \in VT$ , are finite and all have the same number of elements.

Let  $\alpha$  be a vertex in a highly arc transitive digraph  $\Gamma$  and denote with  $c_k$  the number of vertices in  $\operatorname{desc}_k(\alpha)$ . Cameron, Praeger and Wormald in [13, Definition 3.5] define the *out-spread* of a vertex in  $\Gamma$  as  $\limsup_{k\to\infty} c_k^{1/k}$ . One can define the *in-spread* of a highly arc transitive digraph in a similar way. Theorem 3.7 implies the following

**Theorem 3.8.** ([51, Theorem 2]) The out-spread of a locally finite, highly arc transitive digraph is an integer.

The in-spread can be used to characterize the highly arc transitive digraphs treated in Theorem 3.7.

**Theorem 3.9.** ([45, Theorem 2.6]) Let  $\Gamma$  be a locally finite, highly arc transitive digraph. The in-spread of  $\Gamma$  is 1 if and only if there is a line L in  $\Gamma$  such that  $\operatorname{desc}(L) = V\Gamma$ .

**3.2. Tidy subgroups and highly arc transitive digraphs.** Now we turn our attention back to totally disconnected, locally compact groups. The following notation will be used extensively in what follows. Let G be a totally disconnected, locally compact group and x a fixed element in G. Take a compact open subgroup U. We set  $\Omega = G/U$  and let  $\alpha_0$  denote the point in  $\Omega$  that has U as stabilizer. Then define a digraph  $\Gamma = \Gamma_U$  that has  $\Omega$  as a vertex set and edge set  $G(\alpha_0, x\alpha_0)$ – the G-orbit of the ordered pair  $(\alpha_0, x\alpha_0)$ . Note that  $\Gamma$  need not be connected. For an integer i set  $\alpha_i = x^i \alpha_0$ . The vertices  $\alpha_i$  form a line L in  $\Gamma$ . Observe that  $x^i U x^{-i}$  is the stabilizer of  $\alpha_i$  in G and  $U_+$  is the stabilizer of the vertices  $\alpha_0, \alpha_1, \ldots$ and  $U_-$  is the stabilizer of the vertices  $\alpha_0, \alpha_{-1}, \ldots$ 

**Proposition 3.10.** (Cf. [50, Theorem 2.1]) The subgroup U satisfies condition (TA) in Definition 3.1 if and only if G acts highly arc transitively on the digraph  $\Gamma$ .

*Proof.* Let us start by looking at what happens when the digraph  $\Gamma$  is highly arc transitive. Let  $g \in U = G_{\alpha_0}$ . Since G is assumed to act highly arc transitively on  $\Gamma$  and G is a closed in the permutation topology we deduce from Lemma 3.5 that G acts transitively on the set of lines in  $\Gamma$ . For  $i \geq 1$  we set  $\beta_i = g\alpha_i$  and let  $L_1$  denote the line  $\ldots, \alpha_{-1}, \alpha_0, \beta_1, \beta_2, \ldots$  We find an element  $g_-$  that moves the line L to the line  $L_1$  such that  $g_-\alpha_i = \alpha_i$  for  $i \leq 0$  and  $g_-\alpha_i = \beta_i$  for  $i \geq 1$ . Note that  $g_- \in U_-$ . Set  $g_+ = g_-^{-1}g$  and note that  $g_+$  fixes all the vertices  $\alpha_0, \alpha_1, \ldots$  and thus

 $g_+ \in U_+$ . Therefore  $g \in U_-U_+$ . From this we deduce that U satisfies condition TA.

Conversely, assume that U satisfies condition TA. Take a vertex  $\beta$  in  $\operatorname{out}(\alpha_0)$ . Then there must be an element  $g \in U$  such that  $g\alpha_1 = \beta$ . Write  $g = g_-g_+ \in U_-U_+$ and we see that  $g_-\alpha_1 = \beta$ . Thus  $U_-$  acts transitively on  $\operatorname{out}(\alpha_0)$ . Now we use induction over s to show that G acts transitively on the set of s-arcs. Suppose we are given an (s + 1)-arc  $\beta_0, \ldots, \beta_s, \beta_{s+1}$ . Use the induction hypothesis to find an element  $h \in G$  such that  $h\alpha_0 = \beta_s, h\alpha_{-1} = \beta_{s-1}, \ldots, h\alpha_{-s} = \beta_0$ . Then by the above,  $hU_+h^{-1}$  acts transitively on  $\operatorname{out}(h\alpha_0) = \operatorname{out}(\beta_s)$ . We pick an element h'from  $hU_+h^{-1}$  such that  $h'(h\alpha_1) = \beta_{s+1}$ . Now we have found an element h'h that moves the (s + 1)-arc  $\alpha_0, \ldots, \alpha_s, \alpha_{s+1}$  to the (s + 1)-arc  $\beta_0, \ldots, \beta_s, \beta_{s+1}$  and can conclude that G acts transitively on the (s + 1)-arcs in  $\Gamma$  and also that G acts highly arc transitively on  $\Gamma$ .

Condition (TB) can also be translated in to a condition about the graph  $\Gamma$  defined at the start of the section. We use the following lemma.

**Lemma 3.11.** ([76, Lemma 3]) Let G be a totally disconnected, locally compact group and  $x \in G$ . Suppose that U is a compact, open subgroup of G that satisfies condition (TA). Then

(a)  $U_{++}$  is closed if and only if  $U_{++} \cap U = U_+$ .

(b)  $U_{++}$  is closed if and only if  $U_{--}$  is closed.

In our setting  $U_{++}$  is the set of all elements g in G such that there exists a number k such that g fixes  $\alpha_k, \alpha_{k+1}, \ldots$  The condition that  $U_{++} \cap U = U_+$  says that an element in G that fixes  $\alpha_0$  and also  $\alpha_k, \alpha_{k+1}, \ldots$  for some  $k \ge 0$  must also fix  $\alpha_1, \ldots, \alpha_{k-1}$ . If we assume that G acts highly arc transitively on  $\Gamma$  then this implies that  $\alpha_0, \ldots, \alpha_k$  is the unique path in  $\Gamma$  from  $\alpha_0$  to  $\alpha_k$  and we conclude that the subgraph spanned by desc( $\alpha_0$ ) is a tree.

On the other hand, if the subgraph spanned by  $\operatorname{desc}(\alpha_0)$  is a tree then clearly a group element that fixes  $\alpha_0$  and  $\alpha_k$  must fix  $\alpha_1, \ldots, \alpha_{k-1}$  since these vertices lie on the directed path from  $\alpha_0$  to  $\alpha_k$ . Hence  $U_{++} \cap U = U_+$ . Thus we have shown the following result.

**Proposition 3.12.** Suppose U satisfies condition (TA). Then U satisfies condition (TB) if and only if the subgraph spanned by  $\operatorname{desc}(\alpha_0)$  is a tree.

Putting these observation together as a theorem we get.

**Theorem 3.13.** (Cf. [50, Theorem 3.4]) Let G be a totally disconnected, locally compact group and U a compact, open subgroup. Let x be an element in G and define a graph  $\Gamma$  such that the vertex set is G/U and the edge set is  $G(\alpha_0, x\alpha_0)$ where  $\alpha_0$  is the vertex in  $\Gamma$  such that  $U = G_{\alpha}$ . Suppose furthermore that the orbit of  $\alpha_0$  under x is infinite. Then U is tidy for x if and only if G acts highly are transitively on  $\Gamma$  and the subgraph spanned by desc $(\alpha_0)$  is a tree. **3.3. Using the connection.** In this section G denotes a totally disconnected, locally compact group. From the definition of tidy subgroup it is far from obvious that there always is a compact, open subgroup of G that is tidy for a given element x in G. Our first task is thus to construct a compact, open subgroup U that is tidy for x.

First, the case where x is periodic (i.e. the subgroup generated by x is relatively compact). Let U be a compact, open subgroup. Put  $\Omega = G/U$ . Let  $\alpha$  be a point in  $\Omega$  such that  $G_{\alpha} = U$ . Define A as the closure (in the given topology on G) of the subgroup generated by x. By assumption A is compact. Since the permutation topology induced by the action of G on  $\Omega$  is contained in the original topology on G we conclude that A is also compact in the permutation topology. Hence, by Lemma 2.2(ii), all the orbits of the subgroup generated by x are finite. So there is a number N such that  $x^N \alpha = \alpha$  and, therefore,  $x^N \in G_{\alpha} = U$ . The subgroup  $U \cap xUx^{-1} \cap \cdots \cap x^{N-1}Ux^{-(N-1)}$  is compact and open and normalized by x and thus tidy for x. Hence we will assume in what follows that x is not periodic.

Let V be some compact, open subgroup of G. Construct a graph  $\Gamma = \Gamma_V$  as done at the start of the last section. From the proof of Proposition 3.10 we see that G acts highly arc transitively on  $\Gamma$  if and only if  $V_-$  acts transitively on  $\operatorname{out}(\alpha_0)$ . Look at the group  $V_n = \bigcap_{i=0}^n x^{-i}Vx^i = G_{\alpha_0,\alpha_{-1},\dots,\alpha_{-n}}$ . We claim that there is a number n such that  $V_n\alpha_1 = V_-\alpha_1$ . Otherwise one could find an element  $g_i \in V_i$  for each i such that  $g_i\alpha_1 \notin V_-\alpha_1$ . The sequence  $(g_i)_{i\in\mathbb{N}}$  has a convergent subsequence converging to an element g and clearly this element is in  $V_-$ , but  $g\alpha_1 \notin V_-\alpha_1$ , so we have reached a contradiction. Now set  $W = V_n$ . Note that  $W_+ = V_+$ . We can use a similar argument as in the first part of Proposition 3.10 to show that W satisfies condition (TA). Using this compact, open subgroup W to get a compact, open subgroup that also satisfies condition (TB) is more involved, and the details will be left out. By finding a compact, open subgroup W satisfying (TA), we have ensured that G acts highly arc transitively on the digraph  $\Gamma_W$ . What is missing is condition (TB), which would mean that the subgraph spanned by the descendants of a vertex is a tree. To achieve that, Theorem 3.7 is used to produce a highly arc transitive digraph, wherein the graph spanned by the descendants of a vertex is a tree. This will then prove the following theorem of Willis.

**Theorem 3.14.** ([76, Theorem 1], see also [50, Theorem 4.1]) Let G be a totally disconnected, locally compact group and x an element of G. Then there is a compact, open subgroup U of G that is tidy for x.

Now we have ensured that there is something to talk about. We next use digraphs to deduce further facts about tidy subgroups and the scale function. First, we use Lemma 2.4 to deduce the following.

**Theorem 3.15.** ([76, Corollary 1]) Let G be a totally disconnected, locally compact group. Denote by  $\Delta$  the modular function on G and by **s** the scale function on G. Then, for every  $x \in G$ ,

$$\Delta(x) = \frac{\mathbf{s}(x)}{\mathbf{s}(x^{-1})}.$$

*Proof.* Let  $U_1$  and  $U_2$  be compact open subgroups of G such that

 $|U_1: U_1 \cap x^{-1}U_1x| = \mathbf{s}(x)$  and  $|U_2: U_2 \cap xU_2x^{-1}| = \mathbf{s}(x^{-1}).$ 

Note that

$$|U_1: U_1 \cap x U_1 x^{-1}| \ge \mathbf{s}(x^{-1})$$
 and  $|U_2: U_2 \cap x^{-1} U_2 x| \ge \mathbf{s}(x).$ 

Now we use the Remark following Lemma 2.4 and get

$$\frac{\mathbf{s}(x)}{\mathbf{s}(x^{-1})} \ge \frac{|U_1: U_1 \cap x^{-1} U_1 x|}{|U_1: U_1 \cap x U_1 x^{-1}|} = \Delta(x) = \frac{|U_2: U_2 \cap x^{-1} U_2 x|}{|U_2: U_2 \cap x U_2 x^{-1}|} \ge \frac{\mathbf{s}(x)}{\mathbf{s}(x^{-1})}.$$
  
ce  $\Delta(x) = \mathbf{s}(x)/\mathbf{s}(x^{-1}).$ 

Hence  $\Delta(x) = \mathbf{s}(x)/\mathbf{s}(x^{-1})$ .

This also implies the following corollary.

**Corollary 3.16.** ([79, Corollary 3.11]) Let x be an element of a totally disconnected, locally compact group G, and U a compact, open subgroup of G. Then  $|U: U \cap x^{-1}Ux| = \mathbf{s}(x)$  if and only if  $|U: U \cap xUx^{-1}| = \mathbf{s}(x^{-1})$ .

Tidy subgroups are related to the scale function as described in Theorem 3.2. The proof of Theorem 3.2 is involved, and we will only have a look at the proof that compact, open subgroup U such that  $\mathbf{s}(x) = |U: U \cap x^{-1}Ux|$  must be tidy.

Consider a compact, open subgroup U, a fixed element  $x \in G$ , and the digraph  $\Gamma$  defined above. By the above, when trying to minimize  $|U:U\cap x^{-1}Ux|$  in order to find  $\mathbf{s}(x)$ , we could equally try to minimize  $|U: U \cap x U x^{-1}|$ . The latter index is just the out-valency in the digraph  $\Gamma_U$ . When constructing a tidy subgroup for x, we start with an arbitrary compact, open subgroup V and next find a compact, open subgroup V satisfying (TA). The out-valency in  $\Gamma_V$  is at most the out-valency of  $\Gamma_U$ . In the second step, we ensure that condition (TB) is satisfied, and in the process the out-valency does not increase. Thus we can be sure that if U minimizes  $|U: U \cap xUx^{-1}|$  then U must be tidy for x.

Again we look at a compact open subgroup U and the graph  $\Gamma$  as above. Note that  $U\alpha_n = |U: U \cap x^n U x^{-n}|$ . If  $\Gamma$  is highly arc transitive, then this is precisely the number  $b_n$  of vertices  $\beta$  such that there is a directed path of length n from  $\alpha_0$ to  $\beta$ . The out-valency  $d_+$  of  $\Gamma$  is equal to  $|U:U\cap xUx^{-1}|$ . The subgraph spanned by desc( $\alpha_0$ ) is a tree if and only if  $b_n = d_+^n = |U: U \cap xUx^{-1}|^n$  for all natural numbers n. But U is tidy if an only if  $\Gamma$  is highly arc transitive and the subgraph spanned by desc( $\alpha_0$ ) is a tree. Thus we derive the following result.

**Theorem 3.17.** ([50, Corollary 3.5]) Let G be a totally disconnected, locally compact group and x an element in G. Then a compact, open subgroup U is tidy for x if and only if

$$|U: U \cap x^{n}Ux^{-n}| = |U: U \cap xUx^{-1}|^{n}$$

for all  $n \geq 1$ .

**Corollary 3.18.** Let G be a totally disconnected locally compact group and x an element in G. Then  $\mathbf{s}(x^n) = \mathbf{s}(x)^n$ .

*Proof.* Let U be a compact, open subgroup of G that is tidy for x. It is easy to check that if a subgroup U is tidy for x then U is also tidy for  $x^n$  for every integer n. Hence

$$\mathbf{s}(x^{n}) = |U: U \cap x^{-n}Ux^{n}| = |U: U \cap x^{-1}Ux|^{n} = \mathbf{s}(x)^{n}.$$

If  $\Gamma$  is highly arc transitive, the index  $|U : U \cap x^{-n}Ux^n|$  is the number of vertices  $\beta$  such that  $\alpha_0$  is in desc<sub>n</sub>( $\beta$ ). This observation suggests that we compare the scale function and the in-spread of the associated digraph  $\Gamma$ . The following theorem describes their relationship.

**Theorem 3.19.** ([50, Theorem 7.7]) Let G be a totally disconnected, locally compact group and x an element of G. If V is some compact, open subgroup of G, then

$$\mathbf{s}(x) = \lim_{n \to \infty} |V: V \cap x^{-n} V x^n|^{1/n}.$$

For a different formulation and a proof see [6, Lemma 4]. This line of thought also gives us information about the case  $\mathbf{s}(x) = 1$ .

**Theorem 3.20.** ([50, Corollary 7.8]) Let G be a totally disconnected, locally compact group and x an element of G such that  $\mathbf{s}(x) = 1$ . If V is some compact, open subgroup of G, then there is a constant C such that  $|V : V \cap x^{-n}Vx^n| \leq C$  for all  $n \geq 0$ .

These two results can also be formulated as results about permutation groups.

**Theorem 3.21.** Let G be a group acting transitively on a set  $\Omega$ . Assume that all suborbits of G are finite. Let x be an element in G and  $\alpha_0$  a point in  $\Omega$ . Set  $\alpha_i = x^i \alpha_0$ . Then either there is a constant C such that  $|G_{\alpha_0} \alpha_n| \leq C$  for all n or the numbers  $|G_{\alpha_0} \alpha_n|$  grow exponentially with n and  $\lim_{n\to\infty} |G_{\alpha_0} \alpha_n|^{1/n} = s$  for some integer s.

*Remark.* In [72] Trofimov studies generalized x-tracks, which are similar to the directed line ...,  $\alpha_{-1}$ ,  $\alpha_0$ ,  $\alpha_1$ ,  $\alpha_2$ , ... that is fundamental to the graph-theoretical interpretation of Willis' theory in [50]. Theorem 3.21 is clearly related to [72, Theorem 4.1, part 3].

The final illustration of the uses of graphs in Willis' structure theory is a proof of the following theorem.

**Theorem 3.22.** ([77, Theorem 2]) Let G be a totally disconnected, locally compact group. The set P(G) of periodic elements in G is closed. (An element  $x \in G$  is periodic if and only if  $\langle x \rangle$  is compact.)

*Proof.* The trick is to use the fact that that a connected infinite and locally finite highly arc transitive digraph has no directed cycles, see Lemma 3.4.

Suppose x is not periodic, but is in the closure of P(G). Let U be a compact, open subgroup of G, that is tidy for x. Define a digraph  $\Gamma$  as at the start of Section 3.2. If x is not periodic, then the orbit of  $\alpha_0$  under x is infinite, and the connected component of  $\Gamma$  that contains  $\alpha_0$  is infinite. (It must contain the line  $\ldots, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \ldots$ ) The set xU is an open neighbourhood of x, and must therefore contain some periodic element g. The fact that  $g \in xU = xG_{\alpha_0}$  implies  $g\alpha_0 = x\alpha_0 = \alpha_1$ . The element g is periodic, hence the orbit of  $\alpha_0$  under g is finite, and therefore there is an integer n such that  $g^n(\alpha_0) = \alpha_0$ . The sequence  $\alpha_0, \alpha_1 = g\alpha_0, \beta_2 = g^2\alpha_0, \ldots, \beta_n = g^n\alpha_0 = \alpha_0$  is a directed cycle in  $\Gamma$ . This contradicts Lemma 3.4 mentioned above. Hence we conclude that it is impossible that the closure of P(G) contains any elements that are not periodic. Thus P(G)is closed.

## 4. Rough Cayley graphs

Most of the material in this section is taken from a paper by Krön and Möller [40]. Let G be a compactly generated, totally disconnected, locally compact group. In [40] the authors construct a locally finite, connected graph with a transitive G-action, whose vertex stabilizers are compact, open subgroups of G. This graph is called a *rough Cayley graph* of G. As demonstrated in [40], and summarized in this section, a rough Cayley graph can be used to study compactly generated, locally compact groups in a similar way as an ordinary Cayley graph is used to study a finitely generated group. In this article, we illustrate this approach, by using rough Cayley graphs to generalize the concept of ends of groups and to study compactly generated, locally compact groups of polynomial growth.

Below, we explain how to construct a rough Cayley graph and it is also shown that any two rough Cayley graphs for a given group are quasi-isometric. The applications of the rough Cayley graph to the theory of ends of groups and to groups of polynomial growth are only sketched; details can be found in [40], where tools from [19] and [15] are used extensively.

#### 4.1. Definition of a rough Cayley graph.

**Definition 4.1.** ([40, Definition 2.1]) Let G be a topological group. A connected graph  $\Gamma$  is said to be a *rough Cayley graph* of G if G acts transitively on  $\Gamma$  and the stabilizers of vertices are compact, open subgroups of G.

In this section we show that if G is a compactly generated locally compact group that contains a compact open subgroup then G has a locally finite rough Cayley graph and any two rough Cayley graphs are quasi-isometric to each other. The approach here is different from the approach in [40].

Let G be a compactly generated topological group. For a compact generating set S we form the Cayley graph  $\Gamma = \text{Cay}(G, S)$  of G with respect to S. The vertex set of  $\Gamma$  is equal to G and thus carries a topology. The compactness of S and the continuity of multiplication in G implies that if A is a relatively compact set of vertices in  $\Gamma$  then the neighbourhood of A in  $\Gamma$  is contained in the set  $A \cdot S$  and is thus relatively compact also. This can be used to prove that a set A of vertices in  $\Gamma$  is relatively compact if and only if it has finite diameter in the graph metric on  $\Gamma$ , see [1, 2.3 Heine-Borel-Eigenschaft].

**Definition 4.2.** Let G be a group acting transitively on a connected graph  $\Gamma$ . Suppose U is a subgroup of G that contains the stabilizer of some vertex  $\alpha$ . The orbit  $U\alpha$  is a block of imprimitivity. Let  $\Gamma_U$  denote the quotient graph with respect to the G-congruence whose classes are the translates under G of the set  $U\alpha$ .

**Lemma 4.3.** Let G be a compactly generated topological group. Suppose S is a compact generating set and U is a compact open subgroup of G. Then the graph  $Cay(G,S)_U$  is locally finite.

*Proof.* The neighbourhood of a coset gU in  $\Gamma$  is compact and can thus be covered by finitely many right cosets of U. Whence  $\Gamma_U$  is locally finite.

Lemma 4.3 shows that every compactly generated group G that has a compact, open subgroup U, has a locally finite rough Cayley graph, namely,  $Cay(G, S)_U$ .

The proof of a result of Sabidussi [60, Theorem 2], restated below in our own terminology, can be used to show that every locally finite rough Cayley graph for a compactly generated group with a compact, open subgroup can be obtained in this fashion.

**Theorem 4.4.** (Cf. [60, Theorem 2]) Let  $\Gamma$  be a connected graph and G a transitive subgroup of Aut( $\Gamma$ ). Then there is a set S of generators of G such that  $\Gamma \cong Cay(G, S)_U$  where U is the stabilizer in G of some vertex in  $\Gamma$ .

In our setting we are thinking of a topological group G acting on a locally finite rough Cayley graph  $\Gamma$  such that stabilizers of vertices are compact open subgroups but the action is not necessarily faithfully. Take a vertex  $\alpha$  in  $\Gamma$  and let U denote the stabilizer in G of  $\alpha$ . Looking at Sabidussi's proof of his theorem we deduce that if S is the union of U and all elements h in G such that  $\alpha$  and  $h\alpha$  are adjacent in  $\Gamma$  then  $\Gamma \cong \operatorname{Cay}(G, S)_U$ . Note also that the orbit  $S\alpha$  consists precisely of  $\alpha$ and all its neighbours. The graph  $\Gamma$  is by assumption locally finite so S is a finite union of cosets of U and thus compact.

The influential concept of quasi-isometry was introduced by Gromov [31] and has been widely used since.

**Definition 4.5.** Two metric spaces  $(X, d_X)$  and  $(Y, d_Y)$  are said to be quasiisometric if there is a map  $\varphi : X \to Y$  and constants  $a \ge 1$  and  $b \ge 0$  such that for all points  $x_1$  and  $x_2$  in X

 $a^{-1}d_X(x_1, x_2) - a^{-1}b \le d_Y(\varphi(x_1), \varphi(x_2)) \le ad_X(x_1, x_2) + ab,$ 

and for all points  $y \in Y$  we have

$$d_Y(y,\varphi(X)) \le b.$$

A map  $\varphi$  between two metric spaces satisfying the above conditions is called a *quasi-isometry*.

Two connected graphs X and Y are called quasi-isometric if  $(VX, d_X)$  and  $(VY, d_Y)$  are quasi-isometric. Being quasi-isometric is an equivalence relation on the class of metric spaces.

**Theorem 4.6.** Let G be a compactly generated group. Assume that G admits a rough Cayley graph. All rough Cayley graphs for G are quasi-isometric.

The first step in the proof of Theorem 4.6 is the following Lemma. The proof of the Lemma depends on the *Heine-Borel Eigenschaft*, [1, 2.3 Heine-Borel-Eigenschaft], mentioned in the paragraph preceding Definition 4.2.

**Lemma 4.7.** (i) Let G be a compactly generated topological group. Suppose  $S_1$  and  $S_2$  are compact generating sets for G. Then the Cayley-graphs  $Cay(G, S_1)$  and  $Cay(G, S_2)$  are quasi-isometric.

(ii) Suppose S is a compact generating set and U is a compact subgroup of G. Then the graphs  $\operatorname{Cay}(G, S)$  and  $\operatorname{Cay}(G, S)_U$  are quasi-isometric.

*Proof.* (i) There is a constant C, such that the elements in  $S_1$  can be expressed as words of length  $\leq C$  in the elements in  $S_2$  and *vice versa*. For each element of  $S_1$  respectively  $S_2$  fix a word in  $S_2$  respectively  $S_1$  with this property. Using these correspondences, words in  $S_1$  and  $S_2$  may be expressed as words in the other set, that are at most C times longer. This shows that the identity map on G extends to quasi-isometries  $\operatorname{Cay}(G, S_1) \to \operatorname{Cay}(G, S_2)$  and  $\operatorname{Cay}(G, S_2) \to \operatorname{Cay}(G, S_1)$  that are inverse to each other.

(ii) By part (i) we know that  $\operatorname{Cay}(G, S)$  and  $\operatorname{Cay}(G, S \cup U)$  are quasi-isometric. Thus we may assume that S contains U. Under this assumption, each of the right cosets of U has therefore diameter 1. The quotient graph  $\operatorname{Cay}(G, S)_U$  is obtained by contracting each of these cosets. Clearly this operation preserves quasi-isometry and the claim follows.

Proof. (Theorem 4.6) Suppose  $\Gamma_1$  and  $\Gamma_2$  are rough Cayley graphs of G. Then we can find compact generating sets  $S_1$  and  $S_2$  and compact, open subgroups  $U_1$ and  $U_2$  such that  $\Gamma_1 = \operatorname{Cay}(G, S_1)_{U_1}$  and  $\Gamma_2 = \operatorname{Cay}(G, S_2)_{U_2}$ . By part (ii) of Lemma 4.7,  $\Gamma_1$  is quasi-isometric to  $\operatorname{Cay}(G, S_1)$ , which in turn is quasi-isometric to  $\operatorname{Cay}(G, S_2)$  by part (i) of Lemma 4.7, which is again quasi-isometric to  $\Gamma_2 =$  $\operatorname{Cay}(G, S_2)_{U_2}$  by part (ii) of the same result.

Suppose G is a locally compact group with a compact, open subgroup. If the group is compactly generated, Lemma 4.3 gives the existence of a locally finite rough Cayley graph. Conversely, the existence of a rough Cayley graph implies that G is compactly generated.

**Proposition 4.8.** ([52, Corollary 1]) Suppose that G is a locally compact topological group and that G acts transitively on a connected, locally finite graph such that the stabilizers of vertices in G are compact, open subgroups of G. Then G is compactly generated.

The proof of the proposition is based on the Lemma below.

**Lemma 4.9.** Let G be a group acting transitively on a connected, locally finite graph  $\Gamma$ . Then G has a finitely generated, transitive subgroup.

Proof. Fix a reference vertex  $\alpha$ . Denote the neighbours of  $\alpha$  by  $\beta_1, \ldots, \beta_n$ . Choose elements  $h_1, \ldots, h_n$  such that  $h_i \alpha = \beta_i$ . We claim that  $H = \langle h_1, \ldots, h_n \rangle$  is transitive on the vertices of  $\Gamma$ . Note that all the vertices in  $\Gamma$  that are adjacent to the vertex  $\alpha$  are in the *H*-orbit of  $\alpha$ . Suppose that  $\beta = h\alpha$  for some  $h \in H$ . Then  $hh_1h^{-1}\beta, \ldots, hh_nh^{-1}\beta$  is an enumeration of all the neighbours of  $\beta$ . Whence the neighbours of  $\beta$  are also contained in the *H*-orbit of  $\alpha$ . Since our graph is assumed to be connected, we conclude that *H* acts transitively on the vertices.

Proposition 4.8 follows from Lemma 4.9, because the union of the stabilizer of a vertex with a finite generating set for a transitive subgroup forms a compact generating set for G.

The following theorem concludes our basic considerations of rough Cayley graphs. Its first part is well known.

**Theorem 4.10.** ([40, Corollary 2.11]) Let G be a compactly generated topological group that has a compact open subgroup. Assume that H is a cocompact closed subgroup of G. Then H is compactly generated and if  $\Gamma_G$  is a rough Cayley graph for G and  $\Gamma_H$  is a rough Cayley graph for H then  $\Gamma_G$  and  $\Gamma_H$  are quasi-isometric.

Proof. Let X be a locally finite rough Cayley graph for G. By Lemma 2.3 we know H acts with finitely many orbits on X. Choose a vertex  $\alpha$  in  $\Gamma$ . Then there is a number k such that every vertex in  $\Gamma$  is in distance at most k from the orbit  $H\alpha$ . Now form the graph  $\Gamma'$  which has the same vertex set as  $\Gamma$  but two vertices being adjacent if and only if their distance in  $\Gamma$  is at most 2k + 1. Note that  $\Gamma'$  is also locally finite. Consider the subgraph  $\Delta$  of  $\Gamma'$  spanned by the vertices in  $H\alpha$ . Suppose  $\alpha$  and  $\beta$  are vertices in  $\Delta$  and that  $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta$  is a path in  $\Gamma$ (and thus also a path in  $\Gamma'$ ) from  $\alpha$  to  $\beta$ . For each  $\alpha_i$  choose a vertex  $\beta_i$  in  $H\alpha$ such that  $d_{\Gamma}(\alpha_i, \beta_i) \leq k$ . Then  $d(\beta_i, \beta_{i+1}) \leq 2k + 1$  so either  $\beta_i = \beta_{i+1}$  or  $\beta_i$  and  $\beta_{i+1}$  are adjacent in  $\Gamma'$ . Whereupon we conclude that  $\Delta$  is connected. The action of H on the connected, locally finite graph  $\Delta$  is transitive with compact, open vertex-stabilizers. Hence  $\Delta$  is a rough Cayley graph for H and H is compactly generated by Proposition 4.8. From the above it is clear that  $\Delta$  is quasi-isometric to  $\Gamma$ . The second part of the theorem follows by Theorem 4.6.

*Remark.* For the rest of Section 4 we will focus on compactly generated, totally disconnected, locally compact groups. Corresponding results hold for compactly generated, locally compact groups that contain a compact, open subgroup.

**4.2.** Application to  $\mathbf{FC}^-$ -groups. As mentioned in Section 2.3, an element g of a topological group G is called a  $\mathbf{FC}^-$ -element if its conjugacy class in G has compact closure. It is an easy exercise to show that the  $\mathbf{FC}^-$ -elements of G form a normal subgroup B(G) of G. If G = B(G) then G itself is called a  $\mathbf{FC}^-$ -group. These concepts have been extensively studied, see for example the paper [32] by Grosser and Moskowitz and various papers by Wu and his collaborators, e.g. [85] and [86].

A basic question about the subgroup B(G) is whether it is closed or not. This question was discussed by Tits in [66], where it proved that B(G) is closed if G is a connected locally compact group. But, Tits also gives an example of a locally compact, totally disconnected group where B(G) is not closed. Below is another example of such a group described by using graphs.

*Example.* Let  $\Gamma$  denote a directed tree such that each vertex has in-valency 1 and out-valency 2. Choose a directed line  $\ldots, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \ldots$  in  $\Gamma$ . We say that vertices  $\alpha$  and  $\beta$  are in the same *horocycle* if there is a number n such that the unique path in  $\Gamma$  from  $\alpha$  to  $\beta$  starts by going backwards along n arcs and then going forward along n arcs. (This concept is also discussed in Section 3.1.) Membership in the same horocycle is an equivalence relation on vertices. We denote by  $C_i$  the equivalence class of  $\alpha_i$ . Define H as the subgroup of Aut( $\Gamma$ ) that stabilizes  $C_0$ . Let G denote the permutation group that H induces on  $C_0$  and endow G with the permutation topology arising from that action. We think of  $C_0$  as a metric space, the metric being the restriction of the graph metric on  $\Gamma$ . If g is an element of G that fixes all but finitely many vertices in  $C_0$  then by Lemma 2.16 the element g is an FC<sup>-</sup>-element of G. It is also clear that if g and h are elements of G, both with finite support, then there is a conjugate of h whose support is disjoint from the support of g. Let  $g_i$  be an element if G with finite support such that there is a vertex  $\alpha \in C_0$  such that  $d(\alpha, g_i \alpha) = 2i$ . (We could define  $g_i$  explicitly by defining  $\beta_{i+1}$  as the vertex such that  $(\alpha_{-i-2}, \beta_{i+1})$  is an arc in  $\Gamma$  and  $\beta_{i+1} \neq \alpha_{-i-1}$  and then let g transpose the two outward directed arcs from  $\beta_{i+1}$ .) We may assume that  $\operatorname{supp} g_i \cap \operatorname{supp} g_j = \emptyset$  if  $i \neq j$ . All the  $g_i$ 's are contained in B(G) and  $h_i = g_1 \dots g_1$ is also in B(G). Then we define g as the limit of the sequence of the  $h_i$ 's  $(g\alpha = h_i\alpha)$ if  $\alpha$  is in supp  $h_i$  and  $g\alpha = \alpha$  if  $\alpha$  is not in the support of any of the  $h_i$ 's). Clearly g is not a bounded isometry of  $C_0$  and thus g is not in B(G). Hence B(G) is not closed. Indeed, one can easily show that B(G) is dense in G.

The construction of a rough Cayley graph can be used in the study of FC<sup>-</sup>-elements in groups. Trofimov in [70] proved the following.

**Theorem 4.11.** ([70], cf. [83, Theorem 3]) Let  $\Gamma$  be a vertex transitive, connected, locally finite graph. The subgroup of bounded automorphisms in Aut( $\Gamma$ ) is closed in Aut( $\Gamma$ ).

Using a rough Cayley graph for the group and this theorem allows us to prove the following result.

**Theorem 4.12.** ([52, Theorem 2]) Let G be a compactly generated totally disconnected locally compact group. Then the subgroup of  $FC^-$ -elements is closed in

The proof is simple. One constructs a locally finite rough Cayley graph for G. The action of G on  $\Gamma$  gives a continuous homomorphism  $G \to \operatorname{Aut}(\Gamma)$ . It is easy to show that the kernel of this homomorphism is compact and the image is closed in  $\operatorname{Aut}(\Gamma)$  (see [52, Corollary 1]). The subgroup of bounded automorphism in  $\operatorname{Aut}(\Gamma)$  is closed by the theorem of Trofimov above, and the subgroup of FC<sup>-</sup>-elements is closed in G, because it is the preimage of the subgroup of bounded automorphisms in  $\operatorname{Aut}(\Gamma)$ .

**4.3. Rough ends.** The space of ends of a connected, locally finite graph  $\Gamma$  is the boundary of a certain compactification of  $V\Gamma$ , where we think of  $V\Gamma$  as having the discrete topology. One can think of different ends as representing the "different ways of going to infinity" in  $\Gamma$ . Ends of graphs can both be defined by using topological concepts and by purely graph theoretical means. The graph theoretical method extends to graphs that are not locally finite, but then the ends do not give a compactification of the vertex set like in the locally finite case.

Recall that a *ray* in a graph is a sequence  $\alpha_0, \alpha_1, \ldots$  of distinct vertices such that  $\alpha_i$  is adjacent to  $\alpha_{i+1}$  for all *i*. The graph theoretic approach is to define the ends of a graph  $\Gamma$  as equivalence classes of rays. Two rays are equivalent (i.e. are in the same *end*) if there is the third ray that intersects both infinitely often. This definition goes back to the paper [33] by Halin.

Ends can also be defined with reference to connected components when finite sets of edges are removed from the graph. For a finite set  $\Phi$  of edges define  $C_{\Phi}$ as the set of connected components of  $\Gamma \setminus \Phi$ . Suppose  $\Phi_1$  and  $\Phi_2$  are two finite sets of edges such that  $\Phi_1 \subseteq \Phi_2$ . There is a natural map  $C_{\Phi_2} \to C_{\Phi_1}$  that takes a component c of  $\Gamma \setminus \Phi_2$  to the component of  $\Gamma \setminus \Phi_1$  that contains c. The collection of all the sets  $C_{\Phi}$  where  $\Phi$  ranges over all finite sets of edges in  $\Gamma$ , together with the connecting natural maps forms an inverse system. The space of ends  $\Omega\Gamma$  of  $\Gamma$  (with the natural topology of the inverse limit) is then defined as the inverse limit of this system. One could also look at the components of  $\Gamma \setminus \Phi$  where  $\Phi$  is a finite set of vertices, but when the graph is locally finite the inverse limits are homeomorphic. This approach goes back to the thesis of Freudenthal in 1931. In later works Freudenthal and Hopf built up a theory of ends of spaces, see [28], [29] and [34].

The graph theoretic approach to ends also leads to a natural definition of a topology. The *co-boundary* of a set  $c \subseteq V\Gamma$  is the set of all edges in  $\Gamma$  such that one of its end vertices is in c and the other is in  $V\Gamma \setminus c$ . Denote the co-boundary of c with  $\delta c$ . Suppose  $|\delta c| < \infty$ . One sees that if c contains all but finitely many vertices from a ray R then c also contains also all but finitely many vertices from any ray in the same end as R. Thus we can say that the end belongs to c. Define  $\Omega c$  as the set of ends that belong to c. The topology on the space of ends has as a basis the sets  $\Omega c$  where  $c \subset V\Gamma$  and  $|\delta c| < \infty$ . The graph  $\Gamma \setminus \delta c$  is not connected. If  $\omega_1$  and  $\omega_2$  are two ends of  $\Gamma$  and  $\omega_1$  belongs to c and  $\omega_2$  belongs to  $V\Gamma \setminus c$  then we can say that  $\delta c$  separates  $\omega_1$  and  $\omega_2$ .

It is easy to show that a quasi-isometry between two locally finite connected

graphs induces a homeomorphism between the end spaces of the graphs. In particular the number of ends is a quasi-isometry invariant.

For a detailed introduction to ends of graphs the reader can consult [48] or [16].

**4.3.1. Stallings' Theorem.** For a finitely generated group the number of ends is defined as the number of ends of a Cayley graph of the group with respect to a finite generating set. This is well defined, because the number of ends of a space is invariant under quasi-isometry, as noted above. It can be shown that a finitely generated group has 0, 1, 2 or  $\infty$  many ends.

Let G be a compactly generated, totally disconnected, locally compact group. Since G admits a rough Cayley graph  $\Gamma$ , which is unique up to quasi-isometry, we may define the *space of rough ends* of G as the space of ends of  $\Gamma$ . The cardinality of the space of rough ends of G will be called the *number of rough ends* of G. These definitions obviously extend the traditional concepts for finitely generated groups. Because of Lemma 4.9, every compactly generated, totally disconnected, locally compact group has 0, 1, 2 or  $\infty$  many ends.

A compactly generated, totally disconnected, locally compact group has 0 rough ends, if and only if it is compact (in particular, a finitely generated group has 0 ends if and only if it is finite).

Finitely generated groups with precisely two ends are characterized by the following result, which is a conjunction of results of Hopf and C. T. C. Wall.

**Theorem 4.13.** ([34, Satz 5] and [73, Lemma 4.1]) Let G be a finitely generated group. Then the following are equivalent:

(i) G has precisely two ends;

(ii) G has an infinite cyclic subgroup of finite index;

(iii) G has a finite normal subgroup N such that G/N is either isomorphic to the infinite cyclic group or to the infinite dihedral group.

For compactly generated, totally disconnected, locally compact group we have the following analogue.

**Theorem 4.14.** (Cf. [53]) Let G be a compactly generated, totally disconnected, locally compact group. Suppose that the space of rough ends has precisely two points. Then G has a compact, open, normal subgroup N such that G/N is either isomorphic to the infinite cyclic group or to the infinite dihedral group.

Finitely generated groups with more than one end are described in a famous result of Stallings from 1968.

**Theorem 4.15.** [65] Suppose G is a finitely generated group with more than one end. Then G can be written as a non-trivial free product with amalgamation  $A *_C B$ (with  $A \neq C \neq B$ ) where C is finite, or G can be written as a non-trivial HNNextension  $A *_C x$  where C is finite.

The converse also holds, if  $A *_C B$  (with  $A \neq C \neq B$ ) where C is finite, or G can be written as a non-trivial HNN-extension  $A *_C x$  where C is finite, then G has more than one end.

In 1974 Abels [1] proved an analogue of Stallings' Theorem for topological groups. Abels uses similar ideas in his proof as used by Stallings. Essentially the same result as in [1] is proved in [40], using Dunwoody's theory of structure trees (see [19] and [15]) and the Bass-Serre theory of group actions on trees.

**Theorem 4.16.** (Cf. [1]) Let G be a compactly generated, totally disconnected, locally compact group. Suppose that some (equivalently, any) rough Cayley graph of G has infinitely many ends. Then  $G = A *_C B$  (with  $A \neq C \neq B$ ) or  $G = A *_C x$ , where A and B are open, compactly generated subgroups of G, and C is a compact, open subgroup.

That it is possible to write G as either a free product with amalgamation  $G = A *_C B$  or an HNN-extension  $G = A *_C x$  is often expressed by saying that G splits over C. Using this expression, Stallings theorem becomes the statement that a finitely generated group with infinitely many ends splits over a finite subgroup, and Theorem 4.16 above becomes the statement that a compactly generated, totally disconnected, locally compact group with more than one end splits over a compact, open subgroup. If G is totally disconnected, locally compact group with closed, cocompact subgroup H, then a rough Cayley graphs for G and a rough Cayley graph for H are quasi-isometric by Theorem 4.16 has the following corollary.

**Corollary 4.17.** ([40, Corollary 3.22]) Let G be a totally disconnected, locally compact group and H a closed, cocompact subgroup. Then G splits over a compact, open subgroup if and only if H splits over a compact, open subgroup.

**4.3.2. Free subgroups.** A well known theorem of Gromov (also proved by Woess [82]) says that a finitely generated group is quasi-isometric to a tree if and only if it has a finitely generated free subgroup of finite index.

The next theorem provides an analogue of this result for compactly generated, totally disconnected, locally compact groups.

**Theorem 4.18.** ([40, Theorem 3.28]) Let G be a compactly generated, totally disconnected, locally compact group.

(i) Some (hence, every) rough Cayley graph of G is quasi-isometric to a tree if and only if G has an expression as a fundamental group of a finite graph of groups such that all the vertex and edge groups are compact open subgroups of G.

(ii) Assume also that the group G is unimodular. Then some (hence, every) rough Cayley graph of G is quasi-isometric to some tree if and only if G has a finitely generated free subgroup that is cocompact and discrete.

The proof of Theorem 4.18 uses information about ends, quasi-isometries and structure trees from [82], [39], [41], [47] and [67]. The essential result used about graphs quasi-isometric to trees is that a transitive, locally finite, graph is quasi-isometric to a tree if and only if it has no *thick ends* (cf. [82] and [41, Theorem 5.5]). This property can then be used to show that for a locally finite, transitive graph that is quasi-isometric to a tree one can find a locally finite structure tree on which the automorphism group of the original graph acts.

For the proof of existence of a cocompact, discrete, finitely generated, free subgroup in part (ii) of Theorem 4.18 the theory of tree lattices in [3] is used.

**Corollary 4.19.** ([40, Corollary 3.29]) Let G be a totally disconnected, locally compact group. If G has a cocompact, finitely generated, free, discrete subgroup, then G splits over some compact, open subgroup and G can be written as  $G = A *_C B$  (with  $A \neq C \neq B$ ) or  $G = A *_C x$  where A, B and C are compact, open subgroups of G.

The above Corollary implies a special case of a result of Mosher, Sageev and Whyte [54, Theorem 9].

**Corollary 4.20.** ([40, Corollary 3.30]) Let G be a totally disconnected, locally compact group. If G has a cocompact, finitely generated, free, discrete subgroup, then G has an action on a locally finite tree, such that G fixes neither an edge nor a vertex.

Consider a finitely generated group H and a Cayley graph  $\Gamma$  of H. The action of H on the Cayley graph gives an embedding of H as a closed, cocompact subgroup into the totally disconnected, locally compact group  $G = \operatorname{Aut}(\Gamma)$ . Willis asks in [80, Section 6] whether various invariants of G can be bounded in terms of H. For example, he asks if it is possible to deduce that there are only finitely many prime numbers that occur as factors in  $\mathbf{s}(x)$  for  $x \in G$  (the scale function  $\mathbf{s}$  is discussed in Section 3). This question is motivated by the following result.

**Theorem 4.21.** ([78, Theorem 3.4]) Let G be a compactly generated, totally disconnected, locally compact group. Then there are finitely many primes  $p_1, p_2, \ldots, p_t$ such that the number  $\mathbf{s}_G(x)$  for all elements  $x \in G$  can be written in the form  $\mathbf{s}_G(x) = p_1^{s_1} p_2^{s_2} \cdots p_t^{s_t}$ .

Baumgartner [5] has applied the above mentioned result of Mosher, Sageev and Whyte to the program suggested by Willis. Baumgartner has also extended the scope of the program, by considering not only the special type of embedding  $G \to \operatorname{Aut}(\Gamma)$ , where  $\Gamma$  is a Cayley graph of H. A topological group G is called an *envelope* of a group H, if H embeds as a closed, cocompact subgroup of G.

**Theorem 4.22.** [5, Corollary 11]) Let H be a virtually free group of rank at least 2. Then there are finitely many primes  $p_1, p_2, \ldots, p_t$  such that for all totally disconnected, locally compact envelopes G of H all elements  $x \in G$  the number  $\mathbf{s}_G(x)$  can be written in the form  $\mathbf{s}_G(x) = p_1^{s_1} p_2^{s_2} \cdots p_t^{s_t}$ .

The following result comes from a totally different direction, but it also indicates the possible fruitfulness of the study of envelopes and embeddings of a finitely generated group into automorphism groups of its Cayley graphs.

**Theorem 4.23.** [53, Theorem 4.1]) Let N be a finitely generated, torsion free, nilpotent group and  $\Gamma$  a Cayley graph of N with respect to some finite generating set of N. Put  $G = \operatorname{Aut}(\Gamma)$ . Then G is discrete in the permutation topology, and N embeds into G as normal subgroup.

This theorem is proved with the aid of Theorem 2.18.

#### 4.3.3. Accessibility.

**Definition 4.24.** A finitely generated group is said to be *accessible* if it has an action on a tree T such that:

(i) the number of orbits of G on the edges of T is finite;

(ii) the stabilizers of edges in T are finite subgroups of G;

(iii) every stabilizer of a vertex in T is a finitely generated subgroup of G with at most one end.

The question by C. T. C. Wall, [74], of whether or not every finitely generated group is accessible motivated several important developments in combinatorial group theory, among them Dunwoody's theory of structure trees. Wall's question, after being open for a long time, was settled by examples of finitely generated groups that are not accessible, which were constructed by Dunwoody [21].

**Definition 4.25.** ([67, p. 249]) Let  $\Gamma$  be a connected, locally finite graph. If there is a number k such that any two distinct ends can be separated by removing k or fewer vertices from  $\Gamma$  then the graph  $\Gamma$  is said to be *accessible*.

Thomassen and Woess [67, Theorem 1.1] show that a finitely generated group is accessible if and only if every Cayley graph with respect to a finite generating set is accessible.

The notion of accessibility can be generalized to compactly generated, totally disconnected, locally compact groups as follows.

**Definition 4.26.** A compactly generated, totally disconnected, locally compact group is said to be *accessible* if it has an action on a tree T such that:

(i) the number of orbits of G on the edges of T is finite;

(ii) the stabilizers of edges in T are compact, open subgroups of G;

(iii) every stabilizer of a vertex in T is a compactly generated, open subgroup of G with at most one rough end.

Then one can link accessibility of compactly generated totally disconnected locally compact groups to rough Cayley graphs in analogues way as Thomassen and Woess link accessibility of finitely generated groups to Cayley graphs.

**Theorem 4.27.** ([40, Theorem 3.27]) Let G be a compactly generated, totally disconnected, locally compact group. Then G is accessible if and only if every rough Cayley graph of G is accessible.

In fact, the group in Theorem 4.27 is accessible if and only if any of its rough Cayley graphs is accessible, because the property of a graph being accessible is invariant under quasi-isometries by [58, Theorem 0.4]. By the same result, a compactly generated, totally disconnected, locally compact group with a closed, cocompact, accessible subgroup is itself accessible. Since every finitely presentable group is accessible by a result of Dunwoody [20], we deduce the following theorem as a corollary.

**Theorem 4.28.** Let G be a compactly generated, totally disconnected, locally compact group. If G has a cocompact, finitely presented subgroup then G is accessible.

**4.3.4.** Ends of pairs of groups. By Stallings' Theorem the ends of a finitely generated group can be used to detect if the group splits over a finite subgroup. The concept of ends of pairs of groups is an attempt to define a geometric invariant that can be used to detect splittings of G over subgroups that are not finite. This concept was first introduced in the papers [35] by Houghton and [63] by Scott. For a survey of these and related concepts see [75].

**Definition 4.29.** (Cf. [63, Lemma 1.1]) Let G be a finitely generated group and C a subgroup of G. Let  $\Gamma$  be a Cayley graph of G with respect to some finite generating set. The *number of ends of the pair* (G, C), denoted with e(G, C), is defined as the number of ends of the quotient graph  $C \setminus \Gamma$  (quotient with respect to the natural C-action on  $\Gamma$ ).

It can be shown that the number of ends of a pair of groups does not depend on the choice of generating set. While a transitive, locally finite graph has 0, 1, 2 or infinitely many ends, a pair of groups can have any number of ends, see [63, Example 2.1].

The following conjecture generalizing Stallings' Theorem is due to Kropholler, see [57].

**Conjecture 4.30.** Let G be a finitely generated group and C a subgroup of G. If G contains a subset A such that

(i) A = CA;

(ii) for every element  $g \in G$  the symmetric difference of a and Ag is contained in a finite union of right C cosets;

(iii) neither A nor  $G \setminus A$  is contained in any finite union of right C cosets; (iv) A = AC

then G splits over a subgroup that is commensurable with a subgroup of C. (Conditions (i)-(iii) above are equivalent to  $e(G,C) \ge 2$ .)

Here, two subgroups are are said to be *commensurable*, if their intersection has finite index in both subgroups. Furthermore the *commensurator* of a subgroup C of G is the set of elements  $g \in G$  such that C and  $gCg^{-1}$  are commensurable. The commensurator of a subgroup is itself a subgroup. Kropholler's conjecture above has been verified under various additional hypotheses. A sample result is the following theorem.

**Theorem 4.31.** ([22, p. 30]) Let G be a finitely generated group and C a finitely generated subgroup of G. If e(G, C) > 1 and the commensurator of C is the whole group G then G splits over a subgroup commensurable with C.

This result has also been proved in papers by Niblo [56, cf. Theorem B] and Scott and Swarup [64, Theorem 3.12].

In [40, Section 3.7] there is further discussion of how the concepts of ends of pairs of groups, coends and rough ends relate and how these concepts can be interpreted graph theoretically. Amongst other things these considerations lead to a prove of Theorem 4.31 above.

**4.4.** Polynomial growth. Recall from Section 2.3 that a connected, locally finite graph is said to have *polynomial growth* if for every vertex the number of vertices in distance less than or equal to n grows polynomially with n. A finitely generated group is said to have polynomial growth if some (hence, every) Cayley graph with respect to a finite generating set has polynomial growth.

The concept of polynomial growth can be generalized to compactly generated, locally compact groups.

**Definition 4.32.** Let G be a locally compact group generated by a compact symmetric neighbourhood of the identity V. Set  $V^n = \{g_1g_2\cdots g_n \mid g_i \in V\}$ . Let  $\mu$  denote a Haar measure on G. If there are constants c and d such that  $\mu(V^n) \leq cn^d$  for all natural numbers n, G is said to have polynomial growth.

The following theorem characterizes compactly generated, totally disconnected groups of polynomial growth in terms of their rough Cayley graphs.

**Theorem 4.33.** ([40, Theorem 4.4]) Let G be a compactly generated, totally disconnected, locally compact group and  $\Gamma$  some rough Cayley graph for G. Then  $\Gamma$  has polynomial growth if and only if G has polynomial growth (in the sense of Definition 4.32).

Viewed in this context, Trofimov's theorem about automorphism groups of graphs with polynomial growth, Theorem 2.17, can now be seen as a version of Gromov's Theorem for compactly generated, totally disconnected, locally compact groups. As already mentioned, Losert proved a generalization of Gromov's Theorem for topological groups in [42] and Woess deduced Trofimov's theorem from Lostert's result in [83]. The following theorem [40, Theorem 4.6] is a combination of Theorem 4.33 and Trofimov's theorem, but can also be seen as Corollary to Losert's results.

**Theorem 4.34.** (Cf. Trofimov [69] and Losert [42]) Let G be a compactly generated, totally disconnected, locally compact group. Then G has polynomial growth if and only if G has a normal, compact, open subgroup K such that G/K is a finitely generated almost nilpotent group.

### References

- H. Abels, Specker-Kompaktifizierungen von lokal kompakten topologischen Gruppen. Math. Z. 135 (1974), 325–361.
- [2] M. Abert and Y. Glasner, Generic groups acting on regular trees. Trans. Amer. Math. Soc. 361 (2009), 3597–3610.
- [3] H. Bass and R. Kulkarni, Uniform tree lattices. J. Amer. Math. Soc. 3 (1990), 843– 902.
- [4] H. Bass and A. Lubotzky, *Tree lattices*. Progress in Mathematics, 176. Birkhäuser Boston, Boston, 2001.

- [5] U. Baumgartner, Scales for co-compact embeddings of virtually free groups. Geom. Dedicata, 130 (2007), 163–175.
- [6] U. Baumgartner and G. Willis, The direction of an automorphism of a totally disconnected locally compact group. *Math. Z.* 252 (2006), 393–428.
- [7] M. Bhattacharjee, The probability of generating certain profinite groups by two elements. Israel J. Math. 86 (1994), 311–329.
- [8] M. Bhattacharjee, The ubiquity of free subgroups in certain inverse limits of groups. J. Algebra 172 (1995), 134–146.
- [9] M. Bhattacharjee, D. Macpherson, R. G. Möller and P. M. Neumann, Notes on Infinite Permutation Groups. Hindustan Book Agency, Delhi, India, 1997. (Republished as Springer Lecture Note in Mathematics 1698, Springer, 1998.)
- [10] G. M. Bergman and H. W. Lenstra, Subgroups close to normal subgroups. J. Algebra 127 (1989), 80–97.
- [11] P. J. Cameron, Oligomorphic permutation groups. London Mathematical Society Lecture Note Series 152, Cambridge University Press, Cambridge, 1990.
- [12] P. J. Cameron, *Permutation groups*. London Mathematical Society Student Texts 45, Cambridge University Press, Cambridge, 1999.
- [13] P. J. Cameron, C. E. Praeger and N. C. Wormald, Infinite highly arc transitive digraphs and universal covering digraphs. *Combinatorica* 13 (1993), 377–396.
- [14] D. van Dantzig, Zur topologischen Algebra III. Brouwersche und Cantorsche Gruppen. Compos. Math. 3 (1936), 408–426.
- [15] W. Dicks and M. J. Dunwoody, *Groups acting on graphs*. Cambridge University Press, Cambridge, 1989.
- [16] R. Diestel, D. Kühn, Graph-theoretical versus topological ends of graphs. J. Combin. Theory Ser. B 87 (2003), 197–206.
- [17] R. Diestel and I. Leader, A conjecture concerning a limit of non-Cayley graphs. J. Algebraic Combin. 14 (2001), 17–25.
- [18] J. D. Dixon and B. Mortimer, *Permutation Groups*. Graduate text in mathematics 163, Springer 1996.
- [19] M. J. Dunwoody, Cutting up graphs. Combinatorica 2 (1982), 15–23.
- [20] M. J. Dunwoody, The accessibility of finitely presented groups. Invent. Math. 81 (1985), 449–457.
- [21] M. J. Dunwoody, An inaccessible group. In *Geometric Group Theory 1991, Vol. 1* (ed. by G. A. Niblo and M. A. Roller), 75–78, L.M.S. Lecture Notes Series 181, Cambridge University Press, Cambridge, 1993.
- [22] M. J. Dunwoody and M. A. Roller Splitting groups over polycyclic-by-finite subgroups. Bull. London Math. Soc. 25 (1993), 29–36.
- [23] A. Eskin, D. Fisher, K. Whyte, Quasi-isometries and rigidity of solvable groups. Pure Appl. Math. Q. 3 (2007), 927-947.
- [24] A. Eskin, D. Fisher, K. Whyte, Coarse differentiation of quasi-isometries I: spaces not quasi-isometric to Cayley graphs. preprint 2006, arXiv:math/0607207v2.
- [25] D. M. Evans, An infinite highly arc-transitive digraph. Europ. J. Combin. 18 (1997), 281–286.

- [26] D. M. Evans, Suborbits in infinite primitive permutation groups. Bull. London Math. Soc. 33 (2001), 583–590.
- [27] A. Figà-Talamanca and C. Nebbia, Harmonic analysis and representation theory for groups acting on homogeneous trees. London Mathematical Society Lecture Note Series, 162. Cambridge University Press, Cambridge, 1991.
- [28] H. Freudenthal, Neuaufbau der Endentheorie. Ann. of Math. (2) 43 (1942), 261–279.
- [29] H. Freudenthal, Über die Enden diskreter Räume und Gruppen. Comm. Math. Helv. 17 (1945), 1–38.
- [30] M. Gromov, Groups of polynomial growth and expanding maps, *Publ. Math. IHES*, 53 (1981), 53–78.
- [31] M. Gromov, Infinite groups as geometric objects. In Proceedings of the International Congress of Mathematicians Vol. 1, 385–392, PWN, Warsaw, 1984.
- [32] S. Grosser and M. Moskowitz, Compactness conditions in topological groups. J. Reine Angew. Math. 246 (1971), 1–40.
- [33] R. Halin, Über unendliche Wege in Graphen. Math. Ann. 157 (1964), 125–137.
- [34] H. Hopf, Enden offener Räume und unendliche diskontinuierlich Gruppen. Comment. Math. Helv. 16 (1944), 81–100.
- [35] C. H. Houghton, Ends of locally compact groups and their coset spaces. J. Austral. Math. Soc. 17 (1974), 274–284.
- [36] A. A. Ivanov, Bounding the diameter of a distance-regular graph. Soviet Math. Doklady 28 (1983), 149–152. (Translation from Dokl. Akad. Nauk SSSR 271 (1983), 789–792.)
- [37] K. Iwasawa, Topological groups with invariant compact neighborhoods of the identity. Ann. of Math. (2) 54 (1951), 345–348.
- [38] A. Karrass and D. Solitar, Some remarks on the infinite symmetric groups. Math. Z. 66 (1956), 64–69.
- [39] B. Krön, Quasi-isometries between non-locally-finite graphs and structure trees. Abh. Math. Sem. Univ. Hamburg 71(2001), 161–180.
- [40] B. Krön and R. G. Möller, Analogues of Cayley graphs for topological groups, Math. Z. 258 (2008), 637-675.
- [41] B. Krön and R. G. Möller, Quasi-isometries between graphs and trees. J. Combin. Theory Ser. B 98 (2008), 994–1013.
- [42] V. Losert, On the structure of groups with polynomial growth. Math. Z. 195 (1987), 109–117.
- [43] V. Losert, On the structure of groups with polynomial growth II. J. London Math. Soc. (2) 63 (2001), 640–654.
- [44] H. D. Macpherson, Infinite distance transitive graphs of finite valency. *Combinatorica* 2 (1982), 63–69.
- [45] A. Malnič, D. Marušič, R. G. Möller, N. Seifter, V. Trofimov and B. Zgrablič, Highly arc transitive digraphs: reachability, topological groups. *European J. Combin.* 26 (2005), 19–28.
- [46] I. Maurer, Les groupes de permutations infinies. Gaz. Mat. Fiz. Ser. A. 7 (1955), 400–408.

- [47] R. G. Möller, Ends of graphs II. Math. Proc. Camb. Phil. Soc. 111 (1992), 455-460.
- [48] R. G. Möller, Groups acting on locally finite graphs a survey of the infinitely ended case. In *Groups'93 Galway/St Andrews, Vol. 2* (ed. by C. M. Campbell, T. C. Hurley, E. F. Robertson, S. J. Tobin and J. J. Ward), 426–456, L.M.S. Lecture Notes Series 212, Cambridge University Press, Cambridge, 1995.
- [49] R. G. Möller, Topological groups, automorphisms of infinite graphs and a theorem of Trofimov. *Discrete Math.* 178 (1998), 271–275.
- [50] R. G. Möller, Structure theory of totally disconnected locally compact groups via graphs and permutations. *Canad. J. Math.* 54 (2002), 795–827.
- [51] R. G. Möller, Descendants in highly arc transitive digraphs. Discrete Math. 247 (2002), 147–157. (Erratum. Discrete Math. 260 (2003), 321.)
- [52] R. G. Möller, FC<sup>-</sup>-elements in totally disconnected groups and automorphisms of infinite graphs. *Math. Scand.* 92 (2003), 261–268.
- [53] R. G. Möller and N. Seifter, Digraphical regular representations of infinite finitely generated groups. *European J. Combin.* 19 (1998), 597–602.
- [54] L. Mosher, M. Sageev, K. Whyte, Quasi-actions on trees. I. Bounded valence. Ann. of Math. (2), 158 (2003), 115-164.
- [55] C. Nebbia, Minimally almost periodic totally disconnected groups. Proc. Amer. Math. Soc. 128 (2000), 347–351.
- [56] G. A. Niblo, The singularity obstruction for group splittings. Topology Appl. 119 (2002), 17–31.
- [57] G. A. Niblo and M. Sageev, The Kropholler conjecture. In Guido's book of conjectures (ed. by Indira Chatterji), 2006. http://www.math.ohio-state.edu/~indira/GMFinal.pdf.
- [58] P. Papasoglu, K. Whyte, Quasi-isometries between groups with infinitely many ends. Comment. Math. Helv. 77 (2002), 133–144.
- [59] C. E. Praeger, On homomorphic images of edge transitive directed graphs. Australas. J. Combinatorics 3 (1991), 207–210.
- [60] G. Sabidussi, Vertex-transitive graphs. Monatsh. Math. 68 (1964), 426–438.
- [61] G. Schlichting, Polynomidentitäten und Permutationsdarstellungen lokalkompakter Gruppen. Invent. Math. 55 (1979), 97–106.
- [62] G. Schlichting, Operationen mit periodischen Stabilisatoren. Arch. Math. 34 (1980), 97–99.
- [63] P. Scott, Ends of pairs of groups. J. Pure Appl. Algebra 11 (1977/78), 179–198.
- [64] P. Scott and G. A. Swarup, Splittings of groups and intersection numbers. Geom. Topol. 4 (2000), 179–218.
- [65] J. R. Stallings, On torsion free groups with infinitely many ends. Ann. of Math. 88 (1968), 312–334.
- [66] J. Tits, Automorphismes à déplacement borné des groupes de Lie. Topology 3 Suppl. 1 (1964), 97–107.
- [67] C. Thomassen and W. Woess, Vertex-transitive graphs and accessibility. J. Combin. Theory Ser. B 58 (1993), 248–268.

- [68] V. I. Trofimov, Automorphisms of graphs and a characterization of lattices. Math USSR Izv. 22 (1984), 379–392. (Translation from Izv. Akad. Nauk SSSR Ser. Mat. 47 (1983), 407–420.)
- [69] V. I. Trofimov, Graphs with polynomial growth. Math USSR Sb. 51 (1985), 405–417.
   (Translation from Mat. Sb. (N.S.) 123 (1984), 407–421.)
- [70] V. I. Trofimov, Automorphism groups of graphs as topological groups. Math. Notes 38 (1985), 717–720. (Translation from Mat. Zametki 38 (1985), 378–385, 476.)
- [71] V. I. Trofimov, The action of a group on a graph. Math USSR Izv. 29 (1987), 429–447. (Translation from Izv. Akad. Nauk SSSR Ser. Mat. 50 (1986), 1077–1096.)
- [72] V. I. Trofimov, Vertex stabilizers of graphs and tracks. I. European J. Combin. 28 (2007), 613–640.
- [73] C. T. C. Wall, Poincaré complexes. I. Ann. of Math. (2) 86 (1967), 213-245.
- [74] C. T. C. Wall, Pairs of relative cohomological dimension one. J. Pure Appl. Algebra 1 (1971), 141–154.
- [75] C. T. C. Wall, The geometry of abstract groups and their splittings. Rev. Mat. Complut. 16 (2003), 5–101.
- [76] G. Willis, The structure of totally disconnected, locally compact groups. Math. Ann. 300 (1994), 341–363.
- [77] G. Willis, Totally disconnected groups and proofs of conjectures of Hofmann and Mukherjea. Bull. Austral. Math. Soc. 51 (1995), 489–494.
- [78] G. Willis, The number of prime factors of the scale function on a compactly generated group is finite. Bull. London Math. Soc. 33 (2001), 168–174.
- [79] G. Willis, Further properties of the scale function on a totally disconnected group. J. Algebra 237 (2001), 142–164.
- [80] G. Willis, A canonical form for automorphisms of totally disconnected locally compact groups. in *Random walks and geometry* (ed. by V. A. Kaimanovich in collaboration with K. Schmidt and W. Woess), 295–316, Walter de Gruyter GmbH & Co. KG, Berlin, 2004.
- [81] J. A. Wolf, Growth of finitely generated solvable groups and curvature of Riemannian manifolds. J. Differential Geometry 2 (1968), 421–446.
- [82] W. Woess, Graphs and groups with tree-like properties. J. Combin. Theory Ser. B 47 (1989), 361–371.
- [83] W. Woess, Topological groups and infinite graphs. In Directions in Infinite Graph Theory and Combinatorics. (ed. by R. Diestel), Topics in Discrete Math. 3, North Holland, Amsterdam 1992. (Also in Discrete Math. 95 (1991), 373–384.)
- [84] W. Woess, Random walks on infinite graphs and groups. Cambridge Tracts in Mathematics, 138. Cambridge University Press, Cambridge, 2000.
- [85] T. S. Wu and Y. K. Yu, Compactness properties of topological groups. *Michigan Math. J.* 19 (1972), 299–313.
- [86] ] T. S. Wu, On the structure of certain locally compact topological groups. Trans. Amer. Math. Soc. 325 (1991), 413–434.

Rögnvaldur G. Möller, Science Institute, University of Iceland, Dunhaga 3, IS-107 Reykjavk, Iceland

E-mail: roggi@raunvis.hi.is