Exploring students' mathematical meaning making in an upper secondary school classroom with a teacher emphasising dialogues and the use of technology

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EXPLORING STUDENTS' MATHEMATICAL MEANING MAKING IN AN UPPER SECONDARY SCHOOL CLASSROOM WITH A TEACHER EMPHASISING DIALOGUES AND THE USE OF TECHNOLOGY

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Thesis submitted in partial fulfilment of a
Philosophiae Doctor degree in Mathematics Education

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Abstract

Many students experience mathematics as meaningless and irrelevant to their lives outside the school system. In response, mathematics educators have called for more emphasis on meaning making and dialogue in the classroom, and some have argued that technology can transform school mathematics. Still, traditional methods prevail, technology has had a limited impact on teaching, and not much research exists on the daily use of technology for dialogical teaching in classrooms.

This doctoral study addresses the concerns above by reporting on a collaboration with a teacher committed to a pedagogy based on drawing students into mathematical dialogue while exploring, conjecturing, and convincing others. For one semester, the researcher worked with and observed the teacher and his Icelandic 11th-grade class of upper-secondary students, most of whom had not performed well in their previous mathematics classes. The research focused on exploring students' meanings when discussing specific mathematics tasks. These tasks were mainly dynamic geometry software tasks, some of which were designed collaboratively by the teacher and the researcher.

The class was video recorded, and students' communication and actions were analysed from a dialogical viewpoint to interpret the meanings they made of the tasks and to what extent their role as school students influenced their engagement with mathematics. This analysis shows that within the frame of a first-year course in upper secondary school, the teacher can engage students in genuine mathematical dialogue in which students experience the satisfaction of making mathematical meaning, even students who are disaffected with mathematics beforehand. It illustrates how dynamic geometry tasks can be used for this purpose while showing that meaning making is far from linear.

The following tensions that are inherent in mathematics teaching are highlighted in the thesis: between mathematical abstraction and the particularities of real-world context; between mathematical explicitness and the ordinary reliance on the other to do some work to understand what one means; between mathematical exploration-conjecturing-convincing and the completion of required tasks; and between basing conclusions on what seems readily apparent and on reasoning based on properties.

It is argued that a dialogic stance and sensitisation to the complexities of students entering and adopting mathematical discourse are crucial. This involves students developing ways to represent relations with increasing precision when they must make their thinking clear and convincing to others. The teacher needs to connect the students' ways with canonical ways of expressing relations, including those that can be used to create computer screen objects.

Ágrip

Merkingarsköpun nemenda í stærðfræði í framhaldsskóla með kennara sem leggur áherslu á samræður og notkun tölvutækni

Margir nemendur upplifa stærðfræði sem tilgangslausa og óviðkomandi lífi sínu utan skólakerfisins. Því hafa stærðfræðimenntafræðingar kallað eftir meiri áherslu á merkingarsköpun og samræður í stærðfræði og sumir hafa rökstutt að tölvutækni geti umbreytt stærðfræðikennslu. Hefðbundnar aðferðir eru þó enn ríkjandi og tækni hefur haft takmörkuð áhrif á kennslu.

Doktorsrannsóknin beinist að merkingarsköpun íslenskra framhaldsskólanema á fyrsta ári í stærðfræðitímum. Flestir nemendur í hópnum höfðu ekki náð góðum árangri í stærðfræði áður samkvæmt viðmiðum skólakerfisins. Kennslufræði kennarans byggði á því að draga nemendur inn í samræður um stærðfræðiverkefnin. Slíkar samræður ganga út á að rannsaka saman spurningar, setja fram tilgátur og sannfæra aðra um þær. Sum verkefnin voru sérstaklega hönnuð í samstarfi kennara og rannsakanda til að stuðla að merkingarsköpun í samræðum, sér í lagi verkefni þar sem nemendur notuðu kvikt rúmfræðiforrit.

Hópurinn var tekinn upp á myndband yfir eina önn og samskipti og athafnir nemenda voru greind út frá samræðusjónarmiði (dialogism) til að túlka þá merkingu sem nemendur sköpuðu saman í verkefnavinnunni. Greiningin sýnir hvernig kennari getur dregið nemendur inn í stærðfræðilega samræðu þar sem nemendur upplifa ánægju af því að skapa stærðfræðilega merkingu jafnvel þó að nemendur hafi haft neikvæða reynslu af stærðfræði áður. Hún sýnir einnig að slík merkingarsköpun er langt frá því að vera línulegt ferli heldur einkennist hún af ýmsum togstreitum.

Meðal annars takast á sjónarhorn hlutfirringar (abstraction) og sjónarhorn „raunveruleikans", viðmið stærðfræðinnar um skýrleika togast á við hversdagsleg viðmið, þar sem ætlast er til að viðmælendur geti í eyðurnar til að skilja hvað aðrir eiga við, hugmyndir um stærðfræði sem ferli könnunar-tilgátna-sannfæringar takast á við markmið nemenda um að ljúka skylduverkefnum, og tilhneiging fólks til að álykta út frá því sem virðist sjónrænt ljóst togast á við viðmið stærðfræðinnar um að rökstyðja ályktanir út frá skilgreindum eiginleikum.

Samkvæmt niðurstöðum skiptir sköpum fyrir kennara að taka afstöðu með samræðunni. Þeirri afstöðu verður að fylgja næmni á það flókna ferli sem það er að tileinka sér stærðfræðilega orðræðu. Það ferli felur í sér að nemendur þróa sínar eigin leiðir til að lýsa venslum á sífellt nákvæmari hátt vegna þess að þeir þurfa að gera hugsunina öðrum ljósa. Einnig þarf kennari að tengja leiðir nemenda við

hefðbundnar leiðir stærðfræðinnar til að tjá tengsl, þar á meðal þær sem hægt er að nota til að búa til grafísk líkön með stærðfræðihugbúnaði.

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List of Publications

This thesis is based on the following publications, referred to by their Roman numerals:

I **Gíslason, I.** (2019). Centripetal and centrifugal forces in teacher-class dialogues in inquiry-based mathematics. In U. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 1680–1687). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME. https://hal.archives-ouvertes.fr/hal-02435309/

II **Gíslason, I.** (2021). Discussing Dependencies of Variable Points on the Basis of a GeoGebra Task: Meaning Making in a Teacher-Class Dialogue. *Digital Experiences in Mathematics Education, 7*(2), 301–322. https://doi.org/10.1007/s40751-021-00087-7

III **Gíslason, I.** (2022). Persuasive moments, and interaction between authoritative discourse and internally persuasive discourse when using GeoGebra. In J. Hodgen, E. Geraniou, G. Bolondi & F. Ferretti (Eds.), *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12)*. Free University of Bozen-Bolzano and ERME. https://hal.archives-ouvertes.fr/hal-03745960

IV **Gíslason, I.** (2023). Interactions and tensions between mathematical discourses and schoolwork discourses when solving dynamic geometry tasks: what is internally persuasive for students? Accepted for publication in *Research in Mathematics Education*

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Co-authorship: The research project design, data collection, data analysis and paper writing were conducted by me, the doctoral candidate. Ólafur Páll Jónsson and Simon Goodchild supported my writing by discussing my papers with me and giving me written feedback.

Abbreviations

Clarification of terms and abbreviations in this thesis.

Term	Clarification
Students	learners in general, regardless of age or school stage
Teachers	teaching professionals in schools

Abbreviation	Meaning
AD	Authoritative discourse
CERME	Congress of the European Society for Research in Mathematics Education
DGS	Dynamic geometry software
IBME	Inquiry-based mathematics education
IPD	Internally persuasive discourse
NORMA	Nordic Conference on Mathematics Education

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A text is never the production of a single individual, and this thesis is a part and a product of multiple dialogues. It draws on the words and actions of many people, some of which I have the opportunity to thank in this section.

I owe a debt of gratitude to the teacher who participated in the project by initiating our collaboration, allowing me into his classroom, and discussing the challenges of teaching. It is far from a matter of course to open the classroom so entirely to a researcher. Thank you for your fundamental role in making this research possible. I also thank the students at the school, who were willing to be recorded as they lived their classroom lives, with all the complexities of learning mathematics, navigating school, and conducting their social lives.

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1. Introduction

In this chapter, I present the roots and aims of the doctoral study. I argue for the importance of the research and explain the context of the Icelandic school system. Finally, the introduction concludes with an outline of the thesis.

The research comprises a qualitative case study of mathematical meaning making in classroom dialogues. In this dissertation, there is more to be said about the concept of dialogue, as it is embedded in a theory of *dialogism*. Dialogism will be further explained in the section on theoretical perspectives, but there are two aspects of dialogue that might be useful to keep in mind in reading this introduction. First, every interaction is seen as both a response to previous interactions (with those co-present, but also indirectly to more distant sources in time, place and media) and an invitation that anticipates further interactions. Second, meaning is seen as continually being constructed and developed in dialogue by the speakers: there are no final points where the meanings of interactions are settled. These two aspects are part of what in this text is meant by a *dialogical* view on dialogue.

The subject (focus) of the study is a particular class of students and a teacher, with some distinguishing features of interest. The students were in their first year in an Icelandic upper-secondary school (11th grade) and in the course, the students worked to a substantial extent on dynamic geometry tasks. The teacher was explicitly committed to the idea that developing language and other representational means is fundamental to mathematics learning. This idea lead the teacher to emphasise having whole class discussions and pair and small group work, where the point was for students to feel the need for inventing, refining, and mastering mathematical representations. This frequently involved teacher-student dialogues where the teacher tried to build on students' contributions. Thus the case can be seen as an instance of a classroom with a dialogic approach and as a case of a classroom where substantial use is made of dynamic geometry software or both. From another perspective, it can be seen as a case of a classroom of a teacher seeking an alternative to mainstream practice for his struggling students, and a case of a teacher-researcher collaboration aiming for improvements to mainstream practice.

1.1. The roots of the project

In the first years of my own teaching career, I often wondered about things that I viewed as being "wrong" with my students, or at least attitudes that made my work very difficult and painful. For example, I saw my students as lacking in curiosity and wonder and being excessively concerned with practical utility and grades. I could not fathom why they would copy homework solutions from each other, as to me it appeared obviously useless for learning. I did not understand why they wanted clear procedures to follow rather than puzzles to think about. Why did they not realise that they would quickly forget some details of the procedures and therefore make mistakes on exams, even on completely standard calculation tasks?

My own experience as a mathematics student seemed not to help me directly in connecting with my students. This could of course be related to differences in societal positions or home culture. For example, being male and able-bodied and having parents and an extended family where university degrees are common and expected, affects one's approach to studying. But just to make clear that this isn't always so simple, my students were generally not of low socioeconomic status. They were predominantly from the upper echelons of society, all white and spoke Icelandic both in school and at home, the school being one of the elite upper-secondary schools in the country (see section 1.5).

For whatever reasons my own schooling until the university level was rather smooth, and I enjoyed being challenged (in fact this sometimes clashed with sentiments from my classmates in school). At university, mathematics became hard for me, but as with many mathematical people, I saw this as a natural feature of interesting endeavours, not as something to be lamented. Although I sometimes experienced almost debilitating exam anxiety in my university studies, I never lost my mathematical curiosity and belief that struggling with difficult tasks was the only way to truly learn mathematics. In that sense, my vision of mathematics and mathematics learning was always that it was about inquiry than the transfer of information (see more about inquiry below). But I found myself not being able to inspire my own students (except a minority) to enjoy the challenges.

After a few years of puzzlement, I gradually realised that when I focused on truly listening to what my students had to say, rather than explaining and telling them what to do, they were more willing to engage with challenging mathematical tasks. By listening, I also developed much better insight into their thinking, and I found that they often had good ideas, even students who had not done well on tests. These were perhaps the beginnings of my fascination with dialogue in mathematics education and my deep belief in dialogical relationships in teaching.

Still, many things perplexed me. Some students were resistant to the idea that their role was not simply to copy and imitate my methods but to have ideas to express. Frequently, I found it difficult to understand what it was that they didn't understand when we were discussing. At other times, when looking through the textbooks, I found myself agreeing with the students and saw that for novices, the material would be very hard to understand. So much seemed to be assumed about the students' already existing understanding. Thus I both agreed and disagreed with Poincaré (1914, p. 46), who wondered how it could be that there are people who do not understand mathematics since it invokes only rules of logic accepted by everyone? Sometimes I felt like Poincaré, but often I felt exactly the opposite: how on earth can anybody come to understand mathematics at all?

The roots of the project lie not only in my personal experience but also in my readings of research and polemics. In the field of mathematics education and education in general, there is a long history of critique of traditional teaching, teaching that proceeds by directly telling students what is true and how to perform predetermined techniques. Vygotsky (2012, p. 159) for example wrote in 1934:

> Practical experience also shows that direct teaching of concepts is impossible and fruitless. A teacher who tries to do this usually accomplishes nothing but empty verbalism, a parrotlike repetition of words by the child, simulating a knowledge of the corresponding concepts but actually covering up a vacuum.

This type of teaching is often linked to the prevalent unsatisfactory experiences of students in mathematics. The students are alienated from the subject, it has little meaning for them, yet they are forced to sit in mathematics classes. There is a wealth of proposed approaches to teaching that are meant to improve the situation and make mathematics more meaningful to students. For this thesis, the most pertinent ideas are:

1. learning through dialogue

2. working on dynamic representations of mathematical objects and relations

3. inquiry: grappling with problems without obvious recipes for solving.

Learning through dialogue relates to the suggestion that students should talk more and write more and that the teacher should listen better and try to understand the students' thinking rather than rushing to evaluate or correct their contributions. The teacher and the students initiate questions and responses that then give rise to new questions and, through these questions, develop meanings (that are never totally finalised). In this way, students should experience the power of using mathematics as a means of communication. They become mathematicians by entering the discourse of mathematics. This perspective is broadly in line with proposals about *dialogic teaching*, derived from the ideas of Bakhtin on dialogue. According to Bakker et al. (2015), such approaches emphasise teaching *for* dialogue as well as teaching through dialogue.

Working on dynamic representations of mathematical objects and relations relates to the fact that understanding mathematical phenomena involves expressing them in different ways, revealing their different aspects, while simultaneously grasping the relations between these different ways of expression (Duval, 2006). For example, looking at a plotted graph of a function instantly gives the mathematician other information about the function than an algebraic symbolic expression defining the function. Dynamic computer software can show different aspects of mathematical phenomena simultaneously, and varying features in one mode instantly change the corresponding features in other modes. Computer programming (and for me, this includes giving commands to DGS) offers a direct way to use mathematics to create interesting visual objects. Thus it is potentially motivating and an opportunity to immerse oneself in a mathematics-conceptual world. This was my experience when I learned to program in my youth, and it is forcefully argued by Papert (1980) in his book *Mindstorms*.

Inquiry, that is, *grappling with problems without obvious recipes for solving*, for me, relates to the idea that in order to learn mathematics, students need to think hard about the meaning of mathematical concepts and their interrelationships, and that it is not enough to listen to other's explanations and imitate ready-made solution methods (although imitation is often a valuable step in the process of learning). These are points made in countless publications, and as I see it, this emphasis on grappling with problems rather than completing exercises is well aligned with *Inquiry-based mathematics education* (IBME), which refers to

> a student-centred paradigm of teaching mathematics and science, in which students are invited to work in ways similar to how mathematicians and scientists work. This means they have to observe phenomena, ask questions, and look for mathematical and scientific ways to answer these questions (like carrying out experiments, systematically controlling variables, drawing diagrams, calculating, looking for patterns and relationships, and making conjectures and generalizations), interpret and evaluate their solutions, and communicate and discuss their solutions effectively (Dorier & Maass, 2014, p. 300).

IBME can be seen as including learning through dialogue as well as learning through working on dynamic representations of mathematical objects and relations, as I have described. When the main topic of the dialogue is working out ways to answer questions as described above, I call it an *inquiry-focused dialogue*.

In summary, my experience and readings led me to believe in the potential of combining the dialogic and the digital in an inquiry-based pedagogy of mathematics. A fundamental element of such a pedagogy is the eliciting of inquiry-focused dialogues that are meant to foster both the learning of mathematics and learning to participate in mathematical dialogue, that is, teaching both through dialogue and for dialogue.

While there is extensive theoretical and empirical work that suggests that these approaches can support meaningful mathematics learning, their practice has also been viewed as difficult for teachers. Student discussions do not necessarily involve or lead to mathematical meaning making (Sfard et al., 1998). Being offered good questions does not necessarily lead to good thinking. Working on DGS tasks does not necessarily lead to students making connections between different modes of representing mathematical phenomena (Guin & Trouche, 1998). In any case, my personal knowledge of mathematics teaching in Iceland and published research from elsewhere indicate that substantial use (where students work intensively with computers) of DGS in mathematics classrooms is rare (Bozkurt & Ruthven, 2018). Therefore, there is a need for detailed accounts of these approaches being used in the naturalistic settings of a classroom in an ordinary school.

1.2. Teacher-researcher collaboration

This research was only possible because a teacher contacted me around the time I started my doctoral studies, suggesting that we discuss ideas for transforming mathematics teaching. He had been my student in a mathematics education course during his teacher training a few years earlier and had found that the ideas presented there resonated with him. We found that our general outlook and view toward education were well aligned, and we agreed that we would discuss the teaching and learning happening in his classroom while I collected data and researched the classroom. His reflections on his experience as a student and teacher led him to conclude that traditional mathematics instruction did not work well, and he felt that the students that entered his classes were disaffected and alienated from the subject. This resonated with my own thinking, as I had felt the need, after ten years of teaching, to research ways to improve students' experience of mathematics, at least for my own teaching and others who might feel similarly. In light of the paucity of teachers who I felt shared my vision, I was very excited to have the opportunity to do research in his classroom. It should be noted that the descriptions above of IBME are my constructions and not the teacher's, nor did we agree on any explicitly defined approach to teaching. Yet, while our ideas about mathematics teaching may not be perfectly aligned, we consider it fair to say that we shared the broad ideal of mathematics learning as a project of meaning making which happens in substantial part through dialogue.

1.3. Pedagogical designs for mathematical dialogue

To support our aim of creating a space for dialogue and productive use of DGS in the classroom, we selected, adapted, and designed tasks for this purpose to supplement ordinary textbook tasks that were also used in the course. The tasks that the students worked on in the dialogues that are analysed in the papers of this thesis generally emphasise using mathematics as a means of communication. For example, the task might be about describing the position of an object. Then, the mathematical object of a coordinate system can serve as a means for communicating the position. The teacher adapted some of the tasks from Swan (1985, 2005). In these tasks, the point is to use mathematics to describe and analyse situations from the real world, but there is no use of computers involved. In others, designed in teacher-researcher collaboration, the intention was to use mathematics to describe, analyse and create dynamic visual effects with DGS in a computer. In both cases, we intended to create a classroom atmosphere that promoted thoughtful discussion, engaging students in communicating and arguing in mathematical ways about situations presented in various representational modes, that is, in what Mason and Johnston-Wilder (2006) call a *conjecturing atmosphere*. A major goal for the task design was that mathematics would be needed for students to "achieve goals that they find compelling, and made visible to students and expressed in a language with which they can connect" (Confrey et al., 2009, p. 20). Thus, the tasks are intended to elicit and develop inquiry-focused dialogue. Such tasks, when they involve DGS, I call *dialogic DGS tasks*.

As mentioned above, my teaching practice became richer and more fruitful as I moved away from imposing my own mathematics on the students toward listening and making sense of how they understood mathematics. Through dialogue, it became clear to me that students often interpret tasks and content in ways I did not foresee and that it was useful for me to get to know their ways. Later, in reading research literature, I found the research most fascinating which presented detailed analyses of student productions, written or spoken. But I could not find much research of this kind on situations very close to my own: students in an upper-secondary school with a teacher that practised dialogic teaching in the spirit of IBME. I was especially interested in students who were not top students in terms of previous achievement or motivation in mathematics. Moreover, I believed in the potential of DGS as integral to such mathematics classrooms, although research illustrating such classrooms was hard to find. This made the confluence of my research interest and the teacher's teaching intents all the more advantageous.

1.4. Research aims

The aim of the study is to increase knowledge of teaching and learning through dialogue and dialogic DGS tasks. This entails describing

1. the meanings that arise in the classroom dialogues and how they evolve,

2. how dialogic teaching supports mathematical meaning making in a class of students disaffected with mathematics, and

3. the contribution dialogic DGS tasks can make to support mathematical meaning making.

To do this, I present, in my papers, analyses of meaning making in dialogues on mathematical tasks in a classroom that I observed and video recorded over a school semester. My goal is to give theoretically informed interpretations of students' dialogues when working on mathematical tasks using DGS in a classroom where the teacher practices dialogic teaching. By this, I hope to contribute to our understanding of what a dialogic approach to mathematics teaching with technology entails. Furthermore, it is meant to offer some new possibilities for mathematics teaching.

1.5. The Icelandic school system

In Iceland, children usually start school in the year they reach the age of 6, and they progress automatically from one year to the next throughout ten years of compulsory school. After completing compulsory education at 16, almost all students enter upper secondary school, although it is not compulsory. All the upper secondary schools offer academic tracks that end with matriculation, giving access to university studies. Some of them also provide art and vocational tracks.

Since 2000, over 90% of 16-year-olds have enrolled in upper secondary level each year (Blöndal et al., 2011). The use of standardised test results at the end of compulsory school for selection into upper secondary schools was discontinued in Iceland in the year 2008. Instead, schools which receive more applicants than they enrol base their selection on school grades, determined by the classroom teachers. There is no standardisation or external oversight over the methods the teachers use to determine the grades. Nevertheless, there is a hierarchy of prestige among the upper secondary schools, where a few elite schools admit only students with the highest grades while other schools accept most or all applicants (Magnúsdóttir & Garðarsdóttir, 2018).

Upper secondary school studies typically lasted four years until the system was changed in 2016. From then onward, the standard program is three years, but in most schools, there is flexibility, and many students take longer than three years to finish their studies. The Icelandic school year is nine months long, usually split into two semesters, the autumn semester and the spring semester. Most upper secondary schools in Iceland are organised around modules, courses with a certain number of credits, allowing students to transfer between schools or change their lines of study if they so wish. This applies to the school that this study is set in.

The laws specifying the requirements for upper secondary school teaching licences have been frequently changed in Iceland. To acquire a licence, teachers need a master's degree, which must include at least a year of teaching training courses. Teachers also have been required to have considerable qualifications in their taught subjects. However, there is a shortage of qualified secondary school mathematics teachers, as it has been for decades. Many mathematics teachers have backgrounds in other fields, such as lower secondary education, engineering or business. Old and recent reports find that many mathematics teachers at upper secondary schools lack mathematical content knowledge and pedagogical knowledge, especially at the non-elite schools (Jóhannesson, 1987; Jónsdóttir et al., 2014).

The current national curriculum describes the goals in terms of competence criteria, divided into knowledge, skills, and competence, all of which have four possible levels. It states that students should become competent at mathematical practices such as reasoning and problem solving, as well as calculating, and suggests content topics but does not go into detail, as the whole mathematics curriculum is described on only four pages (Mennta- og menningarmálaráðuneytið, 2012). This curriculum was introduced in 2008 in a drastic change from the previous curriculum that mandated a detailed description of courses. In the former curriculum, all courses at all upper secondary schools were standardised, although there was no enforcement either through inspection or standardised examinations. There are still no nationwide examinations for finishing upper-secondary schools and no inspection mechanism for ensuring school compliance with the national curriculum. This means that schools have considerable freedom in their approach, but in practice, mathematics seems to be very standardised, dominated by procedural work and teacher lectures (Jónsdóttir et al., 2014; Sigurgeirsson et al., 2018). Two small-scale studies on upper-secondary classrooms of students with histories of low achievement in Iceland indicate that students tend to show little commitment to learning, and attendance is poor. The curriculum seems misaligned with the goals of the national curriculum, as the focus is much more on algebraic manipulations rather than understanding relationships (Bjarnadóttir, 2011, 2012).

Textbooks and teaching materials for upper-secondary schools in Iceland are not produced nor financed by the state. As the market is tiny, few texts are commercially produced, and therefore books are lacking for many types of courses. This shortage of textbooks has likely increased after the introduction of the 2008 curriculum, as each school now has their particular non-standardised curricula. On the whole, this has led to a situation where many schools produce in-house materials rather than using published books. Yet, this multitude of teaching materials does not mean that they are innovative in their approach. In general, their differences are minor. There is no control or inspection of textbooks or teaching materials.

1.6. Organisation of the thesis

This thesis is organised into five parts, starting with this introduction to the roots and aims of the doctoral project. In chapter 2, I explain the theoretical perspective and the research method is laid out in chapter 3. In chapter 4, I discuss my findings, conclusions, and implications deriving from the study. Finally, the main substance of this dissertation, the scientific papers I have written as a result of the research, is enclosed in chapter 5.

2. Theoretical Perspectives and Concepts

In this chapter, I argue for my choice of *dialogism* as a background theory and position my research in the broader field of mathematics education. I discuss the main theoretical concepts I use, give an account of how my theoretical emphases have developed, and state the values underpinning the research.

My analytical focus is mostly on the spoken word, but in general, I use the word *language* to refer to all modes of human communication, such as spoken and written words, diagrams and gestures. The theoretical concepts about language are mainly drawn from the writings of Bakhtin (1981, 1986) and Linell (1998, 2009), while the researchers that inform my interpretation of Bakhtin with regard to mathematics education, are mainly Wegerif (2007, 2013) and Barwell (2016). I also relate dialogical ideas to concepts that describe aspects of the nature of student participation in mathematics class from Mason (2009) and give some thought to the theoretical role of computers as a tool for dialogic mathematics learning.

2.1. What is a theoretical perspective?

Research is always conducted from some point of view, manifested in specific concepts and assumptions about the relationships of the concepts. For the reader to be able to understand the research, there is a need for an account of the fundamental terms used and of its ontological and epistemological assumptions, that is, assumptions about what the phenomena of interest are and how they are knowable. Some of the more common terms used for such accounts are theories, theoretical frameworks, theoretical perspectives, theoretical approaches, theoretical lenses, conceptual frameworks and even philosophies. What exactly this amounts to, what its scope is and how tightly different concepts and assumptions are integrated, is a matter of debate (see e.g. Bikner-Ahsbahs and Prediger (2010) and Niss (2019)). Often what is meant by *theory* in educational research, is something like a set of concepts used to talk about some related phenomena, with assumptions about relations between the concepts. Some insist that the concept of theory should be preserved for something more elaborated. For example, Niss (2019) states that theories are hierarchically organised networks of connected concepts and statements, where each claim is clearly a fundamental assumption or has been logically deduced from these or empirically tested, or at least testable. Niss, along with many other scholars thus suggests that (what are commonly called) learning theories, such as behaviourism and social constructivism, would more aptly be named learning *philosophies* because they are fundamentally unfalsifiable (Cobb, 2007; Ernest, 2006). In attempting to keep the text as clear as possible, I will proceed to use the term *theoretical perspective* as the subject of the chapter, because I do not wish to imply that I am presenting a comprehensive theory.

2.2. Language in mathematics education

This study can be placed within and as a contribution to, the mathematical educational literature focusing on language, or more generally, communication. This body of literature emphasises the importance of communication (conceived broadly as including all manners of expressions, spoken words, writing, drawing and body language) for mathematics learning. The tradition goes back several decades, as can be seen in Austin's and Howson's (1979) early summarising essay and bibliography, *Language and mathematical education* in Educational Studies in Mathematics, which in turn refers back to the earlier work of Piaget and Vygotsky, who were both concerned with the interrelations and mutual effects between thought and language in general. Much of the research summarised by Austin and Howson can be said to be concerned with linguistic challenges, such as when students speak different languages at home and in school, or about structural aspects of mathematical language that differ from everyday language. This includes vocabulary issues, such as when familiar words are used differently in mathematics, with narrower meaning potential than in ordinary communication. An example, already discussed over a century ago by Poincaré (1914, pp. 122–123), is that the everyday use of the word *circle* is more encompassing than what is allowed by a mathematical definition, such that all its points are in a fixed distance from a fixed point. There are also troubles related to differences in uses of logical connectives, such as the word *or*, which in everyday use often assumes that what is meant is an exclusive or, rather than inclusive or, as is standard in mathematics. Another vocabulary issue is that many words and symbols are new and unfamiliar to the students. In English, middle school students need to grapple with words such as *hypotenuse* and *quadrilateral*, but these matters can play out differently in different languages. There is an ongoing interest in whether and how different natural languages might affect mathematics learning. This type of research also reveals that sentence structures in mathematics are more complicated and subordinate clauses are more used than in everyday language. It also shows that reasoning patterns, such as translating propositions into their contrapositive, especially when used on abstract objects, are a source of difficulties.

Following more recent research, a fundamental assumption of my project is that language is not only a lens through which to observe and understand the challenges of teaching and learning mathematics but that mathematical activity and mathematics education are fundamentally constituted by communication (Morgan, 2016; Planas et al., 2021; Sfard, 2008). Thus, language is *not* considered a neutral vehicle to transport existing meanings between people, be they mental constructions or Platonic mind-independent objects. Rather, mathematical meanings (as all other meanings) only come to exist in and through interactions, in language *use*. Our only access to mathematical objects is through language (which includes diagrams and gestures). Mathematics is thus understood as a fundamentally discursive practice and mathematical objects are understood as discursive constructs.

Discursive research in mathematics education often sees mathematics learning as developing an increased command of mathematical discourse, as a progression from discourse confined to informal everyday language towards the command of a more advanced and formal mathematical discourse—the culturally specific, tool-mediated, historically established ways of communicating that competent users of mathematics use (Chapman, 1997; Roth & Radford, 2011; Sfard, 2008; Vygotsky, 2012). This was indeed a premise of this study, both as an explicitly discussed teaching principle and my original research lens. As the course and data analysis proceeded, I became more aware of the different types of discourses at play in the classroom. The teacher and the students often seemed not to understand each other well, they seemed to speak somewhat different languages, as they did not seem to share each others' assumptions about communication or about mathematics. The students did not seem to me to be en route towards mathematical discourse. This led me to dialogism and Bakhtin's theories about discourse, which seemed appropriate to understand what I observed in the teacher-students dialogues because in his theory no perspective or voice is privileged as owning the correct meaning. During my project, the emphasis shifted from focusing on how students moved from informal (spontaneous, everyday) language towards more formal (scientific, mathematical) languages to interpreting the dialogues that were actually taking place, and how students made meaning in and through the dialogues. So, I became gradually less interested in evaluating the extent to which students used canonical mathematical symbols and concepts towards understanding the dynamics of the dialogues, how the students made meaning of tasks on the basis of their assumptions about school and mathematics, and the participants' practices of exploring, conjecturing, and convincing.

2.3. The basic definitions of dialogism

Dialogism is not one coherent school or theory, not even something that "dialogists" of different persuasions would necessarily agree about (Linell, 2009, p. 8). In this project, I mainly build on the ideas of the Russian scholar Mikhail Bakhtin (1895—1975) on dialogue, who wrote his works in the early 20th century, and the writings of others that build on his approach, especially Linell (1998, 2009). Bakhtin is most often classified as a philosopher or a scholar of literature, while Linell is a linguist who has built further on Bakhtin's work. In their writing, there are no discussions on the teaching and learning of *mathematics* as such. What they have to offer are theoretical concepts and principles about dialogue in general (which constitute dialogism), and useful concepts that apply well to dialogue in school contexts, as will be explicated in section 2.5 (17).

In the most general sense, dialogism signifies a commitment to a basic relational nature of human existence; one becomes a person by participating in dialogue. But what is dialogue? Bakhtin uses this word in many contexts that involve people responding to, and anticipating further, acts (usually words) of people or imagined or abstract entities. I have not found a precise definition in his works and Morson and Emerson (1990, p. 49) write that Bakhtin uses the term in "so many contexts and in such diverse senses that it often seems devoid of clear definition". For me, dialogue means a chain of communicative acts that are invariably also connected to other chains of communicative acts, thus forming a live network of communicative acts. A basic definition (and perhaps an "axiom") is then:

> People make meaning in *dialogue*: a live network of *responsive* and *anticipatory* acts.

A (responsive and anticipatory) communicative act is called an *utterance*: a complete signifying act, a meaningful contribution to dialogue. It is what a person has to say (for now) and is demarcated by a change of speaking subjects.

As mentioned above, an utterance has facets of initiation (asking for something, demanding something, seeking a response) and response (answering, agreeing, protesting, heeding). Also, according to the dialogical perspective, dialogue is not confined to co-present interlocutors. When I say (or write) something, I can also be seen in some respects as responding to "generalised others" (Linell, 2009, p. 103). These can be the "voices" of things such as professions, institutions, political systems, norm systems and imagined entities. I come to know these generalised others through television, films, newspapers, books and other documents I read, the internalised voices of my parents and statements of my teachers in the past. According to Bakhtin, in our dialogues and especially our inner dialogues we assume what he calls a *superaddressee*, which represents a listener who truly understands, and can assume "various ideological expressions (God, absolute truth, the court of dispassionate human conscience, the people, the court of history, science, and so forth)" (Bakhtin, 1986, p. 126). In this dissertation, my analytical focus is on the observable chains of utterances between the persons co-present in the classroom. I use the term dialogue most frequently to talk about these face-to-face interactions through talk, but in order to interpret them, I also keep in mind these more distant or abstract dialogues. A concrete classroom dialogue is in some way a contribution to, and a part of, the larger "dialogue of mathematics", the "dialogue of school" and many other dialogues.

A concrete utterance is always addressed to someone (or some group) in anticipation and expectation of future responses. Even the first utterance in interaction is responsive in the sense that it presupposes a (partially) shared ground with the other. It is a response to a situation and the multiple previous and ongoing dialogues both (or all) parties are assumed to be familiar with. Utterances contain intentions, emotions, and evaluations, which are dependent on context but are not under the sole control of the speaker. There are usually multiple perspectives and intentions present in an utterance and they have potentials for different interpretations. Speakers may realise that interpretations of others that they did not expect are reasonable. Sometimes utterances are consciously or semi-consciously made to have multiple interpretations (for example in irony) and sometimes the speaker wants to know what the other would make of an utterance (for example children learning to speak). Interlocutors may sometimes even clarify the speaker's intentions or assumptions to the speaker themselves or present interpretations that the speaker would contest.

2.4. Dialogism as an epistemological and ontological stance

Without others, there would be no communication and no meaning, and because our means to make meaning are always through interaction with others, Bakhtin says: "I am conscious of myself and become myself only while revealing myself for another, through another, and with the help of another" (Bakhtin, 1984, p. 287). Similar ideas are known in works of other thinkers and in other cultures as for example in "a Zulu phrase, 'Umuntu ngumuntu ngabantu', which means 'A person is a person through other persons'" (Birhane, 2017). Wittgenstein's later philosophy contains arguments for the primacy of human relations above and over understanding people as individuals and argues that language cannot be understood as a system separate from human relations. Wittgenstein (2009, §23) wrote that "the speaking of language is part of an activity, or of a form of life", and argued that the notion of a private language (a language understandable only to a single person) is incoherent. Thus communication is understood as people partaking in different *language games*, embedded in different *forms of life*. His famous dictum on the meaning of words, "In most cases, the meaning of a word is its use" (Wittgenstein, 2009, §43) is one of the major seeds of language-based research into meaning in general.

This contrasts with individualistic approaches where meaning making is studied as something happening inside single minds, as epitomised in Descartes' famous phrase *I think, therefore I am*. Therefore, my objects of study are not the contents or characteristics of individual minds or persons, but rather meanings and meaning making in and through dialogue. Dialogism in this sense serves both as a methodological principle and as an ontological background for my project (Bakhtin, 1981, 1984, 1986; Linell, 2009; Wegerif, 2019).

In accordance with dialogism, I take meaning making to be actively made in interaction through language (spoken language, written symbols, diagrams, gestures and so on). Words (or other types of signs) do not have fixed meanings but have partly open meaning potentials rooted in culture, history and ideology. We, as human beings, use these along with our assumptions about the context and with our personal agency we make sense of the words, a sense that is always incomplete and open to further development. In our interactions, we carry out communicative projects to establish something as mutually understood (for the time being), so that we can go on in our goal-directed activities (Linell, 2009, p. 178).

Dialogism as a contemporary orientation to meaning making owes much to Bakhtin. He focused mainly on literature and on communication and meaning making in general in his writings, but did not leave behind a complete and coherent theory, as his work is somewhat fragmentary (and some of it disputed). His basic aim was to theorise communication as an open process of meaning making that depends on the context and the unique worldviews and experiences of people, rather than as closed and determinable. His focus was not mathematics education, and he did not write extensively on education, but he had a compelling view of how people come about using language, as in the following quote:

> [...] the unique speech experience of each individual is shaped and developed in continuous and constant interaction with others' individual utterances. This experience can be characterized to some degree as the process of assimilation—more or less creative—of others' words (and not the words of a language). Our speech, that is, all our utterances (including creative works), is filled with others' words, varying degrees of otherness or varying degrees of "our-own-ness", varying degrees of awareness and detachment. These words of others carry with them their own expression, their own evaluative tone, which we assimilate, rework, and re-accentuate. (Bakhtin, 1986, p. 89)

Thus, dialogism, as applied to learning, emphasises that learning is not simply copying others' use of words but making them one's own. This is a highly context-dependent, dynamic and creative endeavour, requiring constant interpretation and re-interpretation by speakers and listeners. From this perspective, our concern is not to find what the students may lack in ability or knowledge, or how to optimise their memories, but to interpret their interpretations and understand how they make meaning of their being in mathematics class. This involves values: we value students and their thinking on their own terms instead of, for example, evaluating them according to whether they reproduce or master the curricular content. I consider as fundamental to Bakhtinian thought a "valuing appreciation of the existence of others" (Bazerman, 2004, p. 57).

Linell (1998, 2009) draws on Bakhtin's work and more recent sources to describe dialogism as a general overarching epistemological and ontological orientation. He emphasises the following three principles (Linell, 1998, pp. 85–88), which could be called *the dialogical principles of dialogue*, expressing the *dialogicality* (or dialogicity) of dialogue and, according to dialogism, all human cognition and communication.

(a) *Sequentiality*: meaning of action, contribution or sequence, depends on the position in a sequence, which in real-time interaction is temporal in nature. If you cut a part of an interaction (or text) out of the sequence, you change the meaning, and you recontextualise.

(b) *Joint construction*: a dialogue is a joint construction. No part is a single individual's product or experience although contributions are not necessarily equal. Even monologues are other-oriented. Someone is supposed to listen.

(c) *Act-activity interdependence*: acts, utterances and sequences in discourse are always essentially situated within an embedding activity which the interactants jointly produce. The activity type or genre is usually implicitly shown. People know how to talk/behave and interpret others dependent on different circumstances.

These principles guided my analysis in this doctoral project. As my interest was to observe, describe, and analyse the meaning making in the interactions of students and a teacher in a mathematics classroom, I analysed sequences of utterances. I considered the dialogues as joint constructions, which all participants construct together on the basis of their previous knowledge of how to talk, behave and interpret others in a mathematics classroom.

2.5. Centripetal/centrifugal forces and authoritative/internally persuasive discourses

As I was analysing data and grappling with what I perceived as divergence or gaps between the discourses of the teacher and the students, I found two pairs of concepts from Bakhtin particularly useful for analysis, both of which he explicitly related to teaching in schools. The first pair are the metaphors of a centripetal force and a centrifugal force operating in all discourses, and even on the level of a single utterance. The second is the pair of authoritative discourse and internally persuasive discourse, which can be seen as describing discourses of individuals, but also as descriptors of discourse practices and dialogues. These concept pairs are further explored and explicated in the following subsections.

2.5.1. Centripetal and centrifugal forces

In the inevitable difference (of two or more perspectives) inherent in each utterance Bakhtin identifies a tension between a *centripetal force* towards uniformity, a shared and common meaning, and a *centrifugal force* towards *heteroglossia*, which refers to the diversity of discourse, the diverse meanings made in particular situations.

> Every concrete utterance of a speaking subject serves as a point where centrifugal as well as centripetal forces are brought to bear. The processes of centralization and decentralization, of unification and disunification, intersect in the utterance; the utterance not only answers the requirements of its own language as an individualized embodiment of a speech act, but it answers the requirements of heteroglossia as well; it is in fact an active participant in such speech diversity. And this active participation of every utterance in living heteroglossia determines the linguistic profile and style of the utterance to no less a degree than its inclusion in any normative centralizing system of a unitary language. (Bakhtin, 1981, p. 272)

Sometimes the centripetal pull is enforced by social power. For example, when teachers actively enforce an official language standard, correcting or punishing students for not speaking grammatically ("speaking correctly"), or for using non-standard words or phrases to make statements about mathematical objects or relations. Other times, as Linell (2009, p. 213) points out, we can also see the centripetal force as pulling us towards what language itself prescribes as opposed to what the person wants to say. Language makes it much easier to say some things than others. In society, there are some ways of speaking that have more legitimacy than others. Expressing something new or unusual takes more effort than saying the things people are familiar with. This depends on the particular group of people in dialogue, for example, according to social class, geographical location, age, and profession. Linell (2009, p. 213) also sees the centripetal force operating in dialogue when speakers conform with what the other just said, as opposed to introducing new ideas or divergent perspectives which can be seen as instances of the centrifugal force. And even if the centripetal force is oppressive in some cases, it also can be recognised as necessary, because otherwise, the diversity would overwhelm us, and we would have trouble communicating, lacking a common ground. There would be too much difference between those communicating.

Mathematics itself can be seen through the lens of the metaphor of the centripetal and centrifugal forces. A major project of mathematics is to transform informally expressed insights and intuitions into a more formal discourse that the community of mathematicians agrees on. To go back to Poincaré (1914, pp. 122–123), the word *circle* has both formal definitions (different definitions depending on the surrounding theory in which it is included) and various informal meanings (and further meaning potentials). The formal definitions and axioms about the properties of circles in a mathematical theory are meant to capture intuitive notions of what the community *means* by the word, fixing the meaning and "making it precise" as mathematicians would say. If all participants in the mathematical discourse agree on the formalisation, they can begin to explore and reason about circles, through deductive arguments. This pull towards a fixed agreed definition constitutes a centripetal force. However, in ordinary conversations, and possibly in specialised practices outside mathematics, the word circle may mean things that have some properties in common with the formally defined mathematical circle, but not others. "Going round in circles" may for example mean that we end our journey at the same place as we started while there is no fixed radius to talk about. The various informal meanings represent the centrifugal force. As for teaching, Poincaré describes the situation thus:

> We are in a class of the fourth grade. The teacher is dictating: "A circle is the position of the points in a plane which are the same distance from an interior point called the centre." The good pupil writes this phrase in his copy-book and the bad pupil draws faces, but neither of them understands. Then the teacher takes the chalk and draws a circle on the board. "Ah, " think the pupils, "why didn't he say that at once, a circle is a round [sic], and we should have understood." No doubt it is the teacher who is right. The pupils' definition would have been of no value, because it could not have been used for any demonstration, and chiefly because it could not have given them the salutary habit of analyzing their conceptions. (Poincaré, 1914, pp. 122–123)

Poincaré states that it is the teacher who is right, and I would agree that if the concept of roundness were not to be further explored, it would not be possible to use it in reasoning or to prove statements. But (as was well familiar to Poincaré) it is possible to analyse the conception of roundness and even to use it to define circles, as they are the only *planar curves with constant nonzero curvature*, a formalised way of expressing the intuition that by moving forward while constantly turning evenly, will result in going round in mathematical circles.

Papert (1980, pp. 66–67) even argued that this intuition about circular movement can be developed, via computer programming ("move forward a little, turn a little") into an intuitive grasp of differential equations, and might be a better way (closer to children's sense and knowledge about their own bodies) for young students to understand the circle than the classical Euclidean definition (or its modern descendants). This illustrates how an important project of academic mathematics can be seen as striving for a centripetal pull towards precise and explicitly stated definitions and axioms, while informal, intuitive and imprecise ways of speaking provide a centrifugal force. But it also shows that formalisation draws from the informal, otherwise there would be nothing to formalise and nothing against which to check the viability of the formalisation.

I saw the concepts of a centripetal force and a centrifugal force as apt metaphors to describe the struggle of the mathematics teacher to draw students into canonical mathematical ways of viewing situations and tasks while the students often seemed to be pulling the dialogue away from that centre, towards their own meanings, based on their own different intuitions, concerns, and viewpoints inherent in various ways of speaking about numbers, space, change, and motion, and also their rights and obligations as school students.

2.5.2. Authoritative and internally persuasive discourses

Mathematics students often find that the teachers' discourse does not resonate with them. When the words of the teacher function only as "information, directions, rules, models and so forth" (Bakhtin, 1981, p. 342), without appropriation by the students, they function as what Bakhtin calls *authoritative discourse* (AD). Bakhtin explicitly mentions teachers as conveyors of authoritative discourse, and writes of its character:

> The authoritative word demands that we acknowledge it, that we make it our own; it binds us, quite independent of any power it might have to persuade us internally; we encounter it with its authority already fused to it. The authoritative word is located in a distanced zone, organically connected with a past that is felt to be hierarchically higher. It is, so to speak, the word of the fathers. Its authority was already acknowledged in the past. It is a prior discourse. It is therefore not a question of choosing it from among other possible discourses that are its equal. It is given (it sounds) in lofty spheres, not those of familiar contact. (Bakhtin, 1981, pp. 110–111)

Acknowledged scientific truths and textbooks are good examples of AD, as suggested by Hsu and Roth (2014). In this light, mathematical truths could be considered an especially authoritative discourse, because mathematical truths are generally recognised as the most reliable and least contested of all. This might explain to some extent the high status of mathematics in the school systems, as it is used as a gatekeeper and seen as an instrument to fairly distinguish between people as more or less intelligent. Its frequently unquestioned authority thus leads to AD in teaching where students are to repeat and memorise what the textbooks or teachers say, no matter whether it makes personal sense, as expressed in phrases such as "Ours is not to reason why, just invert and multiply" and "Minus times minus equals plus. The reason for this we will not discuss." The fundamental feature of AD (as I interpret it) is well put by Wegerif (2019) in that "the authoritative word instructs or transmits but does not call us into dialogue."

In contrast to AD, in what Bakhtin calls internally persuasive discourse (IPD), we are called into dialogue—it calls on us to make meaning and respond. Bakhtin describes it thus:

> Internally persuasive discourse—as opposed to one that is externally authoritative—is, as it is affirmed through assimilation, tightly interwoven with "one's own word." In the everyday rounds of our consciousness, the internally persuasive word is half-ours and half-someone else's. Its creativity and productiveness consist precisely in the fact that such a word awakens new and independent words, that it organizes masses of our words from within, and does not remain in an isolated and static condition. It is not so much interpreted by us as it is further, that is, freely, developed, applied to new material, new conditions; it enters into

> interanimating relationships with new contexts. More than that, it enters into an intense interaction, a struggle with other internally persuasive discourses. Our ideological development is just such an intense struggle within us for hegemony among various available verbal and ideological points of view, approaches, directions and values. The semantic structure of an internally persuasive discourse is not finite, it is open; in each of the new contexts that dialogize it, this discourse is able to reveal ever newer ways to mean. (Bakhtin, 1981, pp. 345–346)

Matusov and von Duyke (2010) discussed how the notion of IPD has been conceptualised in different ways by educational researchers. Firstly as *internal to the individual*, as voluntary and deeply committed mastery. This would be indicated for example by the excitement and sincere conviction of a student. Secondly, as *internal to the discourse practice*, as becoming an active insider in a discourse community. This would be indicated by skilful participation as a creative member of the community, for example when students would formulate their own problems, generate their own conjectures, and generally self-initiate (in the context of this project) mathematical work. In Matusov and von Duyke's opinion, these interpretations are well grounded in Bakhtin's writings but do not reach far enough in taking full account of Bakhtin's central notion of dialogue. They find them lacking in expressing truly educational goals, which require a more critical stance. As they see it, doubts, questions and challenges are present and alive in IPD. The ideas are tested in dialogue (and forever open to further development), and aims, values, and different perspectives are always under consideration. This would make the discourse *internal to dialogue* in Matusov and von Duyke's terminology.

In this thesis, I use IPD in a way that sees it as internal to the individual (voluntary and committed mastery) as well as internal to the discourse practice of mathematics. It is debatable whether the classroom interactions showed signs of the level of awareness necessary for discourse to become internal to dialogue as Matusov and von Duyke see it. In my work, I include the important point that IPD is not about believing statements. Rather it is about responding to statements or ways of talking about things in a meaningful way, perhaps accept them, perhaps reject them, or hold as an option to consider further. For example, a student could have their way of thinking about circles (see Poincaré's story above) and then engage with the teacher's definition, finding it compatible or incompatible with their own ways, useful or useless for their purposes and so on. This kind of "interanimation" of ways of thinking would mean that the teacher's discourse was IPD. But then the student also could find it utterly incomprehensible, and it would simply not make any meaningful contact with their thinking. I try not to impose external standards to evaluate the extent to which the students are skilful participants in mathematical discourse. Rather, I focus on what position participants take themselves towards the discourse in their responses, as more or less passive receivers, or as active meaning makers that address the mathematics.

2.6. The dialogic role of computers

In this research project digital geometry software (DGS) is actively and frequently used by the teacher and students. Therefore, the project and its results are relevant for and have a place in, the literature on the use of digital technology for mathematics teaching and learning. Drawing its inspiration from a large extent to Seymour Papert's *Mindstorms* 1980 this line of research has exploded into many directions under different theoretical frameworks filling many volumes published in recent years. Researchers in these traditions have argued that observing students' programming affords a window to their mathematical meaning-making processes (Noss & Hoyles, 1996) and that "computational media make it possible to externalise algorithms and thus make processes of thinking available as explicit objects for reflection" (Shaffer & Kaput, 1999). This, then, links directly to seeing mathematics as discourse, with the representations in the computer being one of the means of expressing mathematical ideas. I consider the construction of objects in dynamic geometry software to be a form of programming (see also arguments in Sinclair and Patterson (2018) and in paper II in this thesis).

Programming, including giving commands to DGS, such as GeoGebra, forces students to express their ideas through symbolic representations. The syntax is close to mathematical text, as in textbooks. Students immediately observe the effects of their input and can compare it with what they expect to see. In this way, operating a computer via mathematical software should support students in increasing their command of mathematical discourse. However, there are at least two challenges for students to learn from programming. First, the visual effects may require mathematical interpretations because the screen does not give direct access to abstract concepts. Rather, the screen shows a particular representation that may reflect mathematical properties but may have other unrelated properties. A mathematician may understand how the computer representation relates to abstract mathematical objects, whereas the link may be opaque to novices. This happens because the mathematical object has necessarily been transposed according to its implementation in the computer system (this is the *computational transposition* (Balacheff, 1993)). Second, the computer only follows rules, it does not contribute any understanding of its own. This means it cannot guess or intuit what the student wants and it cannot ask for clarifications, rephrasing, or any unstated assumptions. The commands given to the computer must be in an extremely restricted form as each system provides a specific syntax that it will accept and it will not accept anything else.

These challenges described above call for dialogue (with other students and a teacher). The teacher needs to bring attention to the possible differences between what is suggested by the visual appearance on a computer screen and the abstract logical relations. Teachers must also support students in expressing their ideas using input syntax that the computer will accept. These difficulties constitute challenges, but they are also opportunities for dialogic teaching and learning. This is because the computer has a role as an interlocutor, albeit a non-human one, which will not "know what you mean", and may also lead you astray if you only look superficially at its productions.

2.7. Concepts to describe mathematical practice

The writers I draw on for my dialogical conception of what happens in mathematics classrooms, Bakhtin and Linell, did not write about mathematics as such. Some have argued that too much emphasis on communication risks losing what is distinctive for mathematics. Amongst others, Sierpinska (2005) has argued that mathematics is more than communication, discourse, or language. This is because mathematics is *about something* and has its own specific epistemological conventions. I see Sierpinska as echoing the words of Gauss (1986), "truths should be drawn from notions rather than from notations" (p. 50). I agree with the critics that mathematics and mathematics learning must be studied based on understanding mathematics as a distinctive subject. However, the nature of mathematics, its practice, and its relations to the natural world have been and still are intensely debated (Hamami & Morris, 2020; Hersh, 1997; Lakatos, 1976). A deep discussion of these matters lies outside the scope of this study. I consider my own position to be close to one described by Hilbert (1950), Poincaré (2010) and, Watson (2021) who each make the point that mathematics can be seen as being about the logical analysis of human intuition about phenomena such as numbers, order, space, change and motion. This disciplined academic endeavour, as exemplified in school curricula and university degrees worldwide, Watson (2021) calls European Heritage School Mathematics in distinction from other ways of organising experience that may be prevalent in various cultures outside of school systems. What I also consider clear is that as a person engages in the above-mentioned logical analysis, they have to have some means of presenting the phenomena to themselves and others. These can take many forms. Some examples are imagined dynamic pictures, strings of symbols, drawings on paper, and spoken words. Some of these forms have become ubiquitous, such as drawing graphs in rectangular coordinate systems. Others may be more idiosyncratic.

I assume that the ability to translate between *representational registers*, such as algebraic equations and the corresponding graphs, is important for participants in mathematical discourse, and that this is a major goal of mathematics teaching (Duval, 2006). The nature of these translations and whether and how they can become internally persuasive for students are discussed in detail in papers II, III, and IV in this thesis.

I made use of some of the concepts that Mason uses in order to label specific mathematical actions and interactions, shortly summarised here:

- specialisation: a specific example of a type of object which nonetheless speaks the generality (Mason & Pimm, 1984, p. 277)

- generalisation: generalising means expressing commonalities, the same underlying structure of different objects. "Expressing generality is the lifeblood of mathematical thinking and of algebra in particular" (Mason et al., 2005, p. 296)

- invariance in the midst of change: mathematical statements are often about some properties not changing while others change. "Looking for *invariance* in the midst of *change* can enrich learners' experience of tasks, as well as inform and guide mathematical exploration." (Mason & Johnston-Wilder, 2006, p. 11)

- assenting and asserting: assenting means passively accepting what is told and doing what is shown how to do, whereas asserting means "actively taking initiative, by making, testing and modifying conjectures, and by taking responsibility for making subject pertinent choices" (Mason, 2009, p. 17).

2.8. School and schoolwork

The particular setting of a school classroom is different from other settings where mathematics is practised and learned. The concerns of engineers solving a manufacturing problem and the concerns of research mathematicians are different from the concerns of school students. My work is informed by research on the goals and rationales that students have for mathematical school work, and how these are influenced by larger societal forces. Teachers and school administrators answer not only to the students but to various societal voices. These voices incorporate different concerns, not all of which put a high value on mathematical inquiry. Some of these voices see schools primarily as instruments for furthering social and economic goals, as exemplified in statements such as "future economic prosperity, social and political cohesion, and the achievement of genuinely democratic societies with full participation all depend on a well-educated population" (OECD, 1996, p. 24). It can be contested whether, or to what extent, existing school systems deliver such benefits to society, but it is at least not clear that schools provide ideal conditions for mathematical inquiry. Critical theorises have long argued that through school education, the current system of production and capital-labour relations is reproduced and legitimised as natural (Althusser, 1971). In mathematics education, Mellin-Olsen (1987) wrote about the fact that for students, school is an instrument to obtain the recognised qualifications that they need or desire. Other researchers have found that typically there is a presupposition that the role of students is to produce things (mainly written work) to their teacher's satisfaction, without consideration for learning, use value outside of school, or meaning of the work (Doyle, 1988; Goodchild, 2001). Many types of common student behaviours make little sense if students' goals are mainly related to the learning of mathematics. For example, Liljedahl and Allan (2013) report students engaging in several types of behaviour that are not in alignment with the teacher's goals and expected actions, but are missed by the teacher, as the students are able to make it look like they are trying to learn. Interpreting mathematics education through critical theory, Pais (2013) relates these observed school behaviours to the role of school mathematics as a crucial element in the school as an accreditation system, as a gatekeeper, and a place of social selection. Students are acutely aware of this role of mathematics and this is reflected in their dialogues, as they speak about passing or failing courses, fair or unfair tests or homework. They also talk about strategies to survive that may have little to do with learning, some of which would be considered "cheating" in the context of school. This is perhaps the most important difference when exploring student dialogue in a naturalistic classroom setting (as I do in this research) as contrasted with experimental conditions outside regular school, where students volunteer, and the pressures of school are not on the students' or the teacher's mind.

2.9. Alternative frameworks

There are other approaches to theorise mathematical communication than those outlined here. For example, a well-known theory of mathematical discourse is Sfard's theory of mathematical commognition (the word is an amalgam of communication and cognition) (Sfard, 2008). She considers mathematics to be a type of discourse, which she describes as a meta-discourse: according to her theory, mathematics begins where the tangible real-life objects end and where reflection on our own discourse about these objects begins. The theory builds on ideas from (amongst others) Bakhtin, Vygotsky and Wittgenstein and develops a framework for characterising different discourses. Her framework identifies four components that distinguish a discourse: they each have their specific vocabularies, visual mediators, routines, and endorsed narratives. Vocabularies refer to words and phrases and visual mediators to things such as graphs, diagrams, and written notation. Routines are repeated patterns in the discourse and endorsed narratives are the sequences of utterances taken to be true in the community of the speakers of the discourse. Routines can refer to procedures for calculating or using proof schemes, while endorsed narratives are mathematical theories, including definitions, proofs and theorems. These can both operate on the object level, when they speak about the behaviour of mathematical objects, but also on a meta-level when they speak about the behaviour of the mathematical speakers. An example of a meta-level endorsed narrative from academic mathematics could be: "In mathematics we need to state exactly what we mean, and in mathematics, we all have to agree about what we mean."

It might have been possible for me to use the theory of commognition, but I felt that the theory did not easily explain my observation of the tensions in the classroom discourse. In any case, I did not want to analyse dialogues from a preconceived framework characterising mathematical discourse, as I wanted to be open to and understand the dialogues as much as possible from the students' own perspectives. I, therefore, decided to use the theory of dialogism complemented with concepts from the mathematics education literature as outlined above.

2.10. Summary

I have now presented my general dialogical epistemology and ontology and discussed two specific concept pairs borrowed from Bakhtin that I find apt to describe aspects of classroom communication and learning. I have also discussed what I consider to be important specifics of mathematical dialogue, in the context of school classrooms, and the role of computers for dialogic mathematical learning. The concept pairs of centripetal force – centrifugal force, authoritative discourse – internally persuasive discourse are lenses through which I interpret the classroom dialogues, while the dialogical principles of sequentiality, joint construction and act-activity interdependence guide my analytical approach, which is described in more depth in the following chapter on methodology.

3. Methodology

In this chapter, I argue for the research methods I used in this qualitative case study and explain my analytical approach. As a whole, the method of analysis could be described as being based on thematic analysis (Braun & Clarke, 2006) incorporating dialogical principles (as described in the theoretical section) and interpretative considerations from conversational analysis.

3.1. Case study

I have chosen to use the term *case study* in describing the research project because it is an in-depth exploration of an instance of a complex phenomenon, namely meaning making in classroom dialogues. There are many definitions of case studies available in the methodological literature, but they generally emphasise that the goal is to understand the complexity of real situations, which are bounded, yet invariably a part of multiple contexts. In discussing the boundaries, that is, what constitutes a particular case, Thomas and Myers (2015) usefully distinguish between a study's *subject* and its *object*. The subject is delimited by concrete boundaries, for example in time and place, while the object is delimited by theoretical frames. As a whole, the subject of this study is a particular mathematics class (a group of people within an institutional and physical setting) with some distinguishing features, described in section 3.3, and the object of the study is meaning making in dialogues about mathematical tasks, as explicated in the theoretical chapter. The study can also be seen as a collection of three smaller embedded case studies, all involving the same group of people, as I selected three distinct clusters of lessons for close study and report through publication. The three clusters have considerable differences between them. They are separated in time, but more important are differences in the nature of the tasks and the form of discussions. In the first cluster (reported on in paper I) the students do no work on computers. In the second cluster (reported on in paper II), there is a teacher-class dialogue on a carefully designed dialogic DGS task. In the third cluster (reported on in papers III and IV), the focus is on dialogues of student pairs (with some input from the teacher or me, the researcher) working on DGS tasks.

3.2. Methodological decisions

The aims of the project are to understand the meaning making that happens in dialogue in a mathematics classroom. In contrast, the aim is neither to find out what the experience "really meant" to the participants as individuals nor to measure their understandings of mathematical concepts in a controlled way. These aims would have called for other data generating methods, such as interviews or think-aloud problem solving experiments. Therefore, as the interactions in a classroom happen fast and are accompanied by gestures and frequently refer to the shared physical context (whiteboards, computer screens etc.) it was essential to video record the participants' dialogues. As understanding of dialogue depends on previous and ongoing interactions in context, I was present in the classroom as I was recording with a small handheld video camera, over the whole semester of the course. I only had one small camera because I did not want to set up any lab-like conditions with many cameras or anything that could disturb the natural on-goings in the classroom. In general, all teacher lectures and whole class discussions were captured, along with some interactions between the teacher and individual, pair, or small groups of students. These were selected at the time of recording more or less randomly or opportunistically: I chose students who seemed engaged in the tasks and were physically positioned in such a way that allowed recording of the things pointed at (such as computer screens). Selecting specific participants to record as opposed to recording the whole classroom was necessary to capture enough detail for fine-grained analysis. This involved a compromise between catching the classroom as a whole and accessing the details of interactions (Heath et al., 2010, p. 45).

3.3. Setting, participants and basic information

The setting for the project was a course in an upper-secondary school, "Mathematics 2A: Functions and graphs". The course was mandatory for graduation and was either the first mathematics course students take or the second (if they had a low mathematics grade from compulsory school they were required to take a preparatory course). It lasted one semester, for about 38 classroom sessions.

There were 30 students enrolled in the course. Of them:

- 9 were in their 16th year of age—which is perfectly on time from the perspective of the school system.
- 12 were in their 17th, 18th, or 19th year of age—which means they were supposed to have already finished the course according to the school curriculum.
- 9 were in their 20th year of age or older—which means they were supposed to have already matriculated from the perspective of the school system. Some of

these students had re-entered education after having been active in the labour market for several years.

- 13 arrived with a final grade 5 out of 10 (the passing grade) or less from compulsory school mathematics (as given by their school, not a standardised measure).

- 8 had national-standardised test results recorded (all of which were at least 19 years of age), their average: 3.1 (out of 10). [1]

- 9 were repeating this "same" course in this school.

While there were a number of students who had performed adequately in their previous schooling, as can be inferred from the above list, the majority of students had histories of not doing well in mathematics at school, although their individual histories may be very diverse. Aside from the preparatory courses mentioned above, the school did not divide students into different groups according to attainment. The broad band of the age of the students may be worth paying attention to, with 16-year-old teenagers working alongside 29-year-old adults. This is not so rare in Icelandic secondary schools, as students have opportunities to move at different paces through the system and also can return to education at this stage if they previously dropped out. As was still common in Iceland at this school stage, the group of students was homogeneous with respect to ethnicity and language, all of them being white and having Icelandic as their first language. This situation is gradually changing in Iceland as in many other countries as group diversity grows with respect to ethnicity and language.

The school is a state school that prepares students for university and matriculation gives automatic right to enter the University of Iceland. It has an explicit policy that students should be active and independent learners, teaching is to be task-focused and grounded on ongoing formative assessment. Summative final examinations have little weight. The use of information technology is strongly encouraged and students have free access to laptop computers, although most students bring their own devices to school. The school takes in students of all backgrounds and the students' previous performances are varied, with many of them having been assigned low grades at the end of their final year in compulsory school and many transfer to the school from other upper-secondary schools. Students transferring from other schools likely do so because they have not been successful and it is possible that the school policies on assessment draw students toward it.

The teacher had been teaching mathematics for four school years, teaching being a second career for him. In his teacher education, he had taken a course on the mathematics of upper-secondary school with me as the instructor. After a year of teaching, he reached out to me and we met to discuss ideas in order for him

[1] In Iceland national standardised tests at the end of compulsory school (10th grade) were discontinued in 2008.

to develop his teaching. His vision of mathematics teaching included a desire to empower his students with mathematics so that they would be able to see through dishonest people and resist all kinds of oppressive forces. He had reflected on his own experience as a mathematics learner and concluded that he did not want to teach in the traditional way which, in his experience, resulted in the students being alienated from the subject, losing confidence, and not enjoying themselves. Thus he wanted to base his teaching on inquiry and formative assessment. He thus saw himself as desiring to teach differently from what he saw as mainstream mathematics teaching. In light of this, the object of study can be described as a case that belongs to what Watson (2021) calls *alternative mathematics education*.

3.4. The data

The primary data consist of recordings of 31 classroom sessions (out of the total 38 sessions) which total 32 hours and 50 minutes of video.

In any pair or small group situations I asked the group whether I could record them working, which was always allowed and did not seem to disrupt the students' normal ways of working or interacting in class. Although it is possible that a researcher's video recording in the classroom has some effect, as far as my experience as a teacher tells me, there was nothing that indicated that my presence or recording had a substantive effect on classroom life. Other researchers have found that children rapidly become accustomed to recording in the classroom, (Ingram & Elliott, 2019, p. 264) and there seems to be little empirical evidence that a camera transforms the ways in which research participants accomplish actions (Heath et al., 2010, p. 49). That being said, it is impossible to be certain that students do not in some way accommodate themselves to the researcher.

The analysis was also supported by auxiliary data, such as the instructional materials, tasks and quizzes, and I also collected the work students handed in, such as written homework, quizzes and computer files. In addition, I was informed by my frequent talks with the teacher on the tasks and the organisation of the teaching, but these were in general not recorded, transcribed, or analysed. All of these supported me in making sense of the primary data by getting a fuller sense of what students were responding to in the classroom, as well as showing their finalised (for the time being) responses.

3.5. Data analysis

The analysis of data may be broken down into phases. At first, my goal was to get an overview, and I had ambitions to analyse the evolving meaning making throughout the course as a whole. This proved challenging for at least three reasons. First, video data of classroom activity is too broad and rich in detail. Over thirty hours of classroom recordings with a multitude of voices and simultaneous actions were too much for me, as an individual researcher, to transcribe and analyse in the kind of depth and detail I was interested in. In my transcripts, I include descriptions of action and contextual information such as what the teacher writes on the whiteboard while speaking to the class and computer screens when students are working on problems in GeoGebra. Second, there were many different types of work in the classroom, such as teacher lectures, teacher-class discussions, individual work and group work, with or without computers. The nature of the mathematical work varied from calculation exercises to explorations, on topics from the use of a coordinate system to trigonometric functions. Third, a number of students dropped out of the course, making it impossible to observe their possible development as participants in mathematical discourse. For example, there were two students who had been very engaged in the class in the first weeks, who quietly disappeared. Also, I did not want to completely lose the big picture, which is a possible problem that researchers using video data must be aware of. I, therefore, worked both on getting a sense of the whole course, by watching every lesson and writing lesson summaries, and also on selecting and marking episodes for deeper analysis that I judged as especially interesting and revealing for my research purposes, which are concentrated on students making mathematical meaning in interaction. In addition to the lesson summaries, I transcribed 18 lessons at various levels of detail. This work will be further described in the sections 3.5.2 and 3.5.3 below.

In order to interpret meaning making in dialogues in a profitable way, it is necessary to look at what really happens in detail. I agree with Morgan (2021, p. 103) that detail can provide "insight into the concepts, values and possible positionings, actions and relationships that constitute the discursive practice," and with Sfard (2008, p. 280) that repeated scrutiny of records give an opportunity to find "interpretations that are not the ones we instinctively produce in real-time while acting as participants." I, therefore, selected a sample of lessons and episodes within lessons for detailed analysis, with different emphases for groups of lessons that had important common characteristics, rather than analysing all the lessons of the course in the same way. For example, in papers III and IV all three lessons analysed had commonalities of the mathematical topic (analytic geometry), mode of work (creating objects in GeoGebra), mode of interaction (pairs working together on a single computer) and special feature, as in all lessons there is an episode where students explicitly display pleasure of having produced a solution. Then, fine grained-analyses provide useful insights into the depth of mathematical dialogues in classrooms, as long as they are properly contextualised. I, therefore, worked by repeated viewing of selected recordings as well as working from transcriptions of the dialogues. The segments were selected, first based on whether they contained mathematical dialogues, that is utterances where a mathematical topic or task was a focus. This is not to say that other kinds of dialogue could not on close inspection be revealed to have connections to mathematics, but it is unlikely that it would have had a great impact on the analysis. Therefore I did not transcribe much dialogue that was solely about matters unrelated to school or mathematics and I did not transcribe all teacher lectures. There was also an element of randomness in this selection, as I considered it impossible to analyse all such segments because it would take too much time.

For analysis, I used the Transana software (Woods & Fassnacht, 2012) to create transcripts that are synchronised with the video. Analysing videos directly is important for two main reasons. First, in the videos, I could see gestures, fingers pointing, and relevant objects pointed at, such as whiteboards and computer screens. Having these in the visual field was often important for understanding what was being talked about. Secondly, it was practically impossible to transcribe all the talk and actions going on simultaneously. For this second reason, and because my participants talked Icelandic, and not English, which is the language in which the research is presented, I did not try to transcribe the dialogues fully according to the strict conventions of conversation analysis (i.e. the Jeffersonian system), where all sounds, pauses, and all variability in intonation and prosody are recorded. I trusted that watching and listening to the episodes I analysed in detail would, along with transcripts, give enough information on which to base interpretation.

I had a special interest in researching the use of a dynamic geometry task design (analysed in paper II) that the teacher and I had worked on together and piloted on a previous occasion. For contrast, I decided to look also at a lesson where dynamic geometry was not used (analysed in paper I), and episodes where students were working in pairs on somewhat different dynamic geometry tasks (analysed in papers III and IV).

3.5.1. First steps in the classroom

The first phase of analysis happened while I was video recording in the classroom. Although I went in with the intention to listen to the students and the teacher, to truly understand the meanings they made in the classroom, it was hard for me not to evaluate the dialogue against standards of correctness and mathematical quality as I saw it as an insider to mathematics and as a mathematics teacher. Thus, my notes that I wrote after lessons would frequently mention low-level disruptions and problems of miscommunication, students not understanding, and what I would have done differently as a teacher. But, as Nemirovsky and Tierney (2001) point out, it is important in research to avoid such diagnostic attitudes. These can cloud the analysis of the students' own meaning making by projecting assumptions on the data. Gradually I realised that an observer perceives things differently than a teacher in action. No one can notice everything, and a teacher will not notice many things an observer can notice, simply because the teacher needs to use their power of attention on interacting with students. I also came to see that students' responses, while not easily interpreted through my insider understanding of mathematical discourse, could usually be seen to make sense upon further examination and especially in light of further responses of others and repetitions of the speakers. For example, as seen in paper II (section 5.3, p. 314-315), the same utterance was understood in several different ways that only subsequently became clear to me. Here both the dialogical principle of multiple meaning potentials and the principle of "the next-turn proof procedure" (attending to how one turn by a participant displays an understanding of the previous turn by another participant, see e.g. Ingram (2021)) supported me in looking for and finding the different meanings made with and of a single phrase.

During the time of recording in the classroom, I also started transcribing lessons and writing more analytical notes and reflections, informed by frequent discussions with the teacher. The teacher and I discussed the potential of tasks, what could be done to improve their effectiveness, and how students might be drawn into mathematical dialogues. I did not analyse these conversations but they clearly inform the way the tasks are presented in the papers.

3.5.2. Getting an overview

Lemke (2007, p. 45) points out that meaning is not only made moment-to-moment but also over longer timescales. It is a danger for video researchers to be overloaded with detail. In order to not lose the big picture after the course, I watched all the recordings, and wrote summaries of all lessons of about 5 pages each (with some photographic stills), as exemplified in the sample summary (Appendix C p. 155). I also wrote more transcripts of segments that I had selected, as described above. From the summaries, I created an overview table (Appendix B, p. 153) where I noted for each session the main topics worked on and the modes of work (whole-class discussion, individual seat work, group work, with or without the use of GeoGebra). This was important for perceiving the course as a whole and getting a sense of its timeline and the order of the episodes I found most interesting. This phase of analysis corresponds to the first step in Braun and Clarke (2006), "Familiarising yourself with your data". From the above-described initial engagement with the data, my sense of the classroom as a site for dialogues deepened. I increasingly came to see it as a space in which the participants made meaning together on basis of different and sometimes conflicting backgrounds and assumptions about communication, school and mathematics.

3.5.3. Coding and interpreting

In the second phase of analysis, I grouped lessons on the basis of commonalities of mathematical topics and mode of work and interaction for further separate analysis. The three groups may be described as follows:

1. Lessons with paper-and-pencil tasks and teacher-class discussions. These became the subject of Paper I.

2. Lessons with a specific teacher-researcher designed GeoGebra task design and teacher-class discussions. These became the subject of Paper II.

3. Lessons with GeoGebra tasks with different features, and student pairs problem solving discussions. These became the subject of Papers III and IV.

I consider these different groups of lessons to represent important but somewhat different types of mathematical dialogues that took place in the classroom and they became the bases for the papers of the dissertation. In what corresponds to the second step in thematic analysis, "Generating initial codes", I coded all interactions as involving at least one of the three following topical criteria: they were about schoolwork, mathematics or about something else, or some combination of the three. This was a predetermined theoretical classification drawing from my research interest. In utterances about mathematics, I further distinguished technical discourse (canonical mathematics words or symbols), visual discourse (appeal to what was seen visually), or deductive discourse (some form of a reasoning step supported by appeal to agreed-upon properties). This resulted from searching for, reviewing, and defining themes from the open coding (Braun & Clarke, 2006) drawing from my interest in what students would talk about when talking about mathematics. In working on Paper I, I tried refining this coding scheme to an analytic framework focused on evaluating how well the utterances and discourses of students aligned with what I saw as the norms of mathematical discourse, mainly precision, explicitness and the giving of reasons with reference to mathematical properties (See Appendix G p. 183). The framework was grounded in the data, but from my assumptions (based on my affection for academic mathematics) about the mathematical quality of discourse with which to compare the data. I coded the transcripts that are the basis for the analysis of Paper I in this way, but then abandoned this refined version of the framework when I felt that it was not conducive enough to understanding meanings that were being made in dialogue. It did not fit the purpose of answering questions such as: What were the students trying to achieve by their utterances? What did they take the tasks to be asking of them and how did they respond to these questions, building on their own backgrounds and assumptions? These questions were closer to my heart than laying out different levels of arguments.

For paper I (p. 51) then, I analysed three teacher-class dialogues on paper-and-pencil real-world tasks. I selected these lessons because they provide a contrast to lessons involving computer work, and they have a unity of topic (interpreting graphs) and mode of work (whole class discussions after individual seatwork). I used thematic coding procedures as described by Charmaz (2006), while keeping in mind the dialogical principles of sequentiality, joint construction and act-activity interdependence (see 2.4 p. 16). This can perhaps be questioned on the grounds that thematic coding is frequently used to find out about people's inner thoughts or feelings and because it is assumed that the researcher can approach the situation relatively unburdened by their own preconceptions, assumptions and ideas. However, as Braun and Clarke (2006) point out, such methods can also be used to examine the ways in which meanings are the effects of different discourses, and that the analysis can be either inductive (coding the data without a preexisting coding frame) or more theoretical (coding at least partly in light of theoretical concepts). In a second round of coding, I focused on *how* students engaged with mathematics, asking the following questions: By their contributions, what do students achieve, in the dialogue? From their utterances, what are the tasks taken to be asking? And how are these responses taken by the teacher and the students to be contributing? The underpinning theoretical idea is that the teacher is trying to pull the students toward an academic mathematical discourse, while the students' contributions may be pushing the dialogue into other directions for whatever reasons (some may be trying to do what they take the teacher to be asking of them, others may knowingly be trying to divert the conversation away from these). My final analysis of the dialogues is in terms of the theoretical concepts of centripetal and centrifugal forces. Rather than labelling utterances as either centripetal or centrifugal from the start, I found in the data that the dialogues seemed to be marked by a difference in how the tasks were understood by the teacher, task designers, and myself (as opportunities for abstraction) and how the students responded (as if the questions were opportunities for discussing real-world situations). I felt that this aligned well with the dialogical theoretical assumptions about opposing centrifugal and centripetal forces and their tension, as well as feeling the struggle of the teacher and the students to establish common ground. I wrote dialogical analyses (see section 3.5.4 below) of a number of segments, and I felt that they could be divided not only into the two categories of opposing forces but into four subtypes (the alien word, irony, authentic real world and emerging mathematical dialogue). I then systematically coded the lessons according to these types as can be seen in the example transcript (see Appendix E, p. 167). Through the described analysis it was my intention that the concepts could be described more fully and take on a deeper meaning in the context of mathematical dialogues.

For paper II (p. 61), I analysed two lessons, while only one of the lessons finally became the paper's subject. I selected these lessons because they were focused on a particular task design that the teacher and I had spent time developing (based on a design described in the literature (Falcade et al., 2007)) and had tried out in an earlier version with a previous class. The analysis proceeded by first creating codes for the two lessons in an open way, aiming at capturing the mathematical meanings of a dynamic geometry situation and characterising the utterances' dialogical functions. I watched the recordings and created a transcript (Appendix F p. 173) where I coded from this evolving coding scheme. As the writing of the paper proceeded, I decided to focus only on the meanings made of the mathematics, and the analysis of the dialogical functions will wait for another day.

For paper III (see p. 85) and paper IV (see p. 95) I selected particular episodes that I had become aware of as contrasting examples of how students convinced themselves that they had solved a dynamic geometry problem. The individual utterances in these sequences were coded as to whether they contained a) explicit expression of joy, b) mathematical words or symbols, c) talk about what could be seen (visual), and d) talk about school work or talk that made sense primarily in the context of school. Then I wrote detailed commentaries on each utterance and their interrelations with each other. Nevertheless, I began the exploration of these episodes at a stage where I had come to see the classroom dialogues in general as characterised by two central tensions, captured by the following conceptual pairs:

- mathematical discourse against schoolwork discourse;
- authoritative discourse against internally persuasive discourse.

In the detailed line-by-line interpretations I then discerned interesting variations in how these tensions played out, resulting in conclusions about how internally persuasive discourse might be inferred and how it seems that schoolwork discourse interferes with mathematical discourse.

3.5.4. Dialogical data analysis

It has been my aim to develop a method for analysing mathematical dialogues that are in accordance with the fundamental principles of dialogism. I can't claim that I have a fully developed method. It borrows from methods of thematic analysis (as done in grounded theory approaches) and conversation analysis. Both of those analytical methods place emphasis on a very careful reading of the data, as openly as possible. Thematic analysis is commonly used for the construction of grounded theory, but then it demands that the initial analysis be done without preconceived theories and concepts, while conversation analysis tends to dismiss or place minimal emphasis on the wider societal (or any other) context, except in so far it is explicitly made relevant by participants. In my research, I can only say that I try to be as open as possible to the participants' own meaning making and understanding of what it is that they are trying to achieve in their interactions while acknowledging that I can't have a God's eye point of view. I am an insider to mathematics education, I have my own specific social background, beliefs and values about mathematics and school, and I work from specific theoretical principles about dialogue (dialogism).

Dialogism entails not looking at utterances in isolation, or as expressing what is inside the speaker's mind. Rather, an utterance needs to be interpreted as a link in a dialogue, where the meaning is seen as evolving and co-produced by all participants, including me, the researcher. The analysis always includes looking at how one turn by a specific participant displays an understanding of the previous turn by another participant. But it also does not overlook the cultural or social contexts, as interactions are in dialogue with these as well. The fact that participants are in a mathematics classroom must be taken into account, and here the analyst might need insider knowledge, because otherwise, it is harder to discern whether and how the participants are making that context relevant and consequential by subtly or indirectly displaying, in their dialogue, an orientation towards it.

3.6. Ethical considerations

I position myself with those who wish for mathematics to be for human flourishing (see Su (2020)) and for teaching mathematics to be based on caring for students and caring for mathematics (see Watson (2021)). Dialogism is also in itself suggestive of an ethical stance. It is attached to a belief in relations, through language. Unfortunately, mathematics is often the source of pain rather than flourishing, as many students go through their mathematics schooling as a Sisyphean work of meaningless labour. Often the teaching makes little attempt to connect to the students' internally persuasive discourse and relations between people and between people and mathematics are disregarded. This is not a new observation. Already in ancient Greece, around 375 B.C.E. Plato wrote (1997, 536d-e):

> Therefore, calculation, geometry, and all the preliminary education required for dialectic must be offered to the future rulers in childhood, and not in the shape of compulsory learning either.
>
> Why's that?
>
> Because no free person should learn anything like a slave. Forced bodily labor does no harm to the body, but nothing taught by force stays in the soul.
>
> That's true.
>
> Then don't use force to train the children in these subjects; use play instead. That way you'll also see better what each of them is naturally fitted for.

Setting aside the context of explicit elitist purposes of education in a society with slavery, alongside an essentialist view of ability (which I absolutely disagree with) we see here a case made against compulsory learning and for learning through play. Unlike Plato, I see myself as ethically committed to playful mathematical education for all students. In my view, it is unethical to subject students to memorising statements or practising procedures that have no meaning for them. The fundamental problem is how to reconcile this with the compulsory nature of the educational system and its gatekeeping functions.

According to Pring (2004, p. 145), the two following ethical principles are important to educational research: "first, the principle which requires respect for the dignity and confidentiality of the 'objects' of research, and, second, the principle which reflects the purpose of the research, namely, the pursuit of truth." The first principle entails ensuring that participants are not harmed and that reasonably informed consent is obtained. Individual participants should not be recognisable from my reports on the research and their names do not appear in my analysis or writing. It is perhaps unrealistic that those most knowledgeable about mathematics education in Iceland could not guess the school's or the teacher's identity from the papers since Iceland is a very small society. However, this research is certainly not intended to evaluate the students, teacher or the school. Researchers should take care when reporting on research not to embarrass the participants or (in this case) students or teachers in general. But they also have a duty to tell the truth and contribute to human knowledge. Here, there is a possibility for tension and contradiction, because the truth may hurt. Additionally, we should not imagine that we, as researchers, are infallible. Therefore, I strove to be careful and modest in any claims and give due consideration to alternative perspectives and interpretations of data. The teacher and I have been in contact since the course and he read the papers before they were published. He has expressed his approval of the publication and believes he benefitted from the collaboration. I also see an ethical dimension to the project as a contribution to the research literature. I depend on researchers that have come before me, as my research is partially a response to theirs. Their works have benefitted me and by sharing my work, I hope to make a contribution from which others can benefit: teachers, students, and other researchers.

In compliance with laws on the protection of privacy and processing of personal data, I contacted Persónuvernd, the Icelandic Data Protection Authority, to inform them about the research project. The teacher and the school principal received each a letter (see Appendix A.1, p. 151) that asked for their permission to carry out the research. In the letter, there was information outlining the research aims, the intended data collection, the treatment of data and how results would be published.

No one has had access to the raw data (videotapes, student work) except me. Students had the right not to participate, and to opt out at any point. I frequently explained to them aspects of the nature and purpose of the research, and they discussed this with me from time to time. To what degree fully informed consent is possible is a matter of debate. How do we for example know whether students or parents fully understand what the research entails, or ensure that they do not feel pressured to give their consent? All students were given letters that gave their parents (or themselves) the choice to opt out of the research, and I always asked students working in pairs or small groups if it was okay that I recorded them at that moment. That said, I do not consider this study to involve serious risks to participants. Therefore, the consent forms were not very elaborate, as I felt this could give rise to inflated and unnecessary worrying.

I believe that participation in this research benefitted both the students and my collaborating teacher. I hope I have shown in the papers that students had valuable experiences while working on the tasks that the teacher and I designed or adapted, and by engaging in dialogue. I have no reason to believe that the experiences were less conducive to mathematics learning than those typically had in a traditional mathematics class. Hopefully, they were sometimes even better. I have no data to prove this, and there was no way to establish this as far as I can see, as the participating students will have learned somewhat different things from what is usually expected in this kind of course. There is no comparison group available.

It is possible that students didn't become as fluent in performing some standard procedures as they could have been if they were intensively trained for that instead of engaging in dialogues about dynamic geometry tasks. In general, I consider this to be an ethical dilemma for mathematics teachers, educators and researchers, crystallised in the dialectic of mathematics having both use-value and exchange-value. On the one hand, it is useful as a tool for the critical, scientific examination of society and nature, it can be used to create new things and it can be enjoyed—mathematics is for human flourishing. On the other hand, "mathematics provides a cultural arbitrary for exchange" (Williams, 2012, p. 70)—mathematics is used for access to careers, educational opportunities (that may have little to do with mathematics) and prestige. These aspects are not always compatible. It may be better for success on some high-stakes tests to train standard algorithms than to strive for conceptual understanding. In this research, the risks associated with this issue were minimal because there were no external examinations for the students to take. Both the teacher and the school principal saw this research as an opportunity to develop the school curriculum and pedagogy in the desired direction. Therefore, I believe my ethical obligation to help students to do well on important tests was not compromised.

4. Conclusions

4.1. Contribution

In the papers that constitute the doctoral work, I present interpretations of mathematical dialogues. They show the potential and the challenges of dialogic mathematics teaching with technology. I show how such a pedagogy can elicit genuine mathematical dialogue in which students can experience the joy and satisfaction of making mathematical meaning, even students who are disaffected with mathematics beforehand. This is important, as these moments are valuable in themselves, as well as being encouraging to students and possibly leading them to change their views of mathematics and their own potential as mathematics learners in a positive way. It is a matter of utmost urgency to develop more ways to organise and teach mathematics such that students can make meaning of it.

The dialogues also show that underperformance in school should not be confused with learning difficulties. The students in the class frequently make clever comments, even meta-remarks on the dialogue itself, questioning if it is really about mathematics. They appear quite sharp and critical thinkers, only that they have never felt that mathematics (and perhaps school in general) has had much to offer them. School has thus functioned overwhelmingly as authoritative discourse for them, and not as internally persuasive discourse.

Through the study of classroom dialogues I have been able to make the following contributions to the literature:

1. A more fine-grained classification of centripetal and centrifugal tendencies in student discourses on contextual tasks.

2. A novel way of analysing and presenting a meaning making trajectory of a teacher-class discussion, with a classification of five types of utterances.

3. A detailed discussion of internally persuasive discourse in mathematics class and how it can be operationalised.

4. A description and interpretation of upper-secondary school practice that contrasts starkly with what has been observed in Iceland in research, which is characterised by monologic teaching and emphasis on performing standard

algorithms (Bjarnadóttir, 2011; Jónsdóttir et al., 2014; Sigurgeirsson et al., 2018).

5. A suggestion of a theoretically grounded pedagogic task design, with a detailed description of how it played out in one classroom.

The thesis also illustrates that meaning making is far from a linear process, as the dialogues contain much more than steadily increasing mathematical engagement. It adds to the field's understanding of how learning to speak mathematically is more complex than learning technical vocabulary and grammatical structures, as it highlights the following tensions between mathematical discourse and the students' discourses:

1. **Between mathematical abstraction and the particularities of real-world context.** For instance, paper I (p. 51) shows the teacher's struggle to bring students to a mathematical perspective, to focus on abstractions and expressing relationships through symbols (which may be connected to pain, failure, and/or meaninglessness in the minds of the students) rather than the particulars of the real world context. This is in contrast with existing research that tends to emphasise the opposite problem when students ignore the actual real-world contents of problems, perform irrelevant calculations and get nonsensical answers. Paper II (p. 61) then also shows a struggle to see something from an abstract mathematical perspective, this time a dynamic graphical computer representation.

2. **Between mathematical explicitness and the ordinary reliance on the other to do some work to understand what you mean.** In paper II (p. 61) some of the students' discourses presupposed that the goal of the dialogue was to establish local meaning, enough to go on and finish the immediate task at hand. However, mathematical discourse makes and creates a need for, distinctions that are not made in everyday talk. This happens because these distinctions are necessary for precise reasoning and representation as was shown by the multiple meanings students made of one and the same phrase "B is half of A" in paper II (p. 61). This thesis thus shows the importance of dialogue to reveal meanings for participants.

3. **Between mathematical exploration-conjecturing-convincing and the completion of required tasks.** Paper III (p. 85) and paper IV (p. 95) bring into focus the tension between the joy of exploring, conjecturing and convincing yourself and others on the one hand, and working on tasks as simply something you must complete, as a necessary evil to be done with on the other. The thesis also demonstrates how working with computers can motivate students to use mathematics as a semantic tool (in exploring, conjecturing and convincing) and bring them satisfaction in being able to control visual objects through mathematics, even when working on required tasks.

4. **Between basing conclusions on appearance or on reasoning based on prop-**

erties. For instance, in Paper III (p. 85) and paper IV (p. 95), the particular designs of the tasks seemed to engender different engagement from students, in terms of whether students experience a need to grasp the link between the symbolic representation and the graphical representation of mathematical objects or whether they can work from appearance and manual manipulation of screen elements alone. It has been suggested before that the physical sense of moving a slider could obscure rather than enhance the understanding of a connection between a parameter value and a visual consequence (Zbiek et al., 2007, p. 1177).

4.2. Dependability and limitations

This work consists of descriptions and interpretations of dialogue. It is not designed to provide laws of causes and effects, nor generalisations about effective ways of teaching. Neither can it describe the inner thinking or understanding of students, as the focus is on what is explicitly communicated. Yet, as in paper IV (p. 85), some inferences are made about internally persuasive discourse, but these are grounded in an argument based on observable communication.

One limitation of the study is that I can't say anything about the long term. For example, did the meaning making in the classroom have a lasting effect on the students' discourses? Perhaps it all fades from awareness as life goes on. Another limitation is that when I was in the classroom, I often had to select a pair or a small group of students to observe and capture on the spot. Therefore, there may have been dialogues occurring that might have revealed some interesting aspects of the meaning making happening if I had captured them. Somewhat related is the limitation of what the video captures visually. It was for example impossible for me to record all facial gestures, pointing fingers, and screens simultaneously. While I believe that I captured enough of the interaction for my purposes, it is not impossible that something has been missed that would have affected my interpretation of the scenario.

From a dialogical viewpoint, the most important limitation is that the interpretations are done by a single person, me. The dialogical stance is that meaning happens in dialogue, between people and between different viewpoints. My interpretations are grounded in what happens in the recording, but unavoidably, these are my interpretations, informed by my mathematical viewpoints and my knowledge of schooling—assumptions that I have tried to be explicit about in my analysis. I discussed many interpretations with my supervisor and to a (much) lesser extent with other researchers in working group 9 at the CERME11 and CERME12 conferences. These discussions provided some alternative interpretations, but they always proceeded from my initial selections of text and my interpretive ideas about these. My interlocutors also had limited access to the data and no first-hand knowledge of the context, weakening their possibilities for interpretation. It would have strengthened the interpretations to have had collaborators looking at more of the data and engaging in dialogue about it. The process of exploring together, and conjecturing, convincing and refuting would have made the interpretations more robust and likely revealed some interesting aspects that I have overlooked.

4.3. Implications

A theme that permeates the study is the tension between control and coercion on the one hand and freedom on the other. School institutions are probably necessary to bring people into contact with scholarly disciplines such as mathematics. In modern societies, everyone is expected (and in effect, forced) to go to school. Students are expected to perform knowledge throughout their schooling on tests and other evaluations according to specified standards. They are threatened (by news media and sometimes parents and teachers) with dire career prospects if they fail. We can see this as a generalised authoritative voice, exhorting students to assimilate sanctioned knowledge, the voice of schoolwork discourse (see paper IV, p. 95). This thesis can help bring about awareness that students (even relatively old students already disaffected by mathematics) *can enjoy* exploring, conjecturing and arguing about mathematics, and how DGS teaching for such engagement can be organised. For a teacher, such teaching involves the difficult task of finding a balance between two aspects of teaching. On the one hand, they need to lead students towards the curricular goals, mastering specific concepts and techniques. On the other hand, they need to listen to students exploring possibilities in dialogue and build on their thinking. Also, the students may describe things in ways that are logical and meaningful, with the potential to be linked with canonical mathematics, yet being difficult to understand for the teacher in the moment. These considerations raise the question of how to prepare teachers for such practice. How do they develop their teaching to create space for exploring, arguing and joyful experiences of grasping mathematics? Here I suggest that a dialogical stance and sensitisation to the complexities of making mathematical discourse internally persuasive are crucial. For example, teachers can be guided that when teaching about functional relationships

- students can and need to add to their way of thinking and talking about motion the possibility to express these relationships in static terms – and this can be supported by moves such as specialisation (what about this specific point?) and generalisation (what always holds?);

- students can and need to add to their way of convincing themselves and others by appealing to pictures the examination of abstract relationships and properties – and this can be supported by moves such as defining new points or other objects according to some specifications, but also needs to be put to the test in dialogue (as pictures and computer representations can be misleading);

- students can develop their own ways to represent situations and relations in increasingly precise ways when they need to make their ways clear to others, both to peers in need of clarifications and to a teacher that acts as a sceptic to push towards increased explicitness. The teacher can then connect the students' ways with canonical systems needed to give commands to a computer.

In such teaching, I suggest that teachers should be truly curious and open to their students' ideas. Great care is needed to not put students in the position that "you are free to say what you want as long as you do it in mathematically acceptable ways".

Task designers could also take something from this research. When designing DGS tasks, care must be taken that tasks can not be completed by manipulation based on visual appearance alone. The reason is that tasks should require thinking about the mathematics in itself but also to make students aware of the difference between specific representations (or presentations) of the same object, which give access to different ways of perceiving them and drawing conclusions. I would also point out the importance of thinking through whole instructional sequences, and trying them out.

On a policy level, we might do good in thinking about what we as a society want from mathematics education. This study describes an alternative practice, based on a dialogical stance, using technology to provide meaning making opportunities. It does not present this practice as an easy solution to be imitated or claim that the teaching is more efficient than what is more typical. It does, however, show that mathematics teaching and learning in upper-secondary school can be more aligned with academic mathematical practices, focused rather on mathematical dialogue than traditional schoolwork mathematics which focuses on reproducing predetermined procedures. And it shows that such practices can lead to sparks of joy related directly to experiences of making mathematical discourse internally persuasive. If we value such moments (and I do) we should put more emphasis on dialogic practices in our curricular policies (such as the national curriculum) and in teacher education.

4.4. Further research

I have in this thesis presented a dialogical interpretation of mathematical communication, from principles derived from the writings of Bakhtin and scholars that have taken his theorising forward. It would be useful to develop a dialogical theory of mathematics education, which would involve networking dialogical theory with theories that are more specifically about mathematical practices and teaching mathematics. This would add depth to the dialogical perspective on mathematics teaching and learning. I have in my papers made use of some of the concepts that Mason uses to label specific mathematical actions and interactions: specialisation, generalisation, discerning invariance in the midst of change, assenting and asserting (see e.g. Mason and Johnston-Wilder (2006). I have also pointed out the importance of balancing listening to students and leading them towards disciplinary mathematics knowledge, which I would like to relate to the concept of mathematics teaching as caring for students and caring for mathematics, as outlined by Watson (2021). There is much more work to be done in combining these (and other) ideas with a dialogical stance.

The digital dialogic tasks that the teacher and I developed need further improvement. Their current state is shown in appendix D (p. 161). Comprehensive descriptions of such tasks, with explanations of their rationales, accompanied with details on possible responses of students, would be an addition to the literature on task design and could support teachers who want to take a dialogical stance and use computers in their teaching. For me, this means that instead of following the same class for an extended period and seeing a task used only once in a group, I am interested in trying out the same task with a number of teachers. This would result in knowledge that could add to and complement the knowledge presented here.

One way forward in researching students' making mathematical discourse internally persuasive, is to ask the question of whether and how it could be productive for a teacher to discuss with students the differences between mathematical and everyday discourse. By this I mean the usually unstated assumptions such as that in mathematical dialogue there comes a point at which it is not the goal to simply make local meaning (that the interlocutor can infer what you mean) but to express yourself so clearly as to convince the most hardened imagined sceptics. Is this something that can only be understood by the experience of being in dialogue with others (who know how to be a sceptic) or can this idea itself become a topic for class discussion?

4.5. Final reflection

My doctoral research, the reading of the literature and writing of the papers and the thesis, has confirmed for me Begle's *second law of mathematics education*:

> Mathematics education is much more complicated than you expected even though you expected it to be more complicated than you expected. (Begle, 1971, p. 30).

The immense diversity of theories and research approaches in the field of education is confusing, but for me, it is also part of what makes it interesting. There is always more to learn and other perspectives to consider. Not only are the questions of how people can learn or not learn mathematics perplexing. The nature of mathematics itself is mysterious and in addition, mathematics education is deeply connected to and has a serious influence on individual lives and the whole of society. My dream of reaching into the heart of what truly constitutes the teaching and learning of mathematics may therefore be too ambitious.

Many things surprised me when I was analysing the recordings, things that I had not noticed (at least not so well as to remember them) at the time that I was regularly in the classroom. For example, there were several moments of joyful outbursts when students had managed to solve tasks (some of which were reported on in papers III and IV) that I had not previously paid attention to. Another thing was that many contributions to dialogue that seemed to me to be nonsensical at first encounter, I could actually make sense of when revisited and thought about in context and by looking back at what they were responses to, and forward to further responses to them by others. The third thing that surprised me was how our language (of teachers and mathematicians) is replete with vagueness and how heavily it relies on unstated assumptions about how we talk about things. This sometimes makes me wonder how on earth students manage to learn mathematics in spite of this.

I have had the privilege of discussing my work at the Nordic/Baltic Doctoral Summer School, and at NORMA and CERME conferences, as well as with my supervisors. These discussions have influenced my research, as they have brought me to see other perspectives and thereby augmented my internally persuasive discourse. My work is in part a response to these voices, a contribution to the long and large dialogues that make up the field of mathematics education.

As I was writing the papers for this thesis, I was very moved by the publication of two books on mathematics education. First, *Mathematics for human flourishing* by Francis Su (2020) gives a rationale for learning and doing mathematics that has nothing to do with its instrumental value as a tool to reach goals outside of mathematics. Mathematics, in itself, is for human flourishing. This resonated with me because I have always felt that school experience should be worthy for its own sake and students should be immersed in activities that they find interesting for their own sake. I don't deny that mathematics can be useful as an instrument, and I am very aware of its exchange value in the form of credentials, as should be clear from my papers. But I see, as Dewey, education first and foremost as "a process of living and not a preparation for future living." (Dewey, 1897, p. 87). The second book is *Care in mathematics education* by Anne Watson (2021) which pictures mathematics teaching as a caring profession. I was aware of the idea from Noddings (2012) and Mason and Hanna (2016), but in the book, Watson develops this thinking and combines the ideas of caring for the student as well as caring for the mathematics into caring for the learning of mathematics. What powerfully resonates with me is her emphasis and appreciation of academic mathematics, its abstractions and reasoning based on properties, alongside a trust that students can think for themselves and a belief in listening to them and engaging in dialogue that seeks to understand their perspective. This is a dialogical stance in the ethical sense.

> Language and listening in the midst of mathematical opportunities are the mechanisms by which care for mathematics and care for learning are fused into care for the learning of mathematics. (Watson, 2021, p. 200)

I hope that this research has shown what a dialogical approach to mathematics teaching with technology can entail, illustrating both its potential and its challenges.

5. Collection of Papers

This chapter introduces the four papers that form the core of the thesis. The papers all deal with different aspects of dialogue in the same mathematics classroom. The papers can be read independently, but together, they form a broader idea of what dialogic teaching entails. Here, I present summaries in order to:

- orient the reader to the temporal place in the sequence of lessons the data for the different papers were taken,
- draw attention to the differences in the basic characteristics of the dialogues (teacher-class or student-student) and the type of tasks worked on in the lessons (computer or pencil-and-paper),
- state the individual research aims of each paper,
- give a short personal opinion on each paper's findings and contribution to the whole of the research.

5.1. Summary of paper I

The data for paper I were drawn from lessons 3, 10 and 13 (see appendix B.1, p. 155). In this paper, I analyse and interpret a teacher-class dialogue on "real-world" problems. Computers are not used in these lessons. The aim of the paper is to describe how everyday, mathematical and schoolwork discourses intertwined and interacted in the dialogue and to discern to what extent mathematical discourses became internally persuasive for students. I see the paper as throwing light on two perspectives that come into tension when such tasks are used: On the one hand, the tasks are interpreted as being about the real world, and students may approach the tasks as such. On the other hand, the task designers (in this case Swan and his collaborators) and the teacher intended the tasks to engender mathematical thinking and bring mathematical ideas in contact with the students' internally persuasive discourses. Building on the Bakhtinian concepts of the centripetal and the centrifugal forces or tendencies in discourse, I constructed the categories of the alien word, irony, authentic real world and emerging mathematical dialogue. The analysis showed how the dialogues were in continual tension, alternately being pulled towards and pushed away from mathematical discourse.

5.2. Summary of paper II

In paper II, the data were drawn from lesson 8 in the course (see appendix B.1, p. 155). It is a fine-grained analysis of a part of a single lesson, in which the teacher tried out a task that we designed in collaboration and had pilot tested with another class in the previous semester. The aim was to identify and characterise the meanings that students made of dynamic geometric functions in GeoGebra, as these meanings evolved in teacher–class dialogue, and as in the first paper, I also asked whether, and to what extent, these meanings can be said to have been appropriated and become internally persuasive for participants. The paper illuminates how the students can start from their own ways of perceiving a graphically represented relationship, and in and through whole-class dialogue, come to more precise descriptions, and an awareness of how they can use their previous knowledge of coordinates to describe and control the graphics. It also shows that in classroom dialogue, there are multiple meanings being made of the same utterances and that while some students (some of whom are active in the spoken dialogue) may be on a progressive trajectory of understanding, others may be stuck.

5.3. Summaries of paper III and paper IV

Paper IV is an elaboration of paper III. For paper III the data were drawn from lessons 19 and 20, while paper IV is also based on data from lesson 18 (see appendix B.1, p. 155). These papers concern students working in pairs on three digital tasks with different characteristics. Although the goal in all of them can be said to be to construct a screen object with some stated properties, the methods by which to achieve this are different. In the first task, the students manipulate sliders to modify parameter values. In the second task, students give symbolic commands, using coordinates and variables, such as of the form $P = (a, a + b)$. In the third task, the students use and type in equations of a predetermined form, $y = ax + b$ but they decide on the parameter values themselves. The aim of the paper is to describe how dialogical teaching centred on these tasks supported students' appropriating mathematical discourse into their internally persuasive discourse and to explore the roles that different types of academic mathematical discourse (technical discourse, visual discourse, and deductive discourses) and schoolwork discourses played in the students' internally persuasive discourse. In all cases, the students explicitly express joy when they consider themselves to have successfully done what is asked of them. Also, in all of these episodes, there are utterances that can only be understood in the context of schoolwork. However, they differ in interesting respects when examined through the question of how and why these students are convinced that they have found a satisfactory solution. In papers III and IV, I relate these differences to the Bakhtinian notions of authoritative discourse and internally persuasive discourse.

Paper I

Paper I

Gíslason, I. (2019).
Centripetal and centrifugal forces in teacher-class dialogues in inquiry-based mathematics
In U. Jankvist, M. van den Heuvel-Panhuizen, & M. Veldhuis (Eds.), *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education* (pp. 1680–1687). Freudenthal Group & Freudenthal Institute, Utrecht University and ERME. https://hal.archives-ouvertes.fr/hal-02435309/
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Centripetal and centrifugal forces in teacher-class dialogues in inquiry-based mathematics

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In this report I examine classroom dialogue in the practice of a teacher who is explicitly committed to a dialogical, inquiry-based approach to teaching mathematics. Of primary interest are the ways in which the dialogues on the mathematical tasks are in continual tension, alternately being pulled towards and pushed away from a proper mathematical discourse, as seen from my perspective as a mathematics educator and an observer of the lessons in the classroom.

Keywords: Classroom communication, discourse analysis, inquiry-based learning, dialogism.

Introduction

If students are to gain a command over mathematical discourse, they need to practice talking mathematics. It is well documented that in traditional classrooms it is predominantly the teacher that talks mathematics, and the IRE (initiation- (short) response-evaluation) format of teacher-student interaction is ubiquitous. Many studies show the meaning making potential of inquiry-based teaching-learning, where the students do more of the mathematical talk (e.g., Boaler, 2002; Goos, 2004; Lampert, 1990) but there is little research on mathematical discourse in inquiry-focused classrooms on the upper-secondary level. I report here on my longitudinal case study of inquiry-based teaching-learning in the practice of an upper-secondary level mathematics teacher working in a challenging classroom in Iceland. I analyse teacher-class dialogue on realistic tasks from a dialogical perspective, showing how the concepts of the centripetal and centrifugal forces (Bakhtin, 1981) may be used to describe and understand the dynamics of the dialogue.

Theoretical perspectives and prior research

In line with discursively focused research, I consider mathematics learning as the increased command of mathematical discourse, a progress from discourse confined to informal everyday language towards the command of a more advanced and formal mathematical discourse – the culturally specific, tool-mediated, historically established ways of communicating that competent users of mathematics use (e.g., Sfard, 2008; Chapman, 1997; Barwell, 2016; Roth & Radford, 2011). Dialogues are the primary objects and units of analysis, and I consider dialogue, following Linell (1998, p. 13) as "interaction through language (or other symbolic means) between two or several individuals who are mutually co-present". A dialogue is a chain of utterances, which are the complete meaningful contributions persons make to the dialogue. An utterance is always a response to what has gone before, addressed to someone(s) in anticipation and expectation of future responses. Utterances contain intentions, emotions, evaluations, which are dependent on context, and sometimes conveyed through means such as tone and gestures. From a dialogical viewpoint, utterances do not have fixed meaning but they do have partly open meaning potentials rooted in culture, history and ideology and the specific speech genres and social languages they belong to. Communication does not presuppose or produce total shared-ness of meaning; rather it consists in people's continued and on-going attempts to expose and test their understandings (Linell, 1998).

In the inevitable difference of two or more perspectives inherent in each utterance Bakhtin identifies a tension between a centripetal force towards uniformity, a shared and common meaning, and a centrifugal force towards diversity of discourse, the meanings made in particular situations by particular persons (Bakhtin, 1981). These forces are depicted in Figure 1 as opposite directed arrows, as we imagine the dialogue happening on a boundary surrounding a shared meaning.

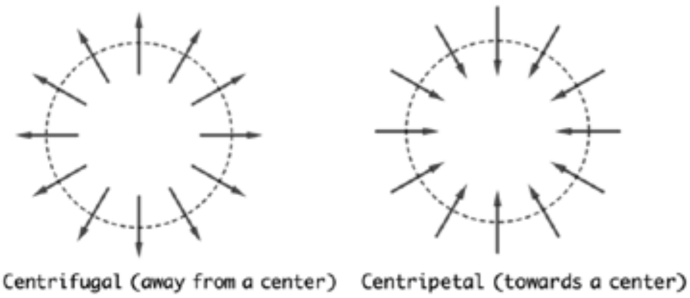

Figure 1: Forces in dialogue

From this Bakhtinian viewpoint I consider the mathematics teacher as working towards the (relative) uniformity of mathematical discourse, while the students make and express more diverse meanings, as they are imbued with language, values and views from their social worlds. When people attend mathematics classes and sense it as a painful, or even meaningless experience, this is because the words of the teacher and the textbooks do not resonate with them and do not move them. Following Bakhtin (1981) we say that the discourse lacks inner persuasiveness. In this all too familiar case, students do not make the discourse their own, it is alien to them. They behave as factory workers although they are not producing things of value for capitalist owners. Rather, they are producing strings of symbols, that are as such of no market value, in order to prove their own value (as good students and future productive citizens). Research published on mathematics in secondary school in Iceland points to a dominance of this kind of alienated form of mathematics work: an emphasis on symbolic manipulations but little emphasis on meaning and no, or very little, exploratory work; the textbook dominates and the class sessions are characterized by teacher transmission and students sitting at desks solving exercises individually (Bjarnadóttir, 2011; Gunnarsdóttir and Pálsdóttir, 2015; Jónsdóttir et al. 2014).

Research method

The setting of the study is an upper-secondary school classroom in which most of the students have a history of very low achievement in mathematics. In order for me to make sense of the classroom dialogues I was present in the classroom the whole semester-long course as a mostly passive observer, concentrating on capturing audio and video of the speaker(s) who have the floor at a given time. I attended the class with a videorecorder in 32 of the 39 classrooms sessions of the course which lasted from late August to December 2012. Although my role was mostly passive, I did interact with students, discussed the situation of me being present in the classroom, asking for permissions to record a pair or small group conversation and sometimes responding (usually rather minimally) to questions about the mathematics tasks. For the purposes of this paper I focus only on the three lessons where there was a public whole-class discussion about realistic mathematics tasks. The tasks are all teacher translations of tasks from Swan, Pitts, Fraser, and Burkhardt (1985). A video analysis tool, Transana, (Fassnacht & Woods, 2012) was used to transcribe video segments synchronously.

I coded the teacher-student dialogues using open coding, drawing on a grounded theory approach (Charmaz, 2006) aided by the dialogical principle that it is the participants' responses that provide evidence for the meaning of a prior utterance. Still, it should be borne in mind, as Bakhtin argues, that "understanding a dialogue as a researcher implies participating in that dialogue as a 'third voice'." (Wegerif, 2007, p. 21). This means that I, as the researcher, make sense as the other participants, anticipating, responding, and creating tentative sense in my own way, based on my cultural background and personal history. After a few iterations, recurrent themes emerged that seemed to capture the different ways the participants responded to the tasks presented, thus contributing to different types of dialogue. In the following I present purposefully selected short sequences of dialogue to illustrate the types. These fragments are then analysed in more detail in order to bring further attention to the centripetal and centrifugal effects of the contributions to the dialogue.

In order to make the reading of the fragments feel natural, the students have all been given unique pseudonyms.

Results and discussion

Inspired by Bakhtin's ideas of learning as adopting words (and gestures, phrases, drawings and all means of communication) of others and using them with one's own intentions with varying degrees of "our-own-ness", and his notions of centripetal and centrifugal forces operating in a dialogue, I classified four types of dialogue. Two of the types refer to distancing in dialogue: *the alien word* (11) and *irony* (29) and two are types of accepting dialogue: *authentic real world* (123) and *emerging mathematical* dialogue (48). The numbers in brackets indicate a count of utterances contributing to a types' appearance to give some indication of their relative frequencies, although an utterance does not always contribute clearly to only a single type of dialogue. The first three types function more centrifugally than centripetally.

To illustrate the most frequent type, *authentic real-world* dialogue, I present a fragment of a dialogue on the question of the nature of a general relationship between the weight of a person and her high-jump ability. The task text states: "Suppose you were to choose, at random, 100 people and measure how heavy they are. You then ask them to perform in 3 sports; High Jumping, Weight Lifting and Darts. Sketch scattergraphs to show how you would expect the results to appear, and explain each graph, underneath. Clearly state any assumptions you make." The following sequence of utterances starts after the teacher has drawn positive coordinate axes on the whiteboard and has exchanged a few turns with students about the nature of the task.

Ari: Like, like, the body weight shouldn't matter crucially, like in high jump. [Eagerly]

Teacher: What.

Ari: Like the guy who is one ninety and a guy who is one sixty. A guy who is one ninety could be heavier than the guy who is one sixty, isn't that right?

Teacher: But if you have two guys that are one ninety and one is heavier and the other is lighter, who

Ari: The lighter guy.

Teacher: What do you think about that? [Addressing the class]

Bjarni: Not necessarily.

Ari: It depends on the technique and where he is muscular.

The request for a relationship immediately provokes a comment from Ari that a functional relationship between body weight and height jumped is not appropriate, body weight is not the most important thing to consider in this context. He also introduces the theme of the height of the imagined people and the relations between those and their body weights. This is not at all mentioned in the task but may be related to the general knowledge that taller people usually are better at the sport of high jump and because the heights of people were a theme of a task these students worked on earlier. In their apparently sincere discussion about the context, they pull the theme of the dialogue from mathematical representations to sports and body types which I understand as a centrifugal force away from the core of mathematical discourse. Ari and the other students are perhaps understandably confused by the realism of the task. They do not know that they are expected to focus on an abstract (and arbitrary) relationship and that for the mathematics teacher it is not important what is important for success in high-jumping. The teacher attempts to pull the dialogue back, trying a what-if question where the real-world factors are controlled, trying to bring the attention to the variables of mathematical interest. The students seem unaffected by the centripetal force of the teacher’s utterance and go off on a further tangent. There followed an animated discussion about important factors relating to the sport of high jump, while the intended task of representing a statistical functional relationship by a scatter graph receded from the dialogue. It is open to question whether some amount of this type of dialogue could actually be a necessary precursor to emerging mathematical dialogue. It can be seen as building emotional connections between teacher and students, showing that their thinking and perspectives are valued and not brushed aside.

The *alien word* was infrequent as a type of sustained dialogue. This is because usually when students experience mathematics as alien to them, they are simply silent, or show their alienation indirectly. They might for example choose to look at irrelevant computer programs, play with their phones, or talk with peers about social matters. Only rarely does the theme of the alien word become explicit in dialogue.

Teacher: Are you all following what we are doing? (...) Do you think this is mathematics?

Einar: I think this is unnecessary bother.

Teacher: Unnecessary bother?

Einar: Yeah.

Gunnar: It’s philosophical mathematics.

Anna: If you think about it in that way, see, that this is the weight plus the weight, but like in weightlifting, I don't feel we are talking about mathematics, rather talking about sports than mathematics.

The first utterance in this sequence was a response of the teacher to the situation that the majority of the student group seemed to be confused by the discussion and had ceased to participate in the dialogue. He tries to pull the class into a meta-discussion about the nature of mathematics and mathematics class, but Einar rejects the question on the grounds that it is unnecessary and a bother. Mathematics to him does not include students' own consideration of real-world complexities. His contribution also indicates that what is important is the amount of bother expected of him. He evaluates bother negatively. Gunnar partially supports Einar in that this is a special type of mathematics. These contributions are centrifugal. Anna makes an interesting comment that resonated with me, as she expressed what I was thinking about the dialogue up till then. It was about sports and not about mathematics. In a way this contribution is centripetal in two ways. She gives a reasonable answer to the teachers' question (she is affected by his pull) and criticizes the discussion on grounds that (in my theoretical terms) it is an authentic real-world dialogue, which she considers (again, in my theoretical terms) to be centrifugal and not conducive to mathematical discourse.

Irony means saying one thing and meaning another. To illustrate irony as a type of dialogue, there follows a dialogue on a task the point of which is to investigate the relationship between the number of workers and the time it would take them to finish a task. The task text asks the student to sketch a graph describing the time it would take to harvest a field of potatoes, depending on the number of potato pickers. The task is presented with unmarked axes in the positive direction, with the label "Total time it will take to finish the job" on the y-axis, and "Number of people picking potatoes" on the x-axis. The students were asked to work on this in their seats, the task being in their text. One student soon exclaims: "I don't understand" and the teacher goes to the whiteboard in front of the class.

Teacher: We are going to think about ... can we maybe assume. What are we going to say that one person would take long to harvest this field?

Siggi: How big is this field?

Teacher: Well, you just decide.

Siggi: Okay, it's just one meter.

Teacher: One meter. [Disbelief in the voice]

Siggi does not answer the teacher's initial question but asks a different one, and I assume he thinks this is an important question to answer in order to finish the task (in fact in the abstract mathematical model this is completely irrelevant and does not need to be answered). The teacher does completely shut down this line of inquiry but indicates a choice available for the student even if it does not really matter what the choice is. Siggi then responds with an extreme example that does not make sense if interpreted in the context of the real-world. Double-voiced discourse is a discourse that has "a twofold direction" (Bakhtin, 1984, p.184); it is directed toward someone else's discourse or toward someone else's position. One interpretation of the student's response is that that the task does not make sense to him or that he is not committed to make sense of it, that the question (and therefore the teacher) is silly and his answer is going to be silly; that the question is too open, and he will test its boundary. It contains an evaluation of the whole situation; it is an ironic rejection to engage authentically in dialogue. The teacher's final utterance in this sequence is an echo, a repetition of the prior utterance, but in a falling tone, with a different pattern of stress on the syllables. I interpret this as an expression of disbelief, and as if to say, "that is ridiculous, you know it, reconsider your suggestion". An alternative interpretation could have been possible: in mathematics it can be informative to specialize (try out specific values in a general expression) with extreme values (not infrequently 1 or 0). The student's ironic assumption could have been taken as an attempt to specialize to see what happens in an extreme instance. However, in light of the effect of the utterance on the teacher and the dialogue closing up at this point, it seems clear that the teacher takes Siggi's contribution to be ironic, and some form of rejection to take the task seriously.

To illustrate the *emerging mathematical* type of dialogue I present a fragment where students are working further on the potato picking task. They have been asked to draw a graph of the relationship between the number of workers (x-axis) and time it takes to harvest a potato field. The teacher has circulated and comes to the whiteboard and says that he has seen "a lot of these" and draws a straight line segment with y-intercept at 100 and a negative slope, but not touching the x-axis. The students' drawings are consistent with the fact that they have not had much (or any) experience with other kinds of graphs in their mathematics in school. Their repertoire of graphs consists of straight lines, as this is what a graph means to them, and they use it to express increasing or decreasing functions. After some students confirm that this is how it is and the teacher acknowledging "in a way, approximately" expecting something more, there follows:

Katie: You need to know how many people you want at the end.

Teacher: Do we?

Katie: Yes.

Teacher: Can't we just…

Katie: If you have the time, you know, the maximum time … how many people are you going to have in the end.

Siggi: We're going to have thirty people.

Katie: Okay, if you are going to have thirty people you just do a hundred divided in thirty, then you have the time a single person has.

Teacher: Okay, you gave us some, what, four hundred eighty minutes? But can we do this like … in mathematics we can add people infinitely. Although there might not be enough room in the field for everyone.

First, Katie expresses that for the task to make sense and be solvable we need to know the maximum number of people. This is reasonable from a real-world perspective, as it would be impossible to have an infinite number of workers, or even a too large number of people. After a contribution from Siggi, Katie tries to express the relationship time divided by number of workers, getting the "time a single person has" where it would be correct to say that this is the total time it takes, but indeed this is also the time each single worker would have to work. My interpretation is that Katie's answer confused the teacher or that the teacher interpreted her as thinking that the workers would work in succession, one at a time. Therefore, he does not build directly on her contribution but returns to a particular number (480) of minutes mentioned long before in the dialogue, and to the question of whether we need to know the maximum number of workers. He indicates that they could be infinitely many, because such is mathematics, even if not realistic.

After the sequence above, there follow over a hundred turns where the teacher and the students slowly clarify the meaning of the graphical representation, the role of the axes, and what the points refer to, they discuss the assumptions needed to make a model, and then gradually move away from a linear model towards a reciprocal one, and at last some students articulated this by themselves, incorporating the model into their own discourse. Here then, the teacher and the student group continually exposed and tested their understandings and perspectives against each other, and the centripetal force was strong enough to keep the dialogue mathematical, bringing students closer and closer to shared mathematical discourse.

Concluding remarks

The students seem to orbit around mathematical concepts while the teacher tries to pull them into a more mathematical discourse. This is the dominating feature of the first three types of dialogue above. The authentic real-world type of dialogue is of particular importance. The students show sincere interest and enthusiasm for the real-world situations in which they recognise important features of the situation. Unfortunately, these features are tangential to the mathematical discourse the teacher tried to draw them into and may result in the dialogue drifting away from mathematical discourse.

To make sense of a mathematical task, the students need to attend to the context and not suspend sense making, while at the same time it also demands that the context is downplayed and not taken too seriously. In that way, the official mathematical discourse itself is double voiced and ironic. It does not really care about the high-jumping ability of people and what factors are most important to consider when estimating or predicting these abilities. It only cares about using graphs and functions to represent idealized phenomena. The teacher needs to convey this and then slowly lead students to productive ways to make mathematical sense, building on the sense made previously. For example, he suggests particular numbers (specializing) to think about or gently challenges the students' answers while often recognizing that they are onto something. In this way the mathematical discourse becomes internally persuasive, at least for some students. We saw how the teacher and the students slowly achieve a more shared understanding in the dialogue in the last example, as it slowly approaches a more mathematical discourse, through constant exposure and testing of understandings.

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Paper II

Paper II

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Discussing Dependencies of Variable Points on the Basis of a *GeoGebra* Task: Meaning Making in a Teacher-Class Dialogue

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Abstract
In this article, I present a meaning-making trajectory of a teacher–class dialogue on digital, dynamic, geometric–functional tasks in an Icelandic upper-secondary classroom of low-attaining students. Learning is seen from a dialogical perspective, as taking up a more mathematical discourse, which here revolves in particular around the language of variables, functions, and Cartesian co-ordinates to describe and construct interactive graphical situations with dynamic geometry software. In the teacher–class dialogue, descriptions of the graphical situations are debated and developed, building on everyday language and intuitions, expanded with more mathematical ways of expression. The main types of discourse identified were (1) in terms of magnitudes and speed, (2) in terms of goal-directed behavior, (3) in terms of vague invariants, (4) in terms of relations between specific points, and (5) in terms of canonical mathematics. The teacher appreciating the students' discourses while prompting for more precision and suggesting specializing was instrumental in eliciting types (4) and (5). The dialogue illustrates the highly non-linear nature of mathematics learning and the complex work it takes to establish shared meanings.

Keywords *Geogebra* · Mathematical classroom discourse · Whole class dialogue · DGS · Meaning-making

If mathematics teaching is to become more about making meaning and less about imitation of procedures, then it is important to understand better how students and teachers make meaning together in authentic classrooms. If the potential of dynamic geometry software for making sense of mathematics is to be realized, we need to explore how such software can support teaching for meaning. The present article addresses these concerns by describing and analyzing mathematical meaning-making

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in the whole-class dialogue of a lesson in an Icelandic upper-secondary school, where students with prior low attainment worked on describing and constructing geometric functional relationships in *GeoGebra*, a dynamic geometry software (DGS). It also provides insights into potentials and difficulties of dialogical teaching with *GeoGebra* tasks, and the specific meanings novices (to mathematical functions) make in regard to dynamic geometric functions, as evidenced by their discourse.

Background

Researchers in mathematics education have argued that student meaning making depends on opportunities for them to articulate their ideas in mathematical conversations and that teachers should build on students' contributions, rather than impose on them ready-made definitions and rules (e.g., Cobb et al. 1997; Kazak et al. 2015; Walshaw and Anthony 2008). This viewpoint has been incorporated into curricular policy documents in many countries, but it is still regularly observed that these ways of teaching do not propagate into regular classroom practice. It remains a challenge for teachers to teach by building on students' ideas. Traditional ways of teaching, where the teacher presents the mathematics for the students to receive without much dialogue, are dominant in Iceland (Jónsdóttir et al. 2014; Sigurgeirsson et al. 2018). In general, the use of technology in mathematics has not been observed to transform traditional pedagogy (e.g., Blikstad-Balas and Klette 2020; Hennessy et al. 2005).

The potential of mathematical software for student meaning-making has been argued for on theoretical grounds (Benton et al. 2017; Feurzeig et al. 1970; Moreno-Armella et al. 2008; Sfard and Leron 1996) and explored in experimental case studies. Many studies on DGS focus on how teachers use the technology or focus on an individual or a pair of students using the computer in experimental conditions (e.g., Ng 2016; Schacht 2018). Research on meaning-making in classrooms where students' use of DGS is integral to daily teaching and learning is much rarer. Sinclair and Yurita (2008) identified changes in teacher–student discourse about geometric shapes ("Is this shape a that shape?"), when moving from static media to using DGS (in this case, *The Geometer's Sketchpad*) in a high-school geometry class, highlighting that teacher expertise made it difficult to appreciate students' difficulties in taking up new discursive routines, as well as their difficulties in understanding the correspondence of dynamic and static geometry. Granberg and Olsson (2015) found that, in an experimental, researcher-taught lesson, *GeoGebra* supported the collaboration and creative reasoning of upper-secondary students learning about linear functions, by providing students with a shared working space and feedback (in the form of immediate *GeoGebra* responses on the screen) that became the subject for students' creative reasoning.

A Dialogical Approach

I take a dialogical approach to meaning-making, both as a methodological principle and as an ontological background (Bakhtin 1981; Linell 1998, 2009; Sfard 2008; Wegerif 2007). Thinking and actions are a response to, and refractions of, the thinking and actions of other human beings, either through direct contact or through cultural artifacts, such as texts, images, and physical tools. Meaning is taken to be actively made in interaction, through spoken language, written symbols, diagrams, and gestures. These communicative actions do not have fixed meanings, but they have *open meaning potentials* rooted in culture, history, and ideology. We use them creatively in ways that depend on the specific situations we find ourselves in, as well as on our emotions and immediate concerns and goals. The meaning of an utterance depends on its embedding in a longer chain of utterances and is always open to be developed further in the course of the dialogue. In the classroom, learners are always responding to the discursive moves of others, which anticipate and call for further responses. Even being unresponsive is a response to a situation.

A Dialogic View of Learning

From a dialogical point of view, learning consists of *appropriating* the words and ways of communicating (by all types of mediational means) of others, which means using them with one's own intentions and increasing degrees of "our-own-ness" and creativity, originating in one's own singular personality, experience, and ideas (Bakhtin 1986, p. 89). This involves becoming aware of others' perspectives, of new (to the learner) properties, relations and distinctions that feature in the others' words (and other signs), and taking up these perspectives in one's own way, using them in one's own thinking and communication. Bakhtin usefully distinguishes two types of discourse: *authoritative discourse* and *internally persuasive discourse*. Authoritative discourse is the discourse of officially accepted knowledge, such as the discourse of teachers and textbooks. Internally persuasive discourse is discourse that resonates with us, is "half-ours and half-someone else's" (Bakhtin 1981, p. 345), and brings to us new ways to make meaning, thereby increasing our creative powers. Discourse can simultaneously have authoritative and internally persuasive qualities, but more frequently, such as in school, the teacher's authoritative discourse stays relatively alien to students and is not appropriated by them for their own purposes.

A dialogical view of learning *mathematics* can thus be seen as appropriating *mathematical* discourse as internally persuasive, increasing one's meaning- and action-potential. It is an expansion of informal everyday talk towards the command of a more mathematical discourse—the historically established ways of communicating that competent users of mathematics employ (Barwell 2016; Pimm 1987; Roth and Radford 2011; Sfard 2008). This does not mean that there is a strict dichotomy between mathematical and informal language, nor that learning to speak mathematically entails shedding one's natural ways of speaking—mathematical discourse is not monolithic (Moschkovich 2018). We speak simultaneously in many voices, reflecting different points of view and ways of conceptualizing the world in words.

The dialogical perspective on mathematics teaching–learning described above does not prescribe any particular approach to teaching and could be used as a lens on any mathematics class. A dialogical pedagogy starts from the premise that dialogue is of utmost importance for learning. A teacher who adopts a dialogical pedagogy teaches mathematics through building on students' ideas through dialogue. The learning of mathematics is then naturally conceptualized as learning to participate in a dialogue, and the teacher teaches *for* dialogue as well as teaching *through* dialogue (Bakker et al. 2015). An important goal for such teaching is that students learn to critique and refine their articulations as they engage in a dialogue, striving to come to shared mathematical meanings (Otten et al. 2015). The teacher would work to meld together canonical mathematical authoritative discourse with the internally persuasive dialogue of the students.

A Dialogic Role of the Computer

Computer programming is often described as "telling the computer what to do," and this useful metaphor has also been used by educational researchers (Papert 1980; Jackiw and Sinclair 2010). I concur with Ball and Barzel (2018), who argue that giving commands to make a technology produce a display *is* communication (with technology), even if it is not exactly like human interaction. From a dialogical viewpoint, programming input can be seen as a communicative action because there is an anticipation made by the learner (based on former interaction with the program as well as with the teacher and peer learners), a response (from the program), an interpretation of the response and then future actions based on this interpretation. An important difference (there are many!) between programming a computer and conversing with a person is that the computer cannot intuit what you mean by the context: it does not have access to any information about goals, needs, prior turns, or the meaning of the situation; it cannot guess what you want it to do, nor does it know about any assumptions. In actuality, to a human a computer may be even less predictable than another human. Nevertheless, its response is completely determined by the input, which means that, to the computer, there are no meaning potentials, only one fixed meaning.

GeoGebra is an interactive mathematics program in which graphics, algebra, and tables are dynamically connected (Hohenwarter and Hohenwarter 2009). The program simultaneously shows the same object in different representational registers and converts between registers, in both directions. Therefore, *GeoGebra* allows for a direct and concrete sensual experience of covariation of independent and dependent values. Most importantly, *GeoGebra* accepts inputs to create objects on screen that represent mathematical objects with fidelity to mathematical structure. For example, if the point A = (x, y) is already present in the system, the command "B = (0.5*x(A), 0.5*y(A))" creates a new point in the system with certain invariant properties in relation to the point A. Another way of creating the same object is to give the command "*B* = Midpoint(Segment((C, A))", which closely aligns with discourse about and in terms of Euclidean geometry. Thus, defining objects and relations in *GeoGebra* is a form of programming and the programming syntax is very close to canonical mathematical discourse—in this case, discourse about points in a co-ordinate system, viewed as pairs of real numbers. Interaction with *GeoGebra* is a type of mathematical discourse.

Mathematicians seek to convey ideas and insights as clearly and precisely as possible, so that arguments are as easy as possible to understand and so that potential weaknesses may be discovered. Sfard (2008), thus described mathematicians as pursuing "perfect, infallible communication" (p. 225). From that aspect, programming seems close to ideal mathematical discourse, clear and explicit enough so that a reader/listener can reconstruct everything without recourse to any implicit knowledge, pictures, or other contextual cues.[1] The unambiguousness of (human) mathematical

[1] Nevertheless, in general, there are many fundamental differences between a mathematical text (spoken or written) for humans and a computer program. Many concepts and symbols are used in subtly different ways.

discourse may often be overstated (see, for instance, Mason and Pimm 1984), but explicitness and preciseness are much more important than in everyday conversation. And they are necessary for creative control over what happens on the computer screen. Therefore, through creating and manipulating screen objects via mathematical commands, students could get a sense of the action potential of mathematical discourse. This might make it likely that mathematical language be incorporated into students' internally persuasive discourse.

In what follows, I identify and characterize the meanings that students made of dynamic geometric functions in *GeoGebra*, as these meanings evolved in teacher–class dialogue. I also ask whether, and to what extent, these meanings can be said to have been appropriated and become internally persuasive for participants.

Method

The lesson reported on here was a part of a larger research project aimed at examining meaning-making in a classroom of upper-secondary school students in Iceland with records of low attainment, where the teacher had as an explicit goal for students to make meaning and to use *GeoGebra* in that endeavor.

Setting and Participants

The school is upper secondary, located in an urban area. It has as an explicit policy that students should be active and independent learners, and teaching is to be "task-focused." The use of information technology is strongly encouraged, and students have free access to laptop computers, although most students bring their own devices to school. The school admits students of all educational backgrounds, and the participant students' prior attainment is varied, with most of them having attained very low final grades in compulsory schooling. As is common in Iceland, the group of students is homogenous with respect to ethnicity and language, all of them being white and having Icelandic as their first language.

Of the 30 learners who enrolled in the course, 21 were older than they would have been if they had done so at the age the school system expects (namely, 16 years old), in the first semester of upper-secondary school. Most of them had not completed (or failed) prior mathematics courses at this school level. The students' educational background and previous attainment in mathematics, therefore suggested that the group would be rather negatively oriented towards mathematics and reluctant to engage in mathematical discussions at the beginning of the course, having an alienated relationship with mathematics. The school policy of using formative assessment and not determining grades by examinations, but rather by coursework and final projects, is a commonly expressed reason learners give for switching to this school from other schools.

The teacher had 4 years of teaching experience, teaching being a second career for him. He and I were acquainted during his studies for an upper-secondary school mathematics teaching license, after which he continued to seek input and ideas from me to develop his teaching. We made an agreement that I would be his conversational partner about his teaching, during which I would be given access to his class for data collection. For the two lessons reported on in this article, I worked with the teacher on creating the tasks, the classroom materials, and the instructional sequence that are reported on here. The teacher's vision of mathematics teaching explicitly included a desire to empower his students with mathematics, teaching through problem solving, and using formative assessment. He had reflected on his own experience as a mathematics learner and concluded that he did not want to teach in the traditional way which, in his experience, resulted in the students not relating to the subject, losing confidence, and not enjoying themselves.

The Tasks

The tasks were collaboratively produced by the teacher and me. The design was inspired by the article of Falcade et al. (2007), in which they described a DGS task (in *Cabri-Géomètre*) where the learners were to move a point directly with the mouse (by dragging) while simultaneously other points also move that were in some way dependent on the first point (the dependent points were defined by geometrical relations). Such use of the dragging tool allows one to experience two interrelated movements (covariation) and sense the functional dependency in space and time (Falcade et al. 2007).

The trace tool displays the trace of the moving points and thereby constructs, in time, a representation of two sets of points: a subset of the domain and the subset's image. Therefore, both global set-wise and local point-wise aspects of a function can be experienced, as it is given a concrete visual form on the screen and can become a topic for dialogue. In their research, no use or connection was made to Cartesian coordinates, but here a curricular goal stated for the course is for students to command the important Cartesian connection between the discourse of graphs (as graphical representations) and the discourse of algebraic expressions.

Often this is done in the context of functions of one variable, but here, we looked at mappings from the plane to the plane, where a point B can be determined from the independent point by a statement such as "if A = (x, y), then B = $(f_1(x), f_2(y))$". The idea is that students can both talk about the function in terms of geometric properties (such as points being collinear or having certain distances from each other) and with algebraic expressions. Mathematical competence includes being able to translate between those different representational registers (Duval 2006), while understanding functions includes becoming aware that they can be expressed through other means than by algebraic symbols, as often is the case.

In the task of the lesson under study, students drag a free point and a dependent point is seen appearing between the free point and a fixed point (see the worksheet https://www.geogebra.org/m/xn5dh8bf and Figure 1).

A worksheet (deliberately not distributed until after a dialogue about the interactive situation) asks students to try dragging each point in the applet, to check which points can be moved and then to say how the other points then move in relation to the point being moved by them. Below a space for writing, there is a second part, asking students to write three different solutions (for how the points move in relation to each other) and to give reasons for one solution to be their favorite. Then, they are asked to add new points to the *GeoGebra* sheet that have a particular (specified) property that the original function has. The prompt was stated as "Add a point to the sheet that will always stay

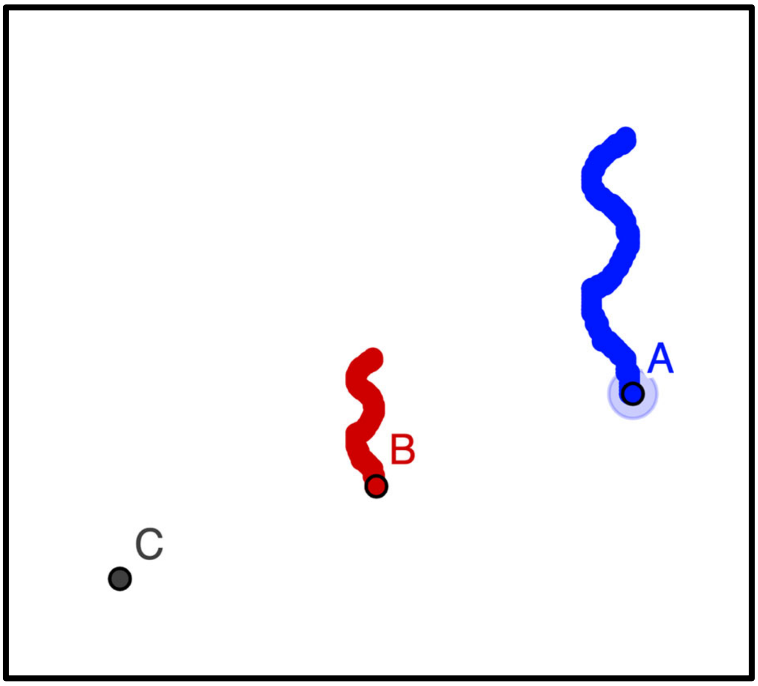

Fig. 1 The applet shows three points A, B, and C: A can be dragged freely, B moves when A moves, but cannot be dragged, and C is fixed

between A and B (no matter what you move). Define it in words and with co-ordinates and then add another point between A and B, and another. What characterizes all points between A and B? In other words, describe how you can always find new points between A and B."

Intended Instructional Sequence

The teaching intention, according to the teacher, was to start with what comes naturally for the students, asking for a more and more precise description until the students agree that the description is satisfying. Ultimately, the goal of the teacher for the class was to create a method that anyone can use to describe situations of this type and use *GeoGebra* (or a related type of mathematical software) to instantiate them. In Bakhtin's terms, we might say that the intention was to start with the students' own internally persuasive discourse and then develop it into an acceptable discourse according to the norms of canonical mathematical (authoritative) discourse. In particular, from our discussion of a pilot study,[2] the teacher expected that, in order to bring students' attention to the invariant relation between the co-ordinates of the points, he would at some point need to specialize. We agreed that he would suggest the point (5, 3), constituting a pair of unequal odd numbers. We wanted the point to be such that the students would be able mentally to divide by two, yet not so easily as to be completely automatic, because we wanted them to be aware of the operation they would have used.

[2] The same teacher had tried out a preliminary version of the task with another group of students in a pilot study a few months before the class reported on in this present article.

Data

The data analyzed in this article was drawn from one lesson in my longitudinal case study of a classroom in an Icelandic upper-secondary school over a whole semester. In the present article, it is the eighth out of 39 lessons that is discussed, and the class had had two prior lessons where they used *GeoGebra*, making most students relative novices in regard to the software. I mostly held a passive role in the classroom as an observer with a videocamera and the video-recordings became the main body of data, being records of communicative actions in the classroom community. The video camera was hand-held and followed my focus of attention: in the public discussion, I captured whomever was speaking (publicly).

The students had their laptop screens in front of them, but in general, I could not see their screens when they held the floor. When the teacher was talking, his whiteboard was also visible, and I took care to capture everything that he put on the board. In the final part of the lesson, when students worked in pairs in their seats, I circulated in the classroom to get an overall sense of what the students were doing, capturing short segments of some of their screens. All verbal utterances were transcribed verbatim in Icelandic and finally excerpts chosen to be presented in this article were translated to English. The analysis was also supported by the pilot study previously mentioned, with the same teacher but another group of students, as well as a subsequent lesson with the same teacher and students working on a similar task.

Dialogical Analysis

When research follows a dialogical approach, the theory of dialogue is used to make sense of the data. Following the dialogical viewpoint, a communicative action (such as an utterance, gesture, or an input to a computer program) can only be interpreted in context, as a response to what has happened before and as an initiation to further development. The interpretation builds as much as possible on the responses the action actually led to, rather than on a guess work on the behalf of the researcher as to what may have been in the mind of the student. Such guesswork, however, is unavoidable to some extent because I, as anyone, make active sense from my ways of conceptualizing the world.

From a dialogical viewpoint, to analyze dialogue is also to analyze thought. Thoughts are not already-existing private objects that are revealed through talk because talking is also thinking. One does not (fully) know what one thinks before one says it, because “verbal articulation mediates and transforms thought” (Roth 2009, p. 23). The meaning that is made of the utterance is not solely under control of the speaker; the responders also have a part in making the meaning. Therefore, it makes no sense to talk about “the true (intended) meaning” of students’ utterances. Rather, thought and meaning emerge and develop in language, through communicative action.

My main interest was on the mathematical ideas and descriptions, on mathematical practices, and on the interactional moves that the teacher and the students made. In the first phase of analysis of the transcript, I coded and categorized individual contributions in an open way, bearing in mind dialogical principles, such as testing possible interpretations of utterances against both immediate and possibly later responses of others. In this way, the meaning of an utterance sometimes became subsequently clearer. As I re-watched the recordings and worked with the transcript to develop the codes, I became more familiar with the data.

It became clear that individual contributions linked together to form longer communicative sequences which could also be analyzed and inductively categorized. These sequences could be demarcated as they fit the following structure: First, the teacher initiated the sequence by requesting a comment or an answer to a question. Then, students made contributions to the discourse, after which the teacher either found a discursive move to continue the dialogue on the same topic or finalized the sequence for the time being, as indicated by a pause in the public conversation until the teacher made another initiating move. Drawing on these sequences, I made dialogically informed interpretations of the classroom mathematical meaning making as it developed over time.

In the next section, I present both an overview of the different meanings that were made and selected sequences of dialogue. The sequences were carefully selected as examples displaying the variety and richness of the student ideas and the interpretational challenges facing the teacher. They exemplify the five main types of meanings that I discerned, although other sequences could also have been chosen.

Results

To show how the meanings developed through the discourse, a table is given, showing utterances in chronological order. The utterances are categorized into five types, as indicated by different borders of the boxes shown in the legend (Fig. 2). Then, the three images in Fig. 3 present utterances publicly made (in that order) in dialogue as the lesson unfolded. The students' utterances are in boxes in the image. Above the students' utterances are typical examples of the teacher's prompts to which the students are responding. The horizontal axis represents chronological order, but not absolute time measure. The vertical axis gives an informal measure going from less to more mathematical student talk. The student utterances are responses to teacher moves and are influenced by his preceding language to various degrees. They are representative for the main thread of the public dialogue in my interpretation, based on the impact on following turns of talk: that is, they are utterances that later utterances address (by referring to them, repeating them, or building on them).

Figure 3 as a whole comprises an interpretation of an analyzed transcript containing 393 turns of talk, out of which 143 were made by the teacher. A recurrent format of interaction was that the teacher initiated the exchange by asking for students'

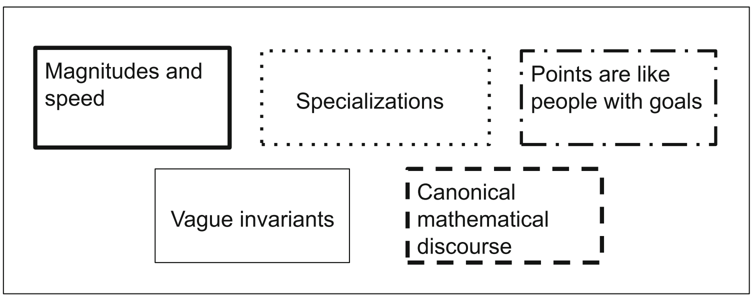

Fig. 2 A key to different box types, that are used in Fig. 3

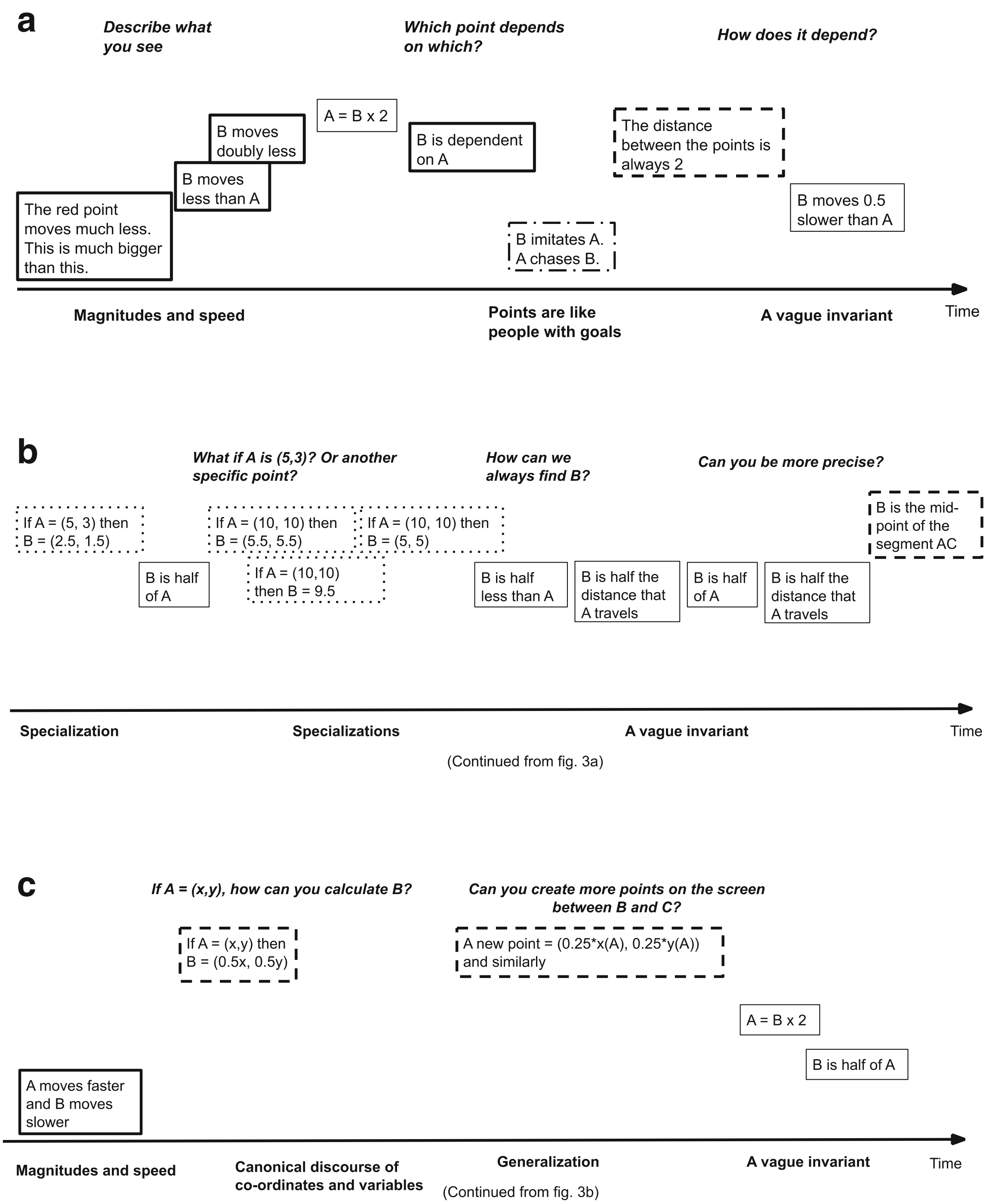

Fig. 3 **a** Part I – from colloquial descriptions towards talk in terms of properties. **b** Part II – fixing the flux: specialization and concluding generalizations in terms of invariant properties. **c** Part III – canonical mathematical descriptions and colloquial descriptions are both alive at the end

description of the situation, then expressed appreciation for their contribution, asked the class for input, and asked the students to clarify a specific point in their answer (e.g., to call for explicit referents, "the distance from *where*" or, in general, to be more precise, in a polite way, "could we be a bit more precise?").

As can be seen in Fig. 3, several utterances recurred after different ones had been endorsed by the teacher as more useful. When he made these endorsements, there were no dissenting opinions or questions voiced and at least some students explicitly assented. From what followed, it is clear that this did not mean that the whole class was internally persuaded.

The more successful learners ended up with a method to create geometric functions, or at least a restricted type of such functions, and it was a method that is extendable.

Some learners seemed to still hang on to the description “B is half of A”, which is ambiguous. It is possible that, if asked what they mean, they would have articulated precisely how the co-ordinates were related, but it may also mean that their conception was still vague. In the following, the five types of meanings are explored in more detail through analysis of exemplary sequences. The sequences are not presented in chronological order, but in the order of the first appearance of a type of meaning. The same types of meanings re-occurred several times, and I did not choose the first instance of a type of meaning if another one was more illustrative as an example.

Magnitudes and Speed

Presented in Sequence 1 below are student responses to an initiating prompt, asking them to describe the situation, without having been taught any methods or vocabulary to do so. The intention here was to elicit what comes naturally for the students to put those ideas up for discussion and development. The students’ first verbal responses to the question were descriptions that mainly focused on stating that the independent point A *moves more* than the dependent point B (all names are pseudonyms):

For someone looking or interacting with the software, the responses of Bjarni and Daníel (turns 15 and 16) can reasonably be interpreted in at least two ways: the motion of one point looks faster than the others or the trace of one point when dragged looks larger than the others. In response, the teacher asked for more detail, and some students attempted to quantify the comparisons: “B moves doubly here” (Bjarni, turn 27).

In further responses to requests for increased precision, the dialogue moved into the territory of relative *speed*, for example “It moves oh point five faster than B” (Anna, turn 84) and “B moves oh point five slower. Because it is half...” (Kristín, turn 89).

These utterances usually did not adhere to accepted syntax of mathematical discourse, although there are reasonable interpretations available, especially for interlocutors who themselves can see the screen and sense the movement of the points. In both Turn 27 and Turn 89, I interpreted students as expressing that the magnitude of speed of point A is twice the magnitude of the speed of point B, or the other way around, that the magnitude of speed of point B is half the magnitude of the speed of point A. The (spontaneous) shift towards talk

Sequence 1 A request to describe answered with terms of magnitude of motion and size

Turn	*Utterance*
1	Teacher: What I am going to ask you to do now is to figure out what can be moved, what cannot be moved. And how do the points move?
	[Student talk not addressed to utterance 1]
9	Teacher: But can you click on any point and drag it around?
10	Multiple students: A.
	[Student talk not addressed to utterances above]
15	Bjarni: The red moves much less.
16	Daníel: This here is much bigger than this here. [Pointing with finger to his screen.]

of speed reveals that the students see the relation between the speeds with which the points move as an important aspect of the dynamic situation.

From the standpoint of mathematical discourse, the utterances are ambiguous—what does it mean to "move doubly"? The utterances (Turn 84) and (Turn 89) are in the zone where mathematical and everyday language overlap: "it moves oh point five slower"—does it mean that one speed is 0.5 times the other (which would be true) or do the speeds have a constant difference of 0.5? This may seem like pedantry, but, as the lesson unfolds, it can be observed later that at least one student did take up this latter meaning. This also illustrates that mathematics needs standards of explicitness and precision, constraining meaning potentials for the time being.

The teacher's first responses to descriptions such as those presented above were to try to direct the student's attention towards the relative *positioning* of the points (in contrast with a focus on their speeds): "Can we say, always, where B is? If we know where A is" (Turn 57).

By so doing, the teacher was following the historical solution of mathematics to the problem of describing motion (see, for example, Stewart 1995, p. 14) by focusing on the (discrete) positions of things (at different points in time), rather than on (continuous) motion as such, a theme explored in the subsections below.

The second thing upon which the teacher tried to focus the students' attention was the idea of determinacy: "Yeah, okay, we can say, if we say that one of the points is dependent on the other. Can we say which one is dependent on which?" (Turn 61).

In everyday Icelandic (and many other languages), this utterance can mean that the position of one point would be in some way influenced by the position of the other, but not necessarily completely determined by it, as in a mathematical functional relationship. Utterances containing some form of the word "dependent" were not heard in student discourse until the teacher introduced it into the dialogue in this way, indicating that the word is not part of their active everyday language prior to the lesson, nor a part of their inner persuasive discourse.

Points Are Like People with Goals

In everyday social discourse, we often describe human movement in space and time by stating what people are trying to achieve, rather than saying where they are. The students used such language of intentionality metaphorically to describe the motion of the points.

In Sequence 2, the mathematical concept discussed is dependency. The common understanding in the classroom that had been publicly established is that B is dependent on A, perhaps meant as one thing being influenced by another rather than being determined in a strictly deterministic sense. But how? Daníel now had responded with a conceptual metaphor: the points were behaving like animated bodies and his description gives an indication of the character of the dependency. Making meaning of one situation (points on screen) in terms of another (people), Daníel talked about mathematical objects as he would talk about living bodies that have agency. The dynamic configuration of points on a screen was described metaphorically, as if the points were like people taking actions: one point is *chasing* or *imitating* another one.[3]

[3] The anthropomorphization of dynamic geometric objects has been documented before, for example, by Healy and Sinclair (2007), who discuss it in terms of a narrative mode of thinking.

Sequence 2 An analogy followed by a request for more precision

Turn	*Utterance*
65	Teacher: But how is it dependent on A? Can we … can we say anything about that?
66	Daníel: It is dependent in that it chases or imitates the movements of A, except it does not move as much or something.
67	Teacher: Okay, this is fine. This is a description of what is happening here. Do we all agree on that?
68	Drífa: Mhm.
69	Teacher: Could we be a bit more precise?

In our everyday speech, this type of description usually captures the most important aspects of what is happening when people's movements and interactions are described (at a certain distance perhaps). They indicate what the students noticed as the essential features of the situation, what captured their attention and what they were aware of at the moment. In effect, the students abstracted away both the relative and absolute positions of the points, as well as their velocity. One could even see both chasing and imitating (perhaps an image is always geometrically similar to the pre-image?) as being mathematically definable classes of functions.

What creates a problem for mathematical discourse is a lack of explicitness—what exactly does chasing or imitating mean here? I also note that Daníel hedged his answer ("or something"), implying that he was not sure, although he hoped to be in the vicinity of a correct answer (and that the teacher would do the work of interpreting it) while not being completely precise. The teacher, in turn, marked the contribution as a good one: he deemed it an appropriate description and sought confirmation for a shared understanding of this from the class. Then, the teacher went on politely to request more precision, again showing an appreciation of Daníel's contribution by saying that they only needed a *little bit* more precision. By this request for more precision (which occurred frequently in this, and other, lessons), the teacher presented a meaning for mathematics that *to be mathematical means to be more precise than in everyday talk.*

An important part of mathematical practice and learning consists of transforming intuitive ideas into explicit and precise statements, thereby incorporating them into mathematical discourse. In contrast, student responses to requests for more precision frequently indicated that, for them, it meant that their contributions were unsatisfactory or even wrong. This would be consistent with students as experiencing the sequence as a classical form of teacher questioning and student answering in traditional mathematics classrooms, where the authoritative teacher discourse dominates.

Vague Invariants ("B is the Half of A")

Much of mathematics, and perhaps other sciences, can be described as being about discerning invariant properties while something changes (Mason 2009). The responses of the students to the teacher asking questions (such as "how is it always?") are mathematical in the sense that they do try to articulate invariant properties. For interlocutors at the scene, their utterances may be sensible, but they are nonetheless vague. The expression "a half of" is used in 34 utterances, of which 19 belong to students and 15 to the teacher. At Turn 121, the teacher wrote from a student's utterance "B is always the half of A", bringing it to the students' attention for further development. Clarifying the meaning of this utterance became the main concern for the teacher. In the following sequence, one student was responding to the teacher's request for more explicitness in the idea "B is the half of A" that had been put forward by a student.

In Sequence 3, Daníel may have been focusing on the distance of the points from the fixed point, C. In fact, as evidenced by later turns, there are reasons to believe that utterances like "B is the half of A" had all of the following five different meanings in this lesson (in the order in which such an interpretation became apparent to the researcher)[4].

1. The speed of B is half the speed of A.
2. The trace of B is "half the size" (a contraction by a factor of 0.5) of the trace of point A.
3. $B = (x/2, y/2)$ if $A = (x, y)$ [The co-ordinates can be calculated by halving.]
4. $d(C, B) = 0.5d(C, A)$ [The distance to B from the origin is half the distance of A to the origin.]
5. $B = (x–0.5, y–0.5)$ if $A = (x, y)$ [This is based on understanding "half of" as "subtracting the number a half from".]

The first four interpretations could all be considered fitting the situation on the screen, whereas only the third one provides a functional relationship. The fifth type of understanding might be more surprising and will be discussed separately.

At Turn 183, the expression "that's just what it says here" is understandable if we think of a goal of everyday discourse as establishing "local meaning," where the assumption "you know what I mean" is often very appropriate. Both in everyday communication, and especially when a child is learning, it is normal to try to say something of which one may only have a vague idea of the meaning. The meaning is then further construed in view of the response. The word is still (and always) "half someone else's". However, mathematical discourse is supposed to be explicit and unambiguous, which is something the teacher was trying to project and cultivate.

Sequence 3 Invariant properties followed by request for referent, followed by a claim that there is no distinction between slightly different elements

Turn	*Utterance*
181	Daníel: B is the half of the distance that A goes.[1]
182	Teacher: Okay, we are getting. Okay, half of?
183	Daníel: But that's just what it says there, B is always the half of A. [Points to the whiteboard where it is written "B is always the half of A".]
184	[Teacher writes on the board and reads slowly: half of the distance that A goes.]

[4] In English, it would be more colloquial to say "B is half of the distance that A goes". In Icelandic, there is no perceivable difference in formality or colloquialness in "helmingur" (half) or "helmingurinn" (the half).

The multitude of meanings that were made of the phrase "B is half of A" illustrates why mathematics calls for explicitness. The students sometimes managed to clarify their utterances, adjusting them to this norm of mathematical discourse, while also sometimes stating that they did not perceive any difference between their original utterances and the ones that emerged later, which the teacher wanted to establish as more precise.

The fifth on the list is based on interpreting the combination "helmingur af" (a half of) as "removal of ½". In Icelandic, the word *af* can be used both as the English word "of" as in "a half of" and as the English word "off" as in "remove this half off". Yet, this second usage would traditionally demand some action (such as "remove") to work grammatically. But this alternative meaning in Icelandic has been documented before (Kristinsdóttir 2016, pp. 231–232), and, interestingly, partly similar phenomena have been observed in English language speakers.

In a dialogue of fifth-grade students, as described in Cazden (2001, pp. 51–54), some students understood "eight minus a half" as 8–½, while others take it to mean 8–4 (as 4 is the half of 8). What is clear is that the subtractive meaning is based on an interpretation of words only, as it does not correspond to what can be perceived when moving the points on the screen. This highlights how difficult it is to have mathematical dialogues with shared understandings and perhaps illustrates why mathematical language is often not internally persuasive.

Specializations

One of the most important purposes of the function concept is to represent how things *change*. A change in the independent variable is often seen as causing a change in the dependent variable and the related variables co-vary. It has been argued that learning about functions and graphs as static objects, rather than as dynamic processes, leads to an impoverished notion of the function concept and may cause difficulties for students in understanding calculus (Ng 2016; Thompson and Carlson 2017). In contrast, modern

Sequence 4 Talk about relative speed followed by a request for specializing

Turn	*Utterance*
84	Kristín: It moves oh point five faster than B.
85	Daníel: And now try to take it down.
86	T: Okay, wait. That A moves oh point five or that B moves oh point five …
87	Kristín and Sara: Slower
88	T: Yes.
89	Ragna: B moves oh point five slower. Because it's half [unclear].
90	DRífa: Both.
91	T: Both?
92	Drífa: Both A moves faster and B moves slower.
93	T: Yes, right? We, of course are moving A. Could you tell me, if I would put the point A in the co-ordinates (5, 3) what, in what, and without moving it, could you tell me where the point B would go?

definitions of mathematical functions do not mention change, cause, or covariation and some mathematicians explicitly emphasize that “nothing varies in mathematics” (e.g., Wu 2011). Sequence 4 illustrates how the teacher shifted the dialogue away from talk about speed towards a correspondence rule (to be expressed as equations later).

Kristín started with a description which contains mathematical terms. Her description proved to be ambiguous to the group and this could possibly be explained by her omitting the arithmetic operation she had in mind. Along with two other students and the teacher, the description was modified and they jointly came to see the situation in terms of the dynamic covariation of the two points. Their dragging of the free point allowed them to experience two interrelated motions: the motion of the point under their direct control with the mouse and that of the dependent point. Through this covariation of the two points, they perceived the functional dependency in space and time. They conjectured about the relation between the speeds, which could be that one of the points moved half as fast as the other. Drífa pointed out that this is relative, that they could just as well say that one point moves faster as they could say that one point moves slower. In this sequence, they do not specify an operation, but it is very tempting to translate “oh point five slower” as “the velocity of B is 0.5 times the velocity of A”, which would be a true statement (if it were made fully mathematically) of the mathematical relationship. Later, it came to light that at least one student had understood the utterance differently, as “the velocity of B is 0.5 subtracted from the velocity of A”.

After Sequence 5, there were 23 turns of talk until a student offered an answer to the specialization question, but in the meantime, another student had expressed that if A were (10, 10), then B would be (5, 5). However, this was not vocalized in a strong voice and was not responded to. The response relevant to Turn 94 came only after the teacher had singled a particular student out, after having observed and listened to him working with his peer. The answer was correct, but further teacher–class talk was then needed to satisfy the teacher that this was generally understood in the classroom, and in the continuation, it was revealed that not at all students understood how to find the co-ordinates of one point, given the co-ordinates of the other.

Sequence 5 A geometric description elicited by the teacher

Turn	*Utterance*
194	Kristín: If C is the starting point, you see, you are in same line, you know then go the starting point C. [Using finger to trace a line in the air.]
195	Teacher: There we bring more precise… You talk about a line.
196	Kristín: Mhm.
197	Teacher: If we imagine a line, between A and B, no, between A and C. Where is then B?
198	Anna: That’s just you know exactly what we are saying in the other.
199	Teacher: What? (pause)
200	Teacher: If we imagine a line between A and B, no between A and C. Where then is B?
201	Bjarni: Right in the middle.
202	Kristín: Yes.
203	Drífa: Right in the middle.

To summarize, after the teacher validated a covariational point of view, he directed the students' attention to a static point of view, to think about a fixed point and the other point in relation to it. This is the first step of the tactic to move from a particular point (5, 3) to a general point (x, y), specializing with the intention to generalize. He used the point (5, 3) as a "specific yet general" point (cf. Mason et al. 2010). He also asked the learners to *imagine* the situation, without actually performing the dragging of the point. This focuses the talk on the *effect* of movement, away from the movement itself. Interestingly, the utterance of the teacher still carried some traces of the dynamic/covariational (process) aspect: he asked not where the dependent point would *be*, but where it would *go*. The covariational interpretation of functional dependency is valuable and important and is the basis for calculus. In the case of our classroom, the weakness in the description is that it did not suffice to determine the position of the dependent point. In mathematical terms, we would also have needed the initial position of B.

Canonical Mathematical Discourse

What counts as mathematical discourse depends on context. In the lesson under analysis, the learning goal can be seen as the appropriation of a mathematical discourse of upper-secondary textbooks (which should be quite close to the discourse of undergraduate mathematics). Mathematical discourse involves reasoning on the basis of properties, not imitation on the basis of syntax alone. Two ways of expressing the functional relationship were established by the teacher as complete and appropriate.

1. The point B is the mid-point of the line segment CA.
2. If A = (x, y), the point B is can be defined as (0.5x, 0.5y).

It was an explicit goal of the lesson to present functions not only as defined by formulas, but also as given solely by geometric descriptions. The geometric description was established first (as a part of the discourse clarifying what "B is half of A" means) in Sequence 5.

In Sequence 6, an algebraic formulation that the teacher considered completely correct was expressed for the first time in the lesson. It was one of the goals of the lesson to have this formulation established and understood. In line with his dialogic teaching approach, the teacher did not immediately express a positive evaluation or

Sequence 6 A Cartesian connection made and the class is asked to respond

Turn	*Utterance*
226	Finnur: Isn't it just B becomes oh point five x comma oh point five y?
	[Teacher is silent for 3 s, looking over the class as if inviting a response.]
227	Ragna (softly): Yeah, makes sense.
228	T: What do you say about this? Is this something?
229	Ragna: Mhm. [Confirming].
230	Daníel: Yes, because x is somehow [unclear].

praise. He kept quiet and looked purposefully over the group of students, awaiting confirmatory responses, questions, or criticism.

Finnur's answer became the official right answer. The teacher's tacit invitation for comment resulted in a statement that the answer made sense, and then, the teacher's explicit request for comments resulted in its further concurring by Ragna and Daníel (229–230) and no dissenting voices or calls for its justification. It was clear that, in an indirect way, the teacher endorsed this description. After this sequence, the teacher then built on Finnur's contribution to introduce a proper syntax for the computer software, which students then used to solve the remaining parts of the task where students are requested to make new points in *GeoGebra* under certain constraints. The teacher wrote the syntax on the whiteboard, which now had the following inscriptions.

B is always the half of the distance that A goes from C.
If A is in (5, 3) then B is in (2.5, 1.5).
If A is in (10, 10) then B is in (5,5).
B is in the middle on the segment between A and C.
If A is in $(x,y) \rightarrow$ B: $(0.5x, 0.5y)$.

The students generally seemed to recognize this last way of expressing the relationship as a natural and useful one. In the next lesson, where another similar task (with another type of geometric function, a projection) was worked on, they immediately sought an expression of this type, relating points through algebraic formulae and co-ordinates, by-passing not only talk in terms of human-like goals or the speed of the points, but also a Euclidean description from geometric properties. This by-passing was not a goal of the teaching as such, but it indicated that this way of describing dynamic geometric functions was internally persuasive to (at least some of) the students, and perhaps, they saw this as "more mathematical" than a geometric description in words.

Conclusions and Implications

The tracing of meaning-making through the duration of the lesson shows that mathematical meaning-making in a classroom discourse does not follow a single path from colloquial discourse to increasingly mathematical discourse. Nonetheless, the five main types of discourse for the points on the screen were identified, in their order of entrance: (1) in terms of magnitudes and speed, (2) in terms of goal-directed behavior, (3) in terms of vague invariants, (4) in terms of specializations, and (5) in terms of canonical mathematics. This can be seen as a progressive development, but it also needs to be stated that some students talked as if they were stuck in (3) when others had moved on to (4) and (5). In the following, I reflect on the five types of discourse and speculate about possibilities for dialogical teaching actions that could be tested in the future.

The initial students' discourses about the points on the screen were in terms of relative motion, of the apparent difference in the speeds of the points and about the behavior of the points in analogies with human action. These formed students' initial inner persuasive discourses and, to them, the teacher must respond in dialogue, recognizing the qualities that could be further developed and connected with more mathematical discourse. In the studied classroom, not much more was done to mathematize the analogies with human action, but this is something that we can now prepare for and see as valuable tasks to work on with students: How can we describe various types of imitation and chasing via mathematical equations, geometric descriptions, or definitions in *GeoGebra* or other mathematical software?

Students' discourse about the relative speeds of the moving points became transformed into a question of static relations, mirroring in a way how the discipline of mathematics tamed motion through a focus on a static correspondence between sets, as there is no talk of motion in modern mathematical definitions of a function. The shift from discourse about relative speeds to static relations came after the teacher had asked for *generalizations* and found the responses vague (or even wrong). In response to these generalizations, he suggested a *specialization*. By the specialization, he directed attention to a single point, a specific yet general point, through which an algebraic formula (a generalization) for the functional relationship became possible.

As many authors have argued, for an understanding of functions, the covariational dynamic viewpoint is crucial (Ng 2016; Thompson and Carlson 2017). Yet, it must be connected to functional expressions, which are static by nature. In a future implementation of the task, it might be worthwhile to try to bring the different discourses to the students' attention, to compare explicitly and contrast a dynamic view and a static view of the situation. This would entail discussing the nature of speed and velocity. For example, the teacher could ask what do the students really mean by speed, how would they take the direction of movement into account, and would other absolute positions of the points be possible with the same relative velocities? This could also lead to a better understanding of the functional relationship as expressed with variables and co-ordinates.

Research on the bridge between everyday language and mathematical language has long recognized that learning to speak mathematically is more than a question of learning a specific vocabulary. It is also learning to flexibly use specific types of sequences of utterances and, more importantly, learning "characteristic modes of arguing" (Pimm 1987, p. 76). Recent research emphasizes that both classroom discourse and the discourse of mathematics experts are hybrids of academic and everyday discourses, so that there is not one correct mathematical discourse that needs to be achieved (see, for instance, Moschkovich 2018, pp. 42–43).

The language students bring to the classroom constitutes a resource that teachers can build on and link with, rather than being an obstacle to be avoided. This includes ambiguity and vagueness (e.g., Barwell et al. 2005), such as was seen in our classroom discourse on "B is the half of A". The teacher did not aim to correct "failures" or misconceptions, but rather to bring into dialogue the students' discourses and more canonical mathematical discourses. Through the dialogue, students had an opportunity to see how the more everyday discourse and the more academic discourse informed and reflected on each other. As Barwell (2016, p. 343) put it, "from a dialogic perspective, mathematics classroom talk is about encountering otherness and the multiplicity of voices, discourses, and languages that can be used to make mathematical meaning".

Some students, in the final part of the lesson, were able to make use of canonical mathematical ways of expressions, both for talking to their teacher and peers, and to gain increased control over the computer. It is impossible to ascertain what was appropriated and internally persuasive over the duration of one or even a few lessons. However, the more successful students could be seen as being in the process of appropriating mathematical language and expanding their meaning potential, having potentially added a discourse in terms of a function concept to their communicative repertoires, their internally persuasive discourse.

As witnessed in the dialogue, some students did not expand their language/thinking in the lesson in this way. This means that the discourse has not resonated with them in the lesson; it was still only an alien authoritative discourse. Those students may have been operating under the norm of everyday discourse that others present would have been able to work out what they meant by using the external representations available to them. For example, they often used mathematical terms with implicit or underdetermined referents. These students likely presupposed that the *goal of the dialogue is to establish local meaning*, enough to go on and finish the immediate task at hand, but not to develop a mathematical discourse. Mathematical discourse makes, and creates a need for, distinctions that are not made in everyday talk, as they are necessary for precise reasoning and representation such as in a DGS (or any mathematical programming). The learners need to become aware of the norms of a more mathematical discourse.

The paradoxical nature of learning to participate in a new discourse has been pointed out before, as the *learning paradox* of Meno: "One's familiarity with what the discourse is all about seems to be a precondition for participation in this discourse, but, at the same time, such familiarity can only emerge from this participation" (Sfard 2008, p. 130). The teacher has the hard work of making sense of what the students might mean by vague utterances (e.g., "B is half of A") (supported by the context, their gestures and what is visually available to the speakers), in order to be able to support students in making explicit what each pronoun refers to and to supply missing information ("the distance of what from where").

In light of the learning paradox, an interesting question is whether and how it could be productive to discuss with students the differences of norms of mathematical and everyday discourse. If they never directly experience a need for more mathematical ways of speaking, they will not appreciate the differences. I argue that the task of making mathematical objects in *GeoGebra* can be an important part in creating such a need, precisely because the computer is not able to do the interpretative work necessary to establish local meaning.

While the focus of this article is not on task design, the structure of the task inevitably affects the dialogue. The task shares similarities with the techno-pedagogic task design framework described by Leung (2011). In that framework, students explore mathematical objects in a DGS (Establishing Practices Mode), describe, and explain them in terms of invariants (Critical Discernment Mode) and then make conjectures, explain, or prove geometric phenomena and construct robust objects (Situated Discourse Mode). What the analysis here shows is that the character of the teaching itself is as crucial as the task design. The dialogical disposition of the teacher and the public dialogue was imperative for the students to make sense of the mathematical objects, in this instance, the functional dependency of the points.

There is no simple way of working with a DGS task that takes the student from ignorance to knowledge. Dialogical teaching requires extensive and challenging interpretational work from the teacher, who must support students in expanding their everyday discourse to incorporate canonical mathematical discourse. Mathematical software such as *GeoGebra* allows the user to "talk" to the computer in ways that are closely aligned with canonical mathematical discourse. The use of techno-dialogical tasks has potential to form the basis for mathematical meaning-making and learning about functional dependency. Teaching through such tasks must both include a responsiveness and a willingness to engage in dialogue, valuing students' thinking and building on it, while also drawing out distinctions and introducing canonical ways of communicating.

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Paper III

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Persuasive moments, and interaction between authoritative discourse and internally persuasive discourse when using GeoGebra

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The purpose of this paper is to explore students' discourse when working in small groups on mathematical problems using GeoGebra. Specifically, the interest lies in to what extent the discourses are internally persuasive and to what extent they are alien to the students themselves. The data analysed are drawn from an upper-secondary class of students with histories of low attainment, focusing on functions and the Cartesian connection between algebra and geometry. Discourses of visual appearance were more present than academic mathematical discourse, and discourses of school as compliance counteracted students' appropriation of mathematical discourses into their own internally persuasive discourses.

Keywords: Classroom communication, discourse analysis, inquiry-based learning, dialogism, dynamic geometry software.

Introduction

Mathematics teachers and educators want students to integrate mathematical ways of thinking and communicating into their own thinking and communicating. According to sociocultural theories, the main goals of learning consist in expanding the students' action- and meaning potentials (e. g. Wells, 1999, p. 48), and other researchers have also argued for the motivating satisfaction that people derive from being able to "do something one could not do before" (Papert, 1980, p. 74). Unfortunately, for many students, mathematical ways of thinking and communicating stay alien to a large extent. Mathematics is only ever thought of as the words of authority, to be imitated in order to satisfy the requirements of the teacher, and ultimately, the school system. Many students thus rarely use mathematics studied in school in order to think or communicate about anything except school tasks.

The affordances offered by dynamic geometry software to visually perceive representations of mathematical objects, including the covariation of variables, have the potential to facilitate student's experiences of being able to do something one could not do before. Yet, there is a lot to learn about how students interact with such software in the classroom and the different types of learning made possible by different didactical designs. I therefore explore dynamic geometry problem solving discourses in an upper-secondary mathematics classroom of students with histories of low attainment. Here *discourses* refer to sequences of utterances.

Learning and authoritative and internally persuasive discourse

When we communicate, we are always responding to, and making use of context, which includes social and physical settings. What we assume to be our common ground, our shared assumptions about the world and the situation we are in and what it is we are trying to achieve, is crucial. When a group of people frequently interact in some sphere of shared activity, they develop patterns and types of utterances that are relatively stable. Bakhtin refers to these types as *speech genres* (1986, p. 60).

The goal of learning mathematics, from a dialogical perspective, is that students expand their discursive repertoires to include *mathematical discourse*—the historically established ways of communicating that competent users of mathematics employ (e.g., Sfard 2008). This constitutes the speech genre of mathematical discourse. But not all talk in mathematics class is directly related to mathematics. An important explicit theme as well as background assumption of communication in classrooms are the demands that the school and the teacher make on students. This I refer to as *schoolwork discourse* – utterances that refer to, or seem interpretable only in the context of the school as an institution that makes demands on students to finish certain work to some standard.

I relate learning to Bakhtin's concepts of *authoritative discourse* and *internally persuasive discourse*. The former is a type of discourse that demands acceptance, and derives its power from social authority, "independent of any power it might have to persuade us internally" (Bakhtin, 1981, pp. 110–111). For example, when mathematics functions as prescriptive rules to be followed, without justifications that are convincing to students, it is an authoritative discourse. On the other hand, a discourse is internally persuasive when it becomes tightly interwoven with "one's own word" (Bakhtin, 1981, p. 345). It is discourse that enters into an interaction and struggle with other internally persuasive discourses, which are all the other "available verbal and ideological points of view, approaches, directions and values" (Bakhtin, 1981, p. 346). It connects with and has an effect on our own discourses, being partly assimilated, partly modified, and always subject to our own creative intentions. In other words, it expands our discursive repertoires.

While a discourse can be internally persuasive without any observable indicators, some types of behaviour would imply that discourse is internally persuasive: when students explicitly make, test and modify mathematical conjectures themselves, they show that the mathematical discourse is interwoven with their own discourse and that mathematics interacts with their other internally persuasive discourses. In contrast, when students apply rules for calculation, without making sense of the rules themselves, after which they ask the teacher "is this right?", the teacher's (or the textbook) discourse has not become internally persuasive, it is only authoritative and remains alien to the students. The research question guiding this study is: How do everyday, mathematical and schoolwork discourses intertwine and interact in the problem solving discourses and to what extent do mathematical discourses become internally persuasive for students?

Method

This paper builds on my longitudinal case study of the classroom. In prior papers I described whole class discussions on contextual (paper-based) tasks (Gíslason, 2019), and a dynamic geometry task (Gíslason, 2021), while here the focus is on students in interaction with peers, while working on tasks.

The setting of the study is an upper-secondary school classroom in which most of the students have a history of low achievement in mathematics. The teacher of the class did not rely on textbooks available on the market, but rather found, translated, and adapted tasks from various sources, and made working with dynamic geometric software (GeoGebra) a centrepiece of classroom work.

The data analysed in this paper is drawn from two lessons, all involving the solving of tasks that are intended to make students aware of the Cartesian connection between algebra and geometry, more precisely on concepts of functional dependency and the interpretations of graphs. I recorded pairs of students selected at random, with a hand-held video camera that followed my focus of attention. My role was generally passive, but I was not invisible and the students sometimes addressed me, both to chat and to ask about mathematical tasks. All verbal utterances were transcribed verbatim in Icelandic and finally excerpts chosen to be presented in this paper were translated to English.

In analysing the data, I followed a dialogical approach. This means that I interpreted communicative actions (such as an utterance, gesture, or an input to a computer program) in context, as responses to what was said and what happened before and as initiations to further responses. The interpretations are also informed by my experience as a mathematics teacher and researcher and I began the exploration with an a priori theoretical distinction in mind, that of discourse being either closer to *everyday discourse* or closer to *mathematical discourse*. In the iterative process of analysis, it became clear that these often were intertwined, and the third major category became apparent, that of *schoolwork discourse*. Finally, I noticed that sometimes students seemed to be convinced by themselves that they had found an answer to a question, while at other times they sought external confirmations from an authority. I took this difference to correspond well to Bakthin's notions of internally persuasive discourse for the former and authoritative discourse for the latter.

Analyses of episodes

In the following I present two sequences of dialogue, selected to illustrate the three main types of discourse, how they intertwine, and function as internally persuasive and authoritative discourse. The students have all been given unique pseudonyms.

Whether to believe one's eyes: everyday speech and alienated mathematics (episode 1)

Ragna and Drífa have created a straight line using the line-tool in addition to the line that is modelled with the parameters *m* and *b* (which may be changed by moving sliders). The intention of the teacher was that students would use the sliders to change the line in accordance with task-directions. The first question asked the students to make it horizontal and go through a fixed point (that they had chosen freely at (4,2)). In this sequence the students address me, as I am recording them, treating me as an authority on the mathematics. Underlining indicates vocal emphasis.

1	Ragna:	So you want this to go [Addressing me, the researcher.]
2	Drífa:	through? [Ragna moves the slider for *m* back and forth, the line changes slope.]
3	Researcher:	We want this line to go through the point, and be … and be horizontal.
4	Drífa:	Okay. [The line now approximately going trough P but it is not horizontal yet.]
5	Ragna:	It isn't horizontal.
6	Researcher:	Then you must change it so that it will become horizontal.
7	Drífa:	You have to move this one. [Drífa points on the slider for *b* (y-intercept).]
8	Ragna:	Oh, okaaay. [Ragna moves the slider for *m* so that the line goes trough P, and then she moves the slider for b which moves the line in parallel off the point. This way does not work.]
9	Ragna:	"No can do". [In English.] [Ragna now adjusts the slope, until m=0, and the line is horizontal, but does not go through P.]
10	Drífa:	Yes! Yes like this. And then you move, no. [Ragna increases and decreases the slope (m), back and forth. Drífa points her finger to the point P.]

11	Drífa:	I think you should do …
12	Ragna:	Aargh. Difficult to own a mac. [Tinkers with the slope until the line is horizontal.]
13	Drífa:	Woah. [Swiftly moves the line, so that it goes through the point P.]
14	Ragna:	What is up!? [A phrase used to express surprise or joy.] [They open the note where they have copied the task-questions.]
15	Drífa:	Tada! ... And then m gives zero and b gives two. Doesn’t it?
16	Ragna:	Yeah [They start typing in the note: m=0] Okay
17	Drífa:	Okay. Isn’t that right?
18	Ragna:	Is it correct? Please tell.
19	Researcher:	Is this a horizontal line through the point P?
20	Drífa:	Yes.
21	Ragna:	It lies!
22	Researcher:	Yes.
23	Ragna:	Totally horizontal.
24	Researcher:	So why are you asking me?

At the beginning of episode 1, the students show that they want to comply with what the authority wants. In turn 1, they seem to assume that I, the researcher, is in the same authoritative position as the teacher and that they are responding to the perceived demand of the schooling situation. I respond with the “we” pronoun, as is common for teachers, possibly to try to frame the task as shared, something to be achieved together. I take this to be an example of schoolwork discourse, as the goal of “finishing the job” is an ubiquitous assumption of schoolwork.

In episode 1, students expressed joy when they managed to adjust the sliders to get a horizontal line (turns 13–15). Their first attempt was to move the slider for the variable b (constant term), until the line coincided with the fixed point, and then change the value of m (slope) to make the line horizontal (turns 4–9). They claim that this is impossible, perhaps perceiving that the “center of rotation” of the line is not in the fixed point. They then try the other way around; change m to make the line horizontal and then change the constant term to translate it (turns 10–14). In turn 15 Drífa reads the parameter values, $m = 0$ and $b = 2$, from the algebra window. By doing so, and noting “the answer”, they show awareness that the mathematical object has both an algebraic representation and a graphical representation. They both express joy that they have found the solution (turns 16–17). However, as far as can be discerned in their talk and interaction with GeoGebra, they relied solely on slider manipulation and visual appearance, without any use of meaningful links between the algebraic and the graphical representations. For example, they did not mention that the slope should be zero to get a line parallel to the x-axis, nor did they express indications of recognising this after the fact. They achieved their goal using mainly everyday speech, more or less bypassing mathematical vocabulary and reasoning based on properties. Having found an answer, they were not fully persuaded by the visual appearance, and rather than linking their work to mathematical concepts themselves, they asked me, in turn 18, as an authority, to confirm the answer. In turn 19 and 24 I strongly imply that they *should trust* their own eyes, which can be interpreted as validating their visual, trial-and-error approach, rather than challenging them to explain their reasoning. It is possible that their experience with mathematics tells them *not to trust* their senses, and it is indeed important to reason on the basis of properties and not only from appearance. In summary, Drífa and Ragna did not make much use of mathematical discourse and it was not present in their internally persuasive discourses. Their everyday discourses were up to the task, and they produced a solution. At the same time they did not *fully* trust that they had an answer that would satisfy the teacher. Perhaps they had some awareness that appearance can be misleading, and therefore they sought additional confirmation for themselves through an authority.

Internally persuasive mathematical discourse (episode 2)

One main goal of the class was to get students to appropriate the language of variables. An experience of covariation can be made possible by creating a variable (represented by a slider) and linking that variable to a screen object, functionally dependent on the variable. The task text was as follows:

> Draw the following in GeoGebra:
>
> a) Make a square that can be enlarged and shrunk with a slider.
>
> b) Add a new slider that moves the square horizontally.
>
> c) Add another slider that moves the square vertically.

The link between a variable and a screen object is not given in the above task, unlike the task in episode 1. It is expected to be created by the student. The variable will create an interactive visual effect, closely linked to mathematical properties.

The teacher assisted students in constructing a dynamic square with vertices (a, a), (a, -a), (-a, a) and (-a, -a). In the following episode, two girls, Lilja and Anna, talk to each other and with the teacher, working on the second question, trying to create a slider that can move the square horizontally on the screen. In the first utterance Lilja suggests a modification, making it possible to move the vertex of the square via a slider determining a variable called *b*.

1	Lilja:	Plus x times b. [Might mean “add one unit, b times” to the x-coordinate, although a more streamlined way is to say “add b to the x coordinate”.]
2	Teacher:	Plus x times b? [Neither affirming nor rejecting, opening for further elaboration. He either does not follow or does not want to make the interpretation for the student.]
3	Lilja:	No [Shakes head, looks at the teacher].
4	Anna:	No... oh … I can’t remember which is x and which is y.
5	Teacher:	Okay the first number is always x and the second number is y.
6	Anna:	Okay should I then do .... aaah [Frustration, both hands waving].
7	Lilja:	We just want the x you know. [Referring to her knowledge that horizontal movement is described by a change in the x-coordinate.]
8	Anna:	Yeah the x is here, I am at the x you know. [Referring to the first coordinate as “x”.]
9	Teacher:	Yes.
10	Anna:	Okay, what should I do just ... plus?
11	Teacher:	Yes, yes what.
12	Lilja:	After the brackets. [Points toward the screen of Anna.]
13	Anna:	After the brackets?

14	Teacher:	Na then you add both to the x coordinate and the y coordinate if you do that. [The teacher knows that (a, a) + b in GeoGebra results in (a + b, a + b). It's unclear that this has an impact on the following turn.]
15	Lilja:	Then not, you should do before the second number ... hooo [Breaths in, throws head back, opens arms, visibly excited.] <u>Before the second number do plus b</u>! [Smiles, increased voice volume and much higher pitch.]
16	Teacher:	Okay that's y, then you move it to the y. [The teacher seems to interpret the suggestion as to write something equivalent to (a, b + a).]
17	Lilja:	No, that, before the first number. [She seems to sense what the teacher meant and the need to make it clear that she means (b + a, a) or equivalently, as she tried to express in turn 15, (a + b, a)].
18	Teacher:	Okay by the first number.
19	Anna:	But why plus b?
20	Lilja:	Because, because when.
21	Anna:	But there is an a there you know.
22	Teacher:	Yes, but yes but it…
23	Lilja:	<u>I got it, I got it</u>! [Visibly excited and joyful.]
24	Teacher:	Okay, show me.
25	Lilja:	Wait, wait.
26	Teacher:	You did a plus b.
27	Anna:	I did something wrong ... I first want to see that she can do it right, then I'll trust you ... first learn to do this plus.
28	Lilja:	Gurrrl gurrrl, gurrrl, gurrrl, gurrrl. Look … oh … gurrrl, gurrrl! [Outburst of joy, smiling and using a higher pitch and volume. Lilja now modifies her coordinates, her suggestion finally implemented as she has meant it, in the software.]
29	Anna:	Okay, uhm, how did you do it?
30	Lilja:	Look, gets bigger and smaller. Just do plus b after the second number.
31	Anna:	Plus b?
32	Lilja:	You know.
33	Anna:	Yes a plus b.
34	Lilja:	Just a plus, ... there ... minus you do also plus a, no, plus b. That's always the first number ... plus b. Now you won't flunk this class! ! [This is accompanied by pointing by her finger on the screen of Anna. She is directing Anna to type, first a + b in the first coordinate place and then -a + b in the first coordinate place of the next vertex. Then she gives a generalisation: always the first number (implied: of the first coordinate-place) plus b.]

Lilja now has a square that can be moved via the slider for variable b. Her square consist of the vertices $(a + b, a)$, $(-a + b, a)$, $(-a + b, -a)$ and $(a + b, -a)$. In episode 2 the students do not immediately solve the problem, which indicates that the mathematical symbolic language of variables and coordinates (as represented in the software) was not initially a part of their internally persuasive discourses. In turn 15 Lilja expresses her excitement in having grasped the nature of the connection between the symbolic slider-controlled variable and the visual behaviour of the screen object. She has not yet implemented her idea, which only happens in turn 28 when she modifies her coordinates and sees the results, verifying her solution. It is as if (this aspect of) symbolic algebraic mathematical discourse suddenly makes sense to her, using the Cartesian link to her own intent, incorporating it into her internally persuasive discourse. At first the teacher does not follow her "before the second number, do plus b" (turn 15), an everyday type of utterance, describing spatial arrangement. Lilja wants to replace (a, a) with (a + b, a) using the variable *b* to control a horizontal movement of the point. She uses everyday discourse to orient to the positions of symbols on the screen, "after the brackets", "before the first number", and the screen object "gets bigger and smaller", yet mathematical discourse is internally persuasive for her and is evident in her input to GeoGebra.

Lilja's partner, Anna, seems not to understand Lilja's description, and she seems to be more or less stuck at trying to imitate (Lilja's) authoritative discourse, grasping for step by step instructions. In turns 6, 8, 10, and 13 she seems to be trying to follow instructions (from Lilja and the teacher) as to what she should type into the software, without consideration of meaning. The mathematical symbolic system seems alien to her, it is only authoritative discourse that does not touch her own internally persuasive discourse. In turn 19, she asks "why plus b", which I see as her attempt to bring the authoritative discourse into contact with her own internally persuasive discourse. She wants the words that she has used to give commands to the computer to have meaning for her. Because what she has typed doesn't work, she expresses doubts as to whether Lilja has really "got it" (turn 27). Lilja describes her solution to Anna only on a syntactical level in everyday language (what symbols to type in and where) but never addresses her why-questions. Instead she tries to encourage Anna that she will not "flunk this class" (turn 34), reminding us that we are in school, talking in a voice from the schoolwork genre. Both Lilja and Anna are concerned to pass the course, but in this episode the mathematical content became internally persuasive only to Lilja.

In contrast with Drífa and Ragna in episode 1, Lilja does not need a confirmation from an authority. She is convinced that she has grasped the symbolic language and the link between that language and the visual representation. She communicates to the computer through the input text: "P = (a + b, a)" and experiences directly an expansion of her action potential. Her outburst of joy *preceded* her actual typing in of the command – it was not a response to seeing it work out (perhaps luckily, through trial and error, as was the case in episode 1). Afterwards, she was quick to also add a functional variable for vertical translation, generalising the method for translation of points in the coordinate system.

Conclusion

In the first episode the students were occupied with the task as a piece of schoolwork to be finished. While they expressed pleasure of having a result (achieved by trial and error), they made no explicit connection between the visual result and mathematical concepts, and also requested confirmation of their answer from an authority. Their satisfaction was due to having finished a job, not with having made mathematical discourse their own. Thus, the schoolwork discourse can be interpreted as being in this case not conducive to learning, or worse, actively working against learning.

In the second episode one student, Lilja, suddenly grasped the relationship between the mathematical symbolism and the visual representation. While she confirmed her answer visually, she was convinced that she knew what needed to be done *before* she gave any input to the software. I interpret her satisfaction as stemming from having made mathematical discourse internally persuasive. Her partner, Anna, did not show any indication of having made the discourse internally persuasive. The schoolwork discourse's assumption that students should finish the tasks set by the teacher frustrates Anna as she was concerned that she might fail the course. Lilja tried to help Anna finishing the task, but described only a step-by-step recipe, without reference to meaning. Lilja, therefore, was not hurt by the schoolwork discourse in this case, while Anna's learning suffered.

The two tasks worked on provide different opportunities to use mathematical discourse to achieve goals. In episode 1, the students manipulated ready made sliders to observe covariation of parameters and a visual representation. This did not make the *mathematical* relationship the center of attention. In episode 2 the students were expected to create sliders for variables and then define the mathematical objects themselves, using the variables. This required students to use mathematical discourse as a semantic tool, which means incorporating it into internally persuasive discourse. One of the students did so, while the other did not.

In their problem solving, students drew on everyday discourse to describe visual elements, both the visual representations of geometric objects and strings of symbols. They also assumed the everyday practice that to be persuaded of something, it is both necessary and sufficient to empirically check its appearence. Mathematical discourse was present in their discourses to a much lesser extent, and in a way that focused more on the surface (the syntax), rather than the conceptual meaning. Schoolwork discourse was always in the background if not explicitly apparent in talk about failing the course. Schoolwork discourse seems push students to imitate authoritative discourse, that is, using mathematical discourse without having made it their own. In other words, schoolwork discourse does not bring authoritative mathematical discourse into contact with internally persuasive discourse. Rather, it functions to keep mathematics only authoritative, and alien to the students themselves.

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Paper IV

Paper IV

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Interactions and tensions between mathematical discourses and schoolwork discourses when solving dynamic geometry tasks: what is internally persuasive for students?
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Interactions and tensions between mathematical discourses and schoolwork discourses in solving dynamic geometry tasks: what is internally persuasive for students?

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Interactions and tensions between mathematical discourses and schoolwork discourses when solving dynamic geometry tasks: what is internally persuasive for students?

I explore students' discourses in small groups working on mathematical problems using GeoGebra, focusing on the Cartesian connection between algebra and geometry. Specifically, the interest lies in what is internally persuasive for students in upper-secondary school (11th grade) with histories of low attainment. Three problem-solving episodes are presented in detail to illuminate different intertwining and interacting discourses that students produce. In all cases, students engaged in visual explorations and expressed joy in the work. Students used discourses of visual appearance and technical symbolism to talk about screen objects and problem solutions. Two cases illustrate how a student makes a conjecture internally before verifying it for themselves and then convincing their peers of the validity of their solution. In a contrasting case, students used visual trial and error and asked an authority to confirm their solution. Discourses concerning the institutional demands of school interfered with students' mathematical dialogue.

Introduction

In this paper, I analyse dialogues from an upper secondary (year 11) classroom that combined two features that have the potential to support meaningful mathematics learning: an emphasis on student dialogue (Bakker et al., 2015; Kazak et al., 2015; Otten et al., 2015) and collaborative use of dynamic geometric software (DGS) for the learning of functions (Baccaglini-Frank, 2021; Gíslason, 2021; Hansen, 2021; Lisarelli 2019; Olsson 2019). A teacher who adopts a dialogic pedagogy teaches mathematics by building on students' ideas through dialogue. Mathematics learning is then naturally conceptualised as learning to participate in a dialogue, and the teacher teaches *for* dialogue and *through* dialogue (Bakker et al. 2015). Students can experience insight in the dialogic act of seeing as if from the point of view of the other when working together (Kazak et al., 2015), and students can realise valued characteristics of mathematical discourse, such as preciseness, because they need to convince each other in dialogue (Otten et al., 2015). Dynamic geometry provides a powerful means for exploring mathematical concepts and relations by visually perceiving dynamic representations of mathematical objects and the covariation of different representations (e.g., Moreno-Armella et al., 2008; Hoyles, 2018). Thus, students working and attending to the screen together is promising as a part of dialogic teaching practice. Dynamic geometry also allows students to create mathematical objects and explore their structures. This creative aspect has been argued to have a motivating effect, as students experience an increase in their action potential (they can do things they could not do before) (e.g., Papert, 1980), giving a satisfying sense of personal ownership of their constructions (e.g. Dalton & Hegedus, 2013).

Research on the collaborative use of DGS for the learning of functions indicates that with careful task designs and teacher support, students can engage in productive explorative mathematical discourse (Baccaglini-Frank, 2021; Gíslason, 2021; Hansen, 2021; Lisarelli, 2019; Olsson, 2019; Oner, 2016). However, these studies also show that working on DGS tasks is not guaranteed to engender productive mathematical dialogue and ensuring the quality of the interactions can be challenging. For example, it is not productive when student groups stick to a one-directional interaction pattern, where only one student leads the conversation by being the primary reasoner and suggestion-maker (Hansen, 2021). It also seems that students need long-term experience in working exploratively with DGS. For example, Baccaglini-Frank (2021) found that a student pair with histories of low achievement in a class that worked extensively with DGS produced more explorative discourse and was more successful in making meaning of mathematical tasks than a higher-achieving pair of students who had little experience with DGS. The latter pair stuck to manipulating symbols without making sense, pointing to the need for sustained and significant student work with DGS. Other researchers have pointed out that in the context of Euclidean geometrical propositions, DGS can be problematic for developing mathematical discourse because it convinces students visually without the need for mathematical reasoning

(Arzarello et al., 2002; Baccaglini-Frank, 2019; Jones, 2000). A similar phenomenon has been identified in research in the context of functions and analytic geometry: Much of the students' dialogue may revolve around the visual appearances on the screen rather than reasoning based on mathematical properties (Gíslason, 2021; Lisarelli, 2019; Olsson, 2019; Oner, 2016). Olsson (2019) compared two task designs on linear functions and showed that tasks with detailed step-by-step instructions to create mathematical objects were less likely to lead to productive reasoning and software utilisation than tasks with less specific guidance. Still, in both designs, students relied mainly on the visual information on the screen rather than engaging in reasoning based on mathematical properties. Oner (2016) studied a different context, that of an online collaborative learning setting, but also found that visual appearance featured heavily in the student discourse, which oscillated between a discourse based on mathematical properties and reasoning about the appearances of screen objects with little regard to their mathematical properties. In Lisarelli's (2019) study, the students communicated about functions and their properties through discourses rich in reference to movement and time, focusing on the processes rather than on the objects, and they found it difficult to verbalise their explorations as they relied heavily on dynamic visual mediators such as gestures and dragging actions to communicate. However, a deeper analysis showed similarities between the students' discourse and a potential expert discourse on functions. Gíslason (2021) also found students focusing on movement, time and processes and trying to guess how to express relationships with algebraic symbolism with little explicit reasoning. Closer analysis revealed that the student discourse mirrored how expert discourse on functions through history developed from a focus on the motion of points to a focus on a static correspondence between sets.

Studies on the collaborative use of DGS for learning about functions, as discussed above, are typically based on data collected outside a regular class with the classroom teacher. Innovative use of DGS (as described in the studies above) is rare in regular classrooms (e.g., Bozkurt &

Ruthven, 2018; Bray & Tangney, 2017; Bretscher, 2021). However, the context of a school classroom greatly influences students' interactions and other behaviour. Studies have shown that students typically prioritise pleasing the teacher above making sense of the subject matter by trying to tell the teacher what they think the teacher wants to hear (Lemke, 1990). They also frequently behave in ways that are poor proxies for learning and understanding, such as faking work or mimicking solution patterns (Liljedahl & Allan, 2013). In general, students focus on producing things (mainly written work) to their teacher's satisfaction without consideration for the meaning of the work (e.g., Doyle, 1988; Goodchild, 2001). At other times students may take up positions of resistance, refusing to do the assigned work, complaining or negotiating work demands, standards, and what will be on the test (e.g., Mellin-Olsen, 1987; Willis, 1977/2006).

Thus, research is needed to understand more about the potential and pitfalls for meaningful mathematics learning in real classrooms with collaborative use of DGS, especially with students with low achievement histories. This paper aims to address this need with analyses of DGS problem solving dialogues in an upper-secondary mathematics classroom through a dialogical lens, explained in the following section.

Theoretical perspectives

To frame the analyses of classroom interactions, I adopt a dialogical theoretical framework, using Bakhtin's concepts of *internally persuasive discourse* and *authoritative discourse*. I define *schoolwork discourse* and *academic mathematical discourse,* and the latter contains the subtypes of technical, visual, and deductive discourse.

The dialogical nature of being

A fundamental assumption of dialogism is that when we, as human beings, communicate, we are always responding to others and their prior actions and communications while simultaneously anticipating possible future responses to our response. Communications also depend on what we assume to be shared assumptions about the world, our physical and social situation, and what we are trying to achieve. Furthermore, we often experience thinking as dialogue, as if our thoughts have both an author and (at least one) addressee, who then may exchange roles as an addressee responding to the author, making thinking and speaking two facets of the same phenomena (Linell, 2009; Sfard, 2008). The meaning of an utterance depends on its temporal position in a longer chain of utterances. However, it is never completely stable, as meanings are constantly renewed in the subsequent chain of responses in future dialogues of the interlocutors (Bakhtin, 1986). These chains of interactions always involve more than one perspective at once, which means that human experiences and actions are fundamentally *dialogical*. It is this chain of questions (something that calls for a response) and answers where each answer gives rise to new questions that define *dialogue* in the Bakhtinian sense (Bakhtin, 1986).

Bakhtin distinguishes two competing characteristics of discourse: authoritative and internally persuasive. *Authoritative discourse* only expresses a single perspective and demands acceptance (Bakhtin, 1981). An example is “traditional” mathematics teaching, where students are expected to accept and imitate the presented content without their perspectives impacting the content or concern about whether their ideas make sense to themselves. Their role is to *follow teachers’ commands, memorise procedures and imitate arguments* (e.g., Civil, 2002; Morgan, 2016; Moschkovich, 2007; Watson, 2008) while the teacher or the textbook verifies the results. *Internally persuasive discourse*, on the other hand, becomes tightly interwoven with “one’s own word” (Bakhtin, 1981, p. 345) through interaction and struggle with all our “available verbal and ideological points of view, approaches, directions and values” (Bakhtin, 1981, p. 346) that constitute our existing internally persuasive discourses. Internally persuasive discourse is not a settled collection of what we believe is true. It contains tensions and differences and is in perpetual development as it is "applied to new material, new conditions; it enters into interanimating relationships with new contexts" (Bakhtin, 1981, p. 346). Its (possibly contradictory) ideas are something we feel the need to respond to (if only for ourselves).

For the context of mathematics learning, I relate the concepts of authoritative and internally persuasive discourse to Mason's (2009) distinction between two modes of student behaviours in the mathematics classroom: assenting and *asserting*. When students assent, they passively accept what they are told and do only what they are shown how to do. They depend on the teacher to evaluate their results and make sense of them. An example is when students ask the teacher, "is this right?" after applying rules for calculations that they were presented with, without any attempt to evaluate their results themselves. In this case, the teacher's discourse is only authoritative and not internally persuasive. When the authoritative mathematical discourse does not connect to internally persuasive discourse, mathematical discourse remains alien to the students. That can be the case even when students complete a task "successfully" by imitation. In contrast, students are assertive when they actively take the initiative "by making, testing and modifying conjectures, and by taking responsibility for making subject pertinent choices" (Mason, 2009, p. 17). Assertive students do not rely on random guessing or trial-and-error to find answers for the teacher to evaluate. Instead, they make personal sense of their work, and reasons guide their explorations. Students' asserting indicates that the academic mathematical discourse has become internally persuasive to a substantial extent for the students, as they use the discourse subject to their intentions and to express their own ideas.

The dialogical nature of mathematics

In defining *academic mathematical discourse,* I build on Lampert's (1990) characterisation of mathematical work as a zig-zag between explorations and verifications. In the *phase of exploration*, people look for structure; they try to grasp how things are and how they work. The explorations can involve empirical explorations, including sketching diagrams or tinkering with graphical interpretations, calculations, abstracting properties, seeking variance and invariance, finding functional relationships, drawing out implications of various assumptions and by all these means, creating conjectures about the structure of our phenomena. In the *phase of verification*, people seek to convince themselves and others of the truth of conjectures created during the phase of exploration. Three types of academic mathematical discourse are operationalised: deductive, technical, and visual.

Deductive discourse

Aiming for conviction, mathematicians use deductive reasoning, verify results, look for errors, clarifications and counterexamples, re-examine their assumptions, and reflect on the goals of the investigation. From a dialogical perspective, as thinking and communicating are two facets of the same phenomena, a significant part of becoming a competent contributor to academic mathematics discourse consists in developing an inner sceptical addressee that engages in an internal mathematical dialogue through which arguments are improved. This "inner sceptic" (Dutilh Novaes, 2020, p. 185) or "inner enemy" (Mason et al., 2010, p. 90) looks for counterexamples, fallacies or unstated assumptions and asks, "why is this true?" of each proposition in the argument. To convince a sceptic, present or imagined, creating a deductive argument demands the highly valued characteristics of academic mathematical discourse of precision and unambiguity, making all assumptions explicit. These characteristics are essential for a convincing argument because they help root out possible errors and objections for the present and future participants in the mathematical dialogue. Discourse involving this type of reasoning is *deductive discourse*.

In contrast, everyday arguments (or most arguments outside academic mathematics or philosophy) are seldom deductive (Dutilh Novaes, 2020, p. 12). For example, we usually assume that although there are premises and justifications for a conclusion, the argument could be defeated by new information, a mark of "defeasible reasoning" (Dutilh Novaes, 2020, p. 3). In everyday interactions, vagueness is typical (people expect that meanings will become more precise or develop in a negotiation as the conversation continues), and to question this impreciseness would usually be considered unreasonable (Wardhaugh, 2006, p. 257). Ethnomethodologists have shown how everyday conversations obey rules of cooperation, trust and turn-taking and that people do not usually confront their interlocutors openly, doubt them, or insist on being perfectly logical (Wardhaugh, 2006, p. 257). "For the person conducting his everyday affairs, objects, for him as he expects for others, are as they appear to be." (Garfinkel, 1964, p. 234). Thus, people usually do not play a demanding sceptic in everyday communication, although we may have, and discuss, different opinions on various matters.

Visual discourse and technical discourse

Both the phases of exploration and verification happen through and in dialogue, whether internally or with others. They both rely on forms of communication whereby the problems and concepts are represented or realised[1]. To facilitate this, people use canonical representational systems (or create new ones). Academic mathematics has developed various representational systems (such as algebraic symbols, geometric diagrams, graphs and tables) in addition to spoken and written words to communicate and think about mathematical objects. Talk in terms and

[1] It is a matter of philosophical contention whether (already existing) mathematical objects are represented or whether there are no independent mathematical objects to represent – they exist only as concrete speech, writings, drawings, imaginations, by which they are instantiated or realised. In Duval's (2006) theory, objects are represented, while in Sfard's (2008) theory, there are visual mediators of realisations instead of representations in light of the fact that there is no privileged thing to be represented. I will use representations because this is a more common word in mathematics literature.

symbols that belong to academic mathematical discourse is *technical discourse*. Such discourse includes writing or enunciating formulas, e.g., "$2x + y$", or technical words, e.g., "parallel line". Technical terms encapsulate specific assumptions about and ways of attending to and thinking about phenomena (Mason, 1998) and are indispensable in mathematics. Without them, deductive arguments and calculations would be impossible. Technical discourses, such as algebraic symbolism, include clear and explicitly laid out rules for transforming one symbolic form into another within the same system (e.g., Duval, 2006).

Talk in terms of the visually apparent, e.g., "look, it goes up", is *visual discourse*. There is considerable literature on the role of pictures in mathematical thinking and reasoning, with various positions as to whether they are to be trusted (Inglis & Mejía-Ramos, 2009). Mathematicians have long argued about the reliability of geometric intuition and pictures in drawing conclusions (e.g., Detlefsen, 2008). What is clear is that pictures and visualising are indispensable in mathematical exploratory work (e.g., Pólya, 1990). Furthermore, an essential part of academic mathematical discourse is the coordination of technical discourse and visual discourse so that one can see or imagine how a technical term corresponds to some visual features of certain types of pictures and be able to transform one into the other. Mathematicians make *graphical perceptual inferences* about whether a mathematical object satisfies a graphical interpretation of some property (Zhen et al., 2016). A crucial example of this is the *Cartesian connection*, through which we relate symbolic algebraic formulae (technical discourse) to graphical representation by the idea that a point belongs to the graph of an equation if and only if the equation is true for the point coordinate values (e. g. Knuth, 2000). Building on the Cartesian connection, mathematicians determine whether a represented function appears one-to-one, continuous, or linear from a graphical representation. This coordination is a significant challenge for mathematics learners (e.g., Duval, 2006). In some cases, we might formulate rules that describe some of these relations, as in the "vertical line test" or in informal talk, as “when the value of the parameter a in the equation $y = ax + b$ increases, the line gets steeper”. These rules are rarely explicit in written mathematical texts and may not be amenable to formalisation. For many, some of these relations are akin to black box processes. They know the rules but do not know the underlying reasons for them and may not be aware of their limitations. Working with calculators and computers may contribute to black box experiences when numbers or graphical objects are shown on the screen without reference to the means or processes underlying their calculation (Boulay et al., 1981; Zbiek et al., 2007).

The dialogical nature of school

In a dialogical interpretation, students are engaged in a dialogic relationship with their school because they must respond to the school’s demands and established behavioural patterns. For this paper, *schoolwork discourse* is defined as communication (by explicit words or action) that *makes the context of schooling relevant*, either directly by being about the organisational practises or requirements of the school or by presupposing these as the context – communication that makes sense only in relation to the school institution. A few examples are that students often speak about how to pass courses, homework, tests and grades. They also say and do things they would likely not do if they were not school students. Examples are copying exercise solutions, asking a teacher if their written answers are correct, or waiting for the teacher to tell them how to solve a set problem. All these acts make little sense outside the institution of schooling.

There are essential differences between being a student in school and being engaged in mathematical inquiry as a mathematician. While students have no choice but to take mathematics classes, mathematicians choose to do mathematics (even if there may be pressures and demands that constrain their freedom), and they choose the problems on which they work. Together with their peers, they debate and determine the solutions and theories to be accepted. Teachers or textbooks usually determine what problems students work on and which solutions are valid. Interactions between teachers and students often follow an initiation-response-evaluation pattern, where a teacher's initiation calls for a short answer that the teacher then evaluates (Cazden, 2001; Mehan, 1979) and then goes on to another question or topic. While the main goals for mathematicians are to look for structure and establish new truths through deductive reasoning, for students, school is an instrument to obtain the recognised qualifications they need or desire (Mellin-Olsen, 1987). Therefore, their primary concern is to meet the demands of the teacher and the school institution to progress through school. Dialogues in school often presuppose that the role of students is to produce things (mainly written work) to their teachers' satisfaction, without consideration for learning, use value outside of school, or meaning of the work (e.g., Doyle, 1988; Goodchild, 2001). Schoolwork discourse does not describe all discourse in all classrooms, but it is normal and expected. It is seen as unusual when schools and teachers give students more responsibility to confirm or disconfirm conjectures.

Research questions

In this research, I explore how students interact when solving problems of a specific type with GeoGebra as a part of their regular schoolwork, with a teacher that stresses teaching through and for dialogue. The research questions in terms of the theoretical concepts introduced are:

1. How can dialogic teaching centred on GeoGebra tasks requiring students to create visual objects support students' appropriating mathematical discourse into their internally persuasive discourse?
2. What roles do academic mathematical discourse (technical discourse, visual discourse, and deductive discourse) and schoolwork discourse play in the students' internally persuasive discourse?

Method

Setting and participants

The study originates in a confluence of my interest in researching alternatives to traditional teaching, especially for students disaffected with mathematics, and a teacher's interest in developing his teaching. The teacher had taken a course on the mathematics of upper-secondary school with me as the instructor. After a year of teaching, he contacted me, and we met to discuss ideas in order for him to develop his teaching. His vision of mathematics teaching was explicitly grounded in an emancipatory concern to make mathematics meaningful and empowering. His reflections on his experience as a mathematics learner led to his conclusion that he did not want to teach traditionally, which he felt typically resulted in the students being alienated from the subject, losing confidence, and not enjoying themselves. Thus, he wanted to base his teaching on student inquiry, discussion and formative assessment. Although his intentions were well aligned with the conclusions of many researchers (e.g., Freeman et al., 2014; Schoenfeld, 2014), policy documents (e.g., Common Core State Standards Initiative, 2010), and the national curriculum, the teacher saw himself as teaching differently from what takes place in the mainstream. Seeing himself as outside the mainstream places him in "alternative mathematics education" (Watson, 2021). He was supported by the school, as it presented (and presents) itself as a progressive school having the policy of organising teaching through task based work and projects and ongoing formative assessment with no final examinations to determine grades. The teacher invited me to numerous informal visits to discuss teaching and observe in his classroom for about three years before I embarked on this study. We agreed that for this study, I would be a conversational partner regarding his teaching, during which I would be given access to his class for data collection with a video camera.

The school is a state Icelandic upper-secondary school that prepares students for university studies, and matriculation gives automatic right to enter the University of Iceland. Like all other upper-secondary schools in Iceland, it has developed its particular school curriculum, approved by the authorities. There are no external examinations for students or mechanisms for ensuring school compliance with the national curriculum, which means that schools have considerable freedom in their approach. However, in practice, mathematics seems to be very standardised, dominated by procedural work and teacher lectures, with results that have been deplored in reports (e.g., Jónsdóttir et al., 2014). The use of information technology is also strongly encouraged in the school, and students have free access to laptop computers, although most students bring their own devices to school. The school takes in students of all backgrounds, and the students' previous performances are varied, with many having been assigned low grades at the end of their final year in compulsory school and many transferring to the school from other schools. Students transferring from other upper-secondary schools likely do so because they have not been successful, and the school policies on assessment may draw students toward it. The teacher's overt ambition and the school's support for exploring alternative ways to teach meaningful mathematics with a diverse student group was thus a unique research opportunity.

The class was for first-year students (the 11th grade), described as a course on functions and graphs. Many of the students had histories of low achievement in mathematics. There were 30 students enrolled in the course, 21 of whom were older than the school system expects (16 years old). For 9 of the students, this was their first mathematics course at the upper-secondary level. Others had either previously failed a course with similarly described content (often at a different school) or taken a preparatory course (required by the school because of their results from compulsory school). Some had re-entered education after being active in the labour market for several years, the oldest being 29. Because of the openness of the Icelandic education system, this type of age mixing is not uncommon in many upper-secondary schools. Aside from the preparatory courses mentioned above, the school did not divide students into groups according to attainment or special educational needs. All students were white and spoke Icelandic as their first language.

Textbooks and teaching materials for upper-secondary schools in Iceland are not produced nor financed by the state. As the market is tiny, few textbooks are commercially produced, so books are lacking for many courses. This lack has led to a situation where many schools produce in-house materials rather than using published books, and this was the case here. The teacher used or adapted tasks from older standard textbooks, made considerable use of his translations of tasks from Swan (1985; 2005), and made GeoGebra central in classroom work.

Data

The data are drawn from a longitudinal case study of a mathematics classroom over a semester. I had a hand-held video camera that followed my focus of attention. If the teacher talked to a group of students, the camera was on him, moving to a student if they contributed to a group discussion. When students were working together in pairs or small groups, I tried to capture the speaker and listener simultaneously, preferably also capturing the computer screen or the paper sheet if these were used, and fingers pointing at the screen or the paper. I did not always succeed at this due to the physical placement of students and laptops in a relatively small and often cramped classroom. I held a primarily passive role in the classroom as an observer. However, the students sometimes drew me into conversations about the research, and I sometimes responded to their questions about mathematical work, as seen in the data presented. In lessons where small groups worked together to solve problems, I video recorded groups of students selected randomly. I tried to capture them from when they seemed to start attempting to solve a problem until they seemed satisfied to have found a solution to a problem or they went on to do something else.

This study focuses on students in pairs or small groups interacting with peers and the teacher, working on dynamic geometry tasks. The research interest lies in classroom dialogues in combination with dynamic geometry. Of all the 31 lessons recorded (out of a total of 38), I transcribed 18 lessons, including the 14 lessons featuring substantial use of GeoGebra, which means that in parts of the lessons, the students themselves worked at problems that were non-routine for them. Out of these, I chose three of the lessons (lessons 18, 19 and 20) to analyse in greater depth and detail based on the commonality of the topic and the mode of work: a) they all involved the solving of tasks that the teacher intended to make students aware of the Cartesian connection between algebra and geometry through achieving a visual effect through mathematical commands, that is, making something appear in a certain way on the screen by using variables, coordinates or equations, and b) students were mainly working together in pairs.

Dialogical analysis of data

Dialogical data analysis aims to uncover the different voices and tensions inherent in individual utterances and sequences of utterances. Utterances are interpreted as responses to what has happened before and as initiations to further development, and interpretations are constantly revised based on subsequent responses in the dialogue.

As the data collection proceeded, I transcribed parts of lessons and wrote analytical notes and reflections, informed by frequent discussions with the teacher. The teacher and I discussed the learning potential of tasks, what we could do to improve their effectiveness, and draw students into mathematical dialogues. After the course, I familiarised myself with the data (as in a first step of a thematic analysis, see Braun & Clarke, 2006) by writing lesson summaries. I noted the type of work done (whole-class discussion, individual seat work, group work, with or without using GeoGebra). I labelled episodes of interactions as involving at least one of the three following topical criteria: they were about schoolwork, mathematics or something else. In utterances about mathematics, I distinguished technical discourse (canonical mathematics words or symbols), visual discourse (appeal to what is seen), or deductive discourse (some form of a reasoning step supported by appeal to agreed-upon properties). I also recorded where I could gauge substantial positive or negative emotional valence in student talk.

In the second analysis phase, I grouped lessons based on commonalities of mathematical topics and mode of work and interaction for further analysis. When analysing episodes of interaction from these lessons, I tried to discern a) the relative roles of academic mathematical discourse and schoolwork discourse and the tensions between the two; b) the relative roles of technical discourse, visual discourse, and deductive discourse and tensions between these; c) the relative roles of authoritative discourse and internally persuasive discourse and tensions between the two. I then selected three episodes for detailed presentation because they feature different aspects of the tensions mentioned above but also have a common feature: a point when a student clearly expresses they had successfully solved the set problem. However, they are not meant to exhaustively show all possible ways the different types of discourses can come into tension.

In interpreting discourse as *academic mathematical discourse,* I especially analyse what representations students use, how they coordinate and use them in their exploring of the problems, as well as how they verify and validate solutions. In interpreting discourse as *schoolwork discourse*, I analyse how discourse relates to the workings of the school as an institution, that is, to the obligations and rights of students and teachers. Interpretations of this kind unavoidably depend on my experience and theoretical view of school and mathematics

In analysing utterances regarding *authoritative* and *internally persuasive discourse*, I kept in mind that these are not necessarily externally observable. Mason (2009, p. 17) notes that assertive behaviour can be internal and need not have visibly overt external expression. However, when students collaborate in pairs or small groups on solving problems, it is possible to observe whether students are more or less assertive. Thus, I looked for students making conjectures, and I looked for what it was that persuaded students that they had found a solution. I realised that sometimes students indicated, by exclamations or statements, that they knew how to solve a problem in GeoGebra *before* implementing their solution method, whereas (more often) they would work by visual trial-and-error until the solution appeared. I also became aware that sometimes students seemed *convinced by themselves* that they had found an answer to a question, while at other times, they *sought external confirmation* from an authority, primarily the teacher, or sometimes me, the researcher. These three observable indicators then operationalise academic mathematical discourse as being internally persuasive discourse for a student: 1) The student is confident in their voice and body language when making a claim or clearly shows that they believe they have a solution, 2) The student must have predicted what would happen on the screen when implementing their solution method and 3) The student verifies the solution themselves, without need to get confirmation from an authority.

In all the analysis, I relied on my own experience of learning and doing mathematics, my teaching experience, and a close and iterated watching of video recordings. Bakhtin argues that a researcher of dialogue always becomes a participant in the dialogue, analogous to an observer of quantum phenomena and “somehow changes [the dialogic system’s] total sense” (Bakhtin, 1986, p. 126). For example, the tensions that I discern in the dialogues in the classroom are not necessarily perceived by the speakers. I see them because I am aware of the conflicting goals students and teachers may have in a classroom and a theoretical perspective of mathematical discourse.

A researcher makes sense of the dialogue as the other participants: anticipating, responding, and creating a tentative sense based on their cultural background and social foreground, their personal history. Therefore, a summary of my background is appropriate. Both of my parents have university-level education. I am white and male and never had trouble with mathematics in my schooling. After academic studies in pure mathematics, I started my teaching career in upper secondary school. There is no universally agreed definition of mathematics. My view of mathematics is strongly affected by the academic mathematical discourse I was inducted into in my university studies, my interpretation of which is described in the theoretical section. I see mathematics as an exploration and analysis of human conceptions about things such as number, space, order and relations. These conceptions develop through dialogues, imagination, visual representation and deductive argument into refined concepts and theories, which function as means to make sense of the world. Perhaps due to my background, it was not easy to respond to my students' difficulties and disaffection with mathematics in my early career. I mainly saw this as a personal choice of students to not pay attention to my instruction, which consisted of explaining things at the front of the class and requiring students to solve textbook problems individually. However, I gradually found that they were more willing to engage with challenging mathematical tasks when I truly listened to what my students had to say rather than explaining and telling them what to do. I also obtained a better insight into their thinking. I came to appreciate that they often had insightful and productive ideas and could discern and describe mathematical relationships, even students who had not done well on tests. Through dialogue, I also realised the school system's significant effects on the students' approaches to learning, for example, when they disclosed that they copied homework to avoid losing points on their final marks. These were perhaps the beginnings of my fascination with dialogue in mathematics education and my deep belief in dialogic relationships in teaching.

I intended to listen to the students and the teacher to understand the meanings made in the classroom from their perspectives. However, it was a challenge not to evaluate the dialogue against standards of correctness and mathematical quality as I saw it as an insider to mathematics and as a mathematics teacher. The students' responses were often not easily interpreted through my insider understanding of mathematical discourse. However, upon further examination, considering subsequent responses of others and repetitions of the speakers, I could almost always find interpretations that made sense (from my perspective). The dialogical principles of multiple meaning potentials and "the next-turn proof procedure" (attending to how one turn by a participant displays an understanding of the previous turn by another participant) (e.g., Ingram, 2021) supported me in looking for and finding the different meanings made with and of a single phrase. My background thus influences the dialogical analysis. However, I aim to give explicitly reasoned and theoretically informed interpretations that can be checked against the transcripts and the theoretical perspectives. In other words, the reader tests my interpretations against their own internally persuasive discourse.

Analyses of episodes

Utterances belonging to schoolwork discourse were a feature in every lesson, although they were more prominent in the first weeks of the course than later as the course proceeded. For example, in the first lesson involving GeoGebra, the fourth lesson of the course, I noted utterances such as "Can we leave when we are finished?", "Can't we just hand in this sheet?" and "Are you happy with our answer?" These utterances address the teacher as an authority that requires certain behaviours and standards with which the students try to comply. In a dialogical interpretation, the last of these could indicate that mathematics functions as authoritative discourse, as it is something to perform for another, who is supposed to take responsibility for the meaning (of an answer). There were also overt moments of joy in most lessons, although these were frequently not directly related to the mathematics content. Instead, they involved interpersonal relationships between students or students and the teacher. For an example that involves mathematics (but not necessarily much mathematical thinking), sometimes students laughed at their DGS constructions when the constructions exhibited surprising visual behaviours. In work involving graphical representations on paper or the screen, students would typically talk about the representations using non-technical words (visual discourse) or describe the mathematical symbols (technical discourse) while looking at the screen and giving commands (by mouse-dragging or typing) to the computer. It was much rarer that they would explicitly connect the visual to the symbolic via mathematical concepts or properties. Visual discourse plays an essential role in all the selected sequences, as does technical discourse, but I discerned no explicit deductive discourse. However, in some cases, as exemplified in episodes 2 and 3 below, I will argue that students appropriated elements of mathematical discourse (graphical inference in the form of the Cartesian connection) into their discourse. In my interpretation, this means that (a part of this dimension of) academic mathematical discourse became internally persuasive for these students.

In each of the three following episodes, a solution to a task was pronounced by students while they explicitly expressed joy. I transcribed all verbal utterances in the selected lessons verbatim in Icelandic and translated the excerpts chosen for this paper into English. The students all have unique pseudonyms. These are only short representative segments of the episodes.

Episode 1: Believe your own eyes (from lesson 15)

The teacher prepared a dynamic line in GeoGebra, modelled with the parameters m and b (adjusted by moving sliders). The equation of the dynamic line, $y = mx + b$, was visible in the algebra window, but the students may not have been aware of this. The stated intention of the teacher was that students would use the sliders to modify the line following the task directions: to make it horizontal and go through a fixed point (that they had chosen freely at (4,2)). Immediately before the first utterance in the following transcript, the students, Ragna and Drífa, created another straight line using the line tool. In this sequence, the students addressed me as I recorded them, treating me as an authority on mathematics. Underlining indicates vocal emphasis, and brackets contain descriptions of actions.

1	Ragna:	So you want this to go [Turning and looking at me, the researcher, while pointing to a line.]
2	Drífa:	through? [Ragna moves the slider for *m* back and forth, the line changes slope.]
3	Researcher:	We want this line to go through the point and be … and be horizontal.
4	Drífa:	Okay. [The line approximately goes through P but is not horizontal.]
5	Ragna:	It isn't horizontal.
6	Researcher:	Then you must change it so that it will become horizontal.
7	Drífa:	You have to move this one. [Drífa points on the slider for *b* (y-intercept).]
8	Ragna:	Oh, okaaay. [Ragna moves the slider for *m* so that the line goes through P, and then she moves the slider for b, which moves the line in parallel off the point.]
9	Ragna:	"No can do". [In English. Adjusts the slope until m=0, and the line is horizontal.]
10	Drífa:	Yes! Yes, like this. And then you move, no. [Ragna changes the slope up and down. Drífa points her finger to the point P.]
11	Drífa:	I think one should do …
12	Ragna:	Aargh. Difficult to own a Mac. [Changes the slope until the line is horizontal.]
13	Drífa:	Woah. [Swiftly moves the line to go through point P.]
14	Ragna:	What is up! [They open the note where they have copied the task questions.]
15	Drífa:	Tada! ... And then m gives zero and b gives two. Doesn't it?
16	Ragna:	Yeah [They start typing in the note: m=0] Okay
17	Drífa:	Okay. Isn't that right? [Turning towards me.]
18	Ragna:	Is it correct? Please tell.
19	Researcher:	Is this a horizontal line through the point P?
20	Drífa:	Yes.
21	Ragna:	It lies [down]!
22	Researcher:	Yes.
23	Ragna:	Totally horizontal.
24	Researcher:	So why are you asking me?

Academic mathematical discourse

Ragna and Drífa seem to have discovered through *empirical exploration* (2-13) a structural property of the links between the sliders and the line: the order in which the sliders are adjusted is essential. They also see that when the line is horizontal, the number *m* is zero. They did not mention this property explicitly, and it is unclear to what extent they were aware of it. Also, they did not mention or describe the slope or its properties, either in everyday or technical terms. They did not explicitly (in words or action) show awareness that the slope should be zero to get a horizontal line, nor made any graphical perceptual inferences ("if we go one to the right, then it goes…"). They achieved their goal by moving sliders and observing the screen, that is, by coordinating slider manipulation with a graphical representation of the line and using descriptive everyday speech between them without substantiating the results by any explicit reasoning.

Schoolwork discourse

At the beginning of the episode (1-2), the students indicated that they aimed to do what the authority wanted ("you want this to …"), positioning myself, the researcher, with the teacher and the school that *wants something*. Therefore, the subsequent dialogue in this episode can be interpreted as a response to the perceived demand of the schooling situation, i.e., schoolwork discourse. They both expressed joy that they had found the solution (14-15), as the phrase "what is up" (hvað er að frétta?) is often used by young people in Iceland to express surprise or joy. Their joy may have stemmed from being able to do something they did not know how to do before. Nevertheless, based on their approach to the task as a whole and their lack of subsequent controlled use of line equations in their problem solving, I interpret their joy as deriving primarily from finishing a required school job.

Authoritative discourse

Despite relying on visual appearance in their exploration phase, Ragna and Drífa were not entirely convinced by the visual appearance. They enunciated and typed in the equations $m = 0$ and $b = 2$ (technical discourse), which are the slider values they saw (15-16). Then they sought an authority to confirm their answer (17-18). I told them to *trust their own eyes* and thereby validated their visual, trial-and-error approach (19-24). Their uncertainty did not lead them to link their work to mathematical concepts or re-examine the task or their exploration process. For example, they did not mention whether they expected or felt that $m = 0$ would be consistent with the line being horizontal. A possible counterpoint to the interpretation that mathematics functions as authoritative discourse is that in turn 21, Drífa showed conviction by her exclamation that “it lies!” where she uses visual appearance to back up their answer. However, she does not refer to how a slope would relate to the graphical object, in effect only basing her claim on the authority of the computer screen simultaneously showing a horizontal line and the values of m and b and disregarding their roles in the equation.

Internally persuasive discourse

In this episode, did academic mathematics function as internally persuasive discourse for these students? I could not discern that Drífa and Ragna made a conjecture to test what would happen on the screen in response to their tinkering with the sliders. They did not coordinate the symbolic representation $y = mx + b$ with the graphical representation of the line except to announce the values for the parameters at the end of their empirical exploration. However, these values can be seen from the slider values, disregarding their roles in the equation. The Cartesian connection itself was thus ignored. They showed that they did not fully trust their solution, asking for an external source of confirmation, thus not being assertive. I conclude that mathematics was not and did not become incorporated into their internally persuasive discourse.

Interactions and tensions between mathematical discourses and schoolwork discourses

My counter-question, “so why are you asking me?” (24) to the request for confirmation, reveals a tension between the schoolwork discourse, which seems to predominate in the episode and my wish for the students to take up mathematical discourse *through a schoolwork* task. My wish was that the students convinced themselves of a mathematical idea, whereas the students saw the goal as finishing the task to my satisfaction. Perhaps the unstated follow-up answer to my question (24) could be, “you set the task, you decided the goals, so you tell me if I am meeting your expectations”, seeing as Ragna ascribed the goal of the task to the researcher (“So you want this to …?”) (1). Such an answer would contradict my desire for the students to desire a mathematical resolution to the task.

Episode 2: Seeing and not seeing the Cartesian connection (from lesson 16)

One goal for the class was to get students to appropriate the language of variables. An experience of covariation can be made possible by creating a variable (represented by a slider) and linking that variable to a screen object functionally dependent on the variable. The task text was as follows:

> Draw the following in GeoGebra:
> a) Make a square that can be enlarged and shrunk with a slider.
> b) Add a new slider that moves the square horizontally.
> c) Add another slider that moves the square vertically.

In this task, the students must create the link between a variable and a screen object. The variable produces an interactive visual effect closely linked to mathematical properties. The teacher assisted the class in constructing a dynamic square with vertices (a, a), (a, -a), (-a, a) and (-a, -a), as shown in figure 1. In the following sequence, Lilja and Anna talk to each other and with the teacher, working on the second question, trying to create a slider that can move the square horizontally on the screen. In the first utterance, Lilja suggests a modification to the vertex (a, a) to make it possible to move it via a variable called b (determined by a slider).

1	Lilja:	Plus x times b.
2	Teacher:	Plus x times b?
3	Lilja:	No [Shakes head, looks at the teacher].
4	Anna:	No... oh … I can’t remember which is x and which is y.
5	Teacher:	Okay the first number is always x and the second number is y.
6	Anna:	Okay should I then do .... aaargh [Both hands waving].
7	Lilja:	We just want the x you know.

8	Anna:	Yeah the x is here. I am at the x you know.
9	Teacher:	Yes.
10	Anna:	Okay, what should I do just ... plus?
11	Teacher:	Yes, yes what.
12	Lilja:	After the brackets. [Points toward Anna's screen.]
13	Anna:	After the brackets?
14	Teacher:	Na then you add both to the x coordinate and the y coordinate if you do that.
15	Lilja:	Then not, you should do before the second number ... hooo [Breaths in, throws head back, opens arms, visibly excited.] Before the second number do plus b! [Smiles, increased voice volume and much higher pitch.]
16	Teacher:	Okay that's y, then you move it to the y.
17	Lilja:	No, that, before the first number.
18	Teacher:	Okay by the first number.
19	Anna:	But why plus b?
20	Lilja:	Because, because when.
21	Anna:	But there is an a there you know.
22	Teacher:	Yes, but yes but it…
23	Lilja:	I got it, I got it! [Visibly excited and joyful.]
24	Teacher:	Okay, show me.
25	Lilja:	Wait, wait. [Lilja types at her keyboard.]
26	Teacher:	You did a plus b.
27	Anna:	I did something wrong ... I first want to see that she can do it right, then I'll trust you ... first learn to do this plus.
28	Lilja:	Gurrrl gurrrl, gurrrl, gurrrl, gurrrl. [Smiling and using a higher pitch and volume. Lilja now modifies her coordinates] Look! … oh … gurrrl, gurrrl!
29	Anna:	Okay, uhm, how did you do it?
30	Lilja:	Look, gets bigger and smaller. Just do plus b after the second number. [She turns her laptop and shows her screen to Anna.]
31	Anna:	Plus b?
32	Lilja:	You know.
33	Anna:	Yes, a plus b.
34	Lilja:	Just a plus, (pause) there (pause) minus you do also plus a, no, plus b. That's always the first number (pause) plus b. Now you won't flunk this class! [She points her finger at Anna's screen.]

Lilja's final square consisted of the vertices ($a + b$, a), ($-a + b$, a), ($-a + b$, $-a$) and ($a + b$, $-a$), and her screen would look as shown in figure 2.

Academic mathematical discourse

Lilja's explicit discourse throughout the whole episode consists, to a large extent, of describing sequences of symbols (technical vocabulary) without reference to anything else, neither what is visually apparent nor to mathematical concepts. Nevertheless, the symbols must mean something to her because she manages to create the dynamic object by using the symbols and not by empirical exploration or tinkering with the symbolic inputs to the computer. Interpreting Lilja's specific mathematical utterances is not easy, but here is my attempt: In (1) by "plus x times b", she might mean "add one unit, b times" to the x-coordinate (there is no variable called x), which could also be expressed by "add b to the x coordinate". This interpretation is supported by the following sequence (4-8), where it becomes apparent that they use the word for x to talk about the first coordinate. In turn 7, Lilja says that they "only want the x", which seems to mean that because they seek horizontal movement, there should only be a change in the x-coordinate. This interpretation is supported by Anna's immediately following turn, where she claims that she is "at the x" (referring to where her screen cursor is) and Lilja's solution in turn 15, which only involves modification to the x-coordinate. In turns 9-14, the students and the teacher consider the suggestion to "do plus after the brackets". The teacher seems to interpret this as a suggestion for the expression (a, a) + b because, in GeoGebra, that would result in (a + b, a + b). It is unclear that this impacts the following turn, where Lilja exclaims that the solution is "before the second number do plus b". By this, she tries to direct Anna (as evident in her successful solution) to type +b after the symbol in the first coordinate place, before the comma symbol that separates the coordinate places – that is, she wants her to modify (a, a) to (a + b, a). However, in the next turn, the teacher seems to interpret the suggestion as writing something equivalent to (a, b + a), which would not work. Lilja responds (17) to this as if she understands the teacher's interpretation and tries to clarify that she meant (b + a, a) or equivalently, as she tried to express in turn 15, (a + b, a). There are still several turns before she has made clear that this is what she meant, and she still has not made her suggested input to the computer. The fact that Lilja knows how to achieve the goal before implementing the solution shows that she does not base her solution on a perceived *empirical* relationship, as the students in episode 1 did, or working by trial and error, using visual appearance to guide her. No, she conjectures about the relationship between the symbolic and the graphical representations. After a few nonspecific (or at least hard to interpret) turns (18-22), Lilja erupts in joy in turn 23 and onward because she is convinced she has found a way. Finally, in turn 28, Lilja visually verifies the construction (30) and shows it to Anna. Anna asks her how she did it, and Lilja responds by providing a purely technical description of how to organise the symbolic input, directing her to type first a + b in the first coordinate place and then -a + b in the first coordinate place of the next vertex. Then she gives a generalisation: always the first number (implied: of the first coordinate place) plus b.

Schoolwork discourse

While Lilja seems to be working things out, her partner Anna is lost because the mathematical symbolism has little meaning for her. In turn 4, she indicated that a problem for her was not remembering mathematics, echoing a common assumption of traditional school mathematics that the student's role is to memorise facts and procedures. In turns 6, 8, 10, and 13, Anna tried to follow instructions (from Lilja and the teacher) about what she should type into the software, suspending consideration of its meaning. In turn 27, she is concerned that she "did something wrong", which I interpret as schoolwork discourse because her main interest is producing a correct answer. Even though she desires meaning, as seen in her questions in turns 19 and 29, Lilja does not explain the meaning of her directions to Anna. Instead, Lilja described her solution to Anna on a syntactical level in everyday language (what symbols to type in and where) but never addressed her why questions. Lilja then tried to encourage Anna by stating that she would not "flunk this class"

(34), talking in a voice from schoolwork discourse as both Lilja and Anna were concerned about passing the course.

Authoritative discourse

For Anna, the symbolic system (technical discourse) seemed alien. It seems to function only as an authoritative discourse that does not touch her internally persuasive discourse, as she neither asks nor describes the connection of the symbols to graphical objects explicitly, nor does she manage to make them work on the computer. She is left trying to follow directions from others. In turn 19, she asked, "why plus b" which I see as her attempt to bring the authoritative discourse into contact with her internally persuasive discourse. She seemed to desire that the words she used to give commands to the computer have meaning for her. Because what she had typed so far did not work, she momentarily gave up and expressed doubts about whether Lilja had really "got it" (27). Throughout the sequence, Anna seems not to make meaningful contact with Lilja's description, seemingly stuck trying to imitate authoritative discourse, grasping for step-by-step instructions from her partner and the teacher.

Internally persuasive discourse

Lilja and Anna did not immediately solve the problem, which indicates that the technical discourse of variables and coordinates to achieve graphical effects was not initially an available part of their internally persuasive discourses. In turn 15, Lilja expresses her excitement that she is grasping the nature of the connection between the symbolic slider-controlled variable and the visual behaviour of the screen object. She had not yet implemented her idea, which only happened in turn 28 when she modified her coordinates and saw the results, verifying that her method worked. It is as if (this aspect of) symbolic algebraic mathematical discourse suddenly made sense to her, and the Cartesian connection became transparent to her. To summarise: She *first* asserted a conjecture and *then* *verified* it *herself* (and showed the result to her partner). She did not ask an authority for confirmation. The conjecture is about the coordination of symbolic forms and graphical representation rather than a deductive argument. However, the fluid coordination of representational systems is a fundamental part of academic mathematical discourse. I, therefore, interpret this as Lilja appropriating academic mathematical discourse into her internally persuasive discourse. In general, Lilja used everyday discourse to refer to the positions of symbols on the screen, "after the brackets", "before the first number", and the screen object "gets bigger and smaller". At the same time, her mathematical discourse was internally persuasive for herself and explicitly present in her input to GeoGebra. The teacher and Anna seem almost overwhelmed by Lilja's emotional energy, and in turn 28, Lilja enters her coordinates as she has described and implements her idea in the software. She shows pride and ownership of the work when she shows the screen to her partner. In addition, she proceeded to solve the next part of the problem with ease.

Interactions and tensions between mathematical discourses and schoolwork discourses

Anna and Lilja show in several ways that they are both concerned about schoolwork – rooted in their need to solve the problem set by the teacher. At the same time, they are both concerned with academic mathematical discourse: while Lilja explores and solves the problem, Anna desires to understand Lilja's solution suggestions (19) – Anna is *not* content to only imitate the solution. The tension lies in that Lilja's explanations (descriptions of syntax), which *for her* belongs (at least partly) to mathematical discourse (a means of conjecturing), do not address Anna's concern for understanding (how the syntax relates to the objects described). They only address her need to finish the task – therefore, *from Anna's perspective,* these explanations function only as schoolwork discourse.

Episode 3: Do not believe your own eyes (from lesson 14)

In this episode, the objective was to create a star on the screen by typing equations for straight lines. The instructions for the first subtask were: Enter an equation of the form y = ☐x into GeoGebra and put some number inside the box. There was a model picture and some further questions, including the question, "why is it not possible to write an equation for a vertical line in this way?" The Starburst task described by Magidson (2005) inspired the task. The teacher and I believed that it had the potential to further students' command over linear equations, allowing them to create interesting visual phenomena through the coordination of symbolic algebraic forms with graphical representations of these forms.

Drífa and Ragna worked together, but they also interacted with Tanja, who sat beside them, working on her laptop. Tanja had, at this point, made a star that Drífa and Ragna claimed was nicer looking than their own because the lines came out more evenly spaced. However, they noted that a vertical line was missing, which Tanja was working on (perhaps not having noticed that the task text stated that this was impossible with the kind of equation given).

1	Tanja:	Wait! [She looks with intensity at the screen. Ragna and Drífa look towards her. She then types on the laptop.] ... I got it! [In a very high pitch. She does a little dance in her seat].
2	Ragna:	No!? You're lying! How did you do that? [Smiling while intensely looking at Tanja's screen.]
3	Tanja:	Ahm, well, a thousand ex.
4	Ragna:	A thousand ex.
5	Drífa:	A thousand, just a thousand ex! How the ...
6	Ragna:	Where do you have the homework? [Ragna gets a notebook from Tanja's bag.]
7	Tanja:	Just on the first pages.
8	Ragna:	What a fat math book, you're really studying for all it's worth? [Shows Drífa the homework task.]
9	Tanja:	It's a thousand ex, the straight line up. [Forcefully, addressing another student passing by, pointing up with her index finger]
…		
10	Teacher:	Okay well done …. there are a few … how did you do it? [The teacher addresses the whole class.]

11	Gunnar:	What? Ex equal to zero.
12	Teacher:	Ex equal to zero. [Flat, neutral intonation.]
13	Drífa:	What?
14	Tanja:	No! It's ex … it's a thousand ex. … It's a thousand ex, he's probably talking about this line [makes her hand and index finger horizontal]. That's zero.
15	Gunnar:	No.
16	Teacher:	Ah, did you do, yes okay, that's a very good point. You did y equals a thousand ex.
17	Tanja:	Yes.
18	Teacher:	Then the slope becomes extremely high.
19	Drífa:	Then it's just
20	Tanja:	Straight down.
21	Teacher:	Is it an absolutely straight line?
22	Tanja:	Yes.
23	Teacher	If you want to go really, really far away … but it would mean that if we went one, notice, if we went one to the side … then we are …
24	Tanja	It's completely straight.
25	Teacher	What? Then we are a thousand up. So it goes a tiny bit to the side. It looks straight but it isn't.

There followed a discussion between Tanja and the teacher where the teacher suggested to (a reluctant) Tanja to try different zooming actions to be able to see that the line would not be perfectly vertical. Tanja did not explicitly modify her conjecture but silently adjusted her GeoGebra sheet. The teacher went on to discuss the task with the whole class. Subsequently, he explained that any line of the form $y = mx$ would not be perfectly vertical, although it would be "closer to it". He explicitly referred to Tanja's idea and asked the class to consider increasing the slope even more, setting it, for example, to a million.

Academic mathematical discourse

In this episode, Tanja seems to *generalise*, from her preceding *empirical exploration* with line equations, that higher numbers for a in the equation $y = ax$ result in steeper lines closer to vertical. She asserts that the equation $y = 1000x$ (a symbolic formula belonging to academic mathematical discourse) will construct the desired object in turns 3, 9 and 14. The teacher engaged with the class (10-18), soliciting answers and getting a correct one in turn 11, which he repeated (12). In a typical classroom, students often take such repetition as either an authoritative confirmation or rejection, depending on the intonation (does it sound like a question, doubtful, or confirmatory). However, in this classroom, the teacher often repeated to the class such contributions, both correct and incorrect, for further consideration. Hence it can be interpreted both as academic mathematical discourse and schoolwork discourse. Against other suggestions (11), Tanja insists, and the teacher validates her contribution ("good point") and restates her idea. He recognises its merit and meaning (16-18), the graphical inference that as the slope increases, the steeper the line appears. It might have been tempting for the teacher to point out and explain why Tanja was wrong, but he recognised the mathematical sense in her idea. In the last turns presented (19-25), Tanja stuck to her answer (which *was* backed up by visual evidence) while the teacher enacted the voice of the sceptic, fundamental to academic mathematical discourse. He hinted at using deductive reasoning ("if … then"). He referred to a blend of the abstract mathematical object and its representation, asking her to imagine what would happen if they "went one to the side" (23) (which converts to asking what the change in the y variable is if one is added to the x variable). Here, I see the teacher and Tanja engaging in academic mathematical discourse (incorporating a prover and a sceptic) in which there was tension between the everyday way of backing up a claim (by empirical appearance) and deductive argument.

Schoolwork discourse

A discourse of schoolwork appeared when Ragna interrupted Tanja and Dría to talk about homework (6-7), finding a physical copy of the homework task and commenting on Tanja's work habits (8). By this, she may have closed on the opportunity for Tanja to explain to Drífa how she solved the problem. In any case, Ragna and Drífa also seemed content to accept Tanja's suggestion without any explanations and copy her method for their solution (which was to be handed in to the teacher). However, they got somewhat upset (13) when doubts emerged in the dialogue. Tanja's defence of her answer and reluctance to revise her approach might indicate a tension between the academic mathematical goal of illuminating phenomena for mutual understanding, improving arguments through interchanging positions between prover and sceptic, and the schoolwork goal of having produced a correct answer.

Authoritative discourse

For Ragna and Drífa, the connection between the equation form and the graphical line seems opaque. This part of mathematics discourse functions as authoritative discourse for them as they do not use it to achieve the desired results. They seem not concerned about whether the mathematical symbols make sense to themselves, only trying to produce acceptable answers by imitating Tanja's suggestion.

Internally persuasive discourse

In (1), Tanja got an idea. Her joyous body movements and confident exclamation indicate that she had formed a conjecture internally that she was convinced would work. Like Lilja in episode 2, Tanja indicated that she had got it *before* she typed her equation. She must have predicted that such a line would appear vertical, and indeed it can be considered vertical in the visual context of the assignment. She *used* symbolic algebraic mathematical discourse (to interact with the software and to create in her mind the image of a vertical line) while empirically confirming the verticality of the line through visual appearance. Like Lilja, she did not seek confirmation from authority and showed her belief in her solution. In the last turns presented (19-25), Tanja defended her answer, backed up by the visual evidence, unwilling to consider Gunnar's suggestion or engage with the teacher's counterexample. She was very reluctant to let the other's discourse influence her internally persuasive discourse.

In summary, Tanja had an aha moment (1), where she used her understanding of the Cartesian connection to make a visibly vertical line. Her explicit assertive behaviour and conjecture indicate that academic mathematical discourse (of graphical inference) was internally persuasive for her. Her following public claim and insistence that her way is right (1-5, 9, 14-17) support this conclusion. Her joy could also be partly due to her finishing the task— following the assumption underlying schoolwork discourse that success is simply finishing each task the teacher sets (which was the salient feature in episode 1).

Interactions and tensions between mathematical discourses and schoolwork discourses

Tanja uses mathematical discourse (graphical inference) to explore and form a conjecture about the problem (1). For her partners, Ragna and Drífa, however, the same discourse seems to function predominantly as schoolwork discourse, although Drífa displays a flash of wanting to connect with it ("how the …", in turn 5). Then, a more heightened tension arises when another student, Gunnar (11, 15), and the teacher (21, 23, 25) disagree with Tanja's solution. It seems her investment in being correct in the eyes of the teacher and her peers (her solution *as* schoolwork discourse) counteracts her engaging with the critical response of the teacher. This response, from the teacher's point of view, I interpret to be a case of an invitation to more mathematical discourse, but from Tanja's point of view, it seems to function more as schoolwork discourse. She could answer his challenge by pointing out that she successfully solved the task as stated (she indicates this by affirming that her line is vertical, in turn 24). If the goal is to produce the visual effect, she has achieved it (the task as schoolwork), while if the goal is mastering a specific mathematical structure (the task as mathematical discourse), she has not reached it yet.

Discussion and conclusion

In this study, I explored problem-solving dialogues of students with histories of low achievement when they worked on GeoGebra tasks together, creating visual objects through technical discourse. I asked 1) whether these types of tasks could support students' appropriating mathematical discourse into their internally persuasive discourse and 2) how academic mathematical and schoolwork discourse were present and how these interacted in students' internally persuasive discourse.

To answer the first research question, I argue that these tasks supported students' appropriating mathematical discourse into their internally persuasive discourse by engaging students in the *phases of exploration* and the *phases of verification*. In all three episodes, students try to grasp how things are and how they work through visual empirical explorations (episode 1) and graphical inference (episodes 2 and 3). The students are also concerned with verifying their solutions visually (all episodes) and by appealing to authority (episode 1). In the result section above, in each of episodes 2 and 3, one student goes through the following: 1) They clearly show that they have got an idea of how to create the visual object through technical discourse *before* trying the idea, that is, *predicting* what will happen on the screen in response to their input; 2) They exhibit assertive behaviour, speaking with confidence; 3) They verify their predictions themselves, without any need for confirmation from an authority. In episode 3, the student, Tanja, even sticks to her solution after the teacher has begun to challenge her solution. I interpret these three features as implying that these students have appropriated (at least to some extent) the Cartesian connection between (algebraic) technical discourse and visual discourse into their internally persuasive discourses, even if they do not verbalise the connection and do not use deductive reasoning in their explicit discourse. In this way, DGS tasks can support students' appropriating mathematical discourse into their internally persuasive discourse in a regular school context. The potential of

DGS tasks for learning outside regular class (e.g., Baccaglini-Frank, 2021; Hansen, 2021; Lisarelli, 2019; Olsson, 2019; Oner, 2016) is also present in the context of regular school.

However, success is not guaranteed. For example, in episode 1, I do not find any sign that the students use the connection between the parameter values and the visual appearance of the line object. They also express uncertainty and request confirmation of their answer from an authority. Therefore, graphical reasoning is not present in their internally persuasive discourses. An alternative interpretation is that although they may have formed internal conjectures about the Cartesian connection, they are concerned that their perception might be deceiving—a natural and essential concern for mathematical discourse, as evidenced by the mismatch between the visually apparent and the mathematical concepts in episode 3, and echoing the worries of mathematicians through the ages about the reliability of geometric intuition and pictures in drawing conclusions (see, e.g. Detlefsen, 2008; Inglis & Mejía-Ramos, 2009). Although, in episodes 2 and 3, while one student seems to have appropriated academic mathematical discourse, their partners do not. Their interactions are one-directional in that only one student leads the conversation and suggests solutions while the others try to follow. Hansen (2021) also observes this pattern in her study of group problem solving with GeoGebra, with participants selected based on high mathematical and verbal competence.

The second research question concerns the roles of academic mathematical and schoolwork discourse in the students' internally persuasive discourse. In all three episodes presented, some students make it explicitly relevant that they are in school, concerned with meeting the teacher's demands. In episode 1, this is evident when the students ask what the teacher/researcher wants and when after finding a solution, they ask an authority to decide on the correctness of the answer. Their satisfaction seems more plausibly connected to finishing a job than expanding their internally persuasive discourse. The goal of the students only to finish the job is not only in tension with the goals of mathematical discourse (to grasp the nature of things) but also creates tension for the teacher and me. We set the task and demand students to work on it, which constitutes authoritative discourse. Nevertheless, our desire is not that the students work on the task but that they appropriate the desire for exploration, that they *want* to explore and verify, making the mathematical discourse internally persuasive.

Academic mathematical discourse is also present in all three episodes, expressed through technical and visual discourse but no explicit deductive discourse. In the first episode, there is little connection between the technical discourse (mentioning and writing the values for two parameters) and the visual discourse (whether the line looked correct). In episodes 2 and 3, students use technical discourse for themselves with deliberation to achieve visual effects. Their explicit discourses concentrate only on describing the sequences of symbols they use or the visual results. Explicitly, the technical discourses and the visual discourses are disconnected. In episode 2, the schoolwork discourse's assumption that students should finish the tasks set by the teacher frustrates Anna as she is concerned that she might fail the course. Lilja tries to help Anna finish the task, but only by giving her step-by-step instructions consistent with dominating practice in schools. Lilja, therefore, seems not to have been hurt by the schoolwork discourse in this case, while Anna's learning suffers. Lilja's technical discourse is in tension with Anna's desire for understanding, that is, a desire to make mathematical discourse internally persuasive for her.

In the case of Tanja, the assumption of schoolwork discourse of avoiding being publicly wrong may have pushed her into stubbornly sticking to her answer against the challenges of the teacher instead of taking his point into serious consideration. It is unclear to what extent she engages with the teacher's argument. Perhaps she takes his point and adjusts her understanding; perhaps she modifies her answer only because of the teacher's authority. Schoolwork discourse also seems to close the thinking of Drífa and Ragna in episode 3, as they do not seek any elaboration from Tanja on her vertical line idea but instead turn to check what is due for homework. These interactions show that, as in other forms of schoolwork, students interpret exploratory work in DGS, to a large extent, as producing work to the satisfaction of a teacher (e.g., Doyle, 1988; Goodchild, 2001).

In episode 1, the students find a visual way to produce a solution. In episodes 2 and 3, a student tries out a conjecture about what would happen and verifies that the solution works from a visual point of view. In episode 2, Lilja's solution is also correct from an expert's point of view. In contrast, in episode 3, the teacher tries to bring to Tanja's attention that her logic is problematic from the academic mathematics point of view. Thus, tension arises between the following two roles of mathematical discourse: a means to describe perceived reality (which Tanja's method did in an arguably acceptable way) and a deductive discourse that does not accept visual perception as a valid mode of verification. The non-alignment of visual computer representations and mathematical objects can be viewed both as a problem and an opportunity. Responding to students in a way that respects their ideas that work according to students' perceptions requires a deep awareness of mathematics, the software and the students' thinking. Framing tasks by appealing to visual appearance makes it tricky for teachers to challenge students to reconsider their ideas in light of counterexamples or by reference to mathematical properties. I suggest that teachers should be careful about their potential double voice here: an open voice for dialogue, inviting and even insisting on drawing out students' ideas, but also a more closed voice that desires convergence of authoritative and inner persuasive discourse, and in effect, says "you are free to say what you want as long as you do it in mathematically acceptable ways, as determined by me".

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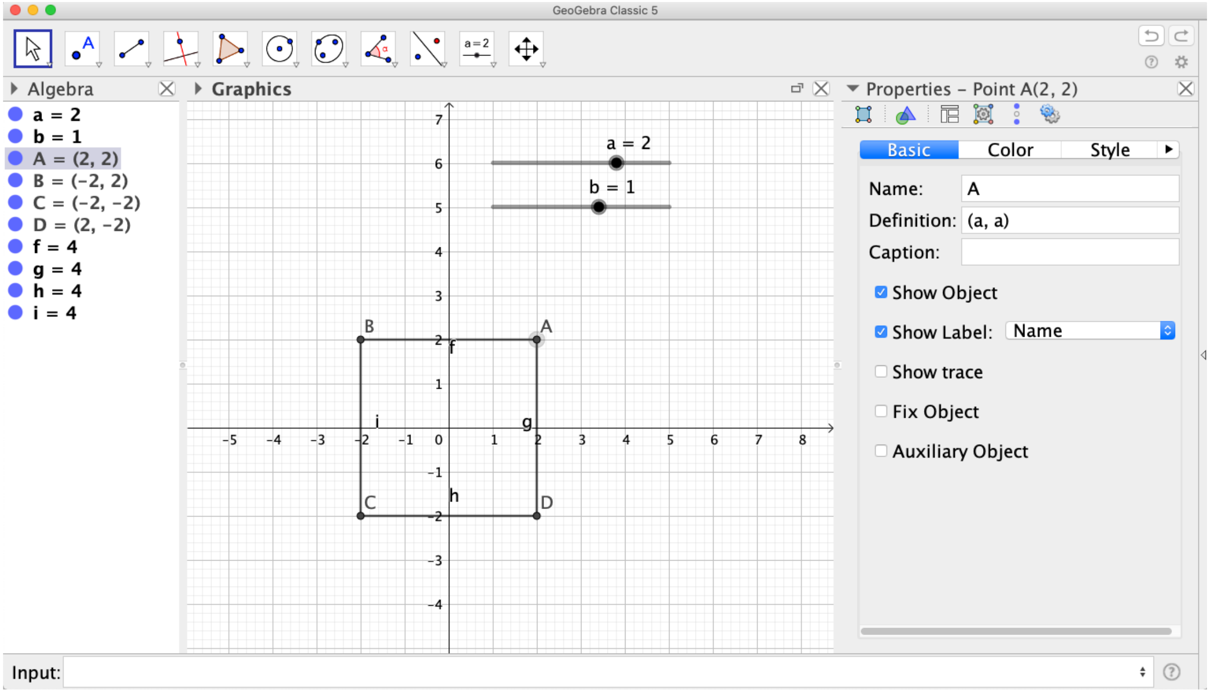

Figure 1

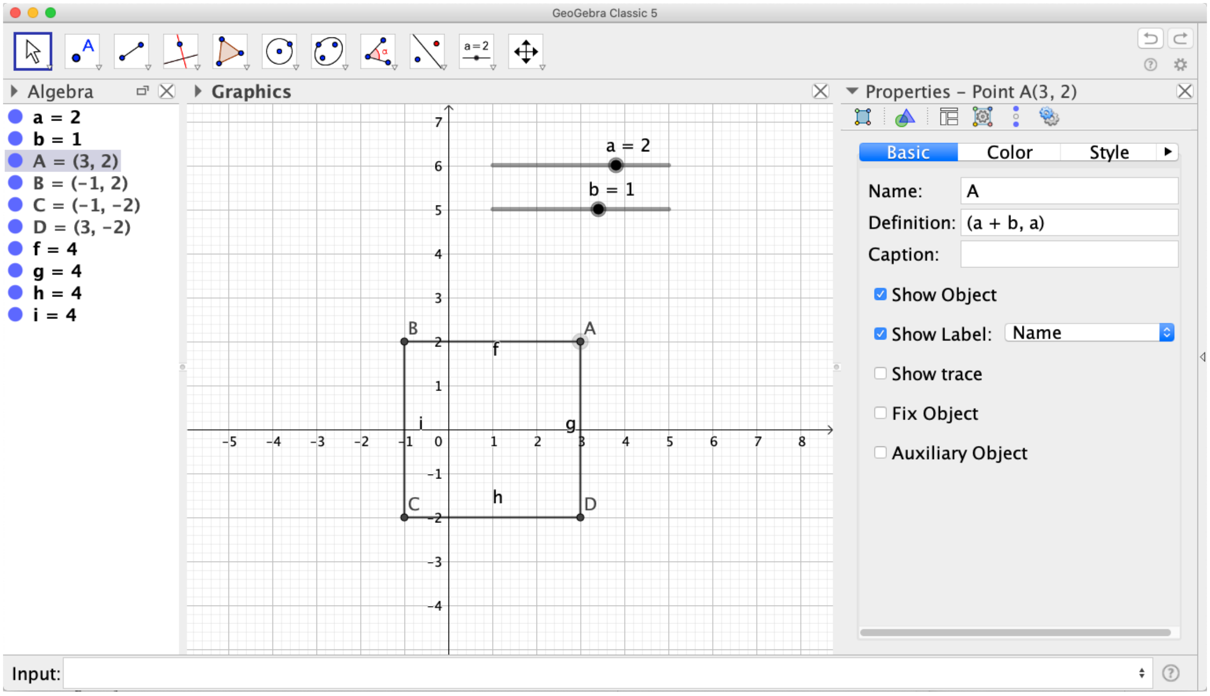

Figure 2

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Appendices

A. Consent and Information Letters

Consent letters and information letters for teachers, students, principals, and guardians.

A.1. Information for principals and the teacher

The school principal and the teacher received a formal letter with information about the research, as well as being informed and consulted in meetings. The letter was identical aside from the addressee. What follows is a translation of the letter they received before the start of the data collection.

Reykjavík, August 2012

Dear teacher[/principal], [name].

With this letter, I would like to ask your permission to do the research described below. I am a research student at the University of Iceland. My doctoral project will focus on mathematics teaching and learning in high school. Ólafur Páll Jónsson, associate professor at the School of Education at the University of Iceland, is my main supervisor for the doctoral dissertation.

The project will be carried out in the school year 2012-2013. I am interested in looking at and researching progressive mathematics teaching in high school where dynamic mathematics software (GeoGebra) is used. I will focus on students' and teachers' experiences of such learning, and the internal and external barriers that need to be overcome in such a learning environment (for example, the framework of the system and the expectations of students and parents.)

The arrangement of the research will be in such a way that I will be in the classroom and get to observe, video record and participate in what is going on (within the limits that the teacher or students want). At the same time, I request to be allowed to interview teachers and students from time to time (if they agree.)

All information will be treated as confidential and anonymity ensured so that it will not be possible to trace information to individuals, but it is worth mentioning that those who are well acquainted with Iceland could possibly speculate about the identity of the school. The dissertation will only be published in English.

The main idea behind this study is not to evaluate the quality of teaching or the school practices, but to gain a deeper understanding and knowledge of mathematics learning as a phenomenon and part of students' and teachers' lives, and especially whether and how students' experiences in mathematics teaching that is somewhat unconventional is different from what happens in a traditional environment. The hope is that a rich description can be given of it, resulting in guidelines on how this can be successful.

If you have any further questions about the aim or plan of the study, the information will be readily available, you can send an e-mail to ingogster@gmail.com

With best regards and a wish that this request will be well received.

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Ingólfur Gíslason

A.2. Information for students

Reykjavík, August 2012

Dear student[/guardian].

This semester a researcher is going to be recording in our classroom. He is interested in how we work in the classroom. He is not here to assess you or to give any information about you to the teacher or anyone else.

He will write scientific papers about the class but your names or the name of the school will not be used. No one else will ever watch the recordings. If you ever feel that something happens in class, involving you, that should not be on record, you can ask, and it will be deleted.

Your agreement is assumed but if you ever do not agree with being on camera let us know and you will not be recorded.

[Teachers' name]

B. Overview of class sessions

The following table contains basic information about all class sessions in the course.

No.	Date	Att.	Film (mins)	Summary
1	21.08	28	32	Introduction to course. Frontal teaching. Teacher-class discussion.
2	23.08	?	0	Not present.
3	24.08	25	52	Frontal teaching, curve drawings, functions, sports. Seatwork on paper.
4	28.08	24	47	Researcher introduced, seatwork on paper, curve drawings (filling vessels).
5	30.08	25	106	Downloading GGB. GGB stick-figure task.
6	31.08	23	49	Seatwork, paper. Tasks: connect table-graph. Pairs and individuals, some frontal teaching.
7	04.09	21	48	Frontal teaching, on using GGB, referring to stick figure task. Seatwork, paper.
8	07.09	19	48	GGB, co-variation task.
9	11.09	20	49	Co-variation task continued, doing the paper-part, teacher whole-class discussion.
10	18.09	17	45	Potato task, whole-class and workbook individual.
11	20.09	22	46	Seatwork, paper, a lot of bantering, no mathematical dialogue captured.
12	25.09	?	0	Not present. Students take a (paper) test.
13	27.09	21	87	Frontal teaching (students self-grade tests from teacher explanations). Papertask: age-height.
14	02.10	?	0	Not present.
15	04.10	18	59	Frontal teaching: "guess the function" game. GeoGebra number line game task in pairs.
16	05.10	?	0	Not present, substitute teacher.
17	09.10	17	44	Frontal teaching, then students in pairs in GeoGebra line-equation game.
18	11.10	20	102	GGB starburst task. Pairwork and then frontal teacher-class discussion.
19	12.10	20	48	Short frontal teaching: shows input y=m*x+b for line-slider task. Pairwork on GGB.
20	18.10	18	101	Short frontal teaching. Pairwork on papertasks using GGB, GGB square control task.
21	19.10	17	62	Pairwork on GGB rectangle perimeter-area task: (a,p(a)).
22	23.10	20	95	Individual and pairwork GGB continuation from previous class.
23	25.10	17	111	Frontal teaching with "number talk". Papertask pairwork: Roots of linear and quadratic polynomials.
24	30.10	16	40	Frontal teaching. Teacher-class discussion. Parabolas, roots. Pairwork on papertasks.
25	01.11	18	93	Frontal teaching. Teacher-class discussion. Pairwork on papertasks: Optimal garden fence.
26	06.11	?	0	Not present
27	08.11	?	0	Not present
28	09.11	?	0	Not present
29	13.11	20	54	Students take a (paper) "test" individually.
30	15.11	14	63	Werewolf (A non-math game). Frontal teaching, introducing periodic functions. Papertask: car tracks.
31	16.11	15	37	Frontal teaching. Seatwork, papertasks: Paris-wheel and cos, sin with unit circle.
32	20.11	15	44	Short frontal teaching: Trigonometry, unit circle, then paper pairwork.
33	22.11	16	101	Short frontal teaching: Trigonometry with GGB then GGB pairwork.
34	23.11	16	47	Short frontal teaching: Trigonometry with GGB then GGB pairwork: periods, amplitude.
35	27.11	15	49	Frontal teaching with GGB, trigonometry. GGB pairwork.
36	29.11	17	38	Teacher-class discussion: summary of functions, intro final project. Paper based seatwork.
37	04.12	14	105	Final assessment task.
38	10.12	14	69	Final assessment task.

Table B.1. Minimal summaries of lesson

C. Sample summary

A sample summary

In the following pages is an example of notes I wrote of a classroom session. These are raw notes, basic descriptions of the phases of the lesson and my initial thoughts. I usually also included a few screenshots from the video recordings as seen in the following.

Classroom session 24.08.2012, 52 minutes.

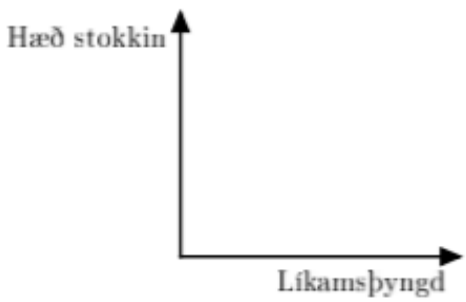
FORMALITIES, ORGANISATION: A SLOW START
(Tape: 2012-08-24a 10 mins)
Tape starts in the teachers lounge.
I follow the teacher up into the classroom.
Teacher tells students to put their computers down.
Asks about "verkefni 3" hand-in.
4 minutes into the class, teacher still handing things out, students coming in.
After 5 mins: Takes attendance.
6 mins into class, students still entering, while taking attendance.
This is work "on paper".
Teacher gently admonishes students who have lost their work.
Students asking what is was that they should have handed in.
After nine minutes teacher delivers consent-forms for the taping.

GRAPHING FUNCTION CURVES FROM "REALISTIC CONTEXTS"

(Tape: 2012-08-24b 10 mins)
Talk about what they are doing, "is it task 1.3b?"
About ten minutes into the session, the teacher asks if anyone would like to share a solution on the board. It's a task about sports, long jump.

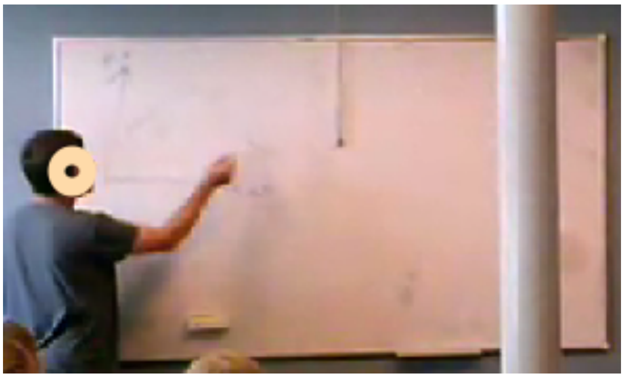
Verkefni 1.4 *Íþróttir*
Segjum að þú veljir 100 manneskjur að handahófi og mælir þynd þeirra. Svo bæðir þú fólkið um að sýna leikni sína í 3 íþróttaviðburðum.
Rissaðu upp punktarit sem sýna hvernig þú býst við að niðurstöðurnar dreifist. Gefðu rökstuðning fyrir hverja mynd. Settu fram allar ályktanir þínar á skýran hátt.

a) Hástökk

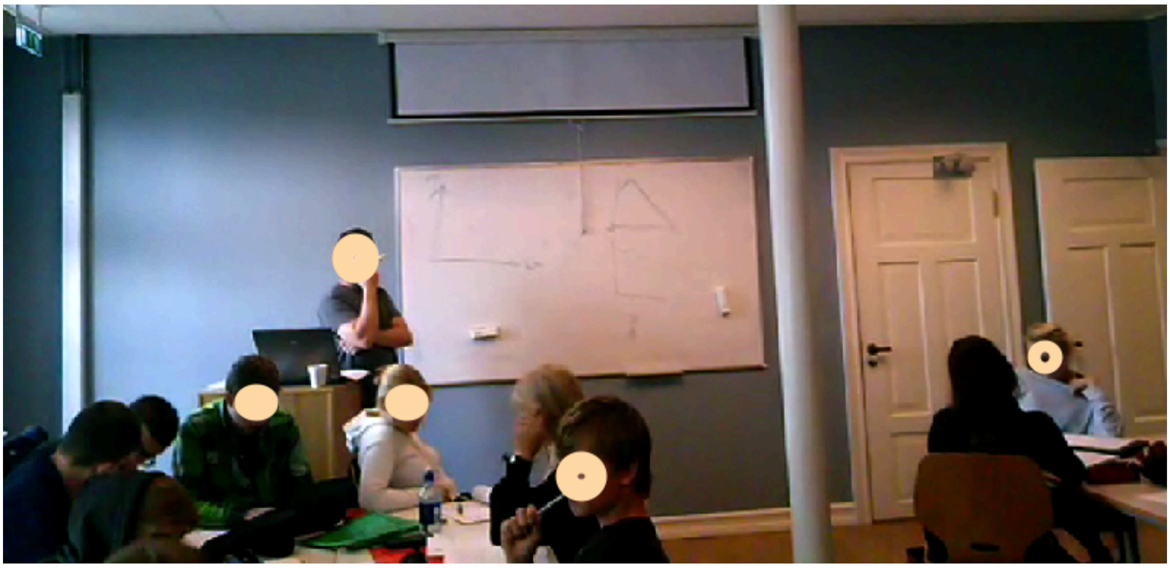


(The task is from "The language of functions and graphs")

The teacher draws coordinate axes (as in the task). There are no volunteers, but students ask to clarify the task. What is meant, they ask, how much success? The teacher responds, "we were wondering, what if we were a hundred people, how high would we jump?" "Added together then?" a student asks.
The teacher tries to clarify, using pointing: "no we imagine someone here, weighing this, he jumps this high".



[SPATIAL METAPHOR OF POSITION, DEIXIS]
Students discuss other variables, such as body height. The teacher struggles to get them on track. It seems difficult to express that this is about an "average" curve. The teacher tries to get them to think: what if many people are of the same height, but different weights, for example. It is not clear whether this helps. It seems there are only about 2 or 3 students participating in the discussion, two boys and one girl. The discussion becomes very much about "the real world", about bodies of athletes, and there really doesn't seem to be any mathematics considered, although it may be interesting to think about how a mathematical model is always an abstraction, what features must be disregarded, and at this level, to realise that the interest is not on a "precise" model. The point is to be able to think about curves as models, not to find the best model. The students (the two-three that are active in the discussion) focus on reality, which is making it very hard for the teacher to do anything about the mathematics.
The teacher tries to explain: we think about a lot of people, therefore it becomes "a line". So he is thinking of a "best curve" to fit hypothesised data which would give a scattergram rather than a connected curve.



When they talk about powerlifting, there is discussion on technique, that it is very important. Yet again the students are "stuck in the real world". But this could be interesting as a discussion about what variables to consider, and which should be "controlled" (in general). This is not utilised by the teacher.

(Tape: 2012-08-24c 10 mins)
In the discussion between the teacher and the few students, the teacher tries to get them to either come to the board to draw or verbally describe the curve, to get them to be more precise and correct. Here he is working on getting them up on the "levels of arguments", where they use words such as "straight" (meaning horizontal), and other imprecise words.
A lot of time is taken on wondering about sports. One wonders if other teaching materials would work, tasks that are focused on modeling and abstraction, that would perhaps center around what such a curve represents that is being asked for here, making the role of variables explicit.

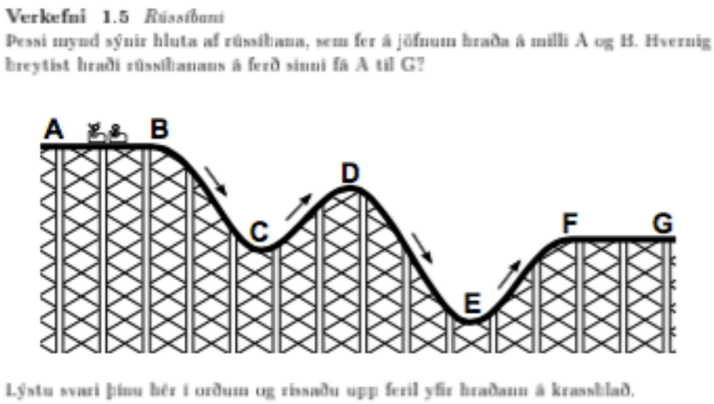
Then the teacher has sensed that only a few students are engaged, and turns the discussion "meta". He asks: is this mathematics? He gets some different views on this and a girl says: No it's about sports. She is not wrong, but the discussion COULD have been mathematical if the relation between reality and math would have been made explicit and discussed AS SUCH.

There is more interesting discussion on the philosophy of mathematics. Students talk about their situation, school mathematics: "It's mathematics since you are teaching it", "everything is mathematics",... Does it matter if there are numbers or not?

Then there is a new problem, "rússíbanadæmið", the roller-coaster.

(Tape: 2012-08-24d 19 mins)

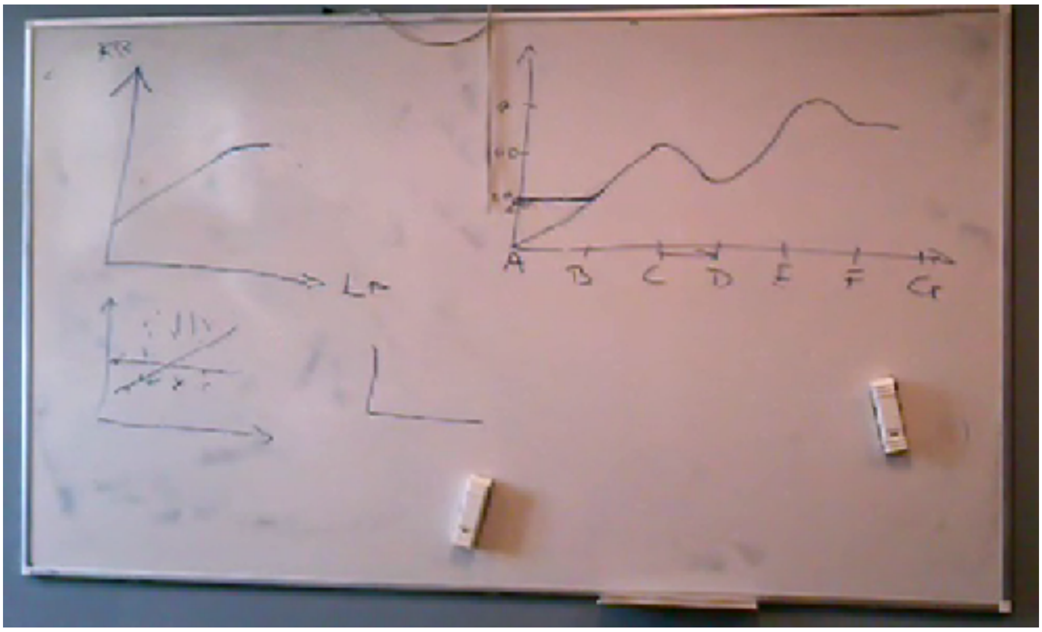
H. and Ka. sit near front and ask some questions, but semi-privately.
From time to time students open computers.
About 4 minutes is just banter talk.
Now the teacher tries to start talking about the roller-coaster task.


Verkefni 1.5 *Rússíbani*
Þessi mynd sýnir hluta af rússíbana, sem fer á jöfnum hraða á milli A og B. Hvernig breytist hraði rússíbanans á ferð sinni fá A til G?

Lýstu svari þínu hér í orðum og rissaðu upp feril yfir hraðann á krassblað.

Here the one girl that has spoken at all, B., talks about the mechanics of roller-coasters.
She describes constant speed at first, then faster when it goes down. She declines drawing.
Here two girls assert themselves and participate in the discussion (B. and Kr.).

For homework: design your own roller-coaster, and draw a time-velocity graph.
Then another student should try to figure out how the roller-coaster looks from the time-velocity graph.



(The board at the end)

SYNOPSIS:

In this session the teacher stood at the whiteboard the whole of 50 minutes, in a discussion with a only few students, about possible graphs of curves describing four different situations. Most students did not participate in the discussion. The discussion was very focused on the particular real-life situations, the features of those and not on mathematics. There was a small amount of mathematics in that students were sometimes trying to describe curves and the teacher was challenging them to be more precise.

Of interest is the language of the teacher when describing curves and the relation between curves and situations. For some reason he focused on smooth connected curves and not scatterplots (as there are in the original materials) which may have made for confusion for the students as they talked of not understanding what the curves were describing (the teacher talked about "a sort of average").

The teacher made no attempts at controlling who spoke, so it was very dominated by one or two students. Even if the teacher said at the start that computers were not to be used, students opened them and their phones at various times.

Judging from those who spoke, the students had a very hard time making sense of the class. They seemed not to understand "the point" of the work.

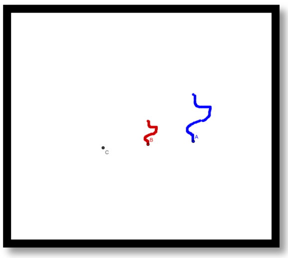
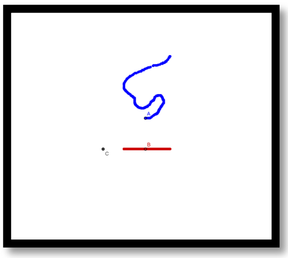
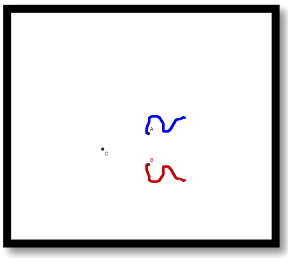
D. DGS tasks featured in paper II

Tasks and a suggestion for the instructional sequence

In the following pages show (paper part of the) translations of the tasks that students worked on in lessons 8 and 9, along with a suggestion for the organisation of the work. The first of the tasks features in paper II.

Functions in coordinate systems

In all of the three following GeoGebra worksheets there are points that you can drag with the mouse. In all of them there are also points that you can not move with the mouse but their movements depend on the other points. In some of them there are points that are immovable.

Worksheet 1. Worksheet 2. Worksheet 3.

Suggested outline of organisation of the work

1. The teacher introduces the GeoGebra document (with a projector). Often there is a simple question (but not necessarily easy to answer) that students should try to answer intuitively. Later, the question is examined in more detail.

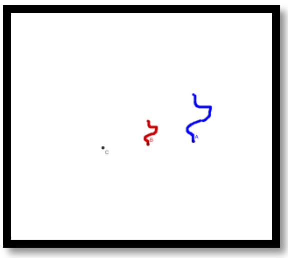
2. Students work in pairs. Talk about and write different descriptions of what is happening on the projected screen.

3. Teacher-class discussion. Listen to the students talk about how they understood and thought about what is happening on the screen. Call on students (for example) "how do you understand this?", "Does anyone want to add this?", "Does it always work?". Challenge the students by following their suggestions in ways that they did not foresee, pushing them to be more precise. The teacher can introduce concepts and notations when necessary. Beware of judging – ask for further explanations instead of saying right/wrong. Write key words and explanations of the students on the board. It may be necessary to point out that "formulas" are not "more accurate" or "more mathematical" than other detailed descriptions.

4. Students work in pairs again. They should modify and improve the GeoGebra document in some way according to the task description (often by adding points with some stated properties.)

5. Students write their conclusions about the task with a mathematical rule (a generalisation). Students should hand in papers after class.

Worksheet 1.

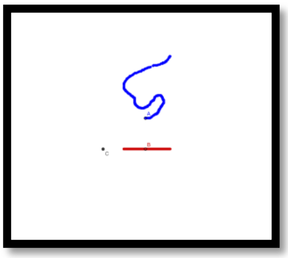


We look at a GeoGebra worksheet together. Check which points can be moved and how the other points move in relation to them.

Write down everything that you notice and try to be as precise as possible.

Add a point to the sheet that will always stay between A and B (no matter what you move). Define it in words and with co-ordinates and then add another point between A and B, and another. What characterizes all points between A and B? In other words, describe how you can always find new points between A and B.

Worksheet 2.



A) Try dragging the points in the picture and check which points can be moved. Which points are dependent on which other points?

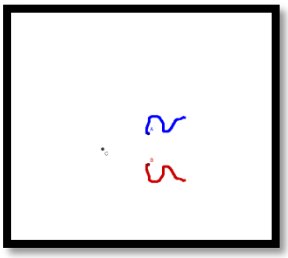
A. Immovable points	B. Points that can be dragged (independent points)	C. Points that move dependent on other points	D. Which points determine the points in column C?

B) How is point B dependent on point A? Write down all you can see and try to be as precise as you can.

C) After the discussion about the task, write down three solutions you consider different and then say what solution you like the most.

D) Add a point to the sheet that also leaves a horizontal trace, but in a different place. Add yet another such point, and another. What characterizes all points that leave such traces, dependent on A? In other words: describe how you can find more and more points that leave such traces when A is moved.

Worksheet 3.



A) Try dragging the points in the picture and check which points can be moved. Which points are dependent on which other points?

A. Immovable points	B. Points that can be dragged (independent points)	C. Points that move dependent on other points	D. Which points determine the points in column C?

B) How is point B dependent on point A? Write down all you can see and try to be as precise as you can.

C) After the discussion about the task, write down three solutions you consider different and then say what solution you like the most.

D) Add a point to the sheet that also leaves a mirror trace, but in a different place. Add yet another such point, and another. What characterizes all points that leave such traces, dependent on A? In other words: describe how you can find more and more points that leave such traces when A is moved.

E. Sample transcript 1

Sample transcript 1

In the following pages is a sample from the analysis work on a lesson reported on in paper I.

Transcript 2012-08-24b

Teacher turns: **140**

Student Covert disregard (Not counted, but mentioned as very distracting after about 20 minutes of class).
Student Overt disregard **3**
Student Inauthentic talk about context (sometimes Over-engagement) **16 turns**
Student Authentic talk about context (sometimes Over-engagement) **84 turns**
Student Developing mathematical discourse (LEVEL IN PARENTHESIS: 0/1/2) **21**
Student School-talk (talk about the rules of school: homework, deadlines, attendance, what page are we on... also meta-comments, about mathematics) **12**

Uncolored for now: **50**
- Private talk btw student and teacher, like asking for answers to some excercises
- Clarification questions about words, yes/no, what?, ...

(xxx) marks: **55**

Lesson	Teacher	Overt disr	Context	Math	Other
Sports	140	3	90	21	71

(0:00:00.2)
[There is talk about "where to begin", but the students seem to be starting "seatwork".]
(0:00:27.2)
B.: Má ég spyrja hérna áður en við byrjum að tala um (xxx)
T: Já.
B.: Hérna, í verkefninu sem við gerðum heima, lið bé. Hvernig (xxx)
T: Geturðu sett inn fleiri, bara einhverja, ef þú hugsar þetta eitthvert, svona. (xxx) símtal.
B.: Ókei.
(0:00:44.8)
T: Getum við talað um íþróttirnar? ... hérna hástökkið.
(0:01:02.9)
T: Er einhver sem treystir sér til að koma og rissa upp ferilinn sinn hérna á töfluna?
G: Ferilinn?
T: Já.
s: Í hverju?
T: Í hástökkinu.
(0:01:17.0)
s: Hversu mikinn árangur þá eða?

T: Nei bara við vorum að velta fyrir okkur, ef við værum hundrað manneskjur. Þá myndum við bara athuga hversu hátt myndu þau stökkva?
s: Samanlagt þá eða?
T: Nei, við kannski hugsum okkur einhvern hérna, sem er svona þungur. Hann hoppar svona hátt. [Points to a point in a coordinate system he has drawn on the whiteboard.]
(0:01:39.0)
B.: Ætlarðu samt að setja hundrað punkta þarna inn?
T: Getum teiknað þetta sem línu er það ekki ... safnað saman punktunum.
s: (xxx) Eins og hérna, eins og ... líkamsþyngdin ætti ekki að skipta neinu krúsíal máli í hástökki til dæmis.
T: Hvað?
s: Eins og hérna... maður sem er einn nítíu (xxx) maður sem er einn sextíu.
T: Já, ókei þannig að þetta er svona nokkurn
s: Maður sem er einn og nítíu getur alveg verið þyngri en maður sem er einn sextíu, er það ekki.
s: Jú.
(xxx)
(0:02:14.1)
T: fiturprósenta?
(xxx)
s: þyngdarstuðull
T: Ókei ef við erum með hundrað manneskjur, og við tökum svona nokkurn veginn meðal ... talið. Ókei, maður sem er einn og nítíu hann er e líklega þyngri en maður sem er einn og sextíu er það ekki?
(0:02:28.4)
s: Jú.
T: Þó það sé ekki algengt.
(xxx)s: Hann ætti þá að stökkva hærra.
T: Er það? Já?
G.: Hann getur svifið samt.
(0:02:35.0)
T: En ef þú ert með tvo menn sem eru einn og nítíu og einn er þyngri og einn er léttari, hver
E.: Léttari maðurinn (xxx)
T: Hvað segið þið um það?
(0:02:42.7)
s: Ekkert endilega.
E.: Það fer eftir tækni, og hvar hann er með vöðva.
s: Já.
E.: Hann þarf ekkert að nota hendurnar í þessu.
s: Hann þarf að hafa hendur til að (vinda upp á sig).
E.: Já hann þarft ekkert að hafa þungar hendur í það sko.
(0:02:56.2)
s: Allir spretthlauparar eru með mjög massaðar hendur.
E.: Já en við erum að tala um hástökk sko.

s: En þeir taka sprettinn áður en þeir stökkva er það ekki.
(0:03:03.2)
E2: Nei, þetta er ekki sprettur. Atrenna bara taka bara stór og ákveðin skref, þeir eiga ekkert að hlaupa hratt. Af því þá hlaupa þeir á slána. Þeir reyna að tipla á jörðinni og í raun stökkva bara.
(0:03:16.9)
T: En hvað ef við einföldum dæmið aðeins. Hugsum okkur ókei, ef við tökum hundrað manns sem allir eru einn og nítíu á hæð.
s: Við vorum að tala um það, er það ekki?
T: Ha?
s: Við vorum að tala um það.
T: Við vorum að tala um tvo.
s: Ja-á.
(0:03:27.6)
T: En ef allir eru einn og nítíu á hæð.
s: Eigum við að miða við að allir séu jafn stórir og þungir?
T: Jafn háir, nei. Hérna aftur á móti, því lengra sem við förum þangað því þyngri erum við. Og léttari ef við förum hingað.
(0:03:40.5)
G.: Ég vill hafa svona boga. (0)
T: negin svona? [draws concave curve]
(0:03:46.1)
(xxx)
G.: Ég vill samt, ég myndi frekar láta byrja ofar semsagt. (0)
[V draws another pair of axes.]
T: Byrja ofar?
G.: Já, hann fer svona aðeins upp (0)
(0:03:58.2)
T: Já, hvar viltu að línan byrji? Má ég biðja þig að teikna línuna?
H.: [Whispering] Hvaða blaðsíðu er þetta?
[The student (G.) who has been conversing with the teacher comes to the whiteboard and takes the felt pen.]
T: Við erum hérna, á undan, þessu hér. [Addressing the girl who asked on which page they were.] Verkefni eitt fjögur.
[The student is drawing a curve on the whiteboard.]
[The curve is concave but is slightly skewed right].
T: Er það viljandi að þú snertir ekki
s: Já
T: byrjunina? Af hverju?
(0:04:33.2)
T: Geturðu sagt mér af hverju
s: Bara ef þú ert ekkert þá hopparðu ekki. (0)
T: Það er enginn ekkert, er það?

T: Það er svona, ágætis punktar að hugsa um sko að ferillinn getur alveg byrjað þú veist án þess að fara alveg hérna [points pen at (0,0) ... niðrí núll, það er enginn núll kíló.
(0:04:52.8)
s: En þarf þetta alltaf byrja á núll... (1)
T: Svo er það líka það, þarf endilega ásinn að byrja á núll? Ef við látum þetta byrja hérna þá þyrftum við kannski að segja það ókei, við ... ásinn byrjar ekkert á núlli.
(0:05:05.5)
s: (xxx)
T: ... allavega þrjátíu kílóum ... þá byrjum við hérna á þrjátíu. En við erum þrjátíu kíló þá kannski hoppum við ekki ...
s: Ekki ef þú ert einn og nítíu og þrjátíu kíló. ... Það væru voðalega þunn bein og ... litla vöðva.
(0:05:25.2)
T: Já, já. ... En, en, en gæti þetta svona nokkurn veginn verið að lýsa því sem er að gerast, hvað segið þið um það? Meikar þetta einhvern.
s: Já, það meikar alveg sens að maður sem er miklu léttari heldur en maður sem er kannski einhverstaðar fyrir miðju ... hann þarf ekki endilega að hoppa hærra.
(0:05:39.9)
T: Voruð þið, hversu margir voru búnir að teikna feril, voruð þið eitthvað. Heyrðu strákar við ætluðum að, þið megið ekki nota tölvurnar. Þið getið bara lagt þeim í dag, það má bara loka þeim.
(0:05:48.6)
T: Eruð þið þá ánægð með þennan feril, eða voruð þið með einhvern annan. Þetta er ekkert endilega neitt rétti...
(0:05:58.6)
s: Svona.
T: Svona bara?
s: Já.
s: En af hverju hoppar léttasti maðurinn þá endilega hæst?
(0:06:09.2)
T: hm?
s: já af hverju(xxx)léttasti maðurinn (xxx)
T: Góð spurning?
E.: Má ég gúgla?
s: (xxx)
T: (xxx)
(0:06:23.0)
T: Hundrað manneskjur, ef þetta væri hundrað íþróttamenn þá kannski ... æ ég veit það ekki ...
s: Flestir þessir atvinnuíþróttamenn eins og á ólympíuleikunum (xxx) ekkert undir áttatíu kíló.
T: Já, kannski var ekkert raunhæft að taka menn á ólympíuleikunum.
(0:06:42.9)
T: Það væri kannski ... af því þá væri enginn hér sem væri að stökkva rosalega lágt. Það væru allir frekar góðir.
(0:06:52.1)

s: Kannski þú veist, ef við myndum taka hundrað íþróttamenn úr öllum mismunandi íþróttum.
E.: Sko hástökkvarar eru ekkert massaðir.
T: Nei, þeir eru yfirleitt mjög grannir.
E.: Ég er að gúgla þetta, þeir eru ekkert massaðir.
(0:07:04.7)
T: Er tölvan komin upp aftur, bæði hjá þér og E2i?
Hva?
E.: Við erum að finna út úr þessu.
B.: Ég veit um alveg hástökkvara sem voru massaðir sko.
(0:07:12.4)
E2: Þeir eru alveg til sko. Það eru líka til horaðir spretthlauparar.
(Many talking)
T: En þeir eru yfirleitt ekkert, ekkert mjög feitir er það?
(0:07:25.7)
s: Nei
T: En hvað með kraftlyftingarnar? Hvernig myndi það líta út?
(0:07:29.8)
s: Það fer náttúrlega eftir því hvaða grein þú ert að tala um í kraftlyftingum.
T: Hvað segirðu hvað (xxx tengja?) ...
E2: Hundrað manns.
T: Þá kannski er sá þyngsti hérna. Kannski bara fullt af punktum hérna. [Teacher's phone rings] Voðalegt er þetta.
(0:07:45.6)
E.: Þeir eru bara með massaðar lappir, þetta er einfalt.
E2: Já.
E.: Þeir hafa ekkert að gera með hendur.
(0:07:54.3)
T: Þannig að við getum hugsað okkur að hérna, þá værum við með allar hundrað manneskjurnar, raðað. Af því þær eru svo margar þá verður þetta bara lína. Meikar það einhvern sens?
(0:08:07.8)
s: Verður þá sá sem stekkur lægst þarna neðst. (0)
T: Já, hann verður þá lægst, nákvæmlega.
T: Sá sem stekkur lægst, hann er neðst. Neðsti punkturinn, hann er neðsta línan.
(0:08:31.7)
T: Já og það sem að G. sagði, og E2 líka. Að þeir voru sammála um það, að það væri þyngsti einstaklingurinn sem stekkur lægst.
E.: Það er alveg öruggt mál.
E2: Nema náttúrlega að þessi létti sé bara lágvaxinn líka.
E.: Já, nei.
G.: Allir einn og nítíu.
Many: Unclear.

F. Sample transcript 2

Sample transcript 2

In the following pages is a sample from the analysis work on a lesson reported on in paper II.

GeoGebra Co-variation (07.09.2012)

Classroom session 07.09.2012, 48 minutes.
Built on tapes FMOS 07.09.2012.0, 44 mins and FMOS 07.09.2012.1, 4 mins.
There are 393 turns of speech. Number of teacher turns are 143.

Summary of structure:

Time (minutes of lesson)	What is going on
0-5	Learners enter classroom and there is the usual unclear start of the lesson.
5-10	The teacher introduces the task with a single prompt, directing the learners to work on a ready made GGB worksheet. It turns out that the sheet does not work correctly. This is found out by the learners and the teacher together in a discussion about what points can be moved
10-16	Teacher at the front, projecting the GGB sheet. A teacher-class discussion about the task: dependency, movement, the cartesian connection. The teacher declares the goal to be to describe the situation depicted with precision.
16-19	The learners sit and use their computers, in pairs.
20-34	Teacher at the front, projecting the GGB sheet. A teacher-class discussion about the task. Some students participate and are successful in approaching mathematical discourse. The teacher writes a lot on the board, which becomes quite messy.
34-46	In this phase the learners are working in pairs on the follow up questions, while the teacher circulates the classroom, but there is also some public teacher-class dialogue.
46-48	Learners leave class, the lesson has no defined ending, but some ask when they are supposed to be finished with the task.

Transcript 07.09.2012: GeoGebra Co-variation task I

Attendance: around 20. Many students sit in a “U”.
Teacher projects GeoGebra. There are three points.
After general introductory talk and loud audible sighs from a female student, the teacher tries to initiate work.

CO: CODE
0 = The teacher asks learners to say what they see, *verbalize what they sense*
1 = A description in everyday words, but from the viewpoint of body-likeness: imitation, chasing, etc.
2 = A dynamic description with an emphasis on the movement and speed of the points.
3 = A description the teacher finds too unclear. It is too vague (undefined pronouns, lack of distinction between e.g. points, traces, distances, inexplicitness).
4 = A mathematically expressed idea, where there is some incorrectness.
5 = A correct mathematically expressed idea.
6 = Refusal to express in words.

In the third position when the teacher speaks, we see the following responsive tactics:

A = Asking for increased precision, explicitly.
B = Giving opportunity for more precision by indirect means, such as by echoing, expecting the speaker to follow up.
C = Suggest a representational system to use, or mathematical ways to express something.
D = Suggest a specific configuration of points to attend to.
E = Re-describing a calculation on a higher level (what you did was to …)
F = Asking for generalizations.
G = Establishing correctness, usually by rewording or emphasizing a definite fact or rule.
H = Ask the group for other viewpoints or consensus.

Nr.	Time	Utterance/action	Analysis	CO
1.	1 (06.33)	TEACHER: Það sem ég ætla að biðja ykkur um að gera núna bara. Er að átta ykkur á því hvað er hægt að draga til (.) hvað er ekki hægt að draga til (..) Og hvernig hreyfast punktarnir?	Directing attention, stressing what can you drag and what can’t you drag. How do the “points” “move”?	0
2.		Kristín: Þeir vilja ekki hreyfast.	Describing: she tries but can’t move points. They won’t MOVE	2
3.		Daníel: Ég færi bara til það sem ...		2
4.		TEACHER: Hvaða punkta getum við hreyft? Engann?	Asking to direct attention again, what points can be dragged. What can we MOVE?	0

5.		Kristín: Jú B.	Simple answer, it's not what teacher supposed would happen.	
6.		Bjarni: B.	Simple answer, it's not what teacher supposed would happen.	
7.		TEACHER: Getum við hreyft B?	Repeats the question, because this is unexpected behavior. Can we MOVE a point?	0
8.	(07.00)	Kristín: Já.		
9.		TEACHER: En getið þið smellt á einhvern punkt og dregið hann til hérna?	Dregið til = "drag" is a metaphor for action. Drag = MOVE.	0
10.		(Multiple): A		
11.		TEACHER: Ég held ég hafi látið ykkur hafa rangt skjal.	There is something wrong with the GGB sheet.	
12.		Gunnar: Já. Það er verra.		
13.		TEACHER: Svona getur tæknin strítt manni.	TECHNICAL PROBLEM	
14.		...		
15.		S: Rauði færist miklu minna.	The red one MOVES a lot less. -> SPATIAL METAPHOR, LESS/MORE	2
16.		S: Þetta hérna er miklu stærra heldur en þetta hérna.	This IS much bigger than this one. -> LESS/MORE	3
17.		Teacher updates the file, creates a new file for students.		
18.		...		
19.	(11.03)	TEACHER: En hérna nú eru náttúrlega einhverjir byrjaðir að fikta. Getið þið séð hvað er að gerast hérna?	ASKING STUDENTS TO DESCRIBE WHAT THEY SEE	0
20.		TEACHER: Ef þið hreyfið til punktana.	Stressing, DIRECTING ATTENTION to moving the points.	0
21.		Daníel: Bé færist styttra eða semsagt (.) minna heldur en A	METAPHOR SHORTER/LONGER MORE/LESS	2
22.		Bjarni: Tvöfalt styttra. [Teacher does not respond to this.]	METAPHOR SHORTER/LONGER MORE/LESS	2
23.		[...]		
24.		Daníel: Nei, vatt eða.	SURPRISE	
25.		Bjarni: Er þetta ekki tvöfalt? ... Eða hvað.	IS it not DOUBLE?	

383	(02.51)	Finnur: [Pointing on the screen, to direct the attention of B.] Það er þú veist, tvisvar sinnum núll komma tuttugu og fimm. Svo er hitt núll komma níutíu. Núll komma níutíu. Og hitt núll komma sjötíu og fimm, núll komma sjötíu og fimm. Þarf alltaf að vera þú veist x og ufsilon þarf alltaf að vera (xxx) (jafnt).	Finnur and Daníel: working (2.46)  GENERALIZING, JOINT ATTENTION Finnur has generalized the “rule” that a point on the line between (0,0) and (x,y) is (a*x,a*y) (not explicit about 0<a<1), but stressing that a is one number (as opposed to any other (a*x,b*y)).	5
384		Daníel: Hvernig er alltaf hægt að finna nýjan og nýjan punkt milli A og B. Bara breyta tölunum fyrir aftan (xxx).	EXPRESSING A RULE, GENERALIZATION. Daníel: reads the task text and verbalizes an answer, which are syntax-oriented (“change the numbers beside”, as opposed to “multiply by ..” or some other mathematical way. This is a good example of talking about algebraic expressions in a syntactical but not semantic way. “abc” = “a and then b and then c” but not “a times b times c”.	
385		Finnur: Já, en x og y þurfa alltaf að vera það sama [writes on paper.]	RE-EMPHASISING, STRESSING, that there is one constant multiplying both coordinates.	
386	(03.21)	Daníel: Hvenær eigum við að vera búin með þetta.	PRODUCTION	
387		TEACHER: Ee, ég vil fá þetta hjá ykkur eftir næsta tíma. Þá ætlum við að klára það sem stendur hinu megin á blaðinu.	PRODUCTION	
388		Daníel: Í næsta tíma, klára í næsta tíma?	PRODUCTION	
389		TEACHER: Þið megið byrja að kíkja (xxx)	PRODUCTION	
390		ss: (xxx)		
391		s: Eigum við að klára þetta í tímanum eða eigum við að skila þessu?	PRODUCTION	
392		TEACHER: E, e, þið ætlið hérna að taka þetta með	PRODUCTION	

		ykkur (gets distracted by someone else).		
393	(04.18)	[Learners leave]		

Notes about the codes

The general principle of interpretation is that something is labelled as A if there is evidence that the participants take it as A.
P-RATIONALE = PRODUCTION -> If it's about the work as work, not about meaning, just about school as production. Eg. handing in details, homework, ...
"WE" -> The use of "we"
"IT" -> The use of "it" to refer ambiguously ... = pronominalization.
LIKE THIS -> Talk move: "it is like this", cf. DEIXIS
DEIXIS -> "pointing with words", "this one", "that"
"What is the it" -> Calling for clarification of referent of "it"
MAKING EXPLICIT -> Clarifying what the "it" is, or some assumptions. cf MAKING VISIBLE, PRECISION
"NORMS" -> Teacher trying to establish or promote socio-mathematical norms
"MAKES SENSE" -> Whether something makes sense
"IF-THEN" -> A proposition or question in if-then format. This is something the teacher tries to promote, often a part of the STATICIZE prompt. “For me, mathematics is the conquest of the continuous by the discrete.” (Thom)
“close the computers” -> The teacher asks students to close the computer in an attempt to direct their attention.
BEHAVIOUR -> Talk about behaviour, cf. NORMS
NORMS -> Talk that affects social norms, cf SETTING NORMS
TECHNICAL PROBLEMS -> Talk about technical problems with computers or the projector or such things
CONDITIONS, DISTRACTION -> Talk about the physical environment such as the light, temperature, noise from outside, font size on slides, etc
SETTING NORMS -> Teacher tries to establish norms, by for example explaining how work is to be done, how they are supposed to talk to each other and so on.
ECHO / REPEAT -> The speaker repeats an utterance of another speaker (often with different tonal stress).
SCHEME OF WORK -> Talk about the scheme of work, f.ex. teacher directions.
"START NOW" -> The teacher commands learners to begin working.
"what do you want", "what do you want from us", "what do you want me to do", "who do you want me to be" -> Learners asking for further details on what they are supposed to do to satisfy the demands of the teacher (as they imagine it)
DESCRIBE WHAT YOU SEE -> Usually the teacher asks learners to describe what they see or perceive.
“what did you do?” -> Usually the teacher, asking learners what they have done.
WORK IN PAIRS -> The teacher commands learners to work in pairs (cf. SCHEME OF WORK)
TECHNICAL -> About the "software-mathematics" boundary, f. ex. zooming issues.
DEPENDENCY -> The mathematics of functional dependency
EMBODIMENT -> Talk where mathematical objects are like bodies. They move, they imitate,

they travel, they turn and so on
MATHEMATICAL -> Talk that I consider as part of mathematical discourse, using some mathematical concept or way of thinking
APPROVING -> A response indicating approval, agreement, assent.
VERBALIZE = ASKING STUDENTS TO DESCRIBE WHAT THEY SEE
ASK TO EXPLAIN -> Someone is asked to explain (something relevant to mathematics)
GENERALIZING -> A generalisation is made or attempted, cf "what is the same"
SPECIALIZING -> The strategy of specializing is used or presented
MODIFYING CONJECTURE -> A speaker modifies a former suggestion of his/her
MOVEMENT -> The talk is focused on things moving
FAST/SLOW -> Talk focused on the movement velocity or speed
LESS/MORE -> Talk focused on magnitude comparison
HALF or DOUBLE -> Talk involving the concept of a half, halving, double, or doubling. Perhaps ambiguously.
SPATIAL -> Descriptions of a spatial nature, where are objects, how are they oriented, in between, up, down, far, near, ...
STATIC -> Description of a situation in terms of fixed elements, not moving or in flux.
REPRESENTATION -> Talk about ways to represent (mathematically)
SOLVING -> Talk involved in solving a problem
CONFIRM? -> Interlocutor asks for confirmation
ASSENT -> Speaker assents (or asks for assent)
ONE-WORD -> A one word response
DIRECTING DIALOGUE -> Talk that serves to direct dialogue such as calling on specific peoples contributions, calling on anyone to make a contribution, ...
"somebody?" -> The move to ask the whole group if anybody would like to say something.
DIRECTING ATTENTION -> Talk to direct the listeners attention to a particular perspective or object, ususally the teacher naming a relevant part of the situation or task, f. ex. suggesting what particular thing to think about.
REASON -> Someone articulates a reason, a part of an argument, something that could come after because...
DEBATE -> Serves to start or keep going a debate. cf "You decide".
"what do you think" -> The teacher asks learners for their opinion, evaluation, cf EVALUATE
"You decide" -> Teacher refuses to evaluate but let's the learners do the evaluation, or offers them a choice
EVALUATE, EVALUATION -> Making a (mathematical) judgement, cf NORMS
SYNTAX = NOTATION -> About the correct/incorrect use of mathematical notation or arbitrary rules, to express something. F. ex. mixing the order of x and y coordinates.
REPEAT? -> Asking someone to repeat.
ESTABLISH THE ANSWER -> Someone declares an answer with confidence and there are no further arguments, it seems taken as the final word cf. INSTITUTIONALIZE.
INSTITUTIONALIZE -> Teacher presents something as definite, the right way, often after an interchange
LIFTING (learner contribution) -> There is a stressing on valuing the individual that contributes, f. ex. naming them: Helgi said ....
MAKING VISIBLE -> Writing or projecting something said or inferred from talk, for others to see.
PRECISION -> The theme of making things precise or calling for more precision.

CLARIFICATION = CLARIFYING -> An utterance to clarify something (on a functional request to do so)
EXTENSION, EXTENDING -> Extension of a task, further questions, going deeper cf ELABORATION.
BUILDING-ON -> Mathematics building on something prior mathematical
BUILDING ON STUDENT CONTRIBUTION -> Teacher picks up learner's expressions and uses them to build on
CO-THINKING -> Showing that people are thinking together, often indicated by structure of talk, f. ex. a new interlocutor joining dialogue with a contribution.
JOINT ATTENTION -> Showing that people are attending together, often indicated by structure of talk or gaze or writing.
CO-PRODUCING -> A dialogue where more than one person produces/achieves does something together
CREATE = CONSTRUCT -> The specific move to ask someone to construct a mathematical object in accordance to some specification.
CARTESIAN CONNECTION -> A use of the connection of mathematical symbolism to the graphical representation of/in a coordinate system.
USE MATH -> A person uses mathematics (mathematical discourse) to express something in another representational medium.
ACTION-POTENTIAL -> About the use of mathematics for action, doing something, achieving something, f. ex. on a computer screen.
GGB -> Referring to specifics of the software, e.g. notation.
EXAMS -> The topic is exams or grades.
META-TALK -> The topic is the being-in-school-learning math-situation, or "what is mathematics about"(?)
"What is mathematics about" -> "Is this mathematics?" expressing concerns or questions about the nature of mathematics, or this class in relation to mathematics
"Can't do it" -> Expressing the self-perceived inability of a speaker to do something, or understand something, express something. (Sbr. Seg þú - Nei nei nei) cf. "I AM LOST"
EMOTIONAL -> Referring to feelings or emotions
SELF -> About "me" the speaker
“That’s what I said” -> A claim that a rephrasing or extension or building-on or an interpretation is simply what the same as what has been said.
PROPERTIES -> Talk about or talk building on mathematical properties.
HESITATION -> Phrases like "you know" or simply not following up or repeating or clarifying when asked, perhaps "hedging", because the speaker is not sure, he hopes that the listener will do the interpreting for him, making the appropriate sense.
“don’t know how to say it” -> Speaker hesitates or tries to get the listener to say or finish a phrase, cf HESITATION
“I don’t know” -> Speaker indicates that s/he does not know, even if indirectly, f. ex. "why not", "sure" in a tone indicating irony
“I don’t matter” -> Indication that speaker's thinking is valueless, that his/her opinion or thoughts are not to be taken seriously
JOKING -> Something said, probably intended as a joke, taken as a joke cf. HUMOUR
HUMOUR -> Something said, using humour - but possibly not to create laughter or joy but to make a point.
AMBIGUITY -> Something ambiguous, taken as ambiguous.

SAVING FACE -> Said to apparently save face
APPREICIATION -> Expression of appreiciation or valuing individual persons.
WRONG -> Someone says something incorrect and it is taken as incorrect, that is, the response indicates that it is taken as wrong, including by ignoring it.
IRONY -> Taken (meant?) as ironic (can be hard to know).
METAPHOR -> Metaphoric language cf EMBODIMENT
STRUCTURAL METAPHOR -> Pimm: metaphoric extensions of ideas from within mathematics itself, so something like extending operations to larger (or other) sets.
SPATIAL-TEMPORAL METAPHOR -> Mathematical objects, such as points or lines are seen as varying in space and time, cf SLOW/FAST, MOVEMENT
SURPRISE -> Indicating surprise.
ELABORATION -> Further explaining and backing up what someone said.
ASSERTING -> A speaker (learner) asserts something mathematical.
CONVINCING -> A speaker is trying to convince someone of something mathematical
CONJECTURING -> A speaker conjectures something
PRAISE -> Someone praises something said or done.
CHECK -> Someone tells or leads someone to check (for correctness, if it works, ...)
"FREEZE" = STATICIZE -> Talk of a static situation instead of a dynamic one. Especially, in regard to a functional relationship.
IMAGINE = VISUALISE -> Ask to imagine, see in the head without an external representation.
OFFERING CHOICE -> Offers a choice of mathematical object or representational register.
"I AM LOST" -> Expression of being lost, totally not understanding or knowing.
“multiply point by number” -> A suggestion that often comes up is multiplying a point by a number, a kind of structural metaphor perhaps (Pimm), as it would work if a point is considered as a vector.
CALLING TEACHER -> A student or students call the teacher, asking him to come (usually a plea for "help")
TELLING -> Teacher tells student how it is.
RESEARCHER INTRUSION -> The researcher does or says something, affecting the situation, f. ex. directing the teacher's attention.
INSISTING & RESISTING -> The teacher insists on the learner answering or doing something, or the learner resists in some way, refusing or iterating a request for the teacher to tell.

G. An alternative analytic framework

A table for kinds of arguments

From informal and descriptive to formal and implicative. I abandoned this idea because it shows levels of mathematical language as a very clear ladder, emphasising "formal" mathematics as the external standard with which to compare students utterances, and because it does not consider dialogue, the fact that people build up ideas in interaction, together, and I came to see it as too restrictive to capture the very many ways in which students expressed their understandings. Their ideas did not come through!

Table G.1. Levels and types of arguments

Level	Type of argument/description
1	Only everyday words (or pointing) that describe how things look and where they are, and are dependent on shared attention (deixis). No implicative phrases (eg. because, therefore). "Naive language".
2	(Mainly) everyday words, (mostly) not dependent on deixis. No implicative phrases. Pre-argumentative language (reasons are not given), there is no argument.
3	A relation is perceived and articulated without reference to mathematical properties. "Because [what can be seen on the screen]". "Pre-mathematical" language. Reasons are given, but they refer to visual relations instead of mathematical relations.
4	Mathematical description but without reference to mathematical properties. Similar to above except with mathematical vocabulary.
5	Mathematical description with reference to mathematical facts (properties in a black box). "Because [mathematical connection]." This may be perfectly satisfactory, referring to known facts, unless it is expected that the learner comes up with arguments for these facts.
6	Mathematical description with reference to mathematical properties (in accord with the norms of the subject). "Because [mathematical deduction]." Similar to above but where the learner needs to make a substantial argument using mathematical properties.